arXiv:2311.14565v1 [math.AP] 24 Nov 2023

Multi-scale analysis of minimizers for a second order regularization of the Perona-Malik functional

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Abstract

We continue the asymptotic analysis of minimizers for the singularly perturbed Perona-Malik functional in dimension one that we started in a previous paper, in which we had shown that the blow-up at a suitable scale of these minimizers converge to a staircase-looking piecewise constant function.

We develop our analysis by considering blow-up at finer scales in both the horizontal and vertical regions of these staircases. In the vertical regime, we show that the transition between consecutive steps resembles a cubic polynomial. In the horizontal regime we show that minimizers are actually flat by providing a quantitative uniform estimate on their derivatives.

Mathematics Subject Classification 2020 (MSC2020): 49J45, 35B25.

Key words: Perona-Malik functional, singular perturbation, higher order regularization, Gamma-convergence, blow-up, multi-scale problem.

1 Introduction

We consider the minimum problem for the one-dimensional functional

$$\mathrm{PMF}(u) := \int_0^1 \log\left(1+u'(x)^2\right)\,dx + \beta\int_0^1 (u(x)-f(x))^2\,dx, \tag{1.1}$$

where $\beta>0$ is a real number, and $f\in L^2((0,1))$ is a given function that we call *forcing term*. The principal part of (1.1) is the functional

$$\mathrm{PM}(u) := \int_0^1 \log\left(1+u'(x)^2\right)\,dx, \tag{1.2}$$

whose Lagrangian $\phi(p) := \log(1+p^2)$ is not convex, while the second integral in (1.1) is a *fidelity term*, tuned by the parameter β, that penalizes the distance between u and the forcing term f.

We refer to (1.2) as the *Perona-Malik functional*, because its formal gradient-flow is (up to a factor 2) the celebrated forward-backward parabolic equation

$$u_t = \left(\frac{u_x}{1+u_x^2}\right)_x = \frac{1-u_x^2}{(1+u_x^2)^2}\,u_{xx}, \tag{1.3}$$

introduced by P. Perona and J. Malik [19].

Despite the initial value problem associated to this equation is in general ill-posed, due to existence of a backward regime (see [13, 18]), numerical experiments suggest that this diffusion process has good stability properties (see for example [5]). However, at the present there is no rigorous theory that explains such phenomenon, that is usually referred to as the *Perona-Malik paradox* after [17].

Singular perturbation of the Perona-Malik functional Various regularized models have been introduced in order to mitigate the ill-posed nature of (1.3). These approximations can be obtained via convolution [9], space discretization [6, 11, 12, 14, 20], time delay [2], fractional derivatives [16], fourth order regularization [10, 4, 7, 15], addition of a dissipative term (see [8] and the references quoted therein).

In this paper we continue the analysis started in [15] (see also [20] for similar results in the discrete setting), so we consider the functional whose gradient flow is the fourth order regularization of (1.3), namely the functional (see also [10, 1, 5, 4, 3])

$$\mathrm{PMF}_\varepsilon(u) := \int_0^1 \left\{\varepsilon^{10}|\log\varepsilon|^2 u''(x)^2 + \log\left(1+u'(x)^2\right) + \beta(u(x)-f(x))^2\right\}dx, \tag{1.4}$$

where the bizarre form of the ε-dependent coefficient is just aimed at preventing the appearance of decay rates defined in an implicit way in the sequel of the paper. For every choice of the parameters $\varepsilon\in(0,1)$ and $\beta>0$ and of the forcing term $f\in L^2((0,1))$ the model is well-posed, in the sense that the minimum problem for (1.4) admits at least one minimizer of class C^2.

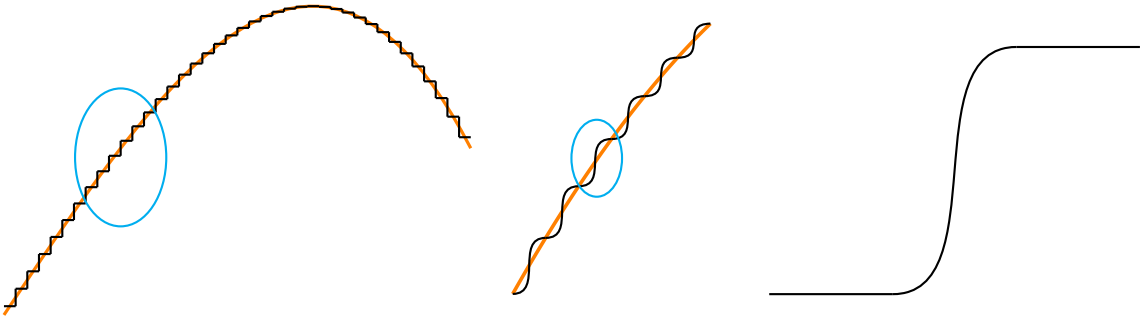


Figure 1: description of the multi-scale problem at three levels of resolution. Left: staircasing effect around the forcing term. Center: zoom of the staircase in a region. Right: cubic transition between two consecutive steps.

First order analysis In [15] we considered any family $\{u_\varepsilon\}$ of minimizers of (1.4) and, for every family of points $\{x_\varepsilon\} \subset (0,1)$ such that $x_\varepsilon \to x_0 \in (0,1)$, we considered the following family of blow-up

$$y \mapsto \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - u_\varepsilon(x_\varepsilon)}{\omega(\varepsilon)}, \tag{1.5}$$

where $\omega(\varepsilon) := \varepsilon|\log\varepsilon|^{1/2}$. We proved that these families converge in a rather strong sense to a piecewise constant function, a sort of staircase with steps whose height and length depend on $f'(x_0)$ (see Theorem A and the central picture in Figure 1).

Main results The aim of this paper is to describe the structure of minimizers at finer scales, in order to reveal the exact structure of the horizontal and vertical parts of each step of the staircase and to justify rigorously the right picture in Figure 1.

We achive this through our two main results.

- In the first main result we consider a family of points $\{x_\varepsilon\}$ such that the functions in (1.5) converge to a staircase with a jump in the origin and we show that, if we rescale them horizontally by a factor ε^2, then they converge in a very strong sense to the unique cubic polynomial which interpolates in a C^1 way the two horizontal steps of the staircase in a neighborhood of the origin whose length depends on the height of the jump, which in turn depends on $f'(x_0)$ (see the first statement in Theorem 2.5).

- In the second main result we show that the derivatives of the functions in (1.5) are of order $\omega(\varepsilon)^2$ in every interval that is contained in an horizontal steps of the limit staircase. This means that minimizers are indeed flat in the horizontal regions.

Overview of the technique The first result is essentially a refinement of the Gamma-convergence result for a suitable rescaling of the functionals (1.4). More precisely, we

show that every recovery sequence for a jumps is necessarily a cubic polynomial, up to a perturbation which is small with respect to the H^2 norm (see Theorem 3.1).

The second result is proved in two steps. First we show that the C^0 norm of the blow-ups in (1.5) is of order $\omega(\varepsilon)^2$ in the horizontal regions by comparing their energies with suitable competitors and estimating the minimal cost of small oscillations. Then we obtain the final C^1 estimate exploiting an auxiliary *convex* variational problem, whose minimizers cannot differ too much from our blow-ups.

Structure of the paper The paper is organized as follows. In section 2 we introduce some notation, we recall the results from [15] that we need in the sequel and we state our main results. Then in section 3 we prove our first result concerning the vertical parts of the steps, while in section 4 we prove the second result, which concerns the horizontal parts. Finally, in section 5 we prove some technical lemmata that we exploited in the previous sections.

2 Statements

In order to state precisely our results, we first need to recall some notation and some results from [15]. To this end, for every $\varepsilon \in (0,1)$ let us set

$$\omega(\varepsilon) := \varepsilon |\log \varepsilon|^{1/2}. \tag{2.6}$$

Let $\beta > 0$ be a real number, let $\Omega \subseteq \mathbb{R}$ be an open set, and let $f \in L^2(\Omega)$ be a function. In order to emphasize the dependence on all the parameters, we write (1.4) in the form

$$\mathbb{PMMF}_\varepsilon(\beta, f, \Omega, u) := \int_\Omega \left\{ \varepsilon^6 \omega(\varepsilon)^4 u''(x)^2 + \log\left(1 + u'(x)^2\right) + \beta (u(x) - f(x))^2 \right\} dx. \tag{2.7}$$

The existence and regularity of minimizers is a standard application of the direct method in the calculus of variation (see [15, Proposition 2.1]).

Let us consider also the following rescaled version of the principal part of (2.7)

$$\mathbb{RPM}_\varepsilon(\Omega, u) := \int_\Omega \left\{ \varepsilon^6 u''(x)^2 + \frac{1}{\omega(\varepsilon)^2} \log\left(1 + u'(x)^2\right) \right\} dx. \tag{2.8}$$

When we add the fidelity term we obtain the functional

$$\mathbb{RPMMF}_\varepsilon(\beta, f, \Omega, u) := \mathbb{RPM}_\varepsilon(\Omega, u) + \beta \int_\Omega (u(x) - f(x))^2 \, dx$$

We recall (see [4, 15] for precise statements) that the Gamma-limit of (2.8) is finite only on pure jump functions, and more precisely it holds that

$$\Gamma - \lim_{\varepsilon \to 0} \mathbb{RPM}_\varepsilon(\Omega, u) = \alpha_0 \cdot \mathbb{J}_{1/2}(\Omega, u)$$

where

$$\alpha_0 := \frac{16}{\sqrt{3}}. \tag{2.9}$$

and

$$\mathbb{J}_{1/2}(\Omega, u) := \sum_{x \in S_u \cap \Omega} |u^+(x) - u^-(x)|^{1/2},$$

for every $u \in BV(\Omega)$ whose derivative is purely atomic, and $\mathbb{J}_{1/2}(\Omega, u) := +\infty$ otherwise.

In order to describe the main result in [15] let us recall the notation used therein for staircase-like functions.

Definition 2.1 (Canonical staircases)**.** Let $S : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$S(x) := 2 \left\lfloor \frac{x+1}{2} \right\rfloor \qquad \forall x \in \mathbb{R},$$

where, for every real number α, the symbol $\lfloor \alpha \rfloor$ denotes the greatest integer less than or equal to α. For every pair (H, V) of real numbers, with $H > 0$, we call *canonical* (H, V)*-staircase* the function $S_{H,V} : \mathbb{R} \to \mathbb{R}$ defined by

$$S_{H,V}(x) := V \cdot S(x/H) \qquad \forall x \in \mathbb{R}. \tag{2.10}$$

Roughly speaking, the graph of $S_{H,V}$ is a staircase with steps of horizontal length $2H$ and vertical height $2V$. The origin is the midpoint of the horizontal part of one of the steps. The staircase degenerates to the null function when $V = 0$, independently of the value of H.

Definition 2.2 (Translations of the canonical staircase)**.** Let (H, V) be a pair of real numbers, with $H > 0$, and let $S_{H,V}$ be the function defined in (2.10). Let $v : \mathbb{R} \to \mathbb{R}$ be a function.

- We say that v is an *oblique translation* of $S_{H,V}$, and we write $v \in \mathrm{Obl}(H, V)$, if there exists a real number $\tau_0 \in [-1, 1]$ such that

$$v(x) = S_{H,V}(x - H\tau_0) + V\tau_0 \qquad \forall x \in \mathbb{R}.$$

- We say that v is a *graph translation of horizontal type* of $S_{H,V}$, and we write $v \in \mathrm{Hor}(H, V)$, if there exists a real number $\tau_0 \in [-1, 1]$ such that

$$v(x) = S_{H,V}(x - H\tau_0) \qquad \forall x \in \mathbb{R}. \tag{2.11}$$

- We say that v is a *graph translation of vertical type* of $S_{H,V}$, and we write $v \in \mathrm{Vert}(H, V)$, if there exists a real number $\tau_0 \in [-1, 1]$ such that

$$v(x) = S_{H,V}(x - H) + V(1 - \tau_0) \qquad \forall x \in \mathbb{R}. \tag{2.12}$$

The main result of [15] is the following.

Theorem A (Blow-up of minimizers at standard resolution)**.** *Let $\omega(\varepsilon)$ be defined by (2.6), and let $\mathbb{PMF}_\varepsilon(\beta, f, (0, 1), u)$ be defined by (2.7), where $\varepsilon \in (0, 1)$ and $\beta > 0$ are two real numbers, and $f \in C^1([0, 1])$ is a given function.*

Let $\{u_\varepsilon\} \subseteq H^2((0,1))$ be a family of functions with

$$u_\varepsilon \in \operatorname{argmin}\left\{\mathbb{PMF}_\varepsilon(\beta, f, (0,1), u) : u \in H^2((0,1))\right\} \qquad \forall \varepsilon \in (0,1),$$

and let $x_\varepsilon \to x_0 \in (0,1)$ be a family of points. Let us consider the canonical (H,V)-staircase with parameters

$$H := \left(\frac{24}{\beta^2 |f'(x_0)|^3}\right)^{1/5}, \qquad V := f'(x_0)H, \tag{2.13}$$

with the agreement that this staircase is identically equal to 0 when $f'(x_0) = 0$.

Then the following statements hold true.

(1) (Compactness of blow-ups). *The family $\{v_\varepsilon(y)\}$ defined by*

$$v_\varepsilon(y) := \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - u_\varepsilon(x_\varepsilon)}{\omega(\varepsilon)}$$

is relatively compact with respect to locally strict convergence, and every limit point is a graph translation of the canonical (H,V)-staircase.

More precisely, for every sequence $\{\varepsilon_n\} \subseteq (0,1)$ with $\varepsilon_n \to 0^+$ there exist an increasing sequence $\{n_k\}$ of positive integers and a function $v_\infty \in \operatorname{Hor}(H,V) \cup \operatorname{Vert}(H,V)$ such that

$$v_{\varepsilon_{n_k}}(y) \rightrightarrows v_\infty(y) \quad \text{in } BV_{\mathrm{loc}}(\mathbb{R}).$$

Moreover, for every bounded interval (a,b) whose endpoints are not jump points of v_∞ it holds that

$$\lim_{n\to+\infty} \mathbb{RPM}_{\varepsilon_{n_k}}((a,b), v_{\varepsilon_{n_k}}) = \alpha_0 \cdot \mathbb{J}_{1/2}((a,b), v_\infty), \tag{2.14}$$

where α_0 is the constant defined in (2.9).

(2) (Realization of all possible graph translations). *Let $v_0 \in \operatorname{Hor}(H,V) \cup \operatorname{Vert}(H,V)$ be any graph translation of the canonical (H,V)-staircase.*

Then there exists a family $\{x'_\varepsilon\} \subseteq (0,1)$ such that

$$\limsup_{\varepsilon\to 0^+} \frac{|x'_\varepsilon - x_\varepsilon|}{\omega(\varepsilon)} \le H,$$

and

$$\frac{u_\varepsilon(x'_\varepsilon + \omega(\varepsilon)y) - u_\varepsilon(x'_\varepsilon)}{\omega(\varepsilon)} \rightrightarrows v_0(y) \quad \text{in } BV_{\mathrm{loc}}(\mathbb{R}).$$

Remark 2.3. In [15], a similar statement is proved also when the family $\{v_\varepsilon\}$ is replaced by the functions

$$y \mapsto \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - f(x_\varepsilon)}{\omega(\varepsilon)},$$

with the difference that in this case one obtains oblique translations of the canonical (H, V)-staircase instead of graph translations.

Here, we do not need this result, but only the fact that

$$\limsup_{\varepsilon\to 0^+} \frac{u_\varepsilon(x_\varepsilon) - f(x_\varepsilon)}{\omega(\varepsilon)} = \limsup_{\varepsilon\to 0^+} \left\{ \frac{u_\varepsilon(x_\varepsilon + \omega(\varepsilon)y) - f(x_\varepsilon)}{\omega(\varepsilon)} - v_\varepsilon(y) \right\} < +\infty.$$

Finally, let us introduce the cubic transitions arising around the jump points of the staircase.

Definition 2.4 (Cubic connection)**.** Let $C : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$C(x) := \begin{cases} -1 & \text{if } x \le -1, \\ \dfrac{3}{2}x - \dfrac{1}{2}x^3 & \text{if } x \in [-1, 1], \\ 1 & \text{if } x \ge 1. \end{cases} \tag{2.15}$$

- For every pair (Λ, V) of positive real numbers we call *canonical* (Λ, V)*-cubic connection* the function $C_{\Lambda,V} : \mathbb{R} \to \mathbb{R}$ defined by

$$C_{\Lambda,V}(x) := VC(x/\Lambda) \qquad \forall x \in \mathbb{R}. \tag{2.16}$$

- For every $\tau_0 \in (-1, 1)$ we call *graph translation* of the canonical (Λ, V)-cubic connection the function $C_{\Lambda,V,\tau_0} : \mathbb{R} \to \mathbb{R}$ defined by

$$C_{\Lambda,V,\tau_0}(x) := C_{\Lambda,V}(x + x_0) - \tau_0 V \qquad \forall x \in \mathbb{R},$$

  where $x_0 \in (-\Lambda, \Lambda)$ is the unique real number such that $C_{\Lambda,V}(x_0) = \tau_0 V$.

In words, the (Λ, V)-cubic connection is the unique polynomial of degree three that interpolates the constants $-V$ and $+V$ in a C^1 way in the interval $[-\Lambda, \Lambda]$. Every graph translation corresponds to taking the graph of $C_{\Lambda,V}(x)$ and moving the origin to a point of the cubic.

The following theorem contains both our main results, concerning higher order blow-ups of minimizers for the functional $\mathbb{PMMF}_\varepsilon(\beta, f, (0, 1), u)$.

Theorem 2.5 (High-resolution blow-ups of minimizers)**.** *Let $\beta > 0$ be a positive number, let $f \in C^1([0, 1])$ be a function, and let $x_0 \in (0, 1)$ be a point with $f'(x_0) \neq 0$. Let $\varepsilon_n \to 0^+$ be a sequence of positive real numbers, and let $\{x_n\} \subseteq (0, 1)$ be a sequence such that $x_n \to x_0 \in (0, 1)$. For every positive integer n, let*

$$u_n \in \operatorname{argmin} \left\{ \mathbb{PMMF}_{\varepsilon_n}(\beta, f, (0, 1), u) : u \in H^2((0, 1)) \right\}, \tag{2.17}$$

where $\mathbb{PMMF}_\varepsilon(\beta, f, (0, 1), u)$ is the family of functionals defined in (2.7). Let

$$H := \left(\frac{24}{\beta^2 |f'(x_0)|^3} \right)^{1/5}, \qquad V := f'(x_0)H,$$

be as in (2.13), and let us set

$$\Lambda := \frac{\sqrt{3}}{2} \cdot \sqrt{2V}. \tag{2.18}$$

Let $\omega(\varepsilon)$ be as in (2.6) and let us assume that the sequence

$$v_n(y) := \frac{u_n(x_n + \omega(\varepsilon_n)y) - u_n(x_n)}{\omega(\varepsilon_n)}$$

of blow-ups at canonical scale converges locally strictly in $BV_{\mathrm{loc}}(\mathbb{R})$ to some v_∞ that, due to Theorem A, belongs to the family $\mathrm{Hor}(H,V) \cup \mathrm{Vert}(H,V)$ of Definition 2.2.

Then the following statements hold true.

(1) (Vertical parts of the steps). *Let us assume that $v_\infty \in \mathrm{Vert}(H,V) \setminus \mathrm{Hor}(H,V)$, and more precisely that $v_\infty(y)$ is given by (2.12) for some $\tau_0 \in (-1,1)$, and let $C_{\Lambda,V,\tau_0}(y)$ be the corresponding translated cubic connection introduced in Definition 2.4. Then it turns out that*

$$v_n(\varepsilon_n^2 y) \to C_{\Lambda,V,\tau_0}(y) \quad \textit{strongly in } H^2_{loc}(\mathbb{R}).$$

(2) (Horizontal parts of the steps). *Let us assume that $v_\infty \in \mathrm{Hor}(H,V) \setminus \mathrm{Vert}(H,V)$, and more precisely that $v_\infty(y)$ is given by (2.11) for some $\tau_0 \in (-1,1)$. Then for every closed interval $[a,b] \subseteq ((\tau_0 - 1)H, (1+\tau_0)H)$ it turns out that*

$$\limsup_{n\to+\infty} \frac{1}{\omega(\varepsilon_n)^2} \cdot \max\big\{|v_n'(y)| : y \in [a,b]\big\} < +\infty. \tag{2.19}$$

As a consequence, for every sequence $\{\alpha_n\} \subset (0,1)$ such that $\alpha_n = o(\omega(\varepsilon_n))$ we deduce that

$$\frac{u_n(x_n + \alpha_n y) - u_n(x_n)}{\alpha_n} \to 0 \quad \textit{uniformly on bounded subsets of } \mathbb{R}.$$

Remark 2.6. We recall that the locally strict convergence of the sequence $\{v_n\}$ to a staircase v_∞ is ensured by Theorem A, at least up to subsequences or up to a slight modifications of the centers $\{x_n\}$.

Therefore, there are only two situations in which higher resolution blow-ups are not characterized by Theorem 2.5: when $v_\infty \in \mathrm{Hor}(H,V) \cap \mathrm{Vert}(H,V)$, namely when $v_\infty(y) = S_{H,V}(y \pm H)$, and when $f'(x_0) = 0$.

In the first case, it can be seen that the behavior of $\{v_n\}$ could be both "horizontal-like" and "vertical-like", because a sequence $\{x_n\}$ such that $v_\infty \in \mathrm{Hor}(H,V) \cap \mathrm{Vert}(H,V)$ can be obtained with a diagonal procedure both starting from sequences generating staircases in $\mathrm{Hor}(H,V) \setminus \mathrm{Vert}(H,V)$ and in $\mathrm{Vert}(H,V) \setminus \mathrm{Hor}(H,V)$. As a consequence, also some intermediate behavior could arise, and we can not exclude that other non trivial blow-ups might exist at a different scale.

The situation in the second case is similar, at least if f' does not vanish identically in a neighborhood of x_0. Indeed, also in this case $\{x_n\}$ can be obtained with a diagonal procedure starting from sequences such that the corresponding blow-ups exhibit different behaviors.

3 Vertical parts (Theorem 2.5, statement (1))

Statement (1) in Theorem 2.5 is a direct consequence of the following result about the rescaled functionals $\mathbb{RPM}_\varepsilon$ defined in (2.8). We point out that the assumption (3.2) follows from 2.14.

Theorem 3.1. *Let $L > 0$ be a positive real number, let $\varepsilon_n \to 0^+$ be a sequence of positive real numbers, and let $\{v_n\} \subseteq H^2((-L, L))$ be a sequence of functions such that $v_n(0) = 0$ for every $n \in \mathbb{N}$. Let us assume that there exist real numbers $V > 0$ and $\tau_0 \in (-1, 1)$ such that*

$$v_n(y) \rightrightarrows V(\operatorname{sign}(y) - \tau_0) \qquad \text{strictly in } BV((-L, L)), \tag{3.1}$$

and

$$\lim_{n\to+\infty} \mathbb{RPM}_{\varepsilon_n}((-L, L), v_n) \le \alpha_0\sqrt{2V}, \tag{3.2}$$

where α_0 is the constant defined in (2.9)

Let us set $w_n(y) := v_n(\varepsilon_n^2 y)$, and let $\Lambda := (\sqrt{3}/2)\sqrt{2V}$ be as in (2.18).

Then $w_n(y)$ converges up to order 2 to the cubic connection $C_{\Lambda,V,\tau_0}(y)$ introduced in Definition 2.4, in the sense that

$$\lim_{n\to+\infty} \max\left\{|w_n(y) - C_{\Lambda,V,\tau_0}(y)| : |y| \le L/\varepsilon_n^2\right\} = 0,$$

and for every bounded interval $(a, b) \subseteq \mathbb{R}$ it turns out that

$$w_n'(y) \to C'_{\Lambda,V,\tau_0}(y) \quad \text{uniformly in } (a, b)$$

and

$$w_n''(y) \to C''_{\Lambda,V,\tau_0}(y) \quad \text{strongly in } L^2((a, b)).$$

Proof. We divide the proof in several steps. First, we introduce some notation, then we show that the transition from $-(1 + \tau_0)V$ and $(1 - \tau_0)V$ occurs in a single interval, and finally we show that in this interval we have some precise estimates that allow us to characterize the limit.

Notation and definitions We observe that $w_n(y)$ is defined for every $|y| < L/\varepsilon_n^2$, and with a variable change in the integral we obtain that it satisfies

$$\mathbb{RPM}_{\varepsilon_n}((-L, L), v_n) = \mathbb{RPMV}_{\varepsilon_n}((-L/\varepsilon_n^2, L/\varepsilon_n^2), w_n), \tag{3.3}$$

where

$$\mathbb{RPMV}_\varepsilon(\Omega, u) := \int_\Omega \left\{u''(x)^2 + \frac{1}{|\log\varepsilon|}\log\left(1 + \frac{u'(x)^2}{\varepsilon^4}\right)\right\} dx.$$

Since strict convergence implies uniform convergence in compact sets that do not contain jump points of the limit, from (3.1) we deduce that for every $a \in (0, L)$ it turns out that

$$\lim_{n\to+\infty} \max\left\{|v_n(y) - V(\operatorname{sign}(y) - \tau_0)| : y \in [-L, L] \setminus (-a, a)\right\} = 0.$$

Since a is arbitrary, with a standard diagonal procedure we can find a sequence $a_n \to 0^+$ such that

$$\lim_{n\to+\infty} \max\big\{|v_n(y) - V(\mathrm{sign}(y) - \tau_0)| : y \in [-L, L] \setminus (-a_n, a_n)\big\} = 0. \tag{3.4}$$

Without loss of generality, we can always assume that $a_n \geq \varepsilon_n$. Therefore, if we set $L_n := a_n/\varepsilon_n^2$, then $L_n \to +\infty$ and (3.4) is equivalent to

$$\lim_{n\to+\infty} \max\big\{|w_n(y) - V(1-\tau_0)| : y \in [L_n, L/\varepsilon_n^2]\big\} = \\ = \lim_{n\to+\infty} \max\big\{|w_n(y) + V(1+\tau_0)| : y \in [-L/\varepsilon_n^2, -L_n]\big\} = 0,$$

so that in particular

$$w_n(-L_n) \to -V(1+\tau_0) \qquad \text{and} \qquad w_n(L_n) \to V(1-\tau_0). \tag{3.5}$$

Now let us choose a sequence $\{M_n\}$ of real numbers such that

$$M_n \to +\infty, \qquad M_n\varepsilon_n^2 \to 0, \qquad L_nM_n\varepsilon_n^2 \to 0. \tag{3.6}$$

We observe that the third condition implies the second one, and a possible choice for this sequence is $M_n := a_n^{-1/2}$. Following [1] (see also [15, Lemma A.2]), for every $n \in \mathbb{N}$ we consider the open set

$$A_n := \left\{ y \in (-L_n, L_n) : |w_n'(y)| > \frac{1}{|\log \varepsilon_n|} \right\},$$

and we write it as a union of intervals of the form

$$A_n = \bigcup_{i\in I_n} (\alpha_{n.i}, \beta_{n,i}),$$

where I_n is a finite or countable set of indices. For every $n \in \mathbb{N}$ and every $i \in I_n$, we observe that $w_n'(y)$ has constant sign in $(\alpha_{n,i}, \beta_{n,i})$, and we set

$$\Delta_{n,i} := |w_n(\beta_{n,i}) - w_n(\alpha_{n,i})| = \int_{\alpha_{n,i}}^{\beta_{n,i}} |w_n'(y)|\, dy,$$

and

$$\widehat{\Delta}_{n,i} := \Delta_{n,i} - \frac{\beta_{n,i} - \alpha_{n,i}}{|\log \varepsilon_n|} = \int_{\alpha_{n,i}}^{\beta_{n,i}} \left(|w_n'(y)| - \frac{1}{|\log \varepsilon_n|} \right) dy \geq 0.$$

Identification of the "big jump" We claim that for every $n \in \mathbb{N}$ there exists an index $i(n) \in I_n$ such that

$$\lim_{n\to+\infty} \Delta_{n,i(n)} = \lim_{n\to+\infty} \int_{-L_n}^{L_n} |w_n'(y)|\, dy = 2V. \tag{3.7}$$

In words, this means that asymptotically the whole total variation of w_n is realized in a single special interval $(\alpha_{n,i(n)}, \beta_{n,i(n)})$. This is the key point of the proof, and requires seven steps.

- In the first step we show that

$$\limsup_{n\to+\infty} |A_n| < +\infty, \tag{3.8}$$

and in particular $|A_n|$ is bounded.

To this end, it is enough to observe that

$$\frac{1}{|\log \varepsilon_n|} \log\left(1 + \frac{1}{|\log \varepsilon_n| \varepsilon_n^4}\right) |A_n| \le \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n),$$

so that from (3.2) and (3.3) we deduce that

$$\begin{aligned} 4 \limsup_{n\to+\infty} |A_n| &\le \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n) \\ &\le \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L, L), v_n) \\ &= \frac{16}{\sqrt{3}} \sqrt{2V}, \end{aligned}$$

which implies (3.8).

As a consequence, up to choosing a slightly larger L_n (between the original L_n and $L_n + 4(2V)^{1/2}$), we can assume that none of the intervals $(\alpha_{n,i}, \beta_{n,i})$ lies at the boundary of $(-L_n, L_n)$, so that $|w_n'(\alpha_{n,i})| = |w_n'(\beta_{n,i})| = 1/|\log \varepsilon_n|$ for every $i \in I_n$, at least if n is large enough.

- In the second step we show that

$$\lim_{n\to+\infty} \int_{A_n^c} |w_n'(y)|\, dy = 0. \tag{3.9}$$

where A_n^c denote the complement set of A_n in $(-L_n, L_n)$.

To this end, we consider the set

$$B_n := \left\{ y \in A_n^c : |w_n'(y)| \le M_n \varepsilon_n^2 \right\},$$

and we observe that

$$\limsup_{n\to+\infty} \frac{\log(1 + M_n^2)}{|\log \varepsilon_n|} |A_n^c \setminus B_n| \le \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n) < +\infty.$$

Therefore we have that

$$\int_{A_n^c} |w_n'(y)|\, dy \le M_n \varepsilon_n^2 \cdot |B_n| + \frac{|A_n^c \setminus B_n|}{|\log \varepsilon_n|} \le 2L_n M_n \varepsilon_n^2 + \frac{|A_n^c \setminus B_n| \log(1 + M_n^2)}{|\log \varepsilon_n| \log(1 + M_n^2)}, \tag{3.10}$$

and the conclusion follows from the first and the third condition in (3.6).

- In the third step we show that

$$\liminf_{n\to+\infty} \sum_{i\in I_n} \Delta_{n,i} \geq 2V. \tag{3.11}$$

To this end, we observe that

$$|w_n(L_n) - w_n(-L_n)| \leq \int_{-L_n}^{L_n} |w_n'(y)|\, dy = \int_{A_n^c} |w_n'(y)|\, dy + \sum_{i\in I_n} \Delta_{n,i},$$

and we conclude by exploiting (3.9) and (3.5).

- In the fourth step we show that

$$\liminf_{n\to+\infty} \sum_{i\in I_n} \widehat{\Delta}_{n,i} \geq 2V. \tag{3.12}$$

Indeed, from the definition we obtain that

$$\begin{aligned}
\sum_{i\in I_n} \widehat{\Delta}_{n,i} &= \sum_{i\in I_n} \left(\Delta_{n,i} - \frac{\beta_{n,i} - \alpha_{n,i}}{|\log \varepsilon_n|} \right) \\
&= \sum_{i\in I_n} \Delta_{n,i} - \frac{1}{|\log \varepsilon_n|} \sum_{i\in I_n} (\beta_{n,i} - \alpha_{n,i}) \\
&= \sum_{i\in I_n} \Delta_{n,i} - \frac{|A_n|}{|\log \varepsilon_n|},
\end{aligned} \tag{3.13}$$

and we conclude by exploiting (3.11) and (3.8).

- In the fifth step we show that

$$\limsup_{n\to+\infty} \sum_{i\in I_n} \left(\widehat{\Delta}_{n,i} \right)^{1/2} \leq \sqrt{2V}. \tag{3.14}$$

To this end, for every $i \in I_n$ we apply Lemma 5.6 in the interval $(\alpha_{n,i}, \beta_{n,i})$ with $M = 1/|\log \varepsilon_n|$, so we obtain that

$$\mathbb{RPMV}_{\varepsilon_n}((\alpha_{n,i}, \beta_{n,i}), w_n) \geq 4\sqrt{\frac{2}{3}} \left(\frac{1}{|\log \varepsilon_n|} \log\left(1 + \frac{1}{\omega(\varepsilon_n)^4} \right) \right)^{3/4} \left(\widehat{\Delta}_{n,i} \right)^{1/2},$$

and therefore

$$\mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n) \geq 4\sqrt{\frac{2}{3}} \left(\frac{1}{|\log \varepsilon_n|} \log\left(1 + \frac{1}{\omega(\varepsilon_n)^4} \right) \right)^{3/4} \sum_{i\in I_n} \left(\widehat{\Delta}_{n,i} \right)^{1/2}.$$

Letting $n \to +\infty$, from (3.2) and (3.3) we conclude that

$$\frac{16}{\sqrt{3}} \sqrt{2V} = \alpha_0 \sqrt{2V} \geq \frac{16}{\sqrt{3}} \limsup_{n\to+\infty} \sum_{i\in I_n} \left(\widehat{\Delta}_{n,i} \right)^{1/2},$$

which is equivalent to (3.14).

- In the sixth step we show that

$$\lim_{n\to+\infty}\sum_{i\in I_n}\Delta_{n,i}=\lim_{n\to+\infty}\sum_{i\in I_n}\widehat{\Delta}_{n,i}=2V \tag{3.15}$$

and

$$\lim_{n\to+\infty}\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}=\sqrt{2V}. \tag{3.16}$$

To this end, we combine (3.12), (3.14), and the subadditivity of the square root, and we obtain that

$$\begin{aligned}\sqrt{2V}&\geq\limsup_{n\to+\infty}\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}\geq\limsup_{n\to+\infty}\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}\\&\geq\liminf_{n\to+\infty}\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}=\left(\liminf_{n\to+\infty}\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}=\sqrt{2V}.\end{aligned}$$

This means that all inequalities are actually equalities, which proves (3.16) and the second equality in (3.15). At this point, the first inequality in (3.15) follows again by letting $n\to+\infty$ in (3.13).

- In the last step we apply Lemma 5.2 to the index set I_n and the function $f(i):=\widehat{\Delta}_{n,i}$. We deduce that

$$0\leq\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}-\max_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}\leq 3\sum_{i\in I_n}\left(\widehat{\Delta}_{n,i}\right)^{1/2}-3\left(\sum_{i\in I_n}\widehat{\Delta}_{n,i}\right)^{1/2}$$

The right-hand side tends to 0 because of (3.16) and the second equality in (3.15). This is enough to establish the existence of $i(n)\in I_n$ such that

$$\lim_{n\to+\infty}\widehat{\Delta}_{n,i(n)}=2V.$$

Now we know that

$$2V=\lim_{n\to+\infty}\sum_{i\in I_n}\Delta_{n,i}\geq\lim_{n\to+\infty}\Delta_{n,i(n)}\geq\lim_{n\to+\infty}\widehat{\Delta}_{n,i(n)}=2V,$$

and therefore all inequalities are actually equalities. This proves the first part of (3.7). The second part follows from the equality

$$\int_{-L_n}^{L_n}|w_n'(y)|\,dy=\Delta_{n,i(n)}+\sum_{i\in I_n\setminus\{i(n)\}}\Delta_{n,i}+\int_{A_n^c}|w_n'(y)|\,dy,$$

since now we know that the last two terms tend to 0.

Uniform estimates in the special interval Let us set for simplicity $\alpha_n := \alpha_{n,i(n)}$ and $\beta_n := \beta_{n,i(n)}$. We claim that

- the sequence $w_n(y)$ tends to $-V(1+\tau_0)$ uniformly in $[-L_n, \alpha_n]$ and to $V(1-\tau_0)$ uniformly in $[\beta_n, L_n]$, in the sense that

$$\lim_{n\to+\infty} \left(\max_{y\in[-L_n,\alpha_n]} |w_n(y) + V(1+\tau_0)| + \max_{y\in[\beta_n,L_n]} |w_n(y) - V(1-\tau_0)| \right) = 0, \quad (3.17)$$

  and in particular

$$w_n(\alpha_n) \to -V(1+\tau_0), \qquad w_n(\beta_n) \to V(1-\tau_0), \quad (3.18)$$

  while we already know that

$$|w_n'(\alpha_n)| = |w_n'(\beta_n)| = \frac{1}{|\log \varepsilon_n|} \to 0. \quad (3.19)$$

- when n is large enough it turns out that

$$-|A_n| < \alpha_n < 0 < \beta_n < |A_n|,$$

  and in particular the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded,

- for every bounded interval $(a, b) \subseteq \mathbb{R}$ it turns out that

$$\{w_n\} \text{ is bounded in } H^2((a,b)), \quad (3.20)$$

  of course with a bound that depends on the interval.

Let us prove these claims. To begin with, from (3.7) we know that

$$\lim_{n\to+\infty} \int_{-L_n}^{\alpha_n} |w_n'(y)|\, dy + \int_{\beta_n}^{L_n} |w_n'(y)|\, dy = 0.$$

Keeping (3.5) into account, this is enough to establish (3.17). In particular, from this uniform convergence we deduce that

$$w_n(y) \le -\frac{V(1+\tau_0)}{2} < 0 \qquad \forall y \in [-L_n, \alpha_n]$$

when n is large enough. Since $w_n(0) = 0$ for every $n \in \mathbb{N}$, we conclude that $\alpha_n < 0$, and also

$$\beta_n = \alpha_n + (\beta_n - \alpha_n) \le \alpha_n + |A_n| < |A_n|.$$

In a symmetric way we obtain that $\beta_n > 0$ and $\alpha_n > -|A_n|$.

Finally, the bound on $\mathbb{RPMMV}_{\varepsilon_n}((-L_n, L_n), w_n)$ yields immediately a uniform bound on the norm of w_n'' in $L^2((-L_n, L_n))$. Together with the pointwise bounds coming from (3.18) and (3.19), this is enough to obtain a uniform bound on w_n in $H^2((a,b))$ for every bounded interval $(a, b) \subseteq \mathbb{R}$.

Passing to the limit We are now ready to prove our convergence results. Since the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded, up to subsequences (not relabeled) we can assume that $\alpha_n \to \alpha_\infty$ and $\beta_n \to \beta_\infty$. Moreover, if we fix an interval $(a,b) \supseteq (\alpha_\infty, \beta_\infty)$, then from (3.20) we can also assume that there exists $w_\infty \in H^2((a,b))$ such that

$$w_n \to w_\infty \quad \text{and} \quad w_n' \to w_\infty' \quad \text{uniformly in } [a,b],$$

and

$$w_n'' \rightharpoonup w_\infty'' \quad \text{weakly in } L^2((a,b)).$$

We claim that w_∞ is the cubic connection. To prove this, we observe that since $w_n(0) = 0$ we have that $w_\infty(0) = 0$, while from (3.18) and (3.19) we obtain that

$$w_\infty(\alpha_\infty) = -V(1+\tau_0), \quad w_\infty(\beta_\infty) = V(1-\tau_0), \quad w_\infty'(\alpha_\infty) = w_\infty'(\beta_\infty) = 0. \tag{3.21}$$

Therefore, from Lemma 5.5 we deduce that

$$\int_{\alpha_\infty}^{\beta_\infty} w_\infty''(y)^2\,dy \geq \frac{12(2V)^2}{(\beta_\infty - \alpha_\infty)^3}.$$

Now we consider the chain of inequalities

$$\begin{aligned}
\frac{16}{\sqrt{3}}\sqrt{2V} &\geq \limsup_{n\to+\infty} \mathbb{RPMV}_{\varepsilon_n}((-L_n, L_n), w_n) \\
&\geq \limsup_{n\to+\infty} \left\{ \int_{\alpha_n}^{\beta_n} \frac{1}{|\log\varepsilon_n|} \log\left(1 + \frac{w_n'(y)^2}{\varepsilon_n^4}\right) dy + \int_a^b w_n''(y)^2\,dy \right\} \\
&\geq \limsup_{n\to+\infty} \left\{ \frac{1}{|\log\varepsilon_n|} \log\left(\frac{1}{\varepsilon_n^4 |\log\varepsilon_n|^2}\right) (\beta_n - \alpha_n) + \int_a^b w_n''(y)^2\,dy \right\} \\
&= 4(\beta_\infty - \alpha_\infty) + \limsup_{n\to+\infty} \int_a^b w_n''(y)^2\,dy \\
&\geq 4(\beta_\infty - \alpha_\infty) + \int_a^b w_\infty''(y)^2\,dy \qquad (3.22) \\
&\geq 4(\beta_\infty - \alpha_\infty) + \int_{\alpha_\infty}^{\beta_\infty} w_\infty''(y)^2\,dy \qquad (3.23) \\
&\geq 4(\beta_\infty - \alpha_\infty) + 12\frac{(2V)^2}{(\beta_\infty - \alpha_\infty)^3} \qquad (3.24) \\
&\geq \frac{16}{\sqrt{3}}\sqrt{2V}. \qquad (3.25)
\end{aligned}$$

Since the first and last term coincide, all inequalities are actually equalities, and each of them gives us some piece of information.

- The equality in (3.25) implies that $\beta_\infty - \alpha_\infty = 2\Lambda$, where Λ is defined by (2.18).
- The equality in (3.24) implies that w_∞ is the unique polynomial of degree three that interpolates the boundary data (3.21). Combining with the previous point and the fact that $w_\infty(0) = 0$, we conclude that w_∞ coincides with C_{Λ,V,τ_0} in the interval $(\alpha_\infty, \beta_\infty) = (-\Lambda - x_0, \Lambda - x_0)$, where x_0 is as in Definition 2.4.

- The equality in (3.23) implies that $w''_\infty(y)$ vanishes outside $(\alpha_\infty, \beta_\infty)$. Since $w'_\infty(\alpha_\infty) = w'_\infty(\beta_\infty) = 0$, this is enough to conclude that w_∞ is constant both in (a, α_∞) and in (β_∞, b).
- The equality in (3.22) implies that actually $w''_n \to w''_\infty$ strongly in $L^2((a,b))$.

Finally, we observe that the previous steps characterize in a unique way the possible limits of subsequences, and this is enough to conclude the convergence of the whole sequence.

□

4 Horizontal parts (Theorem 2.5, statement (2))

Let $[a,b] \subseteq ((-1+\tau_0)H, (1+\tau_0)H)$ be a fixed interval. In the first part of the proof we show that

$$\limsup_{n\to+\infty} \frac{1}{\omega(\varepsilon_n)^2} \max\{|v_n(y)| : y \in [a,b]\} < +\infty, \tag{4.1}$$

and then in the second part we prove (2.19)

4.1 Estimate on functions

Let us set

$$g_n(y) := \frac{f(x_n + \omega(\varepsilon_n)y) - u_n(x_n)}{\omega(\varepsilon_n)} = \frac{f(x_n + \omega(\varepsilon_n)y) - f(x_n)}{\omega(\varepsilon_n)} - \frac{u_n(x_n) - f(x_n)}{\omega(\varepsilon_n)}.$$

We observe that the functions g_n are uniformly bounded on bounded sets because

$$\frac{f(x_n + \omega(\varepsilon_n)y) - f(x_n)}{\omega(\varepsilon_n)} \to f'(x_0)y$$

uniformly on bounded sets, and

$$\frac{u_n(x_n) - f(x_n)}{\omega(\varepsilon_n)}$$

is uniformly bounded because of Remark 2.3.

We also recall that

$$\mathbb{PMF}_{\varepsilon_n}(\beta, f, (0,1), u_n) = \omega(\varepsilon_n)^2\, \mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (-x_n/\omega(\varepsilon_n), (1-x_n)/\omega(\varepsilon_n)), v_n),$$

and hence (2.17) implies that

$$v_n \in \operatorname{argmin}_{\mathrm{loc}}\{\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (a,b), v) : v \in H^2((a,b))\},$$

for every bounded interval $(a,b) \subset \mathbb{R}$.

The proof of (4.1) consists of three main steps.

- (Isolation of horizontal and vertical parts). In the first step we define three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ such that

$$a_n \to (-1+\tau_0)H, \qquad v_n(a_n) \to 0, \qquad |v_n'(a_n)| \le 1, \tag{4.2}$$

$$b_n \to (1+\tau_0)H, \qquad v_n(b_n) \to 0, \qquad |v_n'(b_n)| \le 1, \tag{4.3}$$

$$c_n \to (1+\tau_0)H, \qquad v_n(c_n) \to 2V, \qquad |v_n'(c_n)| \le 1, \tag{4.4}$$

and

$$|c_n - b_n| = O(\omega(\varepsilon_n)^2). \tag{4.5}$$

Moreover, we set

$$m_n := \max\{|v_n(y)| : y \in [a_n, b_n]\},$$

and we prove that $m_n \to 0$ as $n \to +\infty$.

- (Estimate from below). In the second step we prove that there exists a constant $c_0 > 0$ such that

$$\mathbb{RPM}_{\varepsilon_n}((a_n,b_n),v_n) \ge c_0 \min\left\{\sqrt{m_n}, \frac{m_n^2}{\omega(\varepsilon_n)^2}\right\}. \tag{4.6}$$

- (Estimate from above). In the third step we prove that

$$\begin{aligned}\mathbb{RPMF}_{\varepsilon_n}(\beta, g_n, (a_n,c_n), v_n) \;\le\; & \mathbb{RPM}_{\varepsilon_n}((b_n,c_n),v_n) + \beta \int_{a_n}^{b_n} g_n(y)^2\,dy \\ & + O(m_n) + O(\omega(\varepsilon_n)^2),\end{aligned} \tag{4.7}$$

Given the three steps, we can conclude as follows. We observe that

$$\int_{a_n}^{b_n} (v_n(y) - g_n(y))^2\,dy = \int_{a_n}^{b_n} g_n(y)^2\,dy + O(m_n)$$

because $|v_n(y)| \le m_n$ in (a_n, b_n), and that

$$\int_{b_n}^{c_n} (v_n(y) - g_n(y))^2\,dy = O(\omega(\varepsilon_n)^2)$$

because of (4.5) and the uniform boundedness of $g_n(y)$ and $v_n(y)$ on every bounded set, and in particular in (b_n, c_n).

As a consequence, by combining (4.7) and (4.6) we obtain that

$$\begin{aligned}
&\mathbb{RPM}_{\varepsilon_n}((b_n,c_n),v_n)+\beta\int_{a_n}^{b_n} g_n(y)^2\,dy+O(m_n)+O(\omega(\varepsilon_n)^2)\\
&\quad\geq\ \mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(a_n,c_n),v_n)\\
&\quad=\ \mathbb{RPM}_{\varepsilon_n}((a_n,b_n),v_n)+\mathbb{RPM}_{\varepsilon_n}((b_n,c_n),v_n)\\
&\qquad+\beta\int_{a_n}^{b_n}(v_n(y)-g_n(y))^2\,dy+\beta\int_{b_n}^{c_n}(v_n(y)-g_n(y))^2\,dy\\
&\quad\geq\ \Gamma_0\min\left\{\sqrt{m_n},\frac{m_n^2}{\omega(\varepsilon_n)^2}\right\}+\mathbb{RPM}_{\varepsilon_n}((b_n,c_n),v_n)\\
&\qquad+\beta\int_{a_n}^{b_n} g_n(y)^2\,dy+O(m_n)+O(\omega(\varepsilon_n)^2),
\end{aligned}$$

from which we deduce that

$$c_0\min\left\{\sqrt{m_n},\frac{m_n^2}{\omega(\varepsilon_n)^2}\right\}\leq O(m_n)+O(\omega(\varepsilon_n)^2). \tag{4.8}$$

This inequality implies that $m_n = O(\omega(\varepsilon_n)^2)$, which in turn implies (4.1). Indeed, if this is not the case, then there exists a subsequence (not relabeled) such that $m_n/\omega(\varepsilon_n)^2 \to +\infty$. Since eventually $m_n \neq 0$ along this subsequence, we can divide (4.8) by m_n and obtain that

$$\Gamma_0\min\left\{\frac{1}{\sqrt{m_n}},\frac{m_n}{\omega(\varepsilon_n)^2}\right\}\leq\frac{O(m_n)}{m_n}+\frac{O(\omega(\varepsilon_n)^2)}{\omega(\varepsilon_n)^2}\cdot\frac{\omega(\varepsilon_n)^2}{m_n},$$

which is absurd because the left-hand side tends to $+\infty$ when $n\to+\infty$, while the right-hand side remains bounded.

Isolation of horizontal and vertical parts Let us start by constructing b_n and c_n. The idea is that the limit of $v_n(y)$ is a piecewise constant function that jumps from 0 to $2V$ in the point $H(1+\tau_0)$, and we know from Statement (1) of Theorem 2.5 that this jump is entirely achieved in an interval whose length is $O(\varepsilon_n^2)$. In addition, we need a uniform bound on v_n' at the endpoints of this short interval, for which we exploit a bound on the energies to deduce that on every interval whose length is $O(\omega(\varepsilon_n)^2)$ there exists at least a point in which v_n' is bounded.

In order to pursue this path, we begin by observing that $v_n(0)=0$ for every $n\in\mathbb{N}$, and $v_n(2H)\to 2V$. Therefore, there exists $z_n\in(0,2H)$ such that $v_n(z_n)=V$. From the strict convergence we deduce that $z_n\to H(1+\tau_0)$, because the latter is the unique point in $(0,2H)$ where the limit of v_n can be different from an integer multiple of $2V$.

Now we set $y_n := x_n+\omega(\varepsilon_n)z_n$, and we observe that

$$\frac{u_n(y_n+\omega(\varepsilon_n)y)-u_n(y_n)}{\omega(\varepsilon_n)}=v_n(y+z_n)-v_n(z_n)\rightrightarrows S_{H,V}(y+H)-V.$$

Since the limit function is a graph translation of vertical type of the canonical (H, V)-staircase, from the result for the vertical parts of the steps we know that

$$w_n(y) := \frac{u_n(y_n + \omega(\varepsilon_n)\varepsilon_n^2 y) - u_n(y_n)}{\omega(\varepsilon_n)} \to C_{\Lambda,V}(y),$$

where $C_{\Lambda,V}(y)$ is the canonical (Λ, V)-cubic connection introduced in Definition 2.4 with Λ given by (2.18), and the convergence is uniform in the sense that

$$\lim_{n\to+\infty} \max\left\{|w_n(y) - C_{\Lambda,V}(y)| : |y| \le L/\varepsilon_n^2\right\} = 0 \tag{4.9}$$

for every $L \in (0, 2H)$.

Now we recall that

$$\mathbb{RPMMF}_{\varepsilon_n}(\beta, g_n, (-2H, 2H), v_n) \le E_0, \tag{4.10}$$

for some positive constant $E_0 > 0$, because of Theorem A and the boundedness of g_n. Let us consider the interval

$$I_n := \left[z_n - \Lambda\varepsilon_n^2 - \frac{E_0}{\log 2}\omega(\varepsilon_n)^2, z_n - \Lambda\varepsilon_n^2\right].$$

Since $|I_n| = (E_0/\log 2)\omega(\varepsilon_n)^2$, we deduce that

$$\begin{aligned} E_0 &\ge \mathbb{RPMMF}_{\varepsilon_n}(\beta, g_n, I_n, v_n) \\ &\ge \frac{1}{\omega(\varepsilon_n)^2}\int_{I_n} \log(1 + v_n'(y)^2)\, dy \\ &\ge \frac{E_0}{\log 2}\min\left\{\log(1 + v_n'(y)^2) : y \in I_n\right\}. \end{aligned}$$

This implies the existence of $b_n \in I_n$ such that $\log(1+v_n'(b_n)^2) \le \log 2$, and therefore $|v_n'(b_n)| \le 1$. Moreover, $b_n \to H(1 + \tau_0)$ because both endpoints of I_n tend to the same limit of z_n. Finally, we show that $v_n(p_n) \to 0$ for every sequence of points $\{p_n\}$ such that $p_n \in [0, b_n]$, so in particular $v_n(b_n) \to 0$. To this end, we write p_n in the form $z_n + \varepsilon_n^2\beta_n$ for some $\beta_n \in [-z_n/\varepsilon_n^2, -\Lambda]$, and we observe that

$$v_n(p_n) = w_n(\beta_n) + v_n(z_n) = w_n(\beta_n) + V,$$

and we conclude by exploiting (4.9) and by remarking that $C_{\Lambda,V}(y) = -V$ when $y \le -\Lambda$.

In the same way we construct the points c_n satisfying (4.4) starting from the interval

$$I_n := \left[z_n + \Lambda\varepsilon_n^2, z_n + \Lambda\varepsilon_n^2 + \frac{E_0}{\log 2}\omega(\varepsilon_n)^2\right].$$

Of course in this case we exploit that c_n is of the form $y_n + \varepsilon_n^2\gamma_n$ with $\gamma_n \ge \Lambda$, so that $w_n(\gamma_n) \to V$.

Finally, we construct a_n in the same way of b_n, but starting with the point $z_n' \in (-2H, 0)$ such that $v_n(z_n') \to -V$ which implies that $z_n' \to (-1 + \tau_0)H$. As before, we

can show that $|v_n'(a_n)| \le 1$ and that $v_n(p_n) \to 0$ for every sequence of points $\{p_n\}$ such that $p_n \in [a_n, 0]$, and in particular $v_n(a_n) \to 0$.

We conclude the proof of the first step by observing that (4.5) is an immediate consequence of the definition of b_n and c_n, which implies that

$$0 \le c_n - b_n \le \frac{2E_0}{\log 2}\omega(\varepsilon_n)^2,$$

and that $m_n \to 0$ because we have proved that $v_n(p_n) \to 0$ for every sequence $\{p_n\}$ such that $p_n \in [a_n, b_n]$.

Estimate from below We set $M_n := \min\{\sqrt{m_n}, m_n^2/\omega(\varepsilon_n)^2\}$ and we prove that

$$\liminf_{n\to+\infty} \frac{1}{M_n}\,\mathbb{RPM}_{\varepsilon_n}((a_n, b_n), v_n) > 0, \tag{4.11}$$

which in turn implies (4.6). To this end, for every $n \in \mathbb{N}$ we define $\widehat{v}_n(y)$ in such a way that $v_n(y) = m_n\widehat{v}_n(y)$, and we observe that

$$\mathbb{RPM}_{\varepsilon_n}((a_n, b_n), v_n) = \int_{a_n}^{b_n} \left\{ \varepsilon_n^6 m_n^2 \widehat{v}_n''(y)^2 + \frac{1}{\omega(\varepsilon_n)^2}\log\left(1 + m_n^2\widehat{v}_n'(y)^2\right)\right\} dy, \tag{4.12}$$

and that

$$\widehat{v}_n(0) = 0 \qquad \text{and} \qquad \max\{|\widehat{v}_n(y)| : y \in [a_n, b_n]\} = 1.$$

Now we observe that the sequence of intervals (a_n, b_n) and the sequence of functions $\widehat{v}_n \in H^2((a_n, b_n))$ fit into the framework of Lemma 5.7 with $L := 2H$ and $J := 1$, and we claim that the liminf in (4.11) is greater than or equal to the constant $c(2H, 1)$ provided by Lemma 5.7. It is enough to prove this along the two subsequences where M_n takes either of the values in the minimum, and therefore we distinguish two cases (without relabeling subsequences).

- If $M_n = m_n^2/\omega(\varepsilon_n)^2$, namely $\sqrt{m_n} \ge m_n^2/\omega(\varepsilon_n)^2$, then we have that $\omega(\varepsilon_n)^8 \ge m_n^6$. Therefore, at least if n is large enough, we deduce that

  $$|\log m_n| \ge \frac{4}{3}|\log\omega(\varepsilon_n)| = \frac{4}{3}\left(|\log\varepsilon_n| - \frac{1}{2}\log|\log\varepsilon_n|\right) \ge |\log\varepsilon_n|,$$

  which implies that $\varepsilon_n^6\omega(\varepsilon_n)^2 \ge m_n^6/|\log m_n|^3$. As a consequence, from (4.12) we obtain that

  $$\frac{1}{M_n}\,\mathbb{RPM}_{\varepsilon_n}(a_n, b_n), v_n) \ge \mathbb{RPMH}_{m_n}((a_n, b_n), \widehat{v}_n),$$

  where the functional $\mathbb{RPMH}$ is defined according to (5.6).

  Since $m_n \to 0$ and we know that $|\widehat{v}_n'(a_n)| \le 1/m_n$ and $|\widehat{v}_n'(b_n)| \le 1/m_n$, the result follows from Lemma 5.7.

- If $M_n = \sqrt{m_n}$, namely $\sqrt{m_n} \le m_n^2/\omega(\varepsilon_n)^2$, then we have that $m_n^2 \ge \omega(\varepsilon_n)^2\sqrt{m_n}$. Thus, from (4.12) we obtain that

$$\frac{1}{M_n}\,\mathbb{RPM}_{\varepsilon_n}((a_n,b_n),v_n) \ge \min\left\{1, \frac{|\log(\omega(\varepsilon_n)m_n^{1/4})|^3}{|\log\varepsilon_n|^3}\right\}\mathbb{RPMH}_{\omega(\varepsilon_n)m_n^{1/4}}((a_n,b_n),\widehat{v}_n),$$

  and the result follows again from Lemma 5.7 because $\omega(\varepsilon_n)m_n^{1/4} \to 0$, the boundary values satisfy

$$|\widehat{v}_n'(a_n)| \le \frac{1}{m_n} \le \frac{1}{\omega(\varepsilon_n)m_n^{1/4}} \qquad \text{and} \qquad |\widehat{v}_n'(b_n)| \le \frac{1}{m_n} \le \frac{1}{\omega(\varepsilon_n)m_n^{1/4}},$$

  and it holds that

$$\liminf_{n\to+\infty} \frac{|\log(\omega(\varepsilon_n)m_n^{1/4})|^3}{|\log\varepsilon_n|^3} \ge \liminf_{n\to+\infty} \frac{|\log(\omega(\varepsilon_n))|^3}{|\log\varepsilon_n|^3} = 1.$$

Estimate from above In this paragraph we prove (4.7). Since

$$v_n \in \operatorname{argmin}_{\rm loc}\left\{\mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(a_n,c_n),v) : v \in H^2((a_n,c_n))\right\},$$

it is enough to exhibit a function $\widetilde{v}_n \in H^2((a_n,c_n))$, with the same boundary conditions (both on function and on the derivative) of v_n, such that $\mathbb{RPMF}_{\varepsilon_n}(\beta,g_n,(a_n,c_n),\widetilde{v}_n)$ is bounded from above by the right-hand side of (4.7). One would like to choose $\widetilde{v}_n$ as the function identically equal to $v_n(a_n)$ in the horizontal part (a_n,b_n), and equal to a suitable homothety of v_n in the vertical part (b_n,c_n). This choice, however, does not fit the boundary conditions and is not of class H^2. Therefore, we have to smooth out the connections. To this end, we observe that

$$a_n < a_n + \varepsilon_n^3\omega(\varepsilon_n) < b_n - \varepsilon_n^3\omega(\varepsilon_n) < b_n < c_n$$

when n is large enough, so that we can partition $[a_n,c_n]$ into four intervals

$$I_{1,n} := [a_n, a_n+\varepsilon_n^3\omega(\varepsilon_n)], \qquad I_{2,n} := [a_n+\varepsilon_n^3\omega(\varepsilon_n), b_n-\varepsilon_n^3\omega(\varepsilon_n)],$$
$$I_{3,n} := [b_n-\varepsilon_n^3\omega(\varepsilon_n), b_n], \qquad I_{4,n} := [b_n, c_n].$$

Then we consider the constant

$$B_n := \frac{v_n(b_n)-v_n(a_n)}{v_n(c_n)-v_n(b_n)-(c_n-b_n)v_n'(c_n)},$$

and we observe that $B_n = O(m_n)$ because $|v_n(b_n)-v_n(a_n)| \le 2m_n$ and the denominator tends to $2V$. At this point we define $\widetilde{v}_n$ in a piecewise way as follows.

- In $I_{1,n}$ we define $\widetilde{v}_n(x)$ as the cubic polynomial with boundary conditions

$$\widetilde{v}_n(a_n) = \widetilde{v}_n(a_n+\varepsilon_n^3\omega(\varepsilon_n)) = v_n(a_n), \quad \widetilde{v}_n'(a_n) = v_n'(a_n), \quad \widetilde{v}_n'(a_n+\varepsilon_n^3\omega(\varepsilon_n)) = 0.$$

- In $I_{2,n}$ we define $\widetilde{v}_n(x) \equiv v_n(a_n)$.
- In $I_{3,n}$ we define $\widetilde{v}_n(x)$ as the cubic polynomial such that

$$\widetilde{v}_n(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = \widetilde{v}_n(b_n) = v_n(a_n), \qquad \widetilde{v}_n'(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = 0,$$

  and

$$\widetilde{v}_n'(b_n) = v_n'(b_n) + B_n(v_n'(b_n) - v_n'(c_n)).$$

- In $I_{4,n}$ we set

$$\widetilde{v}_n(y) := v_n(y) + B_n\left(v_n(y) + (c_n - y)v_n'(c_n) - v_n(c_n)\right).$$

This definition guarantees that the connections in the intermediate points are of class C^1, so $\widetilde{v}_n \in H^2((a_n, c_n))$, and that the values of $\widetilde{v}_n(y)$ and $\widetilde{v}_n'(y)$ for $y = a_n$ and $y = c_n$ coincide with the corresponding values of $v_n(y)$ and $v_n'(y)$. We claim that the fidelity term satisfies

$$\int_{a_n}^{c_n} |\widetilde{v}_n(y) - g_n(y)|^2\, dy = \int_{a_n}^{b_n} |g_n(y)|^2\, dy + O(\omega(\varepsilon_n)^2) + O(m_n). \tag{4.13}$$

As for the terms of $\mathbb{RPM}_{\varepsilon_n}$ of course in $I_{2,n}$ it holds that $\mathbb{RPM}_{\varepsilon_n}(I_{2,n}, \widetilde{v}_n) = 0$. Moreover, in $I_{1,n}$ and $I_{3,n}$ it turns out that

$$\mathbb{RPM}_{\varepsilon_n}(I_{1,n}, \widetilde{v}_n) = O(\varepsilon_n^3/\omega(\varepsilon_n)) = O(\omega(\varepsilon_n)^2), \tag{4.14}$$

and

$$\mathbb{RPM}_{\varepsilon_n}(I_{3,n}, \widetilde{v}_n) = O(\varepsilon_n^3/\omega(\varepsilon_n)) = O(\omega(\varepsilon_n)^2), \tag{4.15}$$

while in $I_{4,n}$ it turns out that

$$\int_{b_n}^{c_n} \varepsilon_n^6 \widetilde{v}_n''(y)^2\, dy = \int_{b_n}^{c_n} \varepsilon_n^6 v_n''(y)^2\, dy + O(m_n) \tag{4.16}$$

and

$$\int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log\left(1 + \widetilde{v}_n'(y)^2\right) = \int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log\left(1 + v_n'(y)^2\right) + O(m_n) \tag{4.17}$$

These claims, if true, are enough to establish (4.7).

Let us start with (4.13). We recall that the sequence $\{g_n\}$ is uniformly bounded on bounded sets, hence it is uniformly bounded in $[a_n, c_n]$.

On the other hand, also the sequence $\{\widetilde{v}_n(y)\}$ is uniformly bounded in $[a_n, c_n]$. This is trivial in $I_{2,n}$, while in $I_{1,n}$ and in $I_{3,n}$ it follows from Lemma 5.5 and the uniform boundedness of the boundary values, and in $I_{4,n}$ it follows from the uniform bounds on v_n. We point out that in this point it is essential to have an estimate on $v_n'(b_n)$ and

$v_n'(c_n)$. Since the measure of $[a_n, c_n] \setminus I_{2,n}$ is $O(\omega(\varepsilon_n)^2)$ and $|\widetilde{v}_n(y)| = |v_n(a_n)| \le m_n$ in $I_{2,n}$, it follows that

$$\begin{aligned}\int_{a_n}^{c_n} |\widetilde{v}_n(y) - g_n(y)|^2\,dy &= \int_{I_{2,n}} |\widetilde{v}_n(y) - g_n(y)|^2\,dy + O(\omega(\varepsilon_n)^2)\\ &= \int_{I_{2,n}} |g_n(y)|^2\,dy + O(m_n) + O(\omega(\varepsilon_n)^2)\\ &= \int_{a_n}^{b_n} |g_n(y)|^2\,dy + O(m_n) + O(\omega(\varepsilon_n)^2),\end{aligned}$$

that is exactly (4.13).

In order to estimate the terms with second order derivatives, in the intervals $I_{1,n}$ and $I_{3,n}$ we apply again Lemma 5.5, and we obtain that

$$\int_{I_{1,n}} \varepsilon_n^6 \widetilde{v}_n''(y)^2\,dy = 4v_n'(a_n)^2 \frac{\varepsilon_n^3}{\omega(\varepsilon_n)},$$

and

$$\int_{I_{3,n}} \varepsilon_n^6 \widetilde{v}_n''(y)^2\,dy = 4\left[v_n'(b_n) + B_n(v_n'(b_n) - v_n'(c_n))\right]^2 \frac{\varepsilon_n^3}{\omega(\varepsilon_n)},$$

while for first order derivatives it turns out that

$$\int_{I_{1,n}} \frac{1}{\omega(\varepsilon_n)^2} \log(1 + \widetilde{v}_n'(y)^2)\,dy \le \frac{1}{\omega(\varepsilon_n)^2} \log\left(1 + \frac{9}{4}v'(a_n)^2\right) |I_{1,n}| = O(\varepsilon_n^3/\omega(\varepsilon_n)),$$

and similarly in $I_{3,n}$. This is enough to establish (4.14) and (4.15). It remains to consider $I_{4,n} = [b_n, c_n]$, where

$$\widetilde{v}_n'(y) = (1 + B_n)v_n'(y) - B_n v_n'(c_n) \qquad \text{and} \qquad \widetilde{v}_n''(y) = (1 + B_n)v_n''(y).$$

In particular it turns out that

$$\int_{b_n}^{c_n} \varepsilon_n^6 \widetilde{v}_n''(y)^2\,dy = \int_{b_n}^{c_n} \varepsilon_n^6 v_n''(y)^2\,dy + (2B_n + B_n^2) \int_{b_n}^{c_n} \varepsilon_n^6 v_n''(y)^2\,dy,$$

and this implies (4.16) because $B_n = O(m_n)$ and the last integral is bounded by E_0 thanks to (4.10).

Finally, since $|B_n| \le 1/2$ if n is large enough, from Lemma 5.1 applied with

$$x := v_n'(y), \qquad b := B_n, \qquad d := -B_n v_n'(c_n)$$

we obtain that

$$\int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log(1 + \widetilde{v}_n'(y)^2) = \int_{b_n}^{c_n} \frac{1}{\omega(\varepsilon_n)^2} \log(1 + v_n'(y)^2) + R_n,$$

where

$$|R_n| \le \frac{5}{\omega(\varepsilon_n)^2} \cdot |B_n| \cdot (c_n - b_n).$$

Recalling that $B_n = O(m_n)$ and $c_n - b_n = O(\omega(\varepsilon_n)^2)$, we obtain (4.17).

4.2 Estimate on derivatives

In this part of the proof we prove (2.19).

Let $[a,b] \subseteq ((-1+\tau_0)H, (1+\tau_0)H)$ be any interval. Let us choose a' and b' with

$$(-1+\tau_0)H < a' < a < b < b' < (1+\tau_0)H.$$

Let us define $w_n(y)$ in such a way that $v_n(y) = \omega(\varepsilon_n)^2 w_n(y)$. From the results of the previous section we know that there exists a constant Γ_0 such that

$$|w_n(y)| \leq \Gamma_0 \qquad \forall y \in [a',b'], \quad \forall n \in \mathbb{N}. \tag{4.18}$$

In addition, from the mean value theorem applied in the intervals $[a',a]$ and $[b,b']$, we deduce that there exist $a_n \in (a',a)$ and $b_n \in (b,b')$ such that

$$|w_n'(a_n)| \leq \frac{2\Gamma_0}{a-a'} \qquad \text{and} \qquad |w_n'(b_n)| \leq \frac{2\Gamma_0}{b'-b}. \tag{4.19}$$

In the sequel we consider the interval (a_n, b_n), and our claim becomes that

$$\limsup_{n\to+\infty} \max\{|w_n'(y)| : y \in (a_n,b_n)\} < +\infty. \tag{4.20}$$

We observe that

$$\begin{aligned}\mathbb{RPMMF}_{\varepsilon_n}(\beta, g_n, (a_n,b_n), v_n) &= \mathbb{RPM}_{\varepsilon_n}((a_n,b_n), \omega(\varepsilon_n)^2 w_n) \\ &\quad + \int_{a_n}^{b_n} \left(\omega(\varepsilon_n)^4 w_n(y)^2 - 2\omega(\varepsilon_n)^2 g_n(y) w_n(y) + g_n(y)^2\right) dy,\end{aligned}$$

so that we can consider w_n as a minimizer to the right-hand side subject to its own boundary conditions, and we already know from (4.18) and (4.19) that these boundary conditions satisfy a bound of the form

$$|w_n(a_n)| + |w_n(b_n)| + |w_n'(a_n)| + |w_n'(b_n)| \leq \Gamma_1 \tag{4.21}$$

for a suitable real constant Γ_1. Since the integral of $g_n(y)^2$ plays no role in the minimization process, we can neglect it and divide by $\omega(\varepsilon_n)^2$. In this way we obtain that

$$w_n \in \operatorname{argmin}_{\mathrm{loc}} \left\{F_n((a_n,b_n), w) : w \in H^2((a_n,b_n))\right\},$$

where the sequence of functionals F_n is defined by

$$\begin{aligned}F_n(\Omega, w) \;:=\; &\int_\Omega \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 w''(y)^2 + \frac{1}{\omega(\varepsilon_n)^4} \log\left(1 + \omega(\varepsilon_n)^4 w'(y)^2\right)\right) dy \\ &+ \int_\Omega \left(\omega(\varepsilon_n)^2 w(y)^2 - 2 g_n(y) w(y)\right) dy.\end{aligned}$$

One could prove that, if $g_n \to g_\infty$ in L^2, the Gamma-limit of F_n is of the form

$$\int_\Omega \left(w'(y)^2 - 2 g_\infty(y) w(y)\right) dy.$$

Therefore, it is reasonable to expect that $w_n(y)$ behaves for large n as a minimizer to the limit problem, which is a standard quadratic functional. This is what actually happens, but the proof is delicate for many reasons, including the lack of convexity of the approximating functionals, defined on intervals that depend also on n, and the boundary layers due to the loss of the boundary conditions on the derivative when the functionals of order two converge to a functional of order one. In particular, we cannot expect the sequence $\{w_n'(y)\}$ to converge uniformly, and we cannot expect all recovery sequences to have bounded derivatives, which forces us to exploit the minimality of $w_n(y)$ to some extent.

The key tool in our analysis is a comparison between $w_n(y)$ and minimizers to a suitable first order functional with a convex Lagrangian, defined as follows. Let us consider the function

$$\varphi_n(\sigma) := \frac{1}{\omega(\varepsilon_n)^4} \log(1 + \omega(\varepsilon_n)^4 \sigma^2) \qquad \forall \sigma \in \mathbb{R}.$$

An elementary calculation shows that

$$\varphi_n''(\sigma) \geq 1 \qquad \forall \sigma \in \left[-\frac{1}{3\omega(\varepsilon_n)^2}, \frac{1}{3\omega(\varepsilon_n)^2} \right].$$

Now we consider the function $\psi_n : \mathbb{R} \to \mathbb{R}$ such that $\psi_n(0) = \psi_n'(0) = 0$ and

$$\psi_n''(\sigma) := \max\{\varphi_n''(\sigma), 1\} \qquad \forall \sigma \in \mathbb{R}. \tag{4.22}$$

This function satisfies

- $\psi_n \in C^2(\mathbb{R})$,
- $\psi''(\sigma) \geq 1$ for every $\sigma \in \mathbb{R}$,
- $\sigma^2/2 \leq \psi_n(\sigma) \leq \sigma^2$ for every $\sigma \in \mathbb{R}$,
- $\psi_n(\sigma) = \varphi_n(\sigma)$ for every $\sigma \in [-1/3\omega(\varepsilon_n)^2, 1/3\omega(\varepsilon_n)^2]$.

At this point we can consider the functional

$$G_n(\Omega, z) := \int_\Omega \left(\psi_n(z'(y)) + \omega(\varepsilon_n)^2 z(y)^2 - 2 g_n(y) z(y) \right) dy \tag{4.23}$$

and an element

$$z_n \in \operatorname{argmin} \left\{ G_n((a_n, b_n), z) : z \in H^1((a_n, b_n)),\ z_n(a_n) = w_n(a_n),\ z_n(b_n) = w_n(b_n) \right\}. \tag{4.24}$$

We point out that we do not impose any boundary condition on derivatives, because the Lagrangian of G_n is of order one. At this point the proof proceeds as follows.

- (Uniform bounds on z_n). In the first step we show that there exist real numbers Γ_2 and Γ_3 such that

  $$G_n((a_n, b_n), z_n) \leq \Gamma_2, \tag{4.25}$$

  and

  $$|z_n(y)| + |z_n'(y)| + |z_n''(y)| \leq \Gamma_3 \qquad \forall y \in (a_n, b_n). \tag{4.26}$$

- (Estimates from above). In the second step we show that

$$F_n((a_n, b_n), w_n) \leq G_n((a_n, b_n), z_n) + O(\varepsilon_n^3 \omega(\varepsilon_n)). \tag{4.27}$$

- (The contribution of high derivatives is negligible). In the third step we introduce the regions with high derivative

$$A_n := \left\{ y \in (a_n, b_n) : |w_n'(y)| > \frac{1}{4\omega(\varepsilon_n)^2} \right\},$$

and their contribution to the total variation

$$\Delta_n := \int_{A_n} |w_n'(y)|\, dy.$$

We show that they are both negligible, and more precisely that

$$|A_n| = O(\omega(\varepsilon_n)^4) \qquad \text{and} \qquad \Delta_n = O(\varepsilon_n^2 |\log \varepsilon_n|^{5/2}). \tag{4.28}$$

- (Estimates from below in regions with "small" derivative). In the fourth step we show that

$$F_n(A_n^c, w_n) \geq G_n((a_n, b_n), z_n) - O(\omega(\varepsilon_n)^4) - O(\Delta_n), \tag{4.29}$$

where A_n^c denotes the complement of A_n in (a_n, b_n).

- (Reduction to the convexity zone). In the fifth step we show that

$$|w_n'(y)| \leq \frac{1}{3\omega(\varepsilon_n)^2} \qquad \forall y \in (a_n, b_n) \tag{4.30}$$

when n is large enough. This seems to be a weak progress toward (4.20), but actually it is a crucial step, because it implies that eventually w_n stays in the region where $\varphi_n(\sigma)$ coincides with $\psi_n(\sigma)$, and hence in the convex regime.

- (Conclusion). Finally, in the sixth step we show that $w_n(y) - z_n(y)$ tends to zero with respect to the C^0 norm, and it is bounded in C^1 norm (but not necessarily vanishing, due to the boundary layers). Since z_n is bounded in the C^1 norm, this is enough to establish (4.20).

Uniform bounds on z_n The proof of (4.25) and (4.26) is a straightforward application of Lemma 5.3. We just need to check that the functional G_n defined in (4.23) fits into the assumptions, which in turn follow from the properties of ψ_n.

Estimate from above In this paragraph we prove (4.27). To this end, we observe that w_n is a solution to a minimum problem, and therefore it is enough to exhibit a competitor for the minimum problem whose energy is bounded by the right-hand side of (4.27). A natural choice for this competitor would be $z_n(t)$, which however does not necessarily satisfy the boundary conditions on the derivative. Therefore, we need to modify $z_n(t)$ in a neighborhood of the boundary. To this end, we write $[a_n, b_n]$ as the union of three intervals

$$I_{1,n} := [a_n, a_n+\varepsilon_n^3\omega(\varepsilon_n)], \quad I_{2,n} := [a_n+\varepsilon_n^3\omega(\varepsilon_n), b_n-\varepsilon_n^3\omega(\varepsilon_n)], \quad I_{3,n} := [b_n-\varepsilon_n^3\omega(\varepsilon_n), b_n],$$

and we consider the function $\widehat{w}_n : [a_n, b_n] \to \mathbb{R}$ defined as follows.

- In $I_{1,n}$ we define $\widehat{w}_n$ as the cubic polynomial with boundary conditions

$$\begin{aligned} \widehat{w}_n(a_n) = w_n(a_n) = z_n(a_n), \qquad & \widehat{w}_n'(a_n) = w_n'(a_n), \\ \widehat{w}_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)) = z_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)), & \\ \widehat{w}_n'(a_n + \varepsilon_n^3\omega(\varepsilon_n)) = z_n'(a_n + \varepsilon_n^3\omega(\varepsilon_n)). & \end{aligned}$$

- In $I_{2,n}$ we set $\widehat{w}_n(y) := z_n(y)$.

- In $I_{3,n}$ we define $\widehat{w}_n$ as the cubic polynomial with boundary conditions

$$\begin{aligned} \widehat{w}_n(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = z_n(b_n - \varepsilon_n^3\omega(\varepsilon_n)), & \\ \widehat{w}_n'(b_n - \varepsilon_n^3\omega(\varepsilon_n)) = z_n'(b_n - \varepsilon_n^3\omega(\varepsilon_n)), & \\ \widehat{w}_n(b_n) = w_n(b_n) = z_n(b_n), \qquad & \widehat{w}_n'(b_n) = w_n'(b_n). \end{aligned}$$

This definition guarantees that the connections in the intermediate points are of class C^1, so that the resulting function belongs to $H^2((a_n, b_n))$, and that the values of $\widehat{w}_n(y)$ and $\widehat{w}_n'(y)$ for $y = a_n$ and $y = b_n$ coincide with the corresponding values of $w_n(y)$ and $w_n'(y)$.

Now we show that an estimate of the form (4.27) holds true separately in each of the three intervals.

- In the interval $I_{1,n}$ we claim that

$$F_n(I_{1,n}, \widehat{w}_n) = O(\varepsilon_n^3\omega(\varepsilon_n)) \qquad \text{and} \qquad G_n(I_{1,n}, z_n) = O(\varepsilon_n^3\omega(\varepsilon_n)), \quad (4.31)$$

  and hence in particular

$$F_n(I_{1,n}, \widehat{w}_n) - G_n(I_{1,n}, z_n) = O(\varepsilon_n^3\omega(\varepsilon_n)).$$

  Regarding $\widehat{w}_n$, we apply Lemma 5.5 with

$$\begin{aligned} A_0 := w_n(a_n), \quad B_0 := z_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)), \\ A_1 := w_n'(a_n), \quad B_1 := z_n'(a_n + \varepsilon_n^3\omega(\varepsilon_n)). \end{aligned}$$

We observe that (4.21) and (4.26) guarantee that all these values are bounded and that

$$|B_0 - A_0| = |z_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)) - w_n(a_n)| = |z_n(a_n + \varepsilon_n^3\omega(\varepsilon_n)) - z_n(a_n)| = O(\varepsilon_n^3\omega(\varepsilon_n)),$$

Thus from Lemma 5.5 we obtain that

$$\int_{I_{1,n}} \varepsilon_n^6 \omega(\varepsilon_n)^2 \widehat{w}_n''(y)^2 \, dy = O(\varepsilon_n^3\omega(\varepsilon_n)),$$

and the existence of a constant Γ_4 such that

$$|\widehat{w}_n(y)| + |\widehat{w}_n'(y)| \le \Gamma_4 \qquad \forall y \in I_{1,n}, \quad \forall n \in \mathbb{N}.$$

Since $|I_{1,n}| = \varepsilon_n^3\omega(\varepsilon_n)$, this implies that

$$\int_{I_{1,n}} \frac{1}{\omega(\varepsilon_n)^4} \log\left(1 + \omega(\varepsilon_n)^4 \widehat{w}_n'(y)^2\right) \, dy \le \int_{I_{1,n}} \widehat{w}_n'(y)^2 \, dy = O(\varepsilon_n^3\omega(\varepsilon_n)),$$

and

$$\int_{I_{1,n}} \left(\omega(\varepsilon_n)^2 \widehat{w}_n(y)^2 - 2g_n(y)\widehat{w}_n(y)\right) \, dy = O(\varepsilon_n^3 \omega(\varepsilon_n)^3).$$

All these estimates imply the first relation in (4.31).

Regarding z_n, it is enough to observe that

$$\int_{I_{1,n}} \psi_n(z_n'(y)) \, dy \le \left(\max_{\sigma \in \mathbb{R}} \ \psi_n''(\sigma)\right) \int_{I_{1,n}} \frac{z_n'(y)^2}{2} \, dy \le O(\varepsilon_n^3\omega(\varepsilon_n)),$$

and

$$\int_{I_{1,n}} \left(\omega(\varepsilon_n)^2 z_n(y)^2 - 2g_n(y)z_n(y)\right) dy = O(\varepsilon_n^3 \omega(\varepsilon_n)^3),$$

because of the uniform bounds on z_n and g_n.

- In an analogous way, we can show that

$$F_n(I_{3,n}, \widehat{w}_n) - G_n(I_{3,n}, z_n) = O(\varepsilon_n^3\omega(\varepsilon_n)).$$

- In the interval $I_{2,n}$ we claim that

$$F_n(I_{2,n}, \widehat{w}_n) - G_n(I_{2,n}, z_n) = O(\varepsilon_n^6 \omega(\varepsilon_n)^2).$$

Indeed, the uniform bound on $z_n'(t)$ implies that

$$|z_n'(y)| \le \frac{1}{3\omega(\varepsilon_n)^4} \qquad \forall y \in I_{2,n}$$

when n is large enough. For these values of n it follows that

$$\psi_n(z_n'(y)) = \frac{1}{\omega(\varepsilon_n)^4} \log\left(1 + \omega(\varepsilon_n)^4 z_n'(y)^2\right) \qquad \forall y \in I_{2,n},$$

and therefore

$$F_n(I_{2,n}, \widehat{w}_n) = F_n(I_{2,n}, z_n) = \varepsilon_n^6 \omega(\varepsilon_n)^2 \int_{I_{2,n}} z_n''(y)^2 \, dy + G_n(I_{2,n}, z_n),$$

and we conclude by exploiting the uniform bounds on $z_n''(y)$.

The contribution of high derivatives is negligible To begin with, from (4.25), (4.27) and the uniform bounds on w_n and g_n, we deduce that

$$\int_{A_n} \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 w_n''(y)^2 + \frac{1}{\omega(\varepsilon_n)^4} \log\left(1 + \omega(\varepsilon_n)^4 w_n'(y)^2\right) \right) dy \\ \leq F_n((a_n, b_n), w_n) + 2 \left| \int_{a_n}^{b_n} g_n(y) w_n(y)\, dy \right| \leq \Gamma_5, \quad (4.32)$$

for a suitable constant Γ_5. This implies in particular that

$$\Gamma_5 \geq \frac{1}{\omega(\varepsilon_n)^4} \int_{A_n} \log\left(1 + \omega(\varepsilon_n)^4 w_n'(y)^2\right) dy \geq \frac{1}{\omega(\varepsilon_n)^4} \log\left(\frac{17}{16}\right) |A_n|,$$

which in turn implies that $|A_n| = O(\omega(\varepsilon_n)^4)$. In addition, from Lemma 5.6, we deduce that

$$\int_{A_n} \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 w_n''(y)^2 + \frac{1}{\omega(\varepsilon_n)^4} \log\left(1 + \omega(\varepsilon_n)^4 w_n'(y)^2\right) \right) dy \\ \geq \frac{4\sqrt{2/3}}{\varepsilon_n |\log \varepsilon_n|^{5/4}} \left(\log\left(\frac{17}{16}\right) \right)^{3/4} \left(\Delta_n - \frac{|A_n|}{4\omega(\varepsilon_n)^2} \right)^{1/2}.$$

Combining this estimate with (4.32) we obtain that

$$\frac{\Gamma_5}{4\sqrt{2/3}} \left(\log\left(\frac{17}{16}\right) \right)^{-3/4} \geq \frac{1}{\varepsilon_n |\log \varepsilon_n|^{5/4}} \left(\Delta_n - \frac{|A_n|}{4\omega(\varepsilon_n)^2} \right)^{1/2} \geq \frac{(\Delta_n - O(\omega(\varepsilon_n)^2))^{1/2}}{\varepsilon_n |\log \varepsilon_n|^{5/4}},$$

from which we conclude that $\Delta_n = O(\varepsilon_n^2 |\log \varepsilon_n|^{5/2})$.

Estimates from below in regions with "small" derivative In order to prove (4.29), we introduce the functions $\widetilde{w}_n \in H^1((a_n, b_n))$ such that $\widetilde{w}_n(a_n) = w_n(a_n)$ and

$$\widetilde{w}_n'(y) := \begin{cases} w_n'(y) & \text{if } x \in A_n^c, \\ 0 & \text{if } x \in A_n. \end{cases}$$

For every $y \in [a_n, b_n]$ it turns out that

$$|w_n(y) - \widetilde{w}_n(y)| \leq \int_{a_n}^{b_n} |w_n'(s) - \widetilde{w}_n'(s)|\, ds = \int_{A_n} |w_n'(s)|\, ds = \Delta_n,$$

and in particular $|\widetilde{w}_n(y)|$ is bounded, independently of y and n.

Then we introduce the function $\widetilde{z}_n$ that minimizes G_n with respect to the same boundary conditions of $\widetilde{w}_n$, namely

$$\widetilde{z}_n \in \operatorname{argmin}\left\{ G_n((a_n, b_n), z) : z \in H^1((a_n, b_n)),\ z(a_n) = \widetilde{w}(a_n),\ z(b_n) = \widetilde{w}(b_n) \right\}.$$

We claim that

$$F_n(A_n^c, w_n) \geq G_n((a_n, b_n), \widetilde{w}_n) - O(\omega(\varepsilon_n)^4) - O(\Delta_n),$$

and

$$G_n((a_n,b_n),\widetilde{z}_n) = G_n((a_n,b_n),z_n) - O(\Delta_n).$$

Since $G_n((a_n,b_n),\widetilde{w}_n) \geq G_n((a_n,b_n),\widetilde{z}_n)$ due to the minimality of $\widetilde{z}_n$, these two claims, if proved, imply (4.29).

In order to prove the first claim, we begin by observing that

$$\begin{aligned}\int_{A_n^c} \frac{1}{\omega(\varepsilon_n)^4} \log\left(1+\omega(\varepsilon_n)^4 w_n'(y)^2\right)\, dy &= \int_{a_n}^{b_n} \frac{1}{\omega(\varepsilon_n)^4} \log\left(1+\omega(\varepsilon_n)^4 \widetilde{w}_n'(y)^2\right)\, dy \\ &= \int_{a_n}^{b_n} \psi_n(\widetilde{w}_n'(y))\, dy,\end{aligned}$$

and therefore

$$\begin{aligned}F_n(A_n^c, w_n) &\geq \int_{A_n^c} \left[\frac{1}{\omega(\varepsilon_n)^4} \log\left(1+\omega(\varepsilon_n)^4 w_n'(y)^2\right) + \omega(\varepsilon_n)^2 w_n(y)^2 - 2g_n(y)w_n(y)\right] dy \\ &= \int_{a_n}^{b_n} \left(\psi_n(\widetilde{w}_n'(y)) + \omega(\varepsilon_n)^2 \widetilde{w}_n(y)^2 - 2g_n(y)\widetilde{w}_n(y)\right) dy \\ &\quad - \int_{A_n} \left(\omega(\varepsilon_n)^2 \widetilde{w}_n(y)^2 - 2g_n(y)\widetilde{w}_n(y)\right) dy \\ &\quad - \int_{A_n^c} \left[\omega(\varepsilon_n)^2 \left(\widetilde{w}_n(y)^2 - w_n(y)^2\right) - 2g_n(y)\left(\widetilde{w}_n(y) - w_n(y)\right)\right] dy.\end{aligned}$$

The first line after the equality sign is exactly $G_n((a_n,b_n),\widetilde{w}_n)$. The second line is $O(|A_n|)$, and therefore $O(\omega(\varepsilon_n)^4)$, due to the uniform bounds on $g_n(y)$ and $\widetilde{w}_n(y)$. For the same reason, the third line is $O(\|\widetilde{w}_n - w_n\|_\infty)$, and hence $O(\Delta_n)$.

The second claim is a straightforward application of Lemma 5.4 to the functional G_n. We just need to observe that the difference between the boundary values of z_n and $\widetilde{z}_n$ is $O(\Delta_n)$.

Reduction to the convexity zone Let us show that (4.30) holds true whenever n is large enough. Indeed, let us assume by contradiction that this is false. Then along a suitable subsequence (not relabeled) there exists $c_n \in (a_n,b_n)$ such that $|w_n'(c_n)| > (3\omega(\varepsilon_n)^2)^{-1}$. Since $c_n \in A_n$, there exists a point $d_n \in (a_n,b_n)$ such that

$$|w_n'(d_n)| = \frac{1}{4\omega(\varepsilon_n)^2},$$

and that the open interval whose endpoints are c_n and d_n is contained in A_n, so it holds that

$$|d_n - c_n| \leq |A_n| = O(\omega(\varepsilon_n)^4).$$

Let us assume, without loss of generality, that $d_n > c_n$ (the other case is symmetric).

Then it turns out that

$$\begin{aligned}\int_{c_n}^{d_n} \varepsilon_n^6 \omega(\varepsilon_n)^2 w_n''(y)^2\,dy &\geq \varepsilon_n^6 \omega(\varepsilon_n)^2 \frac{1}{d_n - c_n}\left(\int_{c_n}^{d_n} w_n''(y)\,dy\right)^2 \\ &= \varepsilon_n^6 \omega(\varepsilon_n)^2 \frac{1}{d_n - c_n}\left(w_n'(d_n) - w_n'(c_n)\right)^2 \\ &\geq \varepsilon_n^6 \omega(\varepsilon_n)^2 \frac{1}{d_n - c_n}\left(\frac{1}{12\omega(\varepsilon_n)^2}\right)^2.\end{aligned}$$

Recalling the uniform bounds on w_n and g_n, we obtain that

$$\begin{aligned}F_n(A_n, w_n) &\geq \int_{c_n}^{d_n} \varepsilon_n^6 \omega(\varepsilon_n)^2 w_n''(y)^2\,dy - 2\int_{A_n} |g_n(y)| \cdot |w_n(y)|\,dy \\ &\geq \frac{1}{144}\frac{\varepsilon_n^6}{(d_n - c_n)\omega(\varepsilon_n)^2} - O(A_n). \\ &= \frac{1}{144}\frac{\varepsilon_n^6}{(d_n - c_n)\omega(\varepsilon_n)^2} - O(\omega(\varepsilon_n)^4).\end{aligned}$$

Combining with (4.27) and (4.29) we deduce that

$$\begin{aligned}G_n((a_n, b_n), z_n) + O(\varepsilon_n^3\omega(\varepsilon_n)) &\geq F_n((a_n, b_n), w_n) \\ &= F_n(A_n, w_n) + F_n(A_n^c, w_n) \\ &\geq \frac{1}{144}\frac{\varepsilon_n^6}{(d_n - c_n)\omega(\varepsilon_n)^2} - O(\omega(\varepsilon_n)^4) \\ &\quad + G_n((a_n, b_n), z_n) - O(\omega(\varepsilon_n)^4) - O(\Delta_n),\end{aligned}$$

and therefore

$$\frac{\varepsilon_n^6}{O(\omega(\varepsilon_n)^6)} \leq \frac{\varepsilon_n^6}{(d_n - c_n)\omega(\varepsilon_n)^2} \leq O(\Delta_n) + O(\omega(\varepsilon_n)^4).$$

Multiplying both sides by $|\log \varepsilon_n|^3$ this relation leads to a contradiction when $n \to +\infty$, because the left-hand side has a positive liminf, while the right-hand side tends to 0 thanks to the second relation in (4.28).

Conclusion In this final paragraph of the proof we define the reminder $r_n(y)$ in such a way that $w_n(y) = z_n(y) + r_n(y)$, and we show that $r_n'(y)$ is uniformly bounded.

To begin with, from (4.26) and (4.30) we know that both $w_n'(y)$ and $z_n'(y)$ lie in the interval where $\varphi_n(\sigma)$ coincides with the convex function $\psi_n(\sigma)$, at least when n is large enough. Recalling that $\psi''(\sigma) \geq 1$ for every $\sigma \in \mathbb{R}$ we obtain that

$$\varphi_n(w_n'(y)) = \psi_n(z_n'(y) + r_n'(y)) \geq \psi_n(z_n'(y)) + \psi_n'(z_n'(y)) \cdot r_n'(y) + \frac{1}{2} r_n'(y)^2.$$

From this inequality we obtain that

$$\begin{aligned} F_n((a_n,b_n),w_n) \;\geq\; & G_n((a_n,b_n),z_n) \\ & + \int_{a_n}^{b_n} \left(\psi_n'(z_n'(y))\cdot r_n'(y) + 2\omega(\varepsilon_n)^2 z_n(y) r_n(y) - 2g_n(y) r_n(y)\right) dy \\ & + \int_{a_n}^{b_n} \varepsilon_n^6 \omega(\varepsilon_n)^2 \left(z_n''(y)^2 + 2 z_n''(y) r_n''(y)\right) dy \\ & + \int_{a_n}^{b_n} \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 r_n''(y)^2 + \frac{1}{2} r_n'(y)^2 + \omega(\varepsilon_n)^2 r_n(y)^2\right) dy. \end{aligned}$$

The second line vanishes because it is the first variation of the functional G_n in the minimum point z_n, computed with respect to the variation r_n, which is admissible because $r_n(a_n) = r_n(b_n) = 0$. For the third line we exploit the inequality

$$a^2 + 2ab \geq -\frac{b^2}{2} - a^2 \qquad \forall (a,b) \in \mathbb{R}^2,$$

and, recalling that $z_n''(y)$ is uniformly bounded by (4.26), we obtain that

$$\int_{a_n}^{b_n} \varepsilon_n^6 \omega(\varepsilon_n)^2 \left(z_n''(y)^2 + 2 z_n''(y) r_n''(y)\right) dy \geq -\frac{1}{2} \int_{a_n}^{b_n} \varepsilon_n^6 \omega(\varepsilon_n)^2 r_n''(y)^2 \, dy - O(\varepsilon_n^6 \omega(\varepsilon_n)^2).$$

Combining with (4.27) we deduce that

$$\begin{aligned} G_n((a_n,b_n),z_n) + O(\varepsilon_n^3 \omega(\varepsilon_n)) &\geq F_n((a_n,b_n),w_n) \\ &\geq G_n((a_n,b_n),z_n) + \frac{1}{2} \int_{a_n}^{b_n} \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 r_n''(y)^2 + r_n'(y)^2\right) dy - O(\varepsilon_n^6 \omega(\varepsilon_n)^2), \end{aligned}$$

and therefore

$$\int_{a_n}^{b_n} \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 r_n''(y)^2 + r_n'(y)^2\right) dy \leq O(\varepsilon_n^3 \omega(\varepsilon_n)).$$

We are now ready to conclude. Since r_n vanishes at the boundary, there exists $y_n \in (a_n, b_n)$ such that $r_n'(y_n) = 0$. From the inequality between arithmetic mean ad geometric mean we obtain that

$$\varepsilon_n^3 \omega(\varepsilon_n) \frac{d}{ds}[r_n'(s)^2] = 2\varepsilon_n^3 \omega(\varepsilon_n) r_n'(s) r_n''(s) \leq \varepsilon_n^6 \omega(\varepsilon_n)^2 r_n''(y)^2 + r_n'(s)^2,$$

and therefore

$$\begin{aligned} \varepsilon_n^3 \omega(\varepsilon_n) r_n'(y)^2 &= \varepsilon_n^3 \omega(\varepsilon_n) \left(r_n'(y)^2 - r_n'(y_n)^2\right) \\ &= \left| \int_{y_n}^{y} \varepsilon_n^3 \omega(\varepsilon_n) \frac{d}{ds}[r_n'(s)^2] \, ds \right| \\ &\leq \int_{y_n}^{y} \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 r_n''(s)^2 + r_n'(s)^2\right) ds \\ &\leq \int_{a_n}^{b_n} \left(\varepsilon_n^6 \omega(\varepsilon_n)^2 r_n''(s)^2 + r_n'(s)^2\right) ds \\ &\leq O(\varepsilon_n^3 \omega(\varepsilon_n)). \end{aligned}$$

Dividing by $\varepsilon_n^3 \omega(\varepsilon_n)$ we obtain the uniform bound on $r_n'(y)$.

5 Some lemmata

In this section we prove some lemmata that we exploited in the proof of the main results. The first one is an elementary inequality for the logarithm.

Lemma 5.1. *It turns out that*

$$\left|\log\left(1+[(1+b)x+d]^2\right)-\log\left(1+x^2\right)\right|\le 4|b|+|d| \qquad \forall (b,d,x)\in\left[-\frac{1}{2},\frac{1}{2}\right]\times\mathbb{R}^2.$$

Proof. We observe that the function $\psi(\sigma):=\log(1+\sigma^2)$ is Lipschitz continuous, with Lipschitz constant equal to 1, and in particular

$$\left|\log\left(1+[(1+b)x+d]^2\right)-\log\left(1+(1+b)^2x^2\right)\right|\le |d|,$$

so that it is enough to prove that

$$\left|\log\left(1+(1+b)^2x^2\right)-\log\left(1+x^2\right)\right|\le 4|b|. \tag{5.1}$$

To this end we observe that

$$\left|\log\left(1+(1+b)^2x^2\right)-\log\left(1+x^2\right)\right|=\left|\log\left(1+(2b+b^2)\frac{x^2}{1+x^2}\right)\right|$$

and the right-hand side is a nondecreasing function of x^2, for every fixed $b\in\mathbb{R}$. This implies that

$$\left|\log\left(1+(2b+b^2)\frac{x^2}{1+x^2}\right)\right|\le\left|\log\left(1+2b+b^2\right)\right| \qquad \forall b\neq -1 \quad \forall x\in\mathbb{R}.$$

Now we exploit the fact that $b\in[-1/2,1/2]$ and we observe that the function $\phi(\sigma):=\log(1+\sigma)$ is Lipschitz continuous on $[-1/2,1/2]$, with Lipschitz constant equal to 2, so we obtain that

$$\left|\log\left(1+2b+b^2\right)\right|=2\left|\log\left(1+b\right)\right|\le 4|b| \qquad \forall b\in\left[-\frac{1}{2},\frac{1}{2}\right].$$

This completes the proof of (5.1). □

The next lemma is a quantitative version of the subadditivity of the square root.

Lemma 5.2 (Quantitative subadditivity of the square root)**.** *Let I be a finite or countable nonempty set, and let $f: I\to[0,+\infty)$ be a function. Let us assume that the following three quantities are finite:*

$$S:=\sum_{i\in I}f(i), \qquad R:=\sum_{i\in I}\sqrt{f(i)}, \qquad M:=\max\{f(i):i\in I\}.$$

Then it turns out that

$$\sqrt{S-M}\le 3\left(R-\sqrt{S}\right). \tag{5.2}$$

Proof. We distinguish two cases.

Case 1: $M \geq S/2$ Let $i_0 \in I$ be such that $f(i_0) = M$, and let $I' := I \setminus \{i_0\}$. From the subadditivity of the square root we deduce that

$$R = \sqrt{f(i_0)} + \sum_{i \in I'} \sqrt{f(i)} \geq \sqrt{f(i_0)} + \left(\sum_{i \in I'} f(i) \right)^{1/2} = \sqrt{M} + \sqrt{S - M}.$$

Now we observe that the function $\varphi(x) := 1 + \sqrt{x} - \sqrt{1+x}$ is increasing, and therefore

$$1 + \sqrt{x} - \sqrt{1+x} \geq 2 - \sqrt{2} \geq \frac{1}{3} \qquad \forall x \geq 1.$$

Setting $x := M/(S-M)$ (which is greater than or equal to 1 in this case), from the combination of these two inequalities we conclude that

$$R - \sqrt{S} \geq \sqrt{S-M} + \sqrt{M} - \sqrt{S} \geq \frac{1}{3}\sqrt{S-M},$$

which proves (5.2) in this case.

Case 2: $M \leq S/2$ In this case there exists $I_1 \subseteq I$ such that

$$S_1 := \sum_{i \in I_1} f(i) \in \left[\frac{1}{4}S, \frac{3}{4}S \right].$$

Setting $I_2 := I \setminus I_1$, as before from the subadditivity of the square root we obtain that

$$R = \sum_{i \in I_1} \sqrt{f(i)} + \sum_{i \in I_2} \sqrt{f(i)} \geq \left(\sum_{i \in I_1} f(i) \right)^{1/2} + \left(\sum_{i \in I_2} f(i) \right)^{1/2} \geq \sqrt{S_1} + \sqrt{S - S_1}.$$

Now we observe that $\psi(x) := \sqrt{x} + \sqrt{1-x} - 1$ is a concave function with equal values in $1/4$ and $3/4$, and in particular

$$\sqrt{x} + \sqrt{1-x} - 1 \geq \frac{\sqrt{3}-1}{2} \geq \frac{1}{3} \qquad \forall x \in \left[\frac{1}{4}, \frac{3}{4} \right].$$

Setting $x := S_1/S$ (which lies in the interval $[1/4, 3/4]$), from the combination of these two inequalities we conclude that

$$R - \sqrt{S} \geq \sqrt{S_1} + \sqrt{S - S_1} - \sqrt{S} \geq \frac{1}{3}\sqrt{S} \geq \frac{1}{3}\sqrt{S-M},$$

which proves (5.2) in this case. □

Let us consider the minimum problem

$$\min \left\{ \int_a^b L(x, u(x), u'(x))\, dx : u \in H^1((a,b)),\ u(a) = A,\ u(b) = B \right\}, \tag{5.3}$$

where $(a,b) \subseteq \mathbb{R}$ is an interval, $L : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function called Lagrangian, and $(A,B) \in \mathbb{R}^2$ is a pair of Dirichlet data.

Let us assume that there exist positive real numbers $M_1, \ldots, M_9$ such that

(i) the length of the interval satisfies

$$M_1 \le b - a \le M_2,$$

(ii) There exists two functions $\psi \in C^2(\mathbb{R})$ and $f \in C^1([a,b] \times \mathbb{R})$ such that

$$L(x,s,p) = \psi(p) + f(x,s) \qquad \forall (x,s,p) \in [a,b] \times \mathbb{R}^2.$$

(iii) The functions ψ and f satisfy the following conditions

$$M_3 \le \psi''(p) \le M_4 \quad \text{and} \quad M_5 p^2 \le \psi(p) \le M_6 p^2 \qquad \forall p \in \mathbb{R},$$

$$-M_7(1+|s|) \le f(x,s) \le M_7(1+s^2) \quad \text{and} \quad |f_s(x,s)| \le M_8(1+|s|) \quad \forall (x,s) \in [a,b]\times\mathbb{R}.$$

(iv) the boundary conditions satisfy

$$(A,B) \in [-M_9, M_9]^2.$$

The first result is that minima and minimizers are bounded, and the bound depends only on the constants $M_1, \ldots, M_9$.

Lemma 5.3. *Let us consider problem (5.3) under assumptions (i)–(iv) described above.*

Let $u(x) \in H^1((a,b))$ be any minimizer. Then actually $u \in C^2([a,b])$, and there exist real numbers C_1 and C_2, depending only on $M_1, \ldots, M_9$, such that

$$\int_a^b L(x, u(x), u'(x))\, dx \le C_1,$$

and

$$|u(x)| + |u'(x)| + |u''(x)| \le C_2 \qquad \forall x \in [a,b]. \tag{5.4}$$

Proof. The energy estimate follows simply by using the affine function interpolating the boundary conditions as a competitor for the minimum problem.

The regularity and the estimate (5.4) follow from the Euler-Lagrange equation solved by minimizers. □

The second result is that minima depend in a Lipschitz way on the Dirichlet boundary conditions.

Lemma 5.4. *Let us consider problem (5.3) under assumptions (i)–(iv) described above. Let us assume in addition that*

(v) the Lagrangian is locally Lipschitz continuous with respect to the pair (s,p), and more precisely there exists a positive real number M_{10} such that

$$|L(x,s_1,p_1) - L(x,s_2,p_2)| \le M_{10}(|s_1 - s_2| + |p_1 - p_2|)$$

for every

$$(x,s_1,s_2,p_1,p_2) \in [a,b]\times[-C_2-4M_9, C_2+4M_9]^2 \times [-C_2-4M_9/M_1, C_2+4M_9/M_1]^2,$$

where C_2 is the constant in (5.4).

Then there exists a positive real number C_3, depending only on M_1, ..., M_{10}, with the following property. If $u_1(x)$ and $u_2(x)$ are minimizer with boundary data $(A_1, B_1) \in [-M_9, M_9]^2$ and $(A_2, B_2) \in [-M_9, M_9]^2$, respectively, then

$$\left| \int_a^b L(x, u_1(x), u_1'(x))\, dx - \int_a^b L(x, u_2(x), u_2'(x))\, dx \right| \le C_3(|A_1 - A_2| + |B_1 - B_2|).$$

Proof. We observe that the function

$$v(x) = u_2(x) + (A_1 - A_2)\frac{b-x}{b-a} + (B_1 - B_2)\frac{x-a}{b-a}$$

is a competitor for the minimum problem with boundary conditions (A_1, B_1), and that

$$|v(x) - u_2(x)| \le |A_1 - A_2| + |B_1 - B_2|,$$

and

$$|v'(x) - u_2'(x)| \le \frac{|A_1 - A_2|}{b-a} + \frac{|B_1 - B_2|}{b-a}.$$

Therefore from the minimality of u_1 and assumptions (i) and (v) we deduce that

$$\begin{aligned} \int_a^b L(x, u_1(x), u_1'(x))\, dx &\le \int_a^b L(x, v(x), v'(x))\, dx \\ &\le \int_a^b L(x, u_2(x), u_2'(x))\, dx \\ &\quad + M_{10}\left(1 + \frac{1}{M_1}\right)(|A_1 - A_2| + |B_1 - B_2|). \end{aligned}$$

The remaining inequality can be proved by exchanging the role of u_1 and u_2. □

Then next lemma provides the dependence on the boundary conditions of the minimum value of the L^2 norm of the second derivative (see [15, Lemma 6.2]).

Lemma 5.5. *Let $(a, b) \subseteq \mathbb{R}$ be an interval, and let A_0, A_1, B_0, B_1 be four real numbers. Let us consider the minimum problem*

$$\min\left\{ \int_a^b w''(y)^2\, dy : w \in H^2((a,b)),\ \big(w(a), w'(a), w(b), w'(b)\big) = (A_0, A_1, B_0, B_1) \right\}.$$

Then the unique minimum point is the function

$$w_0(y) = P\left(y - \frac{a+b}{2}\right),$$

where $P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ is the polynomial of degree three with coefficients

$$c_0 := \frac{A_0 + B_0}{2} - \frac{B_1 - A_1}{8}(b-a), \qquad c_1 := \frac{3(B_0 - A_0)}{2(b-a)} - \frac{A_1 + B_1}{4},$$

$$c_2 := \frac{B_1 - A_1}{2(b-a)}, \qquad c_3 := -\frac{2(B_0 - A_0)}{(b-a)^3} + \frac{A_1 + B_1}{(b-a)^2}.$$

As a consequence, the minimum value is

$$\frac{(B_1 - A_1)^2}{b-a} + \frac{12}{(b-a)^3}\left[(B_0 - A_0) - \frac{A_1 + B_1}{2}(b-a)\right]^2,$$

and the minimum point satisfies the pointwise estimates

$$|w_0(y)| \le \frac{3(|A_0| + |B_0|)}{2} + \frac{|A_1| + |B_1|}{2}(b-a) \qquad \forall y \in [a,b],$$

and

$$|w_0'(y)| \le \frac{3|B_0 - A_0|}{b-a} + \frac{3(|A_1| + |B_1|)}{2} \qquad \forall y \in [a,b].$$

Finally, we prove two estimates from below for rescaled versions of the singularly perturbed Perona-Malik functionals.

Lemma 5.6. *Let $\alpha, \beta, \gamma, M > 0$ be positive real numbers and let us consider the functional*

$$\mathcal{F}_{\alpha,\beta,\gamma}(\Omega, u) := \int_\Omega \left(\alpha u''(x) + \beta \log(1 + \gamma u'(x)^2)\right)\, dx,$$

defined for every open set $\Omega \subset \mathbb{R}$ and every $u \in H^2(\Omega)$.

Let $(a,b) \subset \mathbb{R}$ be an interval and let $w \in H^2(a,b)$ be a function such that $|w'(a)| \le M$ and $|w'(b)| \le M$. Let us consider the set

$$A_M := \{x \in (a,b) : |w'(x)| > M\},$$

and let us set

$$\Delta_M := \int_{A_M} |w'(x)|\, dx.$$

Then it turns out that

$$\mathcal{F}_{\alpha,\beta,\gamma}(A_M, w) \ge 4\sqrt{\frac{2}{3}}\alpha^{1/4}\beta^{3/4}(\log(1 + \gamma M^2))^{3/4}\sqrt{\Delta_M - M|A_M|} \tag{5.5}$$

Proof. Since A_M is an open set, we can write it as a finite or countable union of open disjoint intervals (its connected components), namely in the form

$$A_M = \bigcup_{i \in I} (a_i, b_i),$$

where I is a suitable index set.

We observe that w' has constant sign in each of the intervals (a_i, b_i), and therefore

$$\Delta_M = \sum_{i \in I} \int_{a_i}^{b_i} |w'(x)|\, dx = \sum_{i \in I} |u(b_i) - u(a_i)|$$

Moreover, we have that $|w'(a_i)| = |w'(b_i)| = M$ for every $i \in I$, because w' is continuous and all the points a_i and b_i are internal to (a,b), since $|w'(a)| < M$ and $|w'(b)| < M$.

As a consequence, from Lemma 5.5 we deduce that

$$\int_{a_i}^{b_i} w''(x)^2\,dx \geq \frac{12}{(b_i-a_i)^3}(|u(b_i)-u(a_i)|-M(b_i-a_i))^2,$$

and hence

$$\begin{aligned}\mathcal{F}_{\alpha,\beta,\gamma}((a_i,b_i),w) &\geq \frac{12\alpha}{(b_i-a_i)^3}(|u(b_i)-u(a_i)|-M(b_i-a_i))^2+\beta\log(1+\gamma M^2)(b_i-a_i)\\ &\geq 4\sqrt{\frac{2}{3}}\alpha^{1/4}\beta^{3/4}(\log(1+\gamma M^2))^{3/4}\sqrt{|u(b_i)-u(a_i)|-M(b_i-a_i)},\end{aligned}$$

where the last inequality follows from the inequality

$$a+b\geq\frac{4}{3^{3/4}}(ab^3)^{1/4}\qquad\forall(a,b)\in[0,+\infty)^2.$$

Summing over i and exploiting the subadditivity of the square root we obtain (5.5). □

Lemma 5.7. *Let (a_n,b_n) be a sequence of intervals such that $b_n-a_n\to L$, for some positive number $L>0$. Let also $\{\delta_n\}\subset(0,+\infty)$ be a sequence of positive numbers such that $\delta_n\to 0$ and $\{w_n\}$ be a sequence of functions such that $w_n\in H^2((a_n,b_n))$ and*

$$\liminf_{n\to+\infty}\operatorname{osc}(w_n,(a_n,b_n))\geq J>0.$$

Let us consider the family of functionals

$$\mathbb{RPMMH}_\delta(\Omega,u):=\int_\Omega\left\{\frac{\delta^6}{|\log\delta|^3}u''(x)^2+\frac{1}{\delta^2}\log\left(1+\delta^2u'(x)^2\right)\right\}dx,\tag{5.6}$$

defined for every positive number $\delta>0$, for every open set $\Omega\subseteq\mathbb{R}$ and every function $u\in H^2(\Omega)$, and let us assume that

$$|w'(a_n)|\leq\frac{1}{\delta_n}\qquad\text{and}\qquad|w'(b_n)|\leq\frac{1}{\delta_n},$$

for every $n\in\mathbb{N}$.

Then there exists a positive constant $c(L,J)$ depending only on L and J such that

$$\liminf_{n\to+\infty}\mathbb{RPMMH}_{\delta_n}((a_n,b_n),w_n)\geq c(L,J).$$

Proof. First of all, up to reducing ourselves to a subsequence, we can assume that

$$\liminf_{n\to+\infty}\mathbb{RPMMH}_{\delta_n}((a_n,b_n),w_n)=\lim_{n\to+\infty}\mathbb{RPMMH}_{\delta_n}((a_n,b_n),w_n)<+\infty.$$

For every $n\in\mathbb{N}$ let us set

$$A_n:=\left\{x\in(a_n,b_n):|w_n'(x)|>\frac{1}{\delta_n^2|\log\delta_n|^3}\right\},\qquad\Delta_n^A:=\int_{A_n}|w_n'(x)|\,dx,$$

$$B_n:=\left\{x\in(a_n,b_n):\frac{1}{\delta_n|\log\delta_n|}\leq|w_n'(x)|\leq\frac{1}{\delta_n^2|\log\delta_n|^3}\right\},$$

$$C_n:=\left\{x\in(a_n,b_n):|w_n'(x)|<\frac{1}{\delta_n|\log\delta_n|}\right\},\qquad\Delta_n^C:=\int_{C_n}|w_n'(x)|\,dx.$$

We observe that

$$|B_n| \le \frac{\delta_n^2}{\log(1+|\log\delta_n|^{-2})}\,\mathbb{RPMMH}_{\delta_n}((a_n,b_n),w_n),$$

and therefore

$$\limsup_{n\to+\infty}\int_{B_n}|w_n'(x)|\,dx \le \frac{|B_n|}{\delta_n^2|\log\delta_n|^3} \le \limsup_{n\to+\infty}\frac{\mathbb{RPMMH}_{\delta_n}((a_n,b_n),w_n)}{|\log\delta_n|^3\log(1+|\log\delta_n|^{-2})} = 0,$$

namely the contribution of B_n to the total variation of w_n is asymptotically negligible.

As a consequence we obtain that

$$\liminf_{n\to+\infty}\Delta_n^A+\Delta_n^C = \liminf_{n\to+\infty}\int_{a_n}^{b_n}|w_n'(x)|\,dx \ge J. \tag{5.7}$$

Concerning A_n, we observe that

$$\limsup_{n\to+\infty}\frac{|A_n|}{\delta_n^2|\log\delta_n|^3} \le \limsup_{n\to+\infty}\frac{\mathbb{RPMMH}_{\delta_n}((a_n,b_n),w_n)}{|\log\delta_n|^3\log(1+\delta_n^{-2}|\log\delta_n|^{-6})} = 0,$$

and that Lemma 5.6 yields

$$\mathbb{RPMMH}_{\delta_n}(A_n,w_n) \ge 4\sqrt{\frac{2}{3}}\left(\frac{1}{|\log\delta_n|}\log\left(1+\frac{1}{\delta_n^2|\log\delta_n|^6}\right)\right)^{3/4}\sqrt{\Delta_n^A-\frac{|A_n|}{\delta_n^2|\log\delta_n|^3}},$$

at least when n is large enough so that $1/\delta_n \le 1/(\delta_n^2|\log\delta_n|^3)$.

Hence it holds that

$$\liminf_{n\to+\infty}\mathbb{RPMMH}_{\delta_n}(A_n,w_n) \ge 8\frac{2^{1/4}}{3^{1/2}}\cdot\liminf_{n\to+\infty}\sqrt{\Delta_n^A}. \tag{5.8}$$

As for C_n, we observe that

$$\begin{aligned}
\liminf_{n\to+\infty}\mathbb{RPMMH}_{\delta_n}(C_n,w_n) &\ge \liminf_{n\to+\infty}\int_{C_n}\frac{\log(1+\delta_n^2w_n'(x)^2)}{\delta_n^2w_n'(x)^2}w_n'(x)^2\,dx\\
&\ge \liminf_{n\to+\infty}|\log\delta_n|^2\log(1+|\log\delta_n|^{-2})\int_{C_n}w_n'(x)^2\,dx\\
&= \liminf_{n\to+\infty}\int_{C_n}w_n'(x)^2\,dx\\
&\ge \liminf_{n\to+\infty}\frac{(\Delta_n^C)^2}{|C_n|}\\
&= \liminf_{n\to+\infty}\frac{(\Delta_n^C)^2}{L},
\end{aligned} \tag{5.9}$$

where the second inequality follows form the fact that the function $t\mapsto\log(1+t)/t$ is nonincreasing on $(0,+\infty)$.

Combining (5.7), (5.8) and (5.9) we obtain that

$$\liminf_{n\to+\infty} \mathbb{RPMMH}_{\delta_n}((a_n,b_n),w_n) \geq \liminf_{n\to+\infty} 8\frac{2^{1/4}}{3^{1/2}}\cdot\sqrt{\Delta_n^A}+\frac{(\Delta_n^C)^2}{L}$$
$$\geq \min\left\{8\frac{2^{1/4}}{3^{1/2}}\sqrt{A}+\frac{C^2}{L} : (A,C)\in[0,+\infty)^2,\ A+C\geq J\right\}.$$

We conclude by remarking that

$$c(L,J) := \min\left\{8\frac{2^{1/4}}{3^{1/2}}\sqrt{A}+\frac{C^2}{L} : (A,C)\in[0,+\infty)^2,\ A+C\geq J\right\} > 0$$

for every $(L,J)\in(0,+\infty)^2$. □

Acknowledgments

The authors are members of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA) of the "Istituto Nazionale di Alta Matematica" (INdAM).

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