A NEGISHI APPROACH TO RECURSIVE CONTRACTS

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Abstract

In this paper we argue that a large class of recursive contracts can be studied by means of the conventional Negishi method. A planner is responsible for prescribing current actions along with a distribution of future utility values to all agents, so as to maximize their weighted sum of utilities. Under convexity the method yields the exact efficient frontier. Otherwise the implementation requires contracts be contingent on publicly observable random signals uncorrelated to fundamentals. We compare our approach with the dual method established in the literature. Finally, considering maxmin-type social welfare functions, we clarify that the dynamics of efficient contracts can be expressed as a stochastic evolution of welfare shares.

Keywords: Recursive contracts, efficiency, Negishi method, dynamic programming, optimal policy.

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1. Introduction

It has been a long tradition in economics to characterize Pareto frontiers of various economic environments through the maximization of weighted sums of individuals' utilities subject to appropriate constraints, a method conventionally attributed to Negishi [31]. Lucas and Stokey [24] showed that this approach has a convenient recursive decomposition in deterministic growth economies with recursive utilities (see also Anderson [4], Dana and LeVan [11] and Kan [16]). We argue that Lucas and Stokey [24]'s approach extends to virtually all structures of interest in dynamic economies. In particular, we develop a Negishi method for economies with recursive utilities and forward-looking constraints. To keep the analysis tight, we abstract from endogenous state variables (such as capital or debt), and we omit a thorough study of implied optimal policies. Both tasks are accomplished by conventional extensions.

We study dynamic economies involving finitely many individuals whose utility is recursively generated by an aggregator. A contract, or a distribution, prescribes a contingent plan of actions for individuals subject to material balance and incentive constraints of various nature. Our framework is general enough to encompass many economic environments of interest such as optimal allocations in asymmetric information economies (*e.g.*, Atkeson and Lucas [6]), efficient distributions with risk-sensitive preferences (*e.g.*, Anderson [4]) and optimal risk-sharing under limited commitment (*e.g.*, Kocherlakota [18]). Furthermore, by reinterpreting one of the individuals as a principal, our formulation also applies to more conventional principal-agent contracts (*e.g.*, Thomas and Worrall [35]).

Following Lucas and Stokey [24], we provide a characterization of efficient contracts by means of an extended Negishi method. A benevolent planner maximizes the weighted sum of utilities by allocating current resources and contingent utility promises subject to participation and incentive constraints. As utility promises are drawn from the set of future attainable utility profiles, the recursive program defines an extended Bellman operator, which we call Negishi operator, mapping future into present attainable utility profiles. Thus, as in conventional dynamic programming, the value of the program obtains as a fixed point of the Negishi operator. However, differently from traditional recursive methods, welfare weights need to adjust over time to reflect contingent rewards, or punishments, enforcing optimal contracts.

The Negishi operator provides a recursive technique to study efficient contracts. It is a widespread belief that this approach is unsatisfactory in non-convex economies. We instead argue that the Negishi approach can be safely applied provided that actions are contingent on some publicly observable random signal, even if the economy remains intrinsically non-convex. This extrinsic uncertainty serves as an intertemporal correlating device without altering non-convex utilities and constraints. Adding lotteries is a common, and to some extent natural, practice in non-convex economies. More importantly, it requires no appeal to any law of large numbers to smooth non-convexities out.

As the Negishi operator is monotone, the existence of (ordered) fixed points obtains plainly by means of Tarsky's Fixed Point Theorem. In general, even in convex economies, the fixed point is not unique. This is an intrinsic feature of recursive methods in economies with incentive constraints, although individual preferences satisfy discounting (see Rustichini [34]): Bellman's principle of optimality only ensures the absence of short-run profitable adjustments, and the additional long-run transversality condition might not be enforced in general. When the economy is convex, the greatest fixed point of the Negishi operator implements efficient contracts exactly and, under a further interiority restriction, this is the only fixed point (by Krasnosel'skiĭ [20]'s theory of monotone concave operators). Unfortunately, non-convexity is more the norm than the exception in the presence of incentive constraints. In general, the greatest fixed point of the Negishi operator might overestimate the actual efficient frontier. However, the original non-convex economy can be expanded by allowing the planner to allocate utility promises contingently on a publicly observable, and purely extrinsic, random signal. The value of the augmented planner's program coincides with the greatest fixed point of the non-augmented Negishi operator.

The sunspot-implementation of efficient contracts requires a distinction between uncertainty and risk. The random device used by the planner must be regarded as a mere risk governed by objective probabilities and evaluated according to the traditional expected utility, even when non-expected utility applies to other sources of uncertainty. This sort of separation between objective risk and subjective uncertainty, inspired by Anscombe and Aumann [5], is commonly adopted in the literature on ambiguity. Thus, in many applications of our theory, the introduction of the auxiliary random device seems innocuous both on a normative and on a positive ground.[1] In some environments, however, the expected-utility evaluation of sunspot uncertainty is more unnatural or can be hardly justified, because it alters the essence of underlying preferences or incentives.[2] In these circumstances, the sunspot-implementation can still be used as a device to estimate the potential error in the determination of the efficient frontier due to the Negishi method.

[1] On a normative ground, why should a social planner abstain from using a public random device when this increases social welfare? On a positive ground, under expected utility, a sufficiently rich uncertainty on fundamentals might mimic the allocative power of a truly extrinsic sunspot signal, uncorrelated to fundamentals, as in the theory of noisy stochastic games (*e.g.*, Duggan [12]).

[2] For instance, as observed by an anonymous reviewer, this happens in our Example 4.2.

A vast literature developed recursive methods for dynamic economies with forward-looking constraints and recursive utilities. A commonly used technique is the promised-utility approach inspired by the work of Abreu *et al.* [1] on repeated games. The limits of this method are extensively discussed in Marcet and Marimon [25] and Pavoni *et al.* [32], which we refer to on this issue for the sake of brevity.[3] This alternative established literature privileges a Lagrangian approach that cannot be, in general, reduced to our more primitive Negishi method. The inceptive observation is that the Lagrangian function of the original welfare program admits a recursive decomposition, in which Lagrange multipliers become state variables. Marcet and Marimon [25] restrict attention to convex environments and appeal to a *saddle point* operator, while Pavoni *et al.* [32] adopt a recursive *dual* formulation.[4] However, these methods are frustrated by duality gaps in non-convex economies, that is, the saddle point operator might be undefined and the dual operator might dramatically overestimate the efficient frontier.

In convex economies, the Negishi, the saddle point and the dual method all coincide by the fundamental theorem of duality. Though adopting one or the other is largely a matter of preference, the Negishi approach requires no explicit appeal to Lagrange multipliers and seems a more natural route towards the determination of efficient contracts. In non-convex economies, instead, the dual value dominates the Negishi value, whereas the saddle point approach fails in general. Therefore, when the Negishi operator overestimates the efficient frontier, so does the dual operator. Contrary to the Negishi gap, however, there is typically no way to reconcile the overestimated dual frontier with an underlying economic mechanism generating it. The dual approach produces a substantial alteration of primitives (a sort of convex envelope), whereas the Negishi method only requires the relatively innocuous assumption that plans are contingent on some public extrinsic uncertainty.

When Pareto utility frontier is not strictly convex, the applicability of both the Negishi and the dual method becomes questionable in terms of optimal policies. Simple examples in Cole and Kubler [10] and Messner and Pavoni [27] clarify that the state variable provides no guidance to the selection of controls consistent with past (incentive compatible) promises. In fact, Cole and Kubler [10] consider environments with flat regions of the Pareto utility frontier and augment the state space in order to make the state a sufficient statistics for the current optimal choices. We instead argue that, notwithstanding these issues

[3] Though published after Pavoni *et al.* [32], versions of Marcet and Marimon [25] have been circulating since 1994, inspiring a branch of the literature on dynamic contracts (see, for instance, Ljungqvist and Sargent [22] and Miao [29]). Apart from occasional references to Pareto welfare weights, neither Negishi [31] nor Lucas and Stokey [24] are cited in this literature on recursive contracts. The original motivation of Lucas and Stokey [24]'s analysis was to encompass time-varying impatience in growth theory, independently of incentive constraints. This might have obscured the fact that the method applies whatever is causing adjustments in welfare weights over time and across states.

[4] Importantly, in Pavoni *et al.* [32], the Bellman operator for the dual program is a contraction in the Thompson metric *under some boundary conditions*, thereby guaranteeing uniqueness of the fixed point even in a non-convex environment. The contraction property under the Thompson metric is related to our approach inspired by Krasnosel'skiĭ [20]'s theory of monotone concave operators. Marcet and Marimon [25] instead use a conventional Contraction Mapping Theorem. This powerful tool is available only in more restrictive environments.

with the optimal policy, first-order conditions can be proficuously derived from the Negishi recursive program, thus permitting a characterization of efficient contracts. Furthermore, we provide a complementary approach to determine optimal policy: an alternative Negishi method based on a maxmin-type social welfare function whose state variables are welfare shares, as opposite to welfare weights. Optimal policy can so be exhaustively expressed in terms of dynamics of welfare shares and, under convexity, long-term dynamics of efficient contracts are governed by an ergodic distribution on this minimal state space.

Incidentally, we point out another potential drawback for recursive methods. For static incentive economies, *ex-ante* efficient allocations may fail to attain *ex-post* efficiency (for instance, Myerson [30]). This sort of time-inconsistency is also a known feature of dynamic contracts, requiring *ex-ante* commitment, and it is commonly revealed by the absence of a natural recursive decomposition of the contract. Yet, time-inconsistent contracts can be studied by means of recursive methods on an enlarged state space (see, *e.g.*, Fernandes and Phelan [14]). We uncover that *ex-post* inefficiency might occur even when the contract admits a natural recursive decomposition, thus limiting the application of recursive methods. In fact, we present an example of *ex-post* inefficiency under private information. We also provide an operational condition on fundamentals ensuring *ex-post* efficiency over time. A thorough study of the implications of the failure of *ex-post* efficiency under private information is left to future research.

The paper is organized as follows. In section 2 we illustrate our method by means of an example and provide a comparison with the alternative approach based on Lagrange multipliers. In section 3 we describe the economic environment, present our assumptions on fundamentals and provide examples of application of our theory. In section 4 we study the Negishi method. In particular, we show that it exactly implements efficient contracts under convexity, whereas it requires a random mechanism for utility promises in non-convex economies. In section 5 we provide simple first-order conditions for the characterization of efficient contracts. In section 6 we compare our method with the dual approach established in the literature. In section 7, finally, we consider a modified Negishi method with maxmin-type social welfare functions and argue that optimal policies can be expressed as a random transition of welfare shares. We conclude with some brief remarks. All proofs are collected in the Appendix.

2. Illustrative example

To illustrate the advantages of our Negishi method upon the dual approach, we abstract from incentive constraints and present a simple example of optimal risk-sharing with non-concave recursive utility. The non-concavity arises due to a preference for early resolution of uncertainty, as established in Kreps and Porteus [21] and further studied in Weil [36]. We argue that the Negishi method is more accurate than the dual approach. In particular, the application of the dual method dramatically alters the temporal preference of the individuals, yielding an erroneous optimal distribution.

Consider an economy populated by two individuals, each endowed with a recursive utility of the form

$$v_t^i = W^i\left(z_t^i, \mathbb{E}_t v_{t+1}^i\right) = f^i\left(z_t^i + \delta {f^i}^{-1}\left(\mathbb{E}_t v_{t+1}^i\right)\right),$$

where δ in $(0,1) \subset \mathbb{R}^+$ is the common discount factor and $f^i : \mathbb{R}^+ \to \mathbb{R}^+$ is a surjective, strictly increasing, strictly concave map with $f^i(0) = 0$. As shown by Kreps and Porteus [21, Theorem 3], individuals exhibit a preference for early (late) resolution of uncertainty over temporal lotteries when $W^i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is convex (concave) in the continuation expected utility, given current consumption z_t^i in $\mathbb{R}^+$. In particular, a preference for early resolution occurs whenever $f^i : \mathbb{R}^+ \to \mathbb{R}^+$ exhibits a constant, or decreasing, coefficient of relative risk-aversion. The resource constraint imposes

$$z_t^a + z_t^b \leq e_t,$$

where z_t^i in $\mathbb{R}^+$ is individual consumption and e_t in $\mathbb{R}^+$ is the uncertain aggregate endowment.

Following Lucas and Stokey [24], the optimal distribution might be determined via a recursive Negishi approach: the planner maximizes the weighted sum of utilities under the resource constraint, conditional on the feasibility of continuation utility values. More formally, given welfare weights θ_t in the (unit) simplex $\Theta \subset \mathbb{R}^+ \times \mathbb{R}^+$, we pose

$$J_t\left(\theta_t^a, \theta_t^b\right) = \max \theta_t^a W^a\left(z_t^a, \mathbb{E}_t v_{t+1}^a\right) + \theta_t^b W^b\left(z_t^b, \mathbb{E}_t v_{t+1}^b\right)$$

subject to the material balance,

$$z_t^a + z_t^b \leq e_t,$$

and the feasibility of continuation utility values,

$$0 \leq \min_{\theta_{t+1} \in \Theta} \left(J_{t+1}\left(\theta_{t+1}^a, \theta_{t+1}^b\right) - \theta_{t+1}^a v_{t+1}^a - \theta_{t+1}^b v_{t+1}^b\right).$$

The latter constraint requires the value distributed by the planner in the continuation not to exceed the maximum social welfare. It is known that, under non-convexity, this constraint might be more permissive than the actual feasibility constraint arising from the primitives. Due to this relaxation, the Negishi method might overestimate the actual value of efficient distributions, as illustrated in Figure 1.

We show in this paper that the value delivered by the Negishi method, even when inaccurate, is always achieved by a feasible distribution of consumptions contingent on an additional purely extrinsic signal. We also establish that the Negishi method is faithful whenever the economy, though non-convex, might be transformed into a convex economy. Thus, accuracy is unaffected by monotone transformations of preferences, unlike the dual method. To verify the implications of this property in the example, consider the transformed recursive utility

$$\tilde{v}_t^i = {f^i}^{-1}\left(v_t^i\right) = z_t^i + \delta {f^i}^{-1}\left(\mathbb{E}_t f^i\left(\tilde{v}_{t+1}^i\right)\right) = z_t^i + \delta C_t^i\left(\tilde{v}_{t+1}^i\right).$$

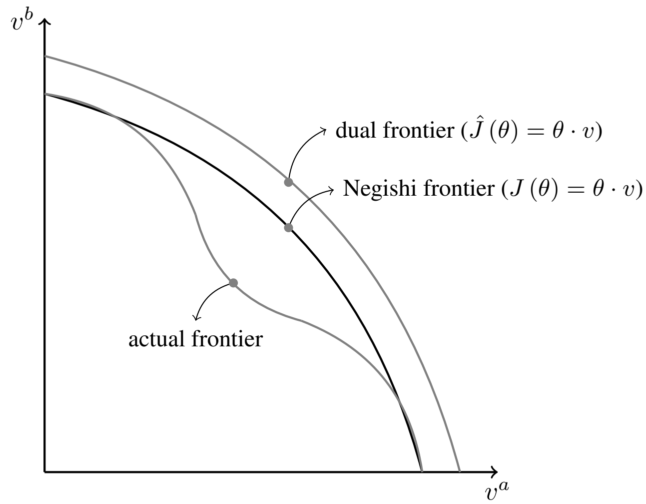

FIGURE 1. Error comparison

As this is a mere monotone transformation of utilities, preferences and, hence, efficient allocations are unaltered. By Hardy *et al.* [15, Theorem 106(i)], the certainty equivalent $C_t^i\left(\tilde{v}_{t+1}^i\right)$ is concave when $f^i : \mathbb{R}^+ \to \mathbb{R}^+$ exhibits constant relative risk-aversion. In this case, the transformed recursive utility is concave and the Negishi method yields the actual value of efficient distributions for the original (untransformed) economy, even if the recursive utility is not concave. We next argue that the dual method is instead disruptive.

In the established literature, Marcet and Marimon [25] and Pavoni *et al.* [32] characterize optimal allocations by means of a recursive Lagrange approach in which the multipliers become auxiliary state variables. This approach corresponds to the dual of our Negishi program. We show in this paper that the application of the dual method distorts the primitives replacing the utility aggregator $W^i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with its concave envelope (*i.e.*, the least map $\tilde{W}^i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that is concave, given z^i in $\mathbb{R}^+$, and satisfies $W^i \leq \tilde{W}^i$). In general, this induces an artificial manipulation of time-preference and, to make our analysis transparent, we set $f^i(v) = v^{1-\sigma^i}$ with σ^i in $(0,1) \subset \mathbb{R}^+$. The Negishi method yields no error in this circumstance by our previous arguments on monotone concave transformations.

Notice that, for any λ in $(0,1)$,

$$\begin{aligned} W^i\left(z_t^i, \mathbb{E}v_{t+1}^i\right) &\leq (1-\lambda) W^i\left(z_t^i, 0\right) + \lambda W^i\left(z_t^i, \lambda^{-1}\mathbb{E}v_{t+1}^i\right) \\ &\leq (1-\lambda) \tilde{W}^i\left(z_t^i, 0\right) + \lambda \tilde{W}^i\left(z_t^i, \lambda^{-1}\mathbb{E}v_{t+1}^i\right). \\ &\leq \tilde{W}^i\left(z_t^i, \mathbb{E}v_{t+1}^i\right), \end{aligned}$$

were we exploit the convexity of $v \mapsto \left(z + \delta v^{\frac{1}{1-\sigma}}\right)^{1-\sigma}$ and the definition of the concave envelope. In the limit, this yields

$$\begin{aligned}\tilde{W}^i\left(z_t^i, \mathbb{E}_t v_{t+1}^i\right) &= \lim_{\lambda \to 0} (1-\lambda) W^i\left(z^i, 0\right) + \lambda W^i\left(z^i, \lambda^{-1}\mathbb{E} v^i\right) \\ &= f^i\left(z^i\right) + \lim_{\lambda \to 0} \lambda f^i\left(z_t^i + \delta {f^i}^{-1}\left(\lambda^{-1}\mathbb{E}_t v_{t+1}^i\right)\right) \\ &= f^i\left(z_t^i\right) + f^i(\delta)\, \mathbb{E}_t v_{t+1}^i,\end{aligned}$$

which exhaustively identifies the concave envelope of the utility aggregator. Thus, the dual method delivers the characterization corresponding to constant relative risk-aversion with indifference for the resolution of uncertainty and with an erroneous rate of impatience.

3. Fundamentals

The economy extends over an infinite set of periods $\mathbb{T} = \{0, 1, \dots, t, \dots\}$. Uncertainty is governed by a Markov transition $\Pi : S \to \Delta(S)$ on a finite state space S. Given an initial state s_0 in S, the transition generates a probability space $(\Omega, \mathcal{F}, \mu)$ and a filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ of Ω corresponding to partial histories of Markov states. We describe all variables as stochastic processes, and we omit the obvious almost-surely qualification. For a given measurable space D, we let $\mathcal{D}$ be the space of all processes $f : \mathbb{T} \times \Omega \to D$ adapted to the filtration, and let $\mathcal{D}_t$ be the space of D-valued $\mathcal{F}_t$-measurable random variables $f_t : \Omega \to D$. When D is endowed with a metric, $\mathcal{D}$ inherits the implied topology of pointwise convergence.This approach will significantly simplify our notation.

The economy consists of a finite set I of agents (one of them might be a principal if this helps the understanding). At every contingency, each agent can take an action in Z^i, with Z being the action space across agents. A *contract* specifies a full contingent plan z in $\mathcal{Z}$ of actions for agents. The nature of these actions will depend on the specific application of our theory.

Each agent evaluates contracts by means of a utility function $U^i : \mathcal{Z} \to \mathcal{V}^i$, where $\mathcal{V}^i$ is the space of $\mathbb{R}$-valued processes, or of $\mathbb{R}^+$-valued processes, depending on applications. We interpret $U_t^i(z)$ as the contingent utility value at period t in $\mathbb{T}$, a random variable in $\mathcal{V}_t^i$. Utility is recursively generated by an aggregator $W_t^i : \mathcal{Z}_t \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$, that is,

$$U_t^i(z) = W_t^i\left(z_t, U_{t+1}^i(z)\right).$$

We assume that the aggregator uniquely identifies stationary preferences (Koopmans [19]). In particular, we provide sufficient conditions for this property based on discounting (Blackwell [7]), and an extension to other (Thompson) aggregators can be found in Marinacci and Montrucchio [26].[5]

[5] Without restricting utility aggregators, individual preferences might be misspecified: none or multiple utility functions might be consistent with the given aggregator, and the very notion of efficiency becomes ambiguous. It is by the Principle of Optimality that the recursive planning program will have a (possibly distinct) value for each profile of individual utilities generated by the aggregators.

All constraints on the contract are captured by contingent feasible sets $\mathcal{G}_t \subset \mathcal{Z}_t \times \mathcal{V}_{t+1}$, that is, a contract z in $\mathcal{Z}$ is *feasible* if

$$(z_t, U_{t+1}(z)) \in \mathcal{G}_t.$$

These feasible sets restrict over time current actions and continuation utility values, or promises. A feasible contract z in $\mathcal{Z}$ is (*weakly*) *efficient* if it is not (strongly) Pareto dominated by another feasible contract $\hat{z}$ in $\mathcal{Z}$. The purpose of this note is to characterize efficient contracts recursively.

Throughout our analysis, fundamentals are restricted by canonical assumptions, all together ensuring that the recursive program is sufficiently regular. Importantly, convexity is not imposed, except when explicitly stated. For a more transparent presentation, we separate assumptions on preferences from assumptions on contractual restrictions, and preliminarily establish that utility functions are unambiguously identified by the aggregators. Notice that our Assumption 3.2 corresponds to Koopmans [19]'s Axiom of Stationarity, and rules out time-inconsistent individual preferences.

Assumption 3.1 (Action space)**.** *Each action space Z^i is a closed set of some Euclidean space.*

Assumption 3.2 (Monotonicity)**.** *Each utility aggregator $W_t^i : \mathcal{Z}_t \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$ is increasing given an action profile z_t in $\mathcal{Z}_t$.*

Assumption 3.3 (Discounting)**.** *Each utility aggregator $W_t^i : \mathcal{Z}_t \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$ satisfies, for some δ^i in $(0,1) \subset \mathbb{R}^+$,*

$$\left|W_t^i\left(z_t, \hat{v}_{t+1}^i\right) - W_t^i\left(z_t, \tilde{v}_{t+1}^i\right)\right| \leq \delta^i \mathbb{E}_t \left|\hat{v}_{t+1}^i - \tilde{v}_{t+1}^i\right|.$$

Assumption 3.4 (Boundedness)**.** *Each utility aggregator $W_t^i : \mathcal{Z}_t \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$ satisfies, for some B^i in $\mathbb{R}^{++}$,*

$$\left|W_t^i\left(z_t, 0\right)\right| \leq B^i.$$

Assumption 3.5 (Continuity)**.** *Each utility aggregator $W_t^i : \mathcal{Z}_t \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$ is jointly continuous.*

Under the stated assumptions, a recursive utility is uniquely generated by the aggregator. In addition, there exist suitable processes bounding utility values over time and across contingencies. We remark that our framework accommodates both non-additive time preferences and non-expected utility.

Proposition 3.1 (Recursive utility)**.** *Under Assumptions 3.1-3.5, there exists a unique bounded and continuous utility function $U^i : \mathcal{Z} \to \mathcal{V}^i$ such that*

$$U_t^i(z) = W_t^i\left(z_t, U_{t+1}^i(z)\right).$$

Furthermore, there are bounded processes $\underline{v}^i$ and $\bar{v}^i$ in $\mathcal{V}^i$ such that, for every contract z in $\mathcal{Z}$,

$$\underline{v}_t^i \leq W_t^i\left(z_t, \underline{v}_{t+1}^i\right) \leq W_t^i\left(z_t, \bar{v}_{t+1}^i\right) \leq \bar{v}_t^i.$$

The remaining assumptions ensure that the contractual framework is well-behaved. Assumption 3.6 is merely technical. The nature of Assumption 3.7 depends on the specific application: it has to be thought as establishing existence of a feasible contract securing some minimal level of utility to all individuals. Assumption 3.8 is rather demanding when actions entail strong complementarities or, more conventionally, in the presence of goods yielding disutility (bads). The bounds appearing in the assumptions below are those of Proposition 3.1.

Assumption 3.6 (Closedness)**.** *Each feasible set $\mathcal{G}_t \subset \mathcal{Z}_t \times \mathcal{V}_{t+1}$ is closed, and it is compact under the additional restriction $\underline{v}_{t+1} \leq v_{t+1} \leq \bar{v}_{t+1}$ for bounds $\underline{v}_{t+1}$ and $\bar{v}_{t+1}$ in $\mathcal{V}_{t+1}$.*

Assumption 3.7 (Viability)**.** *There exists a contract z^0 in $\mathcal{Z}$ such that*

$$\left(z_t^0, U_{t+1}\left(z^0\right)\right) \in \mathcal{G}_t.$$

Assumption 3.8 (Free disposal)**.** *Given any (z_t, v_{t+1}) in $\mathcal{G}_t$, for every $\hat{v}_t$ in $\mathcal{V}_t$ such that $\underline{v}_t \leq \hat{v}_t \leq W_t(z_t, v_{t+1})$, there exists $(\hat{z}_t, \hat{v}_{t+1})$ in $\mathcal{G}_t$ satisfying $\underline{v}_{t+1} \leq \hat{v}_{t+1} \leq v_{t+1}$ and*

$$\hat{v}_t = W_t\left(\hat{z}_t, \hat{v}_{t+1}\right).$$

Finally, as our notational choice might have obscured stationarity, we explicitly state that all fundamentals are measurable with respect to Markov states only. Thus, the current Markov state conveys all the relevant information about the future evolution of fundamentals. As other state variables (such as capital and accumulated assets, or debts) are absent in our simplified framework, the unfolding of welfare distributions over time is the only link across periods. This link is typically referred to in the literature as promise-keeping constraint.

Assumption 3.9 (Markov property)**.** *Feasible sets and utility aggregators are measurable with respect to state space S.*[6]

Our assumptions on fundamentals can be compared with those in Pavoni *et al.* [32, Assumptions 1-2]. Their restrictions for the general analysis are substantially weaker than ours because they do not relate the dual program to the original primal program. Most of the burden in their analysis is carried by primitive assumptions of compactness, supplemented by an added transversality condition [32, Condition (T), Proposition 4]. Our restrictions, instead, are exploited for a direct characterization of efficient contracts.

Our general framework encompasses several well-studied examples of recursive contracts.[7] We describe some of these instances and, in all these examples, we verify existence

[6] That is, up to obvious identifications, feasible sets are generated by a correspondence $G : S \twoheadrightarrow Z \times \mathbb{R}^{I \times S}$, whereas each utility aggregator can be expressed as a map $W^i : Z \times \mathbb{R}^S \times S \to \mathbb{R}$, were $\mathbb{R}^S$ is interpreted as the space of uncertain utility values, or promises, in the next period.

[7] With minor adjustments, our framework could also encompass asymmetric information with history dependence, as in Fernandes and Phelan [14]. We add fictitious individuals serving as (out-of-equilibrium) counterfactuals for untruthful revelation of information. The Negishi method can be applied and efficient contracts can

of a minimal contract z^0 in $\mathcal{Z}$, as required by Assumption 3.7. It is also immediate to check for the validity of our free-disposal condition (Assumption 3.8).

Example 3.1 (Risk-sensitive preferences)**.** This is the economy studied in Anderson [4]. We set $Z^i = \mathbb{R}^+$ and interpret actions as consumption levels. The feasible set $\mathcal{G}_t$ corresponds to material feasibility, that is,

$$(z_t, v_{t+1}) \in \mathcal{G}_t \text{ if and only if } \sum_{i\in I} z_t^i \leq \sum_{i\in I} e_t^i,$$

where the adapted process e^i in $\mathcal{Z}^i$ describes the uncertain evolution of the individual endowment. Finally, the utility aggregator is given by

$$W_t^i\left(z_t, v_{t+1}^i\right) = (1-\delta)\, u^i\left(z_t^i\right) + \delta {f^i}^{-1}\left(\mathbb{E}_t f^i\left(v_{t+1}^i\right)\right),$$

where $f^i : \mathbb{R}^+ \to \mathbb{R}^+$ is a surjective strictly increasing and concave map, δ in $(0,1) \subset \mathbb{R}^+$ is the subjective discount factor and $u^i : \mathbb{R}^+ \to \mathbb{R}^+$ is the Bernoulli utility function. The reservation contract for Assumption 3.7 involves no consumption for all individuals, $z^0 = 0$.

Example 3.2 (Limited enforcement of contracts)**.** This is the economy studied, among others, by Kocherlakota [18] and Kehoe and Levine [17]. Suppose that $Z^i = \mathbb{R}^+$, interpreted as the consumption space, and assume that each agent can ensure a (possibly contingent) external utility value ϕ^i in $\mathcal{V}^i$. The utility aggregator is simply

$$W_t^i\left(z_t, v_{t+1}^i\right) = (1-\delta)\, u^i\left(z_t^i\right) + \delta \mathbb{E}_t v_{t+1}^i.$$

The set $\mathcal{G}_t$ includes all plans (z_t, v_{t+1}) in $\mathcal{Z}_t \times \mathcal{V}_{t+1}$ satisfying material balance,

$$\sum_{i\in I} z_t^i \leq \sum_{i\in I} e_t^i,$$

and participation constraints,

$$W_t^i\left(z_t, v_{t+1}^i\right) \geq \phi_t^i,$$

where the process e^i in $\mathcal{Z}^i$ describes the individual endowment. To ensure viability, we assume that $U_t^i\left(e^i\right) \geq \phi_t^i$. Autarky is thus the reservation contract fulfilling Assumption 3.7.

Example 3.3 (Asymmetric information)**.** This is a finite version of the economy with private information studied by Atkeson and Lucas [6]. Individuals experience privately observable shocks to preferences. Utilities are

$$U_0^i\left(z^i\right) = (1-\delta)\, \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t u^i(z_t^i\left(s_{t+1}\right), s_{t+1}^i),$$

where s_t^i in S^i is the shock to the preferences of individual i in I. The realization of this shock is private information of the individual. The shocks take value into a finite set S^i

be characterized using an exogenous space of welfare weights for truthful and untruthful individuals. To avoid a discontinuity in our narrative, we relegate this extension to Appendix B.

and are identically and independently distributed over time and across individuals. To fit our framework we let z_t^i take values into $\mathbb{R}^S$ and we write the aggregator as

$$W_t^i\left(z_t, v_{t+1}^i\right) = \mathbb{E}_t\left((1-\delta)\, u^i(z_t^i(s_{t+1}), s_{t+1}^i) + \delta v_{t+1}^i\left(s_{t+1}\right)\right),$$

where we use shorthand notation to capture the dependence of consumptions and utility promises on individuals' reported information. Individuals can misreport their types thereby changing the terms of the contracts. Incentive compatibility then reads

$$\begin{aligned} \mathbb{E}_t\left((1-\delta)\, u^i(z_t^i(s_{t+1}), s_{t+1}^i) + \delta v_{t+1}^i\left(s_{t+1}\right) | s_{t+1}^i\right) \quad &\geq \\ \mathbb{E}_t\left((1-\delta)\, u^i(z_t^i\left(\hat{s}_{t+1}^i, s_{t+1}^{-i}\right), s_{t+1}^i) + \delta v_{t+1}^i\left(\hat{s}_{t+1}^i, s_{t+1}^{-i}\right) | s_{t+1}^i\right)&, \end{aligned}$$

that is, misreporting their own type is not profitable given truthful revelation by other individuals. The feasible set $\mathcal{G}_t$ is then defined by the incentive compatibility constraints, along with material feasibility

$$\sum_{i \in I} z_t^i\left(s_{t+1}\right) \leq e_{t+1},$$

where process e in $\mathcal{R}$ describes aggregate resources (and $\mathcal{R}$ is the space of real-valued processes). No consumption is trivially the reservation contract required by Assumption 3.7.

Example 3.4 (Default risk)**.** Our general formulation can encompass Eaton and Gersovitz [13]'s model of sovereign default risk. The economy consists of a principal (a representative creditor) and an agent (a borrower). The action space is $Z = [0, \eta] \times \mathbb{R}$, with typical element $z = (c, b)$. We interpret c in $[0, \eta] \subset \mathbb{R}^+$ as the borrower's consumption, limited by an exogenous upper bound, and b in $\mathbb{R}$ as the amount of uncontingent bonds issued by the borrower. The borrower's preferences are given by

$$W_t^{\mathsf{b}}\left(z_t, v_{t+1}^{\mathsf{b}}\right) = (1-\delta)\, u\left(c_t\right) + \delta \mathbb{E}_t \max\left\{v_{t+1}^{\mathsf{b}}, \phi_{t+1}^{\mathsf{b}}\right\},$$

whereas the creditor's aggregator is

$$W_t^{\mathsf{c}}\left(z_t, v_{t+1}^{\mathsf{c}}\right) = \left(e_t - c_t\right) + \left(\frac{1}{1+r}\right) \mathbb{E}_t v_{t+1}^{\mathsf{c}},$$

where e in $\mathcal{R}$ is the uncertain endowment of the borrower. The exogenous process ϕ^{b} in $\mathcal{V}^{\mathsf{b}}$ identifies the borrower's reservation value upon default. The principal is a representative risk-neutral investor having access to capital markets at a constant rate of interest r in $\mathbb{R}^{++}$. In fact, the role of creditors is to enforce the borrower's budget constraint, as their utility corresponds to the borrower's minimum expenditure. The borrower can only issue a uncontingent bond, so that feasibility imposes

$$v_{t+1}^{\mathsf{c}} = b_t \mathbf{1}_{\left\{v_{t+1}^{\mathsf{b}} \geq \phi_{t+1}^{\mathsf{b}}\right\}},$$

where $\mathbf{1}_E$ is the indicator function of event E in $\mathcal{F}$. This condition captures the fact that the principal receives a flat payment conditional on borrower's utility value being above a given reservation value. Otherwise, the borrower defaults, securing the reservation value, and the creditor receives no payment. The overall construction captures Eaton and Gersovitz [13]'s model of default risk. Any efficient contract is such that the borrower's utility cannot be increased without decreasing the principal's utility. This reformulation of Eaton and Gersovitz [13] is similar to the dual planning program developed in Amador and Aguiar [2].

Example 3.5 (Dynamic Ramsey taxation)**.** Our theory also applies to dynamic Ramsey taxation (*e.g.*, Lucas and Stokey [23]). A government must finance an uncertain stream of expenditures by levying a distortive labor tax and issuing contingent debt. A representative individual is endowed with a utility function $u : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^+$ that is bounded, smoothly strictly increasing on $\mathbb{R}^+$, smoothly strictly decreasing on $[0,1] \subset \mathbb{R}^+$ and smoothly strictly concave, where c in $\mathbb{R}^+$ is consumption and e in $[0,1] \subset \mathbb{R}^+$ is labor supply. A linear technology transforms labor directly into consumption. We assume that

$$\sup_{c \leq e} \left| u_c\left(c, e\right) c + u_e\left(c, e\right) e \right| \text{ is finite,}$$

where u_c and u_e are, respectively, the marginal utility of consumption and the marginal disutility of labor. This assumption is needed to enforce the transversality condition of the representative individual.

To encompass Ramsey taxation in our general framework, we let the action space be $Z = \mathbb{R}^+ \times [0,1]$, and consider an economy composed by the representative individual and the government. The utility aggregator of the representative individual is

$$W_t^{\mathrm{a}}\left(z_t, v_{t+1}^{\mathrm{a}}\right) = \left(1 - \delta\right) u\left(c_t, e_t\right) + \delta \mathbb{E}_t v_{t+1}^{\mathrm{a}},$$

whereas the government's utility aggregator is

$$W_t^{\mathrm{g}}\left(z_t, v_{t+1}^{\mathrm{g}}\right) = u_c\left(c_t, e_t\right) c_t + u_e\left(c_t, e_t\right) e_t + \delta \mathbb{E}_t v_{t+1}^{\mathrm{g}}.$$

This is basically the budget constraint of the government, whose utility increases with the level of its debt. Finally, the feasible set $\mathcal{G}_t$ requires

$$c_t + g_t \leq e_t,$$

where g in $\mathcal{R}$ is the uncertain expenditure of the government.

To verity that this formulation implements an optimal Ramsey taxation plan, consider any efficient allocation. Consolidating the budged constraint of the government, and exploiting the boundedness assumption, we obtain

$$v_0^{\mathrm{g}} = \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t \left(u_c\left(c_t, e_t\right) c_t + u_e\left(c_t, e_t\right) e_t\right).$$

This is the usual implementability constraint (see Lucas and Stokey [23]). Furthermore, by Pareto efficiency, it is not feasible to increase the utility of the representative-individual without reducing v_0^{g} in $\mathbb{R}$, and so without decreasing the initial debt of the government. Hence, the efficient allocation maximizes the welfare of the representative individual subject to the government's budget constraint.

In the proposed formulation the government debt is conveniently denominated in terms of the marginal utility for consumption. This avoids known issues of time-inconsistency and permits a complete recursive decomposition: the debt issued at a future contingency, differently from the initial outstanding debt, involves a commitment by the government to a certain taxation policy, which is embedded in its marginal-utility value. However, in the original Ramsey taxation program, the initial debt of the government is given in consumption units, that is,

$$\tilde{v}_0^{\mathrm{g}} = \frac{v_0^{\mathrm{g}}}{u_c\left(c_0, e_0\right)}.$$

Recovering this additional constraint only requires to replace the government's utility aggregator in the initial period, and *only* in the initial period, with

$$\tilde{W}_0^{\mathrm{g}}\left(z_0, v_1^{\mathrm{g}}\right) = \frac{W_0^{\mathrm{g}}\left(z_0, v_1^{\mathrm{g}}\right)}{u_c\left(c_0, e_0\right)}.$$

4. A Negishi method

4.1. **Recursive program.** We study a recursive decomposition of efficient contracts inspired by Negishi [31].[8] The planner maximizes the weighted sum of utilities by choosing a current action profile along with feasible continuation utility values. In convex economies, this method determines the exact Pareto frontier in utility values. Under non-convexities, instead, the approach generally over-estimates the efficient frontier. However, we show that the Negishi method yields the exact efficient frontier when the planner is allowed to allocate promises contingent on a publicly observable random signal.

The Negishi operator acts on the space $\mathcal{J}$ of all bounded maps $J : \Theta \to \mathcal{R}$ such that

$$(*) \qquad J_t\left(\theta_t\right) \geq \theta_t \cdot U_t\left(z^0\right),$$

where Θ is the space of welfare weights (the unit simplex in $\mathbb{R}^I$), $\mathcal{R}$ denotes the space of $\mathbb{R}$-valued processes and reservation contract z^0 in $\mathcal{Z}$ is given by Assumption 3.7.[9] Elements of $\mathcal{J}$ are called *support maps*. A support map J in $\mathcal{J}$ allows for recovering a convex set of utility values at every contingency, that is,

$$\mathcal{U}_t\left(J_t\right) = \left\{v_t \in \mathcal{V}_t : \theta_t \cdot \underline{v}_t \leq \theta_t \cdot v_t \leq J_t\left(\theta_t\right) \text{ for every } \theta_t \in \Theta\right\}.$$

Following Lucas and Stokey [24]'s recursive decomposition, a given support map J in $\mathcal{J}$ restricts continuation utilities in the planner program, and the planner program itself yields a possibly revised support map $\hat{J}$ in $\mathcal{J}$. The Negishi value of contracts is a rest point of this revision process.

[8] The method is applied by Lucas and Stokey [24] to deterministic optimal growth with recursive utilities. Kan [16] and Anderson [4] provide an extension to efficient distributions under risk with non-expected utility. Miao [29, Chapter 20] presents a textbook illustration. Bloise [8] studies optimal risk-sharing subject to limited commitment. All these applications impose properties of convexity on fundamentals. The Contraction Mapping Theorem can be applied in [4, 16, 24] because incentive constraints are absent. Bloise [8] instead exploits the theory of monotone concave operators of Krasnosel'skiĭ [20].

[9] Restriction (*) is imposed to ensure that the planner's feasible set is always non-empty.

Formally, Negishi operator $T : \mathcal{J} \to \mathcal{J}$ is defined as

$$(TJ)_t(\theta_t) = \sup_{(z_t, v_{t+1}) \in \mathcal{G}_t} \theta_t \cdot W_t(z_t, v_{t+1})$$

subject to

$$v_{t+1} \in \mathcal{U}_{t+1}(J_{t+1}).$$

Thus, the planning program moves from a given support map J in $\mathcal{J}$, and yields a revised support map (TJ) in $\mathcal{J}$. A *Negishi value* is a fixed point of the Negishi operator, that is, a support map J in $\mathcal{J}$ such that $J = (TJ)$.

We compare the Negishi value with the *actual*, or *exact*, *value* of contracts. To this end, consider the space of utility possibilities subject to feasibility, that is,

$$\mathcal{U}_0^* = \{v_0 \in \mathcal{V}_0 : \underline{v}_0 \leq v_0 \leq U_0(z) \text{ for some feasible contract } z \in \mathcal{Z}\}.$$

The actual value of contracts is given by

$$J_0^*(\theta_0) = \sup_{v_0 \in \mathcal{U}_0^*} \theta_0 \cdot v_0.$$

As the economy is recursive, the actual value can be determined at any future contingency, subject to feasibility beginning from that contingency. We so obtain a value J^* in $\mathcal{J}$ reflecting the shape of the actual efficient frontier over time and across contingencies.

By monotonicity, the Negishi operator admits ordered fixed points. Uniqueness, in general, cannot be established without further assumptions, as illustrated by a simple, and non-pathological, example. It is also clear that, in general, the greatest fixed point of the Negishi operator is not a faithful description of the actual efficient frontier when the economy is non-convex. This entails no pathological feature either, and is illustrated by Example 4.2.

Proposition 4.1 (Fixed points)**.** *Negishi operator $T : \mathcal{J} \to \mathcal{J}$ admits a* least *fixed point $\underline{J}$ in $\mathcal{J}$ and a* greatest *fixed point $\bar{J}$ in $\mathcal{J}$. In addition, $J^* \leq \bar{J}$, where J^* in $\mathcal{J}$ is the actual value of contracts.*

Example 4.1 (Multiplicity)**.** This example shows that the Negishi operator might admit multiple fixed points. The recursive method only ensures the absence of feasible Pareto improvements over finitely many periods, without in general preventing efficiency gains over the extended infinite horizon. In the example, a low value is the only current distribution satisfying the incentive constraints when a perpetual low value is expected in the future. By induction, a low value cannot be improved over any arbitrary finite horizon. Over the entire infinite horizon, instead, a higher value satisfies the incentive constraints.

Consider a simple deterministic economy with two identical individuals. The utility aggregator is

$$W_t^i(z_t, v_{t+1}^i) = (1 - \delta)(z_t^i)^2 + \delta v_{t+1}^i,$$

where δ in $(0,1)$ is the discount factor. Feasibility imposes a participation constraint of the form

$$W_t^i\left(z_t, v_{t+1}^i\right) \geq \frac{1}{4}.$$

In addition, allocations are restricted by the material balance constraint

$$z_t^i + z_t^{-i} \leq 1.$$

We first show that $\underline{J}_t\left(\theta_t\right) = 1/4$ is a fixed point of the Negishi operator.

Notice that v_{t+1} in $\mathcal{U}_{t+1}\left(\underline{J}_{t+1}\right)$ necessarily implies that $v_{t+1}^i \leq 1/4$. Therefore, the participation constraint imposes $z_t^i \geq 1/2$ and material balance yields $z_t^i = 1/2$. We so conclude that

$$\left(T\underline{J}\right)_t\left(\theta_t\right) = \theta_t \cdot W_t\left(z_t, v_{t+1}\right) = (1-\delta)\frac{1}{4} + \delta\frac{1}{4} = \frac{1}{4} = \underline{J}_t\left(\theta_t\right).$$

Assuming that $\delta > 1/2$, we now show that $(TJ)_t\left(\theta_t\right) \geq J_t\left(\theta_t\right)$, where $J_t\left(\theta_t\right) = 1/2$. This implies that the Negishi method admits a greatest fixed point satisfying $\bar{J}_t\left(\theta_t\right) \geq 1/2$. To prove this claim, notice that $v_{t+1} = (1/2, 1/2)$ is a feasible distribution for continuation utilities. Furthermore, assuming $\theta_t^i \geq \theta_t^{-i}$, the consumption profile $\left(z_t^i, z_t^{-i}\right) = (1,0)$ implies

$$\theta_t \cdot W_t\left(z_t, v_{t+1}\right) = (1-\delta)\,\theta_t^i + \delta\frac{1}{2} \geq \frac{1}{2}.$$

Hence, we only have to prove that the participation constraint is satisfied. To this purpose, notice that

$$W_t^i\left(z_t, v_{t+1}^i\right) \geq W_t^{-i}\left(z_t, v_{t+1}^{-i}\right) = \delta\frac{1}{2} \geq \frac{1}{4},$$

so establishing our claim.

Example 4.2 (Unfaithful value)**.** This example illustrates that the Negishi operator might overestimate the true Pareto frontier. This is related to the presence of non-convexity in the economy. As a matter of fact, the Negishi planner is allowed to distribute continuation values in the convex hull of the actual Pareto frontier. This constraint is more permissive than the actual utility possibilities frontier and, consequently, the Negishi value increases. The example is convoluted because time-varying distributions permit a rich approximation of the convex hull, rendering hard the estimation of the Negishi gain.

The economy is deterministic and populated by two identical individuals with utility aggregators

$$W_t^i\left(z_t, v_{t+1}^i\right) = (1-\delta)\left(z_t^i\right)^2 + \delta v_{t+1}^i,$$

where the action is thought as consumption, $Z^i = \mathbb{R}^+$. Aggregate endowment is constant and equal to unity. Feasible allocations are further restricted by envy-free constraints on continuation utilities. Therefore, the feasible set $\mathcal{G}_t$ consists of all plans satisfying the constraint of aggregate resources,

$$z_t^i + z_t^{-i} \leq 1,$$

and an envy-free constraint on continuation utilities,

$$v_{t+1}^{i} \geq v_{t+1}^{-i}.$$

We first determine the true value of the contract and then compare it with its Negishi value.

By the envy-free constraints, any feasible contract z in $\mathcal{Z}$ necessarily fulfills the condition

$$U_{t+1}^{i}\left(z^{i}\right) = U_{t+1}^{-i}\left(z^{-i}\right).$$

Material balance thus implies that $z_{t+1} = (1/2, 1/2)$ for all t in $\mathbb{T}$. We finally conclude that

$$J_t^{*}\left(\theta_t\right) = (1-\delta)\max\{\theta_t^{i}, \theta_t^{-i}\} + \frac{\delta}{4}.$$

We show that the Negishi operator instead admits a greater fixed point,

$$J_t\left(\theta_t\right) = (1-\delta)\max\{\theta_t^{i}, \theta_t^{-i}\} + \frac{\delta}{2}.$$

This reveals that the Negishi method increases the value of contracts.

As individuals are identical, continuation utilities v_{t+1} in $\mathcal{V}_{t+1}$ satisfy the envy-free constraint only if $v_{t+1}^{i} = v_{t+1}^{-i}$. In addition, they belong to the restricted set $\mathcal{U}_{t+1}\left(J_{t+1}\right)$ only if $v_{t+1} \leq (1/2, 1/2)$. We thus conclude that

$$(TJ)_t\left(\theta_t\right) = \sup\,(1-\delta)\left(\theta_t^{i}\left(z_t^{i}\right)^2 + \theta_t^{-i}\left(z_t^{-i}\right)^2\right) + \frac{\delta}{2}$$

subject to

$$z_t^{i} + z_t^{-i} \leq 1.$$

Therefore, $(TJ)_t\left(\theta_t\right) = J_t\left(\theta_t\right)$, as claimed.

4.2. **Exact implementation.** In a convex economy, the Negishi operator yields the actual efficient frontier of dynamic contracts. Furthermore, when an additional interiority condition is satisfied, the Negishi operator admits exactly one fixed point and this coincides with the actual value of contracts. The intuition for exact implementation relies on basic principles of convex analysis, as illustrated by Figure 2

Assumption 4.1 (Convexity)**.** *Each feasible set $\mathcal{G}_t \subset \mathcal{Z}_t \times \mathcal{V}_{t+1}$ is convex. Furthermore, each utility aggregator $W_t^{i} : \mathcal{Z}_t \times \mathcal{V}_{t+1}^{i} \to \mathcal{V}_t^{i}$ is concave.*

Proposition 4.2 (Implementation)**.** *Under additional Assumption 4.1, the actual value of contracts J^{*} in $\mathcal{J}$ is the greatest fixed point of the Negishi operator.*

The established implementation exploits the free-disposal property (Assumption 3.8). This is slightly disturbing because the hypothesis is not formulated in terms of primitive principles. We thus identify a more transparent restriction on fundamentals ensuring exact implementation even when free-disposal is dispensed with. Unfortunately, this additional property is *not* innocuous in the presence of incentive constraints. In fact, it is violated by economies in Examples 4.4-4.5.

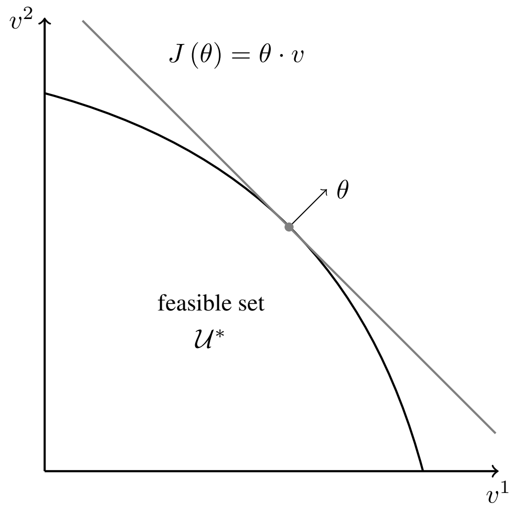

FIGURE 2. Convex feasible set

Assumption 4.2 (Feasibility of welfare increases)**.** *Given any (z_t, v_{t+1}) in $\mathcal{G}_t$, for every $\hat{v}_{t+1}$ in $\mathcal{V}_{t+1}$ such that $v_{t+1} \leq \hat{v}_{t+1}$,*

$$(z_t, \hat{v}_{t+1}) \in \mathcal{G}_t.$$

Proposition 4.3 (Implementation redux)**.** *Under additional Assumption 4.1, and Assumption 3.8 replaced by Assumption 4.2, the actual value of contracts J^* in $\mathcal{J}$ is the greatest fixed point of the Negishi operator.*

We now show that, under an interiority assumption, the Negishi approach delivers unambiguously the value of efficient contracts. The intuition relies on the concave nature of the recursive planning program, and in fact multiplicity persists when concavity fails even under interiority. The additional assumption requires the existence of a feasible contract ensuring a uniform increase in utility with respect to reservation values.

Assumption 4.3 (Interiority)**.** *There exists a plan $\left(z_t, U_{t+1}\left(z^0\right)\right)$ in $\mathcal{G}_t$ such that, for some sufficiently small ϵ in $\mathbb{R}^{++}$,*

$$U_t^i\left(z^0\right) + \epsilon \leq W_t^i\left(z_t, U_{t+1}^i\left(z^0\right)\right),$$

where feasible contract z^0 in $\mathcal{Z}$ is given in Assumption 3.7.

Example 4.3 (Interiority)**.** Reconsider an economy as in Example 4.1 with the modified utility aggregator of the form

$$W_t^i\left(z_t, v_{t+1}^i\right) = (1-\delta)\, z_t^i + \delta v_{t+1}^i.$$

Let the reservation contact be $z_t^0 = (1/4, 1/4)$, so that $U_t^i(z^0) = 1/4$. The contract $z_t = (1/2, 1/2)$ satisfies Assumption 4.3. Indeed,

$$W_t^i\left(z_t, U_{t+1}^i\left(z^0\right)\right) = (1-\delta)\frac{1}{2} + \delta\frac{1}{4} = U_t^i\left(z^0\right) + (1-\delta)\frac{1}{4} \geq \frac{1}{4},$$

as required.

Proposition 4.4 (Uniqueness)**.** *Under additional Assumptions 4.1 and 4.3, the actual value of contracts J^* in $\mathcal{J}$ is the only fixed point of the Negishi operator.*

We finally notice that, without further restrictions, an *ex-ante* efficient contract might not be *ex-post* efficient. In other terms, an efficient contract z in $\mathcal{Z}$ might be such that, at some future contingency, implied utility values are not on the efficient frontier, that is, for all welfare weights θ_t in Θ,

$$\theta_t \cdot U_t(z) < J_t^*(\theta_t).$$

This might happen in the presence of incentive compatibility constraints or of negative consumption externalities. When contracts are restricted by incentive compatibility, the planner might find it profitable to sacrifice future efficiency in order to sustain current incentives. Externalities, on the other side, might interfere with monotonicity, inducing the planner to reduce welfare of some individuals in order to boost welfare of some other individuals. We provide examples of both situations and, preliminarily, we annotate that a failure of *ex post* efficiency can only occur under a violation of Assumption 4.2.

Proposition 4.5 (*Ex post* efficiency)**.** *Under additional Assumption 4.2, every efficiency contract z in $\mathcal{Z}$ satisfies, for some contingent process $(\theta_t)_{t\in\mathbb{T}}$ of welfare weights in Θ,*

$$\theta_t \cdot U_t(z) = J_t^*(\theta_t).$$

Example 4.4 (Private information)**.** We show that, in the presence of incentive constraints, *ex post* efficiency might not be achieved. The logic of the example is simple: destroying resources when types are equal allows the planner to satisfy incentive compatibility for an asymmetric allocation, so privileging the type who values consumption more and thus increasing social welfare.

We consider a simple economy with two *ex-ante* identical individuals, each with utility function

$$U_0^i(z) = (1-\delta)\,\mathbb{E}_0 \sum_{t\in\mathbb{T}} \delta^t \xi^i z_{t+1}^i.$$

The preference shock ξ^i takes values in $\{\alpha, \beta\} \subset \mathbb{R}^{++}$ with $\alpha < \beta$. This shock is private information and affects utility permanently. Each individual action space is $Z^i = [0, 1]$ and a material balance constraint imposes

$$z_t^i + z_t^{-i} \leq 1.$$

The planner devises a contract inducing truthful revelation of private information at the beginning, and assigning contingent consumptions over the entire infinite horizon. Uncertainty is fully resolved after individuals truthfully report their types to the planner.

Consider first the efficient allocation contingent on revealed information. After truthful revelation of types, the planner splits the unit endowment between the two agents, conditional on their preference shocks. The efficient frontier is so given by

$$\frac{v^i\left(\xi^i,\xi^{-i}\right)}{\xi^i}+\frac{v^{-i}\left(\xi^i,\xi^{-i}\right)}{\xi^{-i}}=1. \tag{4.1}$$

To elicit private information, the incentive compatibility constraint imposes

$$\mathbb{E}_{\xi^{-i}|\xi^i}\frac{v^i\left(\xi^i,\xi^{-i}\right)}{\xi^i}\geq\mathbb{E}_{\xi^{-i}|\xi^i}\frac{v^i\left(\hat{\xi}^i,\xi^{-i}\right)}{\hat{\xi}_i}.$$

We show that the planner sacrifices future welfare in order to efficiently extract private information from agents. To this purpose, we assume that, for some sufficiently small ϵ in $\mathbb{R}^{++}$,

$$\pi\left(\alpha,\beta\right)=\pi\left(\beta,\alpha\right)=\frac{1-\epsilon}{2}\text{ and }\pi\left(\alpha,\alpha\right)=\pi\left(\beta,\beta\right)=\frac{\epsilon}{2}. \tag{4.2}$$

Conclusions survive perturbations of these probabilities.

We preliminarily establish that, by condition (4.2), *ex-post* efficient and incentive compatible allocations satisfy

$$\frac{v^i\left(\xi^i,\xi^{-i}\right)}{\xi^i}\equiv z^i,$$

that is, consumption is independent of types. This can be proved by direct inspection of all incentive compatibility constraints under *ex-post* efficiency (4.1):

$$\epsilon\frac{v^1\left(\alpha,\alpha\right)}{\alpha}+\left(1-\epsilon\right)\frac{v^1\left(\alpha,\beta\right)}{\alpha}\geq\epsilon\frac{v^1\left(\beta,\alpha\right)}{\beta}+\left(1-\epsilon\right)\frac{v^1\left(\beta,\beta\right)}{\beta}, \tag{4.3}$$

$$\epsilon\frac{v^1\left(\beta,\beta\right)}{\beta}+\left(1-\epsilon\right)\frac{v^1\left(\beta,\alpha\right)}{\beta}\geq\epsilon\frac{v^1\left(\alpha,\beta\right)}{\alpha}+\left(1-\epsilon\right)\frac{v^1\left(\alpha,\alpha\right)}{\alpha}, \tag{4.4}$$

$$-\epsilon\frac{v^1\left(\alpha,\alpha\right)}{\alpha}-\left(1-\epsilon\right)\frac{v^1\left(\beta,\alpha\right)}{\beta}\geq-\epsilon\frac{v^1\left(\alpha,\beta\right)}{\alpha}-\left(1-\epsilon\right)\frac{v^1\left(\beta,\beta\right)}{\beta}, \tag{4.5}$$

$$-\epsilon\frac{v^1\left(\beta,\beta\right)}{\beta}-\left(1-\epsilon\right)\frac{v^1\left(\alpha,\beta\right)}{\alpha}\geq-\epsilon\frac{v^1\left(\beta,\alpha\right)}{\beta}-\left(1-\epsilon\right)\frac{v^1\left(\alpha,\alpha\right)}{\alpha}. \tag{4.6}$$

Adding up (4.3)-(4.5), and then (4.4)-(4.6), we obtain

$$\frac{v^1\left(\alpha,\beta\right)}{\alpha}=\frac{v^1\left(\beta,\alpha\right)}{\beta}.$$

Adding up (4.3)-(4.6), and then (4.4)-(4.5), we conclude that

$$\frac{v^1\left(\alpha,\alpha\right)}{\alpha}=\frac{v^1\left(\beta,\beta\right)}{\beta}.$$

This suffices to draw our implication.

To see that *ex-ante* efficient allocations are not necessarily *ex-post* efficient, we argue by contradiction. Conditional on *ex-post* efficiency (4.1), the value of the program is determined as

$$J_0\left(\theta_0^i, \theta_0^{-i}\right) = \max \theta_0^i z^i \mathbb{E}_0 \xi^i + \theta_0^{-i} z^{-i} \mathbb{E}_0 \xi^{-i}$$

subject to

$$z^i + z^{-i} = 1.$$

Observing that (4.2) implies

$$\mathbb{E}_0 \xi^i = \mathbb{E}_0 \xi^{-i} = \frac{\alpha + \beta}{2},$$

we obtain

$$J_0\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\alpha + \beta}{4}.$$

We thus argue that, for any sufficiently small ϵ in $\mathbb{R}^{++}$, this value can be increased subject to material balance and incentive compatibility, so delivering a contradiction.

Consider the contingent allocation given by

$$\begin{aligned}
\frac{v^1(\beta, \alpha)}{\beta} = \frac{v^2(\alpha, \beta)}{\beta} &= 1 - \epsilon, \\
\frac{v^1(\alpha, \beta)}{\alpha} = \frac{v^2(\beta, \alpha)}{\alpha} &= \epsilon, \\
\frac{v^i(\alpha, \alpha)}{\alpha} = \frac{v^i(\beta, \beta)}{\beta} &= 0.
\end{aligned}$$

This allocation satisfies incentive compatibility constraints and material balance, though resources are not exhausted in consumption whenever individuals are of the same types. Direct computation shows that the social value of this program is

$$\hat{J}_0\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1 - \epsilon}{2}\left((1 - \epsilon)\beta + \epsilon\alpha\right).$$

As $\beta > \alpha$, for any sufficiently small ϵ in $\mathbb{R}^{++}$,

$$\hat{J}_0\left(\frac{1}{2}, \frac{1}{2}\right) > J_0\left(\frac{1}{2}, \frac{1}{2}\right).$$

This establishes our claim.

Example 4.5 (Consumption externalities)**.** In this example, the utility of each individual decreases with the future consumption of the other individual. For this reason it is efficient to not exhaust aggregate resources. Though extreme, this simple example singles out a mechanism responsible for potential failure of *ex post* efficiency with consumption externalities.

A deterministic economy is populated by two identical individuals with utility aggregator

$$W_t^i\left(z_t, v_{t+1}^i\right) = (1 - \delta) \min\left\{c_t^i, 1\right\} + \delta v_{t+1}^i - \gamma \max\left\{w_t^{-i}, 0\right\},$$

where the individual action space is $Z^i = \mathbb{R}^+ \times \mathbb{R}$, with typical element $z^i = \left(c^i, w^i\right)$. The parameter $\gamma > 0$ captures the externality and we assume that $1 > \gamma > \delta > 0$. Contracts are restricted by feasible sets $\mathcal{G}_t$ containing all plans satisfying consumption feasibility,

$$c_t^i + c_t^{-i} \leq 1,$$

and perfect foresight about future utilities,

$$w_t^i = v_{t+1}^i.$$

Utility values are restricted to the interval $\left[\underline{v}_t^i, \bar{v}_t^i\right] = \left[-\gamma\left(1-\delta\right)^{-1}, 1\right]$. Notice that this economy is convex and presents no relevant pathological features. We claim that the value of the contract is given by

$$J_t\left(\theta_t\right) = \left(1-\delta\right)\max\left\{\theta_t^i, \theta_t^{-i}\right\} + \max\left\{\delta\theta_t^i - \gamma\left(1-\delta\right)^{-1}\theta_t^{-i}, 0\right\}, \tag{4.7}$$

that is, the symmetric Pareto frontier is piecewise linear with three flat regions. This requires us to verify that it is a fixed point of the Negishi operator.

In the Negishi program, the planner's objective is

$$\begin{aligned}\theta_t \cdot W_t\left(z_t, v_{t+1}\right) \quad = \quad & \left(1-\delta\right)\left(\theta_t^i c_t^i + \theta_t^{-i} c_t^{-i}\right) + \left(\delta\theta_t^i - \gamma\theta_t^{-i}\right)\max\left\{v_{t+1}^i, 0\right\} \\ & + \left(\delta\theta_t^{-i} - \gamma\theta_t^i\right)\max\left\{v_{t+1}^{-i}, 0\right\} + \delta\theta_t^i \min\left\{v_{t+1}^i, 0\right\} \\ & + \delta\theta_t^{-i}\min\left\{v_{t+1}^{-i}, 0\right\}.\end{aligned}$$

Whenever $\delta\theta_t^i - \gamma\theta_t^{-i} \leq 0$ and $\delta\theta_t^{-i} - \gamma\theta_t^i \leq 0$, setting $v_{t+1} = 0$ is optimal and

$$\left(TJ\right)_t\left(\theta_t\right) = \left(1-\delta\right)\max\left\{\theta_t^i, \theta_t^{-i}\right\}.$$

Whenever $\delta\theta_t^i - \gamma\theta_t^{-i} > 0$ and, so, $\delta\theta_t^{-i} - \gamma\theta_t^i < 0$, an optimal plan satisfies $v_{t+1}^i \geq 0$ and $v_{t+1}^{-i} \leq 0$. Hence, the planner's objective becomes

$$\theta_t \cdot W_t\left(z_t, v_{t+1}\right) = \left(1-\delta\right)\max\left\{\theta_t^i, \theta_t^{-i}\right\} + \left(\delta\theta_t^i - \gamma\theta_t^{-i}\right)v_{t+1}^i + \delta\theta_t^{-i} v_{t+1}^{-i}.$$

Geometrically, the planner maximizes a linear functional of continuation utilities subject to sign restrictions. This gives as solution either $\left(v_{t+1}^i, v_{t+1}^{-i}\right) = \left(\left(1-\delta\right), 0\right)$ or $\left(v_{t+1}^i, v_{t+1}^{-i}\right) = \left(1, -\left(1-\delta\right)^{-1}\gamma\right)$. In both cases, our conjecture (4.7) is confirmed.

All of the above established, we show that an *ex-ante* efficient contract will not achieve *ex-post* efficiency. Consider the planner program at symmetric welfare weights $\left(\theta_t^i, \theta_t^{-i}\right) = \left(1/2, 1/2\right)$. The planner's objective reduces to

$$\begin{aligned}\theta_t \cdot W_t\left(z_t, v_{t+1}\right) \quad = \quad & \left(\frac{1-\delta}{2}\right)\left(c_t^i + c_t^{-i}\right) + \left(\frac{\delta-\gamma}{2}\right)\max\left\{v_{t+1}^i, 0\right\} \\ & + \left(\frac{\delta-\gamma}{2}\right)\max\left\{v_{t+1}^{-i}, 0\right\} \\ & + \frac{\delta}{2}\min\left\{v_{t+1}^i, 0\right\} + \frac{\delta}{2}\min\left\{v_{t+1}^{-i}, 0\right\}.\end{aligned}$$

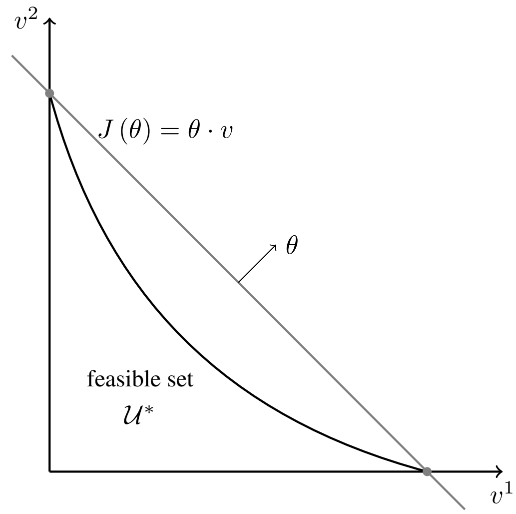

FIGURE 3. Linear social welfare function

Observing that $1 > \gamma > \delta > 0$, we concluse that any optimal plan requires $\left(v_t^i, v_{t+1}^{-i}\right) = (0, 0)$. Thus, continuation utility values are not on the Pareto efficient frontier.

4.3. **Sunspot implementation.** We show that, in the absence of convexity, the Negishi method implements efficient contracts when the planner is allowed to use a random device in order to allocate promises over time. This requires an expansion of the primitive program with the introduction of purely extrinsic uncertainty. Preferences are also extended by means of the expected utility principle to evaluate extrinsic uncertainty, whereas non-expected utility is permitted with respect to sources of intrinsic uncertainty. The logic of sunspot implementation in a non-convex economy is illustrated by Figure 3, where the Pareto frontier refers to continuation utilities: the sunspot relaxes restrictions on *continuation* utilities and so supports possibly higher *current* social welfare.

Uncertainty affecting fundamentals is governed by a Markov transition $P : S \to \Delta(S)$ on the finite state space S. The sunspot consists of a publicly observable signal ϵ uniformly distributed on the interval $E = [0, 1]$. The expanded probability space $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu}\right)$ reflects both intrinsic and extrinsic uncertainty. In the sunspot-expanded economy, a plan $\hat{z}$ in $\hat{\mathcal{Z}}$ is contingent to the observable history of Markov states and sunspot shocks. We clarify how the sunspot-expansion affects the feasible sets and the utility aggregators. To this purpose, we use $(\mathcal{E}_t)_{t \in \mathbb{T}}$ for the filtration reflecting sunspot shocks only, so that available information is captured by $\hat{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{E}_t$. To simplify our presentation, we only admit contingent plans that are measurable, at every t in $\mathbb{T}$, with respect to a *finite* partition of $\hat{\mathcal{F}}_t$. This dispenses us from dealing with issues of integrability.

The sunspot-augmented economy satisfies the following properties. For the feasible set, we assume that

$$(\hat{z}_t, \hat{v}_{t+1}) \in \hat{\mathcal{G}}_t \text{ if and only if } (\hat{z}_t, \mathbb{E}(\hat{v}_{t+1}|\mathcal{F}_{t+1} \otimes \mathcal{E}_t)) \in \mathcal{G}_t.$$

With some abuse of notation, the coincidence between these feasible sets is required conditional on any partial history of sunspot shocks. This condition asserts that a plan is feasible for the sunspot-expanded economy if and only if it is feasible for the primitive economy when utility promises are evaluated in expectation conditional on non-sunspot uncertainty. Utility aggregators are expanded according to

$$\hat{W}_t^i\left(\hat{z}_t, \hat{v}_{t+1}^i\right) = W_t^i\left(\hat{z}_t, \mathbb{E}\left(\hat{v}_{t+1}^i|\mathcal{F}_{t+1} \otimes \mathcal{E}_t\right)\right).$$

This also expresses the idea that the utility derived from sunspot-sensitive continuation values is evaluated in sunspot-expected terms. An example clarifies the complication generated by non-expected utility. When the aggregator is linear in continuation utility, as in the most conventional applications, this qualification is unnecessary.

Example 4.6 (Sunspot expansion with non-expected utility)**.** Consider a utility aggregator of the form

$$W_t^i\left(z_t, v_{t+1}^i\right) = (1-\delta)\, u\,(z_t) + \delta\phi^{-1}\left(\mathbb{E}_t\phi\left(v_{t+1}^i\right)\right),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is surjective, increasing and concave. In the space of expanded plans, the corresponding aggregator would be

$$\hat{W}_t^i\left(\hat{z}_t, \hat{v}_{t+1}^i\right) = (1-\delta)\, u\,(\hat{z}_t) + \delta\phi^{-1}\left(\mathbb{E}_t\phi\left(\mathbb{E}\left(\hat{v}_{t+1}^i|\mathcal{F}_{t+1} \otimes \mathcal{E}_t\right)\right)\right).$$

The added randomness is evaluated using expected utility, though fundamental uncertainty is not.

Proposition 4.6 (Random-device implementation)**.** *The actual value of sunspot-augmented contracts $\hat{J}^*$ in $\mathcal{J}$ is the greatest fixed point of the sunspot-free Negishi operator.*

5. First-order conditions

We present simple first-order conditions arising from the application of the Negishi method. In particular, we argue that first-order conditions allow for an operational characterization of an efficient contract by means of a contingent process of welfare weights. To illustrate the fruitfulness of this approach, we immediately derive certain properties of some efficient contracts established in the previous literature.

We say that feasible contract z^* in $\mathcal{Z}$ satisfies *first-order conditions* if there exists a contingent process $(\theta_t)_{t\in\mathcal{T}}$ of welfare weights in Θ such that

$$\theta_t \cdot U_t\,(z^*) = \max_{(z_t, v_{t+1})\in\mathcal{G}_t} \theta_t \cdot W_t\,(z_t, v_{t+1})$$

subject to

$$(*) \qquad \theta_{t+1} \cdot v_{t+1} \leq \theta_{t+1} \cdot U_{t+1}\,(z^*).$$

In this recursive program, the Negishi constraint on continuation utility values is replaced by its linear approximation. As the Euler equation in a conventional planning program, these first-order conditions rule out welfare-improving readjustments over any arbitrary finite horizon, so enforcing a short-term form of efficiency. Consistently, we say that feasible contract z^* in $\mathcal{Z}$ is *short-term* efficient if it is not Pareto dominated by an alternative feasible contract z in $\mathcal{Z}$ coinciding with contract z^* in $\mathcal{Z}$ at all but finitely many periods t in $\mathbb{T}$. Endowed with this notion, we present our first-order characterization.[10]

Proposition 5.1 (First-order conditions)**.** *Any feasible contract z^* in $\mathcal{Z}$ satisfying first-order conditions for strictly positive initial welfare weights θ_0 in Θ is short-term efficient. Furthermore, under additional Assumptions 4.1-4.2, any efficient contract z^* in $\mathcal{Z}$ satisfies first-order conditions for some contingent process $(\theta_t)_{t\in\mathbb{T}}$ of welfare weights in Θ, provided that there exists a feasible contract z^0 in $\mathcal{Z}$ such that $U_t^i(z) > U_t^i(z^0)$.*

We apply the first-order characterization established in Proposition 5.1 to some well-known instances of recursive contracts. In an economy with limited commitment, we show that the planner increases the welfare weight of a constrained individual, and this fully determines the dynamics with only two individuals. For dynamic Ramsey taxation with contingent government debt, we argue that welfare weights are stationary and, as a consequence, the planner simply maximizes the static surplus subject to legacy debt. Finally, when a principal insures a privately informed agent, we recover the well-known inverted Euler equation in terms of dynamics of welfare weights.

Example 5.1 (Limited commitment)**.** Consider a limited commitment economy described in Example 3.2. Let $\gamma_t^i \geq 0$ in Γ_t be the Lagrange multiplier for the participation constraints and let λ_{t+1} in Λ_{t+1} be the Lagrange multiplier for constraint (*) on continuation utility values, where Lagrange multipliers are represented as random variables. The Lagrangean takes the form

$$\mathcal{L}_t = \Phi_t + \delta \sum_{i\in I} \theta_t^i \mathbb{E}_t v_{t+1}^i + \delta \sum_{i\in I} \gamma_t^i \mathbb{E}_t v_{t+1}^i - \delta \mathbb{E}_t \lambda_{t+1} \sum_{i\in I} \theta_{t+1}^i v_{t+1}^i,$$

where Φ_t in $\mathcal{R}_t$ is short-notation for all terms that do not depend on continuation utility values. Taking the derivative with respect to such values, we obtain

$$\theta_t^i + \gamma_t^i - \lambda_{t+1}\theta_{t+1}^i = 0.$$

We see that the welfare weight of individual i in I is increased whenever the participation constraint is binding.

Example 5.2 (Dynamic Ramsey taxation)**.** Consider dynamic Ramsey taxation with contingent government debt (Example 3.5). Invoking first-order conditions, and constructing

[10] First-order conditions cannot in general implement a fully efficient contract over the entire infinite horizon when not complemented by some sort of transversality condition (see Marcet and Marimon [25, Theorem 2] and Pavoni *et al.* [32, Condition (T), Proposition 4]). However, when the Negishi operator admits a unique value, any efficient contract can be arbitrarily approximated over a sufficiently large finite horizon and our first-order characterization turns accurate.

the associated Laragrangean, we obtain

$$\mathcal{L}_t = \Phi_t + \delta\theta_t^{\mathrm{a}}\mathbb{E}_t v_{t+1}^{\mathrm{a}} + \delta\theta_t^{\mathrm{g}}\mathbb{E}_t v_{t+1}^{\mathrm{g}} - \delta\mathbb{E}_t\lambda_{t+1}\theta_{t+1}^{\mathrm{a}}v_{t+1}^{\mathrm{a}} - \delta\mathbb{E}_t\lambda_{t+1}\theta_{t+1}^{\mathrm{g}}v_{t+1}^{\mathrm{g}},$$

where notation is interpreted as in previous Example 5.1. Taking the derivative with respect to continuation utility values, we conclude that $\lambda_{t+1} = 1$, $\theta_t^{\mathrm{a}} = \theta_{t+1}^{\mathrm{a}}$ and $\theta_t^{\mathrm{g}} = \theta_{t+1}^{\mathrm{g}}$. Thus, welfare weights are stationary and the efficient contract is given by the static program

$$\max \theta_t^{\mathrm{a}} u\left(c_t, e_t\right) + \theta_t^{\mathrm{g}}\left(u_c\left(c_t, e_t\right)c_t + u_e\left(c_t, e_t\right)e_t\right)$$

subject to

$$c_t + g_t \leq e_t.$$

In other terms, the planner distributes the static surplus according to the given welfare weights.

Example 5.3 (Asymmetric information)**.** Consider an amended version of the economy described in Example 3.3 featuring only two individuals: a principal (p) and an agent (a). The agent is privately informed about her own preference shocks, which are unobservable to the principal. The principal has a linear utility and no preference shocks. We let $\gamma_{t+1} \geq 0$ in Γ_{t+1} be the Lagrange multipliers associated with the incentive compatibility constraints. Using first-order conditions, the Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_t &= \Phi_t + \delta\mathbb{E}_t\sum_{i\in I}\theta_t^i v_{t+1}^i\left(s_{t+1}\right) - \delta\mathbb{E}_t\lambda_{t+1}\sum_{i\in I}\theta_{t+1}^i v_{t+1}^i\left(s_{t+1}\right)\\ &\quad -\delta\mathbb{E}_t\sum_{\hat{s}_{t+1}\neq s_{t+1}}\gamma_{t+1}\left(s_{t+1},\hat{s}_{t+1}\right)\left(v_{t+1}^{\mathrm{a}}\left(\hat{s}_{t+1}\right) - v_{t+1}^{\mathrm{a}}\left(s_{t+1}\right)\right),\end{aligned}$$

where Φ_t in $\mathcal{R}_t$ collects all other terms and $\hat{s}_{t+1}$ in S is the untruthful declaration when the true type in s_{t+1} in S. Taking derivatives with respect to continuation utility values, we obtain

$$\theta_t^{\mathrm{p}} = \lambda_{t+1}\theta_{t+1}^{\mathrm{p}}$$

and

$$\theta_t^{\mathrm{a}} = \mathbb{E}_t\lambda_{t+1}\theta_{t+1}^{\mathrm{a}},$$

where this latter condition is obtained supposing an equal increase of continuation utility values of the agent contingent on all types, so that the incentive compatibility constraints remain unaffected. These two equations jointly imply

$$\left(\frac{\theta_t^{\mathrm{a}}}{\theta_t^{\mathrm{p}}}\right) = \mathbb{E}_t\left(\frac{\theta_{t+1}^{\mathrm{a}}}{\theta_{t+1}^{\mathrm{p}}}\right),$$

which is basically the inverted Euler equation appearing in the literature.

6. Dual approach

6.1. **Comparison.** A dual approach to recursive contracts was initially introduced by Marcet and Marimon [25], and it is more recently studied by Pavoni *et al.* [32] (see also

Messner *et al.* [28]).[11] We argue that the dual approach might dramatically overestimate the value of efficient contracts under non-pathological conditions. We discuss this feature in a simple example for which conventional methods work smoothly. The Negishi method delivers the correct value, whereas the dual method yield an erroneous characterization. We also show that, in general, the dual value dominates the Negishi value and we further explore this discrepancy in a class of economies with limited commitment. A conventional Negishi approach seems more reliable than a characterization via dual method.

6.2. **An example.** To uncover the major drawback of the dual method, we consider an environment in which both the Negishi and the dual method are inessential, because conventional tools of dynamic programming can be safely applied and deliver an unambiguous characterization. In addition, as the economy consists of a single individual, the Negishi method is absolutely innocuous, as it exactly coincides with the primitive primal program. So, our thought experiment intentionally uncovers the distortionary action of the dual method. As the program is non-convex, the dual value overshoots dramatically because of the duality wedge.

Consider an economy with a single individual whose utility aggregator is given by

$$W(z, v) = z + \delta f(v),$$

where δ in $(0,1) \subset \mathbb{R}^+$ is the discount factor and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded increasing map such that $|f(v') - f(v'')| \leq |v' - v''|$. The action space is $Z = \mathbb{R}^+$. The trivial planning program consists in allocating available resources to the only consumer in the economy subject to

$$z \leq e.$$

Yet, because of the non-linearity in the utility aggregator, the value of the program has to be determined via a fixed point theorem.

The Negishi method reduces to determining a value J^* in $\mathbb{R}^+$ such that

$$J^* = e + \delta f(J^*).$$

Such a value exists and is unique by the Contraction Mapping Theorem. We now turn to the dual program, and show that it might easily determine a different value due to the non-convex nature of the program. This requires a short digression on conjugate maps (see Blume [9] for a basic introduction).

Consider a bounded map $f : \mathbb{R}^+ \to \mathbb{R}^+$. Its *conjugate* is defined as

$$\hat{f}(\lambda) = \sup_{x \geq 0} f(x) - \lambda x,$$

whereas its *double conjugate* is

$$\hat{\hat{f}}(x) = \inf_{\lambda \geq 0} \hat{f}(\lambda) + \lambda x.$$

[11] More precisely, Marcet and Marimon [25] provide a *saddle point*, rather than a dual, approach to recursive contracts, and they focus on programs in which a saddle point exists.

It is a fundamental result of conventional duality theory that the double conjugate is a bounded concave map such that $\hat{\hat{f}} \geq f$. In fact, it is the least map with this property, that is, it is the concave envelope of the primitive map $f : \mathbb{R}^+ \to \mathbb{R}^+$. We show that the basic action of the dual approach consists in delivering the value of a modified program corresponding to the concave envelope of the planner's objective.

We construct Pavoni *et al.* [32]'s dual operator, though by different arguments.[12] Introducing a Lagrange multiplier, the recursive primal operator $T : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$(TJ) = \sup_{v \geq 0} \inf_{\lambda \geq 0} e + \delta f(v) - \delta \lambda (v - J).$$

The Lagrange multiplier simply accounts for the constraint on the continuation value, $v \leq J$. The dual operator $\hat{T} : \mathbb{R}^+ \to \mathbb{R}^+$ is instead given by

$$\left(\hat{T}J\right) = \inf_{\lambda \geq 0} \sup_{v \geq 0} e + \delta f(v) - \delta \lambda (v - J).$$

Exploiting conjugacy, the dual operator reduces to

$$\begin{aligned}
\left(\hat{T}J\right) &= \inf_{\lambda \geq 0} e + \delta \sup_{v \geq 0} \left(f(v) - \lambda v\right) + \delta \lambda J \\
&= \inf_{\lambda \geq 0} e + \delta \hat{f}(\lambda) + \delta \lambda J \\
&= e + \delta \inf_{\lambda \geq 0} \left(\hat{f}(\lambda) + \lambda J\right) \\
&= e + \delta \hat{\hat{f}}(J).
\end{aligned}$$

By monotone concavity (Krasnosel'skiĭ [20]), the recursive dual operator admits a unique fixed point $\hat{J}^*$ in $\mathbb{R}^+$. This is also established in Pavoni *et al.* [32], because their boundary conditions [32, Assumption 3] are satisfied in this simple example. The value, however, can only fortuitously coincide with the actual value when $f : \mathbb{R}^+ \to \mathbb{R}^+$ is not concave. The logic of this misrepresentation is illustrated by Figure 4.

6.3. **Dominance.** We compare the Negishi value with the value obtained through the dual approach. Marcet and Marimon [25] and Pavoni *et al.* [32] construct an extended Lagrangian accounting for incentive constraints in the program. As this seems inessential for our purposes, we simplify by explicitly considering only the constraints on continuation utilities, and the associated Lagrange multipliers. Thus, for the purpose of this comparison, we concede the most favorable conditions to the dual method: Adding Lagrange multipliers related to the feasible set would amplify the duality gap, increasing the distance between the Negishi value and the dual value.

[12] More precisely, we obtain the dual Bellman operator as in [32, Definition 1] for the trivial program under examination. To simplify, we directly exploit homogeneity and restrict the dual value function on the unit sphere. Their unrestricted value function is implicitly determined everywhere by positive scaling.

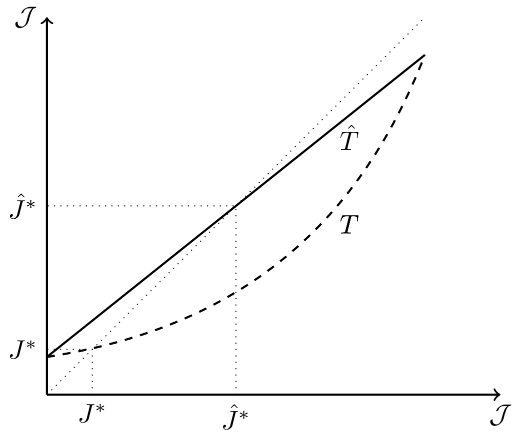

FIGURE 4. Concave envelope

We introduce Lagrange multipliers λ in Λ, the space of processes with values in $\mathbb{R}^+$. Our Negishi operator can be innocuously expressed as

$$(TJ)_t(\theta_t) = \sup \inf \theta_t \cdot W_t(z_t, v_{t+1}) - \mathbb{E}_t \lambda_{t+1}(\theta_{t+1} \cdot v_{t+1} - J_{t+1}(\theta_{t+1})),$$

where the supremum is taken over feasible plans (z_t, v_{t+1}) in $\mathcal{G}_t$ and the infimum over Lagrange multipliers λ_{t+1} in Λ_{t+1} and welfare weights θ_{t+1} in Θ_{t+1}, both interpreted as $\mathcal{F}_{t+1}$-measurable random variables. We obtain the dual operator by reversing the role of infimum and supremum operations. Although derived in a different way, this is precisely the operator studied by Pavoni *et al.* [32].

The dual operator $\hat{T} : \mathcal{J} \to \mathcal{J}$ is given by

$$\left(\hat{T}J\right)_t(\theta_t) = \inf \sup \theta_t \cdot W_t(z_t, v_{t+1}) - \mathbb{E}_t \lambda_{t+1}(\theta_{t+1} \cdot v_{t+1} - J_{t+1}(\theta_{t+1})),$$

where infimum and supremum in the previous primal program are reversed. The *dual value* of contracts is a fixed point of the dual operator, that is, a support map $\hat{J}$ in $\mathcal{J}$ such that $\hat{J} = \left(\hat{T}\hat{J}\right)$. This formulation allows for a simple comparison between the alternative approaches: the Negishi value is in general more accurate. The excess error of the dual method upon the Negishi method is an implication of the added duality gap, that is, a reversal of the infimum and supremum operations. We only need a further restriction ensuring that the dual operator admits a fixed point.[13]

Assumption 6.1 (Bounds)**.** *Utility values v_t are restricted to the interval $[\underline{v}_t, \bar{v}_t] \subset \mathcal{V}_t$, where the bounds are given in Proposition 3.1.*

[13]Pavoni *et al.* [32, Assumption 3] postulate the existence of bounds for the dual operator, in addition to restricting actions and utility values to compact spaces. Their dual operator admits a unique fixed point under a sort of interiority condition [32, Assumption 3 (i)-(iii)] which can be related to our more primitive Assumption 4.3. Our comparison holds true even when interiority fails.

Proposition 6.1 (Comparison)**.** *Under additional Assumption 6.1, for any Negishi value J in $\mathcal{J}$, there exists a dual value $\hat{J}$ in $\mathcal{J}$ such that $J \leq \hat{J}$.*

6.4. **A class of economies.** To further clarify the distortionary nature of the dual approach under non-convexity, we consider a more tractable class of economies with limited commitment. The individual action space Z^i is $\mathbb{R}^+$, interpreted as consumption. Feasible set $\mathcal{G}_t$ only require material balance,

$$\sum_{i \in I} z_t^i \leq \sum_{i \in I} e_t^i,$$

and a participation constraint,

$$W_t^i \left(z_t^i, v_{t+1}^i \right) \geq \phi_t^i,$$

where ϕ^i in $\mathcal{V}^i$ is an exogenously given reservation utility value. The only source of non-convexity in this economy is the utility aggregator, or possibly the non-expected nature of utility. We shall provide an explicit example at the end of our analysis.

In this class of economies, under uncertainty aversion, the computation error due to the dual approach is unambiguously identified: The method returns the value of a transformed economy in which, in the objective of the planner, each utility aggregator is replaced by its concave envelope with respect to continuation utility values. This mirrors the simple characterization obtained in our previous example (§6.2). We remark again that, had we considered the dual operator with a fully expanded set of Lagrange multipliers, as in Pavoni *et al.* [32]'s original analysis, the envelope would have also distorted the full spectrum of non-convex constraints defining feasibility.

Assumption 6.2 (Uncertainty-aversion)**.** *Each utility aggregator is quasi-concave in continuation utility values, that is, given $\tilde{z}_t^i$ in $\mathcal{Z}_t^i$,*

$$\left\{ v_{t+1}^i \in \mathcal{V}_{t+1}^i : W_t^i \left(\tilde{z}_t^i, v_{t+1}^i \right) \geq \eta_t^i \right\} \text{ is convex.}$$

Proposition 6.2 (Dual error)**.** *Under uncertainty-aversion (Assumption 6.2), the dual value of a given economy corresponds to the dual value of a related economy in which feasible sets are unmodified and each utility aggregator in the planner's objective is replaced by its concave envelope with respect to continuation utility values.*[14]

It is in general difficult to single out the distortionary effect entailed in the Negishi value. We argue that the Negishi method is faithful when the economy, though non-convex. can be transformed into a convex economy by a monotone transformation of utility aggregators. As clarified in the initial illustrative example (§2), this might happen with rather conventional aggregators when the effect of the dual method is instead disruptive.

[14] That is, the least upper semicontinuous utility aggregator $\tilde{W}_t^i : \mathcal{Z}_t^i \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$ such that $W_t^i \leq \tilde{W}_t^i$ with the property that, given $\tilde{z}_t^i$ in $\mathcal{Z}_t^i$, it is concave on the restricted domain

$$\left\{ v_{t+1}^i \in \mathcal{V}_{t+1}^i : W_t^i \left(\tilde{z}_t^i, v_{t+1} \right) \geq \phi_t^i \right\}.$$

We say that a utility aggregator is *essentially concave* if, for some strictly increasing continuous map $f^i : V^i \to \mathbb{R}$ on a convex domain $V^i \subset \mathbb{R}$ with $f^i\left(V^i\right) \subset V^i$,

$$\tilde{W}_t^i\left(z_t^i, \tilde{v}_{t+1}^i\right) = f^{i^{-1}}\left(W_t^i\left(z_t^i, f^i\left(\tilde{v}_{t+1}^i\right)\right)\right) \text{ is concave in } \left(z_t^i, \tilde{v}_{t+1}^i\right) \in \mathcal{Z}_t^i \times \mathcal{V}_{t+1}^i,$$

where the utility aggregator takes values in $V^i \subset \mathbb{R}^+$. Whenever this monotone transformation into a concave aggregator is feasible, the Negishi method yields the correct value for the *untransformed* economy. Unlike the dual method, the Negishi distortion depends on intrinsic features of the feasible sets, more than on the representation of preferences by means of a specific utility aggregator.

Example 6.1 (Essential concavity)**.** Consider the aggregator of the form

$$v_t = \sqrt{(1-\delta)\sqrt{z_t} + \delta \mathbb{E}_t v_{t+1}^2}.$$

Using the transformation $v = \sqrt{\tilde{v}}$, we obtain

$$\tilde{v}_t = (1-\delta)\sqrt{z_t} + \delta \mathbb{E}_t \tilde{v}_{t+1},$$

which is a monotone concave utility aggregator. Hence, the initial aggregator is essentially concave and, as shown in Proposition 6.3, the application of the Negishi method is non-distortive.

Proposition 6.3 (Essential concavity)**.** *When each utility aggregator is essentially concave, the greatest fixed point of the Negishi operator is the actual value of contracts.*

We conclude with another example of the extreme distortion created by the dual method. We consider an economy with risk-sensitive preferences under the hypothesis of increasing risk-tolerance. The dual method systematically under-estimate risk-aversion and, assuming risk-aversion vanishes on arbitrarily large consumption, it delivers a risk-neutral efficient contract.

Example 6.2 (Increasing risk-tolerance)**.** Assume $\phi^i = 0$ and consider the utility aggregator given by

$$W_t^i\left(z_t^i, v_{t+1}^i\right) = (1-\delta)\, z_t^i + \delta f^{i^{-1}}\left(\mathbb{E}_t f^i\left(v_{t+1}^i\right)\right),$$

where $f^i : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing and strictly concave map with $f^i\left(\mathbb{R}^+\right) = \mathbb{R}^+$. We assume that relative risk aversion decreases and vanishes as consumption grows unboundedly. In particular, we require

$$f^{i^{-1}}\left(\mathbb{E} f^i\left(\lambda^{-1} v^i\right)\right) \geq \lambda^{-1} f^{i^{-1}}\left(\mathbb{E} f^i\left(v^i\right)\right) \text{ for every } \lambda \in (0,1).$$

This last condition ensures that the certainty equivalence increases no less than proportionally with the expansion of utility values. In addition, for strictly positive utility values,

$$\lim_{\lambda \to 0} \frac{f^{i^{-1}}\left(\mathbb{E} f^i\left(\lambda^{-1} v^i\right)\right)}{\mathbb{E}\left(\lambda^{-1} v^i\right)} = 1,$$

that is, the certainty equivalent approaches the expected value of a lottery as risk aversion disappears on large consumption levels. We then provide a full characterization of the dual error using our Proposition 6.2.

Exploiting decreasing certainty equivalent, we obtain

$$\begin{aligned} W_t^i\left(z_t^i, v_{t+1}^i\right) &\leq (1-\lambda) W_t^i\left(z_t^i, 0\right) + \lambda W_t^i\left(z_t^i, \lambda^{-1} v_{t+1}^i\right) \\ &\leq (1-\lambda) \tilde{W}_t^i\left(z_t^i, 0\right) + \lambda \tilde{W}_t^i\left(z_t^i, \lambda^{-1} v_{t+1}^i\right) \\ &\leq \tilde{W}_t^i\left(z_t^i, v_{t+1}^i\right). \end{aligned}$$

Taking the limit as λ in $(0,1) \subset \mathbb{R}$ vanishes, and recalling asymptotic risk-neutrality, we conclude

$$W_t\left(z_t^i, v_{t+1}^i\right) \leq (1-\delta) z_t^i + \delta \mathbb{E}_t v_{t+1}^i = \tilde{W}_t\left(z_t^i, v_{t+1}^i\right).$$

Hence, the dual method delivers the efficient allocation of consumption under risk-neutrality. We now turn to the examination of the error under the Negishi approach.

We compare the dual and the Negishi value at extreme welfare weights θ_t in Θ with $\theta_t^i = 1$ for some individual i in I. By our previous arguments, the dual value is the utility the risk-neutral individual derives from the aggregate endowment, that is,

$$\hat{v}_t^i = (1-\delta) e_t + \delta \mathbb{E}_t \hat{v}_{t+1}^i. \tag{6.1}$$

The Negishi value is instead given by the utility from aggregate endowment of the risk-averse individual. Hence, the Negishi value under extreme welfare weights is determined by the recursive equation

$$v_t^i = (1-\delta) e_t + \delta {f^i}^{-1}\left(\mathbb{E}_t f^i\left(v_{t+1}^i\right)\right). \tag{6.2}$$

Comparing equations (6.1)-(6.2), we notice that ${f^i}^{-1}\left(\mathbb{E}_t f^i\left(\hat{v}_{t+1}^i\right)\right) < \mathbb{E}_t \hat{v}_{t+1}^i$ by strict concavity and this ensures that the greatest solution to (6.2) is dominated by the only solution to (6.1). We thus obtain

$$J_t\left(\theta_t\right) < \hat{J}_t\left(\theta_t\right).$$

In addition, the greater risk-aversion over the feasible set, the larger the discrepancy between these two values.

7. A MAXMIN-NEGISHI METHOD

We describe an alternative recursive planning program with a maxmin-type social welfare function. This recursive approach allows us to implement efficient contracts exactly even in non-convex economies. Beyond computational (dis)advantages, the method clarifies that an optimal policy can always be expressed in terms of the evolution of welfare *shares*, as opposed to welfare *weights*, along with Markov states, that is, as transitions on the minimally extended state space $S \times \Theta$. This Markov property of the optimal policy only

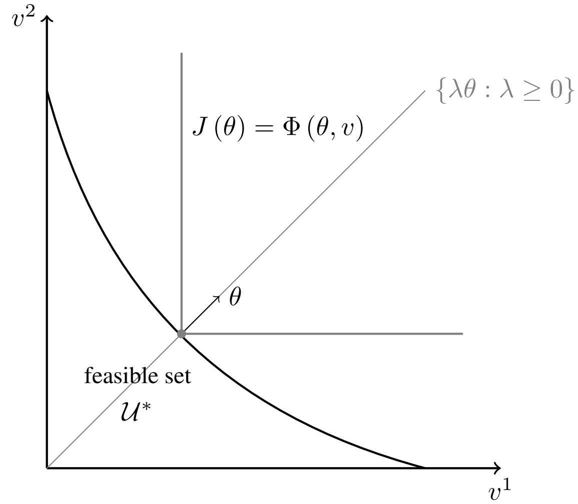

FIGURE 5. Maxmin-type social welfare function

requires that *ex-ante* efficient contracts remain *ex-post* efficient as time and uncertainty unfold. This is certainly the case when the economy satisfies the additional Assumption 4.2.[15] Furthermore, in a convex economy, efficient contracts admits admits an ergodic probability measure on the *minimal* state space $S \times \Theta$.

Given welfare weights θ in Θ, the planner's objective $\Phi : \Theta \times V \to \mathbb{R}$ is given as

$$\Phi(\theta, v) = \max\left\{\lambda \in \mathbb{R}^+ : \lambda\theta \leq v - \underline{v}\right\},$$

where $\underline{v}$ in V is the lower bound on utility values. Here, as in our previous analysis, Θ represents the canonical simplex in $\mathbb{R}^I$, but welfare weights are more properly interpreted as welfare shares. We can equivalently express the planner's objective as a maxmin social welfare function,

$$\Phi(\theta, v) = \min\left\{\ldots, \frac{v^i - \underline{v}^i}{\theta^i}, \ldots\right\}.$$

Maxmin-type social welfare functions support weakly Pareto efficient distributions of utility values even under non-convexity. The advantage of this welfare evaluation, relative to the more traditional weighted sum of utilities, is illustrated by Figure 5.

We modify the Negishi operator consistently, though maintaining the same notation for parsimony. Feasible sets for utility values are now given by

$$\mathcal{U}_t(J_t) = \{v_t \in \mathcal{V}_t : \Phi(\theta_t, \underline{v}_t) \leq \Phi(\theta_t, v_t) \leq J_t(\theta_t) \text{ for every } \theta_t \in \Theta\}.$$

[15] The absence of such a representation for the optimal policy is the major concern in Cole and Kubler [10]. We also notice that Lucas and Stokey [24, Theorem 3]'s statement about recursive optimal policy is slightly deceptive: it does not establish that any plan generated by the optimal policy is a feasible allocation.

The recursive decomposition can so be expressed as

$$(TJ)_t(\theta_t) = \sup_{(z_t, v_{t+1}) \in \mathcal{G}_t} \Phi(\theta_t, W_t(z_t, v_{t+1}))$$

subject to

$$v_{t+1} \in \mathcal{U}_{t+1}(J_{t+1}).$$

The first constraint accounts for feasibility, whereas the second constraint reflects consistency of promised utility values over time.

The maxmin-Negishi value of contracts is compared with the actual maxmin-value of contracts, that is,

$$J_t^*(\theta_t) = \sup_{v_t \in \mathcal{U}_t^*} \Phi(\theta_t, v_t).$$

Not surprisingly, this approach permits the exact determination of the (weakly) efficient frontier. This is due to the fact that any allocation on a non-convex Pareto frontier can be supported by a positive sublinear (as opposed to linear) functional.

Proposition 7.1 (Fixed points)**.** *Maxmin-Negishi operator $T : \mathcal{J} \to \mathcal{J}$ admits a* least *fixed point $\underline{J}$ in $\mathcal{J}$ and a* greatest *fixed point $\bar{J}$ in $\mathcal{J}$. In addition, $J^* = \bar{J}$, where J^* in $\mathcal{J}$ is the actual maxmin-value of contracts.*

We complete our short exploration of the maxmin-Negishi method with a proof of existence of an ergodic distribution on the minimal state space $S \times \Theta$. In other terms, we show that this space exhausts all long-term dynamical properties of efficient contracts. The advantage of the maxmin-type social welfare function is that utility profiles on the efficient frontier are univocally supported by welfare shares θ in Θ. It follows that, subject to *ex post* efficiency (Assumption 4.2), efficient contracts are governed by a Markov correspondence $\Phi : S \times \Theta \twoheadrightarrow \Delta(S \times \Theta)$. Indeed, given a current state (s, θ) in $S \times \Theta$, the recursive optimal plan determines continuation utility values v' in V, contingent on next period state s' in S. As efficient contracts remain on the Pareto frontier as time evolves (by Assumption 4.2), contingent continuation utility values are supported by unique welfare shares θ' in Θ. Hence, the state in the next period can be unambiguously identified with some (s', θ') in $S \times \Theta$. Convexity (Assumption 4.1) guarantees that the Markov correspondence is convex-valued, so that a well-established theorem on ergodic measures can be applied (see Aliprantis and Border [3, Theorem 19.31]).

Proposition 7.2 (Ergodic measure)**.** *Under additional Assumptions 4.1-4.2, efficient contracts are fully described by a Markov correspondence $\Phi : S \times \Theta \twoheadrightarrow \Delta(S \times \Theta)$ admitting an ergodic probability measure.*

8. Conclusion

We have shown that a conventional Negishi method can be used to study recursive contracts. Comparing with the established dual method, a Negishi approach seems more natural and more accurate. In addition, when contractual arrangements can be contingent on purely extrinsic and publicly observable random signals, the Negishi method yields the exact frontier of efficient contracts even in the presence of non-convexity. Finally, a Negishi approach through maxmin social welfare functions reveals that optimal contracts are measurable with respect to a natural state space, consisting of shocks affecting fundamentals augmented with the space of welfare shares.

REFERENCES

[1] Abreu, D., D. Pearce and E. Stacchetti, "Toward a theory of discounted repeated games with imperfect monitoring," *Econometrica*, 58 (1990), 1041-1063.

[2] Amador, M. and M. Aguiar, "A contraction for sovereign debt models," *Journal of Economic Theory*, 183 (2019), 842-875.

[3] Aliprantis, C.D. and K.C. Border, *Infinite Dymensional Analysis: A Hitchhiker's Guide*, third edition (New York: Springer-Verlag, 2006)

[4] Anderson, E.W., "The dynamics of risk-sensitive allocations," *Journal of Economic Theory*, 125 (2005), 93-150.

[5] F. Anscombe and R. Aumann, "A definition of subjective probability," *The Annals of Mathematical Statistics*, 34 (1963), 199-205.

[6] Atkeson, A. and R.E. Lucas, Jr., "On efficient distribution with private information," *Review of Economic Studies*, 59 (1992), 427-453.

[7] Blackwell, D., "Discounted dynamic programming," *Ann. Math. Statist.*, 36 (1965), 226-235.

[8] Bloise, G., "Unique Markov equilibrium under limited commitment," *International Economic Review*, forthcoming (2020).

[9] Blume, L.E., "Duality," in S.N. Durlauf and L.E. Blume (eds), *The New Palgrave Dictionary of Economics* (Palgrave Macmillan, London, 2008).

[10] Cole, H. and F. Kubler, "Recursive contracts, lotteries and weakly concave Pareto sets," *Review of Economic Dynamics*, 15 (2012), 479-500.

[11] Dana, R.-A. and C. LeVan, "Optimal growth and Pareto optimality," *Journal of Mathematical Economics*, 20 (1991), 155-180.

[12] Duggan, J., "Noisy stochastic games," *Econometrica*, 80 (2012), 2017-2045.

[13] Eaton, J. and M. Gersovitz, "Debt with potential repudiation," *Review of Economic Studies*, 48 (1981), 289-309.

[14] Fernandes, A. and C. Phelan, "A recursive formulation for repeated agency with history dependence," *Journal of Economic Theory*, 91 (2000), 223-247.

[15] Hardy, G.H., L.E. Littlewood and G. Pólya, *Inequalities*, second edition, Cambridge University Press, Cambridge, 1934.

[16] Kan, R., "Structure of Pareto optima when agents have stochastic recursive preferences," *Journal of Economic Theory*, 64 (1995), 626-631.

[17] Kehoe, T.J. and D.K. Levine, "Debt-constrained asset markets," *Review of Economic Studies*, 60 (1993), 865-888.

[18] Kocherlakota, N.R., "Implications of efficient risk sharing without commitment," *Review of Economic Studies*, 63 (1996), 595-609.

[19] Koopmans, T.C., "Stationary ordinal utility and impatience," *Econometrica*, 28 (1960), 287-309.

[20] Krasnosel'skiĭ, M.A, *Positive Solutions of Operator Equations*, (Noordhoff, The Netherlands, 1964).

[21] Kreps, D.M. and E.L. Porteus, "Temporal resolution of uncertainty and dynamic choice theory," *Econometrica*, 46 (1978), 185-200.

[22] Ljungqvist, L. and T.J. Sargent, *Recursive Macroeconomic Theory*. second edition, (The MIT Press, 2000).

[23] Lucas, R.E and N.L. Stokey, "Optimal fiscal and monetary policy in an economy without capital," *Journal of Monetary Economics*, 12 (1983), 55-93.

[24] Lucas, R.E. and N.L. Stokey, "Optimal growth with many consumers," *Journal of Economic Theory*, 32 (1984), 139-171.

[25] Marcet, A. and R. Marimon, "Recursive contracts," *Econometrica*, 87 (2019), 1589-1631.

[26] Marinacci, M. and L. Montrucchio, "Unique solutions for stochastic recursive utilities," *Journal of Economic Theory*, 145 (2010), 1776-1804.

[27] Messner, M. and N. Pavoni, "On the recursive saddle point method," *Dynamic Games and Applications*, 6 (2016), 161-173.

[28] Messner, M., N. Pavoni and C. Sleet, "Recursive methods for incentive problems," *Review of Economic Dynamics*, 15 (2012), 501-525.

[29] Miao, J., *Economic Dynamics in Discrete Time*, (The MIT Press, 2014).

[30] Myerson, R.B., *Game Theory: Analysis of Conflict*, (The Harvard University Press, 1997).

[31] Negishi, T., "Welfare economics and existence of equilibrium for a competitive economy," *Metroeconomica*, 12 (1960), 92-97.

[32] Pavoni, N., C. Sleet and M. Messner, "The dual approach to recursive optimization: Theory and examples," *Econometrica*, 86 (2018), 133-172.

[33] Phelan, C., "Repeated moral hazard and one sided commitment," *Journal of Economic Theory*, 66 (1995), 488-506.

[34] Rustichini, A., "Dynamic programming solution of incentive constrained problems," *Journal of Economic Theory*, 78 (1998), 329-354.

[35] Thomas, J. and T. Worrall, "Self-enforcing wage contracts," *Review of Economic Studies*, 55 (1998), 541-554.

[36] Weil, P., "Nonexpected utility in macroeconomics," *Quarterly Journal of Economics*, 105 (1990), 29-42.

Appendix A. Proofs

Proof of Proposition 3.1. To simplify notation, we omit reference to individual i in I and we assume that utility values are in $\mathbb{R}$. Let $C^b(\mathcal{Z},\mathcal{V})$ be the space of bounded and continuous maps $U:\mathcal{Z}\to\mathcal{V}$. Consider the Koopmans operator $T:C^b(\mathcal{Z},\mathcal{V})\to C^b(\mathcal{Z},\mathcal{V})$ defined as

$$(TU)_t(z)=W_t(z_t,U_{t+1}(z)).$$

Clearly, $(TU):\mathcal{Z}\to\mathcal{V}$ is continuous. To see that it is also bounded, notice that

$$(TU)_t(z)\leq W_t(z_t,\|U\|_\infty)\leq W_t(z_t,0)+\delta\|U\|_\infty\leq B+\delta\|U\|_\infty$$

and

$$(TU)_t(z)\geq W_t(z_t,-\|U\|_\infty)\geq W_t(z_t,0)-\delta\|U\|_\infty\geq -B-\delta\|U\|_\infty,$$

where we have exploited Assumptions 3.2-3.4 and

$$\|U\|_\infty=\inf\left\{\lambda\in\mathbb{R}^+:|U(z)|\leq\lambda\mathbf{1}\text{ for every }z\in\mathcal{Z}\right\}.$$

By the discounting property (Assumption 3.3),

$$\left|\left(T\hat{U}\right)_t(z)-\left(T\tilde{U}\right)(z)_t\right|\leq\delta\mathbb{E}_t\left|\hat{U}_{t+1}(z)-\tilde{U}_{t+1}(z)\right|\leq\delta\left\|\hat{U}-\tilde{U}\right\|_\infty.$$

The Contraction Mapping Theorem [3, Theorem 3.48] can be applied, and gives a unique fixed point, thus proving our claim.

To complete our proof, we show that the following process in $\mathcal{V}$ is indeed a lower bound:

$$\underline{v}_t=\inf_{\hat{z}\in\mathcal{Z}}U_t(\hat{z})\geq-\|U\|_\infty.$$

To this purpose, observe that

$$\underline{v}_t=\inf_{\hat{z}\in\mathcal{Z}}U_t(\hat{z})\leq U_t(z)=W_t(z_t,U_{t+1}(z)).$$

Moreover, using monotonicity and continuity of the utility aggregator,

$$\underline{v}_t\leq\inf_{\hat{z}\in\mathcal{Z}}W_t(z_t,U_{t+1}(\hat{z}))\leq W_t\left(z_t,\inf_{\hat{z}\in\mathcal{Z}}U_{t+1}(\hat{z})\right)=W_t(z_t,\underline{v}_{t+1}).$$

This establishes our claim for the lower bound. The upper bound exists by a similar argument. □

Proof of Proposition 4.1. Let $J_t^-(\theta_t)=\theta_t\cdot U_t\left(z^0\right)$ and $J_t^+(\theta_t)=\theta_t\cdot\bar{v}_t$, where contract z^0 in $\mathcal{Z}$ is given in Assumption 3.7 and the bounded processes $\bar{v}_t$ is defined in Proposition 3.1. The interval $[J^-,J^+]\subset\mathcal{J}$ is invariant for the Negishi operator and is a complete lattice. Therefore, the first claim is a direct application of Tarski's Fixed Point Theorem [3, Theorem 1.11].

As for the second claim, consider the following recursive decomposition of the actual value of contracts:

$$J_t^{**}(\theta_t)=\sup\theta_t\cdot W_t(z_t,v_{t+1})$$

subject to

$$(z_t, v_{t+1}) \in \mathcal{G}_t$$

and

$$v_{t+1} \in \mathcal{U}^*_{t+1}.$$

It is immediate to verify that $J^{**}_t(\theta_t) \geq J^*_t(\theta_t)$ (because the feasible set $\mathcal{U}^*_{t+1}$ contains utility values that might not be attained by feasible contracts). The action of the Negishi operator consists in modifying the latter constraint for continuation utilities, which becomes

$$v_{t+1} \in \mathcal{U}_{t+1}\left(J^*_{t+1}\right).$$

This enlarges the feasible set because

$$\mathcal{U}^*_t \subset \mathcal{U}_t\left(J^*_t\right).$$

We so obtain that $(TJ^*) \geq J^*$ and, by Tarski's Fixed Point Theorem, the greatest fixed point exists satisfying $\bar{J} \geq J^*$. □

Proof of Proposition 4.2. Consider the set of utility values $\bar{\mathcal{U}}_t$ containing all v_t in $\mathcal{V}_t$ such that, for some feasible (z_t, v_{t+1}) in $\mathcal{G}_t$,

$$\underline{v}_t \leq v_t \leq W_t(z_t, v_{t+1})$$

and

$$v_{t+1} \in \mathcal{U}_{t+1}\left(\bar{J}_{t+1}\right).$$

It is clear that $\bar{\mathcal{U}}_t \subset \mathcal{U}_t\left(\bar{J}_t\right)$. We thus show that $\mathcal{U}_t\left(\bar{J}_t\right) \subset \bar{\mathcal{U}}_t$.

Fix a contingency and assume that v^*_t lies in $\mathcal{U}_t\left(\bar{J}_t\right)$ but not in $\bar{\mathcal{U}}_t$. Notice that the latter is closed and convex by Assumption 4.1. Therefore, we can strongly separate $\bar{\mathcal{U}}_t$ from $\{v_t \in \mathcal{V}_t : v_t \geq v^*_t\}$. By the Strong Separation Theorem [3, Theorem 5.79], there exists θ_t in Θ such that

$$\sup_{v_t \in \bar{\mathcal{U}}_t} \theta_t \cdot v_t < \theta_t \cdot v^*_t.$$

This, however, reveals the existence of welfare weights θ_t in Θ such that

$$\bar{J}_t(\theta_t) < \theta_t \cdot v^*_t,$$

contradicting the fact that v^*_t lies in $\mathcal{U}_t\left(\bar{J}_t\right)$. Hence, our claim is established.

We now prove that, given welfare weights θ_0 in Θ, $J^*_0(\theta_0) = \bar{J}_0(\theta_0)$. Clearly, there exists v_0 in $\mathcal{U}_0\left(\bar{J}_0\right)$ such that $\bar{J}_0(\theta_0) = \theta_0 \cdot v_0$. By induction, for every v_t in $\mathcal{U}_t\left(\bar{J}_t\right)$, there exists (z_t, v_{t+1}) in $\mathcal{G}_t$ such that $v_t \leq W_t(z_t, v_{t+1})$ and v_{t+1} lies in $\mathcal{U}_{t+1}\left(\bar{J}_{t+1}\right)$. It is here that we exploit the coincidence established in the previous step, which is why convexity of the program is required. Now notice that, by Assumption 3.8, at no loss of generality, the plan recursively constructed satisfies

$$v^i_t = W^i_t\left(z_t, v^i_{t+1}\right).$$

As the utility aggregator uniquely identifies an intertemporal utility function, we also have $v_t^i = U_t^i(z)$. This establishes that, for some feasible contract z in $\mathcal{Z}$,

$$J_0^*(\theta_0) \leq \bar{J}_0(\theta_0) = \theta_0 \cdot v_0 = \theta_0 \cdot U_0(z) \leq J_0^*(\theta_0),$$

thus proving our claim. □

Proof of Proposition 4.3. Arguing as in the proof of Proposition 4.2, we obtain a contract z in $\mathcal{Z}$ and a process v in $\mathcal{V}$ such that

$$\underline{v}_t \leq v_t \leq W_t(z_t, v_{t+1}) \text{ and } (z_t, v_{t+1}) \in \mathcal{G}_t.$$

Also notice that

$$\theta_t \cdot W_t(z_t, v_{t+1}) \leq \bar{J}_t(\theta_t) \text{ only if } W_t(z_t, v_{t+1}) \leq \bar{v}_t.$$

We can thus construct an operator $T_z : [\underline{v}, \bar{v}] \to [\underline{v}, \bar{v}]$ as

$$(T_z \tilde{v})_t = W_t(z_t, \tilde{v}_{t+1}).$$

Arguing as in the proof of Proposition 3.1, this operator is a contraction, and admits the only fixed point $U(z)$ in $[\underline{v}, \bar{v}] \subset \mathcal{V}$. As $v \leq (T_z v)$, and the operator is monotone, we conclude that $U(z) \geq v$. Invoking Assumption 4.2, we establish that contract z in $\mathcal{Z}$ is feasible, and this proves our claim. □

Proof of Proposition 4.4. We adapt traditional arguments due to Krasnosel'skiĭ [20]. Consider the action of the Negishi operator on interval $[J^-, J^+] \subset \mathcal{J}$, where the interval is defined in the proof of Proposition 4.1. Also, consider the greatest μ in $[0,1] \subset \mathbb{R}^+$ such that

$$(1-\mu) J_t^-(\theta_t) + \mu \bar{J}_t(\theta_t) \leq \underline{J}_t(\theta_t).$$

Monotonicity yields

$$\left(T\left((1-\mu)J^- + \mu\bar{J}\right)\right)_t(\theta_t) \leq \underline{J}_t(\theta_t).$$

The convexity of the program implies

$$(1-\mu)\left(TJ^-\right)_t(\theta_t) + \mu \bar{J}_t(\theta_t) \leq \underline{J}_t(\theta_t).$$

This is true because, as it can be verified by direct inspection,

$$\mu \mathcal{U}_t\left(J_t^-\right) + (1-\mu)\mathcal{U}_t\left(\bar{J}_t\right) \subset \mathcal{U}_t\left(\mu J_t^- + (1-\mu)\bar{J}_t\right).$$

By the interiority hypothesis (Assumption 4.3),

$$J_t^-(\theta_t) + \epsilon \leq \left(TJ^-\right)_t(\theta_t),$$

so delivering

$$(1-\mu) J_t^-(\theta_t) + \mu \bar{J}_t(\theta_t) + (1-\mu)\epsilon \leq \underline{J}_t(\theta_t).$$

This can only be consistent with the definition of μ in $[0,1]$ if $\mu = 1$. Thus, $\bar{J} = \underline{J}$, proving our statement. □

Proof of Proposition 5.1. Consider any other feasible contract z in $\mathcal{Z}$ coinciding with contract z^* in $\mathcal{Z}$ at all but finitely many periods t in $\mathbb{T}$. We can assume that, at some large t in $\mathbb{T}$, $\theta_{t+1} \cdot U_{t+1}(z) \leq \theta_{t+1} \cdot U_{t+1}(z^*)$. By the first-order conditions, we conclude that $\theta_t \cdot U_t(z) \leq \theta_t \cdot U_t(z^*)$. Recursively, this implies that

$$\theta_0 \cdot U_0(z) \leq \theta_0 \cdot U_0(z^*) .$$

As θ_0 in Θ is strictly positive, this shows that contract z^* in $\mathcal{Z}$ is short-term efficient. We now turn to the more convoluted argument for necessity.

By *ex post* efficiency (Proposition 4.5), at every t in $\mathbb{T}$, we have contingent welfare weights θ_t in Θ_t such that $J_t^*(\theta_t) = \theta_t \cdot U_t(z^*)$. We determine this contingent process recursively and we prove that it satisfies first-order conditions. To this end, observe that, by the Negishi characterization,

$$\theta_t \cdot U_t(z^*) = \max_{(z_t, v_{t+1}) \in \mathcal{G}_t} \theta_t \cdot W_t(z_t, v_{t+1})$$

subject to

$$v_{t+1} \in \mathcal{U}_{t+1}\left(J_{t+1}^*\right).$$

We need to argue that the latter constraint can be innocuously relaxed for an appropriate choice of contingent welfare weights θ_{t+1} in Θ_{t+1}. Consider the set

$$\mathcal{W}_{t+1} = \{w_{t+1} \in V_{t+1} : \theta_t \cdot W_t(z_t, w_{t+1}) > J_t^*(\theta_t) \text{ for some } (z_t, w_{t+1}) \in \mathcal{G}_t\} .$$

Under Assumption 4.1 this set is convex. By the Separating Theorem, there exists θ_{t+1} in $\mathcal{R}_{t+1}^I$ such that, for every v_{t+1} in $\mathcal{U}_{t+1}\left(J_{t+1}^*\right)$ and every w_{t+1} in $\mathcal{W}_{t+1}$,

$$\theta_{t+1} \cdot v_{t+1} \leq \theta_{t+1} \cdot w_{t+1}.$$

Here, as in the rest of the paper, $\mathcal{R}_t$ denotes the space of $\mathcal{F}_t$-measurable random variables with values in $\mathbb{R}$. By Assumption 4.2, we can assume that θ_{t+1} lies in Θ_{t+1}. As $U_{t+1}(z^*)$ belongs to the closure of $\mathcal{W}_{t+1}$, we obtain

$$\theta_{t+1} \cdot v_{t+1} \leq \theta_{t+1} \cdot U_{t+1}(z^*) ,$$

Suppose there exists (z_t, v_{t+1}) in $\mathcal{G}_t$, subject to these relaxed constraints on continuation utility values, such that $\theta_t \cdot W_t(z_t, v_{t+1}) > J_t^*(\theta_t)$. Setting $v^0 = U\left(z^0\right)$, by convexity, for all sufficiently large λ in $[0, 1] \subset \mathbb{R}^+$,

$$\left(\lambda z_t + (1-\lambda) z_t^0, \lambda v_{t+1} + (1-\lambda) v_{t+1}^0\right) \in \mathcal{G}_t.$$

Furthermore, by continuity,

$$\theta_t \cdot W_t\left(\lambda z_t + (1-\lambda) z_t^0, \lambda v_{t+1}, (1-\lambda) v_{t+1}^0\right) > J_t^*(\theta_t) .$$

By the previous separation argument, we thus conclude that

$$\theta_{t+1} \cdot v_{t+1} = \theta_{t+1} \cdot U_{t+1}(z^*) = \lambda \theta_{t+1} \cdot v_{t+1} + (1-\lambda) \theta_{t+1} \cdot v_{t+1}^0,$$

which in turn implies $\theta_{t+1} \cdot U_{t+1}\left(z^0\right) = \theta_{t+1} \cdot U_{t+1}\left(z^*\right)$, thus revealing a contradiction and establishing our claim. □

Proof of Proposition 4.6. By an adaptation of Proposition 3.1, there exists a unique sunspot-extended utility $\hat{U}^i : \hat{\mathcal{Z}} \to \mathcal{V}^i$. We so consider the sunspot-augmented value of contracts,

$$\hat{J}_t^*\left(\theta_t\right) = \sup_{v_t \in \hat{\mathcal{U}}_t^*} \theta_t \cdot v_t,$$

where $\hat{\mathcal{U}}_t^*$ is the space of utility values that are feasible in the sunspot-augmented economy beginning from contingencies at t in $\mathbb{T}$. As sunspot uncertainty does not affect fundamentals, the sunspot-extended value $\hat{J}^*$ is an element of the sunspot-free space $\mathcal{J}$. We show that $\left(T\hat{J}^*\right) \geq \hat{J}^*$, so proving that $\hat{J}^* \leq \bar{J}$.

As in the proof of Proposition 4.1, consider the following recursive decomposition of the actual (sunspot-augmented) value of contracts:

$$\hat{J}_t^{**}\left(\theta_t\right) = \sup \theta_t \cdot \hat{W}_t\left(z_t, \hat{v}_{t+1}\right)$$

subject to

$$\left(z_t, \hat{v}_{t+1}\right) \in \hat{\mathcal{G}}_t$$

and

$$\hat{v}_{t+1} \in \hat{\mathcal{U}}_{t+1}^*.$$

Exploiting the sunspot-invariance properties of fundamentals, we obtain

$$\hat{J}_t^{**}\left(\theta_t\right) = \sup \theta_t \cdot W_t\left(z_t, v_{t+1}\right)$$

subject to

$$\left(z_t, v_{t+1}\right) \in \mathcal{G}_t$$

and

$$v_{t+1} \in \mathbb{E}\left(\hat{\mathcal{U}}_{t+1}^* | \mathcal{F}_{t+1} \otimes \mathcal{E}_t\right).$$

It is immediate to verify that $\hat{J}_t^{**}\left(\theta_t\right) \geq \hat{J}_t^*\left(\theta_t\right)$. The action of the Negishi operator consists in modifying the latter constraint for continuation utilities, which becomes

$$v_{t+1} \in \mathcal{U}_{t+1}\left(\hat{J}_{t+1}^*\right).$$

This enlarges the feasible set because

$$\mathbb{E}\left(\hat{\mathcal{U}}_{t+1}^* | \mathcal{F}_{t+1} \otimes \mathcal{E}_t\right) \subset \mathcal{U}_{t+1}\left(\hat{J}_{t+1}^*\right).$$

To prove this claim, consider any $\hat{v}_{t+1}$ in $\hat{\mathcal{U}}_{t+1}^*$. We have

$$\hat{J}_{t+1}^*\left(\theta_{t+1}\right) \geq \theta_{t+1} \cdot \hat{v}_{t+1} \text{ only if } \hat{J}_{t+1}^*\left(\theta_{t+1}\right) \geq \theta_{t+1} \cdot \mathbb{E}\left(\hat{v}_{t+1} | \mathcal{F}_{t+1} \otimes \mathcal{E}_t\right),$$

where we use the fact that $\hat{J}^*$ in $\mathcal{J}$ is insensitive to sunspot uncertainty. This show the inclusion. Hence, $\left(T\hat{J}^*\right) \geq \hat{J}^{**} \geq \hat{J}^*$, which proves our claim.

To establish coincidence with the greatest fixed point of the Negishi operator, we argue as in the proof of Proposition 4.2. Due to the lack of convexity, however, we can only verify that

$$\mathcal{U}_t\left(\bar{J}_t\right) \subset \text{convex hull}\left(\bar{\mathcal{U}}_t\right).$$

By Carathéodory Convexity Theorem [3, Theorem 5.32], any v_t in $\mathcal{U}_t\left(\bar{J}_t\right)$ can be expressed as the convex combination of finitely many elements of $\bar{\mathcal{U}}_t$. This permits to construct a sunspot-contingent feasible contract $\hat{z}$ in $\hat{\mathcal{Z}}$, measurable with respect to a finite partition of $\hat{\mathcal{F}}_t$ at every t in $\mathbb{T}$, achieving the value corresponding to the greatest fixed point of the Negishi operator, which completes our proof. □

Proof of Proposition 6.1. This is the traditional duality argument, complemented with a fixed point theorem. Consider a plan $\left(z_t^*, v_{t+1}^*\right)$ in $\mathcal{G}_t$ such that

$$J_t\left(\theta_t\right) = \theta_t \cdot W_t\left(z_t^*, v_{t+1}^*\right),$$

where v_{t+1}^* lies in $\mathcal{U}_{t+1}\left(J_{t+1}\right)$. We have that

$$\begin{aligned}
\left(\hat{T}J\right)_t\left(\theta_t\right) &= \inf\sup \theta_t \cdot W_t\left(z_t, v_{t+1}\right) - \mathbb{E}_t\lambda_{t+1}\left(\theta_{t+1}\cdot v_{t+1} - J_{t+1}\left(\theta_{t+1}\right)\right)\\
&\geq \inf \theta_t \cdot W_t\left(z_t^*, v_{t+1}^*\right) - \mathbb{E}_t\lambda_{t+1}\left(\theta_{t+1}\cdot v_{t+1}^* - J_{t+1}\left(\theta_{t+1}\right)\right)\\
&\geq \theta_t \cdot W_t\left(z_t^*, v_{t+1}^*\right)\\
&= J_t\left(\theta_t\right).
\end{aligned}$$

In addition, Assumption 6.1 guarantees that $\left(\hat{T}J^+\right) \leq J^+$, where J^+ in $\mathcal{J}$ is given in the proof of Proposition 4.1. By Tarski's Fixed Point Theorem [3, Theorem 1.11], the dual operator admits a fixed point $\hat{J}$ in $\mathcal{J}$ such that $J \leq \hat{J}$, so proving the claim. □

Proof of Proposition 6.2. Let $\tilde{W}_t^i : \mathcal{Z}_t^i \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$ be the concave envelope of the original utility aggregator (see Aliprantis and Border [3, Definition 7.4]), that is,

$$\tilde{W}_t^i\left(\tilde{z}_t^i, \tilde{v}_{t+1}^i\right) = \inf \mathbb{E}_t\xi_{t+1}^i\tilde{v}_{t+1}^i + \psi_t^i$$

subject to

$$W_t^i\left(\tilde{z}_t^i, v_{t+1}^i\right) \leq \mathbb{E}_t\xi_{t+1}^i v_{t+1}^i + \psi_t^i \text{ if } W_t^i\left(\tilde{z}_t^i, v_{t+1}^i\right) \geq \phi_t^i,$$

where the infimum is taken over ξ_{t+1}^i in Ξ_{t+1}^i and ψ_t^i in Ψ_t^i, both interpreted as spaces of random variables. Consider the internal supremum of the dual operator, and suppose the claim is false. It follows that, at some contingency, there exists $\epsilon > 0$ such that, for all feasible plans,

$$\begin{aligned}
\epsilon + \sum_{i\in I}\theta_t^i W_t^i\left(\tilde{z}_t^i, v_{t+1}^i\right) - \mathbb{E}_t\lambda_{t+1}\sum_{i\in I}\theta_{t+1}^i v_{t+1}^i &\leq \\
\sum_{i\in I}\theta_t^i\tilde{W}_t^i\left(\tilde{z}_t^i, \tilde{v}_{t+1}^i\right) - \mathbb{E}_t\lambda_{t+1}\sum_{i\in I}\theta_{t+1}^i\tilde{v}_{t+1}^i,&
\end{aligned}$$

where each v_{t+1}^i in $\mathcal{V}_{t+1}^i$ is chosen, subject to feasibility, given $\tilde{z}_t^i$ in $\mathcal{Z}_t^i$. At no loss of generality, we can assume that θ_t in Θ_t is strictly positive, and so define

$$\xi_{t+1}^i = \frac{\lambda_{t+1}\theta_{t+1}^i}{\theta_t^i}.$$

Suppose that, for some individual,

$$\epsilon + W_t^i\left(\tilde{z}_t^i, v_{t+1}^i\right) - \mathbb{E}_t\xi_{t+1}^i v_{t+1}^i \leq \tilde{W}_t^i\left(\tilde{z}_t^i, \tilde{v}_{t+1}^i\right) - \mathbb{E}_t\xi_{t+1}^i \tilde{v}_{t+1}^i.$$

It follows that

$$W_t^i\left(\tilde{z}_t^i, v_{t+1}^i\right) \leq \mathbb{E}_t\xi_{t+1}^i v_{t+1}^i + \psi_t^i \text{ if } W_t^i\left(\tilde{z}_t, v_{t+1}^i\right) \geq \phi_t^i,$$

where

$$\psi_t^i = \tilde{W}_t^i\left(\tilde{z}_t^i, \tilde{v}_{t+1}^i\right) - \mathbb{E}_t\xi_{t+1}^i \tilde{v}_{t+1}^i - \epsilon.$$

This implies

$$\tilde{W}_t^i\left(\tilde{z}_t^i, \tilde{v}_{t+1}^i\right) \leq \mathbb{E}_t\xi_{t+1}^i \tilde{v}_{t+1}^i + \psi_t^i = \tilde{W}_t^i\left(\tilde{z}_t^i, \tilde{v}_{t+1}^i\right) - \epsilon,$$

a contradiction. It follows that the value can be approximated with the degree of accuracy $\epsilon > 0$, that is, for some feasible plan,

$$\epsilon + \sum_{i \in I} \theta_t^i W_t^i\left(\tilde{z}_t^i, v_{t+1}^i\right) - \mathbb{E}_t\lambda_{t+1}\sum_{i \in I}\theta_{t+1}^i v_{t+1}^i \quad > \quad \sum_{i \in I} \theta_t^i \tilde{W}_t^i\left(\tilde{z}_t^i, \tilde{v}_{t+1}^i\right) - \mathbb{E}_t\lambda_{t+1}\sum_{i \in I}\theta_{t+1}^i \tilde{v}_{t+1}^i,$$

thus contradicting our initial statement and establishing our claim. □

Proof of Proposition 6.3. At no loss of generality, we set $\underline{v}_t^i = f^i\left(\underline{\tilde{v}}_t^i\right)$ and $\bar{v}_t^i = f^i\left(\bar{\tilde{v}}_t^i\right)$. Consider an auxiliary planning program in which each utility aggregator is given by the concave transformation $\tilde{W}_t^i : \mathcal{Z}_t^i \times \mathcal{V}_{t+1}^i \to \mathcal{V}_t^i$ and each reservation value is replaced by $\tilde{\phi}_t^i = {f^i}^{-1}\left(\phi_t^i\right)$. Let $\tilde{J}^*$ in $\tilde{\mathcal{J}}$ be the actual value of contracts in this auxiliary economy. As this economy is convex, by Proposition 4.2, $\tilde{J}^*$ in $\tilde{\mathcal{J}}$ coincides with the greatest fixed point of the Negishi operator $\tilde{T} : \tilde{\mathcal{J}} \to \tilde{\mathcal{J}}$.

Given the greatest fixed point $\bar{J}$ in $\mathcal{J}$ of the original Negishi operator $T : \mathcal{J} \to \mathcal{J}$, consider the inverse monotone transformation of the utility feasible set,

$$\tilde{\mathcal{N}}_t = \left\{\tilde{v}_t \in \mathcal{V}_t : \underline{\tilde{v}}_t^i \leq \tilde{v}_t^i \leq {f^i}^{-1}\left(v_t^i\right) \text{ for some } v_t \in \mathcal{U}_t\left(\bar{J}_t\right)\right\},$$

and let $\tilde{J}$ in $\tilde{\mathcal{J}}$ be its support map. As convexity might fail, we can only establish that $\tilde{\mathcal{N}}_t \subset \tilde{\mathcal{U}}_t\left(\tilde{J}_t\right)$, which in turn implies that

$$\left(\tilde{T}\tilde{J}\right)_t(\theta_t) \geq \sup_{(z_t, \tilde{v}_{t+1}) \in \tilde{\mathcal{G}}_t} \theta_t \cdot \tilde{W}_t\left(z_t, \tilde{v}_{t+1}\right)$$

subject to

$$\tilde{v}_{t+1} \in \tilde{\mathcal{N}}_{t+1}.$$

Under the identification $v_t^i = f^i\left(\tilde{v}_t^i\right)$, notice that

$$(z_t, v_{t+1}) \in \mathcal{G}_t \text{ if and only if } (z_t, \tilde{v}_{t+1}) \in \tilde{\mathcal{G}}_t.$$

We conclude that the feasible set in the above recursive program is basically $\tilde{\mathcal{N}}_t$, so that

$$\left(\tilde{T}\tilde{J}\right)_t(\theta_t) \geq \sup_{\tilde{v}_t \in \tilde{\mathcal{N}}_t} \theta_t \cdot \tilde{v}_t = \tilde{J}_t(\theta_t).$$

Hence, by Tarski's Fixed Point Theorem [3, Theorem 1.11], the greatest fixed point $\tilde{J}^*$ in $\tilde{\mathcal{J}}$ of $\tilde{T} : \tilde{\mathcal{J}} \to \tilde{\mathcal{J}}$ satisfies $\tilde{J}^* \geq \tilde{J}$.

For the sake of contraction, suppose that $\bar{J}_t(\theta_t) > J_t^*(\theta_t)$ for some welfare weights θ_t in Θ. It follows that there exists v_t in $\mathcal{U}_t\left(\bar{J}_t\right)$ that is not in $\mathcal{U}_t^*$ and, consequently, by monotone transformations of individual utility values, there exists $\tilde{v}_t$ in $\tilde{\mathcal{N}}_t$ that is not in $\tilde{\mathcal{U}}_t^*$. Therefore, for some welfare weights θ_t in Θ, $\left(T\tilde{J}\right)_t(\theta_t) > \tilde{J}_t^*(\theta_t)$, a contradiction. □

Proof of Proposition 7.1. We argue exactly as in the proof of Proposition 4.1. Let $J_t^-(\theta_t) = \Phi\left(\theta_t, U_t\left(z^0\right)\right)$ and $J_t^+(\theta_t) = \Phi(\theta_t, \bar{v}_t)$, where contract z^0 in $\mathcal{Z}$ is given in Assumption 3.7 and the bounded processes $\bar{v}_t$ is defined in Proposition 3.1. The interval $[J^-, J^+] \subset \mathcal{J}$ is invariant for the maxmin-Negishi operator and is a complete lattice. Therefore, the first claim is a direct application of Tarski's Fixed Point Theorem [3, Theorem 1.11].

As for the second claim, consider the following recursive decomposition of the true value of contracts:

$$J_t^{**}(\theta_t) = \sup_{(z_t, v_{t+1}) \in \mathcal{G}_t} \Phi(\theta_t, W_t(z_t, v_{t+1}))$$

subject to

$$v_{t+1} \in \mathcal{U}_{t+1}^*,$$

where $\mathcal{U}_t^*$ denotes the utility possibilities set, that is, the set of utility values attainable by means of contracts which are feasible beginning from period t in $\mathbb{T}$. We so show that (the closure of) $\mathcal{U}_t^*$ coincides with (the closure of) $\mathcal{U}_t(J_t^*)$. This delivers $(TJ^*) = J^{**} \geq J^*$ and, thus, $\bar{J} \geq J^*$.

It is clear that $\mathcal{U}_t^* \subset \mathcal{U}_t(J_t^*)$, because J^* in $\mathcal{J}$ gives the maximum maxmin-value over feasible contracts. To the purpose of contradiction, at some contingency, assume that $\hat{v}_t$ lies in $\mathcal{U}_t(J_t^*)$, whereas it is not in the closure of $\mathcal{U}_t^*$. Choose $\hat{\lambda}$ in $\mathbb{R}^+$ such that $\hat{\lambda}\hat{\theta}_t = (\hat{v}_t - \underline{v}_t)$ for some welfare weights $\hat{\theta}_t$ in Θ and, at no loss of generality, suppose that $\hat{\lambda} = 1$. As $\hat{v}_t$ is not in the closure of $\mathcal{U}_t^*$, there exists a sufficiently small ϵ in $\mathbb{R}^{++}$ such that v_t is not in $\mathcal{U}_t^*$ whenever $(1-\epsilon)\hat{v}_t + \epsilon\underline{v}_t \leq v_t$. Therefore,

$$J_t^*\left(\hat{\theta}_t\right) = \sup_{v_t \in \mathcal{U}_t^*} \Phi\left(\hat{\theta}_t, v_t\right) \leq 1 - \epsilon < \Phi\left(\hat{\theta}_t, \hat{v}_t\right) \leq J_t^*\left(\hat{\theta}_t\right),$$

thus revealing a contradiction.

Arguing as in the proof of Proposition 4.2, a similar argument also shows that (the closure of) $\bar{\mathcal{U}}_t$ coincides with (the closure of) $\mathcal{U}_t\left(\bar{J}_t\right)$, and we can proceed as in that proof to establish the coincidence $J^* = \bar{J}$. □

Proof of Proposition 7.2. At no loss of generality, assume that $\underline{v} = 0$. In the maximin-Negishi program, an optimal policy correspondence is described as $\gamma_t : \Theta_t \twoheadrightarrow \mathcal{Z}_t \times \Theta_{t+1}$.

Indeed, an optimal plan is of the form (z_t, v_{t+1}) in $\mathcal{G}_t$ and, under Assumption 4.2, v_{t+1} in $\mathcal{V}_{t+1}$ achieves the maxmin social value for welfare weights θ_{t+1} in Θ_{t+1} given by

$$\theta_{t+1} = \frac{v_{t+1}}{\sum_{i\in I} v_{t+1}^i}.$$

Hence, the continuation utility values can be identified with those welfare weights θ_{t+1} in Θ_{t+1}. Under Assumptions 4.1-4.2, the correspondence $\gamma_t : \Theta_t \twoheadrightarrow \mathcal{Z}_t \times \Theta_{t+1}$ is upper semicontinuous. It is also convex valued, because the convex combination of continuation utility values is also optimal. Indeed, supposing v_{t+1}^0 and v_{t+1}^1 in $\mathcal{V}_{t+1}$ are both optimal, for all α_0 and α_1 in $\mathbb{R}^{++}$, we have that the convex combination is also optimal, where

$$v_{t+1} = \frac{\alpha_0}{\alpha_0+\alpha_1} v_{t+1}^0 + \frac{\alpha_1}{\alpha_0+\alpha_1} v_{t+1}^1,$$

Considering weights

$$\alpha_0 = (1-\lambda)\frac{1}{\sum_{i\in I} v_{t+1}^{i,0}} \text{ and } \alpha_1 = \lambda \frac{1}{\sum_{i\in I} v_{t+1}^{i,1}},$$

we obtain

$$\theta_{t+1} = (\alpha_0+\alpha_1)\, v_{t+1} = (1-\lambda)\frac{v_{t+1}^0}{\sum_{i\in I} v_{t+1}^{i,0}} + \lambda \frac{v_{t+1}^1}{\sum_{i\in I} v_{t+1}^{i,1}} = (1-\lambda)\,\theta_{t+1}^0 + \lambda\theta_{t+1}^1.$$

We conclude that efficient contracts are governed by a closed Markov correspondence $\Phi : S\times\Theta \twoheadrightarrow \Delta(S\times\Theta)$ with nonempty convex values. To prove existence of an ergodic measure, we apply Aliprantis and Border [3, Theorem 19.31]. □

Appendix B. History dependence

B.1. **Fundamentals.** We describe an economy in which a principal insures a risk-averse agent experiencing privately observed preference shocks. The unobservable preference shock s in the finite space S is governed by Markov transition $\pi : S \to \Delta(S)$. Consumption z in Z, a transfer from the principal to the agent, is restricted to a compact interval $[0,\eta] \subset \mathbb{R}^+$. Per-period utility of the agent is $u : Z \times S \to \mathbb{R}^+$, and satisfies conventional assumptions. The cost of the principal is $c : Z \to \mathbb{R}^-$, and it is also subject to canonical assumptions. To describe the recursive contract, we adopt a more traditional notation.

Let $\mathcal{S}$ be the space of all partial histories of shocks and, given history s^t in $\mathcal{S}$, let $\mathcal{S}(s^t)$ be the space of all continuation histories (beginning from the next period). Given a contingent plan for consumption, the overall utility of the agent is

$$U(z)\left(s^t, \hat{s}_t\right) = \sum_{s^{t+j}\in\mathcal{S}(s^t)} \delta^j \pi\left(s^{t+j}|\hat{s}_t\right) u\left(z\left(s^{t+j}\right), s_{t+j}\right).$$

We assume that type declaration is truthful in all continuations, whereas the agent has initially declared type s_t in S when in state $\hat{s}_t$ in S. The principal utility (*i.e.*, the negative of the cost) is

$$U^0(z)\left(s^t, \hat{s}_t\right) = -\sum_{s^{t+j}\in\mathcal{S}(s^t)} \delta^j \pi\left(s^{t+j}|\hat{s}_t\right) c\left(z\left(s^{t+j}\right)\right).$$

Finally, we impose the incentive compatibility constraint, enforcing truthful revelation of private information. This takes the form

$$\begin{aligned} u\left(z\left(s^{t+1}\right), s_{t+1}\right) + \delta U\left(z\right)\left(s^{t+1}, s_{t+1}\right) \quad &\geq \\ u\left(z\left(s^{t}, \hat{s}_{t+1}\right), s_{t+1}\right) + \delta U\left(z\right)\left(\left(s^{t}, \hat{s}_{t+1}\right), s_{t+1}\right). \end{aligned}$$

It is a well-known property that preventing a single misreport of type is sufficient to implement truthful revelation over the entire infinite horizon.

B.2. **Efficiency.** The classical formulation features cost-minimization, subject to incentive compatibility, given a sustainable utility level for the truthful agent and for any untruthful agent. Though the agent reports the true type, an untruthful version of the agent serves as a counterfactual. We argue that efficient contracts can be equivalently represented as efficient utility profiles on the utility possibilities frontier, so setting the stage for the application of the Negishi method.

Fix an initial state s_0 in S, and assume initial truthful revelation, that is, $s_0 = \hat{s}_0$. A contract z in $\mathcal{Z}$ is feasible if it satisfies incentive compatibility at all histories s^{t+1} in $\mathcal{S}\left(s^0\right)$. A feasible contract z in $\mathcal{Z}$ is *efficient* it there exists no other feasible contract $\hat{z}$ in $\mathcal{Z}$, such that

$$U_0\left(\hat{z}\right)\left(s^0, s_0\right) \geq U_0\left(z\right)\left(s^0, s_0\right)$$

and, for every $\hat{s}_0$ in S,

$$U\left(\hat{z}\right)\left(s^0, \hat{s}_0\right) \geq U\left(z\right)\left(s^0, \hat{s}_0\right),$$

with at least one strict inequality.

Claim B.1 (Efficiency)**.** *A feasible contract z in $\mathcal{Z}$ is efficient only if it is cost-minimizing subject to incentive compatibility at every history s^{t+1} in $\mathcal{S}\left(s^0\right)$ and subject to the promise-keeping constraints, for every $\hat{s}_0$ in S,*

$$U\left(\hat{z}\right)\left(s^0, \hat{s}_0\right) \geq U\left(z\right)\left(s^0, \hat{s}_0\right).$$

Proof. Otherwise, for some feasible contract $\hat{z}$ in $\mathcal{Z}$, $U_0\left(\hat{z}\right)\left(s^0, s_0\right) > U_0\left(z\right)\left(s^0, s_0\right)$, so violating efficiency. □

Endowed with this simple characterization, we can develop the application of the Negishi method for the determination of efficient contracts. The advantage upon the more traditional approach is that the state space for the recursive program is exogenously given: it consists of (normalized) welfare weights, one for the principal, one for the truthful agent and one for each counterfactual untruthful agent.

B.3. **Recursive decomposition.** Let $v\left(s, \hat{s}\right)$ in $\mathbb{R}^{+}$ be the overall utility of an agent of type $\hat{s}$ in S having declared type s in S. The utility of the agent satisfies the recursive condition

$$\text{(U)} \qquad v\left(s, \hat{s}\right) = \sum_{s' \in S} \pi\left(s' | \hat{s}\right)\left(u\left(z\left(s'\right), s'\right) + \delta v\left(s', s'\right)\right).$$

Similarly, the utility of the principal satisfies the recursive condition

(P) $$v_0(s,\hat{s}) = \sum_{s' \in S} \pi(s'|\hat{s})\left(-c(z(s')) + \delta v_0(s',s')\right).$$

Finally, the incentive compatibility constraint is

(IC) $$u(z(s'),s') + \delta v(s',s') \geq u(z(\hat{s}'),s') + \delta v(\hat{s}',s').$$

Let Θ be the simplex in $\mathbb{R} \times \mathbb{R}^S$. Welfare weights θ in Θ refer to the principal, θ_0, and to each agent conditional on (possible unfaithful) type declaration $\hat{s}$ in S, $\theta(\hat{s})$. Given a truthful state s in S, the objective of the Negishi planner is to maximize the weighted surplus,

$$J(\theta)(s) = \theta_0 v_0(s,s) + \sum_{\hat{s} \in S} \theta(\hat{s},s)\, v(\hat{s},s).$$

Constraints are given by (U), (P) and (IC). Continuation values are chosen subject to the consistency constraint, for every state s' in S,

$$\sup_{\theta' \in \Theta} \theta_0' v_0(s',s') + \sum_{\hat{s}' \in S} \theta'(\hat{s}',s')\, v(\hat{s}',s') - J(\theta')(s') \leq 0.$$

This ensures that values are in the convex envelope of the utility possibilities frontier.