GEOMETRY OF HERMITIAN SYMMETRIC SPACES UNDER THE ACTION OF A MAXIMAL UNIPOTENT GROUP

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Abstract. Let G/K be a non-compact irreducible Hermitian symmetric space of rank r and let NAK be an Iwasawa decomposition of G. By the polydisc theorem, AK/K can be regarded as the base of an r-dimensional tube domain holomorphically embedded in G/K. As every N-orbit in G/K intersects AK/K in a single point, there is a one-to-one correspondence between N-invariant domains in G/K and tube domains in the product of r copies of the upper half-plane in $\mathbb{C}$. In this setting we prove a generalization of Bochner's tube theorem. Namely, an N-invariant domain D in G/K is Stein if and only if the base Ω of the associated tube domain is convex and "cone invariant". We also obtain a precise description of the envelope of holomorphy of an arbitrary holomorphically separable N-invariant domain over G/K.

An important ingredient for the above results is the characterization of several classes of N-invariant plurisubharmonic funtions on D in terms of the corresponding classes of convex functions on Ω. This also leads to an explicit Lie group theoretical description of all N-invariant potentials of the Killing metric on G/K.

1. Introduction

The classical Bochner's tube theorem states that the envelope of holomorphy of a tube domain $\mathbb{R}^n+i\Omega$ in $\mathbb{C}^n$ is univalent and coincides with the convex envelope $\mathbb{R}^n+i\,\mathrm{conv}(\Omega)$. Moreover, there is a one-to-one correspondence between the class of $\mathbb{R}^n$-invariant plurisubharmonic functions on a Stein tube domain in $\mathbb{C}^n$ and the class of convex functions on its base in $\mathbb{R}^n$ (cf. [Gun90]).

Here our goal is to obtain analogous results in the setting of an irreducible Hermitian symmetric space of the non-compact type, under the action of a maximal unipotent group of holomorphic automorphisms.

Any such space can be realized as a quotient G/K, where G is a non-compact real simple Lie group and K is a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{a}\oplus\mathfrak{k}$ be an Iwasawa decomposition of $\mathfrak{g}$, where $\mathfrak{n}$ is a maximal nilpotent

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subalgebra, $\mathfrak{a}$ is a maximally split abelian subalgebra and $\mathfrak{k}$ is the Lie algebra of K. The integer $r := \dim \mathfrak{a}$ is by definition the rank of G/K.

Let NAK be the corresponding Iwasawa decomposition of G, where $A := \exp \mathfrak{a}$ and $N := \exp \mathfrak{n}$. The group N acts on G/K by biholomorphisms and every N-orbit in G/K intersects the smooth, real r-dimensional submanifold $A \cdot eK$ transversally in a single point.

As the space G/K is *Hermitian* symmetric, G contains r pairwise commuting subgroups isomorphic to $SL(2, \mathbb{R})$. The orbit of the base point $eK \in G/K$ under the product of such subgroups is a closed complex submanifold of G/K which contains $A \cdot eK$ and is biholomorphic to $\mathbb{H}^r$, the product of r copies of the upper half-plane in $\mathbb{C}$. Moreover, every N-orbit in G/K intersects $\mathbb{H}^r$ in an $\mathbb{R}^r$-orbit.

This fact is an analogue of the polydisk theorem and determines a one-to-one correspondence between N-invariant domains in G/K and tube domains in $\mathbb{H}^r$ (cf. Prop. 4.1 and Cor. 4.3). If D is an N-invariant domain in G/K, then it is in terms of the base Ω of the associated tube domain in $\mathbb{H}^r$ that the properties of N-invariant objects on D can be best described.

Define the cone

$$C := \begin{cases} (\mathbb{R}^{>0})^r, & \text{in the non-tube case,} \\ (\mathbb{R}^{>0})^{r-1} \times \{0\}, & \text{in the tube case.} \end{cases}$$

A set $\Omega \subset \mathbb{R}^r$ is C-invariant if $\mathbf{y} \in \Omega$ implies $\mathbf{y} + \mathbf{v} \in \Omega$, for all $\mathbf{v} \in C$. Our generalizion of Bochner's tube thorem is as follows

Theorem 4.9. *Let G/K be a non-compact irreducible Hermitian symmetric space of rank r. Let D be an N-invariant domain in G/K and let $\mathbb{R}^r + i\Omega$ be the associated r-dimensional tube domain. Then D is Stein if and only if Ω is convex and C-invariant.*

We also show that a holomorphically separable, N-equivariant, Riemann domain over G/K is necessarily univalent (cf. Prop. 4.13). This implies the following corollary.

Corollary 4.14. *The envelope of holomorphy $\widehat{D}$ of an N-invariant domain D in G/K is the smallest Stein domain in G/K containing D. The base $\widehat{\Omega}$ of the r-dimensional tube domain associated to $\widehat{D}$ is the convex, C-invariant hull of Ω.*

One approach to the proof of the above theorem uses smooth N-invariant functions. There is a one-to-one correspondence between N-invariant functions on D and functions on Ω, and such correspondence preserves regularity. An important ingredient is the computation of the Levi form of a smooth N-invariant function $f \colon D \to \mathbb{R}$ in terms of the Hessian and the gradient of the corresponding function $\widehat{f} \colon \Omega \to \mathbb{R}$. To this end, a simple pluripotential argument enables us to exploit the restricted root decomposition of $\mathfrak{n}$ (cf. Prop. 3.1 and Prop. 4.5).

Then, in the smooth case, the proof of Theorem 4.9 is carried out by showing that D is Levi pseudoconvex, and therefore Stein, if and only if the base Ω of the associated tube domain is convex and C-invariant.

The general case follows from the smooth case by exhausting D with an increasing sequence of Stein, N-invariant domains with smooth boundary. For this we adapt a classical approximation method for convex functions on convex domains to our C-invariant context.

In Section 6, an alternative proof of Theorem 4.9 is carried out by realizing G/K as a Siegel domain and by combining some results from the theory of normal J-algebras with some convexity arguments.

The aformentioned computation of the Levi form leads to a characterization of smooth N-invariant plurisubharmonic functions on N-invariant domains in G/K in terms of the corresponding functions on Ω. By classical approximation methods, a similar characterization is obtained for arbitrary N-invariant (strictly) plurisubharmonic functions on D. In order to formulate such results we need the following definition.

Let $\widehat{f}\colon \Omega \to \mathbb{R}$ be a function defined on a C-invariant domain in $(\mathbb{R}^{>0})^r$ and let $\overline{C}$ be the closure of the cone C. Then $\widehat{f}$ is $\overline{C}$-decreasing if for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C}$ the restriction of $\widehat{f}$ to the half-line $\{\mathbf{y} + t\mathbf{v} \ : \ t \geqslant 0\}$ is decreasing.

Theorem. (see Thm. 5.5) *Let D be a Stein, N-invariant domain in a non-compact, irreducible Hermitian symmetric space G/K of rank r and let Ω be the base of the associated r-dimensional tube domain.*

An N-invariant function $f\colon D \to \mathbb{R}$ is (strictly) plurisubharmonic if and only if the corresponding function $\widehat{f}\colon \Omega \to \mathbb{R}$ is (stably) convex and $\overline{C}$-decreasing.

It follows that every N-invariant plurisubharmonic function on D is continuous.

In fact, the above theorem holds true both in the smooth and non-smooth context, and can be regarded as a generalization of the well known result for $\mathbb{R}^n$-invariant plurisubharmonic functions on tube domains in $\mathbb{C}^n$ (see Sect. 5 for precise definitions and statements).

In the appendix, as an application of our methods we explicitly determine all the N-invariant potentials of the Killing metric on G/K in a Lie group theoretical fashion.

2. Preliminaries

Let G/K be an irreducible Hermitian symmetric space, where G is a real non-compact semisimple Lie group and K is a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the respective Lie lagebras. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, with Cartan involution θ. Denote by $B(\,\cdot\,,\,\cdot\,)$ both the Killing form of $\mathfrak{g}$ and its $\mathbb{C}$-linear extension to $\mathfrak{g}^{\mathbb{C}}$ (which coincides with the Killing form of $\mathfrak{g}^{\mathbb{C}}$).

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. The dimension of $\mathfrak{a}$ is by definition the *rank* r of G/K. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha$ be the restricted root decomposition of $\mathfrak{g}$ determined by the adjoint action of $\mathfrak{a}$, where $\mathfrak{m}$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. For a simple Lie algebra of Hermitian type $\mathfrak{g}$, the restricted root system is either of type C_r (if G/K is of tube type) or of type BC_r (if G/K is not of tube type), i.e. there exists a basis $\{e_1, \ldots, e_r\}$ of $\mathfrak{a}^*$ for which a positive system Σ^+ is given by

$$\Sigma^+ = \{2e_j,\ 1 \leqslant j \leqslant r,\ e_k \pm e_l,\ 1 \leqslant k < l \leqslant r\}, \quad \text{for type } C_r,$$

$$\Sigma^+ = \{e_j,\ 2e_j,\ 1 \leqslant j \leqslant r,\ e_k \pm e_l,\ 1 \leqslant k < l \leqslant r\}, \quad \text{for type } BC_r\,.$$

The roots $2e_1, \ldots, 2e_r$ form a maximal set of long strongly orthogonal positive restricted roots. The root spaces $\mathfrak{g}^{2e_1}, \ldots, \mathfrak{g}^{2e_r}$ are one-dimensional and one can choose generators $E^j \in \mathfrak{g}^{2e_j}$ such that the $\mathfrak{sl}(2)$-triples $\{E^j,\ \theta E^j,\ A_j := [\theta E^j,\ E^j]\}$ are normalized as follows

$$[A_j,\ E^l] = \delta_{jl} 2E^l, \quad \text{for} \quad j, l = 1, \ldots, r. \tag{1}$$

Denote by I_0 the G-invariant complex structure of G/K. We assume that $I_0(E^j - \theta E^j) = A_j$. By the strong orthogonality of $2e_1, \ldots, 2e_r$, the vectors $A_1, \ldots, A_r$ form a B-orthogonal basis of $\mathfrak{a}$, dual to $e_1, \ldots, e_r$ of $\mathfrak{a}^*$, and the associated $\mathfrak{sl}(2)$-triples pairwise commute.

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ be the Iwasawa decomposition subordinated to Σ^+, where $\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$, and let $G = NAK$ be the corresponding Iwasawa decomposition of G. Then $S = NA$ is a real split solvable group acting freely and transitively on G/K. In particular, the tangent space to G/K at the base point eK can be identified with the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$.

The map $\phi \colon \mathfrak{s} \to \mathfrak{p}$, given by $\phi(X) := \frac{1}{2}(X - \theta X)$, is an isomorphism of vector spaces. As a consequence,

$$\langle X\,, Y \rangle := B(\phi(X)\,, \phi(Y)) = -\tfrac{1}{2} B(X\,, \theta Y), \tag{2}$$

for $X,\ Y \in \mathfrak{s}$, defines a positive definite symmetric bilinear form on $\mathfrak{s}$. Moreover, the map $J \colon \mathfrak{s} \to \mathfrak{s}$, given by

$$JX := \phi^{-1} \circ I_0 \circ \phi(X), \tag{3}$$

defines a complex structure on $\mathfrak{s}$, such that $\phi(JX) = I_0 \phi(X)$. The complex structure J permutes the restricted root spaces of $\mathfrak{s}$ (cf. [RoVe73]), namely

$$J\mathfrak{a} = \bigoplus_{j=1}^{r} \mathfrak{g}^{2e_j}, \quad J\mathfrak{g}^{e_j - e_l} = \mathfrak{g}^{e_j + e_l}, \quad J\mathfrak{g}^{e_j} = \mathfrak{g}^{e_j}\,. \tag{4}$$

In order to obtain a precise description of J on $\mathfrak{s}$, we recall a few more facts. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\mu \in \Delta} \mathfrak{g}^\mu$ be the root decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to a maximally split Cartan subalgebra $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$ of $\mathfrak{g}$, where $\mathfrak{b}$ is an abelian subalgebra of $\mathfrak{m}$. Let σ be the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. Let θ denote also the $\mathbb{C}$-linear extension of θ to $\mathfrak{g}^{\mathbb{C}}$. One has $\theta\sigma = \sigma\theta$. Write $\overline{Z} := \sigma Z$, for $Z \in \mathfrak{g}^{\mathbb{C}}$.

As σ and θ stabilize $\mathfrak{h}$, they induce actions on Δ, defined by $\bar{\mu}(H) := \overline{\mu(H)}$ and $\theta\mu(H) := \mu(\theta(H))$, for $H \in \mathfrak{h}$, respectively. Fix a positive root system Δ^+ compatible with Σ^+, meaning that $\mu|_{\mathfrak{a}} = Re(\mu) \in \Sigma^+$ implies $\mu \in \Delta^+$. Then $\sigma\Delta^+ = \Delta^+$.

Given a restricted root $\alpha \in \Sigma$, the corresponding restricted root space $\mathfrak{g}^\alpha$ decomposes into the direct sum of ordinary root spaces with respect to the Cartan subalgebra $\mathfrak{h}$ as follows

$$\mathfrak{g}^\alpha = \Big(\bigoplus_{\substack{\mu\in\Delta,\, \mu\neq\bar{\mu} \\ Re(\mu)=\alpha}} \mathfrak{g}^\mu \oplus \mathfrak{g}^{\bar{\mu}} \quad \oplus \mathfrak{g}^\lambda \Big) \cap \mathfrak{g},$$

where $\lambda \in \Delta$ is possibly a root satisfying $\lambda = \bar{\lambda} = \alpha$. The next lemma is obtained by combining Lemma 2.2 in [GeIa21] with (3).

Lemma 2.1. (the complex structure J on $\mathfrak{s}$).
(a) For $j = 1, \ldots, r$, let $A_j \in \mathfrak{a}$ and $E^j \in \mathfrak{g}^{2e_j}$ be elements normalized as in (1). Then $JE^j = \frac{1}{2}A_j$ and $JA_j = -2E^j$.

(b) Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{e_j-e_l}$, where $\mu \in \Delta^+$ is a root satisfying $Re(\mu) = e_j - e_l$ and $Z^\mu \in \mathfrak{g}^\mu$ (if $\bar{\mu} = \mu$, we may assume $Z^\mu = \overline{Z^\mu}$ and set $X = Z^\mu$). Then $JX = [E^l, X] \in \mathfrak{g}^{e_j+e_l}$.
Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{e_j+e_l}$, where $\mu \in \Delta^+$ is a root satisfying $Re(\mu) = e_j + e_l$ and $Z^\mu \in \mathfrak{g}^\mu$ (if $\bar{\mu} = \mu$, we may assume $Z^\mu = \overline{Z^\mu}$ and set $X = Z^\mu$). Then $JX = [\theta E^l, X] \in \mathfrak{g}^{e_j-e_l}$.

(c) Let $X = Z^\mu + \overline{Z^\mu} \in \mathfrak{g}^{e_j}$, where μ is a root in Δ^+ satisfying $Re(\mu) = e_j$ and $Z^\mu \in \mathfrak{g}^\mu$ (as $\dim \mathfrak{g}^{e_j}$ is even, one necessarily has $\bar{\mu} \neq \mu$). Then $JX = iZ^\mu + \overline{iZ^\mu} \in \mathfrak{g}^{e_j}$.

Remark 2.2. (a J-stable basis of $\mathfrak{s}$) *In view of Lemma 2.1, one can choose a J-stable basis of $\mathfrak{s}$, compatible with the restricted root decomposition.*

(a) As a basis of $\mathfrak{a} \oplus J\mathfrak{a}$, take pairs of elements A_j, $JA_j = -2E^j$, for $j = 1, \ldots, r$, normalized as in (1).

(b) As a basis of $\mathfrak{g}^{e_j-e_l} \oplus \mathfrak{g}^{e_j+e_l}$, take 4-tuples of elements

$$X = Z^\mu + \overline{Z^\mu}, \quad X' = iZ^\mu + \overline{iZ^\mu}, \quad JX = [E^l, X], \quad JX' = [E^l, X'], \tag{5}$$

parametrized by the pairs of roots $\mu \neq \bar{\mu} \in \Delta^+$ satisfying $Re(\mu) = e_j - e_l$ (with no repetition), with Z^μ a root vector in $\mathfrak{g}^\mu$. For $\mu = \bar{\mu}$, one may assume $Z^\mu = \overline{Z^\mu}$ and take the pair $X = Z^\mu$, $JX = [E^l, X]$.

(c) As a basis of $\mathfrak{g}^{e_j}$ (non-tube case), take pairs of elements

$$X = Z^\mu + \overline{Z^\mu}, \quad JX = iZ^\mu + \overline{iZ^\mu},$$

parametrized by the pairs of roots $\mu \neq \bar{\mu} \in \Delta^+$ satisfying $Re(\mu) = e_j$ (with no repetition), with $Z^\mu \in \mathfrak{g}^\mu$.

The next lemma contains some identities which are needed in Section 3. Its proof is essentially contained in [GeIa21], Lemma 2.4.

Lemma 2.3. *Let* $\mu \in \Delta^+$ *be a root satisfying* $Re(\mu) = e_j - e_l$ *and let* Z^μ *a root vector in* $\mathfrak{g}^\mu$. *Let* $X = Z^\mu + \overline{Z}^\mu \in \mathfrak{g}^{e_j - e_l}$ *and* $JX = [E^l, X] \in \mathfrak{g}^{e_j+e_l}$. *If* $\overline{\mu} \neq \mu$, *let* $X' = iZ^\mu + \overline{iZ^\mu}$ *and* $JX' = [E^l, X']$. *Then*

(*a*) $[JX, X] = [JX', X'] = sE^j$, *for some* $s \in \mathbb{R}$, $s \neq 0$*;*

(*b*) $[JX', X] = 0$.

Let μ *be a root in* Δ^+, *with* $Re(\mu) = e_j$ *(non-tube case) and let* Z^μ *be a root vector in* $\mathfrak{g}^\mu$. *Let* $X = Z^\mu + \overline{Z}^\mu$ *and* $JX = iZ^\mu + \overline{iZ^\mu}$. *Then*

(*c*) $[JX, X] = tE^j$, *for some* $t \in \mathbb{R}$, $t \neq 0$.

3. The Levi form of an N-invariant function on G/K

Let G/K be a non-compact, irreducible Hermitian symmetric space of rank r, and let $G = N\exp(\mathfrak{a})K$ be an Iwasawa decomposition of G. Let D be an N-invariant domain in G/K. Then D is uniquely determined by a domain $\mathcal{D}$ in $\mathfrak{a}$ by

$$D := N\exp(\mathcal{D}) \cdot eK\,. \tag{6}$$

Similarly, an N-invariant function $f : D \to \mathbb{R}$ is uniquely determined by the function $\widetilde{f} : \mathcal{D} \to \mathbb{R}$, defined by

$$\widetilde{f}(H) := f(\exp(H)K). \tag{7}$$

The goal of this section is to express the *Levi form*, i.e. the real symmetric J-invariant bilinear form

$$h_f(\,\cdot\,,\,\cdot\,) := -dd^c f(\,\cdot\,, J\,\cdot\,), \tag{8}$$

of a smooth N-invariant function f on D, in terms of the first and second derivatives of the corresponding function $\widetilde{f}$ on $\mathcal{D}$. This will enable us to characterize smooth N-invariant strictly plurisubharmonic functions on a Stein N-invariant domain D in G/K by appropriate conditions on the corresponding functions on $\mathcal{D}$ (Prop. 3.1). As f is N-invariant, h_f is N-invariant as well. Therefore it will be sufficient to carry out the computation along the slice $\exp(\mathcal{D}) \cdot eK$, which meets all N-orbits.

For $X \in \mathfrak{g}$, denote by $\widetilde{X}$ the vector field on G/K induced by the left G-action. Its value at $z \in G/K$ is given by

$$\widetilde{X}_z := \tfrac{d}{ds}\big|_{s=0} \exp sX \cdot z. \tag{9}$$

Let $X \in \mathfrak{g}^\alpha$, for $\alpha \in \Sigma^+ \cup \{0\}$ (here $X \in \mathfrak{a}$, when $\alpha = 0$). If $z = aK$, with $a = \exp H$ and $H \in \mathfrak{a}$, then the vector field $\widetilde{X}$ can also be expressed as

$$\widetilde{X}_z = e^{-\alpha(H)} a_* X. \tag{10}$$

Set

$$\mathbf{b} := B(A_1, A_1) = \ldots = B(A_r, A_r), \tag{11}$$

which is a real positive constant only depending on the Lie algebra $\mathfrak{g}$.

Proposition 3.1. *Let D be an N-invariant domain in G/K and let $f : D \to \mathbb{R}$ be a smooth N-invariant function. Fix $a = \exp H$, with $H = \sum_j a_j A_j \in \mathcal{D}$. Then, in the basis of $\mathfrak{s}$ defined in Remark 2.2, the form h_f at $z = aK \in D$ is given as follows.*

(i) *The spaces $a_*\mathfrak{a}$, $a_*J\mathfrak{a}$, $a_*\mathfrak{g}^{e_j-e_l}$, $a_*\mathfrak{g}^{e_j+e_l}$ and $a_*\mathfrak{g}^{e_j}$ are pairwise h_f-orthogonal.*

(ii) *For $A_j, A_l \in \mathfrak{a}$ one has*
$h_f(a_*A_j, a_*A_l) = -2\delta_{jl}\frac{\partial \widetilde{f}}{\partial a_l}(H) + \frac{\partial^2 \widetilde{f}}{\partial a_j \partial a_l}(H).$

On the blocks $a_*\mathfrak{g}^{e_j-e_l}$ and $a_*\mathfrak{g}^{2e_j}$ the restriction of h_f is diagonal and the only non-zero entries are given as follows.

(iii) *For X, $X' \in \mathfrak{g}^{e_j-e_l}$ as in Remark 2.2(b), one has*

$$h_f(a_*X, a_*X) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \widetilde{f}}{\partial a_j}(H), \qquad h_f(a_*X', a_*X') = -2\tfrac{\|X'\|^2}{\mathbf{b}}\tfrac{\partial \widetilde{f}}{\partial a_j}(H).$$

(iv) *(non-tube case) For $X \in \mathfrak{g}^{e_j}$ as in Remark 2.2(c), one has*

$$h_f(a_*X, a_*X) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \widetilde{f}}{\partial a_j}(H).$$

On the remaining blocks h_f is determined by (4), the J-invariance of h_f, (i) and (iii) above.

Proof. Let $f\colon G/K \to \mathbb{R}$ be a smooth N-invariant function. The computation of h_f uses the fact that, for $X \in \mathfrak{n}$, the function $\mu^X\colon G/K \to \mathbb{R}$, given by $\mu^X(z) := d^c f(\widetilde{X}_z)$, satisfies the identity

$$d\mu^X = -\iota_{\widetilde{X}} dd^c f, \tag{12}$$

where $d^c f := df \circ J$ (see [HeSc07], Lemma 7.1 and [GeIa21], Sect. 2). We begin by determining $d^c f(\widetilde{X}_z)$, for $X \in \mathfrak{n}$ and $z \in G/K$. By the N-invariance of f and of J one has

$$d^c f(\widetilde{X}_{n\cdot z}) = d^c f(\widetilde{\mathrm{Ad}_{n^{-1}} X}_z)\,, \tag{13}$$

for every $z \in G/K$ and $n \in N$. Thus it is sufficient to take $z = aK \in \exp(\mathcal{D})\cdot eK$. Let $H = \sum a_j A_j \in \mathcal{D}$ and $a = \exp H$. Then

$$d^c f(\widetilde{X}_z) = \begin{cases} \frac{1}{2}e^{-2a_j}\frac{\partial \widetilde{f}}{\partial a_j}(H)\,, & \text{for } X = E^j \in \mathfrak{g}^{2e_j} \\ \quad 0\,, & \text{for } X \in \mathfrak{g}^{\alpha}, \text{ with } \alpha \in \Sigma^+ \backslash \{2e_1, \ldots, 2e_r\}. \end{cases} \tag{14}$$

The first part of equation (14) follows from (10) and Lemma 2.1 (a):

$$d^cf((\widetilde{E^j})_z) = e^{-2e_j(H)}df(a_*JE^j) = \tfrac{1}{2}e^{-2a_j}\tfrac{d}{ds}\big|_{s=0}\widetilde{f}(H+sA_j) = \tfrac{1}{2}e^{-2a_j}\tfrac{\partial\widetilde{f}}{\partial a_j}(H).$$

For the second part, let $X\in\mathfrak{g}^\alpha$, with $\alpha\in\Sigma^+\backslash\{2e_1,\dots,2e_r\}$. Then $JX\in\mathfrak{g}^\beta$, with $\beta\in\Sigma^+$. By (10) and the N-invariance of f, one obtains the desired result

$$d^cf(\widetilde{X}_z) = e^{-\alpha(H)+\beta(H)}df(\widetilde{JX}_z) = 0.$$

(i) Orthogonality of the blocks. Let $X\in\mathfrak{g}^\alpha$ and $Y\in\mathfrak{g}^\gamma$, where $\alpha\in\Sigma^+$ and $\gamma\in\{0\}\cup(\Sigma^+\backslash\{2e_1,\dots,2e_r\})$ are distinct restricted roots (here $Y\in\mathfrak{a}$, when $\gamma=0$). Then $JY\in\mathfrak{g}^\beta$, for some $\beta\in\Sigma^+$. By (10) and (12), one has

$$\begin{aligned} h_f(a_*X,a_*Y) &= -dd^cf(a_*X,a_*JY) = -e^{\alpha(H)+\beta(H)}dd^cf(\widetilde{X}_z,\widetilde{JY}_z)\\ &= e^{\alpha(H)+\beta(H)}d\mu^X(\widetilde{JY}_z) = e^{\alpha(H)+\beta(H)}\tfrac{d}{ds}\big|_{s=0}\mu^X(\exp sJY\cdot z)\\ &= e^{\alpha(H)+\beta(H)}\tfrac{d}{ds}\big|_{s=0}d^cf(\widetilde{X}_{\exp sJY\cdot z}) = e^{\alpha(H)+\beta(H)}\tfrac{d}{ds}\big|_{s=0}d^cf(\widetilde{Ad_{\exp(-sJY)}X}_z)\\ &= e^{\alpha(H)+\beta(H)}\tfrac{d}{ds}\big|_{s=0}d^cf(\widetilde{X}_z - s\widetilde{[JY,X]}_z + o(s^2))\\ &= -e^{\alpha(H)+\beta(H)}d^cf(\widetilde{[JY,X]}_z). \end{aligned} \tag{15}$$

The brackets $[JY,X]$ lie in $\mathfrak{g}^{\alpha+\beta}$. Since $\alpha\neq\gamma$, one sees that $\alpha+\beta\neq 2e_1,\dots,2e_r$. Then, by (14), the expression (15) vanishes, proving the orthogonality of $a_*\mathfrak{g}^\alpha$ and $a_*\mathfrak{g}^\gamma$, for all α and γ as above. The J-invariance of h_f implies that $a_*\mathfrak{a}$ is orthogonal to $a_*\mathfrak{g}^\beta$, for all $\beta\in\Sigma^+$, and concludes the proof of (i).

Next we determine the form h_f on the essential blocks.

(ii) The form h_f on $a_*\mathfrak{a}$.

Let $A_j,A_l\in\mathfrak{a}$. Since $JA_l=-2E^l$, one has

$$\begin{aligned} h_f(a_*A_j,a_*A_l) &= -2dd^cf(a_*E^l,a_*A_j) = -2e^{2e_l(H)}dd^cf((\widetilde{E^l})_z,(\widetilde{A_j})_z)\\ &= 2e^{2e_l(H)}d\mu^{E^l}((\widetilde{A_j})_z) = 2e^{2e_l(H)}\tfrac{d}{dt}\big|_{t=0}\mu^{E^l}(\exp tA_j\cdot z)\\ &= 2e^{2e_l(H)}\tfrac{d}{dt}\big|_{t=0}d^cf((\widetilde{E^l})_{\exp tA_j\cdot z}), \end{aligned}$$

which, by (14), becomes

$$= 2e^{2e_l(H)}\tfrac{d}{dt}\big|_{t=0}\tfrac{1}{2}e^{-2e_l(H+tA_j)}\tfrac{\partial\widetilde{f}}{\partial a_l}(H+tA_j) = -2\tfrac{\partial\widetilde{f}}{\partial a_l}(H)\delta_{lj} + \tfrac{\partial^2\widetilde{f}}{\partial a_j\partial a_l}(H).$$

This concludes the proof of (ii).

(iii) The form h_f on $a_*\mathfrak{g}^{e_j-e_l}$.

Let X, $X'\in\mathfrak{g}^{e_j-e_l}$ be elements of the basis given in Remark 2.2 (b). Then JX, $JX'\in\mathfrak{g}^{e_j+e_l}$. From (15), (14) and Lemma 2.3(a) one has

$$\begin{aligned} h_f(a_*X,a_*X) &= -dd^cf(a_*X,a_*JX)\\ &= -e^{(e_j+e_l)(H)}e^{(e_j-e_l)(H)}d^cf(\widetilde{[JX,X]}_z)\\ &= -e^{2e_j(H)}\left(sd^cf((\widetilde{E^j})_z)\right) = -\tfrac{s}{2}\tfrac{\partial\widetilde{f}}{\partial a_j}(H)\,, \end{aligned} \tag{16}$$

for some $s \in \mathbb{R}\backslash\{0\}$. By Remark 6.4, one has $s > 0$. By the comparison of (16) with the formula obtained in Remark 7.2, one deduces the exact value of s, namely $s = \frac{4\|X\|^2}{\mathbf{b}}$. Therefore, one has

$$h_f(a_*X, a_*X) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \tilde{f}}{\partial a_j}(H), \qquad h_f(a_*X', a_*X') = -2\tfrac{\|X'\|^2}{\mathbf{b}}\tfrac{\partial \tilde{f}}{\partial a_j}(H),$$

as stated. From (15) and Lemma 2.3(b), one obtains $h_f(a_*X, a_*X') = 0$. From (15), the skew symmetry of dd^cf and the fact that $2(e_j - e_l) \notin \Sigma^+$, one obtains $h_f(a_*X, a_*JX) = h_f(a_*X, a_*JX') = 0$, respectively. Finally, let $X = Z^\mu + \overline{Z^\mu}$, and $Y = Z^\nu + \overline{Z^\nu}$ be elements of the basis of $\mathfrak{g}^{e_j - e_l}$ given in Remark 2.2 (b), for μ, $\nu \in \Delta^+$ distinct roots satisfying $\nu \neq \mu$, $\bar{\mu}$. Then, by (15) and Lemma 2.1(b) one has

$$h_f(a_*X, a_*Y) = -e^{2e_j(H)}d^cf(\widetilde{[JY, X]}_z) = 0,$$

since no non-real roots in Δ have real part equal to $2e_j$. This completes the proof of (iii).

(iv) The Hermitian form h_f on $a_*\mathfrak{g}^{e_j}$.

Let $X = Z^\mu + \overline{Z^\mu}$ and $JX = iZ^\mu + \overline{iZ^\mu}$ be elements of the basis of $\mathfrak{g}^{e_j}$ given in Remark 2.2 (c). Then, from (15) and Lemma 2.3 (c), one obtains

$$\begin{aligned} h_f(a_*X, a_*X) &= -e^{2e_j(H)}d^cf(\widetilde{[JX, X]}_z) \\ &= -e^{2e_j(H)}t\, d^cf((\widetilde{E^j})_z) = -\tfrac{t}{2}\tfrac{\partial \tilde{f}}{\partial a_j}(H), \end{aligned} \tag{17}$$

for some $t \in \mathbb{R}\backslash\{0\}$. By Remark 6.4, one has $t > 0$. By the comparison of (17) with the formula obtained in Remark 7.2, one deduces the exact value of t, namely $t = \frac{4\|X\|^2}{\mathbf{b}}$ and

$$h_f(a_*X, a_*X) = h_f(a_*JX, a_*JX) = -2\tfrac{\|X\|^2}{\mathbf{b}}\tfrac{\partial \tilde{f}}{\partial a_j}(H).$$

Finally, let $X = Z^\mu + \overline{Z^\mu}$ and $Y = Z^\nu + \overline{Z^\nu}$ be elements of the basis of $\mathfrak{g}^{e_j}$ given in Remark 2.2 (c), for μ, $\nu \in \Delta^+$ distinct roots satisfying $\nu \neq \mu$, $\bar{\mu}$. Then, by (15) and Lemma 2.1(c) one has $h_f(a_*X, a_*Y) = 0$. This concludes the proof of (iv) and of the proposition. □

Remark. The usual Levi form $L_f^{\mathbb{C}}$ of f is given by $L_f^{\mathbb{C}}(Z, \overline{W}) = 2(h_f(X, Y) + ih_f(X, JY))$, where $Z = X - iJX$ and $W = Y - iJY$ are elements of type $(1, 0)$. One easily sees that $L_f^{\mathbb{C}}$ is (strictly) positive definite if and only if h_f is (strictly) positive definite.

4. N-invariant Stein domains in G/K

The main goal of this section is to characterize the Stein N-invariant domains D in G/K in terms of an associated r-dimensional tube domain. We show that D is Stein if and only if the base of the associated tube domain is convex and satisfies an additional geometric condition, arising from the features of the N-invariant plurisubharmonic functions on D.

At the end of the section we also prove a univalence result for N-equivariant Riemann domains over G/K. As a by-product, a precise description of the envelope of holomorphy of N-invariant domains in G/K follows.

Resume the notation introduced in Section 2. Denote by $R := \exp\big(\oplus\, \mathfrak{g}^{2e_j}\big)$ the unipotent abelian subgroup of G, isomorphic to $\mathbb{R}^r$. The orbit of the base point $eK \in G/K$ under the product of the r commuting $SL_2(\mathbb{R})$'s contained in G is the r-dimensional R-invariant closed complex submanifold of G/K

$$R\exp(\mathfrak{a})\cdot eK.$$

By the Iwasawa decomposition of G, such manifold intersects all N-orbits in G/K. Equivalently,

$$N\cdot(R\exp(\mathfrak{a})\cdot eK) = G/K.$$

The above facts together with the next proposition can be regarded as an analogue, for the N-action, of the polydisk theorem (cf. [Wol72], p. 280). Denote by $\mathbb{H}$ the upper half-plane in $\mathbb{C}$, with the usual $\mathbb{R}$-action by translations.

Proposition 4.1. *The map $\mathcal{L}:\mathbb{H}^r \to R\exp\mathfrak{a}\cdot eK$, defined by*

$$(x_1+iy_1,\dots,x_r+iy_r) \to \exp(\textstyle\sum_j x_jE^j)\exp(\frac{1}{2}\sum_j \ln(y_j)A_j)K\,,$$

is an equivariant biholomorphism.

Proof. The map is clearly bijective and equivariant. To prove that is holomorphic, it is sufficient to consider the rank-1 case. Computing separately

$$d\mathcal{L}_zJ\tfrac{d}{dx}\big|_z = d\mathcal{L}_z\tfrac{d}{dy}\big|_z = \tfrac{d}{dt}\big|_{t=0}\mathcal{L}(x+i(y+t)) = \tfrac{d}{dt}\big|_{t=0}\exp(xE)\exp(\tfrac{1}{2}\ln(y+t)A)K$$
$$= \tfrac{d}{dt}\big|_{t=0}\exp(xE)\exp((\tfrac{1}{2}\ln y+\tfrac{t}{2y}+o(t^2))A)K = (\exp(xE)\exp(\tfrac{1}{2}\ln yA))_*\tfrac{1}{2y}A$$

and

$$J\mathcal{L}_z\tfrac{d}{dx}\big|_z = J\tfrac{d}{dt}\big|_{t=0}\mathcal{L}(x+t+iy) = J\tfrac{d}{dt}\big|_{t=0}\exp((x+t)E)\exp(\tfrac{1}{2}\ln yA)K$$
$$= J\tfrac{d}{dt}\big|_{t=0}\exp(xE)\exp(tE)\exp(\tfrac{1}{2}\ln yA)K$$
$$= J\tfrac{d}{dt}\big|_{t=0}\exp(xE)\exp(\tfrac{1}{2}\ln yA)\exp(t\,Ad_{\exp(-\frac{1}{2}\ln yA)}E)K$$
$$= J\exp(xE)_*\exp(\tfrac{1}{2}\ln yA)_*\tfrac{1}{y}E = (\exp(xE)\exp(\tfrac{1}{2}\ln yA))_*\tfrac{1}{2y}A,$$

we obtain the desired identity $d\mathcal{L}_zJ\frac{d}{dx}\big|_z = Jd\mathcal{L}_z\frac{d}{dx}\big|_z$, for all $z\in\mathbb{H}$. □

Remark 4.2. The closed complex submanifold $R\exp(\mathfrak{a})\cdot eK$ can also be regarded as the local orbit of eK under the universal complexification $R^{\mathbb{C}}$ of R. Up to a traslation, $\mathcal{L}$ is the local $R^{\mathbb{C}}$-orbit map through eK.

As a consequence of the above biholomorphism we obtain a one-to-one correspondence between $\mathbb{R}^r$-invariant tube domains in $\mathbb{H}^r$ and N-invariant domains in G/K. Denote by $L:\mathbb{R}^{>0}\times\ldots\times\mathbb{R}^{>0}\to\mathfrak{a}$ the diffeomorphism determined by $\mathcal{L}$

$$L(y_1,\ldots,y_r) := \tfrac{1}{2}\textstyle\sum_j \ln(y_j)A_j. \tag{18}$$

Corollary 4.3. **(N-invariant domains in G/K and tube domains in $\mathbb{C}^r$).**
(i) Let $D=N\exp(\mathcal{D})\cdot eK$ be an N-invariant domain in G/K and let $R\exp(\mathcal{D})\cdot eK$ be its intersection with the closed complex submanifold $R\exp(\mathfrak{a})\cdot eK$. Then the r-dimensional tube domain associated to D is by definition the preimage of $R\exp(\mathfrak{a})\cdot eK$ under $\mathcal{L}$, namely

$$\mathbb{R}^r+i\Omega, \qquad \textit{where } \Omega := L^{-1}(\mathcal{D})\,.$$

(ii) Conversely, a tube domain $\mathbb{R}^r+i\Omega$ in $\mathbb{H}^r$ determines a unique N-invariant domain

$$D=N\exp(\mathcal{D})\cdot eK, \qquad \textit{where } \mathcal{D}=L(\Omega).$$

Remark 4.4. If D is Stein, then the associated tube domain $\mathbb{R}^r+i\Omega\subset\mathbb{C}^r$ is Stein, being biholomorphic to the Stein closed complex submanifold $R\exp(\mathcal{D})\cdot eK$ of D. In particular, the base Ω is an open convex set in $(\mathbb{R}^{>0})^r$.

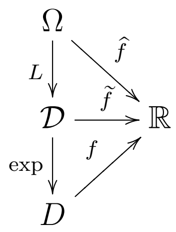
On the other hand, already in the case of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, with $n>1$, one can see that the base Ω of an N-invariant Stein subdomain D must be an entire half-line, and cannot be just an arbitrary convex subset of $\mathbb{R}^{>0}$.

The main goal of this section is to give a precise characterization of the convex sets $\Omega\subset(\mathbb{R}^{>0})^r$ arising from N-invariant Stein domains D in G/K. As we shall see, their shape is determined by the particular features of the Levi form of the N-invariant functions on D, which involve both the Hessian and the gradient of $\widetilde{f}$ (cf. Prop. 3.1).

Let $f:D\to\mathbb{R}$ be an N-invariant plurisubharmonic function. Then f is uniquely determined by the function $\widetilde{f}(H):=f(\exp H\cdot eK)$ on $\mathcal{D}$ (cf. (7)) and also by the function

$$\widehat{f}(\mathbf{y}) := f(\exp(L(\mathbf{y}))K) = \widetilde{f}(L(\mathbf{y})) \tag{19}$$

defined for $\mathbf{y} \in \Omega$, as shown by the following commutative diagram



Since the N-action on D is proper and every N-orbit intersects transversally the smooth slice $\exp(L(\Omega))\cdot eK$ in a single point, it is easy to check that the map $f \to \widehat{f}$ is a bijection from the class $C^0(D)^N$ of continuous N-invariant functions on D and the class $C^0(\Omega)$ of continuous functions on Ω. By Theorem 4.1 in [Fle78], such a map is also a bijection between $C^\infty(D)^N$ and $C^\infty(\Omega)$. Analogous statements hold true for the map $f \to \widetilde{f}$.

Given a non-compact irreducible Hermitian symmetric space, define the cone

$$C := \begin{cases} (\mathbb{R}^{>0})^r, & \text{in the non-tube case,} \\ (\mathbb{R}^{>0})^{r-1} \times \{0\}, & \text{in the tube case.} \end{cases} \tag{20}$$

The next lemma characterizes the plurisubharmonicity of a smooth N-invariant function f in terms of the corresponding functions $\widetilde{f}$ and $\widehat{f}$.

Proposition 4.5. *Let D be an N-invariant domain in G/K and let $f : D \to \mathbb{R}$ be a smooth, N-invariant, plurisubharmonic function. Then the following conditions are equivalent:*

(i) *f is plurisubharmonic (resp. strictly plurisubharmonic) at $z = aK$, with $a = \exp(H)$ and $H \in \mathcal{D}$;*

(ii) *the form*

$$\Big(-2\delta_{jl} \tfrac{\partial \widetilde{f}}{\partial a_l}(H) + \tfrac{\partial^2 \widetilde{f}}{\partial a_j \partial a_l}(H) \Big)_{j,l=1,\dots,r} \tag{21}$$

in Proposition 3.1(ii) is positive semidefinite (resp. positive definite) and

$$\mathrm{grad}\widetilde{f}(H)\cdot \mathbf{v} \leqslant 0 \ \textit{(resp. } < 0\textit{)}, \qquad \textit{for all } \mathbf{v} \in \overline{C}\backslash\{\mathbf{0}\};$$

(iii) *the Hessian of $\widehat{f}$ is positive semidefinite (resp. positive definite) at $\mathbf{y} = (y_1, \dots, y_r) = L^{-1}(H)$ and*

$$\mathrm{grad}\widehat{f}(\mathbf{y})\cdot \mathbf{v} \leqslant 0 \ \textit{(resp. } < 0\textit{)}, \qquad \textit{for all } \mathbf{v} \in \overline{C}\backslash\{\mathbf{0}\}. \tag{22}$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows directly from Proposition 3.1.
$(ii) \Leftrightarrow (iii)$ Since $L(y_1, \dots, y_r) = (\frac{1}{2}\ln(y_1), \dots, \frac{1}{2}\ln(y_r))$ (see (18)), one has $\widetilde{f}(a_1, \dots, a_r) = \widehat{f}(e^{2a_1}, \dots, e^{2a_r})$. Therefore

$$\tfrac{\partial \widetilde{f}}{\partial a_j}(a_1, \dots, a_r) = 2 \tfrac{\partial \widehat{f}}{\partial y_j}(e^{2a_1}, \dots, e^{2a_r}) e^{2a_j} \tag{23}$$

$$\frac{\partial^2 \widetilde f}{\partial a_j \partial a_l}(H) = 4\frac{\partial^2 \widehat f}{\partial y_j \partial y_l}(e^{2a_1},\dots,e^{2a_r})e^{2a_j}e^{2a_l} + 4\frac{\partial \widehat f}{\partial y_j}(e^{2a_1},\dots,e^{2a_r})e^{2a_j}\delta_{jl}\,. \qquad (24)$$

By combining formulas (23) and (24) one obtains

$$\big(4\frac{\partial^2 \widehat f}{\partial y_j \partial y_l}e^{2a_j}e^{2a_l}\big)_{j,l} = \big(\frac{\partial^2 \widetilde f}{\partial a_j \partial a_l} - 2\,\frac{\partial \widetilde f}{\partial a_j}\delta_{jl}\big)_{j,l}. \qquad (25)$$

Also, by (23), the same monotonicity conditions hold both for $\widetilde f$ and for $\widehat f$. □

Definition 4.6. *A smooth function $g\colon \mathbb{R}^r \to \mathbb{R}$ is convex (resp. stably convex) if its Hessian is semidefinite (positive definite).*

Remark 4.7. The above lemma shows that the function $\widehat f$ corresponding to a smooth N-invariant plurisubharmonic function is not just an arbitrary smooth convex function, but it must satisfy the additional monotonicity conditions (22). (cf. Rem. 5.2).

Definition 4.8. *A set $\Omega \subset \mathbb{R}^r$ is C-invariant if $\mathbf{y} \in \Omega$ implies $\mathbf{y} + C \subset \Omega$ Equivalently, if $\mathbf{y} \in \Omega$ implies $\mathbf{y} + \overline{C} \subset \Omega$, where $\overline{C}$ denotes the closure of C.*

Theorem 4.9. *Let G/K be a non-compact irreducible Hermitian symmetric space and let D be an N-invariant domain in G/K. Then D is Stein if and only if the base Ω of the associated tube domain is convex and C-invariant.*

The proof of the above theorem is divided into two parts. If D has smooth boundary, then the argument relies on the computation of the Levi form of smooth, N-invariant functions on D (Prop. 3.1) and some elementary convex-geometric properties of Ω.

In the general case, the proof of the theorem is obtained by realizing D as an increasing union of Stein, N-invariant domains with smooth boundary.

Proof of Theorem 4.9: the smooth case. The rank-1 tube case is trivial, since every $\mathbb{R}$-invariant domain in the upper half-plane $\mathbb{H}$ is Stein. So we deal with the remaining cases: the rank-one non-tube case and the higher rank cases.

We use the notation $\mathbf{y} = (y_1,\dots,y_r)$, for elements in $\mathbb{R}^r$. Let $D \subset G/K$ be a Stein, N-invariant domain with smooth boundary and let $\mathbb{R}^r + i\Omega \subset \mathbb{C}^r$ be its associated tube domain. Then Ω is a convex set with smooth boundary (cf. Rem. 4.4). Assume by contradiction that Ω is not C-invariant, i.e. there exist $\mathbf{y} \in \Omega$ and $\mathbf{z} \in (\mathbf{y} + C) \cap \partial\Omega$. By the convexity of Ω, the open segment from $\mathbf{y}$ to $\mathbf{z}$ is contained in Ω. In addition, the vector $\mathbf{v} = \mathbf{z} - \mathbf{y} \in C$ is transversal to the tangent hyperplane $T_{\mathbf{z}}\partial\Omega$ and points outwards. Therefore, given a smooth local defining function $\widehat f$ of $\partial\Omega$ near $\mathbf{z}$, one has

$$\frac{\partial \widehat f}{\partial \mathbf{v}}(\mathbf{z}) = \mathrm{grad}\widehat f(\mathbf{z}) \cdot \mathbf{v} > 0.$$

In the tube case, the above inequality and (23) imply that $\frac{\partial \widetilde{f}}{\partial a_j}(H) > 0$, for some $j \in \{1, \ldots, r-1\}$. Then, by Proposition 3.1 (iii), the Levi form of the corresponding N-invariant function f is negative definite on the J-invariant subspace $a_* \mathfrak{g}^{e_j - e_l} \oplus a_* \mathfrak{g}^{e_j + e_l}$ of $T_{aK}(\partial D)$, the tangent space to ∂D in aK. In the non-tube case, one has $\frac{\partial \widetilde{f}}{\partial a_j}(H) > 0$, for some $j \in \{1, \ldots, r\}$. By Proposition 3.1 (iv), the Levi form of the corresponding N-invariant function f is negative definite on the J-invariant subspace $a_* \mathfrak{g}^{e_j}$ of $T_{aK}(\partial D)$. This contradicts the fact that f is a defining function of the Stein N-invariant domain D and proves that Ω is C-invariant.

Conversely, assume that Ω is convex and C-invariant. We prove that D is Stein by showing that it is Levi-pseudoconvex, i.e. for all points $aK \in \partial D$ and local defining functions f of D near aK, one has $h_f(X, X) \geqslant 0$, for every tangent vector $X \in T_{aK}\partial D \cap JT_{aK}\partial D$, the complex tangent space to ∂D at aK.

Let $\mathbf{z} \in \partial\Omega$ and let $aK = \mathcal{L}(\mathbf{z})$. Denote by $W := T_{\mathbf{z}}\partial\Omega$ the tangent space to $\partial\Omega$ in $\mathbf{z}$. One can verify that the complex tangent space to ∂D at aK is given by

$$a_*(\bigoplus \mathfrak{g}^{e_j \pm e_l} \oplus \bigoplus \mathfrak{g}^{e_j}) \oplus (\mathcal{L}_*)_{\mathbf{z}} W \oplus J(\mathcal{L}_*)_{\mathbf{z}} W.$$

Let $\mathbf{v} = (v_1, \ldots, v_r)$ be an outer normal vector to W in $\mathbb{R}^r$. The C-invariance and the convexity of Ω imply that $v_j \leqslant 0$, for $j = 1, \ldots, r$ in the non-tube case, and $v_j \leqslant 0$, for $j = 1, \ldots, r-1$ in the tube case. Otherwise the space W would intersect $\mathbf{y} + C$, for every $\mathbf{y} \in \Omega$, yielding a contradiction.

Let $\widehat{f}$ be a smooth local defining function of Ω near $\mathbf{z}$. By the convexity of Ω, the Hessian $Hess(\widehat{f})(\mathbf{z})$ is positive definite on W. Moreover, as the gradient $\mathrm{grad}\widehat{f}(\mathbf{z})$ is a positive multiple of $\mathbf{v}$, one has $\frac{\partial \hat{f}}{\partial y_j}(\mathbf{z}) \leqslant 0$, for all $j = 1, \ldots, r$, in the non-tube case, and $\frac{\partial \hat{f}}{\partial y_j}(\mathbf{z}) \leqslant 0$, for all $j = 1, \ldots, r-1$, in the tube case.

Let f be the corresponding N-invariant local defining function of D near $aK = \exp L(\mathbf{z})K$. By Proposition 4.5, the Levi form of f is positive definite on $(\mathcal{L}_*)_{\mathbf{z}} W \oplus J(\mathcal{L}_*)_{\mathbf{z}} W \subset a_*\mathfrak{a} \oplus a_* J\mathfrak{a}$.

In addition, by (23) and Proposition 3.1, the Levi form of f is positive definite on $a_*(\bigoplus \mathfrak{g}^{e_j \pm e_l} \oplus \bigoplus \mathfrak{g}^{e_j})$. As a result, D is Levi pseudoconvex in $aK = \exp L(\mathbf{z})K$. Since aK is an arbitrary point in $\partial D \cap \exp \mathfrak{a} \cdot eK$ and both D and f are N-invariant, the domain D is Levi-pseudoconvex and therefore Stein, as desired.

In order to prove Theorem 4.9 in the non-smooth case, we need some preliminary Lemmas.

Lemma 4.10. *Let D be a domain in a Stein manifold, let $D' \subset D$ be a subdomain with smooth boundary and let $z \in \partial D \cap \partial D'$. If D' is not Levi pseudoconvex in z, then D is not Stein.*

Proof. Under our assumption, there exists a one dimensional complex submanifold M through z in X with $M \backslash \{z\} \subset D'$ ([Ran86], proof of Thm. 2.11, p. 56).

This implies that D is not Hartogs pseudoconvex ([Ran86], Thm. 2.9, p. 54) and in particular it is not Stein. □

For a domain Ω in $\mathbb{R}^r$, denote by $d_\Omega\colon \Omega \to \mathbb{R}$ the distance function from the boundary (if $\mathbf{z} \in \Omega$, then $d_\Omega(\mathbf{z})$ is by definition the radius of the largest ball centered in $\mathbf{z}$ and contained in Ω). The next lemma is a known characterization of convex domains.

Lemma 4.11. *A proper subdomain Ω of $\mathbb{R}^r$ is convex if and only if the function $-\ln d_\Omega : \Omega \to \mathbb{R}$ is convex.*

In what follows, for a fixed domain Ω in $\mathbb{R}^r$, we denote

$$u := -\ln d_\Omega.$$

Denote by $\mathbb{B}_\rho(\mathbf{y})$ the open ball of center $\mathbf{y} = (y_1, \dots, y_r) \in \mathbb{R}^r$ and radius ρ. Fix a smooth, positive, radial function $\sigma : \mathbb{R}^r \to \mathbb{R}$ (only depending on $R^2 = \|\mathbf{w}\|^2$), with support in $\mathbb{B}_1(\mathbf{0})$, such that $\sigma'(R^2) < 0$ and $\int_{\mathbb{R}^r} \sigma(\mathbf{w})d\mathbf{w} = 1$. For $\varepsilon > 0$, define $\Omega_\varepsilon := \{\mathbf{y} \in \Omega \ : \ d_\Omega(\mathbf{y}) > \varepsilon\}$ and $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$u_\varepsilon(\mathbf{y}) := \tfrac{1}{\epsilon^r} \int_{\mathbb{R}^r} u(\mathbf{z})\sigma(\tfrac{\mathbf{z}-\mathbf{y}}{\epsilon})d\mathbf{z} = \int_{\mathbb{R}^r} u(\mathbf{y} + \varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w}\,.$$

The functions u_ε are clearly smooth. Let $\nu : (\mathbb{R}^{>0})^r \to \mathbb{R}^{>0}$ be the stably convex positive function given by $\nu(\mathbf{y}) := \sum_j \frac{1}{y_j}$. Define $v_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$v_\varepsilon(\mathbf{y}) := u_\varepsilon(\mathbf{y}) + \varepsilon\nu(\mathbf{y})\,.$$

Lemma 4.12. *Let Ω be a convex, C-invariant domain in $(\mathbb{R}^{>0})^r$. Then the following facts hold true:*

(i) *The domain Ω_ε is convex and C-invariant for every $\varepsilon > 0$.*
(ii) *The smooth functions v_ε are stably convex and, for $\varepsilon \searrow 0$, they decrease to u uniformly on the compact subsets of Ω.*
(iii) *Let $\delta_\varepsilon := -\ln 3\varepsilon$. The sublevel set $\widetilde{\Omega}_\varepsilon := \{\mathbf{y} \in \Omega_\varepsilon \ : \ v_\varepsilon(\mathbf{y}) < \delta_\varepsilon\}$ is convex and C-invariant.*
(iv) *The boundary of $\widetilde{\Omega}_\varepsilon$ in $(\mathbb{R}^{>0})^r$ coincides with $\{\mathbf{y} \in \Omega_\varepsilon \ : \ v_\varepsilon(\mathbf{y}) = \delta_\varepsilon\}$ and it is smooth.*
(v) *As $n \in \mathbb{N}$ increases, the sequence of convex, C-invariant subdomains with smooth boundary $\widetilde{\Omega}_{1/n}$ exhausts Ω.*

Proof. (i) Let $\mathbf{y}$ and $\mathbf{y} + \mathbf{v}$ be elements of Ω_ε. Then $\mathbb{B}_\varepsilon(\mathbf{y})$ and $\mathbb{B}_\varepsilon(\mathbf{y} + \mathbf{v})$ are contained in Ω and, by the convexity of Ω, the same is true for $\mathbb{B}_\varepsilon(\mathbf{y} + \mathbf{t}\mathbf{v})$, for every $t \in [0, 1]$. This shows that Ω_ε is convex. Moreover, as Ω is C-invariant, if $\mathbb{B}_\varepsilon(\mathbf{y})$ is contained in Ω and $\mathbf{v}$ is an element of the cone C, then also the open ball $\mathbb{B}_\varepsilon(\mathbf{y} + \mathbf{v})$ is contained in Ω. This shows that Ω_ε is C-invariant.

(ii) As u is convex, for $\mathbf{y}$, $\mathbf{y}+\mathbf{v} \in \Omega$ and $t \in [0,1]$, one has

$$u_\varepsilon(\mathbf{y}+t\mathbf{v}) := \int_{\mathbb{R}^r} u(\mathbf{y}+t\mathbf{v}+\varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w}$$

$$\leqslant \int_{\mathbb{R}^r} \big((1-t)u(\mathbf{y}+\varepsilon\mathbf{w}) + tu(\mathbf{y}+\varepsilon\mathbf{w}+\mathbf{v})\big)\sigma(\mathbf{w})d\mathbf{w} = (1-t)u_\varepsilon(\mathbf{y}) + tu_\varepsilon(\mathbf{y}+\mathbf{v})\,,$$

showing that the smooth function u_ε is convex. Since ν is smooth and stably convex, it follows that $v_\varepsilon := u_\varepsilon + \varepsilon\nu$ is smooth and stably convex. Moreover, as convexity implies subharmonicity, then the last part of statement (ii) follows from [Hör94], Thm 3.2.3(ii), p.143.

(iii) Since the function v_ε is convex, then the domain $\widetilde{\Omega}_\varepsilon$ is convex. In order to show that $\widetilde{\Omega}_\varepsilon$ is C-invariant, we prove that

$$v_\varepsilon(\mathbf{y}+\mathbf{v}) < v_\varepsilon(\mathbf{y})\,, \tag{26}$$

for every $\mathbf{y} \in \Omega_\varepsilon$ and $\mathbf{v} \in C$. Since Ω is C-invariant, if for some $\mathbf{y} \in \Omega$ the ball $\mathbb{B}_r(\mathbf{y})$ is contained in Ω, then also the ball $\mathbb{B}_r(\mathbf{y}+\mathbf{v})$ is contained in Ω, for all $\mathbf{v} \in C$. It follows that $d_\Omega(\mathbf{y}) \leqslant d_\Omega(\mathbf{y}+\mathbf{v})$ and consequently $u(\mathbf{y}+\mathbf{v}+\varepsilon\mathbf{w}) \leqslant u(\mathbf{y}+\varepsilon\mathbf{w})$, for all $\mathbf{v} \in C$. and $\mathbf{w} \in \mathbb{B}_1(\mathbf{0})$. One deduces that

$$u_\varepsilon(\mathbf{y}+\mathbf{v}) = \int_{\mathbb{R}^r} u(\mathbf{y}+\mathbf{v}+\varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w} \leqslant \int_{\mathbb{R}^r} u(\mathbf{y}+\varepsilon\mathbf{w})\sigma(\mathbf{w})d\mathbf{w} = u_\varepsilon(\mathbf{y})\,,$$

for every $\mathbf{y} \in \Omega_\varepsilon$, $\mathbf{v} \in C$. Since $\nu(\mathbf{y}+\mathbf{v}) < \nu(\mathbf{y})$, one concludes that $v_\varepsilon(\mathbf{y}+\mathbf{v}) < v_\varepsilon(\mathbf{y})$, and $\widetilde{\Omega}_\varepsilon$ is C-invariant, as desired.

(iv) For $\mathbf{y}$ close to $\partial\Omega_\varepsilon = \{\mathbf{z} \in \Omega \;:\; d_\Omega(\mathbf{z}) = \varepsilon\,\}$, a rough extimate shows that $d_\Omega(\mathbf{y}+\varepsilon\mathbf{w}) < 3\varepsilon$, for every $\mathbf{w} \in \mathbb{B}_1(\mathbf{0})$. Therefore $v_\varepsilon(\mathbf{y}) > u_\varepsilon(\mathbf{y}) > -\ln 3\varepsilon$, implying that the boundary of $\widetilde{\Omega}_\varepsilon$ is contained in Ω_ε and it is given by $\partial\widetilde{\Omega}_\varepsilon = \{\mathbf{y} \in \Omega_\varepsilon \;:\; v_\varepsilon(\mathbf{y}) = \delta_\varepsilon\,\}$. Concerning the smoothness of $\partial\widetilde{\Omega}_\varepsilon$, the rank one case is trivial. So assume $r > 1$.

Let $\widehat{\mathbf{y}} \in \partial\widetilde{\Omega}_\varepsilon$. Set $\mathbf{v} := (1,\dots,1)$, in the non-tube case, and $\mathbf{v} := (1,\dots,1,0)$, in the tube case. Since $\mathbf{v}$ lies in the cone C, the inequality (26) implies that for γ small enough the real function $g : (-\gamma,\gamma) \to \mathbb{R}$, defined by $g(t) := v_\varepsilon(\widehat{\mathbf{y}}+t\mathbf{v})$, is strictly decreasing. By the stable convexity of v_ε, it is also stricltly convex and $g'(0) < 0$. As $g'(0)$ is a directional derivative of v_ε in $\widehat{\mathbf{y}}$, the differential $dv_\varepsilon|_{\widehat{\mathbf{y}}}$ does not vanish and the boundary of $\widetilde{\Omega}_\varepsilon$ is smooth.

(v) For $m > n$, the inclusion $\Omega_{1/n} \subset \Omega_{1/m}$ and the inequality $v_{1/n} > v_{1/m}$ imply that $\widetilde{\Omega}_{1/n} \subset \widetilde{\Omega}_{1/m}$. This concludes the proof of the lemma. □

Proof of Theorem 4.9: the general case. Let D be an arbitrary Stein, N-invariant domain in G/K. By Remark 4.4, the base Ω of the associated tube domain is necessarily convex. Assume by contradiction that Ω is not C-invariant (cf. Def. 4.8 and (20)), i.e. there exist $\mathbf{y} \in \Omega$ and $\mathbf{z} \in (\mathbf{y}+C) \cap \partial\Omega$. By the convexity of Ω, the open segment from $\mathbf{y}$ to $\mathbf{z}$ is contained in Ω. Moreover, the vector $\mathbf{v} = \mathbf{z} - \mathbf{y}$ lies in the cone C and points to the exterior of Ω. Let $\mathbb{B}_\varepsilon(\mathbf{y})$ be a relatively compact ball in Ω and define

$$t_{\max} := \max\{\, t > 0 \;:\; \mathbb{B}_\varepsilon(\mathbf{y} + t\mathbf{v}) \subset \Omega \,\}\,.$$

Then there exists $\mathbf{w} \in \partial\mathbb{B}_\varepsilon(\mathbf{y} + t_{\max}\mathbf{v}) \cap \partial\Omega$, and by construction

$$\langle \mathbf{w} - (\mathbf{y} + t_{max}\mathbf{v}), \mathbf{v} \rangle \;>\; 0.$$

This implies that the outer normal $\mathbf{n} := \mathbf{w} - (\mathbf{y} + t\mathbf{v})$ to $\partial\mathbb{B}_\varepsilon(\mathbf{y} + t_{\max}\mathbf{v})$ satisfies $n_j > 0$, for some $j \in \{1, \dots, r\}$ in the non-tube case (resp. $n_j > 0$, for some $j \in \{1, \dots, r-1\}$, in the tube case). From the result of the theorem in the smooth case, it follows that the N-invariant subdomain $N\exp(L(\mathbb{B}_\varepsilon(\mathbf{y} + t_{\max}\mathbf{v}))) \cdot eK$, with smooth boundary, is not Levi pseudoconvex in $\exp(L(\mathbf{w}))K$. Then Lemma 4.10 implies that D is not Stein, contradicting the assumption.

Conversely, assume that Ω is convex and C-invariant. By Lemma 4.12, the domain D can be realised as the increasing union of N-invariant domains $D_{1/n} := N\exp(L(\widetilde{\Omega}_{1/n})) \cdot eK$, where the open sets $\widetilde{\Omega}_{1/n} \subset \mathbb{R}^r$ are convex, C-invariant and have smooth boundary. By the result of the theorem in the smooth case, the domains $D_{1/n}$ are Stein and so is their increasing union D. This completes the proof of the theorem. □

We conclude this section with a univalence result for Stein, N-equivariant, Riemann domains over G/K.

Proposition 4.13. *Any holomorphically separable, N-equivariant, Riemann domain over G/K is univalent.*

Proof. Let Z be a holomorphically separable, N-equivariant, Riemann domain over G/K. By [Ros63], Z admits an holomorphic, N-equivariant open embedding into its envelope of holomorphy, which is a Stein N-equivariant, Riemann domain over G/K. Hence, without loss of generality, we may assume that Z is Stein.

Denote by $\pi : Z \to G/K$ the N-equivariant projection and let $\pi(Z) = N\exp(L(\Omega)) \cdot eK$ be the image of Z under π. Define $\Sigma := \exp(L(\Omega)) \cdot eK$ and $\widetilde{\Sigma} := \pi^{-1}(\Sigma)$. Note that $\widetilde{\Sigma}$ is a closed submanifold of Z.

Claim. *The map $\widetilde{\phi} : N \times \widetilde{\Sigma} \to Z$, given by $(n, x) \to n \cdot x$, is a diffeomorphism.*

Proof of the claim. Since $\Sigma = \pi(Z) \cap \exp(\mathfrak{a}) \cdot eK$ is a closed real submanifold of $\pi(Z)$ and π is a local biholomorphism, the restriction $\pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma$ is a local diffeomorphism. Moreover one has the commutative diagram

$$\begin{array}{ccc} N \times \widetilde{\Sigma} & \xrightarrow{\;\widetilde{\phi}\;} & Z \\ {\scriptstyle Id \times (\pi|_{\widetilde{\Sigma}})}\downarrow & & \downarrow{\scriptstyle \pi} \\ N \times \Sigma & \xrightarrow{\;\phi\;} & N\exp L(\Omega) \cdot eK \end{array}$$

where the maps $Id \times (\pi|_{\widetilde{\Sigma}})$, ϕ and π are local diffeomorphisms. Hence so is the map $\widetilde{\phi}$.

To prove that $\widetilde{\phi}$ is surjective, let $z \in Z$ and note that $\pi(z) = n\exp(L(\mathbf{y}))K$, for some $n \in N$ and $\mathbf{y} \in \Omega$. Then the element $w := n^{-1} \cdot z \in \widetilde{\Sigma}$ satisfies $n \cdot w = z$, implying the surjectivity of $\widetilde{\phi}$.

To prove that $\widetilde{\phi}$ is injective, assume that $n \cdot w = n' \cdot w'$, for some $n,\, n' \in N$ and $w,\, w' \in \widetilde{\Sigma}$. From the equivariance of π it follows that $n \cdot \pi(w) = n' \cdot \pi(w')$. As ϕ is bijective, it follows that $n = n'$ and $\pi(w) = \pi(w')$. Thus $w = (n^{-1}n') \cdot w' = w'$, implying the injectivity of $\widetilde{\phi}$ and concluding the proof of the claim.

Now, in order to prove the univalence of π, it is sufficient to show that the restriction $\pi|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \Sigma$ of π to $\widetilde{\Sigma}$ is injective. For this, consider the closed complex submanifold $R \cdot \widetilde{\Sigma} = \pi^{-1}(R \cdot \Sigma)$ of Z. As Z is Stein, so is $R \cdot \widetilde{\Sigma}$. Hence the restriction $\pi|_{R \cdot \widetilde{\Sigma}} : R \cdot \widetilde{\Sigma} \to R \cdot \Sigma$ defines an R-equivariant, Stein, Riemann domain over the Stein tube $R \cdot \Sigma$. As R is isomorphic to $\mathbb{R}^r$, from [CoLo86] it follows that $\pi|_{R \cdot \widetilde{\Sigma}}$ is injective. Hence the same is true for $\pi|_{\widetilde{\Sigma}}$ and π, as wished. □

Corollary 4.14. *The envelope of holomorphy $\widehat{D}$ of an N-invariant domain D in G/K is the smallest Stein domain in G/K containing D. More precisely, $\widehat{D}$ is the tube domain with base $\widehat{\Omega}$, the convex C-invariant hull of Ω.*

5. N-invariant psh functions vs. cvxdec functions

Let D be a Stein, N-invariant domain in a non-compact, irreducible Hermitian symmetric space G/K of rank r and let Ω be the base of the associated r-dimensional tube domain. Then Ω is a convex, C-invariant domain in $(\mathbb{R}^{>0})^r$ (Thm. 4.9). From Proposition 4.5 it follows that there is a one-to-one correspondence between the class of smooth N-invariant plurisubharmonic functions on D and the class of smooth convex functions on Ω satisfying an additional monotonicity condition (cf. Rem. 4.7 and Rem. 5.2). In this section we obtain an analogous result in the non-smooth context.

Let $\overline{C}$ be the closure of the cone defined in (20).

Definition 5.1. *A function $\widehat{f} : \Omega \to \mathbb{R}$ is (strictly) $\overline{C}$-decreasing if for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C} \backslash \{\mathbf{0}\}$ the restriction of $\widehat{f}$ to the half-line $\{\mathbf{y} + t\mathbf{v} \;:\; t \geqslant 0\}$ is (strictly) decreasing.*

Remark 5.2. (i) A smooth function $\widehat{f} : \Omega \to \mathbb{R}$ is $\overline{C}$-decreasing if and only if $\mathrm{grad} f(\mathbf{y}) \cdot \mathbf{v} \leqslant 0$ for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C} \backslash \{\mathbf{0}\}$.

(ii) A smooth, *stably convex* (cf. Def. 4.6) function $\widehat{f} : \Omega \to \mathbb{R}$ is $\overline{C}$-decreasing if and only if $\mathrm{grad} f(\mathbf{y}) \cdot \mathbf{v} < 0$, for every $\mathbf{y} \in \Omega$ and $\mathbf{v} \in \overline{C} \backslash \{\mathbf{0}\}$. This follows from the fact that the directional derivatives $\mathrm{grad} f(\mathbf{y}) \cdot \mathbf{v}$ of a stably convex, $\overline{C}$-decreasing function $\widehat{f}$ never vanish. In particular $\widehat{f}$ is automatically strictly $\overline{C}$-decreasing.

In view of the above observations, we define the following classes of functions:

- $ConvDec^{\infty,+}(\Omega)$: smooth, stably convex, $\overline{C}$-decreasing functions on Ω,
- $ConvDec^{\infty}(\Omega)$: smooth, convex, $\overline{C}$-decreasing functions on Ω,
- $Psh^{\infty,+}(D)^N$: smooth, N-invariant, strictly plurisubharmonic functions on D,
- $Psh^{\infty}(D)^N$: smooth, N-invariant, plurisubharmonic functions on D.

Proposition 4.5 established a one-to-one correspondence between $ConvDec^{\infty,+}(\Omega)$ and $Psh^{\infty,+}(D)^N$, as well as between $ConvDec^{\infty}(\Omega)$ and $Psh^{\infty}(D)^N$. The next goal is to extend such correspondences beyond the smooth context.

Let $\widehat{h} : \Omega \to \mathbb{R}$ be the smooth, stably convex, strictly $\overline{C}$-decreasing function

$$\widehat{h}(\mathbf{y}) := \sum_j \tfrac{1}{y_j}, \qquad \text{for } \mathbf{y} = (y_1, \ldots, y_r) \in \Omega, \tag{27}$$

and let h be the N-invariant strictly plurisubharmonic function on D associated to $\widehat{h}$.

Definition 5.3. *A function $\widehat{f} : \Omega \to \mathbb{R}$ is stably convex and $\overline{C}$-decreasing if every point in Ω admits a convex $\overline{C}$-invariant neighborhood W and $\varepsilon > 0$ such that $\widehat{f} - \varepsilon \widehat{h}$ is a convex, $\overline{C}$-decreasing function on W.*

Definition 5.4. *An N-invariant function $f \colon D \to \mathbb{R}$ is strictly plurisubharmonic if every point in D admits an N-invariant neighborhood U and $\varepsilon > 0$ such that $f - \varepsilon h$ is an N-invariant plurisubharmonic function on U (see also* [Gun90]*, Vol. 1, Def. 1, p. 118).*

In the smooth context the above notions coincide with the ones introduced earlier. Denote by

- $ConvDec^{+}(\Omega)$: stably convex and $\overline{C}$-decreasing functions on Ω;
- $ConvDec(\Omega)$: convex, $\overline{C}$-decreasing functions on Ω;
- $Psh^{+}(D)^N$: strictly plurisubharmonic, N-invariant functions on D;
- $Psh(D)^N$: plurisubharmonic, N-invariant functions on D.

The next theorem summarizes our results.

Theorem 5.5. *Let D be a Stein N-invariant domain in a non-compact, irreducible Hermitian symmetric space G/K of rank r. The map $f \to \widehat{f}$ is a bijection between the following classes of functions*

(i) $Psh^{\infty,+}(D)^N$ *and* $ConvDec^{\infty,+}(\Omega)$,
(ii) $Psh^{\infty}(D)^N$ *and* $ConvDec^{\infty}(\Omega)$,
(iii) $Psh\,(D)^N$ *and* $ConvDec(\Omega)$,
(iv) $Psh^{+}(D)^N$ *and* $ConvDec^{+}(\Omega)$.

In particular, N-invariant plurisubharmonic functions on D are necessarily continuous.

Proof. (i) and (ii) follow from Proposition 4.5 and Remark 5.2.
(iii) Let f be a function in $Psh\,(D)^N$. Since the restriction of f to the embedded r-dimensional Stein tube domain $R\exp(L(\Omega))\cdot eK \cong \mathbb{R}^r \times i\Omega$ (cf. Cor. 4.3) is plurisubharmonic and R-invariant, then $\widehat{f}$ is necessarily convex. Assume by contradiction that $\widehat{f}$ is not $\overline{C}$-decreasing. Then there exists $s \in \mathbb{R}$ such that the sublevel set $\{\widehat{f} < s\}$ is not $\overline{C}$-invariant. By Theorem 4.9, the corresponding N-invariant domain $\{f < s\}$ is not Stein. Since G/K is biholomorphic to a Stein domain in $\mathbb{C}^n$ and f is plurisubharmonic, this contradicts [Car73], Thm. B, p. 419. Hence $\widehat{f}$ belongs to $ConvDec(\Omega)$, as claimed.

In order to prove the converse, as in the previous section, for $\varepsilon > 0$ consider the convex C-invariant set $\Omega_\varepsilon := \{\mathbf{y} \in \Omega \ : \ d_\Omega(\mathbf{y}) > \varepsilon\}$. For $\widehat{f}$ in $ConvDec(\Omega)$, let $\widehat{f}_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ be the function

$$\widehat{f}_\varepsilon(\mathbf{y}) := \int_{\mathbb{R}^r} \widehat{f}(\mathbf{y} + \varepsilon\mathbf{w})\widehat{\sigma}(\mathbf{w})d\mathbf{w} + \varepsilon\widehat{h}\,,$$

where $\widehat{h}$ is the function given in (27) and $\widehat{\sigma} : \mathbb{R}^r \to \mathbb{R}$ is a smooth, positive, radial function (only depending on $R^2 = \|\mathbf{w}\|^2$), with support in $\mathbb{B}_1(\mathbf{0})$, such that $\widehat{\sigma}'(R^2) < 0$ and $\int_{\mathbb{R}^r} \widehat{\sigma}(\mathbf{w})d\mathbf{w} = 1$. Arguments analogous to those used in Lemma 4.12 show that the functions $\widehat{f}_\varepsilon$ are in $ConvDec^{\infty,+}(\Omega_\varepsilon)$. Then (i) implies that the corresponding functions f_ε belong to $Psh^{\infty,+}(D)^N$ and consequently f belongs to $Psh\,(D)^N$.
(iv) follows directly from the definition of $Psh^{+}(D)^N$ and of $ConvDec^{+}(\Omega)$.

Finally, from the inclusions

$$\begin{array}{ccccc} ConvDec^{+}(\Omega) & \subset & ConvDec(\Omega) & \subset & C^0(\Omega) \\ \cup & & \cup & & \\ ConvDec^{\infty,+}(\Omega) & \subset & ConvDec^{\infty}(\Omega) & & \end{array}$$

it follows that all the above functions on Ω are continuous, and so are the corresponding N-invariant plurisubharmonic functions on D. □

6. The Siegel domain point of view

The goal of this section is to present an alternative characterization of Stein N-invariant domains in an irreducible Hermitian symmetric space G/K, realized as a Siegel domain.

Denote by $S = NA$ the real split solvable group arising from the Iwasawa decomposition of G subordinated to Σ^+. With the complex structure J described in (3) and the linear form $f_0 \in \mathfrak{s}^*$ defined by $f_0(X) := B(X, Z_0)$, where $Z_0 \in Z(\mathfrak{k})$ is the element inducing the complex structure on $\mathfrak{p}$, the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ of S has the structure of a *normal J-algebra* (see [GPSV68] and [RoVe73], Sect. 5, A).

This means in particular that $\omega(X, Y) := -f_0([X, Y])$ is a non-degenerate skew-symmetric bilinear form on $\mathfrak{s}$ and that the symmetric bilinear form $\langle X, Y \rangle := -f_0([JX, Y])$ is the J-invariant positive definite inner product on $\mathfrak{s}$ defined in (2).

The adjoint action of $\mathfrak{a}$ on $\mathfrak{s}$ decomposes $\mathfrak{s}$ into the orthogonal direct sum of the restricted root spaces. Moreover, the adjoint action of the element $A_0 = \frac{1}{2}\sum_j A_j \in \mathfrak{a}$ decomposes $\mathfrak{s}$ and $\mathfrak{n}$ as

$$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_{1/2} \oplus \mathfrak{s}_1, \qquad \mathfrak{n}_j = \mathfrak{n} \cap \mathfrak{s}_j$$

where

$$\mathfrak{s}_0 = \mathfrak{a} \oplus \bigoplus_{1 \leqslant j < l \leqslant r} \mathfrak{g}^{e_j - e_l}, \quad \mathfrak{s}_{1/2} = \oplus_{1 \leqslant j \leqslant r} \mathfrak{g}^{e_j}, \quad \mathfrak{s}_1 = \oplus_{1 \leqslant j \leqslant r} \mathfrak{g}^{2e_j} \oplus \bigoplus_{1 \leqslant j < l \leqslant r} \mathfrak{g}^{e_j + e_l}. \tag{28}$$

Let $E_0 := \sum E^j$. The orbit

$$V := Ad_{\exp \mathfrak{s}_0} E_0 \tag{29}$$

is a sharp convex homogeneous selfadjoint cone in $\mathfrak{s}_1$ and

$$F \colon \mathfrak{s}_{1/2} \times \mathfrak{s}_{1/2} \to \mathfrak{s}_1 + i\mathfrak{s}_1, \qquad F(W, W') = \frac{1}{4}([JW', W] - i[W', W]),$$

is a V-valued Hermitian form, i.e. it is sesquilinear and $F(W, W) \in \overline{V}$, for all $W \in \mathfrak{s}_{1/2}$. The Hermitian symmetric space G/K is realized as a Siegel domain in $\mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$ as follows

$$D(V, F) = \{(Z, W) \in \mathfrak{s}_1 \oplus i\mathfrak{s}_1 \oplus \mathfrak{s}_{1/2} \mid Im(Z) - F(W, W) \in V\}.$$

If $\mathfrak{s}_{1/2} = \{0\}$ then G/K is of *tube type*, otherwise it is of *non-tube type*. The group S acts on $D(V, F)$ by the affine transformations

$$(Z, W) \mapsto (Ad_s Z + a + 2iF(Ad_s W, b) + iF(b, b), Ad_s W + b), \tag{30}$$

where $s \in \exp \mathfrak{s}_0$, $a \in \mathfrak{s}_1$, and $b \in \mathfrak{s}_{1/2}$. Recall that $J\mathfrak{a} = \oplus_j \mathfrak{g}^{2e_j}$, (cf. (4)) and denote by $J\mathfrak{a}^+$ the positive octant in $J\mathfrak{a}$. One easily verifies that if $E \in J\mathfrak{a}^+$, then $Ad_{\exp \mathfrak{a}} E = J\mathfrak{a}^+$. This and the fact that S acts freely and transitively on $D(V, F)$ imply that every N-orbit meets the set $J\mathfrak{a}^+$ is a unique point.

Let D be an N-invariant domain in a symmetric Siegel domain. Then

$$D = \{(Z, W) \in D(V, F) \mid Im(Z) - F(W, W) \in V_D\},$$

where V_D is an $Ad_{\exp \mathfrak{n}_0}$-invariant open subset in V, determined by

$$iV_D := D \cap iV.$$

The r-dimensional set

$$v_D := V_D \cap J\mathfrak{a}^+,$$

intersects every N-orbit of D in a unique point, and it is the base of an r-dimensional tube domain in $J\mathfrak{a}\oplus iJ\mathfrak{a}$. The map $R\exp\mathfrak{a}\cdot eK\to R\exp\mathfrak{a}\cdot(iE_0,0)$

$$\exp(\textstyle\sum_j x_jE^j)\exp(\frac{1}{2}\sum_k\ln(y_k)A_k)K\mapsto(iAd_{\exp(\frac{1}{2}\sum_k\ln(y_k)A_k)}E_0+\sum_j x_jE^j,0)$$

is the inverse of the map $\mathcal{L}$ of Proposition 4.1 (cf. Cor. 4.3).

Let C be the cone defined in (20). Then the characterization of N-invariant Stein domains in a symmetric Siegel domain can be formulated as follows.

Proposition 6.1. *Let D be an N-invariant domain in an irreducible symmetric Siegel domain. Then D is Stein if and only if V_D is convex and C-invariant.*

In order to prove the above proposition, we need some preliminary results. For this we separate the tube and the non-tube case.

The tube case. Denote by $conv(V_D)$ the convex hull of V_D in $\mathfrak{s}_1$. Since V_D is $Ad_{\exp\mathfrak{n}_0}$-invariant and the action is linear, then also $conv(V_D)$ is $Ad_{\exp\mathfrak{n}_0}$-invariant. Denote by $p\colon\mathfrak{s}_1\to J\mathfrak{a}$ the projection onto $J\mathfrak{a}$, parallel to $\oplus\mathfrak{g}^{e_j+e_l}$. Denote by

$$(E^1)^*,\ldots,(E^r)^* \tag{31}$$

the elements in the dual $\mathfrak{n}^*$ of $\mathfrak{n}$, with the property that $(E^j)^*(E^l)=\delta_{jl}$ and $(E^j)^*(X^\alpha)=0$, for all $X^\alpha\in\mathfrak{g}^\alpha$, with $\alpha\in\Sigma^+\backslash\{2e_1,\ldots,2e_r\}$.

Lemma 6.2. *One has*

(i) *Let $E=\sum x_kE^k\in J\mathfrak{a}^+$, where $x_k\in\mathbb{R}^{>0}$. Then*

$$p(Ad_{\exp\mathfrak{n}_0}E)=E+C_{r-1}.$$

In particular, $(E^r)^*(Ad_{\exp tX}E)=x_r$, for all $X\in\mathfrak{n}_0$ and $t\in\mathbb{R}$.

(ii) *Let $X\in\mathfrak{g}^{e_j-e_l}$. Then $[[E^l,X],X]=sE^j$, for some $s\in\mathbb{R}^{>0}$.*

(iii) *One has $p(conv(V_D))=conv(p(V_D))$.*

Proof. (i) Let $E\in J\mathfrak{a}^+$ and let $h_0\in\exp\mathfrak{n}_0$, where $\mathfrak{n}_0=\oplus_{1\leqslant i<j\leqslant r}\mathfrak{g}^{e_i-e_j}$. By Theorem 4.10 in [RoVe73], for every $1\leqslant i<j\leqslant r$ there exists a basis $\{E^p_{ij}\}$ of $\mathfrak{g}^{e_i-e_j}$, with coordinates $\{x^p_{ij}\}_p$, such that

$$(E^i)^*(Ad_{h_0}E)=x_i(1+\textstyle\sum_{p,\,j>i}(x^p_{ij})^2)$$

(formula (4.13) in [RoVe73]). Since $i<r$, one has $p(Ad_{\exp X}E)=E+C_{r-1}$, as claimed. In particular the r^{th} coordinate of E does not vary under the $Ad_{\exp\mathfrak{n}_0}$-action.

(ii) Let $X\in\mathfrak{g}^{e_j-e_l}$. Then $\exp tX\in\exp\mathfrak{n}_0$ and the curve

$$Ad_{\exp tX}E_0=\exp ad_{tX}(E_0)=E_0+t[X,E^l]+\tfrac{t^2}{2}[X,[X,E^l]],\ t\in\mathbb{R},$$

is contained in V. By Lemma 2.3 (a), its projection onto $J\mathfrak{a}$ is given by

$$p(Ad_{\exp tX}E_0)=(E^j)^*(Ad_{\exp tX}E_0)E^j=(1+\tfrac{t^2}{2}s)E^j,$$

for some $s \in \mathbb{R}$, $s \neq 0$. Now (i) implies that $1 + \frac{t^2}{2}s > 0$, for all $t \in \mathbb{R}$. Therefore $s > 0$, as claimed.

(iii) We prove the two inclusions. By the linearity of p, the set $p(conv(V_D))$ is convex and contains $p(V_D)$. Hence, $p(conv(V_D)) \supset conv(p(V_D))$. Conversely, let $z \in conv(V_D)$. Then there exist $t_0 \in (0,1)$ and $x, y \in V_D$ such that $z = t_0x + (1 - t_0)y$. Since $p(z) = t_0p(x) + (1 - t_0)p(y)$, one has $p(conv(V_D)) \subset conv(p(V_D))$. □

The non-tube case. Denote by $\widetilde{p}\colon \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2} \to iJ\mathfrak{a}$ the projection onto $iJ\mathfrak{a}$ parallel to $\mathfrak{s}_1 \oplus i(\oplus \mathfrak{g}^{e_j+e_l}) \oplus \mathfrak{s}_{1/2}$.

Lemma 6.3. *Let $E \in J\mathfrak{a}^+$. Then $\widetilde{p}(N \cdot (iE, 0)) = i(E + \overline{C}_r)$.*

Proof. The N-orbit of the point $(iE, 0) \in \mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$ is given by

$$N \cdot (iE, 0) = S_{1/2}S_1Ad_{\exp \mathfrak{n}_0}(iE, 0) = (a + i(Ad_{\exp \mathfrak{n}_0}E + F(b, b)), b), \qquad (32)$$

where $a \in \mathfrak{s}_1$ and $b \in \mathfrak{s}_{1/2}$. By (32) and Lemma 6.2 (i), one has $\widetilde{p}(N \cdot (iE, 0)) = i(E + C_{r-1} + \widetilde{p}(F(\mathfrak{s}_{1/2}, \mathfrak{s}_{1/2})))$. Since in the symmetric case $\{[Jb, b],\ b \in \mathfrak{s}_{1/2}\} = \overline{J\mathfrak{a}^+}$, it follows that $\widetilde{p}(N \cdot (iE, 0)) = i(E + \overline{C}_r)$, as claimed. □

Remark 6.4. (a) *Statement (i) in Lemma 6.2 explains why in Prop.3.1 (iii) no conditions appear on $\frac{\partial \tilde{f}}{\partial a_r}$.*
(b) *Statement (ii) in Lemma 6.2 and the fact that $F(b, b) = [Jb, b]$, for $b \in \mathfrak{s}_{1/2}$, takes values in $\overline{J\mathfrak{a}^+}$, explain why the real constants s and t in Lemma 2.3(a)(b) and later in Proposition 3.1(iii)(iv) are strictly positive.*

Proof of Proposition 6.1. **The tube case**. An N-invariant domain D in a symmetric tube domain $D(V)$ is itself a tube domain with base the $Ad_{\exp \mathfrak{n}_0}$-invariant set V_D. Hence all we have to prove is that V_D is convex if and only if v_D is convex and $v_D + C_{r-1} \subset v_D$.

Assume that V_D is convex. Then v_D is convex, being the intersection of V_D with the positive octant $J\mathfrak{a}^+$. To prove that v_D is C-invariant, let $E = \sum_j x_jE^j \in v_D$, where $x_j > 0$, and let $X \in \mathfrak{g}^{e_j - e_l}$ be a non-zero element. For every $t \in \mathbb{R}$,

$$Ad_{\exp tX}E = E + tx_l[X, E^l] + \tfrac{1}{2}t^2x_l[X, [X, E^l]]$$

lies in V_D and, by the convexity assumption, so does $E + \frac{1}{2}t^2x_l[X, [X, E^l]] = E + t^2sx_lE^j$, where $s > 0$ (cf. Lemma 6.2 (ii)). This argument applied to all $j = 1, \dots, r-1$ and the convexity of v_D show that $v_D + C_{r-1} \subset v_D$, as desired.

Conversely, assume that v_D convex and C-invariant. We prove the convexity of V_D by showing that $conv(V_D) \subset V_D$. From Lemma 6.2 (ii) and the C-invariance of v_D, one has

$$p(V_D) = p(Ad_{\exp \mathfrak{n}_0}v_D) = v_D + C_{r-1} \subset v_D.$$

Moreover, from Lemma 6.2 (iii), the above inclusion and the convexity of v_D, one has

$$conv(V_D) \cap J\mathfrak{a} \subset p(conv(V_D)) = conv(p(V_D)) \subset v_D.$$

Finally, from the $Ad_{\exp \mathfrak{n}_0}$-invariance of $conv(V_D)$ it follows that

$$conv(V_D) = Ad_{\exp \mathfrak{n}_0}(conv(V_D) \cap J\mathfrak{a}) \subset Ad_{\exp \mathfrak{n}_0} v_D = V_D.$$

This completes the proof of the proposition in the tube case.

The non-tube case. Let D be an N-invariant domain in a Siegel domain $D(V,F)$. Denote by $conv(D)$ the convex hull of D in $\mathfrak{s}_1^{\mathbb{C}} \oplus \mathfrak{s}_{1/2}$. As N acts on D by affine transformations, also $conv(D)$ is N-invariant.

If D is Stein, then $D \cap \{W = 0\}$ is a Stein tube domain in $\mathfrak{s}_1^{\mathbb{C}}$ with base V_D. By the result for the tube case and Lemma 6.3, v_D is convex and $v_D + \overline{C}_r \subset v_D$.

Conversely, assume that v_D is convex and C-invariant, i.e. $v_D + \overline{C}_r \subset v_D$ (see Def. 4.8). We are going to prove that D is convex. By Lemma 6.3, one has

$$\widetilde{p}(D) = \widetilde{p}(N \cdot v_D) = i(v_D + \overline{C}_r) \subset i v_D.$$

Moreover,

$$conv(D) \cap iJ\mathfrak{a} \subset \widetilde{p}(conv(D)) = conv(\widetilde{p}(D)) \subset i v_D.$$

By the N-invariance of $conv(D)$, one obtains

$$conv(D) = N \cdot (conv(D) \cap iJ\mathfrak{a}) \subset N \cdot i v_D = D.$$

Hence D is convex and therefore Stein (cf. [Gun90], Vol.1, Thm.10, p. 67). This concludes the proof of the proposition. □

Remark. *The assumption $v_D + C_r \subset v_D$ implies $v_D + C_{r-1} \subset v_D$ and in particular V_D is convex. This means that if $D \subset D(V,F)$ is Stein, then the tube domain $D \cap \{W = 0\}$ is Stein. The converse may not hold true, as $V_D = Ad_{\exp \mathfrak{n}_0} v_D$ convex does not imply $v_D + C_r \subset v_D$.*

7. Appendix: N-invariant potentials for the Killing metric.

Let G/K be a non-compact, irreducible Hermitian symmetric space. The Killing form B of $\mathfrak{g}$, restricted to $\mathfrak{p}$, induces a G-invariant Kähler metric on G/K, which we refered to as the Killing metric. In this section we exhibit an N-invariant potential of the Killing metric and the associated moment map in a Lie theoretical fashion. All the N-invariant potentials of the Killing metric are detemined in Remark 7.5.

Let $f\colon G/K \to \mathbb{R}$ be a smooth N-invariant function. The map $\mu\colon G/K \to \mathfrak{n}^*$, defined by

$$\mu_f(z)(X) := d^c f(\widetilde{X}_z), \tag{33}$$

for $X \in \mathfrak{n}$, is N-equivariant (cf. (13)). If f is strictly plurisubharmonic, then it is referred to as the moment map associated with f.

Proposition 7.1. *Let* $z = naK \in G/K$*, where* $n \in N$*,* $a = \exp H \in A$ *and* $H = \sum_j a_j A_j \in \mathfrak{a}$*. Let* $\mathbf{b}$ *be the constant defined in (11).*
(i) The N*-invariant function* $\rho : G/K \to \mathbb{R}$ *defined by*

$$\rho(naK) := -\tfrac{1}{2}\textstyle\sum_{j=1}^{r} B(H,\, A_j) = -\tfrac{\mathbf{b}}{2}(a_1 + \cdots + a_r)\,,$$

is a potential of the Killing metric.
(ii) The moment map $\mu_\rho : G/K \to \mathfrak{n}^*$ *associated with* ρ *is given by*

$$\mu_\rho(naK)(X) = -\tfrac{\mathbf{b}}{4}\textstyle\sum_{j=1}^{r} e^{-2a_j}(E^j)^*(\mathrm{Ad}_{n^{-1}}X) = B(Ad_{n^{-1}}X, Ad_a Z_0)\,, \tag{34}$$

where $X \in \mathfrak{n}$*, and the* $(E^j)^*$ *are defined in (31).*

Proof. (i) Let $naK \in G/K$, where $a = \exp H$ and $H = \sum_j a_j A_j$. The function $\widetilde{\rho} : \mathfrak{a} \to \mathbb{R}$ associated to ρ is given by $\widetilde{\rho}(H) = -\frac{1}{2}\sum_{j=1}^{r} a_j B(A_j,\, A_j)$ (cf. (7)). In order to obtain (i), we first prove the identities (34). By (33) and (14), one has

$$\mu_\rho(aK)(X) = d^c\rho(\widetilde{X}_{aK}) = -\tfrac{\mathbf{b}}{4}\textstyle\sum_{j=1}^{r} e^{-2a_j}(E^j)^*(X). \tag{35}$$

By (2), one has

$$(E^j)^*(X) = B(X, \theta E^j)/B(E^j, \theta E^j) = 2B(X, \tfrac{1}{2}(E^j + \theta E^j))/B(E^j, \theta E^j).$$

Since

$$\mathbf{b} := B(A_j, A_j) = B(I_0 A_j, I_0 A_j) = B(E^j - \theta E^j, E^j - \theta E^j) = -2B(E^j, \theta E^j)$$

and $Z_0 = S_0 + \frac{1}{2}\sum_j E^j + \theta E^j$, for some $S_0 \in \mathfrak{m}$ (cf.[GeIa21], Sect. 2), one obtains

$$-\tfrac{\mathbf{b}}{4}\textstyle\sum_{j=1}^{r} e^{-2a_j}(E^j)^*(X) = -\tfrac{\mathbf{b}}{2}\sum_{j=1}^{r} e^{-2a_j}\; B(X, \tfrac{1}{2}(E^j + \theta E^j)/B(E^j, \theta E^j)$$

$$= \textstyle\sum_{j=1}^{r}\; B(X,\; Ad_a\; \tfrac{1}{2}(E^j + \theta E^j)) \;=\; B(X,\; Ad_a\; Z_0)\,,$$

and (34) follows from the N-equivariance of μ_ρ.

Next we are going to show that on $\mathfrak{p} \times \mathfrak{p}$ one has

$$h_\rho(\,a_*\cdot\,,\, a_*\cdot\,) = B(\,\cdot\,,\,\cdot\,). \tag{36}$$

Every $X \in \mathfrak{s}$ decomposes as $X = (X - \phi(X)) + \phi(X) \in \mathfrak{k} \oplus \mathfrak{p}$ (see Sect. 2). Since the projection $\phi : \mathfrak{s} \to \mathfrak{p}$ is a linear isomorphism, (36) is equivalent to

$$h_\rho(a_*X, a_*Y) = h_\rho(a_*\phi(X), a_*\phi(Y)) = B\big(\phi(X),\, \phi(Y)\big) = -\tfrac{1}{2}B\big(X,\, \theta Y\big)\,, \tag{37}$$

for all $X,\ Y$ in $\mathfrak{s}$. By Proposition 3.1(i), it is sufficient to consider X, Y both in the same block $a_*\mathfrak{a}$, $a_*\mathfrak{g}^{e_j - e_l}$, and $a_*\mathfrak{g}^{2e_j}$.

Let $A_j, A_l \in \mathfrak{a}$, be as in (1). Then, by (ii) of Proposition 3.1, one has

$$h_\rho(a_*A_j, a_*A_l) = \delta_{jl}B(A_l,\, A_l) = B(A_j, A_l)\,.$$

Let $X,\, Y \in \mathfrak{g}^\alpha$, with $\alpha = e_j - e_l$ or $\alpha = e_j$. Then $JY \in \mathfrak{g}^\beta$, for $\beta = e_j + e_l$ or $\beta = e_j$, respectively. From (15) and (i) one obtains

$$h_\rho(a_*X, a_*Y) = -e^{\alpha(H) + \beta(H)} d^c\rho(\widetilde{[JY, X]}_z)$$

$$= -e^{\alpha(H)+\beta(H)} B([JY, X], Ad_a Z_0). \tag{38}$$

From the invariance properties of the Killing form B, the decomposition of X and JY in $\mathfrak{k} \oplus \mathfrak{p}$ and the identity $\phi(J\cdot) = I_0\phi(\cdot)$ (cf. (3)), one has

$$B([JY, X], Ad_a Z_0) = B(Ad_{a^{-1}}[JY, X], Z_0) = e^{-(\alpha(H)+\beta(H))} B([JY, X], Z_0)$$
$$= e^{-(\alpha(H)+\beta(H))} \left(B([JY - \phi(JY), X - \phi(X)], Z_0) + B([\phi(JY), \phi(X)], Z_0)\right)$$
$$= e^{-(\alpha(H)+\beta(H))} B([[Z_0, \phi(Y)], \phi(X)], Z_0) = e^{-(\alpha(H)+\beta(H))} B(\phi(X), [Z_0, [Z_0, \phi(Y)]])$$
$$= -e^{-(\alpha(H)+\beta(H))} B(\phi(X), \phi(Y)) = \tfrac{1}{2} e^{-(\alpha(H)+\beta(H))} B(X, \theta Y).$$

It follows that

$$h_\rho(a_* X, a_* Y) = -\tfrac{1}{2} B\big(X, \theta Y\big), \tag{39}$$

as desired. This concludes the proof of (i).

(ii) The identity (39) implies that the N-invariant function ρ is strictly plurisubharmonic. Hence μ_ρ is the moment map associated to ρ. □

Remark 7.2. *Combining (16) and (17) in Proposition 3.1 with (37), we obtain the exact value of the positive quantities s and t*

$$s = \tfrac{4\|X\|^2}{\mathbf{b}}, \quad \textit{for } X \in \mathfrak{g}^{e_j - e_l}, \quad \textit{and} \quad t = \tfrac{4\|X\|^2}{\mathbf{b}}, \quad \textit{for } X \in \mathfrak{g}^{2e_j}.$$

Remark 7.3. *The map $\mu_G : G/K \to \mathfrak{g}^*$ given by $\mu_G(gK)(\cdot) := B(\mathrm{Ad}_{g^{-1}} \cdot, Z_0)$ is a moment map for the G-action on G/K. The moment map μ_ρ in (ii) of Proposition 7.1 can be obtained by restricting $\mu_G(naK)$ to $\mathfrak{n}$. Namely, for $X \in \mathfrak{n}$ and $naK \in G/K$ one has*

$$\mu_\rho(naK)(X) = \mu_G(naK)(X) = B(\mathrm{Ad}_{(na)^{-1}} X, Z_0).$$

In the next remark, all possible N-invariant potentials of the Killing metric are determined.

Remark 7.4. *Let $\rho \colon G/K \to \mathbb{R}$ be the potential of the Killing metric given in Proposition 7.1 and let σ be another N-invariant potential. Let $\widehat{\rho}$ and $\widehat{\sigma}$ be the corresponding functions on $(\mathbb{R}^{>0})^r$ defined in (19).*

(a) In the non-tube case, one has $\widehat{\sigma} = \widehat{\rho} + d$, and therefore $\sigma = \rho + d$, for some $d \in \mathbb{R}$;

(b) In the tube case, one has $\widehat{\sigma}(\mathbf{y}) = \widehat{\rho}(\mathbf{y}) + cy_r + d$, for $c, d \in \mathbb{R}$. In particular

$$\sigma(n \exp(L(\mathbf{y}))K) = \rho(n \exp(L(\mathbf{y}))K) + cy_r + d,$$

where $n \in N$, $\mathbf{y} = (y_1, \ldots, y_r) \in (\mathbb{R}^{>0})^r$, and $c, d \in \mathbb{R}$.

Proof. Let $f := \sigma - \rho$ be the difference of the two potentials. Then f is a smooth N-invariant function on G/K such that $dd^c f(\cdot, J\cdot) \equiv 0$. Let $\widehat{f}\colon \Omega \to \mathbb{R}$ be the associated function.
(a) In the non-tube case, by Proposition 3.1 (iv) and (23), the function $\widehat{f}$ satisfies $\frac{\partial \widehat{f}}{\partial y_j} \equiv 0$, for all $j = 1, \dots r$. Hence $\widehat{f}$ is constant on $(\mathbb{R}^{>0})^r$ and f is constant on G/K.
(b) In the tube case, from Proposition 3.1, (25) and (23), it follows that $\frac{\partial \widehat{f}}{\partial y_j} \equiv 0$, for all $j = 1, \dots r-1$, and $\frac{\partial^2 \widehat{f}}{\partial y_r^2} \equiv 0$. Hence $\widehat{f}$ is an affine function of the variable y_r . Equivalently, $\widehat{\sigma}(\mathbf{y}) = \widehat{\rho}(\mathbf{y}) + cy_r + d$, for $c, d \in \mathbb{R}$, as claimed. □

Remark 7.5. *Let $D(V, F)$ be a symmetric Siegel domain. Then the Bergman kernel function $K(z, z)$ is N-invariant and $\ln K(z, z)$ is a potential of the Bergman metric. As both the Killing and the Bergman metric are G-invariant, they differ by a multiplicative constant. It follows that $\ln K(z, z)$ is a multiple of one of the N-invariant potentials of the Killing metric described in the above remark.*

Example 7.6. *As an application of Remark 7.5, we compute all N-invariant potentials of the Killing metric for the upper half-plane in $\mathbb{C}$ and for the Siegel upper half-plane of rank 2.*
(a) Let $G = SL(2, \mathbb{R})$ and let G/K be the corresponding Hermitian symmetric space. Fix an Iwasawa decomposition NAK of G. Since $\mathbf{b} = 8$ and $r = 1$, then the potential of the Killing metric given in Proposition 7.1 is

$$\rho(naK) = -4a_1 \quad \text{and} \quad \widehat{\rho}(y_1) = \rho(\exp L(y_1)K) = \ln \tfrac{1}{y_1^2}.$$

Realize G/K as the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$, i.e. the orbit of $i \in \mathbb{C}$ under the $SL(2, \mathbb{R})$-action by linear fractional transformations. Fix

$$N = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \ : \ m \in \mathbb{R} \right\} \quad \text{and} \quad A = \left\{ \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{-a_1} \end{pmatrix} \ : \ a_1 \in \mathbb{R} \right\},$$

and let $\{x_1 + iy_1 \in \mathbb{C} \ : \ y_1 > 0\}$ be tube associated to G/K. Since

$$x_1 + iy_1 \to \exp(x_1 E^1) \exp(\tfrac{1}{2} \ln y_1 A_1) \cdot i = x_1 + iy_1$$

(cf. Prop. 4.1), then the potential ρ on $\mathbb{H}$ reads as $\rho(z) = \ln \frac{1}{(\mathrm{Im} z)^2}$.
If $\sigma\colon \mathbb{H} \to \mathbb{R}$ is an arbitrary N-invariant potential of the Killing metric, then by Remark 7.5

$$\sigma(z) = \ln \tfrac{1}{(\mathrm{Im} z)^2} + c\mathrm{Im} z + d, \qquad c, d \in \mathbb{R}.$$

(b) The Siegel upper half-plane of rank 2

$$\mathcal{P} = \{W = S + iT \in M(2, 2, \mathbb{C}) \mid {}^tW = W, \ T > 0\},$$

of 2×2 *complex symmetric matrices with positive definite imaginary part, is the orbit of* iI_2 *under the action by linear fractional transformations of the real symplectic group* $Sp(2, \mathbb{R})$. *Fix the Iwasawa decomposition such that*

$$N = \left\{ \begin{pmatrix} \mathbf{n} & \mathbf{m} \\ \mathbf{0} & {}^t\mathbf{n}^{-1} \end{pmatrix} \right\}, \qquad A = \left\{ \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^{-1} \end{pmatrix} \right\},$$

where $\mathbf{n}$ *is unipotent,* $\mathbf{n}\,{}^t\mathbf{m}$ *is symmetric and* $\mathbf{a} = \begin{pmatrix} e^{a_1} & 0 \\ 0 & e^{a_2} \end{pmatrix}$, *with* a_1, a_1 *coordinates in* $\mathfrak{a}$ *with respect to the basis defined in Lemma 2.2.*

As $\mathbf{b} = 12$, *the potential of the Killing metric defined in Proposition 7.1 is given by*

$$\rho(naK) = -6(a_1 + a_2) \quad \text{and} \quad \widehat{\rho}(y_1, y_2) = \rho(\exp L(y_1, y_2)K) = \ln \tfrac{1}{(y_1 y_2)^3}.$$

A matrix $S + iT \in \mathcal{P}$ *can be expressed in a unique way as*

$$na \cdot iI_2 = n \cdot \begin{pmatrix} ie^{2a_1} & 0 \\ 0 & ie^{2a_2} \end{pmatrix}.$$

If $T = \begin{pmatrix} t_1 & t_3 \\ t_3 & t_2 \end{pmatrix}$, *a simple computation shows that* $e^{2a_1} = t_1 - t_3^2/t_2$ *and* $e^{2a_2} = t_2$. *Hence* $y_1 = t_1 - t_3^2/t_2$, $y_2 = t_2$ *and* $\rho(S + iT) = \ln \frac{1}{(t_1 t_2 - t_3^2)^3}$.

If σ *is an arbitrary* N*-invariant potential of the Killing form, then by Remark 7.5*

$$\sigma(S + iT) = \ln \tfrac{1}{(t_1 t_2 - t_3^2)^3} + ct_2 + d, \qquad \text{for some } c, d \in \mathbb{R}.$$

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