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Saddle solution to the complex Ginzburg-Landau system in three dimensions

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Abstract

We construct an entire saddle solution to the non-magnetic complex Ginzburg-Landau system in three dimensions, whose zero set is the union of two perpendicular, intersecting lines, and whose blow-downs concentrate on these lines with multiplicity one.
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1. Introduction

In this work, we prove the existence of a smooth, saddle-type solution $u : \mathbb{R}^3 \to \mathbb{C}$ to the complex Ginzburg-Landau system

$$-\Delta u = u(1 - |u|^2), \tag{GL}$$

whose zero set $\mathcal{Z}(u) := \{u = 0\}$ (also called "vorticity set") is the union of the two perpendicular, intersecting lines

$$X := \{xy = 0\} \cap \{z = 0\}.$$

In addition, this solution enjoys particular symmetries (see Definition 2.1) and its blow-downs concentrate on X with multiplicity one, in the sense that (2) below holds.

Solutions to the system (GL) in a domain $\Omega \subset \mathbb{R}^n$ arise as critical points of the complex Ginzburg-Landau energy

$$E(u, \Omega) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{(1 - |u|^2)^2}{4}, \quad \text{for } u \in H^1(\Omega; \mathbb{C}). \tag{1}$$

With this notation, the main result of our work is the following.

Theorem 1.1. *There exists a smooth solution $u : \mathbb{R}^3 \to \mathbb{C}$ to (GL) with $|u| < 1$ and zero set $\mathcal{Z}(u) = X$. Moreover, u obeys the symmetries in Definition 2.1 and*

$$\lim_{r\to\infty} \frac{E(u, B_r)}{r \log(r)} = 4\pi. \tag{2}$$

In particular, in the varifold viewpoint for the normalized energy densities

$$\mu_r := \frac{e_{1/r}(u_r)}{\pi \log(r)} dx, \quad \text{where } e_{1/r}(u_r) := \frac{1}{2}|\nabla u_r|^2 + r^2 \frac{(1 - |u_r|^2)^2}{4}, \tag{3}$$

of the blow-downs $u_r(x) := u(rx)$, we obtain the following corollary.

Corollary 1.2. *The normalized energy densities* (3) *converge to a limit supported on* X *with unit density. That is*

$$\mu_r \rightharpoonup \mu_\infty := \mathcal{H}^1 \llcorner (X \cap B_1),$$

in $C_c^0(B_1)^*$ *as* $r \to +\infty$.

Our result provides the first construction of an entire saddle solution to the complex Ginzburg-Landau system in three dimensions with a singular zero set and can be thought of as a codimension two analogue of the classical work [12], where the authors build an entire saddle solution $u : \mathbb{R}^2 \to (-1, 1)$ of the Allen-Cahn equation $-\Delta u = u(1 - u^2)$ with zero set $\{xy = 0\}$.

We call our u a "saddle solution" since, as in the case of codimension one in [12], we expect it to be unstable. Indeed, the solution of [12] is unstable and it has index 1 (cf. [25]).

1.1. A brief overview of the Ginzburg-Landau system

The system (*GL*) and the energy functional (1) arise as a simplified version (with a vanishing magnetic field) of the Ginzburg-Landau model for superconductivity, and have attracted considerable attention in the mathematical community following the seminal work [3]. A particular effort has been devoted to the asymptotic analysis of the ε-rescaled versions

$$-\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2), \qquad E_\varepsilon(u, \Omega) := \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2}$$

both as a relaxation for the problem of finding singular $\mathbb{S}^1$-valued harmonic maps when there are no finite energy competitors, and as a codimension two analogue of the Allen-Cahn approximation of the area functional to find minimal surfaces.

After nearly two decades of research, and many highly influential papers [1,5,6,18,19,27], a rather complete comprehension of this problem (and of the pathologies it may display) came with the work of Pigati-Stern [23], whose main result is (partially) quoted in Theorem A.2.

A different but of course strongly related question concerns the characterization of entire solutions to (*GL*), under suitable assumptions on the behavior of u at infinity, either in terms of energy growth or of decay of $1 - |u|$.

In the two dimensional case, it is conjectured that, up to isometries, the only solutions such that $|u| \to 1$ at infinity are the planar degree $d \in \mathbb{Z}$ vortices, namely a sequence of solutions that can be written in polar coordinate in the form $v_d(r, \theta) = \rho_d(r)e^{id\theta}$, where $\rho_d : [0, +\infty) \to [0, 1]$ satisfies a suitable ODE (see [8,15]). This long-standing conjecture is still open, even though partial results under additional assumptions are known (see [7, Section 2]).

In three dimensions, very few nontrivial (that is, neither constant nor the straight extension of the planar vortices v_d) solutions of the complex Ginzburg-Landau system are known. To our knowledge, beyond these trivial examples, the only rigorously constructed entire nontrivial solutions to date are the interacting helical vortex filaments families built by Dávila-del Pino-Medina-Rodiac in [11], featuring a vorticity set of $n \geq 2$ interacting helices. These solutions also furnished a negative answer to the Ginzburg–Landau analogue of the Gibbons conjecture: whether an entire solution u of (*GL*) such that

$$\lim_{|z|\to\infty} |u(z,t)| = 1, \quad \text{uniformly in } t \in \mathbb{R}$$

is necessarily a function of z only.

In addition, regarding solutions in a bounded domain or a compact Riemannian manifold, two more families of solutions are known: the nearly parallel vortex filament solutions in bounded cylindrical domains of [9] and the solutions built in [10] concentrating on a nondegenerate geodesic in a 3-dimensional closed manifold.

Concerning Corollary 1.2, we point out that the integrality and unit-multiplicity of the blow-down varifold in Corollary 1.2 are nontrivial. Indeed, in dimensions $n \geq 3$, Pigati–Stern [23] proved that for general families of Ginzburg–Landau critical points within the natural $O(|\log(\varepsilon)|)$ energy regime, the energy–concentration varifold arising in the limit need not be integral: the limit density can take values in $\{1\} \cup [2, +\infty)$ and they construct examples realizing every prescribed density $\theta \geq 2$. Their examples are obtained suitably blowing down the interacting helical vortex filament solutions of [11]. In contrast, the solution given by our Theorem 1.1 yields a stationary integral varifold of unit density on each line.

We also remark that our solution is not an unconstrained local minimizer (i.e., it is not a minimizer of (1) in compact sets with its own boundary datum), so it does not contradict the conjecture (see [21, Open Problem 5] or [26]) that the only solutions of (GL) which are (unconstrained) local minimizers and satisfy

$$\limsup_{r\to\infty} \frac{E(u, B_r)}{r \log(r)} < +\infty,$$

are 2-dimensional vortices in some direction. That is, up to an isometry on $\mathbb{R}^3$ and an orthogonal transformation in $\mathbb{C}$ they can be written as $u(x, y, z) = v_d(x, y)$. By the symmetries of the zero set $\mathcal{Z}(u) = X$ of our solution, it is clear that it cannot be written in such a form. Nevertheless, by construction, u is a local minimizer in a suitable space with symmetries $\widetilde{H}^1(\mathbb{R}^3; \mathbb{C})$ as defined in Definition 2.1.

Finally, let us mention several possible questions which naturally arise in view of our result: determining the Morse index and stability of our saddle solution, extending the construction to other symmetric line-networks or periodic arrays, and clarifying uniqueness within the symmetry class, together with a classification of entire solutions at the 4π energy growth regime (cf. [26]).

Moreover, we remark that our solution can trivially be extended to higher dimensions, providing examples of entire solutions in $\mathbb{R}^n$ whose zero set consists of two orthogonal $(n-2)$-planes intersecting along an $(n-3)$-plane. We expect that it should be possible to adapt our method to find further classes of solutions in higher dimension (or codimension) whose zero set consists of different arrangements of subspaces (for example, two 2-planes intersecting in a point in $\mathbb{R}^4$).

1.2. *Structure of the proof*

Let us briefly describe the strategy we use to prove the existence of u in Theorem 1.1. The first part of the proof is quite standard, while the second part, in which we prove (2), requires a nontrivial idea due to the lack of a monotonicity formula at the correct rate (see below).

First, in Section 2 we introduce the precise setting of symmetric functions, then in Section 3, working in a suitable Sobolev space with symmetries $\widetilde{H}^1(B_R; \mathbb{C})$ (defined in Definition 2.1) that force every competitor to vanish on X, we minimize the Ginzburg-Landau functional (1) in $B_R := B_R(0) \subset \mathbb{R}^3$ with a generic Dirichlet boundary condition and we prove that a minimizer

u_R exists, solves (GL) in the entire ball B_R, satisfies the symmetries in Definition 2.1, and vanishes exactly on $X \cap B_R$.

Then, in Section 4, we construct a suitable family of boundary data $g_R : \partial B_R \to \mathbb{C}$ for which the minimizers u_R satisfy

$$\limsup_{R\to\infty} \frac{E(u_R, B_R)}{R\log(R)} \leq 4\pi, \tag{4}$$

and in Section 5, we show that a subsequential limit of u_R as $R \to +\infty$ converges to a limit function $u : \mathbb{R}^3 \to \mathbb{C}$ that solves (GL), and we compute its degree around X.

Lastly, in Section 6, which is the most delicate part, we prove the sharp energy bound (2) for our solution u, by a careful analysis of how the energy of u_R propagates at scales smaller than R. This crucial step presents some specific difficulties for the (non-magnetic) Ginzburg-Landau energy (1). Indeed, in our setting the usual monotonicity formula (see, for example, [5, Lemma II.2]) states that

$$\frac{E(u_R, B_r)}{r} \leq \frac{E(u_R, B_R)}{R}, \quad \forall r \leq R,$$

so it propagates energy from a large ball down to smaller scales with the wrong rate, and it cannot recover the optimal $r\log(r)$ growth. More precisely, from (4) one only gets $E(u_R, B_r) \lesssim r\log(R)$; letting $R \to \infty$ makes the right-hand side blow up, so this yields no growth information on the energy of u in B_r. This makes our problem totally different from the codimension-one Ginzburg-Landau or the Yang-Mills-Higgs cases, where the monotonicity formula is compatible with the natural scaling and directly transmits sharp bounds across smaller radii (see [16,17] for monotonicity formulas in the scalar case and [13,22] for the Yang-Mills-Higgs case).

This mismatch forces an ad hoc propagation scheme for minimizers: in an annulus $B_{(1+\delta)r} \setminus B_r$ we patch u_R with competitors that encode the spherical boundary phase of g_R and a carefully chosen modulus, leading to a decay estimate that is (essentially)

$$E(u, B_r) \leq 4\pi r\log(r) + C\sqrt{r\, E(u, B_{(1+\delta)r})}.$$

From here, an elementary iteration lemma (Lemma 6.4) then converts this into the global upper bound $E(u, B_r) \leq 4\pi r\log(r) + o(r\log(r))$.

Finally, once the sharp upper bound is established, the corresponding lower bound and the characterization of the blow-down follow quite easily from some deep results from [23], which we recall in the appendix.

2. Preliminaries and notation

We begin by introducing some notation. For $n \geq 3$, we denote by $B_R(x)$ a ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $R > 0$. When $n = 2$, we denote by $D_R(x)$ the two-dimensional disk of radius $R > 0$ and center $x \in \mathbb{R}^2$. In the case of $x = 0$, we denote for short

$$B_R = B_R(0), \quad D_R = D_R(0).$$

We describe now the setting of the function spaces with symmetries that we will work with. Let the coordinate reflections be

$$\mathcal{R}_x(x,y,z) = (-x,y,z), \qquad \mathcal{R}_y(x,y,z) = (x,-y,z), \qquad \mathcal{R}_z(x,y,z) = (x,y,-z),$$

and let $H^1(B_R;\mathbb{C})$ be the standard Sobolev space $W^{1,2}(B_R;\mathbb{C})$ for functions $u : B_R \subset \mathbb{R}^3 \to \mathbb{C}$.

Definition 2.1. We denote by $\widetilde{H}^1(B_R;\mathbb{C})$ the set of functions $u \in H^1(B_R;\mathbb{C})$ that satisfy the following symmetries almost everywhere in B_R:

$$u \circ \mathcal{R}_x = -\overline{u}, \qquad u \circ \mathcal{R}_y = -\overline{u}, \qquad u \circ \mathcal{R}_z = \overline{u}.$$

Remark 2.2. We point out that any continuous function $u = u_1 + iu_2 \in \widetilde{H}^1(B_R;\mathbb{C})$ necessarily satisfies

$$u_1 = 0 \text{ on } \{x=0\} \cup \{y=0\}, \quad u_2 = 0 \text{ on } \{z=0\}, \tag{5}$$

which implies in particular that $X \cap B_R \subset \mathcal{Z}(u)$. Actually, the same is true also for the precise representative of any $u \in \widetilde{H}^1(B_R;\mathbb{C})$, but we do not need this in the sequel.

Moreover, if $u \in C^1(B_R;\mathbb{C}) \cap \widetilde{H}^1(B_R;\mathbb{C})$, then it necessarily satisfies also

$$\begin{aligned} &\partial_x u_2 = 0, \qquad \text{on} \quad \{x=0\}, \\ &\partial_y u_2 = 0, \qquad \text{on} \quad \{y=0\}, \\ &\partial_z u_1 = 0, \qquad \text{on} \quad \{z=0\}. \end{aligned} \tag{6}$$

Let us set

$$Q := \{x>0\,, y>0\,, z>0\}\,, \text{ and } Q_R := Q \cap B_R,$$

and let also

$$\begin{aligned} \partial^s Q_R &:= \partial B_R \cap \partial Q_R, \\ \partial^x Q_R &:= \{x=0\} \cap \partial Q_R, \\ \partial^y Q_R &:= \{y=0\} \cap \partial Q_R, \\ \partial^z Q_R &:= \{z=0\} \cap \partial Q_R. \end{aligned}$$

Observe that the value of any $u \in \widetilde{H}^1(B_R;\mathbb{C})$ is completely determined by its values on the octant Q_R just by reflecting these values according to the symmetries in Definition 2.1. Vice versa, every map $v \in H^1(Q_R;\mathbb{C})$ that satisfies (5) can be extended to a function in $\widetilde{H}^1(B_R;\mathbb{C})$ by reflecting with our symmetries.

Moreover, for every $u \in \widetilde{H}^1(B_R;\mathbb{C})$ we have that

$$E(u, B_R) = 8E(u, Q_R).$$

3. Minimization in B_R for general boundary datum

This section is entirely dedicated to the proof of the following result.

Proposition 3.1. *Let $g = g_1 + ig_2 \in \widetilde{H}^{1/2}(\partial B_R; \mathbb{C})$ be such that $|g| \leq 1$, $g_1 \geq 0$ and $g_2 \geq 0$ on $\partial^s Q_R$. Then, the minimum*

$$\min \left\{ E(v, B_R) \,:\, v \in \widetilde{H}^1_g(B_R; \mathbb{C}) \right\} \tag{7}$$

is achieved by a smooth map $u \in C^\infty(B_R; \mathbb{C})$ that solves

$$\begin{cases} -\Delta u = u(1 - |u|^2) & \text{in } B_R, \\ u = g & \text{on } \partial B_R. \end{cases} \tag{8}$$

Moreover $u = u_1 + iu_2$ satisfies $|u| < 1$, $u_1, u_2 > 0$ in Q_R and $\mathcal{Z}(u) = X \cap B_R$.

Remark 3.2. It is a simple consequence of the strong maximum principle that every solution of (8), not only the minimizer, with $|g| \leq 1$ on ∂B_R satisfies $|u| \leq 1$. Indeed, the function $v := |u|^2 - 1$ solves

$$-\Delta v + 2|u|^2 v \leq 0, \quad \text{in } B_R.$$

Since $v = |g|^2 - 1 \leq 0$ on ∂B_R and $2|u|^2 \geq 0$, by the strong maximum principle $v = |u|^2 - 1 < 0$ in B_R, unless $|u| \equiv 1$, which is impossible because of (5).

Proof of Proposition 3.1. The proof will be divided into four steps.

Step 1. There exists a minimizer $u = u_1 + iu_2 \in \widetilde{H}^1_g(B_R; \mathbb{C})$ of the minimum problem (7), and satisfies $|u| \leq 1$ and $u_1, u_2 \geq 0$ in Q_R.
Step 2. Any minimizer u is smooth in the octant Q_R and solves $-\Delta u = u(1 - |u|^2)$ in Q_R.
Step 3. Any minimizer u is smooth and is a solution of $-\Delta u = u(1 - |u|^2)$ in all of B_R. That is, it solves $-\Delta u = u(1 - |u|^2)$ also across the faces of Q_R, across the axes, and at the origin.
Step 4. The zero set of u is exactly $X \cap B_R$.

Proof of Step 1. Since $E(\cdot, B_R) \geq 0$ is nonnegative, let $\{v_k\} \subset \widetilde{H}^1_g(B_R; \mathbb{C})$ be a minimizing sequence. First, we claim that, up to replacing v_k with another minimizing sequence, we can assume $|v_k| \leq 1$ and $(v_k)_1, (v_k)_2 \geq 0$ in Q_R.

Indeed, replacing v_k with $w_k := |(v_k)_1| + i|(v_k)_2|$ in Q_R and reflecting with the symmetries in Definition 2.1, we get that $\{w_k\} \subset \widetilde{H}^1_g(B_R; \mathbb{C})$ is still a minimizing sequence since $E(v_k, B_R) = E(w_k, B_R)$. Furthermore, replacing w_k with $\widetilde{w}_k := w_k \min\{1, 1/|w_k|\}$ achieves $|\widetilde{w}_k| \leq 1$ in B_R and is still a minimizing sequence. Indeed, for the potential part of the Ginzburg-Landau energy

$$\int_{B_R} \frac{(1 - |\widetilde{w}_k|^2)^2}{4} \leq \int_{B_R} \frac{(1 - |w_k|^2)^2}{4}, \tag{9}$$

since $|\widetilde{w}_k| = \min\{|w_k|, 1\}$ by definition. For the Dirichlet part

$$\int_{B_R} |\nabla \widetilde{w}_k|^2 = \int_{B_R} |\nabla |\widetilde{w}_k||^2 + |\widetilde{w}_k|^2 \left|\nabla \frac{\widetilde{w}_k}{|\widetilde{w}_k|}\right|^2 \leq \int_{B_R} |\nabla |w_k||^2 + |w_k|^2 \left|\nabla \frac{w_k}{|w_k|}\right|^2 = \int_{B_R} |\nabla w_k|^2. \quad (10)$$

Putting together (10) and (9) we obtain that

$$E(\widetilde{w}_k, B_R) \leq E(w_k, B_R).$$

Hence $\{\widetilde{w}_k\}$ is also a minimizing sequence, and possibly replacing v_k with $\widetilde{w}_k$ we assume that v_k has $|v_k| \leq 1$ and $(v_k)_1, (v_k)_2 \geq 0$ in Q_R.

Since $E(v_k, B_R) \leq C$ and $|v_k| \leq 1$, we have that $\|v_k\|_{H^1(B_R;\mathbb{C})} \leq C(R)$ is uniformly bounded in k. Hence, there exists a subsequence (that we do not relabel) and $u \in H^1(B_R;\mathbb{C})$ such that $v_k \rightharpoonup u$ weakly in $H^1(B_R)$, $v_k \to u$ in $L^2(B_R)$ and almost everywhere. Since $\widetilde{H}^1_g(B_R;\mathbb{C})$ is a closed subspace of $H^1(B_R)$, we also have $u \in \widetilde{H}^1_g(B_R;\mathbb{C})$. By the lower-semicontinuity of the seminorm under weak convergence and Fatou's lemma

$$E(u, B_R) \leq \liminf_{k\to\infty} E(v_k, B_R) = \inf_{v \in \widetilde{H}^1_g(B_R;\mathbb{C})} E(v, B_R),$$

thus u is a minimizer in $\widetilde{H}^1_g(B_R;\mathbb{C})$. Moreover, by the a.e. convergence of v_k to u we also get $|u| \leq 1$ and $u_1, u_2 \geq 0$ in Q_R. □

Proof of Step 2. Choose any $\varphi \in C^\infty_c(Q_R, \mathbb{C})$. Reflecting φ with the symmetries of Definition 2.1 produces a test function $\Phi \in C^\infty_c(B_R, \mathbb{C}) \cap \widetilde{H}^1_0(B_R;\mathbb{C})$. Hence for every $\varepsilon \in (-1, 1)$ we have $u + \varepsilon\Phi \in \widetilde{H}^1_g(B_R;\mathbb{C})$, and by the minimality of u in $\widetilde{H}^1_g(B_R;\mathbb{C})$ we immediately get

$$\begin{aligned} 0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u + \varepsilon\Phi, B_R) &= \int_{B_R} \nabla u \cdot \nabla \Phi + \int_{B_R} (|u|^2 - 1) u \cdot \Phi \\ &= 8 \left(\int_{Q_R} \nabla u \cdot \nabla \varphi + \int_{Q_R} (|u|^2 - 1) u \cdot \varphi \right). \end{aligned}$$

Hence, u is a weak solution of $-\Delta u = u(1 - |u|^2)$ in Q_R, and by standard elliptic regularity, u is smooth and is a strong solution in Q_R. □

Proof of Step 3. Our goal is to show that u satisfies the weak form of the full Ginzburg-Landau system on the whole ball B_R with no symmetry restriction on the test functions. This follows from the Palais's Principle of Symmetric Criticality [20] in the case of a 0-dimensional finite Lie group. Our setting is so concrete and specific that we prefer giving a short proof of this fact explicitly instead of relying on this abstract principle. Once it is proved that u is an unconstrained weak solution in the interior of B_R, standard interior elliptic regularity yields smoothness across the planes.

Fix $\varphi \in C_c^\infty(B_R;\mathbb{C})$. We stress that we do not impose any symmetry on φ. Exploiting the symmetries of u and many changes of variables, we find that

$$\int_{B_R} \nabla u \cdot \nabla \varphi - (1-|u|^2)u \cdot \varphi = \int_{Q_R} \nabla u \cdot \nabla \Phi - (1-|u|^2)u \cdot \Phi, \tag{11}$$

where

$$\Phi := \varphi - \overline{\varphi} \circ \mathcal{R}_x - \overline{\varphi} \circ \mathcal{R}_y + \overline{\varphi} \circ \mathcal{R}_z + \varphi \circ \mathcal{R}_x \circ \mathcal{R}_y - \varphi \circ \mathcal{R}_x \circ \mathcal{R}_z - \varphi \circ \mathcal{R}_y \circ \mathcal{R}_z + \overline{\varphi} \circ \mathcal{R}_x \circ \mathcal{R}_y \circ \mathcal{R}_z.$$

Now we observe that Φ satisfies the symmetries in Definition 2.1, namely $\Phi \in \widetilde{H}^1(B_R;\mathbb{C}) \cap C_c^\infty(B_R;\mathbb{C})$, and hence it is an admissible variation to test the criticality of u. As a consequence, from (11) we deduce that

$$\int_{B_R} \nabla u \cdot \nabla \varphi - (1-|u|^2)u \cdot \varphi = \frac{1}{8} \int_{B_R} \nabla u \cdot \nabla \Phi - (1-|u|^2)u \cdot \Phi = 0,$$

namely, u is a weak solution of (*GL*) in all of B_R, and by standard regularity theory, it is also a strong solution in all of B_R. This finishes the proof of Step 3. □

Proof of Step 4. Recall that every function $v \in \widetilde{H}^1_g(B_R;\mathbb{C})$ vanishes on $X \cap B_R$. Hence, to prove Step 4, it is enough to show that u does not vanish outside this set.

Taking the real part in (8) gives

$$\mathcal{L}u_1 := \Delta u_1 + c(x)u_1 = 0, \quad \text{in } Q_R,$$

where $c(x) = 1 - |u|^2 \geq 0$ since $|u| \leq 1$ in Q_R. Then, by the strong maximum principle $u_1 > 0$ in Q_R unless $u_1 \equiv 0$ in Q_R, but since $u_1 = g_1$ on $\partial^s Q_R$ and g_1 does not vanish identically on $\partial^s Q_R$, we get that $u_1 > 0$. Moreover, since $u_2 \geq 0$ in Q_R, a completely similar argument with the maximum principle implies that also $u_2 > 0$ in Q_R.

Now, assume by contradiction that there exists $x_\circ \in B_R \setminus X$ such that $u(x_\circ) = 0$. We distinguish two cases.

> **Case 1.** $x_\circ \in B_R \setminus \{z = 0\}$. By the symmetries, we can assume $x_\circ \in B_R \cap \{z > 0\}$. Since $u_2 \geq 0$ on $B_R \cap \{z > 0\}$ and $\mathcal{L}u_2 = 0$ on this set, by the strong maximum principle we would get $u_2 \equiv 0$ in $B_R \cap \{z > 0\}$, which is impossible for our boundary datum g.
> **Case 2.** $x_\circ \in B_R \cap \big(\{z = 0\}) \setminus X\big)$. By the symmetries, we can assume that $x_\circ \in \partial^z Q_R \setminus X$. Recall that $u_1 > 0$ in Q_R. By the symmetries in the definition of $\widetilde{H}^1_g(B_R;\mathbb{C})$ we have that u_1 is even across $\partial^z Q_R$ and $\partial_z u_1 = 0$ on $\partial^z Q_R$ by (6). Since $0 = u_1(x_\circ) < u_1(x)$ for every $x \in Q_R$, this contradicts Hopf's boundary point lemma (e.g., Lemma 3.4 in [14] applied to $-u_1$ and $\Omega = Q_R$).

This concludes the proof of Step 4. □

By Steps 1-4 above, we conclude the proof of Proposition 3.1. □

Remark 3.3. We could have obtained u with the same properties as above with a different, unconstrained minimization problem with mixed Dirichlet-Neumann boundary conditions. Indeed, for $u \in H^1(Q_R; \mathbb{C})$ consider the following set of boundary conditions on ∂Q_R:

$$\begin{cases} u_1 = 0, \ \partial_\nu u_2 = 0 & \text{on } \partial^x Q_R, \\ u_1 = 0, \ \partial_\nu u_2 = 0 & \text{on } \partial^y Q_R, \\ u_2 = 0, \ \partial_\nu u_1 = 0 & \text{on } \partial^z Q_R, \\ u = g & \text{on } \partial^s Q_R. \end{cases} \tag{12}$$

Now minimize the energy $E(u, Q_R)$ over $u \in H^1(Q_R; \mathbb{C})$ subject to (12). The standard direct method gives the existence of a minimizer u defined on Q_R, which can be extended to a $\widetilde{u} \in \widetilde{H}^1(B_R; \mathbb{C})$ reflecting according to the symmetries in Definition 2.1. Then, it can be checked that this $\widetilde{u}$ has exactly the same properties stated for u in Proposition 3.1. In particular, $\widetilde{u}$ solves (8) and has boundary datum g on ∂B_R.

Observe that from Remark 2.2 we deduce that our smooth solution u_R obtained in Proposition 3.1 satisfies (12), which is just the combination of (5) and (6).

Hence, both approaches produce a function with the same properties. Nevertheless, being the Ginzburg-Landau energy $E(\cdot, B_R)$ not convex due to the nonconvex potential term, in both minimization problems (the one on $\widetilde{H}^1_g(B_R; \mathbb{C})$ and the one on $H^1(Q_R; \mathbb{C})$ with mixed boundary conditions and then reflected according to the symmetries) the minimizer is generally not unique.

4. Construction of the boundary datum

In this section, we need to work with differential operators on the sphere, so we first introduce some notation. We denote by $\widetilde{D}_R(\theta)$ a spherical disk on the sphere $\mathbb{S}^2$ with center $\theta \in \mathbb{S}^2$ and geodesic radius R, and by $\nabla^\top$ the Riemannian gradient on the sphere.

Let us set

$$V := \mathbb{S}^2 \cap X = \{\pm e_1, \pm e_2\},$$
$$d_V(\theta) := \operatorname{dist}_{\mathbb{S}^2}(\theta, V) \quad \forall \theta \in \mathbb{S}^2 \qquad \text{and} \qquad d_X(x) := \operatorname{dist}_{\mathbb{R}^3}(x, X) \quad \forall x \in \mathbb{R}^3.$$

The objective of this section is to construct a family of boundary data

$$g_R : S_R := \partial B_R \to \mathbb{R}^2,$$

such that $|g_R| \leq 1$ and $(g_R)_1, (g_R)_2 > 0$ in $\partial^s Q_R$ (as required by Proposition 3.1), and satisfies the following properties.

- (i) g_R is Lipschitz continuous on S_R and satisfies the symmetries in Definition 2.1.
- (ii) $g_R(x) = \min\{d_X(x), 1\} g(x/|x|)$ for some $g \in \operatorname{Lip}_{\text{loc}}(\mathbb{S}^2 \setminus V; \mathbb{S}^1)$.
- (iii) The degree of g around $\pm e_1$ is $+1$, and the degree of g around $\pm e_2$ is -1.
- (iv) There exists a universal constant $C > 0$ such that

$$|\nabla^\top g(\theta)|^2 \leq \frac{1}{d_V(\theta)^2} + C \qquad \forall \theta \in \mathbb{S}^2. \tag{13}$$

Let $\psi_1 : \widetilde{D}_r(e_1) \to D_r \subset \mathbb{C}$ be geodesic coordinates such that

$$\psi_1(\widetilde{D}_r(e_1) \cap \{y = 0, z > 0\}) = D_r \cap \{\Re = 0, \Im > 0\},$$

and

$$\psi_1(\widetilde{D}_r(e_1) \cap \{z = 0, y > 0\}) = D_r \cap \{\Im = 0, \Re > 0\}.$$

We define g in $\widetilde{D}_r(e_1)$ by

$$g(\theta) = \frac{\psi_1(\theta)}{|\psi_1(\theta)|} \qquad \forall\, \theta \in \widetilde{D}_r(e_1),$$

and we observe that it satisfies $g \circ \mathcal{R}_z = \overline{g}$ and $g \circ \mathcal{R}_y = -\overline{g}$ (we remark that $\widetilde{D}_r(e_1)$ is invariant under $\mathcal{R}_z$ and $\mathcal{R}_y$). Then we define g also in $\widetilde{D}_r(-e_1) = -\widetilde{D}_r(e_1)$ by imposing the symmetry, namely by $g(\theta) = -\overline{g(\mathcal{R}_x\theta)}$ for every $\theta \in \widetilde{D}_r(-e_1)$. We observe that the degree of g around $\pm e_1$ is equal to $+1$.

Similarly, we now fix geodesic coordinates $\psi_2 : \widetilde{D}_r(e_2) \to D_r \subset \mathbb{C}$ such that

$$\psi_2(\widetilde{D}_r(e_2) \cap \{x = 0, z > 0\}) = D_r \cap \{\Re = 0, \Im > 0\},$$

and

$$\psi_2(\widetilde{D}_r(e_2) \cap \{z = 0, x < 0\}) = D_r \cap \{\Im = 0, \Re > 0\}.$$

Then we set

$$g(\theta) := -\frac{\overline{\psi_2(\theta)}}{|\psi_2(\theta)|} \qquad \forall\, \theta \in \widetilde{D}_r(e_2),$$

and we observe that it satisfies $g \circ \mathcal{R}_z = \overline{g}$ and $g \circ \mathcal{R}_x = -\overline{g}$ (we remark that $\widetilde{D}_r(e_2)$ is invariant under $\mathcal{R}_z$ and $\mathcal{R}_x$). Again, we define g also in $\widetilde{D}_r(-e_2) = -\widetilde{D}_r(e_2)$ by imposing the symmetry, namely by $g(\theta) = -\overline{g(\mathcal{R}_y\theta)}$ for every $\theta \in \widetilde{D}_r(-e_2)$. We observe that the degree of g around $\pm e_2$ is equal to -1.

At this point the function g is defined in four small balls around the four points in V, and it satisfies the symmetries in Definition 2.1.

Now we extend g to $\mathbb{S}^2$ in such a way that it is Lipschitz continuous away from V and still satisfies the symmetries. This can be done, for example, in the following way.

We consider the (closed) spherical octant $\overline{\partial^s Q_1}$, and we observe that g has already been defined in $(\overline{\partial^s Q_1} \cap \widetilde{D}_r(e_1)) \cup (\overline{\partial^s Q_1} \cap \widetilde{D}_r(e_2))$, that is the disjoint union of two quarters of disk in which g is smooth away from $\{e_1, e_2\}$, and hence can be written as $g = e^{i\phi}$ for some real-valued phase ϕ that is smooth away from $\{e_1, e_2\}$. From the definition of g we see that ϕ can be chosen to take values only in $[0, \pi/2]$, with $\phi = 0$ where $z = 0$ and $\phi = \pi/2$ where $x = 0$ or $y = 0$.

Now we observe that the boundary Γ of the set $\overline{\partial^s Q_1} \setminus (\widetilde{D}_r(e_1) \cup \widetilde{D}_r(e_2))$ in which g still needs to be defined consists of the two small arcs $\gamma_1 := \Gamma \cap \partial\widetilde{D}_r(e_1)$ and $\gamma_2 := \Gamma \cap \partial\widetilde{D}_r(e_2)$, on

which $g = e^{i\phi}$ has already been defined and is smooth, plus the three arcs $\gamma_x := \Gamma \cap \{x = 0\}$, $\gamma_y := \Gamma \cap \{y = 0\}$ and $\gamma_z := \Gamma \cap \{z = 0\}$.

If we now set

$$g = e^{i\phi} = \begin{cases} i = e^{i\pi/2} & \text{on } \gamma_x \cup \gamma_y, \\ 1 = e^{i0} & \text{on } \gamma_z, \end{cases}$$

then it turns out that ϕ is Lipschitz continuous on Γ, hence it can be extended to a Lipschitz continuous function $\phi : \overline{\partial^s Q_1} \setminus (\widetilde{D}_r(e_1) \cup \widetilde{D}_r(e_2)) \to [0, \pi/2]$, so that the map $g = e^{i\phi}$ belongs to $\mathrm{Lip}_{\mathrm{loc}}(\overline{\partial^s Q_1} \setminus \{e_1, e_2\})$. Moreover, this extension can be made so that $0 < \phi < \pi/2$ inside $\partial^s Q_1$, so that $g_1, g_2 > 0$ in $\partial^s Q_1$.

Finally, we extend g to $\mathbb{S}^2 \setminus V$ by reflection, imposing the symmetries of Definition 2.1. Since $g = i$ on $\overline{\partial^s Q_1} \cap (\{x = 0\} \cup \{y = 0\})$ and $g = 1$ on $\overline{\partial^s Q_1} \cap \{z = 0\}$, these reflections preserve the continuity across the planes $\{xyz = 0\}$, so the resulting function is still Lipschitz continuous away from V, and in particular $|\nabla^\top g|$ is bounded outside any neighborhood of V.

Moreover, inside the four disks $\widetilde{D}_r(e_1)$, $\widetilde{D}_r(e_2)$, $\widetilde{D}_r(-e_1)$, $\widetilde{D}_r(-e_2)$, the map g behaves like the standard vortex map in the plane, up to a small error due to the geodesic coordinates. Therefore, since in the plane

$$\left| \nabla \left(\frac{\zeta}{|\zeta|} \right) \right| = \frac{1}{|\zeta|} \qquad \forall \zeta \in \mathbb{C},$$

we deduce that

$$|\nabla^\top g(\theta)|^2 = \frac{1}{|\psi(\theta)|^2} + O(1) = \frac{1}{d_V(\theta)^2} + O(1),$$

for $\theta \in \mathbb{S}^2$ near V, and this proves (13).

We can now set $g_R(x) = \min\{d_X(x), 1\} g(x/|x|)$ and we observe that g_R satisfies the symmetries in Definition 2.1. Moreover, since $d_X(x) \le C R d_V(x/|x|)$ near $S_R \cap X$, then g_R is Lipschitz continuous around V, and hence on the whole sphere S_R. Therefore, the maps g_R and g satisfy all the required properties.

5. The entire solution

Now we consider the solution u_R provided by Proposition 3.1 with the boundary datum $g_R : \partial B_R \to \mathbb{C}$ constructed in the previous section, and we wish to send $R \to +\infty$ to obtain a solution in all of $\mathbb{R}^3$. The next result states that this actually works.

Proposition 5.1. *Let $u_R : B_R \to \mathbb{C}$ be the smooth minimizer of*

$$\min \left\{ E(v, B_R) : v \in \widetilde{H}^1_{g_R}(B_R; \mathbb{C}) \right\}$$

given by Proposition 3.1. Then, a discrete subsequence (not relabeled) satisfies

$$u_R \to u \ \text{ in } C^2_{\mathrm{loc}}(\mathbb{R}^3),$$

where u is a solution of (GL) in $\mathbb{R}^3$. Moreover

(i) $|u| < 1$ *in* $\mathbb{R}^3$ *and* $u_1, u_2 \geq 0$ *in* Q.
(ii) $u \not\equiv 0$ *in* $\mathbb{R}^3$ *and* $|u| > 0$ *in* $\mathbb{R}^3 \setminus X$.

Proof. By interior elliptic estimates (see for example Lemma A1 in [2]) in B_R we have

$$\|\nabla u_R\|^2_{L^\infty(B_{R/2})} \leq C\|u_R\|^2_{L^\infty(B_R)} \leq C, \tag{14}$$

for some universal constant $C > 0$. Here we have used that $|u_R| \leq 1$. Then, by Sobolev's embedding for some $p > 3$ and interior Schauder estimates

$$\|u_R\|_{C^{2,\alpha}(B_k)} \leq C\|u_R\|_{C^{0,\alpha}(B_{2k})} \leq C\|u_R\|_{W^{1,p}(B_{2k})} \leq C, \quad \forall k \in \mathbb{N} \quad \forall R \geq 4k,$$

for some $C = C(k) > 0$ independent of R. By the Arzelà-Ascoli theorem, a subsequence $\{u_{R_j}\}$ converges in $C^2(B_k)$ to a solution u_k in B_k. Taking a further diagonal subsequence that we do not relabel, we obtain a subsequence converging in $C^2_{\text{loc}}(\mathbb{R}^3)$ to a smooth solution $u \in C^2(\mathbb{R}^3)$ of (*GL*). Note also that, since in (14) the constant C is independent of R, letting $R \to \infty$ in (14) gives

$$|\nabla u(x)| \leq C, \quad \text{for all } x \in \mathbb{R}^3,$$

where C is an absolute constant.

Since every u_R given by Proposition 3.1 satisfies $|u_R| < 1$ in B_R and $(u_R)_1, (u_R)_2 > 0$ in Q_R, it follows that $|u| \leq 1$ and $u_1, u_2 \geq 0$ in Q. The fact that $|u| < 1$ easily follows by the maximum principle. Indeed, arguing as in Remark 3.2 by the strong maximum principle if $|u(x_\circ)| = 1$ for some $x_\circ \in \mathbb{R}^3$, then $|u| \equiv 1$ everywhere. But since u satisfies the symmetries in Definition 2.1 then $u = 0$ on X, this is impossible if $|u| \equiv 1$. This proves (i).

Now we prove (ii). Since the Ginzburg-Landau energy of $u \equiv 0$ in B_ρ is $C\rho^3$, to show that $u \not\equiv 0$ it is enough to prove that $E(u, B_\rho) = o(\rho^3)$. This follows by Proposition 6.3 below, which is proved without using any information on u apart from the fact that it is the limit (along a subsequence) in $C^2_{\text{loc}}(\mathbb{R}^3)$ of u_R; hence $u \not\equiv 0$.

We are left to prove that $|u| > 0$ in $\mathbb{R}^3 \setminus X$; but this follows by an argument identical to Step 4 in the proof of Proposition 3.1. □

5.1. *Further properties of the entire solution*

By (ii) in Proposition 5.1, we know that

$$\deg\left(\frac{u}{|u|}, \sigma\right)$$

is well-defined for loops σ not intersecting X.

Lemma 5.2. *Let* $\sigma : \mathbb{S}^1 \to \mathbb{R}^3 \setminus X$ *be an oriented circle orthogonal to the* x*-axis or* y*-axis. Then*

$$\left|\deg\left(\frac{u}{|u|}, \sigma\right)\right| = 1.$$

Proof. By the symmetries of u and g, it suffices to consider the case where σ is orthogonal to the x-axis; using the symmetry $x \mapsto -x$, we may also assume $\sigma \subset \{x > 0\}$.

Fix R large so that $\sigma \Subset B_R$, and let u_R be the sequence in Proposition 5.1 converging to the entire solution u. Since $u_R \to u$ in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$, we have uniform convergence on σ; together with $|u|, |u_R| > 0$ on $(\mathbb{R}^3 \setminus X) \cap B_R$ gives that

$$\deg\left(\frac{u}{|u|}, \sigma\right) = \deg\left(\frac{u_R}{|u_R|}, \sigma\right),$$

for R large.

Let

$$\sigma_R := \{ tp : \ t > 0, \ \ p \in \sigma \} \cap \partial B_R$$

be the radial (conical) projection of σ to ∂B_R. The cone joining σ to σ_R lies in $\{x > 0\} \setminus X$, so by homotopy invariance of the degree in the zero-free region of u_R, and by the continuity of u_R up to the boundary (which follows from the continuity of g_R) we have

$$\deg\left(\frac{u_R}{|u_R|}, \sigma\right) = \deg\left(\frac{u_R}{|u_R|}, \sigma_R\right) = \deg\left(\frac{g}{|g|}, \sigma_R\right) = \pm 1,$$

based on the orientation of σ_R, for R sufficiently large. This concludes the proof. □

A similar argument to the proof of Lemma 5.2 provides the degree of u for every loop $\sigma \subset \mathbb{R}^3 \setminus X$, and results in

$$\left|\deg\left(\frac{u}{|u|}, \sigma\right)\right| = \mathrm{link}(\sigma, \Gamma_1) + \mathrm{link}(\sigma, \Gamma_2),$$

where $\mathrm{link}(\cdot, \cdot)$ denotes the linking number and $\Gamma_1, \Gamma_2 : \mathbb{R} \to \mathbb{R}^3$ are the lines (with a corner)

$$\Gamma_1(t) := \begin{cases} (0, t, 0) & \text{if } t \le 0, \\ (t, 0, 0) & \text{if } t \ge 0, \end{cases} \qquad \text{and} \qquad \Gamma_2(t) := \begin{cases} (0, -t, 0) & \text{if } t \le 0, \\ (-t, 0, 0) & \text{if } t \ge 0. \end{cases}.$$

Equivalently, one could prove that the distributional Jacobian of $u/|u|$ is (up to the usual identification of 2-forms with 1-currents via the Hodge $\star$ operator) the 1-current induced by Γ_1 and Γ_2, with constant multiplicity equal to π.

Lemma 5.3. *The entire solution given by Theorem 1.1 is a local minimizer (i.e. in compact subsets with its own boundary datum) among competitors with the symmetries in Definition 2.1.*

Proof. Clearly it suffices to show that u is locally minimizing in every ball B_r for $r > 0$. In what follows, $C > 0$ denotes a general constant that depends only on r.

Recall that u given by Theorem 1.1 is the limit in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$ of the minimizers u_R of Proposition 3.1. Since u_R is a minimizer among symmetric competitors that agree with u_R on ∂B_r, for $R \ge r$ we have that

$$E(u_R, B_r) \leq E(w, B_r), \qquad \forall\, w \in \widetilde{H}^1(B_r; \mathbb{C}) \text{ with } w = u_R \text{ on } \partial B_r. \tag{15}$$

Let v be any symmetric competitor of u on B_r with $v = u$ on ∂B_r. Let U_R be the componentwise harmonic extension of $(u_R - u)|_{\partial B_r}$ to B_r. It can be checked that U_R satisfies the correct symmetries since the harmonic extension is unique. In particular, since $u_R \to u$ in $C^2(B_r)$ we have that $u_R \to u$ in $H^{1/2}(\partial B_r)$, which implies

$$\int_{B_r} |\nabla U_R|^2 = C[u_R - u]^2_{H^{1/2}(\partial B_r)} \to 0.$$

Moreover, by the Poincaré inequality

$$\int_{B_r} |U_R|^2 \leq 2\int_{B_r} |U_R - (u_R - u)|^2 + 2\int_{B_r} |u_R - u|^2 \leq C\int_{B_r} |\nabla U_R|^2 + C\|u_R - u\|^2_{H^1(B_r)} \to 0.$$

Hence $U_R \to 0$ in $\widetilde{H}^1(B_r; \mathbb{C})$ as $R \to \infty$.

Observe that $v + U_R$ equals u on ∂B_r by construction. Testing (15) with $w = v + U_R$ gives

$$E(u_R, B_r) \leq E(v + U_R, B_r).$$

By C^2 convergence, $E(u_R, B_r) \to E(u, B_r)$. Moreover, by H^1 convergence of U_R to zero and continuity of the potential term, we have that $E(v + U_R, B_r) \to E(v, B_r)$. Letting $R \to \infty$ gives $E(u, B_r) \leq E(v, B_r)$ as desired. □

Remark 5.4. We point out that u is not an unconstrained local minimizer. Indeed, if u were a local minimizer, then the same would be true for the blow-downs $u_r(x) := u(rx)$, so the convergence results for local minimizers in [1,18] would imply that the energy densities (3) concentrate on the support of an area minimizing current.

Instead, Corollary 1.2 states that the limit density is supported on $X \cap B_1$, which is not the support of an area minimizing current, since two oblique segments with the same boundary have a lower length.

6. Sharp energy asymptotics

In this section, we aim to prove (2), that is

$$\lim_{r\to\infty} \frac{E(u, B_r)}{r\log(r)} = 4\pi,$$

for the solution $u : \mathbb{R}^3 \to \mathbb{C}$ constructed in Section 5. In terms of the normalized energy densities, this implies that the limit in B_1 is the cross $X \cap B_1$ with density $\theta = 1$, namely

$$\frac{e_{1/r}(u_r)}{\pi\log(r)} \rightharpoonup \mathcal{H}^1 \llcorner (X \cap B_1) \quad \text{in } C_c^0(B_1)^*.$$

6.1. Upper bound

Before proving the optimal upper bound for the energy of u, we need an estimate for the energy of a natural competitor for the minimization problem (7) solved by u_R. This also provides a sharp estimate for the total energy of u_R in B_R.

Proposition 6.1. *Let $g : \mathbb{S}^2 \to \mathbb{C}$ be the map constructed in Section 4, let $m > 0$ be a positive real number and let us set*

$$v_m(x) := \min\{m \cdot d_X(x), 1\} g\left(\frac{x}{|x|}\right) \qquad \forall x \in \mathbb{R}^3.$$

Then, for every $r_2 > r_1 \geq 0$ it holds that

$$E(v_m, B_{r_2} \setminus B_{r_1}) \leq 4\pi(r_2 - r_1)\log r_2 + C(m) r_2,$$

where $C(m) > 0$ is a constant depending only on m.

Proof. Let us set, for brevity, $A := B_{r_2} \setminus B_{r_1}$ and, for every $\delta > 0$, let $\mathcal{N}_\delta := \{d_X \leq \delta\}$.

Then

$$\begin{aligned}
E(v_m, A) &= E(v_m, A \setminus \mathcal{N}_{1/m}) + E(v_m, A \cap \mathcal{N}_{1/m}) \\
&= \frac{1}{2} \int_{A \setminus \mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right)}{|x|^2} + \frac{1}{2} \int_{A \cap \mathcal{N}_{1/m}} m^2 |\nabla g d_X + g \nabla d_X|^2 + \int_{A \cap \mathcal{N}_{1/m}} \frac{(1 - m^2 d_X^2)^2}{4} \\
&\leq \frac{1}{2} \int_{A \setminus \mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right)}{|x|^2} + m^2 \int_{A \cap \mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right) d_X^2}{|x|^2} + \left(m^2 + \frac{1}{4}\right) |A \cap \mathcal{N}_{1/m}| \\
&\leq \frac{1}{2} \int_{A \setminus \mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right)}{|x|^2} + m^2 \int_{A \cap \mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right) d_X^2}{|x|^2} + C r_2, \qquad (16)
\end{aligned}$$

where here and in the rest of the proof C denotes a positive constant that may depend only on m, and whose precise value may vary from line to line.

We observe that

$$\frac{1}{|x|^2} |\nabla^\top g|^2\left(\frac{x}{|x|}\right) d_X^2(x) = |\nabla^\top g|^2\left(\frac{x}{|x|}\right) d_X^2\left(\frac{x}{|x|}\right) \leq \left(\frac{1}{d_V^2(x/|x|)} + C\right) d_X^2\left(\frac{x}{|x|}\right) \leq C, \quad (17)$$

where we have used (13) and that

$$\frac{d_X(\theta)}{d_V(\theta)} \leq C, \quad \text{for every } \theta \in \mathbb{S}^2.$$

Hence

$$\int_{A\cap\mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right) d_X^2}{|x|^2} \le C|A\cap\mathcal{N}_{1/m}| \le Cr_2.$$

Lastly, we have to estimate the first term in (16), which accounts for the highest order term in the energy. In polar coordinates, write

$$\frac{1}{2}\int_{A\setminus\mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right)}{|x|^2} = \frac{1}{2}\int_{r_1}^{r_2} d\rho \int_{\partial B_1\setminus\mathcal{N}_{1/(m\rho)}} |\nabla^\top g|^2(\theta)\, d\mathcal{H}^2(\theta). \tag{18}$$

By (13) and the coarea formula for every $\rho > 2/(m\pi)$ we have

$$\begin{aligned}\int_{\partial B_1\setminus\mathcal{N}_{1/(m\rho)}} |\nabla^\top g|^2(\theta)\, d\mathcal{H}^2(\theta) &\le \int_{\partial B_1\setminus\mathcal{N}_{1/(m\rho)}} \frac{1}{d_V(\theta)^2}\, d\mathcal{H}^2(\theta) + C\\ &\le \int_{1/(m\rho)}^{\pi/2} \frac{1}{t^2}\mathcal{H}^1(\{\theta\in\mathbb{S}^2 : d_V(\theta)=t\})\, dt + C,\end{aligned}$$

while of course the left-hand side vanish if $\rho < 2/(m\pi)$ (actually it is enough that $\rho < 1/m$).

Since the sphere $\mathbb{S}^2$ is positively curved, for $t\in(0,\pi/2)$ it holds

$$\mathcal{H}^1(\{\theta\in\mathbb{S}^2 : d_V(\theta)=t\}) \le 2\pi t\cdot 4 = 8\pi t.$$

Thus

$$\int_{\partial B_1\setminus\mathcal{N}_{1/(m\rho)}} |\nabla^\top g|^2(\theta)\, d\mathcal{H}^2(\theta) \le 8\pi \int_{1/(m\rho)}^{\pi/2} \frac{1}{t}\, dt + C = 8\pi\log(m\rho) + C \le 8\pi\log(\rho) + C,$$

and using this bound in (18) gives

$$\frac{1}{2}\int_{A\setminus\mathcal{N}_{1/m}} \frac{|\nabla^\top g|^2\left(\frac{x}{|x|}\right)}{|x|^2} \le \frac{1}{2}\int_{\max\{r_1,2/(m\pi)\}}^{r_2} \left(8\pi\log(\rho)+C\right) d\rho \le 4\pi(r_2-r_1)\log(r_2) + Cr_2.$$

Putting all the estimates together, we obtain the desired bound. □

Since $v_1\in\widetilde{H}^1_{g_R}(B_R;\mathbb{C})$, the previous result immediately yields the following.

Corollary 6.2. *Let $g_R\in\widetilde{H}^{1/2}(\partial B_R;\mathbb{C})$ be the boundary datum constructed in Section 4, and let u_R be a minimizer provided by Proposition 3.1. Then*

$$E(u_R, B_R) = \min\left\{E(v, B_R) \,:\, v\in\widetilde{H}^1_{g_R}(B_R;\mathbb{C})\right\} \le 4\pi R\log(R) + O(R). \tag{19}$$

As we explained in the introduction, the estimate (19) does not propagate directly to smaller scales, because of the different dependence on the radius in the monotonicity formula.

However, we can obtain the sharp upper estimate by a direct comparison argument for the minimizers u_R on B_R which, after letting $R \to \infty$ along the subsequence provided by Proposition 5.1, gives the claimed bound for the entire solution u. Of course, this argument exploits in an essential way the local minimality property of u_R, so it could not be applied to general critical points, as opposed to the monotonicity formula, which works for every solution of (GL).

Proposition 6.3. *Let u be the entire solution of Theorem 1.1 (that is, the one given by Proposition 5.1). Then*

$$\limsup_{r\to\infty} \frac{E(u, B_r)}{r \log r} \leq 4\pi.$$

Proof. Throughout this proof, $C > 0$ denotes an absolute constant independent of r and R (it may vary from line to line).

Since u_R is smooth and has $(u_R)_1, (u_R)_2 \geq 0$ in Q_R, we can write $u_R = \rho_R e^{i\varphi_R}$ in Q_R, for some smooth functions $\rho_R, \varphi_R : Q_R \to \mathbb{R}$. Similarly, we write $g = e^{i\phi}$, where $\phi : \partial^s Q_1 \to \mathbb{R}$ is the map constructed in Section 4, that we also extend to Q_R by $\phi(x) = \phi(x/|x|)$. We observe that $0 \leq \phi \leq \pi/2$ by construction, and also $0 \leq \varphi_R \leq \pi/2$ since u_R has nonnegative real and imaginary part in Q_R.

Moreover, since we know that $\rho_R = 0$ on X and (14) yields $|\nabla \rho_R| \leq |\nabla u_R| \leq m$ in $B_{R/2}$ for some absolute constant $m > 0$ (independent of R), we deduce that $\rho_R \leq \min\{1, m \cdot d_X\}$ in $B_{R/2}$.

Set

$$w(x) := \rho_R(x) g\left(\frac{x}{|x|}\right) \qquad \forall x \in B_R,$$

so in particular $w \in \widetilde{H}^1(B_R; \mathbb{C})$ and

$$w(x) = \rho_R(x) e^{i\phi(x)} \qquad \forall x \in Q_R.$$

Let us fix $r > 1$ and $\delta \in (0, 1)$. Let $\eta_1 : \mathbb{R}^3 \to [0, 1]$ be a smooth radial cutoff satisfying

$$\eta_1(x) = 1 \quad \forall x \in B_{r+\delta r}, \qquad \eta_1(x) = 0 \quad \forall x \in \mathbb{R}^3 \setminus B_{r+2\delta r}, \qquad |\nabla \eta_1(x)| \leq \frac{2}{\delta r} \quad \forall x \in \mathbb{R}^3.$$

First, we interpolate u_R and w. For $R > 2r + 4\delta r$, we consider the polar interpolation

$$Z_1(x) := \rho_R(x)\, e^{i\Phi(x)} \quad \text{on } A_1 := (B_{r+2\delta r} \setminus B_{r+\delta r}) \cap Q_R,$$

with

$$\Phi(x) := (1 - \eta_1(x))\varphi_R(x) + \eta_1(x)\phi(x),$$

and we extend Z_1 to a $B_{r+2\delta r} \setminus B_{r+\delta r}$ by the usual symmetries.

Using the polar form and $|\nabla(\rho e^{i\theta})|^2 = |\nabla\rho|^2 + \rho^2 |\nabla\theta|^2$ we have for the Sobolev part of the energy

$$\frac{1}{2}\int_{A_1}|\nabla Z_1|^2=\frac{1}{2}\int_{A_1}|\nabla\rho_R|^2+\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\Phi|^2. \tag{20}$$

Regarding the phase, we have

$$\nabla\Phi=(1-\eta_1)\nabla\varphi_R+\eta_1\nabla\phi+(\phi-\varphi_R)\nabla\eta_1,$$

hence

$$\begin{aligned}|\nabla\Phi|^2&=(1-\eta_1)^2|\nabla\varphi_R|^2+\eta_1^2|\nabla\phi|^2+|\nabla\eta_1|^2|\phi-\varphi_R|^2\\&\quad+2(1-\eta_1)\eta_1\nabla\varphi_R\cdot\nabla\phi+2(1-\eta_1)(\phi-\varphi_R)\nabla\varphi_R\cdot\nabla\eta_1+2\eta_1(\phi-\varphi_R)\nabla\phi\cdot\nabla\eta_1.\end{aligned}$$

Exploiting the estimate

$$2(1-\eta_1)\eta_1\nabla\varphi_R\cdot\nabla\phi\le\eta_1^2|\nabla\varphi_R|^2+(1-\eta_1)^2|\nabla\phi|^2,$$

and recalling that $|\nabla\eta_1|\le 2/(\delta r)$ and $|\phi-\varphi_R|\le\pi$, we find that

$$\begin{aligned}|\nabla\Phi|^2&\le\big((1-\eta_1)^2+\eta_1^2\big)|\nabla\varphi_R|^2+\big(\eta_1^2+(1-\eta_1)^2\big)|\nabla\phi|^2+\frac{C}{\delta^2r^2}+\frac{C}{\delta r}|\nabla\varphi_R|+\frac{C}{\delta r}|\nabla\phi|\\&\le|\nabla\varphi_R|^2+|\nabla\phi|^2+\frac{C}{\delta^2r^2}+\frac{C}{\delta r}|\nabla\varphi_R|+\frac{C}{\delta r}|\nabla\phi|,\end{aligned}$$

where we have used that $(1-t)^2+t^2\le 1$ for $t\in[0,1]$.

Then we have

$$\begin{aligned}\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\Phi|^2&\le\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\varphi_R|^2+\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\phi|^2+\frac{Cr}{\delta}+\frac{C}{\delta r}\int_{A_1}\rho_R^2|\nabla\varphi_R|+\frac{C}{\delta r}\int_{A_1}\rho_R^2|\nabla\phi|\\&\le\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\varphi_R|^2+\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\phi|^2+\frac{Cr}{\delta}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{\int_{A_1}\rho_R^2|\nabla\varphi_R|^2}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{\int_{A_1}\rho_R^2|\nabla\phi|^2},\end{aligned}$$

where we have used that $\mathcal{H}^3(A_1)=Cr^3\delta$, that $\rho_R\le 1$, and Jensen's inequality. Plugging this estimate in (20) gives

$$\frac{1}{2}\int_{A_1}|\nabla Z_1|^2\le\frac{1}{2}\int_{A_1}|\nabla u_R|^2+\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\phi|^2+\frac{Cr}{\delta}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{E(u_R,A_1)}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{\int_{A_1}\rho_R^2|\nabla\phi|^2}$$

Now we observe that

$$\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\phi|^2\le\frac{1}{2}\int_{A_1}\min\{1,m^2d_X^2\}|\nabla\phi|^2\le\frac{1}{2}\int_{A_1}|\nabla v_m|^2=\frac{1}{8}\cdot\frac{1}{2}\int_{B_{r+2\delta r}\setminus B_{r+\delta r}}|\nabla v_m|^2,$$

so Proposition 6.1 yields

$$\frac{1}{2}\int_{A_1}\rho_R^2|\nabla\phi|^2 \leq \frac{\pi}{2}\delta r\log(r+2\delta r)+C(r+2\delta r)\leq \frac{\pi}{2}\delta r\log r+Cr,$$

where C is independent of R, r and δ. Therefore, we obtain that

$$\begin{aligned}\frac{1}{2}\int_{A_1}|\nabla Z_1|^2 &\leq \frac{1}{2}\int_{A_1}|\nabla u_R|^2+\frac{\pi}{2}\delta r\log r+Cr+\frac{Cr}{\delta}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{E(u_R,A_1)}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{r\delta\log r+r}\\ &\leq \frac{1}{2}\int_{A_1}|\nabla u_R|^2+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{E(u_R,A_1)}+C\delta r\log r, \qquad (21)\end{aligned}$$

for some constant $C>0$ which is still independent of r and δ, provided $r>r_0=r_0(\delta)$, where r_0 is chosen so that

$$Cr+\frac{Cr}{\delta}+\frac{Cr}{\sqrt{\delta}}\sqrt{\delta\log r+1}\leq \delta r\log r \qquad \forall r>r_0.$$

For the potential term, we simply have

$$\int_{A_1}\frac{(1-|Z_1|^2)^2}{4}=\int_{A_1}\frac{(1-\rho_R^2)^2}{4}=\int_{A_1}\frac{(1-|u_R|^2)^2}{4} \qquad (22)$$

Combining (21) and (22) we obtain that

$$E(Z_1,A_1)=\frac{1}{2}\int_{A_1}|\nabla Z_1|^2+\int_{A_1}\frac{(1-|Z_1|^2)^2}{4}\leq E(u_R,A_1)+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{E(u_R,A_1)}+C\delta r\log r, \qquad (23)$$

for every $\delta\in(0,1)$, every $r>r_0(\delta)$ and every $R>2r+4\delta r$

Now we set

$$d_m(x):=\min\{1,m\cdot d_X(x)\},$$

and we consider the map introduced in Proposition 6.1

$$v_m(x)=d_m(x)g\left(\frac{x}{|x|}\right)=d_m(x)e^{i\phi(x)},$$

where the second equality holds for $x\in Q_R$.

Let $\eta_2:\mathbb{R}^3\to[0,1]$ be a smooth radial cutoff satisfying

$$\eta_1(x)=1 \quad \forall x\in B_r, \qquad \eta_2(x)=0 \quad \forall x\in\mathbb{R}^3\setminus B_{r+\delta r}, \qquad |\nabla\eta_1(x)|\leq\frac{2}{\delta r} \quad \forall x\in\mathbb{R}^3.$$

We now consider the interpolation between v_m and w given by

$$Z_2(x):=((1-\eta_2(x))\rho_R(x)+\eta_2(x)d_m(x))e^{i\phi(x)} \quad \text{on } A_2:=(B_{r+\delta r}\setminus B_r)\cap Q_R,$$

and we extend Z_2 to $B_{r+\delta r} \setminus B_r$ by the usual symmetries

We have for the Sobolev part

$$\begin{aligned}\frac{1}{2}\int_{A_2}|\nabla Z_2|^2 &= \frac{1}{2}\int_{A_2}|\nabla((1-\eta_2)\rho_R+\eta_2 d_m)|^2+\frac{1}{2}\int_{A_2}((1-\eta_2)\rho_R+\eta_2 d_m)^2|\nabla\phi|^2\\ &\leq \frac{1}{2}\int_{A_2}|\nabla((1-\eta_2)\rho_R+\eta_2 d_m)|^2+\frac{1}{2}\int_{A_2} d_m^2|\nabla\phi|^2,\end{aligned}$$

since $\rho_R \leq d_m$.

By an estimate similar to the one for Φ above

$$|\nabla(1-\eta_2)\rho_R+\eta_2 d_m)|^2 \leq |\nabla\rho_R|^2+|\nabla d_m|^2+\frac{C}{\delta^2 r^2}+\frac{C}{\delta r}|\nabla\rho_R|+\frac{C}{\delta r}|\nabla d_m|,$$

hence

$$\begin{aligned}\frac{1}{2}\int_{A_2}|\nabla Z_2|^2 &\leq \frac{1}{2}\int_{A_2}|\nabla\rho_R|^2+\frac{1}{2}\int_{A_2}\left(|\nabla d_m|^2+d_m^2|\nabla\phi|^2\right)+\frac{Cr}{\delta}+\frac{C}{\delta r}\int_{A_2}|\nabla\rho_R|+\frac{C}{\delta r}\int_{A_2}|\nabla d_m|\\ &\leq \frac{1}{2}\int_{A_2}|\nabla u_R|^2+\frac{1}{2}\int_{A_2}|\nabla v_m|^2+\frac{Cr}{\delta}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{\int_{A_2}|\nabla u_R|^2}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{\int_{A_2}|\nabla d_m|^2}.\end{aligned}$$

From the computations of Proposition 6.1 we find that

$$\int_{A_2}|\nabla d_m|^2 \leq C\delta r \quad \text{and} \quad \int_{A_2}|\nabla v_m|^2 \leq \frac{\pi}{2}\delta r\log r + Cr,$$

so, as above, we obtain

$$\begin{aligned}\frac{1}{2}\int_{A_2}|\nabla Z_2|^2 &\leq \frac{1}{2}\int_{A_2}|\nabla u_R|^2+\frac{Cr}{\delta}+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{\int_{A_2}|\nabla u_R|^2}+C\delta r\log r+Cr\\ &\leq \frac{1}{2}\int_{A_2}|\nabla u_R|^2+\frac{C\sqrt{r}}{\sqrt{\delta}}\sqrt{\int_{A_2}|\nabla u_R|^2}+C\delta r\log r\end{aligned}$$

for every $r > r_0(\delta)$, possibly for a larger r_0 than above (in case the constants C here are larger).

For the potential part, since $d_m \geq \rho_R$ implies that $|Z_2| \geq \rho_R$, now we have

$$\int_{A_2}\frac{(1-|Z_2|^2)^2}{4} \leq \int_{A_2}\frac{(1-\rho_R^2)^2}{4} = \int_{A_2}\frac{(1-|u_R|^2)^2}{4}.$$

Hence

$$E(Z_2, A_2) \leq E(u_R, A_2) + \frac{C\sqrt{r}}{\sqrt{\delta}} \sqrt{\int_{A_2} |\nabla u_R|^2} + C\delta r \log r$$
$$\leq E(u_R, A_2) + \frac{C\sqrt{r}}{\sqrt{\delta}} \sqrt{E(u_R, A_2)} + C\delta r \log r. \tag{24}$$

Finally, by Proposition 6.1 we find that

$$E(v_m, B_r) \leq 4\pi r \log r + Cr \leq (4\pi + \delta) r \log r, \tag{25}$$

for every $r > r_0$.

For $r > r_0$ and $R > 2r + 4\delta r$ define $Z \in \widetilde{H}^1(B_R; \mathbb{C})$ by

$$Z = \begin{cases} v_m & \text{in } B_r, \\ Z_2 & \text{in } B_{r+\delta r} \setminus B_r, \\ Z_1 & \text{in } B_{r+2\delta r} \setminus B_{r+\delta r}, \\ u_R & \text{in } B_R \setminus B_{r+2\delta r}. \end{cases}$$

Minimality of u_R yields

$$E(u_R, B_R) \leq E(Z, B_R) = E(v, B_r) + 8E(Z_2, A_2) + 8E(Z_1, A_1) + E(u_R, B_R \setminus B_{r+2\delta}),$$

so from (23), (24) and (25) we deduce that for every $r > r_0$ and every $R > r + 2\delta r$ it holds

$$E(u_R, B_{r+2\delta}) \leq (4\pi + C\delta) r \log r + 8E(u_R, A_2) + \frac{C\sqrt{r}}{\sqrt{\delta}} \sqrt{E(u_R, A_2)}$$
$$+ 8E(u_R, A_1) + \frac{C\sqrt{r}}{\sqrt{\delta}} \sqrt{E(u_R, A_1)},$$

which implies

$$E(u_R, B_r) \leq (4\pi + C\delta) r \log r + \frac{C\sqrt{r}}{\sqrt{\delta}} \sqrt{E(u_R, B_{r+2\delta r})}.$$

Let $R_j \to +\infty$ be the subsequence in Proposition 5.1 such that $u_{R_j} \to u$ in $C^2_{\mathrm{loc}}(\mathbb{R}^3)$. Letting $j \to +\infty$ yields

$$E(u, B_r) \leq (4\pi + C\delta) r \log r + \frac{C\sqrt{r}}{\sqrt{\delta}} \sqrt{E(u, B_{(1+2\delta)r})},$$

for every $r > r_0(\delta)$.

On the other hand, the Lipschitz bound on u_R immediately yields

$$E(u, B_r) = \lim_{j\to+\infty} E(u_{R_j}, B_r) \leq \int_{B_r} \left(\frac{m^2}{2} + \frac{1}{4}\right) \leq Cr^3.$$

As a consequence, from Lemma 6.4 applied with

$$f(r) = E(u, B_r), \quad A = 4\pi + C\delta, \quad B = \frac{C}{\sqrt{\delta}}, \quad \lambda = 1 + \delta, \quad K = C,$$

we deduce that

$$\limsup_{r\to+\infty} \frac{E(u, B_r)}{r \log r} \leq 4\pi + C\delta.$$

Since C is a universal constant independent of δ, letting $\delta \to 0$, we conclude the proof. □

Lemma 6.4. *Let $r_0 \geq 1$ be a positive number and let $f : (r_0, +\infty) \to (0, +\infty)$ be a function such that*

$$f(r) \leq Ar \log r + B\sqrt{r}\sqrt{f(\lambda r)}, \quad \forall r > r_0,$$

and

$$f(r) \leq Kr^3, \quad \forall r > r_0,$$

for some $A, B, K > 0$ and $\lambda > 1$. Then there exists $r_1 = r_1(A, B, \lambda) \geq r_0$ such that

$$f(r) \leq Ar \log r + r(\log r)^{2/3} + B^2\lambda^2 r \qquad \forall r > r_1.$$

In particular, it holds that

$$\limsup_{r\to+\infty} \frac{f(r)}{r \log r} \leq A.$$

Proof. Let us fix $r_1 > r_0$ sufficiently large so that

$$B\sqrt{\lambda}\sqrt{A \log(\lambda r) + (\log(\lambda r))^{2/3}} \leq (\log r)^{2/3} \qquad \forall r > r_1. \tag{26}$$

Now we claim that for every $n \in \mathbb{N}$ it holds that

$$f(r) \leq Ar \log r + r(\log r)^{2/3} + B^{2-2^{-n}} K^{2^{-n-1}} \lambda^2 r^{1+2^{-n}} \qquad \forall r > r_1. \tag{27}$$

We prove this by induction. First of all, we observe that our assumptions on f yield

$$f(r) \leq Ar\log r + B\sqrt{r}\sqrt{K\lambda^3 r^3} = Ar\log r + BK^{1/2}\lambda^{3/2}r^2 \qquad \forall r > r_0,$$

and, since $\lambda > 1$, this implies (27) with $n = 0$.

Now let us assume that (27) holds for some $n \in \mathbb{N}$, so we deduce that

$$\begin{aligned} f(r) &\leq Ar\log r + B\sqrt{r}\sqrt{f(\lambda r)} \\ &\leq Ar\log r + B\sqrt{r}\sqrt{A\lambda r\log(\lambda r) + \lambda r(\log(\lambda r))^{2/3} + B^{2-2^{-n}}K^{2^{-n-1}}\lambda^2\lambda^{1+2^{-n}}r^{1+2^{-n}}} \\ &\leq Ar\log r + B\sqrt{r}\sqrt{A\lambda r\log(\lambda r) + \lambda r(\log(\lambda r))^{2/3}} + B\sqrt{r}\sqrt{B^{2-2^{-n}}K^{2^{-n-1}}\lambda^{3+2^{-n}}r^{1+2^{-n}}} \\ &= Ar\log r + rB\sqrt{\lambda}\sqrt{A\log(\lambda r) + (\log(\lambda r))^{2/3}} + B^{2-2^{-n-1}}K^{2^{-n-2}}\lambda^{3/2+2^{-n-1}}r^{1+2^{-n-1}} \end{aligned}$$

for every $r > r_1$.

Now we recall that $\lambda > 1$ and we observe that $3/2 + 2^{-n-1} \leq 2$ so, plugging (26) into the last line, we obtain (27) with $n+1$ in place of n. Hence (27) holds for every $n \in \mathbb{N}$.

Therefore, letting $n \to +\infty$ in (27), we conclude the proof. □

6.2. *Lower bound*

From here, the sharp energy bound (2) easily follows by the clearing-out lemma (i.e., Theorem A.1) applied to

$$u_r(x) = u(rx)$$

and a simple covering argument. Fix $\delta < 1/10$ and $\tau < 1$. It is easy to see that there exists a family of N disjoint balls $\{B_\delta(x_i)\}_{i=1}^N$ contained in B_1 with $x_i \in X$ and

$$N \geq \frac{2}{\delta} - 10.$$

Since $u(x_i) = 0$ for every i, by Theorem A.1 we get

$$\frac{E(u, B_r)}{r\log(r)} = \frac{E_{1/r}(u_r, B_1)}{\log(r)} \geq \sum_i \frac{E_{1/r}(u_r, B_\delta(x_i))}{\log(r)} \geq (2 - 10\delta)(2\pi - \kappa)\frac{\log(\delta r)}{\log(r)},$$

when r is large enough, hence

$$\liminf_{r\to\infty} \frac{E(u, B_r)}{r\log(r)} \geq (2 - 10\delta)(2\pi - \kappa).$$

Letting $\kappa \to 0^+$ and $\delta \to 0^+$ gives

$$\liminf_{r\to\infty} \frac{E(u, B_r)}{r\log(r)} \geq 4\pi$$

which, together with Proposition 6.3, concludes the proof of (2).

6.3. Concentration of the normalized energy densities

We apply Theorem A.2 with $\varepsilon = 1/r$ to the blow-downs u_r. Then, up to extracting a subsequence, the normalized energy measures

$$\mu_r := \frac{e_{1/r}(u_r)}{\pi \log r}$$

converge weakly in $C_c^0(B_1)^*$ to a limit of the form

$$\mu_\infty = |V| + f \cdot dx \llcorner B_1,$$

where $f \geq 0$ and V is a stationary and rectifiable $(n-2)$-varifold with density $\theta \geq 1$, whose support contains $X \cap B_1$, so in particular $|V| \geq \theta \mathcal{H}^1 \llcorner (X \cap B_1)$. Since $\mathcal{H}^1(X \cap B_1) = 4$ (two orthogonal diameter segments in B_1), we have

$$|V|(B_1) \geq \theta\, \mathcal{H}^1(X \cap B_1) \geq 4,$$

and hence

$$4 + \int_{B_1} f \leq \mu_\infty(B_1) \leq \liminf_{r\to\infty} \mu_r(B_1) = 4,$$

where we have used (2) in the last equality. Therefore $\int_{B_1} f \leq 0$, which, together with $f \geq 0$, yields $f \equiv 0$. Moreover, $|V|(B_1) = 4$ forces $\theta = 1$ for $\mathcal{H}^1$-a.e. point of $X \cap B_1$, and $V = \mathcal{H}^1 \llcorner (X \cap B_1)$.

We conclude that

$$\mu_\infty = \mathcal{H}^1 \llcorner (X \cap B_1).$$

Since the above argument applies to any convergent subsequence, the whole sequence μ_r converges to this limit as $r \to \infty$.

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Appendix A

We collect here two important results for solutions of the Ginzburg-Landau system

$$-\Delta u = \frac{1}{\varepsilon^2} u(1-|u|^2), \tag{GL_ε}$$

with parameter $\varepsilon > 0$.

The first one is the sharp version of the clearing-out lemma for $n = 3$. This was first proved in [26] in dimension three in the case of local minimizers, and then recently extended to every dimension [23, Corollary 1.7]. Previous non-sharp versions of this result were proved in the pioneering works [4,18,19,24]. In the statement ω_k denotes the Lebesgue measure of the unit ball in $\mathbb{R}^k$.

Theorem A.1 *(Clearing-out lemma). Let $\varepsilon > 0$, $x \in \mathbb{R}^n$ and $u_\varepsilon : B_\delta(x) \to \mathbb{C}$ be a solution to (GL_ε). Then, for every $\kappa \in (0, \pi\omega_{n-2})$ there exists $c(\kappa, n) > 0$ such that if $\varepsilon/\delta \le c$ and*

$$\frac{E_\varepsilon(u_\varepsilon, B_\delta(x))}{\delta^{n-2}} \le (\pi\omega_{n-2} - \kappa)\log(\delta/\varepsilon),$$

then $|u(x)| \ge 1/2$.

Following the combination of the results [2,6,23,27], we recall the asymptotic description of solutions to the Ginzburg-Landau system with parameter ε as $\varepsilon \to 0$.

Theorem A.2. *Let $n \ge 3$ and $u_\varepsilon : B_1 \to \mathbb{C}$ be a sequence of solutions to (GL_ε) with*

$$E_\varepsilon(u_\varepsilon, B_1) \le C|\log(\varepsilon)|,$$

for some constant $C > 0$ independent of ε. Then, up to subsequences, the normalized energy densities

$$\mu_\varepsilon := \frac{e_\varepsilon(u_\varepsilon)}{\pi|\log(\varepsilon)|}\, dx \llcorner B_1$$

converge in $C_c^0(B_1)^*$ to a Radon measure μ which decomposes as

$$\mu = |V| + f \cdot dx \llcorner B_1,$$

where $f : B_1 \to \mathbb{R}$ is a smooth, nonnegative function and V is a stationary, rectifiable $(n-2)$-varifold with density bounded below by 1 on its support. Moreover, $\mathrm{supp}(|V|)$ *is the limit of the sets $\{|u_\varepsilon| < 1/2\}$ in the local Hausdorff topology.*

Data availability

No data was used for the research described in the article.

References

[1] G. Alberti, S. Baldo, G. Orlandi, Variational convergence for functionals of Ginzburg-Landau type, Indiana Univ. Math. J. 54 (5) (2005) 1411–1472.

[2] Fabrice Bethuel, Haïm Brezis, Frédéric Hélein, Asymptotics for the minimization of a Ginzburg-Landau functional, Calc. Var. Partial Differ. Equ. 1 (2) (1993) 123–148.

[3] Fabrice Bethuel, Haïm Brezis, Frédéric Hélein, Ginzburg-Landau Vortices, Progress in Nonlinear Differential Equations and Their Applications, vol. 13, Birkhäuser Boston, Inc., Boston, MA, 1994.

[4] Fabrice Bethuel, Haïm Brezis, Giandomenico Orlandi, Small energy solutions to the Ginzburg-Landau equation, C. R. Acad. Sci. Paris, Ser. I Math. 331 (10) (2000) 763–770.

[5] F. Bethuel, H. Brezis, G. Orlandi, Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions, J. Funct. Anal. 186 (2) (2001) 432–520.

[6] F. Bethuel, G. Orlandi, D. Smets, Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature, Ann. Math. (2) 163 (1) (2006) 37–163.

[7] Haïm Brezis, Some of my favorite open problems, Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. 34 (2) (2023) 307–335.

[8] Xinfu Chen, Charles M. Elliott, Tang Qi, Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation, Proc. R. Soc. Edinb., Sect. A 124 (6) (1994) 1075–1088.

[9] Andres Contreras, Robert L. Jerrard, Nearly parallel vortex filaments in the 3D Ginzburg-Landau equations, Geom. Funct. Anal. 27 (5) (2017) 1161–1230.

[10] Andrew Colinet, Robert Jerrard, Peter Sternberg, Solutions of the Ginzburg-Landau equations with vorticity concentrating near a nondegenerate geodesic, J. Eur. Math. Soc. 27 (4) (2025) 1527–1561.

[11] Juan Dávila, Manuel del Pino, Maria Medina, Rémy Rodiac, Interacting helical vortex filaments in the three-dimensional Ginzburg-Landau equation, J. Eur. Math. Soc. 24 (12) (2022) 4143–4199.

[12] Ha Dang, Paul C. Fife, L.A. Peletier, Saddle solutions of the bistable diffusion equation, Z. Angew. Math. Phys. 43 (6) (1992) 984–998.

[13] Guido De Philippis, Aria Halavati, Alessandro Pigati, Decay of excess for the abelian higgs model, preprint, 2024.

[14] David Gilbarg, Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin-New York, 1977.

[15] Rose-Marie Hervé, Michel Hervé, étude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg-Landau, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 11 (4) (1994) 427–440.

[16] John E. Hutchinson, Yoshihiro Tonegawa, Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory, Calc. Var. Partial Differ. Equ. 10 (1) (2000) 49–84.

[17] Tom Ilmanen, Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature, J. Differ. Geom. 38 (2) (1993) 417–461.

[18] Fanghua Lin, Tristan Rivière, Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents, J. Eur. Math. Soc. 1 (3) (1999) 237–311.

[19] Fang-Hua Lin, Tristan Rivière, A quantization property for static Ginzburg-Landau vortices, Commun. Pure Appl. Math. 54 (2) (2001) 206–228.

[20] Richard S. Palais, The principle of symmetric criticality, Commun. Math. Phys. 69 (1) (1979) 19–30.

[21] Adriano Pisante, Symmetry in nonlinear PDEs: some open problems, J. Fixed Point Theory Appl. 15 (2) (2014) 299–320.

[22] Alessandro Pigati, Daniel Stern, Minimal submanifolds from the abelian Higgs model, Invent. Math. 223 (3) (2021) 1027–1095.

[23] Alessandro Pigati, Daniel Stern, Quantization and non-quantization of energy for higher-dimensional Ginzburg-Landau vortices, Ars Inven. Anal. (2023) 3.

[24] Rivière Tristan, Line vortices in the $u(1)$ - Higgs model, ESAIM Control Optim. Calc. Var. 1 (1996) 77–167.

[25] M. Schatzman, On the stability of the saddle solution of Allen-Cahn's equation, Proc. R. Soc. Edinb., Sect. A 125 (6) (1995) 1241–1275.

[26] Etienne Sandier, Itai Shafrir, Small energy Ginzburg-Landau minimizers in $\mathbb{R}^3$, J. Funct. Anal. 272 (9) (2017) 3946–3964.

[27] Daniel Stern, Existence and limiting behavior of min-max solutions of the Ginzburg-Landau equations on compact manifolds, J. Differ. Geom. 118 (2) (2021) 335–371.