Criticality of measures on 2-d Ising configurations: from square to hexagonal graphs

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Abstract

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1 Introduction and definitions

Let Λ be a two-dimensional $2L \times 2L$ square box centered in the origin in $\mathbb{Z}^2$ and $\mathcal{B}_\Lambda$ denotes the set of all nearest neighbours in Λ, i.e. $\{\langle x, y\rangle : \ x, y \in \Lambda, \ |x - y| = 1\}$ with $|x-y|$ being the usual lattice distance in $\mathbb{Z}^d$, plus the pairs of sites at opposite faces of the square Λ, so that the pair $(\Lambda, \mathcal{B}_\Lambda)$ is homeomorphic to the two-dimensional discrete torus $(\mathbb{Z}/L\mathbb{Z})^2$. We denote by $\mathcal{X}_\Lambda$ the set of spin configurations in Λ., i.e., $\mathcal{X}_\Lambda = \{-1, 1\}^\Lambda$. On this spin configuration space we can consider the simplest Ising Hamiltonian

$$H(\sigma) = - \sum_{\langle x,y\rangle \in \mathcal{B}_\Lambda} J\sigma_x\sigma_y \tag{1}$$

with $J > 0$, and the associated Gibbs measure

$$\pi^G(\sigma, \tau) := \frac{1}{Z^G} e^{-H(\sigma)} \qquad \text{with} \qquad Z^G = \sum_{\sigma \in \mathcal{X}_\Lambda} e^{-H(\sigma)}. \tag{2}$$

Looking for efficient algorithm for sampling from this measure, we introduced in [5] an approximate sampling by means of a pair Hamiltonian, adaptable to general pair interaction. The main idea was indeed to define a parallel dynamics, i.e., a Markov chain updating all the spin at each time, with an invariant measure strictly related to π^G. In a second paper [6], following these ideas, we defined a non reversible parallel dynamics with polynomial mixing time in the size of the system. The main ingredient was the combination of parallel updating and non symmetric interaction.

Goal of the present paper is to study pair Hamiltonians from a static point of view. Actually pair Hamiltonians turn out to be an important tool to relate Ising models on different lattices. Using the standard coupling between Ising model and Random Cluster Model (RCM) we can compare the correlations on different lattices and discuss the efficacy of the approach with pair Hamiltonian.

Define the space of pairs of configurations

$$\mathcal{X}^2_\Lambda = \mathcal{X}_\Lambda \times \mathcal{X}_\Lambda.$$

For each pair $(\sigma, \tau) \in \mathcal{X}^2_\Lambda$ we define the Hamiltonian with asymmetric interaction

$$H(\sigma,\tau) = -\sum_{x\in\Lambda}[J\sigma_x(\tau_{x^\uparrow}+\tau_{x^\rightarrow})+q\sigma_x\tau_x] = -\sum_{x\in\Lambda}[J\tau_x(\sigma_{x^\downarrow}+\sigma_{x^\leftarrow})+q\tau_x\sigma_x] \tag{3}$$

where $x^\uparrow, x^\rightarrow, x^\downarrow, x^\leftarrow$ are respectively the up, right, down, left neighbours of the site x on the torus $(\Lambda, \mathcal{B}_\Lambda^{per})$, $J > 0$ is the ferromagnetic interaction and $q > 0$ is an inertial constant. We have $H(\sigma,\sigma) = H(\sigma) - q|\Lambda|$ where $H(\sigma)$ is the Ising Hamiltonian given in (1). Note also that $H(\sigma,\tau) \neq H(\tau,\sigma)$.

On the configuration space $\mathcal{X}_\Lambda$ we define the following measure

$$\pi(\sigma) = \frac{1}{Z}\sum_{\tau\in\mathcal{X}_\Lambda} e^{-H(\sigma,\tau)} \qquad \text{with} \qquad Z = \sum_{(\sigma,\tau)\in\mathcal{X}_\Lambda^2} e^{-H(\sigma,\tau)}. \tag{4}$$

This measure has been considered in the previous papers [5], [6], [15] and turns out to be the invariant measure of parallel irreversible dynamics defined in that papers. In a more recent paper [**?**] the measure $\pi(\sigma)$ is the invariant measure of a reversible parallel dynamics, the “shaken dynamics”, modelling geological dynamics related to earthquakes.

The usual Gibbs measure (2) and the measure $\pi(\sigma)$ defined above are connected by the following result obtained in [5], [15] (Actually this is an extension of Theorem 1.2 in [15]****):

Theorem 1.1 *Define the total variation distance, or L_1 distance, between π and π^G as*

$$\|\pi - \pi^G\|_{TV} = \frac{1}{2}\sum_{\sigma\in\mathcal{X}_\Lambda}|\pi(\sigma) - \pi^G(\sigma)|. \tag{5}$$

Set $\delta = e^{-2q}$, and let δ be such that

$$\lim_{|\Lambda|\to\infty} \delta^2|\Lambda| = 0, \tag{6}$$

then there exists $\bar{J}$ such that for any $J > \bar{J}$

$$\lim_{|\Lambda|\to\infty} \|\pi - \pi^G\|_{TV} = 0 \tag{7}$$

Let us observe that the pair Hamiltonian (1), considering only half of the interactions

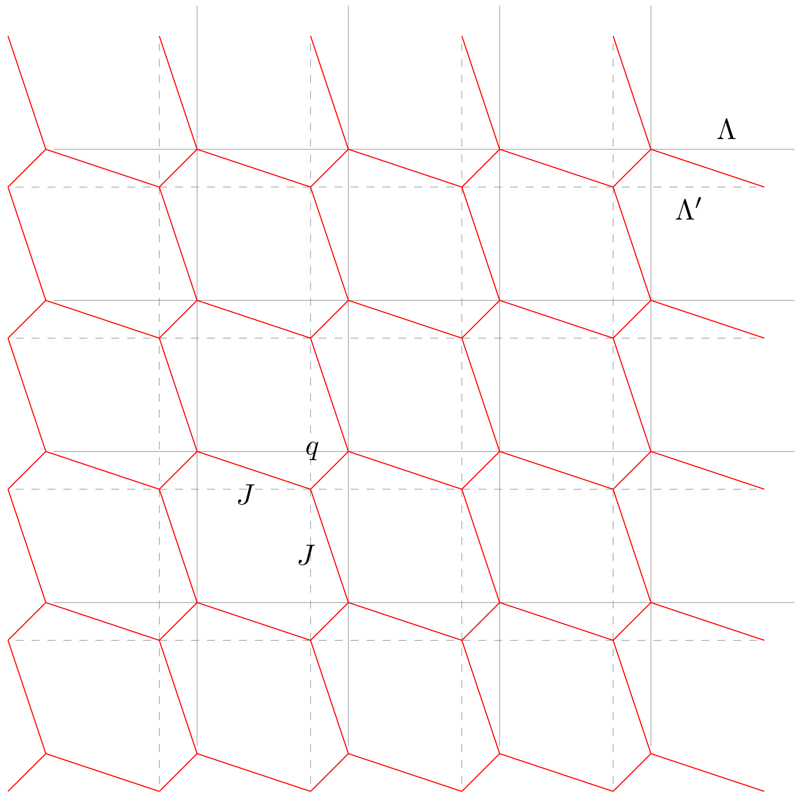

Figure 1: Interaction in the pair Hamiltonian

(down-left), allows to interpolate between different lattices. Indeed, as already shown in [ADSST], the space of pairs of configurations with interaction given by $H(\sigma, \tau)$ can be represented as the configuration space $\mathcal{X}_{\mathbb{H}}$ for the Ising model on an hexagonal lattice $\mathbb{H} = (V, E)$. Since $\mathbb{H}$ is a bipartite graph, the vertex set V of $\mathbb{H}$ can be decomposed into two layers $V = \Lambda^1 \cup \Lambda^2$, with $|\Lambda^i| = |\Lambda|,\ i = 1, 2$ and each $\boldsymbol{\sigma} \in \mathcal{X}_{\mathbb{H}}$ can be written as $\boldsymbol{\sigma} = (\sigma^1, \sigma^2)$ with $\sigma^i \in \mathcal{X}_{\Lambda^i, B},\ i = 1, 2$. We distinguish two type of edges, $E = E_J \cup E_q$, indeed two of the three edges exiting from each site correspond to the left and downwards interactions of strength J (in the set E_J), while the third corresponds to the self-interaction q (in the set E_q).

In other words we associate to each edge e a weight

$$J_e = \begin{cases} J & \text{if } \ e \in E_J \\ q & \text{if } \ e \in E_q \end{cases}$$

We can apply to our model the powerful connection between Ising model and Random Cluster Model.

Define $\Omega := \{0,1\}^E$, for any $\omega \in \Omega$ the edge e is opened (or present) if $\omega(e) = 1$, let $\eta(\omega) := \{e \in E : \ \omega(e) = 1\}$. We will assume periodic boundary conditions. Let $k(\omega)$ denotes the number of connected components (or open clusters) of the graph $(V, \eta(\omega))$. Given now two parameters $p_J, p_q \in [0,1]$, by defining

$$p_e = \begin{cases} p_J & \text{if } \ e \in E_J \\ p_q & \text{if } \ e \in E_q \end{cases}$$

we introduce the measure on Ω:

$$\Phi_{p_J,p_q}(\omega) = \frac{1}{Z^{RC}} \Big\{ \prod_{e \in E} p_e^{\omega(e)} (1-p_e)^{1-\omega(e)} \Big\} 2^{k(\omega)} \tag{8}$$

with partition function

$$Z^{RC} = \sum_{\omega \in \Omega} \Big\{ \prod_{e \in E} p_e^{\omega(e)} (1-p_e)^{1-\omega(e)} \Big\} 2^{k(\omega)}.$$

Following the general theory (see for instance [9]) we define now a coupling between our pairs of configurations $\boldsymbol{\sigma} = (\sigma^1, \sigma^2) \in \mathcal{X}_\Lambda^2$ and the random cluster by the following probability mass on $\mathcal{X}_\Lambda^2 \times \Omega$:

$$\mu(\boldsymbol{\sigma}, \omega) \propto \prod_{e \in E} \Big\{ (1-p_e) \delta_{\omega(e),0} + p_e \delta_{\omega(e),1} \delta_e(\boldsymbol{\sigma}) \Big\} \tag{9}$$

where

$$\delta_e(\boldsymbol{\sigma}) = \delta_{\sigma_x^1 = \sigma_y^2} \quad \text{ for } \quad e = (x,y), \text{ with } x \in \Lambda^1, y \in \Lambda^2$$

We have the following result:

Proposition 1.2 *If* $p_J = 1 - e^{-2J}$ *and* $p_q = 1 - e^{-2q}$

1) the marginal on $\mathcal{X}_\Lambda^2$ *of* $\mu(\boldsymbol{\sigma}, \omega)$ *is*

$$\mu_1(\boldsymbol{\sigma}) = \sum_{\omega \in \Omega} \mu((\boldsymbol{\sigma}), \omega) = \frac{e^{-H(\sigma^1, \sigma^2)}}{Z}$$

2) the marginal on Ω *of* $\mu(\boldsymbol{\sigma}, \omega)$ *is*

$$\mu_2(\omega) = \sum_{\boldsymbol{\sigma} \in \mathcal{X}^2_{\Lambda,B}} \mu(\boldsymbol{\sigma}, \omega) = \Phi_{p_J, p_q}(\omega)$$

3) the conditional measure on $\mathcal{X}^2_\Lambda$ *given* ω *is obtained by putting uniformly random spins on entire clusters of* ω. *These spins are constant on given clusters, are independent between clusters and each is uniformly distributed on the set* $\{-1, +1\}$.

4) the conditional measure on Ω *given* $\boldsymbol{\sigma}$ *is obtained by setting* $\omega(e) = 0$ *if* $\delta_e(\boldsymbol{\sigma}) = 0$ *and otherwise* $\omega(e) = 1$ *with probability* p_J *(*p_q*) for* $e \in E_J$ *(*$e \in E_q$*).*

We refer to the clear review by Grimmett [11] (see also [12]) of the Fortuin-Kasteleyn construction [**?**],[**?**], and to the rich paper [**?**] for further developments. The coupling between these two models is robust and of wide applicability, in particular in [11] the infinite-volume random-cluster measure and phase transitions are widely discussed. With this construction we can easily prove the existence of a phase transition in our model and compute the correlation function and its strong anisotropy.

For any x, $y \in V$ we will denote by $\{x \leftrightarrow y\}$ the set of $\omega \in \Omega$ for which there exists an open path joining the vertex x with the vertex y.

2 Results

The measure π, even though not Gibbsian, turns out to be the marginal of a Gibbs measure and inherits from it the thermodynamics. **da migliorare**

Theorem 2.1 *The measure* π*, defined in* (4)*, is not Gibbsian but it is the marginal of the Gibbs measure on the hexagonal lattice*

$$\pi_2(\sigma^1, \sigma^2) := \frac{1}{Z} e^{-H(\sigma^1, \sigma^2)} \qquad \textit{with} \qquad Z = \sum_{(\sigma^1, \sigma^2) \in \mathcal{X}^2_\Lambda} e^{-H(\sigma^1, \sigma^2)}. \tag{10}$$

The following relations hold:

1)

$$m := \pi\Big(\frac{\sum_{x\in\Lambda}\sigma_x}{|\Lambda|}\Big) = m_2 := \pi_2\Big(\frac{\sum_{x\in\Lambda^1\cup\Lambda^2}\boldsymbol{\sigma}_x}{2|\Lambda|}\Big)$$

2) Let π^+ (π^-) and π_2^+ (π_2^-) be the previous measures with plus (minus) boundary conditions, then for any $x \in \Lambda$

$$\pi^{\pm}(\sigma_x) = \pi_2^{\pm}(\sigma_x^1) = \pm\Phi_{p_J,p_q}(x^1 \leftrightarrow \partial\Lambda)$$

3)

$$\pi(\sigma_x\sigma_y) = \Phi_{p_J,p_q}(x^1 \leftrightarrow y^1)$$

with the obvious notation x^1, $y^1 \in \Lambda^1$.

It is well known that the Gibbs measure π_G exhibits a phase transition at

$$J_c^G = \tanh^{-1}\big(\sqrt{2}-1\big) = 0.441...$$

As far as the measure π is concerned, its critical behavior is described by the following:

Corollary 2.2 *The critical equation relating the parameters J and q in the measure π is given by the equation:*

$$J_c(q) = \tanh^{-1}\Big(-\tanh q + \sqrt{\tanh^2 q + 1}\Big) \tag{11}$$

In particular

$$\lim_{q\to\infty} J_c(q) = J_c^G$$

*and the curve $J_c(q)$ (see figure ***) intersects the line $J = q$ for $J = \tanh^{-1}\big(\frac{\sqrt{3}}{3}\big)$, corresponding to the critical value of J in the homogeneous hexagonal lattice.*

Note that the parameter q tunes the geometry of the system. Infact the limit $q \to 0$ corresponds to erasing the q-edges obtaining, from the hexagonal lattice, independent copies of 1-d Ising model. The opposite limit, $q \to \infty$, corresponds to the collapse of the hexagonal lattice into the square one, by identifying the sites connected by the q-edges. The case $J = q$ corresponds to the homogeneuous hexagonal graph.

The next and last result is about correlation functions and reflects the strong anisotropy of the model.

Theorem 2.3 *If the parameters p_J and p_q satisfy the following inequality*

$$\frac{4p_q p_J}{1-p_J} < 1$$

then there exist two constants $c_2 < c_1$ such that for any integer $\ell \in (0, L)$

$$\pi(\sigma_{x'}\sigma_{y'}) \le c_2 < c_1 \le \pi(\sigma_x \sigma_y)$$

where $x' = (0,0)$, $y' = (\ell,\ell)$, $x = (0,\ell)$, $y = (\ell, 0)$.

3 Proof of the results

3.1 Proof of theorem 2.1

1)
$$m = \sum_{(\sigma,\tau)} \frac{\sum_{x\in\Lambda}\sigma_x}{|\Lambda|} \cdot \frac{e^{-H(\sigma,\tau)}}{Z} = \frac{1}{2}\sum_{(\sigma,\tau)} \frac{\sum_{x\in\Lambda}(\sigma_x+\tau_x)}{|\Lambda|} \cdot \frac{e^{-H(\sigma,\tau)}}{Z} = m_2$$

where the second equality follows by the translation invariance of the lattice.

□

2)

$$\begin{aligned}
\pi^+(\sigma_x) &= \sum_\sigma \sum_\tau \sigma_x \pi^+(\sigma) = \sum_{\boldsymbol{\sigma}} \sigma_x^1 \pi_2^+(\boldsymbol{\sigma}) = \pi_2^+(\sigma_x^1) = \\
&= \sum_{\omega\in\Omega}\sum_{\boldsymbol{\sigma}} \mu(\boldsymbol{\sigma},\omega)\sigma_x^1\big(\mathbb{1}_{x^1\leftrightarrow\partial\Lambda} + \mathbb{1}_{x^1\not\leftrightarrow\partial\Lambda}\big) = \\
&= \Phi_{p_J,p_q}(x^1 \leftrightarrow \partial\Lambda) + \sum_{\omega\in\Omega}\sum_{\boldsymbol{\sigma}}\Big[\mu(\boldsymbol{\sigma},\omega|\omega)\sigma_x^1 \mathbb{1}_{x^1\not\leftrightarrow\partial\Lambda}\Big]\Phi_{p_J,p_q}(\omega) = \\
&= \Phi_{p_J,p_q}(x^1 \leftrightarrow \partial\Lambda)
\end{aligned}$$

since by proposition 1.2 the square bracket vanishes. The minus boundary conditions can be treated in the same way.

□

3) The proof of point (3) can be obtained following the same argument.

□

3.2 Proof of corollary 2.2

*** commenti Roberto su Cimasoni e cella elementare ***

Since the hexagonal lattice $\mathbb{H}$ induced by the interaction (1) is a planar, non-degenerate locally-finite doubly periodic weighted graph we can use theorem 1.1 in [3] to derive the following critical equation relating the parameters J and q in the measure π_2

$$J_c(q) = \tanh^{-1}\left(-\tanh q + \sqrt{\tanh^2 q + 1} \right) \tag{12}$$

Observing that π_2 and π have the same partition function Z we can also argue that the same critical equation still applies for the measure π. This can be proved rigorously in terms of the Random Cluster Model. By theorem 2.1 this result can be immediately extended to the measure π.

□

3.3 Proof of theorem 2.3

Let $\gamma \subset E$ be a path of open edges between two vertices x, $y \in \Lambda$. We introduce the notation $\omega \supset \gamma$ to identify all the configurations $\omega \in \Omega$ such that $\omega(e) = 1$, $\forall e \in \gamma$. By definition

$$\begin{aligned}
\Phi_{p_J,p_q}(x \leftrightarrow y) &= \sum_{\gamma : x \leftrightarrow y} \sum_{\substack{\omega \in \Omega : \\ \omega \supset \gamma}} \Phi_{p_J,p_q}(\omega) = \\
&= \frac{1}{Z^{RC}} \sum_{\gamma : x \leftrightarrow y} \left(\prod_{e \in \gamma} p_e \right) \sum_{\omega' \in \{0,1\}^{E \setminus \gamma}} \left(\prod_{e \in E \setminus \gamma} p_e^{\omega'(e)} (1 - p_e)^{1 - \omega'(e)} \right) 2^{k(\omega' \cup \gamma)} = \\
&= \frac{1}{Z^{RC}} \sum_{\gamma : x \leftrightarrow y} \left(\prod_{e \in \gamma} p_e \right) Z_{\gamma'}
\end{aligned}$$

where we express a configuration $\omega \supset \gamma$ in terms of the union between the path γ itself, that is fixed, and a configuration $\omega' \in \{0,1\}^{E \setminus \gamma}$ and we introduce

$$Z_{\gamma'} = \sum_{\omega' \in \{0,1\}^{E \setminus \gamma}} \left(\prod_{e \in E \setminus \gamma} p_e^{\omega'(e)} (1 - p_e)^{1 - \omega'(e)} \right) 2^{k(\omega' \cup \gamma)}$$

Upperbound

Since $k(\omega) \geq k(\omega' \cup \gamma)$ we can state the following inequality for the partition function

$$Z^{RC} \geq \sum_{\omega\in\Omega}\left(\prod_{e\in\gamma} p_e^{\omega(e)}(1-p_e)^{1-\omega(e)}\right)\left(\prod_{e\in E\setminus\gamma} p_e^{\omega(e)}(1-p_e)^{1-\omega(e)}\right)2^{k(\omega\supset\gamma)} =$$
$$= \left(\prod_{e\in\gamma}\sum_{\omega\in\Omega} p_e^{\omega(e)}(1-p_e)^{1-\omega(e)}\right)Z_{\gamma'} = Z_{\gamma'}$$

This observation implies

$$\Phi_{p_J,p_q}(x \leftrightarrow y) \leq \sum_{\gamma:x\leftrightarrow y}\left(\prod_{e\in\gamma} p_e\right) \tag{13}$$

fino a qui x e y sono generici, da qui in poi x e y diventano proprio quelli del teorema

Now let us suppose to slice the lattice $\mathbb{H}$ as in the figure (**see figure**). It is easy to see that each path $\gamma : x' \leftrightarrow y'$ crosses a fixed number of slices separating x' and y'. We give an upper bound for the sum in (13) in terms of possible crossing-paths η that start in x' and stop in the slice which contains y'. Let $d = d(x', y')$ be the classical distance between x' and y' on the lattice. Note that each crossing-path is uniquely defined by the number of crossed slices l, that coincides with the number of edges $e \in E_q \cap \eta$, and by the set of steps $\eta_{x',y'} \subset \eta$ along the diagonal direction, or equivalently, along the edges $e \in E_J$. Therefore we can write

$$\Phi_{p_J,p_q}(x' \leftrightarrow y') \leq \sum_{l=d}^{\infty}\sum_{\eta_{x',y'}}(2p_q)^l\left(\sum_{n_l=1}^{\infty} p_J^{n_l}\right)^l = \sum_{l=d}^{\infty}\sum_{\eta_{x',y'}}(2p_q)^l\left(\frac{p_J}{1-p_J}\right)^l =$$
$$= \sum_{l=d}^{\infty}\binom{l}{\frac{l+d}{2}}\left(\frac{2p_qp_J}{1-p_J}\right)^l \leq \sum_{l=d}^{\infty}\left(\frac{4p_qp_J}{1-p_J}\right)^l$$

The last sum converges if and only if the parameters p_J and p_q satisfy the following condition

$$\frac{4p_qp_J}{1-p_J} < 1$$

that is

$$4(1-e^{-2q})(1-e^{-2J}) < 1$$

Lowerbound

We introduce the diagonal path γ^* in the figure (**see figure**) and $\bar{\gamma} = \gamma^* \cup \partial\gamma^*$. Let $d = d(x, y)$ be the classical distance between x and y on the lattice. Using Theorem (3.66) in [12] we can give a lower bound for the correlation function as follows

$$\Phi_{p_J,p_q}(x \leftrightarrow y) \geq \frac{1}{Z^{RC}} \left(\prod_{e \in \gamma^*} p_e \right) \sum_{\omega'' \in \{0,1\}^{E \setminus \bar{\gamma}}} \left(\prod_{e \in \partial\gamma^*} (1 - p_e) \right) \left(\prod_{e \in E \setminus \bar{\gamma}} p_e^{\omega''(e)} (1 - p_e)^{1-\omega''(e)} \right) 2^{k(\omega'')+1} =$$

$$= \frac{1}{Z^{RC}} \left(\prod_{e \in \gamma^*} p_e \right) \left(\prod_{e \in \partial\gamma^*} (1 - p_e) \right) 2 Z_{E \setminus \bar{\gamma}} \geq$$

$$\geq 2 \left(\prod_{e \in \gamma^*} p_e \right) \left(\prod_{e \in \partial\gamma^*} (1 - p_e) \right) = 2 e^{-4J} (1 - e^{-2J})^d e^{-2q(d+1)} = c_1$$

where $Z_{E \setminus \bar{\gamma}}$ is the partition function of the Random Cluster Model defined on the graph $\mathbb{H}_{\bar{\gamma}} = (V, E \setminus \bar{\gamma})$.

□

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