



Bifurcation of resonances in a ring model

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Abstract

We consider a particle in a ring system around a primary body, which is modeled as an ellipsoid. We study the occurrence of bifurcations associated to resonances, which arise at different values of the orbital eccentricity of the particle. The analysis of the bifurcations is first conducted by establishing theoretical conditions ensuring the existence of equilibria. Then, each specific resonance is examined through the study of the associated linearized system; their stability is obtained by computing the eigenvalues of the corresponding Jacobian matrix. Our results reveal a variety of dynamical behaviours characterizing the resonances, including pitchfork, saddle-node and transcritical bifurcations.

Keywords Bifurcation · Resonance · Normal form · Ring model

Mathematics Subject Classification 70F15 · 37N05 · 34C23

1 Introduction

Within the Solar system, it has recently been established that ring structures are not confined to the giant planets (Jupiter, Saturn, Uranus, Neptune), but are also present around small bodies, specifically Chariklo, Haumea, and Quaoar ([1–3]). The main difference between ring systems around planets and those around minor bodies lies in

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the shape of the central object: the former are nearly spherical, whereas the latter exhibit highly irregular shapes. This irregularity significantly complicates the construction of models describing the dynamics of particles in such ring systems, as it needs the explicit consideration of a central body with a complex, non-spherical geometry. At the same time, the departure from spherical symmetry introduces a variety of non-trivial dynamical effects, making the ring systems particularly rich laboratories for investigating diverse dynamical phenomena (see, e.g., [4–9]).

An important aspect to be investigated from a dynamical perspective is the occurrence of resonances. More specifically, a planetary ring can be modeled as a set of massless test particles orbiting a primary body, whose figure is approximated by a triaxial ellipsoid. The primary is therefore characterized by two shape parameters, the oblateness and the elongation, both expressed as functions of the ellipsoid's semi-axes. Throughout this analysis, we assume that a generic ring particle moves within the equatorial plane of the ellipsoid. A corotation resonance occurs when the mean motion of an orbiting particle becomes equal to the rotational frequency of the central body. Lindblad resonances are induced by a commensurability between two fundamental frequencies that characterize the particle's motion: the epicyclic frequency and the synodic frequency ([10]). Both corotation and Lindblad resonances thus establish a direct coupling between the orbital dynamics of the particle and the rotational dynamics of the central object.

Resonances play a fundamental role in the dynamical structure of planetary rings. Some resonant locations are largely devoid of material; a classical example is the Cassini Division within Saturn's ring system. Other resonances, by contrast, are significantly populated by ring particles. Understanding the physical mechanisms that lead to the depletion or accumulation of material at specific resonances remains one of the central open problems in the study of ring dynamics.

In this work, we investigate the emergence of bifurcations associated to the main resonances; bifurcations occur at different values of the orbital eccentricity of a particle orbiting a primary body. The bifurcation analysis is first conducted by establishing theoretical conditions that ensure the existence of equilibrium points; these conditions, in turn, yield parameter values at which bifurcations arise. Subsequently, each specific resonance is examined by identifying bifurcations through the study of the associated linearized system and the computation of the eigenvalues of the corresponding Jacobian matrix (see also [4, 5]). This approach enables us to characterize the stability of the equilibria as a function of the eccentricity. Our results reveal a variety of dynamical behaviours distinguishing different resonances, including pitchfork, saddle-node, and transcritical bifurcations. Overall, the analysis highlights that the orbital eccentricity plays a crucial role in shaping the dynamical behaviour of the resonances and must be carefully accounted for in studies of the existence, persistence, and bifurcation of equilibrium configurations.

This work is organized as follows. The model is introduced in Sect. 2 together with the corresponding Hamiltonian and the definition of corotation and Lindblad resonances. Analytical conditions on the existence of equilibria and, hence, of bifurcations are given in Sect. 3. A numerical investigation of the stability of the equilibria for several resonances is presented in Sect. 4.

2 The model around an ellipsoid

We consider a body, say $\mathcal{P}$, with mass M_P and the shape of an ellipsoid with symmetric mass distribution with respect to the equatorial plane; we denote by O the center of mass and by $a > b > c > 0$ the semi-axes of the ellipsoid. We assume that the ellipsoid rotates about the shortest physical axis, belonging to the symmetry plane, and we denote by Ω_P the angular velocity, while T_{rot} is the rotation period. Let $\mathcal{S}$ be a massless particle, which moves in the equatorial plane around $\mathcal{P}$ with its position given by polar coordinates (r, L) in an inertial frame centered in O ; hence, r is the distance from the origin and L denotes the true longitude with respect to a fixed direction. In a synodic frame rotating with angular velocity Ω_P , we introduce the angle $\theta = L - \Omega_P t$. The potential describing the attraction of $\mathcal{P}$ on $\mathcal{S}$ is given by (see [11])

$$U(r, \theta) = -\frac{\mathcal{G}M_P}{r} \sum_{p=0}^{\infty} \cos(2p\theta) \sum_{\ell=p}^{\infty} \left(\frac{R}{r}\right)^{2\ell} C_{2\ell,2p} P_{2\ell,2p}(0), \quad (1)$$

where $\mathcal{G}$ is the gravitational constant, the coefficients $C_{2\ell,2p}$ are given by

$$C_{2\ell,2p} = \frac{3}{R^{2\ell}} \frac{\ell!(2\ell-2p)!}{2^{2p}(2\ell+3)(2\ell+1)!} (2 - \delta_{0,p}) \times \sum_{k=0}^{\lfloor \frac{\ell-p}{2} \rfloor} \frac{(a^2 - b^2)^{p+2k} (c^2 - \frac{1}{2}(a^2 + b^2))^{\ell-p-2k}}{16^k (\ell-p-2k)!(p+k)!k!} \quad (2)$$

and the numbers $P_{2\ell,2p}(0)$ are the values at $z = 0$ of the associated Legendre polynomials $P_{2\ell,2p}(z)$:

$$P_{2\ell,2p}(0) = (-1)^{\ell-p} \frac{(2\ell+2p)!}{2^{2\ell}(\ell+p)!(\ell-p)!}, \quad \ell \geq p. \quad (3)$$

As remarked in [12], the series in (1) is uniformly convergent under the following condition on the semi-axis: $c > a/\sqrt{2}$. Finally, we introduce the quantities defining the *oblateness* and *elongation* of the ellipsoid:

$$\mathcal{O}_b = \frac{a^2 + b^2 - 2c^2}{4R^2}, \quad \mathcal{E}_\ell = \frac{a^2 - b^2}{2R^2}, \quad (4)$$

where R is the reference radius defined as

$$R = \left(\frac{3}{a^{-2} + b^{-2} + c^{-2}} \right)^{\frac{1}{2}}.$$

We denote by p_r, p_θ the momenta conjugated to r, θ : $p_r = \dot{r}$, $p_\theta = r^2(\dot{\theta} + \Omega_P)$. We can now write the Hamiltonian as

$$\mathcal{H}(p_r, p_\theta, r, \theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \Omega_P p_\theta + U(r, \theta), \quad (5)$$

where the gravitational potential can be split as the sum of an axisymmetric and non-axisymmetric parts:

$$\begin{aligned} U_s(r) &= -\frac{\mathcal{G}M_P}{r} \sum_{\ell=0}^{\infty} \left(\frac{R}{r}\right)^{2\ell} C_{2\ell,0} P_{2\ell,0}(0) \\ U_{ns}(r, \theta) &= -\frac{\mathcal{G}M_P}{r} \sum_{\ell=1}^{\infty} \left(\frac{R}{r}\right)^{2\ell} \sum_{p=1}^{\ell} \cos(2p\theta) C_{2\ell,2p} P_{2\ell,2p}(0). \end{aligned} \quad (6)$$

Next, we introduce the mean motion n as

$$n(r) = \left(\frac{1}{r} \frac{dU_s(r)}{dr}\right)^{\frac{1}{2}}, \quad (7)$$

and the epicyclic frequency κ as

$$\kappa(r) = \left(\frac{3}{r} \frac{dU_s(r)}{dr} + \frac{d^2 U_s(r)}{dr^2}\right)^{\frac{1}{2}}. \quad (8)$$

We also introduce the synodic frequency as $n - \Omega_P$. These quantities allow us to introduce the definition of resonances as follows.

Definition 1 A Lindblad resonance is defined by the equation

$$j\kappa(r) = m(n(r) - \Omega_P) \quad (9)$$

for $j, m \in \mathbb{Z} \setminus \{0\}$. Equation (9) defines the Lindblad resonant radius, say $r_{lin} = r_{lin}(\Omega_P; m, j)$.

A corotation resonance is defined by the equation (9) for $j = 0$, which implies that the synodic frequency is zero: $n(r) - \Omega_P = 0$; the solution of such equation gives the corotation radius $r_{cor} = r_{cor}(\Omega_P)$.

Close to a resonance, we introduce the quantities $\rho = r - r_*$ with r_* the resonant radius, and $I = p_\theta - p_*$ with $p_* = n_* r_*^2$ and $n_* = n(r_*)$. We also introduce $p_\rho = p_r$ the momentum conjugated to ρ and set $\kappa_* = \kappa(r_*)$. Next, we compute the Taylor expansion of the Hamiltonian (5) around $I = 0$ and $\rho = 0$; finally, we introduce the epicyclic action-angle variables (J, φ) related to (ρ, p_ρ) by means of the canonical transformation:

$$\begin{cases} \rho = \sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi \\ p_\rho = \sqrt{2|\kappa_*|J} \cos \varphi. \end{cases} \quad (10)$$

Then, the Hamiltonian (5) is transformed as

$$\mathcal{H}(J, I, \varphi, \theta) = |\kappa_*|J + (n_* - \Omega_P)I + \tilde{F}_s(J, I, \varphi) + \tilde{F}_{ns}(J, \varphi, \theta), \quad (11)$$

where $\tilde{F}_s$ is the axisymmetric part of the potential:

$$\begin{aligned} \tilde{F}_s(J, I, \varphi) &= -2\frac{n_*}{r_*}I\sqrt{\frac{2J}{|\kappa_*|}}\sin\varphi + \frac{I^2}{2r_*^2} + F_0\left(I, \sqrt{\frac{2J}{|\kappa_*|}}\sin\varphi\right) \\ &= -2\frac{n_*}{r_*}I\sqrt{\frac{2J}{|\kappa_*|}}\sin\varphi + \frac{I^2}{2r_*^2} + \sum_{i=0}^2 \sum_{j=3-i}^{\infty} d_{i,j} I^i J^{j/2} \sin^j\varphi \end{aligned} \quad (12)$$

with

$$d_{i,j} = c_{i,j} \left(\frac{2}{|\kappa_*|}\right)^{j/2}, \quad (13)$$

and where $\tilde{F}_{ns}$ is the non-axisymmetric part:

$$\begin{aligned} \tilde{F}_{ns}(J, \varphi, \theta) &= U_{ns}\left(\sqrt{\frac{2J}{|\kappa_*|}}\sin\varphi + r_*, \theta\right) \\ &= \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \tilde{V}_{j,i} J^{j/2} \cos(2i\theta) \sin^j\varphi \end{aligned} \quad (14)$$

with

$$\tilde{V}_{j,i} = V_{j,i} \left(\frac{2}{|\kappa_*|}\right)^{j/2} \quad (15)$$

(see [13] for the explicit expressions of the coefficients $c_{i,j}$ and $V_{j,i}$).

It is convenient to write the sum of (12) and (14) as

$$\begin{aligned} &\tilde{F}_s(J, I, \varphi) + \tilde{F}_{ns}(J, \varphi, \theta) \\ &= \sum_{|i|=0}^{\infty} \sum_{j=0}^{\infty} (\alpha_{2i,j}(J, I) \cos(2i\theta + j\varphi) + \beta_{2i,j}(J, I) \sin(2i\theta + j\varphi)), \end{aligned} \quad (16)$$

where explicit expressions for $\alpha_{2i,j}$, $\beta_{2i,j}$ are given in Appendix A for $i = 0, 1$ and $j = 0, 1, 2$. In general, $\alpha_{2i,j} = \alpha_{-2i,j}$, $\beta_{2i,j} = \beta_{-2i,j}$; $\alpha_{2i,j} = 0$ if j is odd and $\beta_{2i,j} = 0$ if j is even (included zero). According to Definition 1, we have that

- (i) a corotation or 1 : 1 resonance is obtained for $j = 0$ and any i ;
- (ii) a Lindblad resonance of order $m : (m - j)$ is obtained for $2i = -m$ and any j .

3 Existence and bifurcation of resonances

Let $n_1, n_2 \in \mathbb{Z}$ be coprime; we consider a resonance of order (n_1, n_2) , corresponding to the resonance degree $n_1/(n_1 - n_2)$ and to the resonant angle $-n_1\theta + n_2\varphi$. We denote by r_* the resonant radius, which is the solution of the equation

$$n_1(n(r) - \Omega_P) = n_2\kappa(r). \quad (17)$$

We denote by $\mathcal{H}_{res}$ the resonant Hamiltonian, that we write as

$$\mathcal{H}_{res}(J, I, \varphi, \theta) = Z(J, I) + \mathcal{R}(J, I, -n_1\theta + n_2\varphi), \quad (18)$$

where $Z = Z(J, I)$ denotes the integrable part depending just on the actions and $\mathcal{R}$ is the resonant part, depending also on the resonant angle. Let us expand $\mathcal{R}$ in Fourier series; according to the nature of the coefficients $\alpha_{2i,i}$, $\beta_{2i,j}$ (see Appendix A), we distinguish the following cases, where we denote by N the order of the trigonometric approximation of the function $\mathcal{R}$:

(i) if n_1 is even and n_2 is odd, then we write the Hamiltonian in (18) as

$$\begin{aligned} \mathcal{H}_{res}(J, I, \varphi, \theta) = Z(J, I) + \sum_{k=1}^{N/2} \{ & R_{2k-1}(\sqrt{J}, I) \sin[(2k-1)(-n_1\theta + n_2\varphi)] \\ & + R_{2k}(J, I) \cos[2k(-n_1\theta + n_2\varphi)] \} \end{aligned}$$

for suitable polynomials R_{2k-1} , R_{2k} which are, respectively, functions of $(\sqrt{J}, I)$, (J, I) ;

(ii) if n_1 is odd and n_2 is even or odd, or otherwise if n_1 and n_2 are both even, then we can write

$$\mathcal{H}_{res}(J, I, \varphi, \theta) = Z(J, I) + \sum_{k=1}^{N/2} R_{2k}(J, I) \cos[2k(-n_1\theta + n_2\varphi)]$$

for suitable polynomials $R_{2k} = R_{2k}(J, I)$.

Next, we make the change of coordinates

$$(J, I, \varphi, \theta) \rightarrow (G, L, \psi, \mu)$$

with

$$\psi = -n_1\theta + n_2\varphi$$

the resonant angle and G, L, μ given by

$$\begin{aligned} G &= aJ + bI \\ L &= cJ + dI \end{aligned}$$

$$\mu = f\varphi + g\theta, \quad (19)$$

where a, b, c, d, e, f, g are real numbers; we choose $f = 1, g = -1$, and determine a, b, c, d , so that the transformation of coordinates is canonical, namely

$$Gd\psi + Ld\mu = Jd\varphi + Id\theta.$$

The transformed Hamiltonian, say $\tilde{\mathcal{H}}_{res}$, in terms of the new variables takes the form

$$\tilde{\mathcal{H}}_{res}(G, L, \psi) = \tilde{Z}(G, L) + \tilde{\mathcal{R}}(G, L, \psi), \quad (20)$$

where $\tilde{Z}, \tilde{\mathcal{R}}$ are the transformed normal form and resonant part. From (20) we obtain that $\dot{L} = 0$, hence L is constant.

Now, we express L, G in terms of the eccentricity as follows. Using the expressions

$$\rho = r_*e, \quad p_* = n(r_*)r_*^2, \quad p_\theta = (\rho + r_*)^2n(\rho + r_*),$$

we obtain that the displacement $I_0 = p_\theta - p_*$ is a function of the eccentricity, say

$$I_0 = I_0(e). \quad (21)$$

Similarly, using $p_\rho = \rho n(r_*)$ and

$$J_0 = \frac{p_\rho^2}{2\kappa(r_*)} + \frac{\kappa(r_*)\rho^2}{2},$$

we obtain that J_0 is a function of the eccentricity, say

$$J_0 = J_0(e). \quad (22)$$

From (19), (21), (22), we get $G_0 = aJ_0 + bI_0, L_0 = cJ_0 + dI_0$ as a function of the eccentricity, say

$$L_0 = L_0(e), \quad G_0 = G_0(e). \quad (23)$$

Hamilton's equations associated to (20) are

$$\begin{aligned} \dot{G} &= -\frac{\partial \tilde{\mathcal{R}}(G, L, \psi)}{\partial \psi} \\ \dot{\psi} &= \frac{\partial \tilde{Z}(G, L)}{\partial G} + \frac{\partial \tilde{\mathcal{R}}(G, L, \psi)}{\partial G}. \end{aligned} \quad (24)$$

Correspondingly, under the coordinate change (19), we have that the Hamiltonian (20) becomes

$$\begin{aligned}\tilde{\mathcal{H}}_{res}(G, L, \psi) = \tilde{Z}(G, L) + \sum_{k=1}^{N/2} \{ \tilde{R}_{2k-1}(G, L) \sin[(2k-1)\psi] \\ + \tilde{R}_{2k}(G, L) \cos[2k\psi] \}\end{aligned}\quad (25)$$

for n_1 even, n_2 odd, and

$$\tilde{\mathcal{H}}_{res}(G, L, \psi) = \tilde{Z}(G, L) + \sum_{k=1}^{N/2} \tilde{R}_{2k}(G, L) \cos(2k\psi) \quad (26)$$

for n_1 odd, n_2 even or odd, or otherwise n_1 and n_2 both even. Accordingly, the equations of motion are given by

$$\begin{aligned}\dot{G} &= -\frac{\partial \tilde{\mathcal{H}}_{res}(G, L, \psi)}{\partial \psi} \\ \dot{\psi} &= \frac{\partial \tilde{\mathcal{H}}_{res}(G, L, \psi)}{\partial G}.\end{aligned}$$

We notice that, when n_1 is even and n_2 odd, the derivative appearing in $\dot{\psi}$, $\frac{\partial \tilde{R}_{2k-1}(G, L)}{\partial G}$, contains divisors generated by the derivative of $J = J(G, L)$ with respect to G , since

$$J = \frac{dG - bL}{ad - bc}$$

appears with exponent $1/2$ in the resonant Hamiltonian due to the presence of the coefficients $\beta_{2i,j}$ in the polynomials $\tilde{R}_{2k-1}(G, L)$ appearing in (25). Before giving the domain of existence of the equilibria, we need the following result.

Lemma 2 *For all $k \geq 1$, the following relations hold:*

$$\begin{aligned}\cos((2k-1)\psi) &= \cos \psi \, A_k(\sin \psi, \cos \psi) \\ \sin(2k\psi) &= \cos \psi \, B_k(\sin \psi, \cos \psi)\end{aligned}\quad (27)$$

for suitable functions A_k, B_k .

Proof We proceed by induction on k . For $k = 1$, we have that the first in (27) holds with $A_1 = 1$ and $B_1 = 2 \sin \psi$. Assuming (27) true for any $j < k$, we have:

$$\begin{aligned}\cos((2k-1)\psi) &= \cos((2k-2)\psi) \cos \psi - \sin((2k-2)\psi) \sin \psi \\ &= \cos((2k-2)\psi) \cos \psi - \cos \psi \, B_{k-1}(\sin \psi, \cos \psi) \sin \psi \\ &= \cos \psi \, A_k(\sin \psi, \cos \psi)\end{aligned}$$

with

$$A_k(\sin \psi, \cos \psi) = \cos((2k-2)\psi) - B_{k-1}(\sin \psi, \cos \psi) \sin \psi.$$

Similarly, for the second equation in (27) we have:

$$\sin(2k\psi) = 2 \sin(k\psi) \cos(k\psi) ,$$

so that for k even:

$$\sin(2k\psi) = 2 \cos \psi \cos(k\psi) B_{\frac{k}{2}}(\sin \psi, \cos \psi)$$

and for k odd:

$$\sin(2k\psi) = 2 \cos \psi \sin(k\psi) A_{\frac{k+1}{2}}(\sin \psi, \cos \psi) ;$$

we conclude that

$$\sin(2k\psi) = \cos(k\psi) B_k(\sin \psi, \cos \psi)$$

with

$$B_k(\sin \psi, \cos \psi) = 2 \cos(k\psi) B_{\frac{k}{2}}(\sin \psi, \cos \psi)$$

for k odd and

$$B_k(\sin \psi, \cos \psi) = 2 \sin(k\psi) A_{\frac{k+1}{2}}(\sin \psi, \cos \psi)$$

for k even. $\square$

To compute the equilibrium points, we start by solving the equation $\dot{G} = 0$, namely $\partial \tilde{\mathcal{H}}_{res} / \partial \psi = 0$. For n_1 odd, we denote by $\psi = \psi_{eq}$ the solution of $\dot{G} = 0$, which amounts to solve $\sin(k_{min} \psi) = 0$, where k_{min} is the smallest index such that $\tilde{R}_{k_{min}} \neq 0$. We denote by $G = G_{eq}$ the solution of $\dot{\psi} = 0$, computed for $\psi = \psi_{eq}$ and taking into account that L is constant.

Proposition 3 *The values of the eccentricity at which an equilibrium point associated to the resonance of order (n_1, n_2) , with n_1 odd and n_2 even or odd, or otherwise both n_1 and n_2 even, bifurcates are solutions of the equation*

$$\frac{dG_{eq}(e) - bL_0(e)}{ad - bc} \geq 0 , \quad (28)$$

where a, b, c, d are determined as in (19), $G_{eq} = G_{eq}(e)$ is the equilibrium solution, depending on the eccentricity as in (23) and $L_0(e)$ is as in (23).

Proof Reminding that ρ is related to J by

$$\rho = \sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi ,$$

we need to have $J \geq 0$. Let us write the first two equations in (19) as

$$\begin{pmatrix} G \\ L \end{pmatrix} = M \begin{pmatrix} J \\ I \end{pmatrix} ,$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, we have

$$\begin{pmatrix} J \\ I \end{pmatrix} = M^{-1} \begin{pmatrix} G \\ L \end{pmatrix},$$

which provides

$$J = \frac{1}{ad - bc}(dG - bL).$$

Hence, the requirement $J \geq 0$ is equivalent to (28). $\square$

Let us now consider the case n_1 even and n_2 odd; then, the solutions of $\cos \psi = 0$ are equilibria, which give $\psi = \pi/2$, $\psi = 3\pi/2$. Next, we solve the equation $\dot{\psi} = 0$, which corresponds to find the solution of

$$\begin{aligned} \frac{\partial \tilde{Z}(G, L)}{\partial G} + \sum_{k=1}^{N/2} \left\{ \frac{\partial \tilde{R}_{2k-1}(G, L)}{\partial G} \sin((2k-1)\psi) \right. \\ \left. + \frac{\partial \tilde{R}_{2k}(G, L)}{\partial G} \cos(2k\psi) \right\} = 0. \end{aligned} \quad (29)$$

We notice that for $\psi = \pi/2$:

$$\cos(2k\psi) = (-1)^k, \quad \sin((2k-1)\psi) = (-1)^{k+1},$$

while for $\psi = 3\pi/2$:

$$\cos(2k\psi) = (-1)^k, \quad \sin((2k-1)\psi) = (-1)^k.$$

Hence, from (29) we have:

$$\frac{\partial \tilde{Z}(G, L)}{\partial G} + \sum_{k=1}^{N/2} \left\{ \frac{\partial \tilde{R}_{2k-1}(G, L)}{\partial G} (-1)^K + \frac{\partial \tilde{R}_{2k}(G, L)}{\partial G} (-1)^k \right\} = 0$$

with $K = k + 1$ for $\psi = \pi/2$ and $K = k$ for $\psi = 3\pi/2$.

Proposition 4 *The values of the eccentricity at which an equilibrium point associated to the resonance of order (n_1, n_2) , with n_1 even and n_2 odd, bifurcates are given by the solutions of*

$$\frac{dG_{eq}(e) - bL_0(e)}{ad - bc} > 0, \quad (30)$$

where a, b, c, d are determined as in (19), $G_{eq} = G_{eq}(e)$ is the equilibrium solution of (30) and $L_0(e)$ is defined in (23).

Proof Let $Q = Q(G, L)$ be the function

$$Q(G, L) = \sum_{k=1}^{N/2} \left\{ \frac{\partial \tilde{R}_{2k-1}(G, L)}{\partial G} (-1)^K + \frac{\partial \tilde{R}_{2k}(G, L)}{\partial G} (-1)^k \right\},$$

which is a rational function with the numerator given by a polynomial of G , L , $\sqrt{J(G, L)}$ and denominator $\sqrt{J(G, L)}$. Computing (30) at $L_0 = L_0(e)$, we denote by $G_{eq} = G_{eq}(e)$ the solution of

$$\frac{\partial \tilde{Z}(G_{eq}(e), L_0(e))}{\partial G} + Q(G_{eq}(e), L_0(e)) = 0.$$

Hence, the existence condition $J \geq 0$, together with the fact that $\sqrt{J}$ appears at the denominator, implies that one needs to require $J > 0$, namely, condition (30). $\square$

As an example, let us consider the resonance of order $(n_1, n_2) = (-1, 1)$ corresponding to the 1:2 resonance. The resonant angle is $\psi = \varphi + \theta$ and the transformation of coordinates (19) becomes

$$\begin{aligned} G &= \frac{1}{2}J + \frac{1}{2}I \\ L &= \frac{1}{2}J - \frac{1}{2}I \\ \mu &= \varphi - \theta. \end{aligned} \quad (31)$$

The equilibrium positions such that $\dot{G} = 0$ are the solutions of $\sin(2\psi) = 0$, namely, $\psi = 0$ and $\psi = \pi/2$. Since $J = G + L$ from (31), condition (28) amounts to require that the equilibrium solution $G = G_{eq}(e)$ satisfies the condition

$$C(e) = G_{eq}(e) + L_0(e) \geq 0.$$

We report in Figure 1, left, the curves $C(e) = G_{eq}(e) + L_0(e)$ as a function of the eccentricity in the interval $[0, 0.5]$ for $\psi = 0$ and $\psi = \pi/2$.

4 A zoo of resonances

The aim of this Section is to apply the results presented in Sect. 3 and to compute the bifurcation curves as a function of the eccentricity for a concrete case associated to the model presented in Sect. 2. We investigate low-order orbital resonances in the motion of a satellite around a central body, with the physical and dynamical parameters selected to be representative of small bodies in the Solar system. In particular, we set the semi-axes of the ellipsoid as $a = 1000$ km, $b = 900$ km, $c = 800$ km, the mass of the primary as $M_P = 10^{21}$ kg and its rotation period as $T_{rot} = 10$ hours. We specify that in practical computations the units of measure are normalized so that the semi-axis a is the unit of length, the mass M_P is the unit of mass, and one hour is the unit of

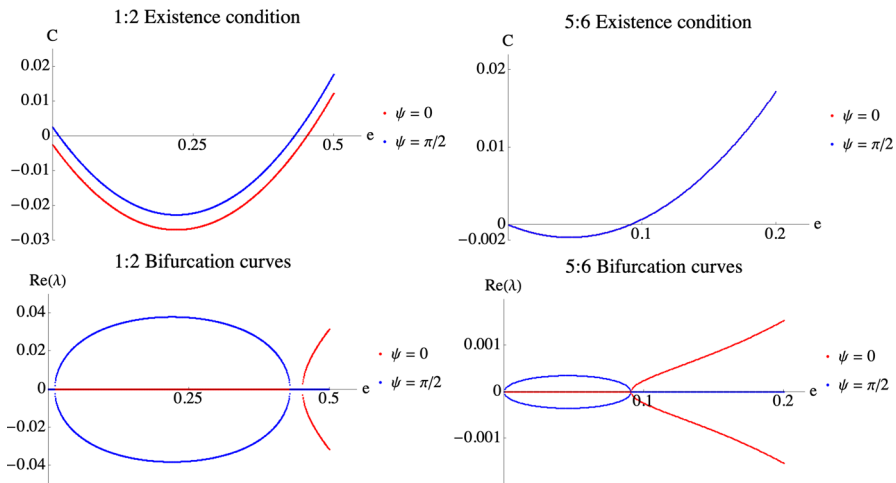


Fig. 1 Left panels: 1:2 resonance. Top: function $C(e) = G_{eq}(e) + L_0(e)$ in terms of the eccentricity for the equilibria $\psi = 0$ (red) and $\psi = \pi/2$ (blue). Bottom: bifurcation curves in the same eccentricity interval. Right panels: 5:6 resonance. Top: function $C(e) = G_{eq}(e) + L_0(e)$ in terms of the eccentricity for the equilibria $\psi = 0$ (red) and $\psi = \pi/2$ (blue). Bottom: bifurcation curves in the same eccentricity interval (color figure online)

time. As a consequence, the gravitational constant $\mathcal{G}$ is expressed in the normalized units. We use Hamiltonian (11) with the function (16) truncated at $i = 16$ and $j = 16$.

We analyse all resonances of order (n_1, n_2) with $-n_1, n_2 = 1, \dots, 7$ coprime; this choice corresponds to resonance degree $n_1/(n_1 - n_2)$ such that $\max(|n_1 - n_2|) = 8$. The resonances studied in this work are listed in Table 1.

The procedure to draw the bifurcation curves can be summarized as follows:

1. for a resonance of order (n_1, n_2) , we consider $\mathcal{H}_{res}$ as in (18), selecting r_* as the solution of the resonance equation (17);
2. we pass to the coordinates (G, L, ψ) by means of the transformation (19), obtaining the Hamiltonian $\tilde{\mathcal{H}}_{res}$;
3. we get a 1-DOF Hamiltonian system governed by Eqs. (24), being $L = L_0$ a constant of motion;
4. we compute L_0 as in (23) and substitute this value in the Hamiltonian, obtaining a Hamiltonian function depending on the eccentricity as a parameter;
5. we fix the eccentricity by considering a discretized interval for $e \in [0, e_{max}]$ with e_{max} suitably chosen. Once the eccentricity is fixed, we compute the corresponding value of L_0 from (23);
6. for every value of the eccentricity in the discretized interval $[0, e_{max}]$, we analyse the number and stability of the equilibria, computing the eigenvalues of the associated Jacobian matrix (see [14]). As the eccentricity varies, we search for possible bifurcations, checking whether they are associated to conditions (28) or (30).

We specify that the value of e_{max} is chosen in such a way that, in the interval $[0, e_{max}]$, the dynamics induced by the resonant Hamiltonian $\mathcal{H}_{res}$ is an actual approximation

of that induced by $\mathcal{H}$ (see [13] for more details). As for the classification of the bifurcations, we will make use of the definitions provided in [14, Chapter 8]. In particular, we classify the bifurcations as follows:

- (i) a transcritical bifurcation occurs when two equilibria always exist and interchange their stability at the same value of the parameter e ; we refer to it as “TRAN BIF” in Table 1;
- (ii) a pitchfork bifurcation occurs when the change in stability of a single equilibrium is associated to the appearance/disappearance of secondary fixed points; we refer to it as “PIT BIF” in Table 1;
- (iii) a saddle-node bifurcation occurs when two equilibria of opposite stability branch at a certain value of the eccentricity; we refer to it as “S-N BIF” in Table 1.

In Table 1 we present the results concerning several resonances. For each order (n_1, n_2) , we provide the degree $n_1/(n_1 - n_2)$ of the resonance, the type of bifurcation (whenever it occurs) and the resonant radius in km computed as solution of (17). For each kind of bifurcation we provide two plots: the first one showing conditions (28) or (30) (according to the order (n_1, n_2)) for the equilibria, and the second plot showing the occurrence of bifurcations, obtained by computing the eigenvalues of the Jacobian matrix at equilibrium.

Figure 1 shows two samples with n_1 odd; the corresponding resonant Hamiltonian (see (26)) has two equilibria in $\psi = 0$ and $\psi = \pi/2$. In the left panels, we show the results of the 1:2 resonance already described in Section 3: the top panel shows the curve $G_{eq}(e) + L_0(e)$ which has zeros at $e = 0.4515$ for $\psi = 0$, while for $\psi = \frac{\pi}{2}$ two solutions are found at $e = 0.011$ and $e = 0.4295$. These values of the eccentricity perfectly agree with the three pitchfork bifurcation points shown in the bottom panel (one bifurcation point for $\psi = 0$ and two for $\psi = \pi/2$). The right panels of Figure 1 show the occurrence of a transcritical bifurcation for the 5:6 resonance. The top panel shows that the condition $G_{eq}(e) + L_0(e) = 0$ has solution at $e = 0.09$ for both the equilibria $\psi = 0$ and $\psi = \pi/2$ (the two curves indeed overlap); this value of the eccentricity perfectly agrees with that shown in the bottom panel, where the two equilibria interchange their stability.

Figure 2 shows two cases with n_1 even and n_2 odd, where the corresponding resonant Hamiltonian is given by a function of the form (25). Equilibria exist for $\psi = \pi/2$ and $\psi = 3/2\pi$. In the left panels, we show the 4:5 resonance where a saddle-node bifurcation occurs in $\psi = \pi/2$. The top panel provides the curves $C(e) = G_{eq}(e) + L_0(e)$ in agreement with (30). The two curves branch at the same value of the eccentricity $e = 0.126$, which agrees with the values shown in the bottom panel giving the stability of the equilibria. The right panels of Figure 2 show the 4:7 resonance, which presents a peculiarity. For values of the eccentricity $e \leq 0.3345$ the Hamiltonian has no equilibria, while for $e > 0.3345$ two equilibria appear in $\psi = \pi/2$ and $\psi = 3/2\pi$. The top panel shows the bifurcation condition given by $3G_{eq}(e) + L_0(e) > 0$, which agrees with the appearance of the equilibria shown in the bottom panel.

The dynamical implications of the occurrence of bifurcations are especially significant when examined within the framework of the evolutionary history of the ring system. In fact, during the formation phase of the ring, the orbital eccentricity may have experienced slight variations induced by multiple mechanisms, such as the gravi-

Table 1 Table of resonances according to the order (n_1, n_2) ; each cell provides the indication of the degree of the resonance, the occurrence of bifurcations (“BIF”) and the resonant radius

n_1 n_2	1	2	3	4	5	6	7
-1	1:2 3 PIT BIF 2061	1:3 NO BIF 2698	1:4 NO BIF 3268	1:5 NO BIF 3793	1:6 NO BIF 4284	1:7 NO BIF 4748	1:8 TRAN BIF 5191
-2	2:3 S-N BIF 1706		2:5 no equilibria 2389		2:7 no equil. 2989		
-3	3:4 TRAN BIF 1581	3:5 NO BIF 1828		3:7 NO BIF 2282	3:8 NO BIF 2494		
-4	4:5 S-N BIF 1517		4:7 no eq. $e < 0.3$ 2 eq. $e \geq 0.3$ 1887				
-5	5:6 TRAN BIF 1478	5:7 NO BIF 1632	5:8 TRAN BIF 1779				
-6	6:7 S-N BIF 1452						
-7	7:8 TRAN BIF 1434						

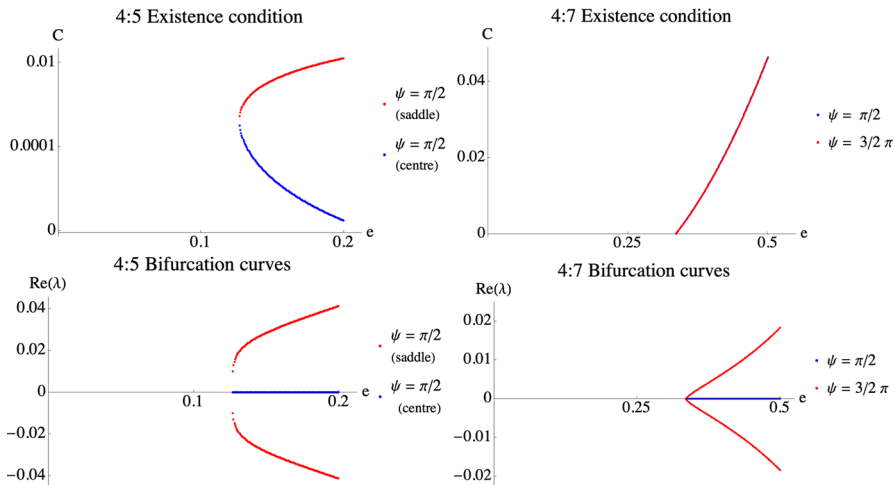


Fig. 2 Left panels: 4:5 resonance. Top: function $C(e) = G_{eq}(e) + L_0(e)$ in terms of the eccentricity for the two equilibria in $\psi = \frac{\pi}{2}$ (the saddle in red and the centre in blue) shown in logarithmic scale. Bottom: bifurcation curves in the same eccentricity interval. Right panels: 4:7 resonance. Top: function $C(e) = 3 G_{eq}(e) + L_0(e)$ in terms of the eccentricity for the equilibria $\psi = \frac{\pi}{2}$ (blue) and $\psi = \frac{3}{2}\pi$ (red). Bottom: bifurcation curves in the same eccentricity interval (color figure online)

tational perturbations exerted by nearby satellites or the action of dissipative processes. Whenever such variations arise in the vicinity of the critical parameter values at which bifurcations occur, the system may undergo transitions between stable and unstable regimes, implying a substantial modification of its dynamical behavior. This scenario should be examined within the framework of more physically realistic models that include effects neglected in the present study, which will be addressed in future works.

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Declarations

Conflicts of Interest The author declare no conflicts of interest.

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A Explicit expression of the coefficients $\alpha_{i,j}$, $\beta_{i,j}$

We provide explicit expressions for the non-zero coefficients $\alpha_{2i,j}$, $\beta_{2i,j}$ appearing in Eq. (16) for $i = 0, 1$ and $j = 0, \dots, 2$.

$$\begin{aligned}
 \beta_{0,1} &= \frac{105\sqrt{2} P_{6,0} C_{6,0} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} + \frac{75 P_{4,0} C_{4,0} \mathcal{G} M_P R^4 J^{3/2}}{\sqrt{2} |\kappa_*|^{3/2} r_*^8} \\
 &\quad + \frac{6\sqrt{2} P_{2,0} C_{2,0} \mathcal{G} M_P R^2 J^{3/2}}{|\kappa_*|^{3/2} r_*^6} \\
 &\quad - \frac{3 P_{0,0} C_{0,0} \mathcal{G} M_P J^{3/2}}{\sqrt{2} |\kappa_*|^{3/2} r_*^4} - \frac{3\sqrt{2} I^2 J^{3/2}}{|\kappa_*|^{3/2} r_*^5} - \frac{6\sqrt{2} n_* I J^{3/2}}{|\kappa_*|^{3/2} r_*^3} - \frac{\sqrt{2} I^2 J^{1/2}}{|\kappa_*|^{1/2} r_*^3} - \frac{2\sqrt{2} n_* I J^{1/2}}{|\kappa_*|^{1/2} r_*} \\
 \alpha_{0,2} &= \frac{385 P_{6,0} C_{6,0} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{115 P_{4,0} C_{4,0} \mathcal{G} M_P R^4 J^2}{\kappa_*^2 r_*^9} + \frac{15 P_{2,0} C_{2,0} \mathcal{G} M_P R^2 J^2}{\kappa_*^2 r_*^7} \\
 &\quad - \frac{3 P_{0,0} C_{0,0} \mathcal{G} M_P J^2}{\kappa_*^2 r_*^5} - \frac{5 I^2 J^2}{\kappa_*^2 r_*^6} - \frac{10 n_* I J^2}{\kappa_*^2 r_*^4} - \frac{3 I^2 J}{2 |\kappa_*| r_*^4} - \frac{3 n_* I J}{|\kappa_*| r_*^2} \\
 \alpha_{2,0} &= - \frac{315 P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} - \frac{105 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J^2}{\kappa_*^2 r_*^9} - \frac{45 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J^2}{2 \kappa_*^2 r_*^7} \\
 &\quad - \frac{28 P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} - \frac{15 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J}{|\kappa_*| r_*^7} - \frac{6 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J}{|\kappa_*| r_*^5} \\
 &\quad - \frac{P_{6,2} C_{6,2} \mathcal{G} M_P R^6}{r_*^7} - \frac{P_{4,2} C_{4,2} \mathcal{G} M_P R^4}{r_*^5} - \frac{P_{2,2} C_{2,2} \mathcal{G} M_P R^2}{r_*^3} \\
 \beta_{2,1} &= \frac{63\sqrt{2} P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} + \frac{105 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J^{3/2}}{2\sqrt{2} |\kappa_*|^{3/2} r_*^8} + \frac{7 P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^8} \\
 &\quad + \frac{15 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J^{3/2}}{\sqrt{2} |\kappa_*|^{3/2} r_*^6} + \frac{5 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^6} + \frac{3 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^4} \\
 \alpha_{2,2} &= \frac{210 P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{70 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J^2}{\kappa_*^2 r_*^9} + \frac{14 P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} \\
 &\quad + \frac{15 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J^2}{\kappa_*^2 r_*^7} + \frac{15 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J}{2 |\kappa_*| r_*^7} + \frac{3 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J}{|\kappa_*| r_*^5}
 \end{aligned}$$

References

1. Sicardy, B., et al.: Ring dynamics around non-axisymmetric bodies with application to Chariklo and Haumea. *Nat. Astron.* **3**, 146–153 (2019). <https://doi.org/10.1038/s41550-018-0616-8>
2. Braga-Ribas, F., et al.: A ring system detected around the Centaur (10199) Chariklo. *Nature* **508**(7494), 72–75 (2014). <https://doi.org/10.1038/nature13155>
3. Morgado, B.E., et al.: A dense ring of the trans-Neptunian object Quaoar outside its Roche limit. *Nature* **614**(7947), 239–243 (2023). <https://doi.org/10.1038/s41586-022-05629-6>

4. Madeira, G., Giuliatti Winter, S.M., Ribeiro, T., Winter, O.C.: Dynamics around non-spherical symmetric bodies - I. The case of a spherical body with mass anomaly. *MNRAS* **510**(1), 1450–1469 (2022). <https://doi.org/10.1093/mnras/stab3552>
5. Ribeiro, T., Winter, O.C., Madeira, G., Giuliatti Winter, S.M.: Dynamics around non-spherical symmetric bodies: II. The case of a prolate body. *MNRAS* **525**(1), 44–56 (2023). <https://doi.org/10.1093/mnras/stad2362>
6. Sicardy, B.: Dynamics of Planetary Rings. In: Souchay, J. (ed.) *Dynamics of Extended Celestial Bodies and Rings* vol. 682, p. 183 (2006). https://doi.org/10.1007/3-540-32455-0_9
7. Sicardy, B.: Resonances in Nonaxisymmetric Gravitational Potentials. *Astron. J.* **159**(3), 102 (2020). <https://doi.org/10.3847/1538-3881/ab6d06>
8. Sicardy, B., Salo, H., El Moutamid, M., Renner, S., Souami, D.: Rings around irregular bodies: I. Structure of the resonance mesh, and applications to Chariklo, Haumea, and Quaoar. *Astron. and Astroph* **704**, 23 (2025)
9. Salo, H., Sicardy, B.: Rings around irregular bodies: II. Numerical simulations of the 1/3 spin-orbit resonance confinement and applications to Chariklo. *Astron. and Astroph* **708**, 9 (2026)
10. Murray, C.D., Dermott, S.F.: *Solar System Dynamics* (1999). <https://doi.org/10.1017/CBO9781139174817>
11. Balmino, G.: Gravitational Potential Harmonics from the Shape of an Homogeneous Body. *Celest. Mech. Dyn. Astron.* **60**(3), 331–364 (1994). <https://doi.org/10.1007/BF00691901>
12. Pick, M., Picha, J., Vyskocil, V.: *Theory of the Earth's Gravity Field*. Elsevier Science Ltd (1973)
13. Celletti, A., De Blasi, I., Di Ruzza, S.: Stability and bifurcation of resonances in ring's dynamics. Preprint (2026)
14. Hirsch, M.W., Smale, S., Devaney, R.L.: *Differential Equations, Dynamical Systems, and an Introduction to Chaos*. Academic Press (2013)

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