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Discontinuous Solutions of Hamilton–Jacobi Equations Versus Radon Measure-Valued Solutions of Scalar Conservation Laws: Disappearance of Singularities

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Abstract

Let H be a bounded and Lipschitz continuous function. We consider discontinuous viscosity solutions of the Hamilton–Jacobi equation $U_t + H(U_x) = 0$ and signed Radon measure valued entropy solutions of the conservation law $u_t + [H(u)]_x = 0$. After having proved a precise statement of the formal relation $U_x = u$, we establish estimates for the (strictly positive!) times at which singularities of the solutions disappear. Here singularities are jump discontinuities in case of the Hamilton–Jacobi equation and signed singular measures in case of the conservation law.

Keywords Hamilton–Jacobi equation · First order hyperbolic conservation laws · Singular boundary conditions · Waiting time

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1 Introduction

Consider the Cauchy problem for the first order Hamilton–Jacobi equation

$$\begin{cases} U_t + H(U_x) = 0 & \text{in } S := \mathbb{R} \times \mathbb{R}^+ \\ U = U_0 & \text{in } \mathbb{R} \times \{0\}, \end{cases} \tag{HJ}$$

where

$$H \in W^{1,\infty}(\mathbb{R}), \tag{H_1}$$

and

$$U_0 \text{ is piecewise continuous in } \mathbb{R}, \quad \text{with jump points } x_1 < \cdots < x_p \,. \tag{1.1}$$

In spite of the apparent simplicity, investigating (HJ) under the above assumptions (as suggested by a mathematical model for the process of ion etching; see [15,24,25]) is mathematically challenging. Firstly, assumption (H_1) is uncommon in the theory of Hamilton–Jacobi equations. Secondly, in view of (1.1) *discontinuous* solutions of (HJ) must be considered (solutions of this kind are also important on other grounds, e.g. in optimal control problems and differential games theory). Let us recall that:

- Starting from the pioneering papers [17,18], where the basis for a systematic theory of *discontinuous viscosity solutions* were laid, the important issue of their *uniqueness* remained open.
- Examples of nonuniqueness of solutions to the Cauchy problem for Hamilton–Jacobi equations with discontinuous initial data are known, if the Hamiltonian is non-convex and explicitly depends on space and/or time [2,16];
- Several concepts of discontinuous solutions of Hamilton–Jacobi equations have been proposed [3,4,16,26], proving related existence, comparison and uniqueness results (e.g., if the Hamiltonian is convex). However, the relationships between these different notions are still partially unclear (see [11,16]);

In the light of the above situation, the main result of our paper [8] was the proof of uniqueness of discontinuous viscosity solutions (in the spirit of [18]) for problem (HJ), assuming (H_1) and (1.1). The proof, which required a detailed investigation of some qualitative features of these solutions, can be described as follows (see Theorem 3.5):
(a) *discontinuities* of viscosity solutions of (HJ) cannot appear instantaneously, and discontinuity jumps do not increase in time. Due to the boundedness of H, the discontinuity at each x_j survives for a positive *waiting time* τ_j (which essentially means that the discontinuity at x_j disappears at time τ_j if $\tau_j < \infty$). Hence there exists $\theta > 0$ such that the strip $S_\theta := \mathbb{R} \times (0, \theta)$ is the disjoint union of rectangular subdomains $Q_1, \ldots, Q_{p+1}$ (Q_1 and Q_{p+1} unbounded), whose boundaries consist of segments $\{x_j\} \times (0, \theta)$ $(j = 1, \ldots, p)$;
(b) it is proven that, if U and V are discontinuous viscosity solutions of (HJ), their restrictions to each $\mathring{Q}_k$ coincide and are continuous viscosity solutions of a *singular* Cauchy–Neumann problem for $U_t + H(U_x) = 0$, with initial data given by the proper restriction of U_0 and boundary condition $\pm\infty$, depending on the sign of the jump discontinuity of U_0 at x_k and x_{k+1}. It follows that $U = V$ a.e. in S_θ. If some discontinuity jump vanishes at $t = \theta$, iterating the procedure a finite number of times proves uniqueness.

The above overview points out the deep link between *regularity* and uniqueness of discontinuous viscosity solutions. In fact, the nonincreasing character of discontinuities and their persistence for a positive time are regularity features, and it is persistence that makes each region $\mathring{Q}_k$ *isolated* from the others. Equivalently, following [13] we say that each segment

$\{x_j\} \times (0, \theta)$ is a *barrier* for the solution - a concept to which the above use of singular Cauchy–Neumann problems gives a sound meaning.

By formal differentiation with respect to x, problem (HJ) is transformed in the Cauchy problem for a scalar conservation law,

$$\begin{cases} u_t + [H(u)]_x = 0 \text{ in } S \\ u = u_0 \qquad\qquad \text{in } \mathbb{R} \times \{0\}, \end{cases} \tag{CL}$$

where $u_0 := U_0'$ is a signed *Radon measure* on $\mathbb{R}$ such that

$$u_{0r} \in L^1(\mathbb{R}), \quad u_{0s} = \sum_{j=1}^{p} c_j \, \delta_{x_j}, \quad c_j \in \mathbb{R} \setminus \{0\} \quad (p \in \mathbb{N}). \tag{1.2}$$

Here u_{0r} denotes the density of the absolutely continuous part and u_{0s} the singular part of u_0 with respect to the Lebesgue measure on $\mathbb{R}$.

In this formal way, piecewise continuous solutions of (HJ) correspond to *Radon measure-valued solutions* of (CL). *Entropy* solutions of this kind to (CL) have been introduced and investigated in [5–7] assuming (H_1) and (1.2). In particular, it was proven that (see Sect. 3.1):

(a') *singularities* of entropy solutions cannot appear spontaneously, and their size is nonincreasing in time. Since H is bounded, each Dirac mass δ_{x_j} survives for a positive *waiting time* (incidentally, this proves that *Radon measure-valued* entropy solutions must be considered);

(b') as long as δ_{x_j} persists, it acts as a *barrier* for the solution. Accordingly, for some $\theta > 0$ the strip S_θ is split into a finite number of *isolated regions*, in each of which there exists a unique entropy solution of a *singular* Cauchy-Dirichlet problem for $u_t + [H(u)]_x = 0$ which satisfies suitable *compatibility conditions* at the lateral boundary and the initial condition in the sense of narrow topology;

(c') "gluing" properly the solutions in (b') gives a Radon measure-valued entropy solution of (CL), whose uniqueness is proven adapting the Kružkov method of doubling variables; in doing so, the above referred compatibility conditions play a crucial role.

The correspondence between the situations depicted for (HJ) and (CL) strongly suggests that the formal link $u = U_x$ can be made rigorous. Theorem 4.1 below proves that this is indeed the case. In proving this result our main motivation comes from the search for estimates of the waiting times, which are the same for both problems since their solutions are in one-to-one correspondence.

Typically, the main tool to prove such estimates is the construction of comparison functions. In particular thanks to the correspondence between (HJ) and (CL) solutions we have two distinct tools to find estimates of the waiting times.

A comparison principle for viscosity sub- and supersolutions of problem (HJ) is known from [8]. In Sect. 4.2 we prove a new comparison result for entropy solutions of (CL) which satisfy the compatibility conditions. This result seems to be of independent interest: since we compare measures with different singular parts (possibly with different supports), the uniqueness techniques used in [6,7] need to be refined.

It is easy to prove that the waiting times τ_j are always finite, if H has no limit at $\pm\infty$ (Theorem 4.4). Otherwise, it can happen that $\tau_j = \infty$. It is trivial to see that this is the case if $H(\xi)$ is constant for sufficiently large ξ. On the other hand, it is an open problem whether waiting times are finite under the following assumption:

$$\begin{cases} (i)\ \exists \lim_{\xi\to\infty} H(\xi), \text{ and } \nexists\, c>0 \text{ such that } H \text{ is constant in } (c,\infty); \\ (ii)\ \exists \lim_{\xi\to-\infty} H(\xi), \text{ and } \nexists\, d<0 \text{ such that } H \text{ is constant in } (-\infty,d)\,. \end{cases}$$

We conjecture that this is always the case, since Theorems 4.5, 4.7 give a strong indication in this sense (see Sect. 4.3).

It is worth placing the above results in the broader context of the study of evolution equations with singular initial data. Whether or not solutions of these equations become *function-valued* for positive times depends both on the dynamics inherent to the equation and on the properties of the initial singularity. For the conservation law in (CL) the dynamics crucially depends on the behaviour of H at infinity. If H has superlinear growth and $u_0 \geq 0$ is a finite Radon measure, the unique entropy solution of (CL) is a function for all positive times, namely the regularizing effect is *instantaneous* [20]. Instead, as outlined before, if H is bounded and u_0 satisfies (1.2) regularization can only take place after a positive time. Similar phenomena occur for parabolic equations, also depending on the concentration of the initial singularity with respect to suitable *capacities* related to the given equation (e.g., see [9,22,23] and references therein), and expectedly for scalar conservation laws in higher space dimension.

Let us add some comments concerning Theorem 4.1, whose correspondence result is central for the above considerations. Let assumptions (H_1) and (1.2) be satisfied, and let u be a Radon measure-valued solution of (CL) which satisfies the compatibility condition (see Sect. 3.1). Let U be a suitably defined viscosity solution of (HJ) with initial data U_0 satisfying $U_0' = u_0$ in distributional sense (see Sect. 3.2); observe that by (1.2) U_0' is a Radon measure without singular continuous part:

$$U_0' = \sum_{j=1}^{p} \left[U_0(x_j^+) - U_0(x_j^-)\right] \delta_{x_j} + (U_0')_{ac}\,. \tag{1.3}$$

Then there holds

$$U(x,t) = -\int_0^t H(u_r(x,s))\,ds + U_0(x) \quad \text{a.e. in } \mathbb{R} \ \text{ for all } t \geq 0\,, \tag{1.4}$$

$$U_x = u \ \text{ in } \mathcal{D}'(S)\,, \quad u_s(\cdot,t) = \sum_{j=1}^{p} \left[U(x_j^+,t) - U(x_j^-,t)\right] \delta_{x_j} \ \text{ for all } t \geq 0\,; \tag{1.5}$$

here u_r is the density of the absolutely continuous part and u_s is the singular part of u.

The proof of the above result is indirect and based on the uniqueness theory for problems (CL) and (HJ). More precisely, choosing suitable approximating problems with smooth initial data u_{0n} and U_{0n} (with $U_{0n}' = u_{0n}$) and smooth solutions u_n and U_n, the relation $U_{nx} = u_n$ is trivial. Letting $n \to \infty$, the main tool consists in proving that the sequences u_n and U_n approach a measure-valued solution of problem (CL) and a discontinuous viscosity solution of (HJ), respectively. In this way, the formal relation between *constructed* solutions u and U can be made rigorous. To complete the argument, it is enough to use the uniqueness part of Theorems 3.2 and 3.5 for both (CL) and (HJ), which were proven in [6] and [8], respectively. Observe that the above construction of solutions to (HJ) is different from that in [8], which is based on Perron's method but inappropriate for our purposes.

To our knowledge, even in the non-singular case a direct proof, merely based on the definitions of entropy and viscosity solutions, of the correspondence between (CL) and (HJ)

is not available in the literature. We refer to [19] for the indirect approach if $U_0 \in BV(\mathbb{R})$, and to [10] for the direct approach in the stationary case. Stimulating remarks about the above correspondence when H is convex can be found in the pioneering paper [12].

The paper is organized as follows. In Sect. 2 we introduce the basic notations. In Sect. 3 we review some known results. In Sect. 4 we present the main results, which are proven in the remaining sections.

2 Notation

2.1 Radon Measures

For every open subset $\Omega \subseteq \mathbb{R}$ we denote by $C_c(\Omega)$ the space of continuous real functions with compact support in Ω and by $\mathcal{M}^+(\Omega)$ the cone of the nonnegative Radon measures on Ω. Following [14, Section 1.3] we say that μ is a (signed) Radon measure on Ω, if there exists $\nu \in \mathcal{M}^+(\Omega)$ and a locally ν-summable function $f : \Omega \to \mathbb{R}$ such that

$$\mu(K) = \int_K f \, d\nu$$

for all compact sets $K \subset \Omega$. The space of (signed) Radon measures on Ω is denoted by $\mathcal{M}(\Omega)$. The measure $\mu \in \mathcal{M}(\Omega)$ is finite if its total variation $|\mu|(\Omega)$ is finite.

If $\mu, \nu \in \mathcal{M}(\Omega)$, we say that $\mu \leq \nu$ in $\mathcal{M}(\Omega)$ if $\nu - \mu \in \mathcal{M}^+(\Omega)$. We denote by $\langle \cdot, \cdot \rangle_\Omega$ the duality map between $\mathcal{M}(\Omega)$ and $C_c(\Omega)$. For any open set $\tilde{\Omega} \subset\subset \Omega$, $\mathcal{M}(\tilde{\Omega})$ is a Banach space with norm $\|\mu\|_{\mathcal{M}(\tilde{\Omega})} := |\mu|(\tilde{\Omega})$. Similar definitions are used for Radon measures on any subset of $Q := \Omega \times (0, T)$.

Every $\mu \in \mathcal{M}(\Omega)$ has a unique decomposition $\mu = \mu_{ac} + \mu_s$, with $\mu_{ac} \in \mathcal{M}(\Omega)$ absolutely continuous and $\mu_s \in \mathcal{M}(\Omega)$ singular with respect to the Lebesgue measure. We denote by $\mu_r \in L^1_{\text{loc}}(\Omega)$ the density of μ_{ac}. Every function $f \in L^1_{\text{loc}}(\Omega)$ can be identified to an absolutely continuous Radon measure on Ω; we shall denote this measure by the same symbol f used for the function.

For every open subset $\Omega \subseteq \mathbb{R}$ we denote by $BV(\Omega)$ the Banach space of functions of bounded variation in Ω:

$$BV(\Omega) := \{z \in L^1(\Omega) \,|\, z' \in \mathcal{M}(\Omega), \|z'\|_{\mathcal{M}(\Omega)} < \infty\},$$
$$\|z\|_{BV(\Omega)} := \|z\|_{L^1(\Omega)} + \|z'\|_{\mathcal{M}(\Omega)},$$

where z' is the first order distributional derivative. The total variation in Ω of z is $TV(z; \Omega) := \|z'\|_{\mathcal{M}(\Omega)}$. We say that $z \in BV_{\text{loc}}(\Omega)$ if $z \in BV(\tilde{\Omega})$ for every open subset $\tilde{\Omega} \subset\subset \Omega$. Similar notions hold if $z \in BV(Q)$; in this case we denote by z_x, z_t the first order distributional derivatives of z.

By $C([0, T]; \mathcal{M}(\Omega))$ we denote the set of strongly continuous mappings from $[0, T]$ into $\mathcal{M}(\Omega)$ - namely, $u \in C([0, T]; \mathcal{M}(\Omega))$ if for all $t_0 \in [0, T]$ and for every compact $K \subset \Omega$ there holds $\|u(\cdot, t) - u(\cdot, t_0)\|_{\mathcal{M}(K)} \to 0$ as $t \to t_0$.

We denote by $L^\infty_{w*}(0, T; \mathcal{M}^+(\Omega))$ the set of nonnegative Radon measures $u \in \mathcal{M}^+(S)$ such that for a.e. $t \in (0, T)$ there is a measure $u(\cdot, t) \in \mathcal{M}^+(\Omega)$ such that
(i) if $\zeta \in C([0, T]; C_c(\Omega))$ the map $t \mapsto \langle u(\cdot, t), \zeta(\cdot, t)\rangle_\Omega$ belongs to $L^1(0, T)$ and

$$\langle u, \zeta \rangle_S = \int_0^T \langle u(\cdot, t), \zeta(\cdot, t)\rangle_\Omega \, dt\,; \tag{2.1}$$

(ii) the map $t \mapsto \|u(\cdot,t)\|_{\mathcal{M}(K)}$ belongs to $L^\infty(0,T)$ for every compact $K \Subset \Omega$.

By the definition of $L^\infty_{w*}(0,T;\mathcal{M}^+(\Omega))$, for all $\rho \in C_c(\Omega)$ the map $t \mapsto \langle u(\cdot,t),\rho\rangle_\Omega$ is measurable, thus the map $u:(0,T)\to\mathcal{M}^+(\Omega)$ is weakly* measurable.

If $u \in L^\infty_{w*}(0,T;\mathcal{M}^+(\Omega))$, then $u_{ac}, u_s \in L^\infty_{w*}(0,T;\mathcal{M}^+(\Omega))$, $u_r \in L^\infty(0,T;L^1_{\mathrm{loc}}(\Omega))$ and, by (2.1), for all $\zeta \in C([0,T];C_c(\Omega))$

$$\langle u_{ac},\zeta\rangle_S = \iint_S u_r\,\zeta\,dxdt, \qquad \langle u_s,\zeta\rangle_S = \int_0^T \langle u_s(\cdot,t),\zeta(\cdot,t)\rangle_\Omega\,dt.$$

Denoting by $[u(\cdot,t)]_{ac},\ [u(\cdot,t)]_s \in \mathcal{M}^+(\Omega)$ the absolutely continuous and singular parts of the measure $u(\cdot,t)\in\mathcal{M}^+(\Omega)$, a routine proof shows that for a.e. $t\in(0,T)$

$$u_s(\cdot,t) = [u(\cdot,t)]_s\,, \quad u_{ac}(\cdot,t) = [u(\cdot,t)]_{ac}\,, \quad u_r(\cdot,t) = [u(\cdot,t)]_r\,, \tag{2.2}$$

where $[u(\cdot,t)]_r$ denotes the density of the measure $[u(\cdot,t)]_{ac}$.

We say that a (signed) Radon measure $u\in\mathcal{M}(S)$ belongs to $L^\infty_{w*}(0,T;\mathcal{M}(\Omega))$ if both its positive and negative parts u^+ and u^- belong to $L^\infty_{w*}(0,T;\mathcal{M}^+(\Omega))$. In particular, this implies that the total variation $|u|$ of the measure u belongs to $L^\infty_{w*}(0,T;\mathcal{M}^+(\Omega))$, and that conditions (i) and (ii) in the definition of $L^\infty_{w*}(0,T;\mathcal{M}^+(\Omega))$ hold with $u(\cdot,t) := u^+(\cdot,t) - u^-(\cdot,t)$ for a.e. $t\in(0,T)$.

Since u^+ and u^- are mutually singular, it follows that for a.e. t the nonnegative measures $u^+(\cdot,t)$ and $u^-(\cdot,t)$ are mutually singular, whence

$$u^\pm(\cdot,t) = [u(\cdot,t)]^\pm\,, \quad |u(\cdot,t)| = |u|(\cdot,t) \quad \text{for a.e. } t\in(0,T)\,, \tag{2.3}$$

$$u_s^\pm(\cdot,t) = [u(\cdot,t)]_s^\pm, \quad |u_s|(\cdot,t) = |[u(\cdot,t)]_s| \quad \text{for a.e. } t\in(0,T)\,. \tag{2.4}$$

2.2 Functions and Envelopes

Let χ_E be the characteristic function of $E\subseteq\mathbb{R}$. For every $u\in\mathbb{R}$ we set

$$[u]_\pm := \max\{\pm u, 0\}, \quad \mathrm{sgn}_\pm(u) := \pm\chi_{\mathbb{R}_\pm}(u), \quad \mathrm{sgn}(u) := \mathrm{sgn}_-(u) + \mathrm{sgn}_+(u)\,.$$

Let $\Omega=(a,b)$ $(-\infty<a<b<\infty)$. We say that a function $f:\Omega\to\mathbb{R}$, $f\in L^\infty(\Omega)$, is *piecewise continuous* if:

- $\Omega = \bigcup_{j=1}^{p+1} I_j$ $(p\in\mathbb{N})$ with $I_1 := (a,x_1)$, $I_j := (x_{j-1},x_j)$ for $j=2,\dots,p$, $I_{p+1} := (x_p,b)$;

- $f_j := f\llcorner I_j$ admits a representative (denoted again f_j for simplicity) which belongs to $C(\overline{I}_j)$ $(j=1,\dots,p+1)$; $f_j(x_j)\neq f_{j+1}(x_j)$ $(j=1,\dots,p)$.

If Ω is unbounded, $f\in L^\infty_{\mathrm{loc}}(\overline{\Omega})$ is piecewise continuous in Ω if it is piecewise continuous in every bounded interval $(a_0,b_0)\subset\Omega$.

Let $Q\subseteq\mathbb{R}^2$ be open, $g:Q\mapsto\mathbb{R}$ be a measurable function, $(x_0,t_0)\in\overline{Q}$. We set

$$\begin{aligned}
\mathrm{ess}\limsup_{Q\ni(x,t)\to(x_0,t_0)} g(x,t) &:= \inf_{\delta>0}\left(\mathrm{ess}\sup_{(x,t)\in B_\delta(x_0,t_0)\cap Q} g(x,t)\right) \\
&= \lim_{\delta\to0^+}\left(\mathrm{ess}\sup_{(x,t)\in B_\delta(x_0,t_0)\cap Q} g(x,t)\right), \\
\mathrm{ess}\liminf_{Q\ni(x,t)\to(x_0,t_0)} g(x,t) &:= \sup_{\delta>0}\left(\mathrm{ess}\inf_{(x,t)\in B_\delta(x_0,t_0)\cap Q} g(x,t)\right) \\
&= \lim_{\delta\to0^+}\left(\mathrm{ess}\inf_{(x,t)\in B_\delta(x_0,t_0)\cap Q} g(x,t)\right),
\end{aligned}$$

where

$$B_r(x_0,t_0) := \{(x,t) \in \mathbb{R}^2 \mid (x-x_0)^2 + (t-t_0)^2 < r^2\} \qquad (r>0)\,.$$

If $\operatorname{ess\,lim\,sup}_{Q\ni(x,t)\to(x_0,t_0)} g(x,t) = \operatorname{ess\,lim\,inf}_{Q\ni(x,t)\to(x_0,t_0)} g(x,t)$, the *essential limit* of g at (x_0,t_0) is defined as

$$\text{ess} \lim_{Q\ni(x,t)\to(x_0,t_0)} g(x,t) := \text{ess} \limsup_{Q\ni(x,t)\to(x_0,t_0)} g(x,t) = \text{ess} \liminf_{Q\ni(x,t)\to(x_0,t_0)} g(x,t)\,.$$

The quantities

$$\text{ess} \limsup_{Q\ni(x,t)\to(x_0,t_0^+)} g(x,t), \quad \text{ess} \liminf_{Q\ni(x,t)\to(x_0,t_0^+)} g(x,t)$$

are defined by replacing $B_r(x_0,t_0)$ by $B_r(x_0,t_0) \cap \{(x,t) \in \mathbb{R}^2 \mid t > t_0\}$. Similarly,

$$\text{ess} \limsup_{Q\ni(x,t)\to(x_0^\pm,t_0)} g(x,t), \quad \text{ess} \liminf_{Q\ni(x,t)\to(x_0^\pm,t_0)} g(x,t)$$

are defined by replacing $B_r(x_0,t_0)$ by $B_r(x_0,t_0) \cap \{(x,t) \in \mathbb{R}^2 \mid x > x_0\}$, respectively by $B_r(x_0,t_0) \cap \{(x,t) \in \mathbb{R}^2 \mid x < x_0\}$.

Let $g \in L^\infty(Q)$. By the *essential upper semicontinuous envelope* (shortly, upper envelope) of g we mean the function $g^* : \overline{Q} \to \mathbb{R}$,

$$g^*(x_0,t_0) := \text{ess} \limsup_{Q\ni(x,t)\to(x_0,t_0)} g(x,t) \quad \text{for any } (x_0,t_0) \in \overline{Q}\,. \tag{2.5}$$

Similarly, the *essential lower semicontinuous envelope* (shortly, lower envelope) of g is the function $g_* : \overline{Q} \to \mathbb{R}$,

$$g_*(x_0,t_0) := \text{ess} \liminf_{Q\ni(x,t)\to(x_0,t_0)} g(x,t) \quad \text{for any } (x_0,t_0) \in \overline{Q}\,. \tag{2.6}$$

Similar definitions hold for measurable functions $f : \mathbb{R} \mapsto \mathbb{R}$.

3 Definitions and Preliminary Results

3.1 Conservation Law

Definition 3.1 Let $-\infty \le a < b \le \infty$, $\Omega = (a,b)$, $u_0 \in \mathcal{M}(\Omega)$ and $H \in W^{1,\infty}(\mathbb{R})$. A measure $u \in L^\infty_{w*}(0,T;\mathcal{M}(\Omega))$ is called a *solution* of

$$u_t + [H(u)]_x = 0 \quad \text{in } Q := \Omega \times (0,T), \qquad u = u_0 \quad \text{in } \Omega \times \{0\} \tag{3.1}$$

in Q if for all $\zeta \in C^1([0,T];C^1_c(\Omega))$, $\zeta(\cdot,T) = 0$ in Ω there holds

$$\iint_Q \left[u_r\zeta_t + H(u_r)\,\zeta_x\right] dxdt + \int_0^T \langle u_s(\cdot,t), \zeta_t(\cdot,t)\rangle_\Omega\, dt = -\langle u_0, \zeta(\cdot,0)\rangle_\Omega\,. \tag{3.2}$$

A solution of (3.1) in Q is called an *entropy solution* if it satisfies the *entropy inequality*: for all $k \in \mathbb{R}$ and $\zeta \in C^1([0,T]; C_c^1(\Omega))$, $\zeta \geq 0$, $\zeta(\cdot, T) = 0$ in Ω,

$$\iint_Q \{|u_r - k|\,\zeta_t + \operatorname{sgn}(u_r - k)\,[H(u_r) - H(k)]\,\zeta_x\}\,dxdt$$
$$+ \int_0^T \langle |u_s(\cdot,t)|, \zeta_t(\cdot,t)\rangle_\Omega \, dt \geq -\int_\Omega |u_{0r}(x) - k|\,\zeta(x,0)\,dx - \langle |u_{0s}|, \zeta(\cdot,0)\rangle_\Omega \,. \tag{3.3}$$

Global (entropy) solutions of (3.1) are (entropy) solutions in $\Omega \times (0,T)$ for all $T > 0$.

In particular, setting $\Omega = \mathbb{R}$, we have defined a (global) entropy solution of the Cauchy problem (CL). Summing and subtracting (3.2) and (3.3), we find that entropy solutions u in Q of (3.1) satisfy

$$\iint_Q \{[u_r - k]_\pm\,\zeta_t + \operatorname{sgn}_\pm(u_r - k)\,[H(u_r) - H(k)]\,\zeta_x\}\,dxdt$$
$$+ \int_0^T \left\langle u_s^\pm(\cdot,t), \zeta_t(\cdot,t)\right\rangle_\Omega \, dt \geq -\int_\Omega [u_{0r}(x) - k]_\pm\,\zeta(x,0)\,dx - \left\langle u_{0s}^\pm, \zeta(\cdot,0)\right\rangle_\Omega \tag{3.4}$$

for all $k \in \mathbb{R}$ and $\zeta \in C^1([0,T]; C_c^1(\Omega))$, $\zeta \geq 0$, $\zeta(\cdot, T) = 0$ in Ω.

Entropy solutions satisfy the following monotonicity result (see [7, Theorem 3.3]).

Theorem 3.1 *Let (H_1) hold, let $u_0 \in \mathcal{M}(\Omega)$ and let u be an entropy solution of* (3.1) *in Q. Then for a.e.* $0 \leq t_1 \leq t_2 \leq T$

$$[u(\cdot,t_2)]_s^\pm \leq [u(\cdot,t_1)]_s^\pm \leq u_{0s}^\pm \quad in\ \mathcal{M}(\Omega)\,. \tag{3.5}$$

From now on we consider entropy solutions of (3.1) with initial data u_0 which satisfy

$$\begin{cases} u_0 \ \text{ is a Radon measure on } \ \Omega, \ \text{finite if } \Omega \ \text{ is bounded;} \\ u_{0s} = \sum_{j=1}^p c_j \delta_{x_j} \ \text{ with } x_1 < x_2 < \cdots < x_p,\ c_j \in \mathbb{R} \setminus \{0\} \ \text{ for } \ 1 \leq j \leq p. \end{cases} \tag{H_2}$$

We shall indicate the support of u_{0s} by $\mathcal{J} := \{x_1, x_2, \ldots, x_p\}$.

Let (H_1) and (H_2) be satisfied. If u is an entropy solution of (3.1) in Q, it follows from the proof of [5, Proposition 3.20] that $u \in C([0,T]; \mathcal{M}(\Omega))$. This implies that if u is a global entropy solution of (3.1) in Q, then

$$t_j = \sup\left\{t > 0 \,|\, u_s(\cdot,t)(\{x_j\}) \neq 0\right\} > 0 \quad \text{for all } x_j \in \mathcal{J} = \{x_1, x_2, \ldots, x_p\}. \tag{3.6}$$

More precisely, t_j can be estimated from below (see the proof of [7, Corollary 1]):

$$t_j \geq \frac{|u_{0s}|\left(\{x_j\}\right)}{2\|H\|_\infty}. \tag{3.7}$$

In addition it follows from (3.5) that $\operatorname{supp} u_s \subseteq \mathcal{J} \times [0,T]$ and, for all $t \in (0, t_j)$,

$$u_s(\cdot,t)(\{x_j\}) \begin{cases} > 0 & \text{if } c_j = u_{0s}(\{x_j\}) > 0 \\ < 0 & \text{if } c_j = u_{0s}(\{x_j\}) < 0. \end{cases} \tag{3.8}$$

Definition 3.2 Let (H_1)-(H_2) hold. An entropy solution u of (3.1) in Q is said to satisfy the *compatibility condition* at $x_j \in \mathcal{J}$ if

$$\operatorname{ess}\lim_{x\to x_j^+} \int_0^{t_j} \operatorname{sgn}_\pm(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \le 0 \quad \text{if } \pm c_j < 0 \tag{3.9a}$$

$$\operatorname{ess}\lim_{x\to x_j^-} \int_0^{t_j} \operatorname{sgn}_\pm(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \ge 0 \quad \text{if } \pm c_j < 0 \tag{3.9b}$$

for all $k \in \mathbb{R}$ and $\beta \in C_c^1(0, t_j)$, $\beta \ge 0$, where $t_j \in (0, T]$ is defined by (3.6).

By [7, Remark 7] the limits in (3.9a)–(3.9b) exist and are finite.

Before stating the basic well-posedness result for the Cauchy problem, we introduce the following *singular* Cauchy-Dirichlet problems, where $m_1, m_2 = \pm\infty$:

- If $\Omega = (a, b)$ with $-\infty < a < b < \infty$,

$$\begin{cases} u_t + [H(u)]_x = 0 & \text{in } Q \\ u = m_1 & \text{in } \{a\} \times (0, T) \\ u = m_2 & \text{in } \{b\} \times (0, T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,; \end{cases} \tag{D}$$

- If $\Omega = (-\infty, b)$ with $b < \infty$,

$$\begin{cases} u_t + [H(u)]_x = 0 & \text{in } Q \\ u = m_2 & \text{in } \{b\} \times (0, T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,; \end{cases} \qquad (D)_-$$

- If $\Omega = (a, \infty)$ with $a > -\infty$,

$$\begin{cases} u_t + [H(u)]_x = 0 & \text{in } Q \\ u = m_1 & \text{in } \{a\} \times (0, T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \qquad (D)_+$$

Definition 3.3 Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$. Let (H_1) hold, and let $u_0 \in \mathcal{M}(\Omega)$. An *entropy solution* u of (D) in Q with $m_1, m_2 = \pm\infty$ is an entropy solution of (3.1) in Q such that for all $k \in \mathbb{R}$ and $\beta \in C_c^1(0, T)$, $\beta \ge 0$ there holds

$$\operatorname{ess}\lim_{x\to a^+} \int_0^{T} \operatorname{sgn}_+(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \le 0 \quad \text{if } m_1 = -\infty, \tag{3.10a}$$

$$\operatorname{ess}\lim_{x\to a^+} \int_0^{T} \operatorname{sgn}_-(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \le 0 \quad \text{if } m_1 = \infty, \tag{3.10b}$$

$$\operatorname{ess}\lim_{x\to b^-} \int_0^{T} \operatorname{sgn}_+(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \ge 0 \quad \text{if } m_2 = -\infty, \tag{3.10c}$$

$$\operatorname{ess}\lim_{x\to b^-} \int_0^{T} \operatorname{sgn}_-(u_r(x,t)-k)\left[H(u_r(x,t))-H(k)\right]\beta(t)\,dt \ge 0 \quad \text{if } m_2 = \infty. \tag{3.10d}$$

Entropy solutions of $(D)_-$ and $(D)_+$ are defined by dropping conditions (3.10a)–(3.10b) at $x = a$ (resp. (3.10c)–(3.10d) at $x = b$).

Again it follows from [7, Remark 7] that the limits in (3.10) exist and are finite.

The proof of the following well-posedness result is basically the same as in the case of problem (CL) (see [7, Theorem 3.5]; for the existence part, see also the proof of Theorem 4.2 below).

Theorem 3.2 *Let* (H_1) *and* (H_2) *be satisfied. Then the following problems have a unique global entropy solution which satisfies the compatibility condition at all* $x_j \in \mathcal{J}$:

(i) *Problem* (D), *with* $m_1 = \pm\infty$, $m_2 = \pm\infty$;
(ii) *Problem* $(D)_-$, *with* $m_2 = \pm\infty$;
(iii) *Problem* $(D)_+$ *with* $m_1 = \pm\infty$;
(iv) *Problem* (CL).

The following results follow from the proofs of [7, Theorem 3.5 and Proposition 5.8]. The first one states that at the singularities, the one-sided traces of $H(u) = H(u_r)$ at $x_j \in \mathcal{J}$ exist in a weak sense:

Proposition 3.3 *Let* (H_1) *and* (H_2) *be satisfied and let* u *be the global entropy solution of* (D) *satisfying the compatibility conditions at all* $x_j \in \mathcal{J}$. *Let* $t_j \in (0, \infty]$ *be defined by* (3.6). *For all* x_j *there exists* $f_{x_j^\pm} \in L^\infty(0, t_j)$ *such that*

$$\operatorname*{ess\,lim}_{x \to x_j^\pm} \int_0^{t_j} H(u(x,t))\,\beta(t)\,dt = \int_0^{t_j} f_{x_j^\pm}(t)\,\beta(t)\,dt \quad \textit{for all } \beta \in C_c([0,\infty)). \tag{3.11}$$

Moreover, for a.e. $t \in (0, t_j)$ *there holds*

$$\limsup_{u\to\infty} H(u) \le f_{x_j^+}(t) \le \sup_{u\in\mathbb{R}} H(u) \quad \textit{if } c_j > 0\,, \tag{3.12}$$

$$\inf_{u\in\mathbb{R}} H(u) \le f_{x_j^+}(t) \le \liminf_{u\to-\infty} H(u) \quad \textit{if } c_j < 0\,, \tag{3.13}$$

$$\inf_{u\in\mathbb{R}} H(u) \le f_{x_j^-}(t) \le \liminf_{u\to\infty} H(u) \quad \textit{if } c_j > 0\,, \tag{3.14}$$

$$\limsup_{u\to-\infty} H(u) \le f_{x_j^-}(t) \le \sup_{u\in\mathbb{R}} H(u) \quad \textit{if } c_j < 0\,. \tag{3.15}$$

The weak traces $f_{x_j^\pm}$ determine the evolution of the Dirac masses. In fact, since the solution u satisfies the weak formulation (3.2), we have:

Proposition 3.4 *Under the assumptions of* Proposition 3.3, *for all* $x_j \in \mathcal{J}$,

$$u_s(t) \llcorner \{x_j\} = C_j(t)\delta_{x_j}\,, \quad C_j(t) := \begin{cases} c_j - \int_0^t \left[f_{x_j^+}(s) - f_{x_j^-}(s) \right] ds & \textit{if } 0 \le t < t_j \\ 0 & \textit{if } t \ge t_j, \end{cases} \tag{3.16}$$

$$C_j(t) := \begin{cases} > 0 & \textit{if } c_j > 0 \\ < 0 & \textit{if } c_j < 0 \end{cases} \quad \textit{for every } 0 \le t < t_j. \tag{3.17}$$

Similar results hold for problems $(D)_-$ and $(D)_+$ when Ω is an half-line, and for the Cauchy problem (CL) when $\Omega = \mathbb{R}$.

3.2 Hamilton–Jacobi Equation

Definition 3.4 Let $H \in W^{1,\infty}(\mathbb{R})$, $E \subseteq \mathbb{R}^2$ an open set and $U \in L^\infty_{\text{loc}}(\overline{E})$. U is a *viscosity solution* of the equation $U_t + H(u_x) = 0$ in E, if for all $\varphi \in C^1(E)$:

$$\varphi_t(x,t) + H(\varphi_x(x,t)) \le 0 \text{ if } (x,t) \text{ is a local maximum point of } U^* - \varphi \text{ in } E; \tag{3.18}$$

$$\varphi_t(x,t) + H(\varphi_x(x,t)) \ge 0 \text{ if } (x,t) \text{ is a local minimum point of } U_* - \varphi \text{ in } E. \tag{3.19}$$

Definition 3.5 Let $-\infty \le a < b \le \infty$, $\Omega = (a, b)$, $U_0 \in L^\infty_{\rm loc}(\overline{\Omega})$ and $H \in W^{1,\infty}(\mathbb{R})$. A *viscosity solution* of

$$\begin{cases} U_t(x,t) + H(U_x(x,t)) = 0 & \text{in } Q = \Omega \times (0,T) \\ U(\cdot,0) = U_0 & \text{in } \Omega \end{cases} \tag{3.20}$$

is a viscosity solution of $U_t + H(u_x) = 0$ in Q such that

$$U^*(\cdot,0) = (U_0)^*\,, \quad U_*(\cdot,0) = (U_0)_* \quad \text{in } \overline{\Omega}\,. \tag{3.21}$$

Global viscosity solutions of (3.20) are viscosity solutions in $\Omega \times (0,T)$ for all $T > 0$.

In particular we have defined a viscosity solution of the Cauchy problem (HJ).

The singular Dirichlet problems for the conservation law naturally correspond to singular Neumann problems for the Hamilton–Jacobi equation, where $m_1, m_2 = \pm\infty$:

- If $\Omega = (a, b)$ with $-\infty < a < b < \infty$,

$$\begin{cases} U_t + H(U_x) = 0 & \text{in } Q \\ U_x = m_1 & \text{in } \{a\} \times (0,T) \\ U_x = m_2 & \text{in } \{b\} \times (0,T) \\ U = U_0 & \text{in } \Omega \times \{0\}\,; \end{cases} \tag{N}$$

- If $\Omega = (-\infty, b)$ with $b < \infty$,

$$\begin{cases} U_t + H(U_x) = 0 & \text{in } Q \\ U_x = m_2 & \text{in } \{b\} \times (0,T) \\ U = U_0 & \text{in } \Omega \times \{0\}\,; \end{cases} \qquad (N)_-$$

- If $\Omega = (a, \infty)$ with $a > -\infty$,

$$\begin{cases} U_t + H(U_x) = 0 & \text{in } Q \\ U_x = m_1 & \text{in } \{a\} \times (0,T) \\ U = U_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \qquad (N)_+$$

Definition 3.6 Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$ and $\hat{Q} := \overline{\Omega} \times (0, T]$. Let (H_1) hold, and let $U_0 \in L^\infty_{\rm loc}(\overline{\Omega})$. A *viscosity solution* U of (N) with $m_1 = \pm\infty$, $m_2 = \pm\infty$ is a viscosity solution of (3.20) in Q such that for all $\varphi \in C^1(\hat{Q})$ there holds:

(i) If $m_1 = m_2 = \infty$:

$$\varphi_t(a,t) + H(\varphi_x(a,t)) \le 0 \text{ if } (a,t) \text{ is a local maximum point of } U^* - \varphi \text{ in } \hat{Q}, \tag{3.22}$$

$$\varphi_t(b,t) + H(\varphi_x(b,t)) \ge 0 \text{ if } (b,t) \text{ is a local minimum point of } U_* - \varphi \text{ in } \hat{Q}; \tag{3.23}$$

(ii) If $m_1 = m_2 = -\infty$:

$$\varphi_t(a,t) + H(\varphi_x(a,t)) \ge 0 \text{ if } (a,t) \text{ is a local minimum point of } U_* - \varphi \text{ in } \hat{Q}, \tag{3.24}$$

$$\varphi_t(b,t) + H(\varphi_x(b,t)) \le 0 \text{ if } (b,t) \text{ is a local maximum point of } U^* - \varphi \text{ in } \hat{Q}; \tag{3.25}$$

(iii) If $m_1 = \infty$ and $m_2 = -\infty$ and (a, t) and/or (b, t) are local maximum points of $U^* - \varphi$ in $\hat{Q}$, then

$$\begin{cases} \varphi_t(a,t) + H(\varphi_x(a,t)) \le 0\,, \\ \varphi_t(b,t) + H(\varphi_x(b,t)) \le 0\,; \end{cases} \tag{3.26}$$

(iv) If $m_1 = -\infty$ and $m_2 = \infty$ and (a, t) and/or (b, t) are local minimum points of $U_* - \varphi$ in $\hat{Q}$, then

$$\begin{cases} \varphi_t(a,t) + H(\varphi_x(a,t)) \ge 0\,, \\ \varphi_t(b,t) + H(\varphi_x(b,t)) \ge 0\,. \end{cases} \tag{3.27}$$

Viscosity solutions of $(N)_-$ and $(N)_+$ are defined as above, dropping conditions at $x = a$, respectively at $x = b$ in Definition 3.6.

The following well-posedness result holds for (N) ([8, Theorem 3.3 and 3.4]).

Theorem 3.5 *Let $\Omega = (a, b)$. Let (H_1) hold, and let $U_0 \in L^\infty_{\rm loc}(\overline{\Omega})$ be piecewise continuous in Ω with $\mathcal{J} = \{x_1, \ldots, x_p\}$ as the set of jump discontinuities. Then there exists a unique global viscosity solution U of problem (N), with $m_1 = \pm\infty$, $m_2 = \pm\infty$. Moreover:*

(a) *For every $j = 1, \ldots, p+1$ the restriction $U \llcorner \overline{S_j}$ has a continuous representative $\tilde{U}_j$ in $\overline{S_j}$, with $S_j := I_j \times \mathbb{R}^+$, $I_j := (x_{j-1}, x_j)$, $x_0 := a$, $x_{p+1} := b$;*
(b) *For every $j = 1, \ldots, p$ there exists a unique waiting time $\tau_j \in (0, \infty]$ such that*

$$\tilde{U}_j(x_j, t) \neq \tilde{U}_{j+1}(x_j, t) \quad \Leftrightarrow \quad t \in [0, \tau_j)\,.$$

Similar statements hold for $(N)_-$ with $m_2 = \pm\infty$ if $\Omega = (-\infty, b)$ with $b < \infty$, for $(N)_+$ with $m_1 = \pm\infty$ if $\Omega = (a, \infty)$ with $a > -\infty$, and for (HJ) if $\Omega = \mathbb{R}$.

Remark 3.1 Let U be the global viscosity solution of (N) with initial datum U_0 as in Theorem 3.5. For all $x_j \in \mathcal{J}$ we consider the jumps

$$J_0(x_j) := U_0(x_j^+) - U_0(x_j^-)\,, \quad J_t(x_j) := U(x_j^+, t) - U(x_j^-, t) \quad (t > 0) \tag{3.28}$$

(here $U(x_j^+, t) = \tilde{U}_{j+1}(x_j, t)$ and $U(x_j^-, t) = \tilde{U}_j(x_j, t)$; see Theorem 3.5$(a)$). By Theorem 3.5$(b)$ the jump $J_t(x_j)$ persists until the strictly positive waiting time

$$\tau_j = \sup\left\{t \in \mathbb{R}^+ \mid J_t(x_j) \neq 0\right\} \in (0, \infty]\,. \tag{3.29}$$

Moreover, as observed in [8, Remark 3.2], jumps cannot change sign,

$$J_t(x_j) \begin{cases} > 0 & \text{if } J_0(x_j) > 0 \\ < 0 & \text{if } J_0(x_j) < 0 \end{cases} \qquad \text{for all } t \in [0, \tau_j), \tag{3.30}$$

and are nonincreasing (in absolute value, [8, Theorem 3.4-(d)]): for $0 \le t_0 < t_1 < \tau_j$

$$|J_{t_1}(x_j)| \le \begin{cases} |J_{t_0}(x_j)| - \left[\limsup\limits_{\xi\to\infty} H(\xi) - \liminf\limits_{\xi\to\infty} H(\xi)\right](t_1 - t_0) \text{ if } \mathrm{J_0(x_j)} > 0 \\ |J_{t_0}(x_j)| - \left[\limsup\limits_{\xi\to-\infty} H(\xi) - \liminf\limits_{\xi\to-\infty} H(\xi)\right](t_1 - t_0) \text{ if } \mathrm{J_0(x_j)} < 0. \end{cases} \tag{3.31}$$

4 Results

4.1 Conservation Law Versus Hamilton–Jacobi Equation

The correspondence between the solutions u of (CL) and U of (HJ), with $u_0 = U_0'$, is a special case (set $\Omega = \mathbb{R}$) of the following result. Observe that, in terms of U_0, hypothesis (H_2) on u_0 becomes

$$\begin{cases} U_0 \in BV_{\mathrm{loc}}(\overline{\Omega});\ U_0 \in C(\overline{\Omega}) \text{ or } \exists\, x_1 < \cdots < x_p :\ U_0(x_j^+) \neq U_0(x_j^-)\ \forall\, x_j, \\ U_0 \in W^{1,1}_{\mathrm{loc}}(\overline{I}_j),\ I_j = (x_{j-1}, x_j)\ (1 \le j \le p+1;\ x_0 = a,\ x_{p+1} = b). \end{cases} \tag{H_3}$$

Theorem 4.1 *Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$, let (H_1)-(H_3) be satisfied and let $\mathcal{J} = \{x_1, x_2, \ldots, x_p\}$.*

(i) *Let u be the unique entropy solution of (D) with initial data $u_0 = U_0'$ as in (1.3), which satisfies the compatibility condition at all $x_j \in \mathcal{J}$. Set*

$$U(\cdot, t) := -\int_0^t H(u_r(\cdot, s))\, ds + U_0 \quad a.e.\ in \Omega \qquad (t \in (0, T))\,. \tag{4.1}$$

Then U is the unique viscosity solution of (N), and u and U satisfy (1.5).

(ii) *Let U be the unique viscosity solution of (N). Then the distributional derivative U_x belongs to $C([0, T]; \mathcal{M}(\Omega))$, the measure $u := U_x$ is the unique entropy solution of problem (D) with initial data $u_0 := U_0'$ which satisfies the compatibility condition at all $x_j \in \mathcal{J}$, and u and U satisfy (1.4) and (1.5).*

Similar statements hold if Ω is unbounded.

4.2 Comparison

We shall prove the following:

Theorem 4.2 *Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$, and let (H_1) hold. Let $u_0, v_0 \in \mathcal{M}(\Omega)$ satisfy*

$$\begin{cases} u_{0s} = \sum_{j=1}^{p} c_j \delta_{x_j} \ \textit{with}\ x_1 < x_2 < \ldots x_p,\ c_j \in \mathbb{R} \setminus \{0\}\ \textit{for}\ 1 \le j \le p\,, \\ v_{0s} = \sum_{j=1}^{q} d_j \delta_{x_j'} \ \textit{with}\ x_1' < x_2' < \ldots x_q',\ d_j \in \mathbb{R} \setminus \{0\}\ \textit{for}\ 1 \le j \le q\,, \end{cases}$$

and let $u_0 \le v_0$ in $\mathcal{M}(\Omega)$. Let u, v be the entropy solutions of (D) with initial data u_0, v_0 given by Theorem 3.2 *(in particular u and v satisfy the compatibility condition). Then $u(\cdot, t) \le v(\cdot, t)$ in $\mathcal{M}(\Omega)$ for all $t \in [0, T]$.*

Similar statements hold if Ω is unbounded.

The companion result for solutions of (N) is known ([8, Corollary 3.5]):

Theorem 4.3 *Let $\Omega = (a, b)$ with $-\infty \le a < b \le \infty$, and let (H_1) hold. Let $U_0, V_0 \in L^\infty(\Omega)$, U_0 and V_0 piecewise continuous in Ω with a finite number of discontinuities. If U and V are viscosity solutions of problem (N) in Q with initial data $U_0 \le V_0$ a.e. in Ω, then $U \le V$ a.e. in Q. Similar statements hold if Ω is unbounded.*

Observe that the above assumptions on U_0 and V_0 are satisfied if (H_3) holds.

4.3 Waiting Time for Global Solutions of (*HJ*) and (*CL*)

The first result is an upper bound for the waiting times of solutions of problem (HJ) if the Hamiltonian $H(\xi)$ does not have a limit as $\xi \to \pm\infty$.

Theorem 4.4 *Let $H \in W^{1,\infty}(\mathbb{R})$ and let $U_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ be piecewise continuous in $\mathbb{R}$ with a finite number of discontinuities: $\mathcal{J} = \{x_1, \dots, x_p\}$. Let*

$$(H^*)_\pm := \limsup_{\xi \to \pm\infty} H(\xi)\,, \quad (H_*)_\pm := \liminf_{\xi \to \pm\infty} H(\xi)\,,$$

and let U be the unique global viscosity solution of (HJ). Let the initial jump $J_0(x_j)$ and the waiting time $\tau_j \in (0, +\infty]$ at $x_j \in \mathcal{J}$ be defined by (3.28) and (3.29). Then

$$\tau_j \le \begin{cases} \dfrac{J_0(x_j)}{(H^*)_+ - (H_*)_+} & \text{if } J_0(x_j) > 0 \text{ and } (H^*)_+ > (H_*)_+ \\ \dfrac{|J_0(x_j)|}{(H^*)_- - (H_*)_-} & \text{if } J_0(x_j) < 0 \text{ and } (H^*)_- > (H_*)_- . \end{cases} \tag{4.2}$$

By assumption (H_1), both $(H^*)_\pm$ and $(H_*)_\pm$ are finite.

In view of Theorem 4.4, it is natural to seek estimates of τ_j from above assuming that the limits $\lim_{\xi\to\pm\infty} H(\xi)$ exist. However, if there exist $c, d \in \mathbb{R}$ such that H is constant either in $(-\infty, d)$, or in (c, ∞), it is easy to construct examples with $\tau_j = \infty$. Hence we make the following assumption:

$$\begin{cases} (i)\ \exists\, H^+ := \lim\limits_{\xi\to\infty} H(\xi);\ \nexists\, c > 0 \text{ such that } H \text{ is constant in } (c, \infty); \\ (ii)\ \exists\, H^- := \lim\limits_{\xi\to-\infty} H(\xi);\ \nexists\, d < 0 \text{ such that } H \text{ is constant in } (-\infty, d). \end{cases} \tag{H_4}$$

Theorem 4.5 *Let (H_1) hold. Let $U_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ be piecewise continuous in $\mathbb{R}$, let $\mathcal{J}$ be the finite set of its discontinuities, and let $A, B > 0$ be such that*

$$|U_0(x)| \le A + B|x| \quad \text{for all } x \in \mathbb{R}\,. \tag{A_1}$$

Let U be the unique global viscosity solution of (HJ) with initial data U_0. Then for every $x_j \in \mathcal{J}$ the waiting time τ_j is finite if either $J_0(x_j) > 0$ and H satisfies (H_4)-(i), or $J_0(x_j) < 0$ and H satisfies (H_4)-(ii).

In view of the correspondence between problems (HJ) and (CL) stated in Theorem 4.1, the above results concerning the waiting time have a counterpart for global entropy solutions of (CL). For every $U_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and $u_0 \in \mathcal{M}(\mathbb{R})$ as in assumptions (H_2)-(H_3), with $U_0' = u_0$ in $\mathcal{M}(\mathbb{R})$, let $U \in L^\infty_{\mathrm{loc}}(\overline{S})$ and $u \in C([0,\infty); \mathcal{M}(\mathbb{R}))$ be the global viscosity solution of (HJ), respectively the global entropy solution of (CL) satisfying the compatibility condition at every $x_j \in \mathcal{J} = \operatorname{supp} u_{0s}$. Then for every $x_j \in \mathcal{J}$

$$J_0(x_j) = u_{0s}(\{x_j\}) = c_j \tag{4.3}$$

and the waiting times for the persistence of jumps in (HJ) (see (3.29)) and of the singular part in (CL) (see (3.6)) coincide, namely

$$t_j = \tau_j\,, \tag{4.4}$$

$$u_s(\cdot, t)(\{x_j\}) = J_t(x_j) \quad \text{for every } 0 \le t \le t_j \tag{4.5}$$

(see (1.5) and (3.28)). Therefore, as a by-product of Theorems 4.1, 4.4 and 4.5 we have the following statements.

Corollary 4.6 *Let* (H_1)*-*(H_2) *hold. Let* $u \in C([0,\infty); \mathcal{M}(\mathbb{R}))$ *be the unique global entropy solution of* (CL) *with initial data* u_0*, which satisfies the compatibility condition at all* $x_j \in \mathcal{J}$*. Let* t_j *be the waiting time defined by* (3.6)*. Then*

$$t_j \le \begin{cases} \dfrac{c_j}{(H^*)_+ - (H_*)_+} & \textit{if } c_j > 0 \textit{ and } (H^*)_+ > (H_*)_+ \\ \dfrac{|c_j|}{(H^*)_- - (H_*)_-} & \textit{if } c_j < 0 \textit{ and } (H^*)_- > (H_*)_- . \end{cases} \tag{4.6}$$

In addition, if $\bar{A},\ \bar{B} > 0$ *are such that*

$$\left| \int_0^x u_{0r}(s)\, ds \right| \le \bar{A} + \bar{B}|x| \quad \textit{for } x \in \mathbb{R}\,, \tag{A_2}$$

then the waiting time t_j *is finite if either* $c_j > 0$ *and* H *satisfies* (H_4)*-*(i) *or* $c_j < 0$ *and* H *satisfies* (H_4)*-*(ii)*.*

Remark 4.1 Clearly, assumption (A_2) is satisfied if $u_{0r} \in L^1(\mathbb{R})$ or $u_{0r} \in L^\infty(\mathbb{R})$.

By strengthening the assumptions on H, the conclusions in the second part of Corollary 4.6 still hold under very weak assumptions on the initial data. Set

$$M_k^+ := \|H'\|_{L^\infty(k,\infty)}\,, \quad M_k^- := \|H'\|_{L^\infty(-\infty,k)}$$

(observe that $M_k^\pm > 0$ by (H_4)). We introduce the following assumptions:

$$\begin{cases} (i)\ H \text{ satisfies } (H_4)-(i), \quad \lim\limits_{k\to\infty} M_k^+ = 0, \quad \limsup\limits_{k\to\infty} \frac{|H(k)-H^+|}{M_k^+} \ge C_0^+ > 0\,; \\ (ii)\ H \text{ satisfies } (H_4)-(ii), \quad \lim\limits_{k\to-\infty} M_k^- = 0, \quad \limsup\limits_{k\to-\infty} \frac{|H(k)-H^-|}{M_k^-} \ge C_0^- > 0 \end{cases} \tag{H_5}$$

(an example of function H satisfying (H_5)-(i) is $H(s) = e^{-s}\sin s$), and

$$\begin{cases} (i)\ \ \exists\, \overline{k} > 0 \text{ such that either } H(\xi) > H^+, \text{ or } H(\xi) < H^+ \text{ for any } \xi \ge \overline{k}\,; \\ (ii)\ \exists\, \underline{k} < 0 \text{ such that either } H(\xi) > H^-, \text{ or } H(\xi) < H^- \text{ for any } \xi \le \underline{k}\,. \end{cases} \tag{H_6}$$

Theorem 4.7 *Let* (H_1)*-*(H_2) *hold, and let* $u \in C([0,\infty); \mathcal{M}(\mathbb{R}))$ *be the unique global entropy solution of* (CL) *with initial data* u_0*, which satisfies the compatibility condition at all* $x_j \in \mathcal{J}$*. Then the waiting time* t_j *is finite if either* $c_j > 0$ *and* H *satisfies* (H_5)*-*(i) *or* (H_6)*-*(i)*, or* $c_j < 0$ *and* H *satisfies* (H_5)*-*(ii) *or* (H_6)*-*(ii)*.*

Again, by Theorem 4.1 these results for (CL) can be translated to problem (HJ).

Corollary 4.8 *Let* (H_1)*-*(H_3) *hold, and let* U *be the unique global viscosity solution of* (HJ) *with initial data* U_0*. Then for every* $x_j \in \mathcal{J}$ *the waiting time* τ_j *is finite if either* $J_0(x_j) > 0$ *and* H *satisfies* (H_5)*-*(i) *or* (H_6)*-*(i)*, or* $J_0(x_j) < 0$ *and* H *satisfies* (H_5)*-*(ii) *or* (H_6)*-*(ii)*.*

5 (*D*) Versus (*N*): Proof of Theorem 4.1

5.1 Preliminary Definitions and Notations

Let $\Omega = (a, b)$, $-\infty \le a < b \le \infty$. Below we generalize problem (N) to the case that $m_1, m_2 \in \overline{\mathbb{R}} := [-\infty, \infty]$:

$$\begin{cases} U_t + H(U_x) = 0 & \text{in } Q := \Omega \times (0, T) \\ U_x = m_1 & \text{in } \{a\} \times (0, T) \\ U_x = m_2 & \text{in } \{b\} \times (0, T)\,, \end{cases} \tag{5.1}$$

with initial condition

$$U = U_0 \quad \text{in } \Omega \times \{0\}\,. \tag{5.2}$$

Definition 5.1 Let $\hat{Q} := \overline{\Omega} \times (0, T]$ and $m_1,\ m_2 \in \overline{\mathbb{R}}$.
(i) By a *viscosity subsolution of* (5.1) in Q we mean any viscosity subsolution U of $U_t + H(U_x) = 0$ in Q such that if (a, t) and/or (b, t) are local maximum points of $U^* - \varphi$ in $\hat{Q}$ for some $\varphi \in C^1(\hat{Q})$, then

$$\begin{cases} \varphi_t(a, t) + H(\varphi_x(a, t)) \le 0 & \text{if } \varphi_x(a, t) \le m_1, \\ \varphi_t(b, t) + H(\varphi_x(b, t)) \le 0 & \text{if } \varphi_x(b, t) \ge m_2\,. \end{cases} \tag{5.3}$$

(ii) By a *viscosity supersolution of* (5.1) in Q we mean any viscosity supersolution U of $U_t + H(U_x) = 0$ in Q such that if (a, t) and/or (b, t) are local minimum points of $U_* - \varphi$ in $\hat{Q}$ for some $\varphi \in C^1(\hat{Q})$, then

$$\begin{cases} \varphi_t(a, t) + H(\varphi_x(a, t)) \ge 0 & \text{if } \varphi_x(a, t) \ge m_1, \\ \varphi_t(b, t) + H(\varphi_x(b, t)) \ge 0 & \text{if } \varphi_x(b, t) \le m_2\,. \end{cases} \tag{5.4}$$

(iii) A function U is called a *viscosity solution* of (5.1) in Q, if it is both a viscosity subsolution and a viscosity supersolution.
(iv) Let $U_0 \in L^\infty_{\text{loc}}(\overline{\Omega})$. A *viscosity solution of* (5.1) *in* Q *with initial condition* (5.2) is a viscosity solution of (5.1) satisfying (3.21).

Remark 5.1 Formally, conditions (5.3) for viscosity subsolutions of (5.1) are void when $m_1 = -\infty$, $m_2 = \infty$; conditions (5.4) for viscosity supersolutions of (5.1) are void when $m_1 = \infty$, $m_2 = -\infty$. Analogously, the boundary conditions at $x = a$ and $x = b$ are dropped if $a = -\infty$ and $b = \infty$, respectively.

5.2 Parabolic Approximation

Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$. Let $f_{1,\epsilon}, f_{2,\epsilon}, f_{3,\epsilon} \in C^\infty(\mathbb{R})$ ($\epsilon \in (0, 1)$) be a partition of unity:

$$\begin{cases} 0 \le f_{i,\epsilon} \le 1\,, \quad \sum_{i=1}^3 f_{i,\epsilon} = 1 \quad \text{in } \mathbb{R}\,, \\ f_{1,\epsilon} = 1 \quad \text{in } (-\infty, a + 2\sqrt{\epsilon}]\,, & \text{supp } f_{1,\epsilon} \subseteq (-\infty, a + 3\sqrt{\epsilon}]\,, \\ f_{2,\epsilon} = 1 \quad \text{in } [a + 3\sqrt{\epsilon}, b - 3\sqrt{\epsilon}]\,, & \text{supp } f_{2,\epsilon} \subseteq [a + 2\sqrt{\epsilon}, b - 2\sqrt{\epsilon}]\,, \\ f_{3,\epsilon} = 1 \quad \text{in } [b - 2\sqrt{\epsilon}, \infty)\,, & \text{supp } f_{3,\epsilon} \subseteq [b - 3\sqrt{\epsilon}, \infty)\,, \end{cases}$$

such that for $i = 1, 2, 3$

$$\sup_{\epsilon\in(0,1)} \|f'_{i,\epsilon}\|_{L^1(\mathbb{R})} < \infty\,, \qquad \sup_{\epsilon\in(0,1)} \sqrt{\epsilon}\,\|f''_{i,\epsilon}\|_{L^1(\mathbb{R})} < \infty\,.$$

Let $U_0 \in C^\infty(\overline{\Omega})$ and $m_1,\ m_2 \in \mathbb{R}$. For every $x \in \overline{\Omega}$, we set

$$u_{0,\epsilon} := m_1 f_{1,\epsilon} + f_{2,\epsilon} U'_0 + m_2 f_{3,\epsilon}\,, \qquad U_{0,\epsilon}(x) := U_0(a) + \int_a^x u_{0,\epsilon}(s)ds \tag{5.5}$$

(to keep notation as simple as possible we suppress the dependence of $u_{0,\epsilon}$ on m_1, m_2). Then $U_{0,\epsilon} \in C^\infty(\overline{\Omega})$, $u_{0,\epsilon} = m_1$ in $[a, a+\sqrt{\epsilon}]$, $u_{0,\epsilon} = m_2$ in $[b-\sqrt{\epsilon}, b]$,

$$\begin{aligned} & U'_{0,\epsilon} = u_{0,\epsilon} \text{ in } \overline{\Omega}, \qquad \|u_{0,\epsilon}\|_{L^\infty(\Omega)} \le \max\left\{|m_1|, |m_2|, \|U'_0\|_{L^\infty(\Omega)}\right\} \quad \text{for } \epsilon \in (0,1)\,, \\ & \sup_{\epsilon\in(0,1)} \|u'_{0,\epsilon}\|_{L^1(\Omega)} < \infty\,, \qquad \sup_{\epsilon\in(0,1)} \sqrt{\epsilon}\,\|u''_{0,\epsilon}\|_{L^1(\Omega)} < \infty\,, \end{aligned} \tag{5.6}$$

$$\begin{aligned} & u_{0,\epsilon}(x) \to U'_0(x) \quad \text{for all } x \in \Omega, \qquad U_{0,\epsilon} \to U_0 \quad \text{in } C(\overline{\Omega})\,, \\ & u_{0,\epsilon} \overset{*}{\rightharpoonup} U'_0 \text{ in } L^\infty(\Omega) \text{ and } u_{0,\epsilon} \to U'_0 \text{ in } L^p(\Omega) \text{ for all } 1 \le p < \infty. \end{aligned} \tag{5.7}$$

Let H satisfy (H_1). We set

$$H_\epsilon(u) := g_\epsilon(u)\left([\eta_\epsilon * H](u) - [\eta_\epsilon * H](0)\right) \qquad (u \in \mathbb{R})\,,$$

where $\{\eta_\epsilon\} \subseteq C_c^\infty(\mathbb{R})$ is a sequence of standard mollifiers and the family $\{g_\epsilon\} \subset C_c^\infty(\mathbb{R})$ satisfies $g_\epsilon = 1$ in $(-1/\epsilon, 1/\epsilon)$, supp $g_\epsilon \subseteq (-2/\epsilon, 2/\epsilon)$, and $0 \le g_\epsilon \le 1$, $|g'_\epsilon| \le 1$ in $\mathbb{R}$. It is easily seen that

$$\sup_{\epsilon\in(0,1)} \|H_\epsilon\|_{W^{1,\infty}(\mathbb{R})} < \infty\,, \qquad H_\epsilon \to H \text{ uniformly on compact subsets of } \mathbb{R}\,. \tag{5.8}$$

Let $m_1, m_2 \in \mathbb{R}$ and let $u_\epsilon \in C^{2,1}(\overline{Q})$ be the unique classical solution (*e.g.*, see [21] of the parabolic problem

$$\begin{cases} u_{\epsilon t} + [H_\epsilon(u_\epsilon)]_x = \epsilon u_{\epsilon xx} & \text{in } Q \\ u_\epsilon = m_1 & \text{in } \{a\} \times (0,T) \\ u_\epsilon = m_2 & \text{in } \{b\} \times (0,T) \\ u_\epsilon = u_{0,\epsilon} & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{D_ϵ}$$

By the maximum principle and (5.5) we have

$$\|u_\epsilon\|_{L^\infty(Q)} \le \max\left\{|m_1|, |m_2|, \|U'_0\|_{L^\infty(\Omega)}\right\} \quad \text{for any } \epsilon \in (0,1)\,. \tag{5.9}$$

Moreover, there exists $c > 0$ such that for any $\epsilon \in (0,1)$

$$\|u_{\epsilon x}\|_{L^\infty(0,T;L^1(\Omega))} \le c\,, \qquad \|u_{\epsilon t}\|_{L^\infty(0,T;L^1(\Omega))} \le c\,, \qquad \epsilon\,\|u_{\epsilon x}\|_{L^\infty(Q)} \le c\,. \tag{5.10}$$

In fact, arguing as in the proof of [27, Proposition 3.1] (see also [1]) and using (5.6) we obtain the first two estimates, and the third one easily follows (see [7, Lemma 6.2] for details).

By (5.10) the family $\{u_\epsilon\}$ is bounded in $L^\infty(Q)$, and $\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{W^{1,1}(Q)} \le M$ for some $M > 0$. It follows from embedding theorems and the uniqueness of the entropy solution $u \in L^\infty(0,T; L^1(\Omega))$ of

$$\begin{cases} u_t + [H(u)]_x = 0 & \text{in } Q \\ u = m_1 & \text{in } \{a\} \times (0,T) \\ u = m_2 & \text{in } \{b\} \times (0,T) \\ u = U'_0 & \text{in } \Omega \times \{0\} \end{cases}, \tag{D_R}$$

that

$$u_\epsilon \to u \quad \text{in } L^1(Q) \quad \text{as } \epsilon \to 0. \tag{5.11}$$

The following result will be used (see [7, Lemma 5.9]).

Lemma 5.1 *Let u be given by (5.11). Then for every $t \in (0, T]$*

$$\|u(\cdot, t)\|_{L^1(\Omega)} \le \|U_0'\|_{L^1(\Omega)} + 2\,\|H\|_\infty t\,. \tag{5.12}$$

It is easily seen that the function

$$U_\epsilon(x,t) := -\int_0^t \{H_\epsilon(u_\epsilon(x,s)) - \epsilon u_{\epsilon x}(x,s)\}\,ds + U_{0,\epsilon}(x) \qquad ((x,t) \in \overline{Q}) \tag{5.13}$$

satisfies $U_{\epsilon x} = u_\epsilon$ in $\overline{Q}$ and is the unique classical solution of

$$\begin{cases} U_{\epsilon t} + H_\epsilon(U_{\epsilon x}) = \epsilon U_{\epsilon xx} & \text{in } Q \\ U_{\epsilon x} = m_1 & \text{in } \{a\} \times (0,T) \\ U_{\epsilon x} = m_2 & \text{in } \{b\} \times (0,T) \\ U_\epsilon = U_{0,\epsilon} & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{N_ϵ}$$

Then, by (5.10), for all $\epsilon \in (0,1)$ there holds

$$\|U_{\epsilon x}\|_{L^\infty(Q)} \le \max\left\{|m_1|, |m_2|, \|U_0'\|_{L^\infty(\Omega)}\right\}, \qquad \|U_{\epsilon xx}\|_{L^\infty(0,T;L^1(\Omega))} \le c\,,$$
$$\|U_{\epsilon xt}\|_{L^\infty(0,T;L^1(\Omega))} \le c\,, \qquad \epsilon\,\|U_{\epsilon xx}\|_{L^\infty(Q)} \le c\,, \qquad \|U_{\epsilon t}\|_{L^\infty(Q)} \le c + \|H\|_\infty \tag{5.14}$$

(the latter estimate follows from the previous one and the equality $U_{\epsilon t} = \epsilon U_{\epsilon xx} - H_\epsilon(U_{\epsilon x})$).

Proposition 5.2 *Let $\Omega = (a,b)$ with $-\infty < a < b < \infty$, $m_1, m_2 \in \mathbb{R}$, and let (H_1) be satisfied. Then for every $U_0 \in C^\infty(\overline{\Omega})$ there exists a viscosity solution of problem (5.1) with initial condition (5.2). Moreover:*
(i) *$U \in W^{1,\infty}(Q)$ and*

$$\|U_x\|_{L^\infty(Q)} \le \max\left\{|m_1|, |m_2|, \|U_0'\|_{L^\infty(\Omega)}\right\}, \tag{5.15a}$$
$$\|U_t\|_{L^\infty(Q)} \le \|H\|_\infty\,. \tag{5.15b}$$

(ii) *$U(x,t) = -\int_0^t H(u(x,s))\,ds + U_0(x)$ and $U_x(x,t) = u(x,t)$ for a.e. $(x,t) \in Q$, where u is the unique entropy solution of problem (D_R).*

Proof By the estimates for $U_{\epsilon x}$ and $U_{\epsilon t}$ in (5.14), the family $\{U_\epsilon\}$ is bounded in $W^{1,\infty}(Q)$. Hence there exist $\{U_{\epsilon_k}\} \subseteq \{U_\epsilon\}$ and $U \in C(\overline{Q})$, with $U_t, U_x \in L^\infty(Q)$, such that $U_{\epsilon_k} \to U$ in $C(\overline{Q})$ (in particular, $U_{\epsilon_k}(0) = U_{0,\epsilon_k} \to U_0$ in $C(\overline{\Omega})$; see (5.7)), and (5.15a) follows at once from (5.14). Claim (ii) follows from (5.13), the equality $U_{\epsilon x} = u_\epsilon$ in $\overline{Q}$, (5.11) and the uniform convergence of U_{ϵ_k} to U in $\overline{Q}$ (observe that, by (5.11) and the last estimate in (5.10), $\epsilon_k u_{\epsilon_k x} \overset{*}{\rightharpoonup} 0$ in $L^\infty(Q)$).

Finally, (5.15b) will follow from (see [8, Proposition 3.2])

$$\inf_{s\in\mathbb{R}}[-H(s)] \le \frac{U(x,t_1) - U(x,t_2)}{t_1 - t_2} \le \sup_{s\in\mathbb{R}}[-H(s)] \qquad (0 < t_1 < t_2 < T)\,, \tag{5.16}$$

as soon as we prove that U is a (continuous) viscosity solution of the equation $U_t + H(U_x) = 0$ in Q. To this purpose, we shall only check conditions (3.18) and (5.3) (checking (3.19) and (5.4) is similar). We distinguish 3 cases: (α), (β), (γ).

(α) Let $(x, t) \in \Omega \times (0, T]$ be a point where $U - \varphi$, with $\varphi \in C^2(\hat{Q})$, has a local maximum. Without loss of generality we may assume that the maximum is strict. Since $U_{\epsilon_k} \to U$ in $C(\overline{Q})$, there exists a sequence $\{(x_k, t_k)\} \subseteq \Omega \times (0, T]$ such that $(x_k, t_k) \to (x, t)$ as $k \to \infty$, and the function $U_{\epsilon_k} - \varphi$ assumes a local maximum at $(x_k, t_k) \in \Omega \times (0, T]$. Combined with the regularity of U_{ϵ_k}, this implies that

$$U_{\epsilon_k x}(x_k, t_k) = \varphi_x(x_k, t_k), \quad U_{\epsilon_k t}(x_k, t_k) \geq \varphi_t(x_k, t_k), \quad U_{\epsilon_k xx}(x_k, t_k) \leq \varphi_{xx}(x_k, t_k),$$

whence

$$\begin{aligned} \varphi_t(x_k, t_k) + H_{\epsilon_k}(\varphi_x(x_k, t_k)) \leq U_{\epsilon_k t}(x_k, t_k) + H_{\epsilon_k}(U_{\epsilon_k x}(x_k, t_k)) = \\ = \epsilon_k U_{\epsilon_k xx}(x_k, t_k) \leq \epsilon_k \varphi_{xx}(x_k, t_k). \end{aligned} \tag{5.17}$$

Letting $k \to \infty$ and using (5.8), we obtain (3.18).

(β) Let $U - \varphi$ ($\varphi \in C^2(\hat{Q})$) assume a strict local maximum at (a, t), $t \in (0, T]$, and let $\varphi_x(a, t) \leq m_1$. Suppose first that $\varphi_x(a, t) < m_1$. Arguing as in (α), there exists a sequence $\{(x_k, t_k)\} \subseteq [a, b) \times (0, T]$ such that $(x_k, t_k) \to (a, t)$ as $k \to \infty$ and $U_{\epsilon_k} - \varphi$ assumes a local maximum at (x_k, t_k). Observe that $x_k > a$ for all k, since otherwise $m_1 = U_{\epsilon_k x}(a, t_k) \leq \varphi_x(a, t_k) < m_1$. So also in this case (5.17) holds, and letting $k \to \infty$ we obtain the first inequality in (5.3): $\varphi_t(a, t) + H(\varphi_x(a, t)) \leq 0$.

Next, let $\varphi_x(a, t) = m_1$. Set

$$\varphi_\delta(x, t) := \varphi(x, t) - \delta(x - a) \qquad ((x, t) \in \hat{Q},\ \delta > 0); \tag{5.18}$$

notice that $\varphi_{\delta t} = \varphi_t$, $\varphi_{\delta x} = \varphi_x - \delta$, and $\varphi_\delta \to \varphi$ in $C(\overline{Q})$ as $\delta \to 0^+$. Since $U - \varphi$ has a strict maximum at (a, t), there exists $\{(x_{\delta_j}, t_{\delta_j})\} \subset [a, b) \times (0, T]$ such that

$$(x_{\delta_j}, t_{\delta_j}) \to (a, t), \qquad U - \varphi_{\delta_j} \text{ has a local maximum at } (x_{\delta_j}, t_{\delta_j}). \tag{5.19}$$

If $x_{\delta_j} \in (a, b)$, as in (α) we obtain that

$$\varphi_t(x_{\delta_j}, t_{\delta_j}) + H(\varphi_x(x_{\delta_j}, t_{\delta_j}) - \delta_j) \leq 0. \tag{5.20}$$

On the other hand, if $x_{\delta_j} = a$, for all sufficiently large j we get $t_{\delta_j} = t$ (recall that $U - \varphi$ achieves a strict local maximum at the point (a, t)), hence $U - \varphi_{\delta_j}$ admits a local maximum at the point (a, t). Since $\varphi_{\delta_j x}(a, t) = \varphi_x(a, t) - \delta_j < m_1$, by the first part of case (β), we get inequality (5.20) in (a, t), namely

$$\varphi_t(a, t) + H(\varphi_x(a, t) - \delta_j) \leq 0. \tag{5.21}$$

Letting $j \to \infty$ in (5.20)–(5.21), the conclusion follows from the continuity of H.

(γ) If $U - \varphi$ achieves a local maximum at (b, t), with $t \in (0, T]$ and $\varphi_x(b, t) \geq m_2$, we argue as in step (β) and distinguish the cases $\varphi_x(b, t) > m_2$ and $\varphi_x(b, t) = m_2$ (we omit the details). □

5.3 Proof of the Correspondence Between Problems (*D*) and (*N*)

We prove Theorem 4.1 first in the case that $u_{0s} = 0$ and $U_0 \in W^{1,1}_{\text{loc}}(\overline{\Omega})$.

Proposition 5.3 *Let (H_1) hold. Let $\Omega = (a, b)$, $-\infty < a < b < \infty$, $U_0 \in W^{1,1}(\Omega)$, $u_0 = U_0'$, $m_1 = \pm\infty$ and $m_2 = \pm\infty$. Let $U \in C(\overline{Q})$ be the unique viscosity solution of problem (N) and let $u \in C([0, T]; L^1(\Omega))$ be the unique entropy solution of problem (D). Then $U \in W^{1,1}(Q)$ and for a.e. $(x, t) \in Q$*

$$U(x, t) = -\int_0^t H(u(x, s))\, ds + U_0(x), \quad U_x(x, t) = u(x, t). \tag{5.22}$$

Similar statements hold if Ω *is unbounded and* $U_0 \in W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$*, with* $U \in W^{1,1}_{\mathrm{loc}}(\overline{Q})$.

Proof of Proposition 5.3 The proof consists of several steps.
(α_1) Let $-\infty < a < b < \infty$, $U_0 \in C^\infty(\overline{\Omega})$, $m_1 = \infty$ and $m_2 = -\infty$ (if $m_1, m_2 = \pm\infty$ the proof is similar). Let $n, p \in \mathbb{N}$ and let $U_{n,p} \in W^{1,\infty}(Q)$ be the viscosity solution of

$$\begin{cases} U_t + H(U_x) = 0 & \text{in } Q \\ U_x(a,t) = n, \ U_x(b,t) = -p & \text{if } t \in (0,T) \\ U = U_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{$N_{n,p}$}$$

constructed in Proposition 5.2. Then,

$$U_{n,p}(x,t) = -\int_0^t H(u_{n,p}(x,s))\,ds + U_0(x)\,, \quad [U_{n,p}]_x(x,t) = u_{n,p}(x,t) \tag{5.23}$$

for a.e. $(x,t) \in Q$, where $u_{n,p}$ is the unique entropy solution of

$$\begin{cases} [u_{n,p}]_t + [H(u_{n,p})]_x = 0 & \text{in } Q \\ u_{n,p}(a,t) = n, \ u_{n,p}(b,t) = -p & \text{if } t \in (0,T) \\ u_{n,p} = U_0' & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{$D_{n,p}$}$$

We first let $n \to \infty$ in the above problems. Observe that

$$u_{n,p} \to u_p \quad \text{in } L^1(Q) \quad \text{as } n \to \infty\,, \tag{5.24}$$

where $u_p \in C([0,T]; L^1(\Omega))$ is an entropy solution ([7, proof of Theorem 6.3]) of

$$\begin{cases} [u_p]_t + [H(u_p)]_x = 0 & \text{in } Q \\ u_p(a,t) = \infty, \ u_p(b,t) = -p & \text{if } t \in (0,T) \\ u_p = U_0' & \text{in } \Omega \times \{0\}. \end{cases} \tag{$D_{\infty,p}$}$$

In view of $(5.23)_1$ and (5.15b), $\{U_{n,p}\}_n$ and $\{(U_{n,p})_t\}_n$ are bounded in $L^\infty(Q)$. It follows from $(5.23)_2$ and (5.24) that $\{(U_{n,p})_x\}_n$ is bounded in $L^1(Q)$ and uniformly integrable. Hence $\{U_{n,p}\}_n$ is uniformly equicontinuous and, possibly up to a subsequence, there exists $U_p \in W^{1,1}(Q)$ with $(U_p)_t \in L^\infty(Q)$ such that

$$U_{n,p} \to U_p \quad \text{in } C(\overline{Q}) \quad \text{as } n \to \infty\,. \tag{5.25}$$

Moreover, by construction, $U_p(\cdot,0) = U_0$ in Ω, $(U_{n,p})_x = u_{n,p} \to u_p$ in $L^1(Q)$,

$$U_p(x,t) = -\int_0^t H(u_p(x,s))\,ds + U_0(x)\,, \quad (U_p)_x(x,t) = u_p(x,t) \tag{5.26}$$

for a.e. $(x,t) \in Q$ (see (5.23)–(5.24)), and, by (5.15b),

$$\|(U_p)_t\|_{L^\infty(Q)} \le \|H\|_\infty\,. \tag{5.27}$$

We claim that U_p is a viscosity solution of problem (5.1) with $m_1 = \infty$, $m_2 = -p$, *i.e.*

$$\begin{cases} (U_p)_t + H((U_p)_x) = 0 & \text{in } Q\,, \\ (U_p)_x(a,t) = \infty, \ (U_p)_x(b,t) = -p & \text{if } t \in (0,T)\,, \\ U_p = U_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{$N_{\infty,p}$}$$

We only check conditions (3.18) and (5.3) (for (3.19) and (5.4) the proof is similar). If $U_p - \varphi$ has a strict local maximum at $(x,t) \in \Omega \times (0,T)$, by (5.25) there exists $\{(x_n,t_n)\} \subseteq \Omega \times (0,T)$

such that $(x_n, t_n) \to (x, t)$ and $U_{n,p} - \varphi$ has a local maximum at $(x_n, t_n) \in \Omega \times (0, T)$. Since $U_{n,p}$ is a viscosity solution of problem $(N_{n,p})$,

$$\varphi_t(x_n, t_n) + H(\varphi_x(x_n, t_n)) \le 0\,. \tag{5.28}$$

If instead $U_p - \varphi$ assume a strict local maximum at (a, t), $t \in (0, T)$, we fix a sufficiently small $\delta > 0$. Then there exists $\{(x_n, t_n)\} \subseteq [a, b) \times (0, T)$ such that: (i) $(x_n, t_n) \to (a, t)$ as $n \to \infty$, $0 < t - \delta \le t_n \le t + \delta < T$ for all sufficiently large n; (ii) $U_{n,p} - \varphi$ achieves a local maximum at (x_n, t_n); (iii) $\varphi_x(x, t) < n$ for all $(x, t) \in \overline{\Omega} \times [t - \delta, t + \delta]$. Since $U_{n,p}$ is a viscosity solution of $(N_{n,p})$ and $\varphi_x(x_n, t_n) < n$, we obtain again (5.28). Letting $n \to \infty$ in (5.28) we obtain the claim. Finally, if $U_p - \varphi$ achieves a local maximum at (b, t), with $t \in (0, T)$, the proof is similar.

To conclude step (α_1), we argue as above and let $p \to \infty$ in problems $(D_{\infty,p})$ and $(N_{\infty,p})$. More precisely, it can be easily checked that $u_p \to u$ in $L^1(Q)$, where $u \in C([0, T]; L^1(\Omega))$ is the unique entropy solution of problem (D) with $m_1 = \infty$, $m_2 = -\infty$ and $u_0 = U_0'$ (see the proof of [7, Theorem 6.3]), and $U_p \to U$ in $C(\overline{Q})$, where U_p is the (unique) viscosity solution of the corresponding (singular) Neumann problem (N) with initial condition U_0. Clearly, by (5.26) and (5.27), it follows that the limiting functions u and U satisfy both (5.22) and the estimate in (5.15b).

(α_2) Let $\Omega = (a, b)$ with $-\infty < a < b < \infty$ and $U_0 \in W^{1,1}(\Omega)$. Let $\{U_{0,k}\} \subseteq C^\infty(\overline{\Omega})$, $U_{0,k} \to U_0$ in $C(\overline{\Omega})$ as $k \to \infty$. Let U_k be the viscosity solution of problem (N) with $m_1 = \pm\infty$, $m_2 = \pm\infty$ and initial condition $U_k(\cdot, 0) = U_{0,k}$, given in step (α_1). Moreover, let $u_{0,k} := U_{0,k}'$, thus $\{u_{0,k}\} \subseteq BV(\Omega)$, $u_{0,k} \to U_0'$ in $L^1(\Omega)$ as $k \to \infty$. Let $\{u_k\}$ be the sequence of entropy solutions to problem (D) with the same boundary conditions $m_1 = \pm\infty$, $m_2 = \pm\infty$ and initial data $u_{0,k}$ considered in step (α_1).

Arguing as in the proof of [7, Theorem 6.3], it can be seen that $u_k \to u$ in $L^1(Q)$ as $k \to \infty$, where u is the entropy solution of problem (D) with initial data $u_0 = U_0'$. On the other hand, by [8, Theorem 3.1] there holds

$$\max_{\overline{Q}} |U_k - U_h| \le \max_{\overline{\Omega}} |U_{0,k} - U_{0,h}| \quad \text{for all } k, h \in \mathbb{N}\,.$$

Hence $\{U_k\}$ is a Cauchy sequence in $C(\overline{Q})$ and there exists $U \in C(\overline{Q})$ such that $U_k \to U$ in $C(\overline{Q})$. Arguing as in step (α_1) we conclude that U is a viscosity solution of problem (N) with initial condition U_0.

Finally we observe that (5.22) and (5.15b) are satisfied by u_k, U_k and $U_{0,k}$ for all $k \in \mathbb{N}$, and so, letting $k \to \infty$, also by u and U. In particular, there holds $U \in W^{1,1}(Q)$. This completes the proof of Proposition 5.3 if Ω is bounded.

(α_3) If Ω is unbounded, we only the consider the case $\Omega = (a, \infty)$, $a \in \mathbb{R}$ (the other cases are similar). Let $\Omega_j := (a, b_j)$, $b_j \le b_{j+1}$ for every $j \in \mathbb{N}$, $b_j \to \infty$ as $j \to \infty$. Let $U_0 \in C(\overline{\Omega})$, $U_{0,j} \in C(\overline{\Omega}_j)$, $\operatorname{supp} U_{0,j} = \Omega_j$, and let $U_{0,j} \to U_0$ uniformly on compact subsets of $[a, \infty)$. Let U_j be the viscosity solution of (N) in $Q_j := \Omega_j \times (0, T)$ with initial condition $U_j(\cdot, 0) = U_{0,j}$ in Ω_j, with the given boundary condition $m_1 = \pm\infty$ at $\{a\} \times (0, T)$ and arbitrary boundary condition $m_2 = \pm\infty$ at $\{b_j\} \times (0, T)$. For every $b > a$ set $K := [a, b] \times [0, T]$, and let $j_0 \in \mathbb{N}$ be fixed such that $b_j > b + \|H'\|_\infty T$ for all $j \ge j_0$. Applying [8, inequality (3.10) in Theorem 3.1] we obtain, for every $i, j \ge j_0$,

$$\max_K |U_j - U_i| \le \max_{[a, b + \|H'\|_\infty T]} |U_{0,j} - U_{0,i}|\,.$$

By the above inequality $\{U_j\}$ is a Cauchy sequence, thus a converging sequence in $C(K)$. Then from the arbitrariness of K, by diagonal and separability arguments, there exists a subsequence of $\{U_j\}$ (not relabelled) and $U \in C(\overline{Q})$ such that $U_j \to U$ uniformly on the compact subsets of $\overline{Q}$. Arguing as in step (α_1) it is shown that U is a viscosity solution of problem (N_+) with initial data U_0.

Similarly, let $u \in C([0,T]; L^1(\Omega))$ be the unique entropy solution of problem $(D)_+$ with the same m_1 as in $(N)_+$ and initial data $u_0 = U_0' \in L^1_{\mathrm{loc}}(\overline{\Omega})$. Let $u_{0,j} = U_{0,j}'$, thus $u_{0,j} \to U_0'$ in $L^1_{\mathrm{loc}}(\overline{\Omega})$ as $j \to \infty$. Let u_j be the entropy solution of

$$\begin{cases} u_t + [H(u)]_x = 0 & \text{in } (a, b_j) \times (0,T) \\ u(a,t) = m_1, \ u(b_j,t) = m_2 & \text{if } t \in (0,T) \\ u = u_{0,j} & \text{in } (a, b_j) \times \{0\} \end{cases}$$

with $m_1 = \pm\infty$ given and $m_2 = \pm\infty$ fixed as above. Then (up to subsequences) $u_j \to u$ in $L^\infty(0,T; L^1(\tilde{\Omega}))$ for all open intervals $\tilde{\Omega} \subset\subset \overline{\Omega}$ (see the proof of [7, Theorem 6.3]). Since $\tilde{\Omega}$ is bounded, it follows from step (α_2) that for all j large enough there holds

$$U_j(x,t) = -\int_0^t H(u_j(x,s))\,ds + U_{0,j}(x)\,, \quad (U_j)_x(x,t) = u_j(x,t)$$

for a.e. $(x,t) \in \tilde{\Omega} \times (0,T))$, and $\|(U_j)_t\|_{L^\infty(Q)} \le \|H\|_\infty$. Then letting $j \to \infty$, it is easily seen that $U \in W^{1,1}_{\mathrm{loc}}(\overline{Q})$ and equality (5.22) follows. □

When (H_2)-(H_3) hold, we set $I_j = (x_{j-1}, x_j)$ for $j = 2, \dots, p$, $I_1 = (a, x_1)$, $I_{p+1} = (x_p, b)$, $Q_j = I_j \times (0,T)$ $(j = 1, \dots, p+1)$. We denote by (D_j) problem (D) stated in Q_j with initial data $u_{0,j} = u_0 \llcorner I_j \in L^1(\overline{I}_j)$, and by (N_j) problem (N) stated in Q_j with initial data $U_{0,j} = U_0 \llcorner I_j \in C(\overline{I_j})$. The proof of the following result can be found in [7, Proposition 5.8].

Proposition 5.4 *Let (H_1)-(H_3) hold.*
(i) *For every $j = 2, \dots, p+1$, let u_j be the entropy solution of (D_j) with $m_1 = \pm\infty$. Then there exists $f^{\pm}_{x^+_{j-1}} \in L^\infty(0,T)$ such that for any $\beta \in C_c(0,T)$*

$$\operatorname*{ess\,lim}_{x \to x^+_{j-1}} \int_0^T H(u_j(x,t))\,\beta(t)\,dt = \int_0^T f^{\pm}_{x^+_{j-1}}(t)\,\beta(t)\,dt\,. \tag{5.29}$$

(ii) *For every $j = 1, \dots, p$ let u_j be the entropy solution of (D_j) with $m_2 = \pm\infty$. Then there exists $f^{\pm}_{x^-_j} \in L^\infty(0,T)$ such that for any $\beta \in C_c(0,T)$*

$$\operatorname*{ess\,lim}_{x \to x^-_j} \int_0^T H(u_j(x,t))\,\beta(t)\,dt = \int_0^T f^{\pm}_{x^-_j}(t)\,\beta(t)\,dt\,. \tag{5.30}$$

Moreover, for a.e. $t \in (0,T)$ there holds

$$\limsup_{u \to \infty} H(u) \le f^{+}_{x^+_{j-1}}(t) \le \sup_{u \in \mathbb{R}} H(u), \tag{5.31a}$$

$$\inf_{u \in \mathbb{R}} H(u) \le f^{-}_{x^+_{j-1}}(t) \le \liminf_{u \to -\infty} H(u), \tag{5.31b}$$

$$\inf_{u \in \mathbb{R}} H(u) \le f^{+}_{x^-_j}(t) \le \liminf_{u \to \infty} H(u), \tag{5.31c}$$

$$\limsup_{u \to -\infty} H(u) \le f^{-}_{x^-_j}(t) \le \sup_{u \in \mathbb{R}} H(u). \tag{5.31d}$$

Remark 5.2 By standard density arguments, from (5.29)–(5.30) we get

$$\text{ess} \lim_{x \to x_{j-1}^+} \int_0^T H(u_j(x,t))\zeta(x,t)\,dt = \int_0^T f^{\pm}_{x_{j-1}^+}(t)\zeta(x_{j-1},t)\,dt \tag{5.32}$$

for all $\zeta \in C^1([0,T]; C_c^1([x_{j-1}, x_j)), \zeta(\cdot, 0) = \zeta(\cdot, T) = 0$ in I_j, and

$$\text{ess} \lim_{x \to x_j^+} \int_0^T H(u_j(x,t))\zeta(x,t)\,dt = \int_0^T f^{\pm}_{x_j^-}(t)\zeta(x_j,t)\,dt \tag{5.33}$$

for all $\zeta \in C^1([0,T]; C_c^1((x_{j-1}, x_j]), \zeta(\cdot, 0) = \zeta(\cdot, T) = 0$ in I_j.

The following result is an easy consequence of Propositions 5.3–5.4.

Lemma 5.5 *Let* (H_1)-(H_3) *hold.*
(i) *Let* $j = 2, \dots, p+1$, *let* U_j *be the viscosity solution of* (N_j) *with* $m_1 = \pm\infty$ *(and* $m_2 = \pm\infty$ *if* $j = 2, \dots, p$*) and initial condition* $U_j(\cdot, 0) = U_{0,j}$. *Let* u_j *be the entropy solution of problem* (D_j) *with the same boundary conditions and initial data* $u_{0,j} = U'_{0,j}$. *Let* $f^{\pm}_{x_{j-1}^+} \in L^\infty(0,T)$ *be given by* Proposition 5.4. *Then*

$$U_j(x_{j-1},t) = -\int_0^t f^{\pm}_{x_{j-1}^+}(s)\,ds + U_{0,j}(x_{j-1}) \quad \textit{for all } t \in (0,T]. \tag{5.34}$$

(ii) *Let* $j = 1, \dots, p$, *let* U_j *be the viscosity solution of* (N_j) *with* $m_2 = \pm\infty$ *(and* $m_1 = \pm\infty$ *if* $j = 2, \dots, p$*) and initial condition* $U_j(\cdot, 0) = U_{0,j}$. *Let* u_j *be the entropy solution of problem* (D_j) *with the same boundary conditions and initial data* $u_{0,j} = U'_{0,j}$. *Let* $f^{\pm}_{x_j^-} \in L^\infty(0,T)$ *be given by* Proposition 5.4. *Then*

$$U_j(x_j,t) = -\int_0^t f^{\pm}_{x_j^-}(s)\,ds + U_{0,j}(x_j) \quad \textit{for all } t \in (0,T]. \tag{5.35}$$

Proof We only prove (i) with $m_1 = \infty$. Since $U_{0,j} \in C(\overline{I}_j)$ and $u_{0,j} \in L^1(\overline{I}_j)$, (5.34) follows from Proposition 5.3, (5.29) and the essential limit $x \to x_{j-1}^+$ in (see (5.22))

$$U_j(x,t) = -\int_0^t H(u_j(x,s))\,ds + U_{0,j}(x) \quad \text{for a.e. } x \in (x_{j-1}, x_j).$$

□

Proof of Theorem 4.1 We rewrite (H_2) as follows:

$$u_{0s} = \sum_{j=1}^{p_+} c_j^+ \,\delta_{x'_j} - \sum_{j=1}^{p_-} c_j^- \,\delta_{x''_j} \qquad (c_j^{\pm} \equiv [c_j]_{\pm} > 0,\ p_+ + p_- = p)\,.$$

Since $u_0 = U'_0$, by (H_3) there holds (see (1.3))

$$c_j = \mathcal{J}_0(x_j) := U_0(x_j^+) - U_0(x_j^-) = U_{0,j+1}(x_j) - U_{0,j}(x_j) \qquad (j = 1, \dots, p)\,.$$

For every $j = 1, \dots, p$ such that $c_j = \mathcal{J}_0(x_j) > 0$ set

$$C_j^+(t) := \left[c_j - \int_0^t \left(f^+_{x_j^+}(s) - f^+_{x_j^-}(s) \right) ds \right]_+ \qquad (t \in [0,T])\,, \tag{5.36}$$

with $f^+_{x_j^+}$ satisfying (5.29) and $f^+_{x_j^-}$ satisfying (5.30); observe that by (5.31a) and (5.31c)

$$f^+_{x_j^+}(s) - f^+_{x_j^-}(s) \geq 0 \quad \text{for a.e. } s \in (0, T)\,. \tag{5.37}$$

Similarly, for every $j = 1, \ldots, p$ such that $c_j = \mathcal{J}_0(x_j) < 0$ set

$$C_j^-(t) := \Big[\, c_j - \int_0^t \Big(f^-_{x_j^+}(s) - f^-_{x_j^-}(s) \Big)\, ds \,\Big]_- \qquad (t \in [0, T])\,, \tag{5.38}$$

with $f^-_{x_j^+}$ satisfying (5.29) and $f^-_{x_j^-}$ satisfying (5.30); observe that by (5.31b) and (5.31d)

$$f^-_{x_j^+}(s) - f^-_{x_j^-}(s) \leq 0 \quad \text{for a.e. } s \in (0, T)\,. \tag{5.39}$$

Moreover, by Proposition 5.3 and (5.34)–(5.35) there holds

$$C_j^{\pm}(t) = \big[\, U_{j+1}(x_j, t) - U_j(x_j, t) \,\big]_{\pm} \qquad (t \in [0, T])\,. \tag{5.40}$$

Let $j = 1, \ldots, p$ and set

$$\tau_1 := \min\{\bar{t}_1, \ldots, \bar{t}_p\}\,, \quad \text{where } \; \bar{t}_j := \sup\{t \in [0, T] \,|\, C_j^{\pm}(t) > 0\}. \tag{5.41}$$

Then $\tau_1 > 0$, since $\bar{t}_j > 0$ and $C_j^{\pm}(0) = c_j^{\pm} > 0$. By (5.37)–(5.39) $C_j^{\pm}$ is nonincreasing in $(0, T)$, whence $C_j^{\pm} > 0$ in $[0, \bar{t}_j)$ and, if $\bar{t}_j < T$, there holds $C_j^{\pm} = 0$ in $[\bar{t}_j, T]$.

Set $Q_{\tau_1} := \Omega \times (0, \tau_1)$, $Q_{j,\tau_1} := I_j \times (0, \tau_1)$. Arguing as in the proof of Theorem 3.2 (see [7, Theorem 3.5]) shows that the unique entropy solution $u \in C([0, \tau_1];\, \mathcal{M}(\Omega))$ of problem (D) in Q_{τ_1} has the following features:

$$\begin{cases} \text{in } \; Q_{1,\tau_1} u_r \; \text{ is the entropy solution of } \; (D_1) \text{with } \; m_2 = \pm\infty \; \text{ if} c_1 \gtrless 0\,; \\ \text{in } \; Q_{j,\tau_1} (j = 2, \ldots, p) u_r \; \text{ is the entropy solution of } \; (D_j) : \\ \text{- with } \; m_1 = m_2 = \infty \; \text{ if } \; \min\{c_{j-1}, c_j\} > 0, \\ \text{- with } \; m_1 = m_2 = -\infty \; \text{ if } \; \max\{c_{j-1}, c_j\} < 0, \\ \text{- with } \; m_1 = \infty, m_2 = -\infty \; \text{ if } \; c_{j-1} > 0 > c_j, \\ \text{- with } \; m_1 = -\infty, m_2 = \infty \, \text{if } \; c_{j-1} < 0 < c_j; \\ \text{in } \; Q_{p+1,\tau_1} u_r \; \text{ is the entropy solution of} \, (D_{p+1}) \; \text{ with } \; m_1 = \pm\infty \; \text{ if } \; c_p \gtrless 0\,; \end{cases}$$

$$u_s(\cdot, t) = \sum_{j=1}^{r} C_j^+(t) \delta_{x_j'} - \sum_{j=1}^{s} C_j^-(t) \delta_{x_j''} = \sum_{j=1}^{p} \big[\, U_{j+1}(x_j, t) - U_j(x_j, t) \,\big] \delta_{x_j} \tag{5.42}$$

(see (5.40)). Similarly, by the proof of [8, Theorem 3.4] (see also [8, Lemma 5.2]), the unique viscosity solution U of problem (N) in Q_{τ_1} with the same boundary conditions has the following features:

$$\begin{cases} \text{in } Q_{1,\tau_1} U \; \text{ is the viscosity solution of} \, (N_1) \; \text{ with } \; m_2 = \pm\infty \; \text{ if } \; J_0(x_1) \gtrless 0\,; \\ \text{in } \; Q_{j,\tau_1} (j = 2, \ldots, p) U \; \text{ is the viscosity solution of } \; (D_j) : \\ \text{- with} \, m_1 = m_2 = \infty \; \text{ if } \; \min\{J_0(x_{j-1}), J_0(x_j)\} > 0, \\ \text{- with } \; m_1 = m_2 = -\infty \; \text{ if } \; \max\{J_0(x_{j-1}), J_0(x_j)\} < 0, \\ \text{- with } \; m_1 = \infty, m_2 = -\infty \; \text{ if } \; J_0(x_{j-1}) > 0 > J_0(x_j), \\ \text{- with } \; m_1 = -\infty, m_2 = \infty \; \text{ if } \; J_0(x_{j-1}) < 0 < J_0(x_j); \\ \text{in } \; Q_{p+1,\tau_1} U \; \text{ is the viscosity solution of} \, (D_{p+1}) \; \text{ with } \; m_1 = \pm\infty \; \text{ if } \; J_0(x_p) \gtrless 0\,. \end{cases}$$

Then, by Proposition 5.3 and (5.42),

- Equality (1.4) holds a.e. in Ω for any $t \in [0, \tau_1]$,
- The second equality in (1.5) holds for any $t \in [0, \tau_1]$.

Let $\rho \in C_c^1(\Omega)$ and $t \in (0, \tau_1)$. Since

$$\int_\Omega U(x,t)\rho'(x)\,dx = -\int_0^t \int_\Omega H(u_r(x,s))\rho'(x)\,dxds - \langle u_0, \rho\rangle_\Omega$$

(see (1.4)) and

$$\langle u_0 - u(t), \rho\rangle_\Omega = -\int_0^t \int_\Omega H(u_r(x,s))\rho'(x)\,dxds$$

(the above equality easily follows by a proper choice of the test function ζ in the weak formulation (3.2)), we get $\int_\Omega U(x,t)\rho'(x)\,dx = -\langle u(t), \rho\rangle_\Omega$. Hence

$$\iint_{Q_{\tau_1}} U(x,t)\rho'(x)h(t)\,dxdt = -\int_0^{\tau_1} h(t)\,\langle u(t), \rho\rangle_\Omega\,dt = -\langle u, h\rho\rangle_{Q_{\tau_1}}$$

for all $h \in C_c^1((0, \tau_1))$, which implies that $U_x = u$ in $\mathcal{D}'(Q_{\tau_1})$. If $\tau_1 = T$, the proof is complete. Otherwise, we can repeat the above argument with a lesser number of discontinuities (possibly zero). Hence the conclusion follows. □

6 Comparison: Proof of Theorem 4.2

The proof of Theorem 4.2 relies on some preliminary definitions and results.

6.1 Sub- and Supersolutions of (D) with Regular Initial Data

We introduce the notions of sub and supersolutions of problem (D) if u_0 is a summable function. If $\Omega = (a, b)$ and $-\infty < a < b < \infty$, problem (D) stands for four different initial-boundary value problems, which we denote by (D_+^+), (D_-^-), (D_+^-) and (D_-^+) according to the four choices $m_1 = m_2 = \infty$, $m_1 = m_2 = -\infty$, $m_1 = \infty$, $m_2 = -\infty$ and $m_1 = -\infty$, $m_2 = \infty$.

Definition 6.1 Let $-\infty < a < b < \infty$, $\Omega = (a, b)$ and $u_0 \in L^1(\Omega)$, and let (H_1) hold. Let $\underline{u} \in C([0, T]; L^1(\Omega))$ satisfy

$$\lim_{t\to 0^+} \int_\Omega [\underline{u}(x,t) - u_0(x)]_+\,dx = 0$$

and, for all $k \in \mathbb{R}$ and $\zeta \in C_c^1(Q)$, $\zeta \geq 0$ in Q,

$$\iint_Q \left\{[\underline{u} - k]_+\zeta_t + \operatorname{sgn}_+(\underline{u} - k)\,[H(\underline{u}) - H(k)]\zeta_x\right\}\,dxdt \geq 0.$$

Then $\underline{u}$ is an *entropy subsolution* of:
(i) problem (D_+^+);

(ii) problem (D_-^-) if for all $k \in \mathbb{R}$, $\beta \in C_c^1(0,T)$, $\beta \geq 0$,

$$\operatorname{ess}\lim_{\xi \to a^+} \int_0^T \operatorname{sgn}_+(\underline{u}(\xi,t)-k)\left[H(\underline{u}(\xi,t))-H(k)\right]\beta(t)\,dt \leq 0\,, \tag{6.1a}$$

$$\operatorname{ess}\lim_{\eta \to b^-} \int_0^T \operatorname{sgn}_+(\underline{u}(\eta,t)-k)\left[H(\underline{u}(\eta,t))-H(k)\right]\beta(t)\,dt \geq 0\,; \tag{6.1b}$$

(iii) problem (D_+^-) if (6.1b) holds for all $k \in \mathbb{R}$, $\beta \in C_c^1(0,T)$, $\beta \geq 0$;
(iv) problem (D_-^+) if (6.1a) holds for all $k \in \mathbb{R}$, $\beta \in C_c^1(0,T)$, $\beta \geq 0$.

Definition 6.2 Let $-\infty < a < b < \infty$, $\Omega = (a,b)$ and $u_0 \in L^1(\Omega)$, and let (H_1) hold. Let $\overline{u} \in C([0,T];L^1(\Omega))$ satisfy

$$\lim_{t \to 0^+} \int_\Omega [\overline{u}(x,t)-u_0(x)]_+\,dx = 0$$

and, for all $k \in \mathbb{R}$ and $\zeta \in C_c^1(Q)$, $\zeta \geq 0$ in Q,

$$\iint_Q \left\{[\overline{u}-k]_-\zeta_t \;+\; \operatorname{sgn}_-(\overline{u}-k)\,[H(\overline{u})-H(k)]\zeta_x\right\}\,dxdt \geq 0.$$

Then $\overline{u}$ is an *entropy supersolution* of:
(i) problem (D_-^-);
(ii) problem (D_+^+) if for all $k \in \mathbb{R}$ and $\beta \in C_c^1(0,T)$, $\beta \geq 0$,

$$\operatorname{ess}\lim_{\xi \to a^+} \int_0^T \operatorname{sgn}_-(\overline{u}(\xi,t)-k)\left[H(\overline{u}(\xi,t))-H(k)\right]\beta(t)\,dt \leq 0\,, \tag{6.2a}$$

$$\operatorname{ess}\lim_{\eta \to b^-} \int_0^T \operatorname{sgn}_-(\overline{u}(\eta,t)-k)\left[H(\overline{u}(\eta,t))-H(k)\right]\beta(t)\,dt \geq 0\,; \tag{6.2b}$$

(iii) problem(D_+^-) if (6.2a) holds for all $k \in \mathbb{R}$, $\beta \in C_c^1(0,T)$, $\beta \geq 0$;
(iv) problem (D_-^+) if (6.2b) holds for all $k \in \mathbb{R}$, $\beta \in C_c^1(0,T)$, $\beta \geq 0$.

If $u \in C([0,T];L^1(\Omega))$ is both an entropy subsolution and supersolution of (D), it is an entropy solution in the sense of Definition 3.3. In fact u satisfies the entropy inequalities and it is also a weak solution (see [7, Remark 5]).

Similar definitions hold when Ω is a half-line and $u_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})$ (see [7]).

For problem (D) with locally L^1-initial data the following comparison result holds (see [7, Theorem 5.7]).

Theorem 6.1 *Let (H_1) hold and let $u_0 \in L^1_{\mathrm{loc}}(\overline{\Omega})$. Let $\underline{u}$, $\overline{u} \in C([0,T];L^1_{\mathrm{loc}}(\overline{\Omega}))$ be an entropy sub- and supersolution of (D) with the same boundary conditions. Then $\underline{u} \leq \overline{u}$ a.e. in Q. In particular, there exists at most one entropy solution of (D).*

6.2 Proof of the Main Result

We prove Theorem 4.2 for problem (D). The proofs for problems $(D)_\pm$ and (CL) are similar.

Proposition 6.2 *Let (H_1) hold. Let $u_0, v_0 \in \mathcal{M}(\Omega)$ satisfy (H_2), and let* $\operatorname{supp} u_{0s}^\pm = \operatorname{supp} v_{0s}^\pm$. *Let $u, v \in C([0,T];\mathcal{M}(\Omega))$ be the entropy solutions of (D) with initial data u_0, v_0 which satisfy the compatibility condition and given by* Theorem 3.2. *Let $\tau \in (0,T]$ be so small that*

$$\operatorname{supp} u_s^\pm(\cdot,t) = \operatorname{supp} v_s^\pm(\cdot,t) = \operatorname{supp} u_0^\pm = \operatorname{supp} v_0^\pm \quad \textit{if}\; 0 \leq t < \tau. \tag{6.3}$$

(i) *If* $u_{0r} \le v_{0r}$ *a.e. in* Ω*, then* $u_r \le v_r$ *a.e. in* $Q_\tau = \Omega \times (0,\tau)$*.*
(ii) *Let* $f_{x_j^\pm}$, $g_{x_j^\pm} \in L^\infty(0,\tau)$ *be the functions in* Proposition 3.3*, related to* u *and* v*, respectively. If* $u_{0r} \le v_{0r}$ *a.e. in* I_j $(j = 1,\dots,p+1)$*, then*

$$f_{x_{j-1}^+} \ge g_{x_{j-1}^+} \text{ for } j = 2,\dots,p+1\,,\ f_{x_j^-} \le g_{x_j^-} \text{ for } \ j = 1,\dots,p, \ \text{ a.e. in } (0,\tau)\,. \tag{6.4}$$

Proof (i) By the compatibility conditions (3.9), in each $Q_{j,\tau} := I_j \times (0,\tau)$, with $I_j = (x_{j-1}, x_j)$ $(j = 1,\dots,p+1;\ x_0 = a,\ x_{p+1} = b)$, $u_{r,j} := u_r \llcorner Q_{j,\tau}$ (resp. $v_{r,j} := v_r \llcorner Q_{j,\tau}$) is the unique entropy solution of (D) with initial data $u_{0r,j} := u_{0r} \llcorner I_j$ (resp. $v_{0r,j} := v_{0r} \llcorner I_j$) and $m_1 = \pm\infty$, $m_2 = \pm\infty$ according to the sign of the initial Dirac masses at x_{j-1} and x_j $(j = 2,\dots,p)$. Since, by (6.3), $u_{r,j}$ and $v_{r,j}$ satisfy the same boundary conditions and $u_{0r,j} \le v_{0r,j}$ a.e. in I_j, the conclusion follows from Theorem 6.1.
(ii) First we prove that $f_{x_{j-1}^+} \ge g_{x_{j-1}^+}$ a.e. in $(0,\tau)$. Let $\zeta \in C^1([0,\tau]; C_c^1([x_{j-1}, x_j))$, $\zeta(\cdot,0) = \zeta(\cdot,\tau) = 0$ in I_j. Arguing as in the proof of [6, Lemma 4.4], we find that

$$\iint_{Q_{j,\tau}} \left\{(u_r - k)\,\zeta_t + [H(u_r) - H(k)]\zeta_x\right\} dxdt = -\int_0^\tau \left[f_{x_{j-1}^+}(t) - H(k)\right] \zeta(x_{j-1}, t)dt. \tag{6.5}$$

Similarly, if $\zeta \ge 0$ in $Q_{j,\tau}$ it follows from the entropy inequality that

$$\begin{aligned}&\iint_{Q_{j,\tau}} \{|u_r - k|\,\zeta_t + \operatorname{sgn}(u_r - k)\,[H(u_r) - H(k)]\,\zeta_x\}\,dxdt \ \ge \\ &\ge -\text{ess}\lim_{x\to x_{j-1}^+} \int_0^\tau \operatorname{sgn}(u_r(x,t) - k)\,[H(u_r(x,t)) - H(k)]\,\zeta(x,t)\,dt\,.\end{aligned} \tag{6.6}$$

for all $k \in \mathbb{R}$. Analogous inequalities hold for v_r.

Since $\operatorname{sgn}(u) = 1 + 2\operatorname{sgn}_-(u)$ and $\operatorname{sgn}(u) = -1 + 2\operatorname{sgn}_+(u)$, summing (6.5) and (6.6) it follows from Remark 5.2 that

$$\begin{aligned}&\iint_{Q_{j,\tau}} \{\,[u_r - k]_+\,\zeta_t + \operatorname{sgn}_+(u_r - k)[H(u_r) - H(k)]\,\zeta_x\}\,dxdt \ \ge \\ &\ge -\frac{1}{2}\Big(\text{ess}\lim_{x\to x_{j-1}^+} \int_0^\tau \operatorname{sgn}(u_r(x,t) - k)\,[H(u_r(x,t)) - H(k)]\,\zeta(x,t)\,dt \ + \\ &\quad + \int_0^\tau \left[f_{x_{j-1}^+}(t) - H(k)\right]\zeta(x_{j-1},t)\,dt\Big) = \\ &= -\,\text{ess}\lim_{x\to x_{j-1}^+} \int_0^\tau \operatorname{sgn}_-(u_r(x,t) - k)\,[H(u_r(x,t)) - H(k)]\,\zeta(x,t)\,dt \ - \\ &\quad - \int_0^\tau \left[f_{x_{j-1}^+}(t) - H(k)\right]\zeta(x_{j-1},t)\,dt\,.\end{aligned} \tag{6.7}$$

Similarly, using again that $\operatorname{sgn}(u) = -1 + 2\operatorname{sgn}_+(u)$, we obtain

$$\begin{aligned}&\iint_{Q_{j,\tau}} \{\,[u_r - k]_+\,\zeta_t + \operatorname{sgn}_+(u_r - k)[H(u_r) - H(k)]\,\zeta_x\}\,dxdt \\ &\quad \ge -\,\text{ess}\lim_{x\to x_{j-1}^+} \int_0^\tau \operatorname{sgn}_+(u_r(x,t) - k)\,[H(u_r(x,t)) - H(k)]\,\zeta(x,t)\,dt\,.\end{aligned} \tag{6.8}$$

On the other hand, if we subtract (6.5) from (6.6), we get

$$\iint_{Q_{j,\tau}} \{ [u_r - k]_- \, \zeta_t + \operatorname{sgn}_-(u_r - k)[H(u_r) - H(k)] \, \zeta_x \} \, dxdt$$
$$\geq - \operatorname{ess} \lim_{x \to x_{j-1}^+} \int_0^\tau \operatorname{sgn}_-(u_r(x,t) - k) \, [H(u_r(x,t)) - H(k)] \, \zeta(x,t) \, dt \,, \quad (6.9)$$

and

$$\iint_{Q_{j,\tau}} \{ [u_r - k]_- \, \zeta_t + \operatorname{sgn}_-(u_r - k)[H(u_r) - H(k)] \, \zeta_x \} \, dxdt \geq$$
$$\geq - \operatorname{ess} \lim_{x \to x_{j-1}^+} \int_0^\tau \operatorname{sgn}_+(u_r(x,t) - k) \, [H(u_r(x,t)) - H(k)] \, \zeta(x,t) \, dt \, +$$
$$+ \int_0^\tau \left[f_{x_{j-1}^+}(t) - H(k) \right] \zeta(x_{j-1}, t) \, dt \,. \quad (6.10)$$

Now let $c_{j-1} > 0$. From (6.7), (6.9) and the compatibility condition (3.9a) (with $j-1$ instead of j) we get

$$\iint_{Q_{j,\tau}} \{ [u_r - k]_+ \, \zeta_t + \operatorname{sgn}_+(u_r - k)[H(u_r) - H(k)] \, \zeta_x \} \, dxdt$$
$$\geq - \int_0^\tau \left[f_{x_{j-1}^+}(t) - H(k) \right] \zeta(x_{j-1}, t) \, dt \,, \quad (6.11a)$$
$$\iint_{Q_{j,\tau}} \{ [u_r - k]_- \, \zeta_t + \operatorname{sgn}_-(u_r - k)[H(u_r) - H(k)] \, \zeta_x \} \, dxdt \geq 0 \,. \quad (6.11b)$$

Suppose instead that $c_{j-1} < 0$. Then from (6.8), (6.10) and the compatibility condition (3.9a) (with $j-1$ instead of j) we get

$$\iint_{Q_{j,\tau}} \{ [u_r - k]_+ \, \zeta_t + \operatorname{sgn}_+(u_r - k)[H(u_r) - H(k)] \, \zeta_x \} \, dxdt \geq 0 \,, \quad (6.12a)$$
$$\iint_{Q_{j,\tau}} \{ [u_r - k]_- \, \zeta_t + \operatorname{sgn}_-(u_r - k)[H(u_r) - H(k)] \, \zeta_x \} \, dxdt$$
$$\geq \int_0^\tau \left[f_{x_{j-1}^+}(t) - H(k) \right] \zeta(x_{j-1}, t) \, dt \,. \quad (6.12b)$$

Obviously, analogous inequalities hold for v_r and $g_{x_{j-1}^+}$.

Now we proceed as in the proof of [6, Theorem 3.2] using the Kružkov method of doubling variables. If $c_{j-1} > 0$ we use (6.11a) and the inequality for $v_r = v_r(y,s)$ analogous to (6.11b), namely

$$\iint_{Q_{j,\tau}} \left\{ [v_r - l]_- \, \xi_s + \operatorname{sgn}_-(v_r - l)[H(v_r) - H(l)] \, \xi_y \right\} dyds \geq 0 \quad (6.13)$$

with $l \in \mathbb{R}$ and $\xi \in C^1([0,\tau]; C^1_c([x_{j-1}, x_j))$, $\xi(\cdot, 0) = \xi(\cdot, \tau) = 0$ in I_j, $\xi \geq 0$ in $Q_{j,\tau}$. Choose $\psi = \psi(x,t,y,s)$, $\psi \geq 0$ such that $\psi(\cdot,\cdot,y,s), \psi(x,t,\cdot,\cdot) \in C^1([0,\tau]; C^1_c([x_{j-1}, x_j))$, and $\psi(\cdot,0,\cdot,\cdot) = \psi(\cdot,\tau,\cdot,\cdot) = \psi(\cdot,\cdot,\cdot,0) = \psi(\cdot,\cdot,\cdot,\tau) = 0$

in I_j. Setting in (6.11a) $k = v_r(y,s)$, $\zeta = \psi(\cdot,\cdot,y,s)$ we have

$$\iint_{Q_{j,\tau}} \big\{\operatorname{sgn}_+(u_r(x,t) - v_r(y,s))[H(u_r(x,t)) - H(v_r(y,s))]\,\psi_x(x,t,y,s)$$
$$+[u_r(x,t) - v_r(y,s)]_+\,\psi_t(x,t,y,s)\big\}\,dxdt$$
$$\geq -\int_0^\tau \Big[f_{x_{j-1}^+}(t) - H(v_r(y,s))\Big]\,\psi(x_{j-1},t,y,s)dt\,,$$

whereas from (6.13) with $l = u_r(x,t)$, $\xi = \psi(x,t\cdot,\cdot)$, using the identities $[u]_- = [-u]_+$, $\operatorname{sgn}_-(-u) = -\operatorname{sgn}_+(u)$ we get

$$\iint_{Q_{j,\tau}} \big\{\operatorname{sgn}_+(u_r(x,t) - v_r(y,s))[H(u_r(x,t)) - H(v_r(y,s))]\,\psi_y(x,t,y,s)$$
$$+[u_r(x,t) - v_r(y,s)]_+\,\psi_s(x,t,y,s)\big\}\,dyds\ \geq 0\,.$$

Now choose
$$\psi(x,t,y,s) = \eta\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\rho_\epsilon(x-y)\,\rho_\epsilon(t-s)$$

where $\eta \in C^1([0,\tau]; C_c^1([x_{j-1},x_j)))$, $\eta \geq 0$, $\eta(\cdot,0) = \eta(\cdot,\tau) = 0$ in I_j, and ρ_ϵ ($\epsilon > 0$) is a symmetric mollifier in $\mathbb{R}$. Arguing as in the proof of [6, Theorem 3.2], from the above inequalities we get

$$\iint_{Q_{j,\tau}} \big\{\operatorname{sgn}_+(u_r(x,t) - v_r(x,t))[H(u_r(x,t)) - H(v_r(x,t))]\,\eta_x$$
$$+[u_r(x,t) - v_r(x,t)]_+\,\eta_t\big\}\,dxdt\ \geq -\frac{1}{2}\int_0^\tau \Big[f_{x_{j-1}^+}(t) - g_{x_{j-1}^+}(t))\Big]\,\eta(x_{j-1},t)dt\,. \tag{6.14}$$

Recalling that if $u_{0r,j+1} \leq v_{0r,j+1}$ a.e. in I_j then, by part (i), $u_{r,j+1} \leq v_{r,j+1}$ a.e. in $Q_{j,\tau}$, we obtain from (6.14) and the arbitrariness of η that $f_{x_{j-1}^+} \geq g_{x_{j-1}^+}$ a.e. in $(0,\tau)$.

If $c_{j-1} < 0$ we use (6.12a) and the inequality for $v_r = v_r(y,s)$ analogous to (6.12b),

$$\iint_{Q_{j,\tau}} \big\{[v_r - l]_-\,\xi_s + \operatorname{sgn}_-(v_r - l)[H(v_r) - H(l)]\,\xi_y\big\}\,dyds$$
$$\geq \int_0^\tau \Big[g_{x_{j-1}^+}(s) - H(l)\Big]\,\xi(x_{j-1},s)\,ds \tag{6.15}$$

with $l \in \mathbb{R}$ and ξ as above. Choosing in (6.12a) $k = v_r(y,s)$, $\zeta = \psi(\cdot,\cdot,y,s)$ with ψ as above gives

$$\iint_{Q_{j,\tau}} \big\{\operatorname{sgn}_+(u_r(x,t) - v_r(y,s))[H(u_r(x,t)) - H(v_r(y,s))]\,\psi_x(x,t,y,s)$$
$$+[u_r(x,t) - v_r(y,s)]_+\,\psi_t(x,t,y,s)\big\}\,dxdt\ \geq 0\,.$$

On the other hand, from (6.15) with $l = u_r(x,t)$, $\xi = \psi(x,t\cdot,\cdot)$, using again the identities $[u]_- = [-u]_+$, $\text{sgn}_-(-u) = -\text{sgn}_+(u)$ we get

$$\iint_{Q_{j,\tau}} \big\{\text{sgn}_+(u_r(x,t)-v_r(y,s))[H(u_r(x,t))-H(v_r(y,s))]\,\psi_y(x,t,y,s)$$
$$+[u_r(x,t)-v_r(y,s)]_+\,\psi_s(x,t,y,s)\big\}\,dyds$$
$$\geq \int_0^\tau \Big[g_{x^+_{j-1}}(s) - H(u_r(x,t))\Big]\,\psi(x_{j-1},t,y,s)\,ds\,.$$

Then arguing as in the proof of (6.14) we get inequality (6.14) for any η as above, whence $f_{x^+_{j-1}} \geq g_{x^+_{j-1}}$ a.e. in $(0,\tau)$.

Concerning the inequalities $f_{x^-_j} \leq g_{x^-_j}$ $(j = 1,\dots,p)$ a.e. in $(0,\tau)$, the proof relies on the following counterpart of (6.5)–(6.6):

$$\iint_{Q_{j,\tau}} \big\{(u_r-k)\,\zeta_t\;+\;[H(u_r)-H(k)]\,\zeta_x\big\}dxdt = \int_0^\tau \Big[f_{x^-_j}(t) - H(k)\Big]\,\zeta(x_j,t)\,dt\,,$$
$$\iint_{Q_{j,\tau}} \{|u_r-k|\,\zeta_t + \text{sgn}\,(u_r-k)\,[H(u_r)-H(k)]\,\zeta_x\}\,dxdt$$
$$\geq \text{ess}\lim_{x\to x^-_j}\int_0^\tau \text{sgn}\,(u_r(x,t)-k)\,[H(u_r(x,t))-H(k)]\,\zeta(x,t)\,dt$$

where $\zeta \in C^1([0,\tau];C^1_c((x_{j-1},x_j]), \zeta \geq 0, \zeta(\cdot,0) = \zeta(\cdot,\tau) = 0$ in I_j, and on the compatibility condition (3.9b). We leave the details to the reader. □

Now we can prove Theorem 4.2.

Proof of Theorem 4.2 Let

$$\tau = \sup\{t\in(0,T);\;\text{supp}\,u_s(t) = \text{supp}\,u_{0s},\;\text{supp}\,v_s(t) = \text{supp}\,v_{0s}\}.$$

Set

$$\text{supp}\,u_{0s}\,\cup\,\text{supp}\,v_{0s} \equiv \{y_1,\dots,y_r\}\;\text{ with }\;y_1<y_2<\dots<y_r,$$
$$u_{0s} = \sum_{k=1}^r \hat{c}_k\delta_{y_k}\,,\quad v_{0s} = \sum_{k=1}^r \hat{d}_k\delta_{y_k}$$

with $\hat{c}_k, \hat{d}_k \in \mathbb{R}$, at least one of $\hat{c}_k, \hat{d}_k$ different from zero, $\hat{c}_k \leq \hat{d}_k$; observe that

$$\hat{c}_k\hat{d}_k \neq 0 \quad\Leftrightarrow\quad y_k \in \text{supp}\,u_{0s}\,\cap\,\text{supp}\,v_{0s} \qquad (k=1,\dots,r)\,.$$

Also set $I_k = (y_{k-1},y_k)$, with $y_0 = a$, $y_{r+1} = b$, $Q_{k,\tau} = I_k\times(0,\tau)$, and $u_{0r,k} = u_{0r}\llcorner I_k$, $v_{0r,k} = v_{0r}\llcorner I_k$, $u_{r,k} = u_r\llcorner Q_{k,\tau}$, $v_{r,k} = v_r\llcorner Q_{k,\tau}$ $(k=1,\dots,r+1)$.

By assumption there holds $u_{0r} \leq v_{0r}$ a.e. in I_k for any k. We claim that

$$u_r \leq v_r \quad\text{in } Q_{k,\tau} \quad\text{for all } k = 1,\dots,r+1. \tag{6.16}$$

Observe that at each point y_k there holds either $\hat{c}_k\hat{d}_k \leq 0$, or $\hat{c}_k\hat{d}_k > 0$. If $\hat{c}_k\hat{d}_k = u_{0s}(\{y_k\})\,v_{0s}(\{y_k\}) \leq 0$, by (3.5) there holds $u_s(\cdot,t)(\{y_k\}) \leq 0 \leq v_s(\cdot,t)(\{y_k\})$ for any $t\in(0,\tau)$, thus in this case

$$u_s(\cdot,t)\llcorner\{y_k\} \leq v_s(\cdot,t)\llcorner\{y_k\}\;\text{ for any }\;t\in(0,\tau)\,. \tag{6.17}$$

On the other hand, if $\hat{c}_k\hat{d}_k > 0$, there holds either $\hat{c}_k > 0$, $\hat{d}_k > 0$, or $\hat{c}_k < 0$, $\hat{d}_k < 0$. By Proposition 3.4, for any $t \in (0, \tau)$ there holds

$$u_s(\cdot, t)\llcorner\{y_k\} = C_k(t)\delta_{y_k}\,, \quad v_s(\cdot, t)\llcorner\{y_k\} = D_k(t)\delta_{y_k}\,, \tag{6.18}$$

where C_k are defined by (3.16), and D_k are the analogous quantities for v_s. Assuming $u_r \le v_r$ in $Q_{k,\tau}$ and arguing as in the proof of Proposition 6.2(ii), it is easily seen that inequalities (6.4) hold (with x_k^+ instead of x_{j-1}^+) for any $t \in (0, \tau)$, whence in both cases $\hat{c}_k, \hat{d}_k > 0$ or $\hat{c}_k, \hat{d}_k < 0$ we get

$$C_k(t) \le D_k(t) \quad \text{for all } t \in [0, \tau)\,. \tag{6.19}$$

From (6.18) and (6.19) we obtain (6.17) also in this case. Then by (6.16) and (6.17) there holds $u(\cdot, t) \le v(\cdot, t)$ in $\mathcal{M}(\Omega)$ for any $t \in [0, \tau]$.

If $\tau = T$ the proof is complete. Otherwise, we can repeat the above arguments in $\Omega \times [\tau, T]$, since we proved that $u(\cdot, \tau) \le v(\cdot, \tau)$ in $\mathcal{M}(\Omega)$. In a finite time of steps the conclusion follows.

It remains to prove the claim (6.16). We only consider the case that $k = 2, \dots, r$, the proof being simpler for $k = 1$ or $r + 1$. We distinguish the following cases:

(a) $\hat{c}_{k-1}\hat{d}_{k-1} > 0$, $\hat{c}_k\hat{d}_k > 0$. In this case u_r and v_r are solutions of the same problem $(D_k) \equiv (D)$ in $Q_{k,\tau}$. Since by assumption there holds $u_{0r} \le v_{0r}$ a.e. in I_k, (6.16) follows from Proposition 6.2.

(b) $\hat{c}_{k-1}\hat{d}_{k-1} > 0$, $\hat{c}_k\hat{d}_k \le 0$. We consider two subcases:

- (b_1) $\hat{c}_k < 0$, $\hat{d}_k \ge 0$. In this case u_r solves problem $(D^-_\pm)$ in $Q_{k,\tau}$, depending on $\pm\hat{c}_{k-1} > 0$. Since in both cases $\hat{d}_k > 0$ or $\hat{d}_k = 0$ it can be easily checked that v_r is an entropy supersolution of problem $(D^-_\pm)$ in $Q_{k,\tau}$, depending on $\pm\hat{c}_{k-1} > 0$ (see Definition 6.2(ii) and (iii)), hence (6.16) follows from Theorem 6.1.
- (b_2) $\hat{c}_k \le 0$, $\hat{d}_k > 0$. In this case v_r solves problem $(D^+_\pm)$ in $Q_{k,\tau}$, depending on $\pm\hat{c}_{k-1} > 0$. In both cases $\hat{c}_k < 0$ or $\hat{c}_k = 0$, we get that u_r is an entropy subsolution of problem $(D^+_\pm)$ in $Q_{k,\tau}$, depending on $\pm\hat{c}_{k-1} > 0$ (see Definition 6.1(i) and (iv)), and (6.16) follows from Theorem 6.1.

(c) $\hat{c}_{k-1}\hat{d}_{k-1} \le 0$, $\hat{c}_k\hat{d}_k > 0$. This case is analogous to (b); we omit the details.

(d) $\hat{c}_{k-1} < 0, \hat{d}_{k-1} = 0, \hat{c}_k = 0, \hat{d}_k > 0$. It is easily checked that u_r is an entropy subsolution and v_r is an entropy supersolution of problem (D^+_-) in $Q_{k,\tau}$ (see Definitions 6.1(iv) and 6.2(iv)). Again (6.16) follows from Theorem 6.1.

(e) $\hat{c}_{k-1} = 0$, $\hat{d}_{k-1} > 0$, $\hat{c}_k < 0$, $\hat{d}_k = 0$. This case is analogous to (d).

□

7 Waiting Time for Global Solutions of (*HJ*) and (*CL*): Proofs

In this section we prove the results about the waiting times listed in Sect. 4.3. We observe that Theorem 4.4 is an immediate consequence of (3.31).

Proof of Theorem 4.5 We only address the case that $J_0(x_j) > 0$. As outlined in the Introduction, until the waiting time $\tau_j \in (0, +\infty]$, the jump discontinuity at x_j has a barrier effect in the following sense: by [8, Lemma 5.2], $U_1 = U\llcorner((x_j, \infty) \times (0, \tau_j))$ and

$U_2 = U\llcorner((-\infty, x_j) \times (0, \tau_j))$ are the viscosity solutions of the problems

$$\begin{cases} U_{1t} + H(U_{1x}) = 0 & \text{in } (x_j, \infty) \times (0, \tau_j) \\ U_{1x} = \infty & \text{in } \{x_j\} \times (0, \tau_j) \\ U_1 = U_0\llcorner(x_j, \infty) & \text{in } (x_j, \infty) \times \{0\} \end{cases} \tag{7.1}$$

and

$$\begin{cases} U_{2t} + H(U_{2x}) = 0 & \text{in } (-\infty, x_j) \times (0, \tau_j) \\ U_{2x} = \infty & \text{in } \{x_j\} \times (0, \tau_j) \\ U_1 = U_0\llcorner(-\infty, x_j) & \text{in } (-\infty, x_j) \times \{0\}. \end{cases} \tag{7.2}$$

In view of assumption (H_4)-(i), we consider the case that for all $M > 0$ there exists $k_M > M$ such that $H(k_M) > H^+$ (if $H(k_M) < H^+$ the proof is similar). By (A_1) we have that $|U_0(x)| \leq A_j + B|x - x_j|$, where $A_j = A + B|x_j|$). We set, for all $k > B$ such that $H(k) > H^+$,

$$v(x,t) := C_k + k(x - x_j) - H(k)t \qquad \text{for } (x,t) \in (x_j, \infty) \times (0, \tau_j)\,,$$

where C_k is chosen such that

$$v(x,0) \geq A_j + B(x - x_j) \geq (U_0)^*(x) \quad \text{for all } x \geq x_j\,. \tag{7.3}$$

By (3.21) and the envelope properties we have that $(U_0)^*(x) = U^*(x,0) \geq U_1^*(x,0)$ for all $x \geq x_j$, thus inequality (7.3) gives

$$v(x,0) \geq U_1^*(x,0) \quad \text{for all } x \geq x_j\,. \tag{7.4}$$

Since v is a viscosity supersolution of (7.1) (see [8, Definition 3.2]), by the comparison principle in [8, Theorem 3.1] and (7.4) we get

$$(U_1)^*(x,t) \leq v(x,t) \quad \text{for all } (x,t) \in [x_j, \infty) \times [0, \tau_j)\,. \tag{7.5}$$

Next, observe that Theorem 3.5(a) ensures that $U_1^*(x,t) = U(x,t)$ for all $x > x_j$ sufficiently close to x_j; here, as in Remark 3.1, we have identified U with its continuous representative $\tilde{U}_{j+1}$ in the rectangle $Q_{j+1} = (x_j, x_{j+1}) \times (0, \tau_j)$. Therefore letting $x \to x_j^+$ in (7.5) gives

$$U(x_j^+, t) \leq C_k - H(k)t \quad \text{for any } t \in (0, \tau_j)\,. \tag{7.6}$$

For all t as above there also holds

$$U(x_j^-, t) \geq U_0(x_j^-) - H^+ t \tag{7.7}$$

(see inequalities (5.21) in [8] for details). Then from (7.6)–(7.7) we obtain

$$(H(k) - H^+)t \leq \underbrace{U(x_j^-, t) - U(x_j^+, t)}_{<0 \text{ by } (3.30)} + C_k - U_0(x_j^-) \quad \text{for any } t \in (0, \tau_j)\,.$$

Therefore, letting $t \to \tau_j^-$, the claim follows from the estimate $\tau_j \leq \dfrac{C_k - U_0(x_j^-)}{H(k) - H^+}$. □

Proof of Corollary 4.6 We first prove (4.6). For every $x \in \mathbb{R}$, set $U_0(x) = u_0([0,x])$, and let U be the global viscosity solution of (HJ) with initial datum U_0. Since U_0 satisfies assumption (H_3), we can apply the correspondence between u and U stated in Theorem 4.1. Then (4.6) follows from (4.2) and the identifications in (4.3)–(4.4).

It remains to prove that the waiting time is finite if (A_2) is satisfied. Observe that $U_0(x) = u_0([0, x])$ $(x \in \mathbb{R})$ satisfies (H_3) and (A_1), as $\|u_{0s}\|_{\mathcal{M}(\mathbb{R})} \leq C$ (see (H_2)) and u_{0r} satisfies (A_2). Applying Theorem 4.5 to the global viscosity solution U of (HJ) with initial datum U_0, the desired results follow from (4.3)–(4.4). □

It remains to prove Theorem 4.7, which immediately implies Corollary 4.8. In the proof we distinguish the two different hypotheses, (H_5) and (H_6).

Proof of Theorem 4.7 the case of hypothesis (H_5). We only address the case that $c_j > 0$ and (H_5)-(i) is satisfied (when $c_j < 0$ and (H_5)-(ii) holds the proof is similar). Let $\{k_n\}$ be a sequence diverging to ∞ such that

$$\lim_{n\to\infty} \frac{|H(k_n) - H^+|}{M_{k_n}} = \limsup_{k\to\infty} \frac{|H(k) - H^+|}{M_k} \geq C_0^+ > 0\,. \tag{7.8}$$

Since $M_k = \|H'\|_{L^\infty(k,\infty)} \to 0$ as $k \to \infty$, we have that

$$\lim_{n\to\infty} M_{k_n} = 0\,, \tag{7.9}$$

whereas by assumption (H_4)-(i), possibly up to a subsequence (not relabeled), there holds either $H(k_n) > H^+$ or $H(k_n) < H^+$ for every n. Without loss of generality, we may assume that $H(k_n) > H^+$ for all n.

Let $\operatorname{supp} u_{0s}^+ \equiv \{x_1, \ldots, x_q\}$ $(x_1 < x_2 < \cdots < x_q)$. Below we prove that the waiting time t_q associated to x_q is finite. By a recursive argument, it follows that all Dirac masses of u_{0s}^+ disappear in finite time.

By contradiction, suppose that $t_q = \infty$. Let $T > 0$ be fixed arbitrarily. Arguing as in the proof of Proposition 6.2(ii) (in particular, see (6.11a)), for every $k > 0$ and $\zeta \in C^1([0, T]; C_c^1([x_q, \infty)))$, $\zeta \geq 0$, $\zeta(\cdot, T) = 0$, we get

$$\begin{aligned}&\int_0^T \int_{x_q}^\infty \left\{[u_q - k]_+ \zeta_t + \operatorname{sgn}_+(u_q - k)[H(u_q) - H(k)]\zeta_x\right\} dxdt \geq \\ &\geq -\int_{\mathbb{R}} [u_{0r} - k]_+ \zeta(x, 0)\, dx - \int_0^T [f_{x_q^+} - H(k)]\zeta(x_q, t)\, dt\,.\end{aligned} \tag{7.10}$$

Let $\gamma > x_q$ be arbitrarily fixed. For every $k > 0$ and $p \in \mathbb{N}$ large enough we set

$$\beta_p(t) := \chi_{[0,T-1/p]}(t) + p(T-t)\chi_{(T-1/p,T](t)} \qquad (t \in (0, T))$$

$$\zeta_{k,p}(x,t) = \begin{cases} 1 & \text{if } x_q \leq x \leq \gamma + M_k(T-t) - \frac{1}{p}, \\ p\left[\gamma + M_k(T-t) - x\right] & \text{if } \gamma + M_k(T-t) - \frac{1}{p} < x < \gamma + M_k(T-t), \\ 0 & \text{if } x \geq \gamma + M_k(T-t) \end{cases}$$

for $(x, t) \in \mathbb{R} \times (0, T)$. One easily sees that, by the definitions of M_k and $\zeta_{k,p}$,

$$\int_0^T \int_{x_q}^\infty \underbrace{\left\{[u_q - k]_+ \partial_t \zeta_{k,p} + \operatorname{sgn}_+(u_q - k)[H(u_q) - H(k)]\partial_x \zeta_{k,p}\right\}}_{\leq 0} \beta_p(t)\, dxdt \leq 0.$$

Choosing $\zeta(x,t) = \zeta_{k,p}(x,t)\beta_p(t)$ in (7.10) and letting $p \to \infty$, this implies that

$$\int_0^T [f_{x_q^+}(t) - H(k)]\, dt + \int_{x_q}^{\gamma + M_k T} [u_{0r} - k]_+\, dx \geq \int_{x_q}^{\gamma} [u_q(x, T) - k]_+\, dx \geq 0\,,$$

whence, by the second inequality in (3.14),

$$\int_0^T \Big[f_{x_q^+}(t) - f_{x_q^-}(t) \Big]\, dt + \int_{x_q}^{\gamma+M_k T} [u_{0r} - k]_+\, dx \geq$$
$$\geq \int_0^T \Big[H(k) - f_{x_q^-}(t) \Big]\, dt \geq [H(k) - H^+]\, T. \tag{7.11}$$

Since $t_q = \infty$, it follows from (3.16)–(3.17) that

$$\int_0^T \Big[f_{x_q^+}(t) - f_{x_q^-}(t) \Big]\, dt \leq u_{0s}^+(\{x_q\}) \quad \text{for all } T > 0. \tag{7.12}$$

Let $\{k_n\}$ be any sequence satisfying (7.8)–(7.9) and $H(k_n) > H^+$ for all n. From (7.11)–(7.12) (written with $k = k_n$), for every $T > 0$ and $\gamma > x_q$ we get

$$[H(k_n) - H^+]T \leq u_{0s}^+(\{x_q\}) + \int_{x_q}^{\gamma+M_{k_n} T} [u_{0r} - k_n]_+\, dx\,. \tag{7.13}$$

Set $T_n := \dfrac{2u_{0s}^+(\{x_q\})}{C_0^+ M_{k_n}}$. Then from (7.8) we obtain

$$\lim_{n\to\infty} [H(k_n) - H^+]T_n = \lim_{n\to\infty} \frac{2u_{0s}^+(\{x_q\})|H(k_n) - H^+|}{C_0^+ M_{k_n}} \geq 2u_{0s}^+(\{x_q\})\,. \tag{7.14}$$

Moreover, there holds

$$\lim_{n\to\infty} \int_{x_q}^{\gamma+M_{k_n} T_n} [u_{0r} - k_n]_+\, dx = 0\,, \tag{7.15}$$

since $\gamma + M_{k_n} T_n = \gamma + 2u_{0s}^+(\{x_q\})/C_0^+$ and $u_{0r} \in L^1_{loc}(\mathbb{R})$. By (7.14)–(7.15), choosing $T = T_n$ in (7.13) and letting $n \to \infty$ we obtain $u_{0s}^+(\{x_q\}) \leq 0$, a contradiction. □

Proof of Theorem 4.7 the case of hypothesis (H_6). Let (H_6)-(i) be satisfied and

$$H(k) < H^+ \ \text{ for } k \geq \overline{k} \qquad (\overline{k} > 0) \tag{7.16}$$

(in case of (H_6)-(ii) the proof is similar). Fix $x_j \in \operatorname{supp} u_{0s}^+$ and let $w \in C([0,\infty); \mathcal{M}^+(\mathbb{R}))$ be the global entropy solution of problem (CL) with initial data

$$w_0 := \max\{u_{0r}, \overline{k}\} + u_{0s}^+\,,$$

satisfying the compatibility conditions in $\operatorname{supp} w_{0s} = \operatorname{supp} u_{0s}^+ = \{x_1, \ldots, x_q\}$. By the comparison principle (see Theorem 4.2), it suffices to prove that the waiting time $\tilde{t}_j$ associated to each x_j $(j = 1, \ldots, q)$ is finite.

Since $w_{0r} \geq \overline{k}$ a.e. in $\mathbb{R}$ and $w_{0s} \geq 0$ in $\mathcal{M}(\mathbb{R})$, it follows from (3.4), using a proper sequence of test functions, that $w_r \geq \overline{k}$ a.e. in S. Hence w also is the global entropy solution of the Cauchy problem

$$\begin{cases} w_t + [\tilde{H}(w)]_x = 0 & \text{in } S = \mathbb{R} \times \mathbb{R}^+ \\ w = w_0 & \text{in } \mathbb{R} \times \{0\}\,, \end{cases}$$

where $\tilde{H}(w) := H\big((w - \overline{k})^+ + \overline{k}\big)$, satisfying the compatibility conditions at every $x_j \in \operatorname{supp} w_{0s} = \operatorname{supp} u_{0s}^+$. By the definition of $\tilde{H}$ and assumption (7.16), there holds

$$\lim_{u\to\infty} \tilde{H}(u) = \sup_{u\in\mathbb{R}} \tilde{H}(u) = H^+\,. \tag{7.17}$$

For every $j = 1, \ldots, q$ let $h_{x_j^\pm} \in L^\infty_{\mathrm{loc}}(0,\infty)$ be the functions relative to w given by Proposition 3.3. Then by (3.12) and (7.17) we get

$$h_{x_j^+}(t) = H^+ \quad \text{for a.e. } t \in (0, t_j). \tag{7.18}$$

By contradiction, let $\tilde{t}_j = \infty$. Then by (3.16) and (7.18) we get

$$\int_0^\infty [H^+ - h_{x_j^-}(t)]\, dt \le c_j\,. \tag{7.19}$$

Fix any $\gamma < x_j$ such that $u^+_{0s} \llcorner I = 0$, where $I \equiv (\gamma, x_j)$. Consider the singular Cauchy-Dirichlet problem

$$\begin{cases} v_t + [\tilde{H}(v)]_x = 0 & \text{in } I \times (0,\infty) \\ v = \infty & \text{in } \{\gamma, x_j\} \times (0,\infty) \\ v = w_{0r} & \text{in } I \times \{0\}\,. \end{cases} \tag{7.20}$$

By Definition 6.1(i) the restriction $w \llcorner (I \times (0,\infty))$ is a subsolution of (7.20), whereas by Theorem 3.2(i) there exists a unique global entropy solution $v \in C([0,\infty); L^1(I))$, $v \ge 0$ of (7.20). Then by Theorem 6.1 we get

$$w \le v \quad \text{a.e. in } I \times (0,\infty)\,. \tag{7.21}$$

Let $g_{x_j^-},\ g_{\gamma^+} \in L^\infty_{\mathrm{loc}}(0,\infty)$ be the functions relative to v given by Proposition 5.4. Arguing as for (7.18), from (5.31a) we get

$$g_{\gamma^+}(t) = H^+ \ge g_{x_j^-}(t) \quad \text{for a.e. } t > 0\,. \tag{7.22}$$

On the other hand, in view of (7.21), arguing as in the proof of Proposition 6.2(ii) gives

$$h_{x_j^-}(t) \le g_{x_j^-}(t) \quad \text{for a.e. } t > 0\,,$$

whence by inequality (7.19)

$$\int_0^\infty [H^+ - g_{x_j^-}(t)]\, dt \le c_j\,. \tag{7.23}$$

Fix any $T > 0$. From the weak formulation (3.2), by a standard argument we get

$$\int_I v(x,T)\rho(x)\, dx = \int_I w_{0r}(x)\rho(x)\, dx + \iint_{I\times(0,T)} \tilde{H}(v(x,t))\rho'(x)\, dxdt \tag{7.24}$$

for every $\rho \in C^1_c(I)$. By a proper choice of $\rho = \rho_n \to \chi_I$ as $n \to \infty$, we get

$$\|v(\cdot,T)\|_{L^1(I)} = \int_I w_{0r}(x)\, dx + \int_0^T [H^+ - g_{x_j^-}(t)]\, dt \le \|w_{0r}\|_{L^1(I)} + c_j =: D_0\,; \tag{7.25}$$

here we used inequalities (7.22)–(7.23) and the fact that for all $\beta \in C_c(0,\infty)$ (see (5.29)–(5.30)) there holds

$$\lim_{x\to x_j^-} \int_0^\infty \tilde{H}(v(x,t))\beta(t)\, dt = \int_0^\infty g_{x_j^-}(t)\beta(t)\, dt\,,$$

$$\lim_{x\to \gamma^+} \int_0^\infty \tilde{H}(v(x,t))\beta(t)\, dt = \int_0^\infty g_{\gamma^+}(t)\beta(t)\, dt\,.$$

Similarly, for a.e. $y \in (\gamma, x_j)$, a suitable choice of $\rho = \rho_n \to \chi_{(\gamma,y)}$ in (7.24) implies

$$\int_\gamma^y v(x,T)\,dx = \int_\gamma^y w_{0r}(x)\,dx + \int_0^T \Big[H^+ - \tilde{H}(v(y,t))\Big]\,dt\,,$$

whence, by integration with respect to y and (7.25),

$$\int_0^T \left(\int_I \Big[H^+ - \tilde{H}(v(y,t))\Big]\,dy\right)dt \le \int_I \left(\int_\gamma^y v(x,T)\,dx\right)dy \le D_0\,|I|\,.$$

By (7.17), this implies that

$$\int_0^T \|\tilde{H}(v(\cdot,t)) - H^+\|_{L^1(I)}\,dt \le D_0\,|I|\,.$$

By the arbitrariness of T, there exists a sequence $T_k \to \infty$ such that

$$\|\tilde{H}(v(\cdot,T_k)) - H^+\|_{L^1(I)} \to 0\,,$$

whence (possibly up to a subsequence, not relabeled)

$$\tilde{H}(v(x,T_k)) \to H^+ \quad \text{for a.e. } x \in I\,.$$

In view of (7.17), this implies that

$$v(x,T_k) \to \infty \quad \text{for a.e. } x \in I\,,$$

whence $\|v(\cdot,T_k)\|_{L^1(I)} \to \infty$. However, this contradicts estimate (7.25). □

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