

MULTIPARAMETER QUANTUM SUPERGROUPS, DEFORMATIONS AND SPECIALIZATIONS

GASTÓN ANDRÉS GARCÍA^b, FABIO GAVARINI[#], MARGHERITA PAOLINI[‡]

ABSTRACT. In this paper we introduce a multiparameter version of the quantum universal enveloping superalgebras introduced by Yamane in [Ya1]. For these objects we consider:

- (1) their deformations by twist and by 2-cocycle (both of “toral type”); in particular, we prove that this family is stable under both types of deformations;
- (2) their semiclassical limits, which are multiparameter Lie superbialgebras;
- (3) the deformations by twist and by 2-cocycle (of “toral type”) of these multiparameter Lie superbialgebras: in particular, we prove that this family is stable under these deformations, and that “quantization commutes with deformation”.

*To the memory of Pierre Cartier,
with our deepest admiration.*

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1. INTRODUCTION

In the last forty years, quantum groups — mainly in the form of quantized universal enveloping algebras, hereafter shortened into QUEA's — have been thoroughly studied, both in their *formal* version (à la Drinfeld, say), and in their *polynomial* formulation (following Jimbo and Lusztig). The semiclassical limit underlying them is given by Lie bialgebras (infinitesimally) and Poisson-Lie groups (globally): conversely, the “quantization problem” for a Lie bialgebra consists in the quest for some QUEA whose semiclassical limit yields back the given Lie bialgebra.

Quite soon, *multiparameter* quantum groups have been introduced too, including those coming from the FRT-construction, as well as multiparameter QUEA's — or MpQUEA's, in short. A standard method for that uses the technique of *deformation* of Hopf algebras, which comes in two versions: deformation by twist — where only the comultiplication and the antipode are modified — and deformation by 2-cocycle — where one changes only the multiplication and the antipode. The twist deformation was applied by Reshetikhin to Drinfeld's formal QUEA's $U_h(\mathfrak{g})$, thus finding some multiparameter QUEA's in which the new, “additional” parameters show up to describe the new coalgebra structure. The second method instead was applied by several people, all working with Jimbo & Lusztig's polynomial QUEA $U_q(\mathfrak{g})$ — in a suitable “quantum double version” — thus providing some other multiparameter QUEA's whose parameters enter in the description of the (new) algebra structure. A common, *unified* point of view on this matter was presented in [GG3] (see also [GG1]): the key message there is that, although the two families of MpQUEA's arising from deformations by twist or by 2-cocycle seem to be entirely disconnected, they form instead one and the same family. In addition, a second key result from [GG3] is that deformations of either kind of any MpQUEA still provide some new MpQUEA's, in this same family.

Looking at MpQUEA's, it was quickly realized that their many parameters could always be re-arranged so that only one of them kept the role of “quantization parameter”, while all the other took instead a “geometrical” meaning. Namely, when one specializes the single “quantum parameter” and thus gets the semiclassical limit, the remaining parameters enter in the description of the Poisson structure (in the form of a Lie cobracket) that is induced onto the underlying Lie algebra. In this way, specialization of MpQUEA's yields *multiparameter* structures onto the semiclassical limit, so one ends up with multiparameter Lie bialgebras (or MpLbA's, in short). On the other hand, one can also introduce this family of MpLbA's via an independent construction, and prove that this family is stable under deformations (in the sense of Lie bialgebra theory) by twist or by 2-cocycle; see [GG3] for details.

The present paper extends the previous world-building process to the setup of quantum *supergroups*: namely, we introduce *formal multiparameter quantized enveloping superalgebras* (in short *formal MpQUESA's*, or just *FoMpQUESA's*) and we study their deformations, by twist and by 2-cocycle.

Both Lie superalgebras and their quantizations, the quantized universal enveloping superalgebras (QUESA), can be regarded as Lie algebras and quantum universal enveloping algebras in the category of super vector spaces. As noted by Majid [Mj], both classical and super objects are related via the process of the so-called *bosonization* which yields a functor from one category to the other. Despite this strong connection, Lie superalgebras and QUESA's are interesting on their own, mainly

because of their wide range of applications. Simple Lie superalgebras over an algebraically closed field of characteristic zero were classified by Kac in [Ka2]: like for simple Lie algebras, this classification is given in terms of Cartan matrices and (generalized) Dynkin diagrams. Nevertheless, a single Lie superbialgebra may have non-isomorphic Dynkin diagrams. As the study of MpQUEA's (and their semiclassical limit) was mostly bounded to those whose semiclassical limits were Lie algebras of contragredient type with symmetrizable Cartan matrices, we focus onto MpQUESA's whose semiclassical limit are simple contragredient Lie superalgebras of type $A-G$ (w.r.t. the classification in [Ka2]). These QUESA's were first introduced by Yamane, in [Ya1], who defined formal QUESA's of type $A-G$: for them, relations are not only commutation relations and super quantum Serre relations, as one also needs to add higher order relations. Likewise, the MpLSbA's that we construct are, as Lie superalgebras, the contragredient ones of type $A-G$, again.

Specifically, we introduce multiparameter versions of Yamane's formal QUESA's (in short, FoMpQUESA's), see Definition 4.2.2, by mimicking the strategy we developed in [GG3] for the non-super case. The very definition is based upon the connection between the multiparameters and the action of a fixed commutative subsuperalgebra: this is encoded in the notion of *realization* of a multiparameter matrix P , much like for Kac-Moody algebras. This allows us to relate the quantum objects with their semiclassical limit, and also multiparameter objects with Yamane's standard ones: the latter step goes through deformation(s). To do this, we consider deformations by toral twists of Yamane's uniparameter QUESA's, just like Reshetikhin did with Drinfeld's uniparameter QUEA's, thus finding suitable new, deformed QUESA's that are now "multiparameter" ones, as they depend on the many parameters coming from the twist: these parameters, though, only show up in the description of the coalgebra structure, while the algebra structure is independent of them. As a consequence, one may endow these deformed QUESA's with a Hopf superalgebra structure. The second, key idea is to perform on these MpQUESA's a careful "change of presentation", modifying the original generators (from Yamane's presentation) into new ones: the outcome is a new presentation where the parameters appear in the defining relations, whereas the description of the coalgebra structure now is entirely parameter-free. Finally, we take this last presentation as "the" pattern for our definition of MpQUESA, given again by generators and relations, mimicking the last presentation of the deformed MpQUEA's that we found after the "change-the-generators" step. The arguments used in this analysis apply again to study, more in general, the deformations by twist of *any* one of our MpQUESA's: thus we find that any such deformation is yet another MpQUESA — this is proved in Theorem 5.1.3. Moreover, we prove in Proposition 4.2.8 that any FoMpQUESA can be obtained from a Yamane's uniparameter QUESA by a proper choice of the twist. Then we draw our attention to an extension of the dual notion of twist, *polar-2-cocycles*, and show in Theorem 5.2.15 that actually the family of FoMpQUESA is again stable by deformations of the superalgebra structure by a polar-2-cocycle. In conclusion, any FoMpQUESA can be obtained from Yamane's (uniparameter) QUESA either by a twist deformation or by a polar-2-cocycle deformation — the former fact is fairly natural, but the latter instead is quite surprising indeed. So, for every FoMpQUESA one can decide to focus the dependence on the discrete multiparameters either on the supercoalgebra structure or on the superalgebra structure: in fact, we chose the second option. As in the non-super setting [GG3], everything is encoded in the notion of "realization" of multiparameters.

Quite naturally, the same strategy used in the quantum setup can be adopted as well in the semiclassical framework, even with some simplifications. Therefore, we develop our construction of MpLSbA's, and the study of their deformations, *independently* of the quantum setup — though everything about that might be deduced by specialization from the quantum level (cf. Theorem 6.1.2). We obtain then a self-contained “chapter” of Lie superbialgebra theory, that introduces a new family of multiparameter Lie superbialgebras and their deformations, the key fact being that this family is stable under deformations (by twist and by 2-cocycle), see Theorem 3.3.3 and Theorem 3.4.3. These MpLSbA's come equipped with a presentation “à la Serre”, in which the parameters rule the Lie superalgebra structure (cf. §3.2). In addition, every such MpLSbA admits an alternative presentation, in which the Lie algebra structure stands fixed, while the Lie coalgebra structure varies according to the multiparameter matrix P . Like in the quantum setup, the isomorphism between the two presentations is quite meaningful, as it boils down to a well-chosen change of generators. In short, we might say that our multiparameters are encoded in realizations, FoMpQUESA's are quantizations of MpLSbA's, and multiparameter objects are given by deformations of either the (super)algebra or the (super)coalgebra structure of uniparameter objects — see a diagrammatic summary in §6.3. In all this, *the underlying Lie superalgebras are the (well-known) simple, contragredient ones of type A–G (in Kac' classification)*: we are thus providing a new contribution to a classical topic of utmost importance. Even more, indeed, *all of our construction also extends, verbatim, to the setup of affine Lie superalgebras and their quantum counterparts*, as introduced in [Ya2] and [Ya3].

Finally, we briefly introduce *polynomial* MpQUESA's: mimicking the recipe of Lusztig & Jimbo for polynomial QUEA's, these are certain subalgebras (up to modifying the ground ring) of our FoMpQUESA's. We also study both types of deformations for these objects. Now the polar-2-cocycles become actually Hopf 2-cocycles, and the deformation of the algebraic structure is well-understood. On the other hand, the twist deformations need to satisfy an arithmetic condition in order to be meaningful; nevertheless, these types of deformations cannot be seen, *strictu sensu*, as genuine “twist deformations” of our polynomial MpQUESA's. In fact, “formal” quantum groups are better suited to deal with deformations (up to resorting to the broader notion of “polar-2-cocycles”), so we focused mainly onto them.

One can also consider multiparameter versions of quantum formal series superalgebras on supergroups: in [GP] this is done by constructing such an object as dual to the FoMpQUESA's of the present paper. The same goal was pursued in [Ma] via an FRT-construction: the link between [Ma] and [GP] is discussed in [GP] as well.

A last word about the organization of the paper, and its detailed content.

In Section 2 we introduce the “combinatorial data” underlying our constructions of MpLSbA's and FoMpQUESA's alike: the notion of multiparameters, their realizations and deformations. Then in Section 3 we present our MpLSbA's, given by an explicit presentation (involving parameters) and bearing a Lie bialgebra structure which arise as deformation of one arising from Yamane's QUESA's. We also study their deformations, both by (toral) twists and by (toral) 2-cocycles: in particular, this proves that the family of these Lie superbialgebras is *stable w.r.t. deformations*. In addition, before presenting our new results about multiparameter objects and their deformations, we quickly resume the general theory of Lie superbialgebras and their deformations, as the literature on the subject seems to be quite poor.

Section 4 is dedicated to our newly minted notion of FoMpQUESA's. We begin by introducing them as superalgebras with a given presentation (by generators and relations), which closely mimics that of Yamane's QUESA's in [Ya1] but for involving more general parameters. Then we prove that they do bear a structure of Hopf superalgebra, which is obtained from that of a usual Yamane's QUESA through deformation by a suitable (toral) twist followed by a suitable "change of variables". Finally, we prove the functoriality (w.r.t. the underlying, Cartan-like combinatorial data) of our construction of FoMpQUESA's, and their triangular decomposition.

We spend Section 5 to discuss deformations of FoMpQUESA's by (toral) twists and by (toral) polar-2-cocycles. In short, we prove that these deformations turn FoMpQUESA's into new FoMpQUESA's again, hence the family of all FoMpQUESA's is *stable by such deformations*. Moreover, under mild assumptions things go the other way around too, so that any FoMpQUESA's can be realized as a suitable deformation — both by (toral) twist and by (toral) polar-2-cocycle — of one of Yamane's uniparameter QUESA's.

In Section 6 we compare the classical and quantum setups. Namely, we perform specializations of FoMpQUESA's, showing that their semiclassical limit is always a MpLSbA, with the same underlying "Cartan datum". Conversely, any possible MpLSbA arises as such a limit: in other words, for any MpLSbA there is a suitable FoMpQUESA which is quantization of it. As a second step, we compare deformations before and after specialization/quantization: the outcome is, in a nutshell, that "specialization/quantization and deformation commute with each other".

Finally, in Section 7 we introduce the notion of *polynomial MpQUESA's*, and we study their behavior under the processes of both types of deformations.

ACKNOWLEDGEMENTS

The authors thankfully acknowledge support by: (1) the *Research-in-Pairs* program of *CIMPA-ICTP*; (2) CONICET, ANPCyT, Secyt (Argentina), (3) the MIUR *Excellence Department Project MatMod@TOV (CUP E83C23000330006)* awarded to the Department of Mathematics, University of Rome "Tor Vergata" (Italy), (4) the research branch *GNSAGA* of INdAM (Italy), (5) the European research network *COST Action CaLISTA CA21109*.

2. MULTIPARAMETERS AND THEIR REALIZATIONS

In this section we fix the basic combinatorial data that we need. The definition of our multiparameter Lie superbialgebras and formal multiparameter quantum super groups requires a full lot of related material that we now present. In particular, $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$, while $\mathbb{k}$ will be a field of characteristic zero. This material is actually taken from [GG3], §2, up to minimal changes: we shall then skip many details (and proofs), as the reader can find them in [GG3].

2.1. Cartan data, multiparameter matrices, and realizations.

We introduce hereafter the "multiparameters", which we will use to construct (semi)classical and quantum objects as well. The theory can be developed in wide generality, yet we restrict ourselves to the setup more closely related with (contragredient) Lie superalgebras of simple type — the so-called *basic* ones.

2.1.1. Cartan data. Let $\mathfrak{g}$ be a complex, finite-dimensional simple Lie superalgebra. By the classification results due to Kac — cf. [Ka2] — we know that the

isomorphism class of $\mathfrak{g}$ falls within a family in a finite list, namely A_N , B_N , C_N , D_N , $D(2, 1; \alpha)$, F_4 and G_3 : we discard hereafter the case $D(2, 1; \alpha)$, as we build upon [Ya1], which also drops it. In short, we shall refer to these as “*basic Lie superalgebras*” — of type A , B , etc. up to G , in short $A - G$.

Every basic Lie superalgebra $\mathfrak{g}$ can be realized as a *contragredient Lie superalgebra*, or a *suitable quotient of it*, following Kac’ recipe in [Ka2, §2.5.1]: this involves fixing a set of (positive) simple roots, say $\alpha_1, \dots, \alpha_n$, with their parities — given by a map $p : I \longrightarrow \mathbb{Z}_2$ — and an associated “Cartan matrix” $A = (a_{ij})_{i,j \in I} \in M_n(\mathbb{Z})$ — where $n \in \mathbb{N}_+$ is the *rank* of $\mathfrak{g}$ and $I := \{1, \dots, n\}$ — as the starting point of the construction. Note that the matrix A , besides having integral coefficients, is also symmetrizable: indeed, we can (and will) also fix “minimal” elements $d_i \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$ ($i = 1, \dots, n$) such that DA is symmetric with integral entries, where D is the diagonal matrix $D := (d_i \delta_{ij})_{i,j \in I}$.

We stress the fact that for the same basic Lie superalgebra $\mathfrak{g}$ one has different pairs (A, p) that yield a realization of $\mathfrak{g}$ as the contragredient Lie superalgebra associated with that pair. In turn, every such pair corresponds (bijectively) to a suitable *Dynkin diagram*, after the recipe explained in [Ka2, §2.5 (Tables IV–V)]: in particular, by default we “incorporate” the parity in the Dynkin diagram itself, by labelling each vertex as explained in [Ka2], so that the label itself yields the parity of the corresponding (vertex, hence) simple root. Namely, the label for a vertex in a Dynkin diagram is a “colour”, which is “white”, “black” or “grey” respectively for the vertex “ $\circ$ ”, “ $\bullet$ ” or “ $\otimes$ ”: then the parity is $\bar{0}$ for white vertices and it is $\bar{1}$ for black and for grey vertices. Therefore, we can construct our basic Lie superalgebras directly from these (labelled) Dynkin diagrams. Still following [Ya1], we assume that in cases $F(4)$ and $G(3)$ the Dynkin is *distinguished*, i.e. it has exactly one vertex which is odd.

In the end, the list of all Dynkin diagrams we will deal with (arranged by type A , B , etc.) is the following (with the symbol “ $\times$ ” standing for a white or grey vertex):

$$\begin{array}{ll}
A_n & \begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \quad n \\ \times \text{---} \times \text{---} \times \text{---} \text{---} \text{---} \text{---} \times \end{array} \\
B_n (I) & \begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \quad n \\ \times \text{---} \times \text{---} \times \text{---} \text{---} \text{---} \Rightarrow \circ \end{array} \\
B_n (II) & \begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \quad n \\ \times \text{---} \times \text{---} \times \text{---} \text{---} \text{---} \Rightarrow \bullet \end{array} \\
C_n & \begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \quad n \\ \times \text{---} \times \text{---} \times \text{---} \text{---} \text{---} \Leftarrow \circ \end{array} \\
D_n (I) & \begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \quad \begin{array}{l} \nearrow \circ \\ \searrow \circ \end{array} \\ \times \text{---} \times \text{---} \times \text{---} \text{---} \text{---} \end{array} \\
D_n (II) & \begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \quad \begin{array}{l} \nearrow \otimes \\ \searrow \otimes \end{array} \\ \times \text{---} \times \text{---} \times \text{---} \text{---} \text{---} \end{array}
\end{array}$$

$$F_4 \quad \begin{array}{cccc} 1 & 4 & 3 & 2 \\ \circ & \text{---} & \circ \Rightarrow \Rightarrow & \circ \text{---} \otimes \end{array}$$

$$G_3 \quad \begin{array}{ccc} 1 & 3 & 2 \\ \otimes & \text{---} & \circ \Leftarrow \Leftarrow \Leftarrow \circ \end{array}$$

Note that every Dynkin diagram as above defines a parity function $p : I \longrightarrow \mathbb{Z}_2$, given by $p(i) := \bar{0}$ if the vertex $i \in I$ is white and $p(i) := \bar{1}$ if i is grey or black.

Definition 2.1.2. We will call *Cartan super-datum* any pair (A, p) where A is a Cartan matrix associated with one of the Dynkin diagrams listed in §2.1.1 above, and $p : I \longrightarrow \mathbb{Z}_2$ is the parity function defined by that Dynkin diagram. $\diamond$

A basic tool in the subsequent construction is the following, taken from [GG3], §2 (see also [Ka1], Ch. 1):

Definition 2.1.3. Let $\hbar$ be a formal variable, and $\mathbb{k}[[\hbar]]$ the ring of formal power series in $\hbar$ with coefficients in $\mathbb{k}$. Let $\mathfrak{h}$ be a free $\mathbb{k}[[\hbar]]$ -module of finite rank, and pick subsets $\Pi^\vee := \{T_i^+, T_i^-\}_{i \in I} \subseteq \mathfrak{h}$, and $\Pi := \{\alpha_i\}_{i \in I} \subseteq \mathfrak{h}^* := \text{Hom}_{\mathbb{k}[[\hbar]]}(\mathfrak{h}, \mathbb{k}[[\hbar]])$. For later use, we also introduce the elements $S_i := 2^{-1}(T_i^+ + T_i^-)$ and $\Lambda_i := 2^{-1}(T_i^+ - T_i^-)$ (for $i \in I$) and the sets $\Sigma := \{S_i\}_{i \in I} \subseteq \mathfrak{h}$ and $\Lambda := \{\Lambda_i\}_{i \in I} \subseteq \mathfrak{h}$.

Let $P \in M_n(\mathbb{k}[[\hbar]])$ be any $(n \times n)$ -matrix with entries in $\mathbb{k}[[\hbar]]$.

(a) We call the triple $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ a *realization* of P over $\mathbb{k}[[\hbar]]$, with *rank* defined as $\text{rk}(\mathcal{R}) := \text{rk}_{\mathbb{k}[[\hbar]]}(\mathfrak{h})$, if:

$$(a.1) \quad \alpha_j(T_i^+) = p_{ij}, \quad \alpha_j(T_i^-) = p_{ji}, \quad \text{for all } i, j \in I;$$

(a.2) the set $\bar{\Sigma} := \{\bar{S}_i := S_i \pmod{\hbar \mathfrak{h}}\}_{i \in I}$ is $\mathbb{k}$ -linearly independent in $\bar{\mathfrak{h}} := \mathfrak{h} / \hbar \mathfrak{h}$ — N.B.: this is equivalent to saying that Σ itself can be completed to a $\mathbb{k}[[\hbar]]$ -basis of $\mathfrak{h}$, hence in particular Σ is $\mathbb{k}[[\hbar]]$ -linearly independent in $\mathfrak{h}$.

(b) We call a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of the matrix P , respectively,

(b.1) *straight* if the set $\bar{\Pi} := \{\bar{\alpha}_i := \alpha_i \pmod{\hbar \mathfrak{h}^*}\}_{i \in I}$ is $\mathbb{k}$ -linearly independent in $\bar{\mathfrak{h}}^* := \mathfrak{h}^* / \hbar \mathfrak{h}^*$ — N.B.: this is equivalent to saying that Π can be completed to a $\mathbb{k}[[\hbar]]$ -basis of $\mathfrak{h}^*$, thus in particular Π is $\mathbb{k}[[\hbar]]$ -linearly independent in $\mathfrak{h}^*$;

$$(b.2) \quad \text{small if } \text{Span}_{\mathbb{k}[[\hbar]]}(\{S_i\}_{i \in I}) = \text{Span}_{\mathbb{k}[[\hbar]]}(\{T_i^+, T_i^-\}_{i \in I});$$

(b.3) *split* if the set $\bar{\Pi}^\vee := \{\bar{T}_i^\pm := T_i^\pm \pmod{\hbar \mathfrak{h}}\}_{i \in I}$ is $\mathbb{k}$ -linearly independent in $\bar{\mathfrak{h}} := \mathfrak{h} / \hbar \mathfrak{h}$ — N.B.: this is equivalent to saying that $\Pi^\vee$ can be completed to a $\mathbb{k}[[\hbar]]$ -basis of $\mathfrak{h}$, hence in particular it is $\mathbb{k}[[\hbar]]$ -linearly independent in $\mathfrak{h}$;

(b.4) *minimal* if $\text{Span}_{\mathbb{k}[[\hbar]]}(\{T_i^+, T_i^-\}_{i \in I}) = \mathfrak{h}$ — N.B.: in particular, $\mathcal{R}$ is *split* and *minimal* if and only if $\{T_i^+, T_i^-\}_{i \in I}$ is a $\mathbb{k}[[\hbar]]$ -basis of $\mathfrak{h}$.

(c) For any pair of realizations $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ and $\dot{\mathcal{R}} := (\dot{\mathfrak{h}}, \dot{\Pi}, \dot{\Pi}^\vee)$ of the same matrix P , a (homo)morphism $\underline{\phi} : \mathcal{R} \longrightarrow \dot{\mathcal{R}}$ is the datum of any $\mathbb{k}[[\hbar]]$ -module morphism $\phi : \mathfrak{h} \longrightarrow \dot{\mathfrak{h}}$ such that $\phi(T_i^\pm) = \dot{T}_{\sigma(i)}^\pm$ and $\phi^*(\dot{\alpha}_{\sigma(i)}) = \alpha_i$ (for all $i \in I$) for some permutation $\sigma \in \mathbb{S}_I$ — the symmetric group over I — hence, in particular, $\phi(\Pi^\vee) = \dot{\Pi}^\vee$, and also that $\phi^*(\dot{\Pi}) = \Pi$ — N.B.: realizations along with

their morphisms form a category, in which the iso-/epi-/mono-morphisms are those morphisms ϕ as above that actually are $\mathbb{k}[[\hbar]]$ -module iso-/epi-/mono-morphisms.

(d) Let A be any Cartan matrix associated with a Dynkin diagram from the list given in §2.1.1, and let $D := (d_i \delta_{ij})_{i,j \in I}$ be the diagonal matrix associated with A , as in §2.1.1. We say that a square matrix $P \in M_n(\mathbb{k}[[\hbar]])$ is of *Cartan type*, with associated Cartan matrix A , if $P_s := 2^{-1}(P + P^T) = DA$.

N.B.: condition (b.3) is equivalent to requiring that $\bar{\Sigma} \cup \bar{\Lambda}$ be $\mathbb{k}$ -linearly independent in $\bar{\mathfrak{h}} := \mathfrak{h} / \hbar \mathfrak{h}$; in turn, this is equivalent to saying that $\Sigma \cup \Lambda$ itself can be completed to a $\mathbb{k}[[\hbar]]$ -basis of $\mathfrak{h}$, hence in particular it is $\mathbb{k}[[\hbar]]$ -linearly independent. Similarly, the condition $\phi(T_i^\pm) = \dot{T}_{\sigma(i)}^\pm$ — $i \in I$, for some permutation $\sigma \in \mathbb{S}(I)$ — in (c) can be replaced by $\phi(S_i) = \dot{S}_{\sigma(i)}$ and $\phi(A_i) = \dot{A}_{\sigma(i)}$.

(e) In an entirely similar way, one may define realizations of a matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k})$ over a ground field $\mathbb{k}$. Such a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ consists of a $\mathbb{k}$ -vector space $\mathfrak{h}$ and distinguished subsets $\Pi^\vee := \{T_i^+, T_i^-\}_{i \in I} \subseteq \mathfrak{h}$ and $\Pi := \{\alpha_i\}_{i \in I} \subseteq \mathfrak{h}^* := \text{Hom}_{\mathbb{k}}(\mathfrak{h}, \mathbb{k})$: then condition (a.1) reads the same, while (a.2) instead says that the set $\{S_i = 2^{-1}(T_i^+ + T_i^-)\}_{i \in I}$ is linearly independent, and the *rank* of the realization is $\text{rk}(\mathcal{R}) := \dim_{\mathbb{k}}(\mathfrak{h})$. Also, $\mathcal{R}$ is *straight*, resp. *split*, if Π , resp. $\Pi^\vee$, is linearly independent. $\diamond$

The following consequence of the definitions yields a close link with Kac' notion:

Lemma 2.1.4. (cf. [GG3], Lemma 2.1.4) *Let $P \in M_n(\mathbb{k}[[\hbar]])$ be a matrix as above. If $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ is a straight realization of P , then the triple $(\mathfrak{h}, \Pi, \Pi_S^\vee)$ — with $\Pi_S^\vee := \{S_i\}_{i \in I}$ — is a realization of $P_s := 2^{-1}(P + P^T)$ — over the ring $\mathbb{k}[[\hbar]]$ — in the sense of [Ka1], Ch. 1, §1.1, but for condition (1.1.3). $\square$*

Note that condition (1.1.3) in [Ka1, Ch. 1, §1.1], is fulfilled whenever $\text{rk}(\mathfrak{h}) = 2n - \text{rk}(P_s)$; in particular, we can always achieve that condition up to suitably enlarging or restricting $\mathfrak{h}$. In any case, from now on with any straight realization of a matrix P of Cartan type, for some Cartan matrix A , we shall always associate the realization of $P_s = DA$ given by Lemma 2.1.4, hence also the corresponding realization of A and then all the related data and machinery mentioned in §2.1.

We collect now a few technical results:

Proposition 2.1.5. (cf. [GG3], Proposition 2.1.5)

(a) *For every $P \in M_n(\mathbb{k}[[\hbar]])$ and every $\ell \geq 3n - \text{rk}(P + P^T)$, there exists a straight split realization of P with $\text{rk}(\mathfrak{h}) = \ell$, which is unique up to isomorphisms.*

(b) *Claim (a) still holds true if we drop the word “straight” and pick $\ell \geq 2n$. $\square$*

Remark 2.1.6. It follows from definitions that a *necessary* condition for a small realization of any $P \in M_n(\mathbb{k}[[\hbar]])$ to exist is $\text{rk}(P_s | P_a) = \text{rk}(P_s)$. Conversely, with much the same arguments used in the proof of Proposition 2.1.5, we can prove that such a condition is also *sufficient*, as the following holds true, indeed:

Proposition 2.1.7. (cf. [GG3], Proposition 2.1.7)

If $P \in M_n(\mathbb{k}[[\hbar]])$ is such that $\text{rk}(P_s | P_a) = \text{rk}(P_s)$, then, for all $\ell \geq 2n - \text{rk}(P_s)$, there exists a straight small realization of P with $\text{rk}(\mathfrak{h}) = \ell$, and such a realization is unique up to isomorphisms. $\square$

After this existence results concerning realizations of special type, we can achieve a more general result with two additional steps. The first one tells in short that every realization can be “lifted” to a split one:

Lemma 2.1.8. (cf. [GG3], Lemma 2.1.8)

Let $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of $P \in M_n(\mathbb{k}[[\hbar]])$. Then:

(a) There exists a split realization $\dot{\mathcal{R}} := (\dot{\mathfrak{h}}, \dot{\Pi}, \dot{\Pi}^\vee)$ of the same matrix P and an epimorphism of realizations $\underline{\pi} : \dot{\mathcal{R}} \longrightarrow \mathcal{R}$ such that, if $\mathfrak{h}_T := \text{Span}(\{T_i^\pm\}_{i \in I})$ and $\dot{\mathfrak{h}}_T := \text{Span}(\{\dot{T}_i^\pm\}_{i \in I})$, then $\underline{\pi}$ induces an isomorphism $\dot{\pi} : \dot{\mathfrak{h}} / \dot{\mathfrak{h}}_T \cong \mathfrak{h} / \mathfrak{h}_T$.

If in addition $\mathcal{R}$ is straight, resp. minimal, then a split realization $\dot{\mathcal{R}}$ as above can be found that is straight, resp. minimal, as well.

(b) There exists a straight realization $\hat{\mathcal{R}} := (\hat{\mathfrak{h}}, \hat{\Pi}, \hat{\Pi}^\vee)$ of the same matrix P and an epimorphism of realizations $\underline{\pi} : \hat{\mathcal{R}} \longrightarrow \mathcal{R}$ such that, if $\mathfrak{h}_T := \text{Span}(\{T_i^\pm\}_{i \in I})$ and $\hat{\mathfrak{h}}_T := \text{Span}(\{\hat{T}_i^\pm\}_{i \in I})$, then $\underline{\pi}$ induces an isomorphism $\hat{\pi} : \hat{\mathfrak{h}}_T \cong \mathfrak{h}_T$.

If in addition $\mathcal{R}$ is split, resp. minimal, then a straight realization $\hat{\mathcal{R}}$ as above can be found that is split, resp. minimal, as well.

Proof. (a) This is nothing but Lemma 2.1.8 in [GG3].

(b) Take $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathfrak{h}'$ for some $\mathbb{k}[[\hbar]]$ -module $\mathfrak{h}'$ such that the α_i 's extend from $\mathfrak{h}$ to all of $\hat{\mathfrak{h}}$ so to give linearly independent elements $\hat{\alpha}_i$'s of $\hat{\mathfrak{h}}$: e.g., choose $\mathfrak{h}'$ to be a free $\mathbb{k}[[\hbar]]$ -module of rank $|I|$ with $\mathbb{k}[[\hbar]]$ -basis $\{e_j\}_{j \in I}$ and set $\hat{\alpha}_i|_{\mathfrak{h}} := \alpha_i$ and $\hat{\alpha}_i(e_j) := \delta_{i,j}$ for all $i, j \in I$. Then $\hat{\mathcal{R}} := (\hat{\mathfrak{h}}, \hat{\Pi} := \{\hat{\alpha}_i\}_{i \in I}, \hat{\Pi}^\vee := \Pi^\vee)$ is a straight realization of P as required, which is even split, respectively minimal, if so is $\mathcal{R}$; moreover, an epimorphism $\underline{\pi} : \hat{\mathcal{R}} \longrightarrow \mathcal{R}$ is given by the natural projection $\pi : \hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathfrak{h}' \longrightarrow \mathfrak{h}$, which induces an isomorphism $\hat{\pi} : \hat{\mathfrak{h}}_T \cong \mathfrak{h}_T$ as required. $\square$

A last, significant result concerns morphisms between realizations.

Lemma 2.1.9. (cf. [GG3], Lemma 2.1.8)

Let $\hat{\mathcal{R}} := (\hat{\mathfrak{h}}, \hat{\Pi}, \hat{\Pi}^\vee)$ and $\check{\mathcal{R}} := (\check{\mathfrak{h}}, \check{\Pi}, \check{\Pi}^\vee)$ be two realizations of a same $P \in M_n(\mathbb{k}[[\hbar]])$, and let $\underline{\phi} : \hat{\mathcal{R}} \longrightarrow \check{\mathcal{R}}$ be a morphism between them. Then $\text{Ker}(\phi : \hat{\mathfrak{h}} \longrightarrow \check{\mathfrak{h}}) \subseteq \bigcap_{j \in I} \text{Ker}(\hat{\alpha}_j)$. $\square$

Remark 2.1.10. Working with matrices in $M_n(\mathbb{k})$ and realizations of them over $\mathbb{k}$ (for some field $\mathbb{k}$), all the previous constructions still make sense; some results (e.g., Proposition 2.1.5 and Lemma 2.1.8) even get stronger and/or have simpler proofs.

2.2. Twist deformations of multiparameters and realizations.

In this subsection we introduce the notion of *deformation by twist* of realizations, which will be needed later when dealing with (quantum) super-Lie theoretical objects. Like before, we are essentially recollecting results from [GG3], §2.2, so we skip those details that can be found there.

2.2.1. Deforming realizations (and matrices) by twist. Fix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$, possibly — up to changing minimal details in what follows — over $\mathbb{k}$ if $P \in M_n(\mathbb{k})$.

Recall that $\mathfrak{h}$ is, by assumption, a free $\mathbb{k}[[\hbar]]$ -module of finite rank $t := \text{rk}(\mathfrak{h})$. We fix in $\mathfrak{h}$ any $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, where $\mathcal{G}$ is an index set with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Fix an antisymmetric square matrix $\Phi = (\phi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ — in fact, we might work with any $\Phi \in M_t(\mathbb{k}[[\hbar]])$, but at some point we should single out its antisymmetric part $\Phi_a := 2^{-1}(\Phi - \Phi^T)$ which would be all that matters. We define the twisted “distinguished toral elements” (or “coroots”)

$$T_{\Phi,\ell}^\pm := T_\ell^\pm \pm \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k \quad (2.1)$$

As a matter of notation, let $\mathbf{T} := \begin{pmatrix} \underline{T}^+ \\ \underline{T}^- \end{pmatrix}$ be the $(2n \times 1)$ -matrix given by the column vectors $\underline{T}^\pm = (T_i^\pm)_{i \in I}$. Similarly, let $\mathbf{T}_\Phi := \begin{pmatrix} \underline{T}_\Phi^+ \\ \underline{T}_\Phi^- \end{pmatrix}$ be the $(2n \times 1)$ -matrix given by the (superposed) two column vectors $\underline{T}_\Phi^\pm = (T_{\Phi,i}^\pm)_{i \in I}$, and $\mathbf{H}$ the column vector $\mathbf{H} := (H_g)_{g \in \mathcal{G}}$. Moreover, denote by $\mathfrak{A}$ the $(n \times t)$ -matrix with entries in $\mathbb{k}[[\hbar]]$ given by $\mathfrak{A} := (\alpha_\ell(H_g))_{\ell \in I}^{g \in \mathcal{G}}$, and set $\mathfrak{A}_\bullet := \begin{pmatrix} +\mathfrak{A} \\ -\mathfrak{A} \end{pmatrix}$ — a matrix of size $(2n \times t)$.

Now, using matrix notation we have $\mathbf{T}_\Phi = \mathbf{T} - \mathfrak{A}_\bullet \Phi \mathbf{H}$. Eventually, define also

$$P_\Phi = (p_{i,j}^\Phi)_{i,j \in I} := \left(p_{i,j} + \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k) \right)_{i,j \in I} = P - \mathfrak{A} \Phi \mathfrak{A}^T \quad (2.2)$$

Now, using the above notation, a direct computation yields

$$\begin{aligned} S_{\Phi,i} &:= 2^{-1}(T_{\Phi,i}^+ + T_{\Phi,i}^-) = 2^{-1}(T_i^+ + T_i^-) = S_i & \forall i \in I \\ \alpha_j(T_{\Phi,i}^+) &= p_{i,j}^\Phi, & \alpha_j(T_{\Phi,i}^-) &= p_{j,i}^\Phi & \forall i, j \in I \end{aligned}$$

so that the triple $\mathcal{R}_\Phi := (\mathfrak{h}_\Phi, \Pi_\Phi, \Pi_\Phi^\vee)$ with $\mathfrak{h}_\Phi := \mathfrak{h}$, $\Pi_\Phi := \Pi$ and $\Pi_\Phi^\vee := \{T_{\Phi,i}^+, T_{\Phi,i}^- \mid i \in I\}$, is a realization of the matrix $P_\Phi = (p_{i,j}^\Phi)_{i,j \in I}$ — as in Definition 2.1.3; also, by construction $\mathcal{R}_\Phi$ is also *straight*, resp. *small*, if and only if such is $\mathcal{R}$. Moreover, P_Φ is the sum of P plus an antisymmetric matrix, so *the symmetric part of P_Φ is the same as P* , i.e. $(P_\Phi)_s = P_s$. In particular, if P is of Cartan type, then so is P_Φ , and they are associated with the same Cartan matrix. In short, we get:

Proposition 2.2.2. (cf. [GG3], Proposition 2.2.2)

With notation as above, the following holds true:

(a) *the matrix $P_\Phi := P - \mathfrak{A} \Phi \mathfrak{A}^T$ obeys $(P_\Phi)_s = P_s$; in particular, if P is of Cartan type, then so is P_Φ , and they are associated with the same Cartan matrix.*

(b) *the triple $\mathcal{R}_\Phi := (\mathfrak{h}_\Phi := \mathfrak{h}, \Pi_\Phi := \Pi, \Pi_\Phi^\vee := \{T_{\Phi,i}^+, T_{\Phi,i}^- \}_{i \in I})$ is a realization of P_Φ ; moreover, $\mathcal{R}_\Phi$ is *straight*, resp. *small*, if and only if such is $\mathcal{R}$. $\square$*

Definition 2.2.3. The realization $\mathcal{R}_\Phi := (\mathfrak{h}, \Pi, \Pi_\Phi^\vee)$ of the matrix $P_\Phi = (p_{i,j}^\Phi)_{i,j \in I}$ is called a *twist deformation* (via Φ) of the realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P .

Similarly, the matrix P_Φ is called a *twist deformation* of the matrix P . $\diamond$

Remarks 2.2.4. (a) Observe that, by the very definition of twisting one has that

$$(P_\Phi)_{\Phi'} = P_{\Phi+\Phi'} \quad \text{and} \quad (\mathcal{R}_\Phi)_{\Phi'} = \mathcal{R}_{\Phi+\Phi'} \quad \text{for all} \quad \Phi, \Phi' \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$$

Therefore, the additive group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ acts on the set of (multiparameter) matrices of size $n := |I|$ with fixed symmetric part, as well as on the set of their realizations of (any) fixed rank. When two matrices $P, P' \in M_n(\mathbb{k}[[\hbar]])$ belong to the same orbit of this $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ -action we say that P and P' are *twist equivalent*.

(b) It follows from Proposition 2.2.2(a) that if two multiparameter matrices P and P' are twist equivalent, then their symmetric part is the same, that is $P_s = P'_s$. Next result shows the converse holds true as well, *up to taking the group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ big enough, namely with $t \geq 3n - \text{rk}(P_s)$* .

Lemma 2.2.5. (cf. [GG3], Lemma 2.2.5)

With notation as above, let $P, P' \in M_n(\mathbb{k}[[\hbar]])$, and consider the aforementioned action by twist on $M_n(\mathbb{k}[[\hbar]])$ by any additive group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$ with $t \geq 3n - \text{rk}(P_s)$. Then P and P' are twist equivalent if and only if $P_s = P'_s$. $\square$

Next proposition “upgrades” the previous result to the level of realizations.

Proposition 2.2.6. (cf. [GG3], Proposition 2.2.7) Let P and P' be two matrices in $M_n(\mathbb{k}[[\hbar]])$ with the same symmetric part, i.e. such that $P_s = P'_s$.

(a) Let $\mathcal{R}$ be a straight realization of P of rank $t \geq 3n - \text{rk}(P_s)$. Then there is $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that $P' = P_\Phi$ and the corresponding realization $\mathcal{R}_\Phi$ is straight.

In short, if $P'_s = P_s$ then from any straight realization of P we can obtain by twist deformation a straight realization (of the same rank) for P' , and viceversa.

(b) Let $\mathcal{R}$ and $\mathcal{R}'$ be straight small realizations of P and P' , such that $\text{rk}(\mathcal{R}) = \text{rk}(\mathcal{R}') =: t \geq 3n - \text{rk}(P_s)$. Then there exists $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that $\mathcal{R}' \cong \mathcal{R}_\Phi$.

In short, if $P'_s = P_s$ then any straight small realization of P' is isomorphic to a twist deformation of a straight small realization of P of same rank, and viceversa.

(c) Every straight small realization $\mathcal{R}$ of P is isomorphic to some twist deformation of the standard realization of P_s of the same rank as $\mathcal{R}$, as in Lemma 2.1.4. $\square$

2.2.7. Stability issues for twisted realizations. Applying deformation by twist upon a given realization provides a new realization (for a different matrix, in general). If the initial realization is *split*, however, the new realization achieved through deformation is *not necessarily split*, in general. This issue is discussed at length in [GG3], §§2.2.8–9, where examples and counterexamples are given as well.

2.3. 2-cocycle deformations of multiparameters and realizations.

In this subsection we introduce the notion of *deformation by 2-cocycles* of realizations (and of multiparameters), which is dual to deformation by twist. Here again, we directly report material from [GG3], §2.3, so we skip all secondary details that can be found there.

2.3.1. Deforming realizations (and matrices) by 2-cocycles. Fix again $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it. We consider special deformations of realizations, called “2-cocycle deformations”. To this end, like in §2.2, we fix in $\mathfrak{h}$ a $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, with $\mathcal{G}$ an index set, $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Let $\chi : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{k}[[\hbar]]$ be any $\mathbb{k}[[\hbar]]$ -bilinear map: note that it bijectively corresponds to some $X = (\chi_{g\gamma})_{g,\gamma \in \mathcal{G}} \in M_t(\mathbb{k}[[\hbar]])$ via $\chi_{g\gamma} = \chi(H_g, H_\gamma)$. We assume that χ is antisymmetric, which means $\chi^T(x, y) = -\chi(x, y)$ where $\chi^T(x, y) := \chi(y, x)$, for all $x, y \in \mathfrak{h}$; this is equivalent to saying that X is antisymmetric, i.e. $X \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$. We denote by $\text{Alt}_{\mathbb{k}[[\hbar]]}(\mathfrak{h} \times \mathfrak{h}, \mathbb{k}[[\hbar]])$ the set of all antisymmetric, $\mathbb{k}[[\hbar]]$ -bilinear maps from $\mathfrak{h} \times \mathfrak{h}$ to $\mathbb{k}[[\hbar]]$. We assume also that χ obeys

$$\chi(S_i, -) = 0 = \chi(-, S_i) \quad \forall i \in I \quad (2.3)$$

where $S_i := 2^{-1}(T_i^+ + T_i^-)$ for all $i \in I$. In particular, this implies (for $i \in I$, $T \in \mathfrak{h}$) that $\chi(+T_i^+, T) = \chi(-T_i^-, T)$, $\chi(T, +T_i^+) = \chi(T, -T_i^-)$, hence

$$+\chi(-, T_i^+) = -\chi(-, T_i^-) \quad \forall i \in I \quad (2.4)$$

For later use, we introduce also the notation

$$\text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h}) := \left\{ \chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}(\mathfrak{h} \times \mathfrak{h}, \mathbb{k}[[\hbar]]) \mid \chi \text{ obeys (2.3)} \right\} \quad (2.5)$$

and to each $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ we associate $\mathring{X} := (\mathring{\chi}_{ij} = \chi(T_i^+, T_j^+))_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$.

Basing on the above, we define

$$P_{(\chi)} := P + \mathring{X} = \left(p_{ij}^{(\chi)} := p_{ij} + \mathring{\chi}_{ij} \right)_{i,j \in I}, \quad \Pi_{(\chi)} := \left\{ \alpha_i^{(\chi)} := \alpha_i \pm \chi(-, T_i^\pm) \right\}_{i \in I}$$

We are now ready for our key result on 2-cocycle deformations.

Proposition 2.3.2. (cf. [GG3], Proposition 2.3.2) *Keep notation as above. Then:*

(a) $P_{(\chi)} := P + \mathring{X}$ obeys $(P_{(\chi)})_s = P_s$; in particular, if P is of Cartan type, then so is $P_{(\chi)}$, and they are associated with the same Cartan matrix.

(b) the triple $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ is a realization of the matrix $P_{(\chi)}$, which is minimal, resp. split, if so is $\mathcal{R}$. $\square$

Definition 2.3.3. The realization $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ of $P_{(\chi)} = (p_{ij}^{(\chi)})_{i,j \in I}$ is called a 2-cocycle deformation of the realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P .

Similarly, the matrix $P_{(\chi)}$ is called a 2-cocycle deformation of the matrix P . $\diamond$

Remarks 2.3.4. (a) The very definitions give

$$(P_{(\chi)})_{(\chi')} = P_{(\chi+\chi')} \quad \text{and} \quad (\mathcal{R}_{(\chi)})_{(\chi')} = \mathcal{R}_{(\chi+\chi')} \quad \text{for all } \chi, \chi' \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$$

Thus, the additive group $\text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ acts on the set of (multiparameter) matrices of size $n := |I|$ with fixed symmetric part, as well as on the set of their realizations of (any) fixed rank. When two matrices P and P' in $M_n(\mathbb{k}[[\hbar]])$ belong to the same orbit of this $\text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ -action, we say that P and P' are 2-cocycle equivalent.

(b) It follows from Proposition 2.3.2(a) that if two multiparameter matrices P and P' are 2-cocycle equivalent, then their symmetric part is the same, i.e. $P_s = P'_s$. As a consequence of the next result, the converse holds true as well (cf. Lemma 2.3.6 below), under mild, additional assumptions.

Next result concerns the aforementioned $\text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ -action on realizations; indeed, up to minor details it can be seen as the “2-cocycle analogue” of Proposition 2.2.6:

Proposition 2.3.5. (cf. [GG3], Proposition 2.3.2)

Let $P, P' \in M_n(\mathbb{k}[[\hbar]])$ be two matrices with the same symmetric part, i.e. such that $P_s = P'_s$. Moreover, let $\mathcal{R}$ be a split realization of P .

(a) There exists a map $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that $P' = P_{(\chi)}$ and the realization $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ of $P' = P_{(\chi)}$ is split. In a nutshell, if $P'_s = P_s$ then from any split realization of P we can obtain a split realization (of the same rank) of P' by 2-cocycle deformation, and viceversa.

(b) Assume in addition that $\mathcal{R}$ be minimal. Then $\mathcal{R}$ is isomorphic to a 2-cocycle deformation of the split minimal realization of P_s . $\square$

As a byproduct, we find the following “2-cocycle counterpart” of Lemma 2.2.5:

Lemma 2.3.6. (cf. [GG3, Lemma 2.3.6]) With notation as above, let $P, P' \in M_n(\mathbb{k}[[\hbar]])$. Then P and P' are 2-cocycle equivalent (for the above 2-cocycle action on $M_n(\mathbb{k}[[\hbar]])$ of an additive group $\mathfrak{so}_t(\mathbb{k}[[\hbar]])$) if and only if $P_s = P'_s$. $\square$

Observation 2.3.7. To sum up, we wish to stress the following, remarkable fact. Consider two matrices $P, P' \in M_n(\mathbb{k}[[\hbar]])$ with the same symmetric part $P_s = P'_s$, and a realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P that is *split* and *straight*. Then, by Proposition 2.2.6 and Proposition 2.3.5, one can construct two realizations $\mathcal{R}_\Phi$ and $\mathcal{R}_{(\chi)}$ of P' by a twist deformation, respectively a 2-cocycle deformation, of $\mathcal{R}$ that affects only the coroot set $\Pi^\vee$ or the root set Π , respectively; in particular, $\mathcal{R}_\Phi$ is still straight (yet possibly not split) and $\mathcal{R}_{(\chi)}$ is still split (yet possibly not straight), while both have the same rank as $\mathcal{R}$.

3. MULTIPARAMETER LIE SUPERBIALGEBRAS AND THEIR DEFORMATIONS

In this section we first recall the notion of Lie superbialgebra, and we explain their theory of deformations, both by twist and by 2-cocycles. Then we introduce our *multiparameter Lie superbialgebras* (MpLSbAs in short). Finally, we study deformations of MpLSbAs, both by toral twists and by toral 2-cocycles, proving in particular that our family of MpLSbAs is *stable* for deformations of either type.

3.1. Lie superbialgebras and their deformations.

We recall hereafter a few notions — including some partially new, to the best of our knowledge — concerning Lie superbialgebras and their deformations. For basic definitions and results see [An, Ka2]; as to notation, we adopt the same used for Lie bialgebras in [GG3]. We work over any field of characteristic different from 2.

3.1.1. Generalities on Lie superbialgebras. We begin by giving some basic and (more or less) well-known definitions.

A *super vector space* (or “vector superspace”) is a $\mathbb{Z}_2$ -graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$; the homogeneous component $V_{\bar{0}}$ is called the even part and $V_{\bar{1}}$ the odd part.

For any homogeneous element $v \in V$, we denote by $|v| \in \{\bar{0}, \bar{1}\} =: \mathbb{Z}_2$ the degree of v , which is defined by $|v| = \bar{a} \iff v \in V_{\bar{a}}$. By a common abuse of notation, in the sequel whenever a formula involves parity of some elements, one assumes that the concerned elements be homogeneous, and then extends the validity of that formula to all elements by linearity.

Any super vector space has a *braiding* map $\sigma : V \otimes V \longrightarrow V \otimes V$ defined by $\sigma(v \otimes w) := (-1)^{|v||w|} w \otimes v$ for all $v, w \in V$, which satisfies the braid equation $(\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) = (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma)$. As $\sigma^2 = \text{id}$, the map σ induces a representation of the symmetric group $\mathbb{S}_n$ on the n -th tensor power $T^n(V)$ of V . In particular, one has the super symmetrizer operator $\text{Sym}_n : T^n(V) \longrightarrow T^n(V)$ given by $\text{Sym}_n := \sum_{\mu \in \mathbb{S}_n} \mu$ and the super antisymmetrizer operator $\text{Alt}_n : T^n(V) \longrightarrow T^n(V)$ given by $\text{Alt}_n := \sum_{\mu \in \mathbb{S}_n} (-1)^{\ell(\mu)} \mu$; note that, in order to compute the action of the operators, one has to write each permutation as a product of transpositions. With this action in mind, the exterior powers $\Lambda^n(V)$ of V are defined, as usual, as the subspace of $T^n(V)$ given by the image of Alt_n , or more explicitly as

$$\Lambda^n(V) := \{ v_{(n)} \in T^n(V) \mid \mu \cdot v_{(n)} = (-1)^{\ell(\mu)} v_{(n)} \ \forall \mu \in \mathbb{S}_n \}$$

In particular, $\Lambda^2(V) = \{ v_{(2)} \in V \otimes V \mid \sigma(v_{(2)}) = -v_{(2)} \} = \text{Ker}(\text{id} + \sigma)$.

Similarly one defines the symmetric powers $S^n(V) := \text{Im}(\text{Sym}_n)$.

A *Lie superalgebra* is a pair $(\mathfrak{g}, [\ , \])$ where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a super vector space and $[\ , \] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is a $\mathbb{Z}_2$ -graded bilinear map, i.e. $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$, called a (Lie) superbracket, satisfying the following conditions for any choice of homogeneous elements $x, y, z \in \mathfrak{g}$:

$$\begin{aligned} 0 &= [x, y] + (-1)^{|x||y|} [y, x] \\ 0 &= (-1)^{|x||z|} [x, [y, z]] + (-1)^{|y||x|} [y, [z, x]] + (-1)^{|z||y|} [z, [x, y]] \end{aligned} \quad (3.1)$$

The first condition in (3.1) is called *super antisymmetry*: it can be expressed in element-free form by saying that $[\ , \]$ is a map from $\mathfrak{g} \otimes \mathfrak{g}$ to $\mathfrak{g}$ such that

$$0 = [\ , \] \circ (\text{id} + \sigma) = [\ , \] \circ \text{Sym}_2$$

Note that in general one has that $(\text{id} - (1, 2)) \circ \text{Sym}_{\mathbb{A}_n} = \text{Alt}_n$, where $\text{Sym}_{\mathbb{A}_n}$ is the symmetrizer of the alternating subgroup $\mathbb{A}_n$. In case $n = 3$ this reads

$$\text{Alt}_3 = (\text{id} - (1, 2)) \circ \text{Sym}_{\mathbb{A}_3} = (\text{id} - (1, 2)) \circ (\text{id} + (1, 2, 3) + (3, 2, 1))$$

The action of the operator $\text{Sym}_{\mathbb{A}_3}$ on $T(\mathfrak{g})$ is also usually called “(action by) cyclic permutations” and written as “c.p.”

The second condition in (3.1) is called the *super Jacobi* identity. One can express the latter by stating that the adjoint action of any element is a *super derivation*, i.e.

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} [y, [x, z]]$$

or by using the antisymmetrizer operator

$$[[\ , \], \] \circ \text{Alt}_3 = 0 \quad \text{which is equivalent to} \quad [[\ , \], \] \circ \text{Sym}_{\mathbb{A}_3} = 0$$

It is worth noting that the operators $\text{Sym}_{\mathbb{A}_n}$ and Alt_n actually commute with the action of $\mathfrak{g}$ on tensor powers $T^n(\mathfrak{g})$. Indeed, $\mathfrak{g}$ acts on itself by the adjoint action $x \cdot y := \text{ad}(x)(y) = [x, y]$ for all $x, y \in \mathfrak{g}$. This action extends to the super vector space $T^n(\mathfrak{g})$ by

$$x \cdot (y_1 \otimes y_2 \otimes \cdots \otimes y_n) := \sum_{i=1}^n (-1)^{|x| \sum_{k=1}^{i-1} |y_k|} y_1 \otimes \cdots \otimes [x, y_i] \otimes \cdots \otimes y_n \quad (3.2)$$

for all elementary tensors $y_1 \otimes y_2 \otimes \cdots \otimes y_n \in T^n(\mathfrak{g})$ with homogeneous factors. For example, the adjoint action on $\mathfrak{g} \otimes \mathfrak{g}$ reads

$$x \cdot (y \otimes z) = [x, y] \otimes z + (-1)^{|x||y|} y \otimes [x, z]$$

One can show that this action commutes with the action of the elementary transposition, whence with the operators $\text{Sym}_{\mathbb{A}_n}$ and Alt_n .

A *Lie supercoalgebra* is a pair $(\mathfrak{g}, \delta)$ where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a super vector space and $\delta : \mathfrak{g} \rightarrow \Lambda^2(\mathfrak{g})$ is a morphism of super vector spaces, called a *super (Lie) cobracket*, that satisfies the *super co-Jacobi* identity, namely

$$\text{Alt}_3 \circ ((\text{id} \otimes \delta) \circ \delta) = 0 \quad \text{which is equivalent to} \quad \text{Sym}_{\mathbb{A}_3} \circ ((\text{id} \otimes \delta) \circ \delta) = 0$$

Equivalently again, this amounts to be the same as requiring that the pair $(\mathfrak{g}^*, \delta^*)$ be a Lie superalgebra. Indeed, let $\langle -, - \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{k}$ be the canonical evaluation map. If we identify $\mathfrak{g}^* \otimes \mathfrak{g}^*$ with a subspace of $(\mathfrak{g} \otimes \mathfrak{g})^*$ via

$$\langle \alpha \otimes \beta, x \otimes y \rangle = (-1)^{|\beta||x|} \langle \alpha, x \rangle \langle \beta, y \rangle$$

for all (homogeneous) $x, y \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{g}^*$, we have $\langle \alpha \otimes \beta, \delta(x) \rangle = \langle \delta^*(\alpha \otimes \beta), x \rangle$. Then, the image of δ is contained in $\Lambda^2(\mathfrak{g})$ if and only if $(\text{id} + \sigma) \circ \delta = 0$ if and only if $\delta^* \circ (\text{id} + \sigma) = 0$ if and only if δ^* satisfies the super antisymmetry condition. Moreover, δ satisfies the super co-Jacobi if and only if δ^* satisfies the super Jacobi.

To express the equations above on elements, it is better to consider $\Lambda^2(\mathfrak{g})$ as the subspace of $V \otimes V$ given by the image of Alt_2 . Then, for all (homogeneous) $x, y \in \mathfrak{g}$, we set

$$x \wedge y := 2^{-1} \text{Alt}_2(x \otimes y) = 2^{-1}(x \otimes y - (-1)^{|x||y|} y \otimes x)$$

Finally, as a matter of notation, all along the paper we shall use a Sweedler's-like notation $\delta(x) = x_{[1]} \wedge x_{[2]}$ (summation being understood) for any $x \in \mathfrak{g}$.

A *Lie superbialgebra* is a triple $(\mathfrak{g}, [\ , \], \delta)$ where $(\mathfrak{g}, [\ , \])$ is a Lie superalgebra, $(\mathfrak{g}, \delta)$ is a Lie supercoalgebra, and δ is a 1-cocycle for the Lie superalgebra cohomology, cf. [Le]. This is a compatibility condition that, for any homogeneous $x, y \in \mathfrak{g}$, reads as follows

$$\delta([x, y]) = x \cdot \delta(y) - (-1)^{|x||y|} y \cdot \delta(x)$$

where $x \cdot \delta(y)$ denotes the action of $\mathfrak{g}$ onto $\Lambda^2(\mathfrak{g})$ induced by the adjoint action on $\mathfrak{g} \otimes \mathfrak{g}$ as in (3.2). Taking this into account, the compatibility condition above reads

$$\begin{aligned} \delta([x, y]) &= [x, y_{[1]}] \wedge y_{[2]} + (-1)^{|x||y_1|} y_{[1]} \wedge [x, y_{[2]}] - \\ &\quad - (-1)^{|x||y|} [y, x_{[1]}] \wedge x_{[2]} - (-1)^{|x_2||y|} x_{[1]} \wedge [y, x_{[2]}] \end{aligned}$$

If we write the action above by $x \cdot (y \otimes z) = [\Delta(x), y \otimes z]$, we have

$$\delta([x, y]) = [\Delta(x), \delta(y)] - (-1)^{|x||y|} [\Delta(y), \delta(x)] = [\Delta(x), \delta(y)] + [\delta(x), \Delta(y)]$$

So, the element-free formulation of the compatibility condition reads

$$\delta \circ [\ , \] = [\ , \] \circ (\Delta \otimes \delta) \circ \text{Alt}_2 \quad \text{inside } \text{End}(\mathfrak{g} \otimes \mathfrak{g})$$

When $(\mathfrak{g}, [\ , \], \delta)$ is a Lie superbialgebra, the same holds for $(\mathfrak{g}^*, \delta^*, [\ , \]^*)$ — up to topological technicalities, if $\mathfrak{g}$ is infinite-dimensional — which is called the *dual* Lie superbialgebra to $(\mathfrak{g}, [\ , \], \delta)$. We simply write $\mathfrak{g}$ for a Lie superbialgebra if the structure maps are clear from the context.

As claimed before, the super Lie cobracket in a Lie superbialgebra satisfies a 1-cocycle condition. Thus, as in the non-super case, one may define super cobrackets by taking a 1-cochain: one then says that $(\mathfrak{g}, [\cdot, \cdot], \delta)$ is a *coboundary* Lie superbialgebra if there exists $c \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta = \partial c$, that is, for all (homogeneous) $x \in \mathfrak{g}$ one has

$$\delta(x) = (\partial c)(x) = x \cdot c = [x, c_1] \otimes c_2 + (-1)^{|x||c_1|} c_1 \otimes [x, c_2]$$

Dually, $(\mathfrak{g}, [\cdot, \cdot], \delta)$ is a *boundary* Lie superbialgebra if there exists $\chi \in (\mathfrak{g} \otimes \mathfrak{g})^*$ such that $[\cdot, \cdot] = \partial_* \chi$, that is, for all (homogeneous) $x, y \in \mathfrak{g}$, one has

$$[x, y] = (\partial_* \chi)(x \otimes y) = x_{[1]} \chi(x_{[2]}, y) + (-1)^{|x||y_{[1]}|} y_{[1]} \chi(x, y_{[2]})$$

where ∂_* is the coboundary map for the Lie algebra $\mathfrak{g}^*$.

Finally, given $r = r_1 \otimes r_2$ (summation understood) and $s = s_1 \otimes s_2$ in $\mathfrak{g} \otimes \mathfrak{g}$, set

$$\llbracket r, s \rrbracket := (-1)^{|s_1||r_2|} [r_1, s_1] \otimes r_2 \otimes s_2 + r_1 \otimes [r_2, s_1] \otimes s_2 + (-1)^{|s_1||r_2|} r_1 \otimes s_1 \otimes [r_2, s_2]$$

which in compact form reads

$$\llbracket r, s \rrbracket := (-1)^{|s_1||r_2|} [r_{1,2}, s_{1,3}] + [r_{1,2}, s_{2,3}] + (-1)^{|s_1||r_2|} [r_{1,3}, s_{2,3}]$$

The following proposition is a *super*-version of [Mj, Proposition 8.1.3]:

Proposition 3.1.2.

(a) Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra and $c \in \mathfrak{g} \otimes \mathfrak{g}$. Then $(\mathfrak{g}, [\cdot, \cdot], \partial c)$ is a Lie superbialgebra if and only if for all $x \in \mathfrak{g}$ one has

$$x \cdot ((\text{id} + \sigma)(c)) = 0 \quad \text{and} \quad x \cdot \llbracket c, c \rrbracket = 0$$

(b) Let $(\mathfrak{g}, \delta)$ be a Lie supercoalgebra and $\chi \in (\mathfrak{g} \otimes \mathfrak{g})^*$. Then $(\mathfrak{g}, \partial_* \chi, \delta)$ is a Lie superbialgebra if and only if for all $\alpha \in \mathfrak{g}^*$ one has

$$\alpha \cdot (\chi(\text{id} + \sigma)) = 0 \quad \text{and} \quad \alpha \cdot \llbracket \chi, \chi \rrbracket_* = 0$$

where $\llbracket \alpha, \beta \rrbracket_*$ stands for the corresponding object defined above in $(\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g})^*$ associated with any pair $\alpha, \beta \in \mathfrak{g}^*$.

Proof. We prove only item (a), and we leave instead (b) to the reader as it follows directly from duality.

The first equality is the one that guarantees that the image of ∂c is in $\Lambda^2(\mathfrak{g})$: for any $x \in \mathfrak{g}$ we have

$$(\text{id} + \sigma)((\partial c)(x)) = \text{Sym}_2(x \cdot c) = x \cdot \text{Sym}_2(c) = x \cdot ((\text{id} + \sigma)(c))$$

Hence, $(\partial c)(x) \in \text{Ker}(\text{id} + \sigma) = \Lambda^2(\mathfrak{g})$ if and only if $x \cdot ((\text{id} + \sigma)(c)) = 0$.

To check the super co-Jacobi identity, we proceed as follows. Write $c = c^+ + c^-$ with $c^+ \in S^2(\mathfrak{g})$ being the symmetric part of c and $c^- \in \Lambda^2(\mathfrak{g})$ the antisymmetric part. Then one has that $(\partial c)(x) = (\partial c^-)(x)$ for all $x \in \mathfrak{g}$. Indeed, as $\text{Sym}_2(c^+) = 2c^+$ and $\text{Sym}_2(c^-) = 0$, the equality above tells us that $x \cdot c^+ = 0$. Thus we have that $(\partial c)(x) = x \cdot c = x \cdot c^- = (\partial c^-)(x)$. Now we show that

$$(\text{Sym}_{\mathbb{A}_3} \circ (\text{id} \otimes (\partial c)) \circ (\partial c))(x) = x \cdot \llbracket c^-, c^- \rrbracket$$

for all $x \in \mathfrak{g}$; the claim then will follow from this.

Take $x \in \mathfrak{g}$. We begin by checking the following:

$$\begin{aligned}
((\text{id} \otimes (\partial c^-)) \circ (\partial c^-))(x) &= (\text{id} \otimes (\partial c^-))(x \cdot c^-) = \\
&= (\text{id} \otimes (\partial c^-))\left([x, c_1^-] \otimes c_2^- + (-1)^{|x||c_1^-|} c_1^- \otimes [x, c_2^-]\right) = \\
&= [x, c_1^-] \otimes (c_2^- \cdot c^-) + (-1)^{|x||c_1^-|} c_1^- \otimes ([x, c_2^-] \cdot c^-) = \\
&= [x, c_1^-] \otimes [c_2^-, c_1'^-] \otimes c_2'^- + (-1)^{|c_2^-||c_1'^-|} [x, c_1^-] \otimes c_1'^- \otimes [c_2^-, c_2'^-] + \\
&\quad + (-1)^{|x||c_1^-|} c_1^- \otimes [[x, c_2^-], c_1'^-] \otimes c_2'^- + \\
&\quad + (-1)^{|x||c_1^-|+|x||c_1'^-|+|c_2^-||c_1'^-|} c_1^- \otimes c_1'^- \otimes [[x, c_2^-], c_2'^-]
\end{aligned}$$

Then we get

$$\begin{aligned}
&(\text{Sym}_{\mathbb{A}_3} \circ (\text{id} \otimes (\partial c^-)) \circ (\partial c^-))(x) = \\
&= \text{Sym}_{\mathbb{A}_3} \left([x, c_1^-] \otimes [c_2^-, c_1'^-] \otimes c_2'^- + (-1)^{|c_2^-||c_1'^-|} [x, c_1^-] \otimes c_1'^- \otimes [c_2^-, c_2'^-] + \right. \\
&\quad + (-1)^{|x||c_1^-|} c_1^- \otimes [[x, c_2^-], c_1'^-] \otimes c_2'^- + \\
&\quad \left. + (-1)^{|x||c_1^-|+|x||c_1'^-|+|c_2^-||c_1'^-|} c_1^- \otimes c_1'^- \otimes [[x, c_2^-], c_2'^-] \right) = \\
&= \text{Sym}_{\mathbb{A}_3} \left([x, c_1^-] \otimes [c_2^-, c_1'^-] \otimes c_2'^- + (-1)^{|c_2^-||c_1'^-|} [x, c_1^-] \otimes c_1'^- \otimes [c_2^-, c_2'^-] + \right. \\
&\quad + (-1)^{|x||c_1^-|+|c_1^-|(|x|+|c_2^-|+|c_1'^-|+|c_2'^-|)} [[x, c_2^-], c_1'^-] \otimes c_2'^- \otimes c_1^- + \\
&\quad \left. + (-1)^{|x|(|c_1^-|+|c_1'^-|)+|c_2^-||c_1'^-|+|x|+|c_2^-|+|c_2'^-|)(|c_1'^-|+|c_1^-|)} [[x, c_2^-], c_2'^-] \otimes c_1^- \otimes c_1'^- \right) = \\
&= \text{Sym}_{\mathbb{A}_3} \left([x, c_1^-] \otimes [c_2^-, c_1'^-] \otimes c_2'^- + (-1)^{|c_2^-||c_1'^-|} [x, c_1^-] \otimes c_1'^- \otimes [c_2^-, c_2'^-] + \right. \\
&\quad + (-1)^{|c_1^-||c_2^-|} [[x, c_2^-], c_1'^-] \otimes c_2'^- \otimes c_1^- + \\
&\quad \left. + (-1)^{|c_2^-||c_1'^-|+|c_2^-||c_1^-|+|c_2'^-||c_1^-|} [[x, c_2^-], c_2'^-] \otimes c_1^- \otimes c_1'^- \right) = \\
&= \text{Sym}_{\mathbb{A}_3} \left([x, c_1^-] \otimes [c_2^-, c_1'^-] \otimes c_2'^- + (-1)^{|c_2^-||c_1'^-|} [x, c_1^-] \otimes c_1'^- \otimes [c_2^-, c_2'^-] + \right. \\
&\quad + (-1)^{|c_1^-||c_2^-|} [x, [c_2^-, c_1'^-]] \otimes c_2'^- \otimes c_1^- - \\
&\quad - (-1)^{|c_1^-||c_2^-|+|x||c_2^-|} [c_2^-, [x, c_1'^-]] \otimes c_2'^- \otimes c_1^- + \\
&\quad \left. + (-1)^{|x||c_2'^-|+|c_1'^-||c_2'^-|} [c_2'^-, [x, c_1^-]] \otimes c_2^- \otimes c_1'^- \right) = \\
&= \text{Sym}_{\mathbb{A}_3} \left([x, c_1^-] \otimes [c_2^-, c_1'^-] \otimes c_2'^- + \right. \\
&\quad + (-1)^{|c_2^-||c_1'^-|} [x, c_1^-] \otimes c_1'^- \otimes [c_2^-, c_2'^-] + \\
&\quad \left. + (-1)^{|c_1^-||c_2^-|} [x, [c_2^-, c_1'^-]] \otimes c_2'^- \otimes c_1^- \right) = \\
&= \left(\text{Sym}_{\mathbb{A}_3} \circ ([x, -] \otimes \text{id} \otimes \text{id}) \right) (c_1^- \otimes [c_2^-, c_1'^-] \otimes c_2'^- + \\
&\quad + (-1)^{|c_2^-||c_1'^-|} c_1^- \otimes c_1'^- \otimes [c_2^-, c_2'^-] + (-1)^{|c_1^-||c_1'^-|} [c_1'^-, c_1^-] \otimes c_2'^- \otimes c_2^-) =
\end{aligned}$$

$$\begin{aligned}
&= \left(\text{Sym}_{\mathbb{A}_3} \circ ([x, -] \otimes \text{id} \otimes \text{id}) \right) \left((-1)^{|c_1^-||c_2^-|} [c_1^-, c_1^-] \otimes c_2^- \otimes c_2^- \right) + \\
&\quad + c_1^- \otimes [c_2^-, c_1^-] \otimes c_2^- + (-1)^{|c_2^-||c_1^-|} c_1^- \otimes c_1^- \otimes [c_2^-, c_2^-] = \\
&\quad = \left(\text{Sym}_{\mathbb{A}_3} \circ ([x, -] \otimes \text{id} \otimes \text{id}) \right) ([c^-, c^-]) = x \cdot [c^-, c^-]
\end{aligned}$$

Note that, although it is written in a compact form, the last equality involves a long computation where a lot of signs are involved.

Finally, the cocycle condition follows from the fact that ∂c is a coboundary. Indeed, by using the co-Jacobi identity, for all $x, y \in \mathfrak{g}$ we have

$$\begin{aligned}
(\partial c)([x, y]) &= [x, y] \cdot c = [[x, y], c_1] \otimes c_2 + (-1)^{(|x|+|y|)|c_1|} c_1 \otimes [[x, y], c_2] = \\
&= [x, [y, c_1]] \otimes c_2 - (-1)^{|x||y|} [y, [x, c_1]] \otimes c_2 + \\
&\quad + (-1)^{(|x|+|y|)|c_1|} c_1 \otimes [x, [y, c_2]] - (-1)^{(|x|+|y|)|c_1|+|x||y|} c_1 \otimes [y, [x, c_2]] = \\
&= x \cdot ((\partial c)(y)) - (-1)^{|x||y|} y \cdot ((\partial c)(x)) \quad \square
\end{aligned}$$

3.1.3. Deformations of Lie superbialgebras. To the best of our knowledge, there is a lack of reference and bibliography about a general theory of deformations for Lie superbialgebras. The reason may lie in the fact that Lie superbialgebras may be related to Lie bialgebras through the process of bosonization. Moreover, the theory of deformations of Lie bialgebras can be considered as a sub-theory of that of Lie algebras. For the latter, we refer to [CG], [MW], and references therein. In fact, as in the present work we are mainly interested in two special kinds of deformations performed by *even* objects, where either the Lie cobracket or the Lie bracket alone is deformed, leaving the “other side” of the overall structure untouched, we rely heavily on definitions and basic results on Lie bialgebra cohomology.

We study first a deformation of the super cobracket. Let $(\mathfrak{g}, [\ , \], \delta)$ be a Lie superbialgebra. A homogeneous even element $c \in \mathfrak{g} \otimes \mathfrak{g}$ is called a *twist* if

$$x \cdot ((\text{id} + \sigma)(c)) = 0 \quad \text{and} \quad x \cdot (\text{Sym}_{\mathbb{A}_3}(\text{id} \otimes \delta)(c) - [c, c]) = 0 \quad (3.3)$$

for all $x \in \mathfrak{g}$; here the (adjoint) action of $x \in \mathfrak{g}$ is onto $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ — cf. the action onto $\mathfrak{g} \otimes \mathfrak{g}$ above. As a side note, observe that this definition coincides with that of twist for Lie bialgebras, but with respect to a different symmetric braiding.

Proposition 3.1.4. *Let $(\mathfrak{g}, [\ , \], \delta)$ be a Lie superbialgebra and $c \in \mathfrak{g} \otimes \mathfrak{g}$ a twist. Then the map $\delta^c : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$ defined by*

$$\delta^c(x) := \delta(x) - (\partial c)(x) = \delta(x) - x \cdot c \quad \forall x \in \mathfrak{g} \quad (3.4)$$

is a new super cobracket on the Lie superalgebra $(\mathfrak{g}; [\ , \])$ making $(\mathfrak{g}; [\ , \], \delta^c)$ into a new Lie superbialgebra.

Proof. The proof follows [Mj, Theorem 8.1.7], mutatis mutandis, taking into account the sign changes.

As the image of δ lies in $\Lambda^2(\mathfrak{g})$, the first condition in (3.3) (which is the same as the first in Proposition 3.1.2) ensures that also the image of δ^c is contained in $\Lambda^2(\mathfrak{g})$.

In order to prove that δ^c is a Lie super-cobracket, we have to check the identity $\text{Sym}_{\mathbb{A}_3}((\text{id} \otimes \delta^c) \circ \delta^c)(x) = 0$ for all $x \in \mathfrak{g}$. But $\text{Sym}_{\mathbb{A}_3}((\text{id} \otimes \delta) \circ \delta)(x) = 0$, hence

$$\begin{aligned}
(\text{id} \otimes \delta^c) \circ \delta^c &= (\text{id} \otimes (\delta - \partial c)) \circ (\delta - \partial c) = \\
&= (\text{id} \otimes \delta) \circ \delta - (\text{id} \otimes \delta) \circ (\partial c) - (\text{id} \otimes \partial c) \circ \delta + (\text{id} \otimes \partial c) \circ (\partial c)
\end{aligned}$$

and we have $\text{Sym}_{\mathbb{A}_3}((\text{id} \otimes \partial c) \circ (\partial c))(x) = x.\llbracket c, c \rrbracket$, by the proof of Proposition 3.1.2: thus it is enough to prove that

$$\text{Sym}_{\mathbb{A}_3}((\text{id} \otimes \delta) \circ (\partial c) + (\text{id} \otimes \partial c) \circ \delta)(x) = x.(\text{Sym}_{\mathbb{A}_3}(\text{id} \otimes \delta)(c))$$

for all $x \in \mathfrak{g}$. Set $\delta(x) = x_{[1]} \otimes x_{[2]} \in \Lambda^2(\mathfrak{g})$: then by the cocycle condition we have

$$\begin{aligned} & ((\text{id} \otimes \delta) \circ (\partial c) + (\text{id} \otimes \partial c) \circ \delta)(x) = \\ &= [x, c_1] \otimes \delta(c_2) + (-1)^{|x||c_1|} c_1 \otimes \delta([x, c_2]) + x_{[1]} \otimes [x_{[2]}, c_1] \otimes c_2 + \\ &+ (-1)^{|x_{[2]}||c_1|} x_{[1]} \otimes c_1 \otimes [x_{[2]}, c_2] = \\ &= [x, c_1] \otimes \delta(c_2) + (-1)^{|x||c_1|} c_1 \otimes (x. \delta(c_2)) - c_1 \otimes (c_2. \delta(x)) + \\ &+ x_{[1]} \otimes [x_{[2]}, c_1] \otimes c_2 + (-1)^{|x_{[2]}||c_1|} x_{[1]} \otimes c_1 \otimes [x_{[2]}, c_2] = \\ &= x. (c_1 \otimes \delta(c_2)) - c_1 \otimes (c_2. \delta(x)) + x_{[1]} \otimes [x_{[2]}, c_1] \otimes c_2 + \\ &+ (-1)^{|x_{[2]}||c_1|} x_{[1]} \otimes c_1 \otimes [x_{[2]}, c_2] = \\ &= x. ((\text{id} \otimes \delta)(c)) - c_1 \otimes (c_2. \delta(x)) + x_{[1]} \otimes [x_{[2]}, c_1] \otimes c_2 + \\ &+ (-1)^{|x_{[2]}||c_1|} x_{[1]} \otimes c_1 \otimes [x_{[2]}, c_2] \end{aligned}$$

Since $\text{Sym}_{\mathbb{A}_3}(x.((\text{id} \otimes \delta)(c))) = x.(\text{Sym}_{\mathbb{A}_3}(\text{id} \otimes \delta)(c))$, to finish this calculation it is enough to prove that

$$\text{Sym}_{\mathbb{A}_3}(-c_1 \otimes (c_2. \delta(x)) + x_{[1]} \otimes [x_{[2]}, c_1] \otimes c_2 + (-1)^{|x_{[2]}||c_1|} x_{[1]} \otimes c_1 \otimes [x_{[2]}, c_2]) = 0$$

which follows directly from a straightforward computation. To finish the proof we just note that the super cobracket $\delta^c = \delta - \partial c$ satisfies the cocycle condition since it is a difference of two functions that they do, δ because it is a super cobracket for a Lie superbialgebra and ∂c because it is a coboundary. $\square$

Definition 3.1.5. Let $(\mathfrak{g}, [\ , \], \delta)$ be a Lie superbialgebra and $c \in \mathfrak{g} \otimes \mathfrak{g}$ a twist. The associated Lie superbialgebra $\mathfrak{g}^c := (\mathfrak{g}; [\ , \], \delta^c)$ is called a *deformation by twist* (or “*twist deformation*”) of the original Lie superbialgebra $\mathfrak{g}$. $\diamond$

We introduce now the dual recipe of *deformation by 2-cocycle*, where one deforms the Lie superbracket while keeping the Lie supercobracket untouched. Let again $(\mathfrak{g}; [\ , \], \delta)$ be a Lie superbialgebra.

A homogeneous even element $\chi \in (\mathfrak{g} \otimes \mathfrak{g})^*$ is called a *2-cocycle* if

$$\alpha.(\chi \circ (\text{id} + \sigma)) = 0 \quad \text{and} \quad \alpha.(\chi \circ (\text{id} \otimes [\ , \])) \circ \text{Sym}_{\mathbb{A}_3} - \llbracket \chi, \chi \rrbracket_* = 0 \quad (3.5)$$

for all $\alpha \in \mathfrak{g}^*$. The dual version of Proposition 3.1.4 reads as follows.

Proposition 3.1.6. Let $(\mathfrak{g}, [\ , \], \delta)$ be a Lie superbialgebra and $\chi \in (\mathfrak{g} \otimes \mathfrak{g})^*$ a 2-cocycle. Then the map $[\ , \]_\chi : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ defined for all $x, y \in \mathfrak{g}$ by

$$[x, y]_\chi := [x, y] - (\partial_* \chi)(x \otimes y) = [x, y] - x_{[1]} \chi(x_{[2]}, y) - (-1)^{|x||y_{[1]}|} y_{[1]} \chi(x, y_{[2]}) \quad (3.6)$$

is a new Lie superbracket on the Lie supercoalgebra $(\mathfrak{g}; \delta)$ making $(\mathfrak{g}; [\ , \]_\chi, \delta)$ into a (new) Lie superbialgebra. $\square$

The proof is a sheer dualization of that for Proposition 3.1.4, hence is left to the reader. The relevant terminology is fixed as follows:

Definition 3.1.7. Let $(\mathfrak{g}, [\cdot, \cdot], \delta)$ be a Lie superbialgebra and $\chi \in (\mathfrak{g} \otimes \mathfrak{g})^*$ be a 2-cocycle for it. Then the Lie superbialgebra $\mathfrak{g}_\chi := (\mathfrak{g}; [\cdot, \cdot]_\chi, \delta)$ is called a *deformation by 2-cocycle* (or “2-cocycle deformation”) of the Lie superbialgebra $\mathfrak{g}$. $\diamond$

At last, we point out that the two notions of “twist” and of “2-cocycle” for Lie superbialgebras, as well as the associated deformations, are so devised as to be dual to each other. The following result then holds, whose proof is left to the reader:

Proposition 3.1.8. *Let $\mathfrak{g}$ be a Lie superbialgebra, $\mathfrak{g}^*$ the dual Lie superbialgebra.*

(a) *Let c be a twist for $\mathfrak{g}$, and χ_c the image of c in $(\mathfrak{g}^* \otimes \mathfrak{g}^*)^*$ for the natural composed embedding $\mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathfrak{g}^{**} \otimes \mathfrak{g}^{**} \hookrightarrow (\mathfrak{g}^* \otimes \mathfrak{g}^*)^*$. Then χ_c is a 2-cocycle for $\mathfrak{g}^*$, and there exists a canonical isomorphism $(\mathfrak{g}^*)_{\chi_c} \cong (\mathfrak{g}^c)^*$.*

(b) *Let χ be a 2-cocycle for $\mathfrak{g}$; assume that $\mathfrak{g}$ is finite-dimensional, and let c_χ be the image of χ in the natural identification $(\mathfrak{g} \otimes \mathfrak{g})^* = \mathfrak{g}^* \otimes \mathfrak{g}^*$. Then c_χ is a twist for $\mathfrak{g}^*$, and there exists a canonical isomorphism $(\mathfrak{g}^*)^{c_\chi} \cong (\mathfrak{g}_\chi)^*$. $\square$*

3.2. Multiparameter Lie superbialgebras (=MpLSbA’s).

Let (A, p) be a fixed Cartan super-datum (cf. Definition 2.1.2): we fix a matrix of Cartan type $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k})$ associated with A , i.e. $P_s := 2^{-1}(P + P^T) = DA =: P_A$, and we let $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of P .

With these data, we are going to associate some Lie superbialgebra $\mathfrak{g}_P^{\mathcal{R}, p}$, to be called “multiparameter Lie superbialgebra” (or MpLSbA’s in short): these will depend explicitly on the parameters in P (i.e. its entries), but also on the chosen realization $\mathcal{R}$. Up to technicalities, these MpLSbA’s arise as deformations by twist of Lie superbialgebras of a special family, which comes from quantum counterparts introduced by Yamane — hence we call them “of Yamane’s type”.

3.2.1. Yamane’s Lie superbialgebras. In the present subsection, we refer to the notions in Definition 2.1.3 and all what follows in §2, *but* working now with $\mathbb{k}$ as ground ring instead of $\mathbb{k}[[\hbar]]$; then, since we are dealing with $\mathbb{k}$ rather than $\mathbb{k}[[\hbar]]$, we identify the space $\mathfrak{h}$ with $\bar{\mathfrak{h}}$, the roots α_j with $\bar{\alpha}_j$, etc. (cf. §2.1.3). Following §2.1, let (A, p) be a Cartan super-datum, let $P_A = DA$ be the symmetrization of A , and fix a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P_A which is *split, straight* and *minimal*.

We define the Lie superalgebra $\mathfrak{g}_A^{\mathcal{R}, p}$ as being the one generated over $\mathbb{k}$ by the $\mathbb{k}$ -subspace $\mathfrak{h}$ together with elements E_i and F_i ($i \in I$), whose parity is fixed by $|T| := \bar{0}$ and $|E_i| := p(i) =: |F_i|$ (for all $T \in \mathfrak{h}$ and $i \in I$), with relations as follows (for all $T, T', T'' \in \mathfrak{h}$, $i, j, \ell, k \in I$, $i \neq \ell$):

$$\begin{aligned} [T', T''] &= 0, & [T, E_j] - \alpha_j(T) E_j &= 0, & [T, F_j] + \alpha_j(T) F_j &= 0 \\ [E_i, F_\ell] &= \delta_{i\ell} \frac{T_i^+ + T_i^-}{2d_i}, & [E_i, E_j] &= 0, & [F_i, F_j] &= 0 \quad \text{if } a_{i,j} = 0 \\ (\text{ad}(E_i))^{1-a_{ij}}(E_j) &= 0, & (\text{ad}(F_i))^{1-a_{ij}}(F_j) &= 0 & \text{with } p(i) = \bar{0}, \quad i \neq j \end{aligned}$$

and then also all of the following, with $X \in \{E, F\}$:

$$\begin{aligned} & \begin{array}{c} i \qquad j \qquad k \\ \times \text{---} \bigotimes \text{---} \times \end{array} & \begin{array}{c} i \qquad j \qquad k \\ \times \text{---} \bigotimes \Rightarrow \Rightarrow \bullet \end{array} & \begin{array}{c} i \qquad j \qquad k \\ \times \text{---} \bigotimes \Rightarrow \Rightarrow \times \end{array} \end{aligned}$$

$$\begin{aligned}
 & \left[\left[[X_{n-1}, X_n], X_n \right], X_n \right] = 0 \\
 & \text{with } \begin{array}{c} n-1 \\ \times \rightrightarrows \end{array} \begin{array}{c} n \\ \bullet \end{array} \quad \text{and } A \text{ of type } B_n \\
 & \left[[X_{n-2}, X_{n-1}], X_n \right] - \left[[X_{n-2}, X_n], X_{n-1} \right] = 0 \\
 & \text{with } \begin{array}{c} n-2 \\ \times \end{array} \begin{array}{c} n \\ \otimes \\ \parallel \\ \otimes \\ n-1 \end{array} \quad \text{and } A \text{ of type } D_n(II) \\
 & \left[\left[[X_{n-2}, X_{n-1}], X_n \right], [X_{n-2}, X_{n-1}] \right], X_{n-1} \right] = 0 \\
 & \text{for } \begin{array}{c} n-2 \quad n-1 \quad n \\ \otimes \longrightarrow \otimes \rightrightarrows \circ \end{array} \quad \text{and } A \text{ of type } C_n \\
 & \left[\left[\left[[X_{n-3}, X_{n-2}], X_{n-1} \right], X_n \right], X_{n-1} \right], X_{n-2} \right], X_{n-1} \right] = 0 \\
 & \text{for } \begin{array}{c} n-3 \quad n-2 \quad n-1 \quad n \\ \times \longrightarrow \otimes \longrightarrow \otimes \rightrightarrows \circ \end{array} \quad \text{and } A \text{ of type } C_n
 \end{aligned}$$

This $\mathfrak{g}_A^{\mathcal{R},p}$ is nothing but the simple Lie superalgebra associated with (A, p) : its presentation given above comes from the presentation of a suitable quantization of $U(\mathfrak{g}_A^{\mathcal{R},p})$ — in the sense of §4.1.2 — introduced in [Ya1] — in a “quantum double version”, denoted in [loc. cit.] by $\mathcal{D}'$. Moreover, out of that quantization the Lie superalgebra $\mathfrak{g}_A^{\mathcal{R},p}$ inherits an additional Lie cobracket, which makes it into a Lie superbialgebra, given on generators by (for $T \in \mathfrak{h}$ and $i \in I$)

$$\delta(T) = 0 \quad , \quad \delta(E_i) = 2 T_i^+ \wedge E_i \quad , \quad \delta(F_i) = 2 T_i^- \wedge F_i \quad (3.7)$$

Let us now pick a realization $\mathcal{R}'$ of P_A that is still straight and minimal but not necessarily split; then we can still define a Lie superalgebra $\mathfrak{g}_A^{\mathcal{R}',p}$ with the same presentation as $\mathfrak{g}_A^{\mathcal{R},p}$ above (with a “Cartan subalgebra” $\mathfrak{h}'$ replacing $\mathfrak{h}$): indeed, the former will just be a quotient of the latter. Moreover, the Lie cobracket of $\mathfrak{g}_A^{\mathcal{R},p}$ will clearly be pushed forward onto $\mathfrak{g}_A^{\mathcal{R}',p}$, so that the latter is indeed a Lie superbialgebra with the same description as $\mathfrak{g}_A^{\mathcal{R},p}$, but for changing $\mathcal{R}$ into $\mathcal{R}'$. Finally, consider a realization $\mathcal{R}''$ of P_A that is still straight but not necessarily split nor minimal. Then we can again define a Lie superalgebra $\mathfrak{g}_A^{\mathcal{R}'',p}$ with the same presentation as $\mathfrak{g}_A^{\mathcal{R}',p}$ (with $\mathfrak{h}''$ replacing $\mathfrak{h}'$), the former being an extension of the latter, i.e. $\mathfrak{g}_A^{\mathcal{R}',p}$ is a Lie sub-superalgebra of $\mathfrak{g}_A^{\mathcal{R}'',p}$; moreover, the Lie cobracket of $\mathfrak{g}_A^{\mathcal{R}',p}$ uniquely extends to all of $\mathfrak{g}_A^{\mathcal{R}'',p}$, so the latter is also a Lie superbialgebra with Lie cobracket as in (3.7) again.

In the following, we will refer to any of the Lie superalgebras $\mathfrak{g}_A^{\mathcal{R},p}$ just described — for every straight realization $\mathcal{R}$ of P_A — as “Yamane’s Lie superbialgebras”.

3.2.2. Definition of multiparameter Lie superbialgebras. Let (A, p) be a fixed Cartan super-datum (as in Definition 2.1.2), and let $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k})$ be a matrix of Cartan type associated with A , i.e. $P_s := 2^{-1}(P + P^T) = DA =: P_A$.

With these assumptions, let $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of P .

We define the Lie $\mathbb{k}$ -superalgebra $\mathfrak{g}_P^{\mathcal{R},p}$ as the one having the very same presentation as $\mathfrak{g}_A^{\mathcal{R},p}$ above, but with respect to the realization $\mathcal{R}$. Then we prove that this $\mathfrak{g}_P^{\mathcal{R},p}$ actually bears a structure of Lie superbialgebra, through a series of steps.

First of all, let $\mathcal{R}$ be a straight realization of $P_A := DA$; then Yamane's Lie superbialgebra $\mathfrak{g}_A^{\mathcal{R},p}$ is defined (see §3.2.1 above).

Second, for any antisymmetric matrix $\Theta = (\theta_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k})$ we consider

$$j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k \in \mathfrak{h} \otimes \mathfrak{h} \subseteq \mathfrak{g} \otimes \mathfrak{g} \quad (3.8)$$

This is indeed a twist element (cf. (3.3) and Definition 3.1.5) for Yamane's Lie superbialgebra $\mathfrak{g}_A^{\mathcal{R},p}$: we call this j_Θ the toral twist (or “twist of toral type”) associated with Θ . Then out of Θ we can consider the deformed Lie superalgebra $(\mathfrak{g}_A^{\mathcal{R},p})^{j_\Theta}$ — as in §3.1.3 — the “deformed” multiparameter matrix $P_\Theta = (p_{i,j}^\Theta)_{i,j \in I} := P_A - \mathfrak{A} \Theta \mathfrak{A}^T$ as in (2.2) — again of Cartan type A , the same as P — and its corresponding “deformed” straight realization $\mathcal{R}_\Theta := (\mathfrak{h}, \Pi, \Pi_\Theta^\vee := \{T_{\Theta,i}^+, T_{\Theta,i}^-\}_{i \in I})$ — see Definition 2.2.3. Accordingly, we write $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ for the Lie superalgebra associated with the realization $\mathcal{R}_\Theta$ of P_Θ .

Third, we find the following, strict link between all these “deformed objects”:

Proposition 3.2.3. *With assumptions as above, the following hold:*

(a) *there exists a Lie superalgebra isomorphism*

$$\phi_P^\Theta : \mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p} \xrightarrow{\cong} (\mathfrak{g}_A^{\mathcal{R},p})^{j_\Theta}, \quad E_i \mapsto E_i, T \mapsto T, F_i \mapsto F_i \quad (i \in I, T \in \mathfrak{h})$$

(b) *there exists a unique structure of Lie superbialgebra onto $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ such that the map ϕ_P^Θ in (a) above is an isomorphism of Lie superbialgebras;*

(c) *the Lie cobracket of the Lie superbialgebra structure of $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ mentioned in (b) above is given by the formulas (for all $T \in \mathfrak{h}$, $i \in I$)*

$$\delta(T) = 0, \quad \delta(E_i) = 2 T_{\Theta,i}^+ \wedge E_i, \quad \delta(F_i) = 2 T_{\Theta,i}^- \wedge F_i$$

Proof. (a) By construction, the Lie algebra structure in $(\mathfrak{g}_A^{\mathcal{R},p})^{j_\Theta}$ is the same of $\mathfrak{g}_A^{\mathcal{R},p}$, and the latter only depends on the α_j 's and the sums $S_j := 2^{-1}(T_j^+ + T_j^-)$ ($j \in I$). Now, both the α_j 's and the S_j 's do not change (by construction) when we pass from $\mathfrak{g}_A^{\mathcal{R},p}$ to $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ or viceversa; therefore, the formulas in the claim — mapping each generator of $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ onto the same name generator of $\mathfrak{g}_A^{\mathcal{R},p} = (\mathfrak{g}_A^{\mathcal{R},p})^{j_\Theta}$ — actually do provide an isomorphism ϕ_P^Θ of Lie superalgebras, as claimed.

(b) By pull-back through ϕ_P^Θ one gets onto $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ the unique Lie cobracket which makes the former into a Lie superbialgebra such that the map ϕ_P^Θ is also an isomorphism of Lie superbialgebras.

(c) For the toral twist $j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k$ in (3.8), formula (3.4) yields

$$\delta^{j_\Theta}(x) := \delta(x) - x \cdot j_\Theta = \delta(x) - \sum_{g,k=1}^t \theta_{gk} ([x, H_g] \otimes H_k + H_g \otimes [x, H_k])$$

for all $x \in \mathfrak{g}_{P_A}^{\mathcal{R},p}$. Now take $x := E_\ell$ ($\ell \in I$): then computations give

$$\begin{aligned} \delta^{j_\Theta}(E_\ell) &:= \delta(E_\ell) - \sum_{g,k=1}^t \theta_{gk} ([E_\ell, H_g] \otimes H_k + H_g \otimes [E_\ell, H_k]) = \\ &= \delta(E_\ell) - \sum_{g,k=1}^t \theta_{gk} (-\alpha_\ell(H_g) E_\ell \otimes H_k - H_g \otimes \alpha_\ell(H_k) E_\ell) = \\ &= T_\ell^+ \otimes E_\ell - E_\ell \otimes T_\ell^+ + \sum_{g,k=1}^t \theta_{gk} (\alpha_\ell(H_g) E_\ell \otimes H_k + \alpha_\ell(H_k) H_g \otimes E_\ell) = \\ &= \left(T_\ell^+ + \sum_{g,k=1}^t \theta_{kg} \alpha_\ell(H_g) H_k \right) \otimes E_\ell - E_\ell \otimes \left(T_\ell^+ + \sum_{g,k=1}^t \theta_{kg} \alpha_\ell(H_g) H_k \right) = \\ &= T_{\Theta,\ell}^+ \otimes E_\ell - E_\ell \otimes T_{\Theta,\ell}^+ = 2 T_{\Theta,\ell}^+ \wedge E_\ell \end{aligned}$$

hence in short we get $\delta^{j_\Theta}(E_\ell) = 2 T_{\Theta,\ell}^+ \wedge E_\ell$. Similar computations give

$$\delta^{j_\Theta}(T) = 0, \quad \delta^{j_\Theta}(F_\ell) = 2 T_{\Theta,\ell}^- \wedge F_\ell, \quad \forall \ell \in I, T \in \mathfrak{h}$$

Therefore, pulling back through ϕ_P^Θ onto $\mathfrak{g}_P^{\mathcal{R},p}$ the Lie cobracket of $(\mathfrak{g}_{P_A}^{\mathcal{R},p})^{j_\Theta}$, we find that this Lie cobracket is described by the formulas in claim (c), q.e.d. $\square$

With some extra effort we can get the following, key result:

Theorem 3.2.4. *Every Lie superalgebra $\mathfrak{g}_P^{\mathcal{R},p}$ is indeed a Lie superbialgebra, whose Lie cobracket is given on generators by the formulas (for all $T \in \mathfrak{h}$, $i \in I$)*

$$\delta(T) = 0, \quad \delta(E_i) = 2 T_i^+ \wedge E_i, \quad \delta(F_i) = 2 T_i^- \wedge F_i \quad (3.9)$$

Proof. The result follows from the discussion above when the realization $\mathcal{R}$ of P is straight. To prove the statement for general realizations $\mathcal{R}$, we proceed as follows. By assumption, we have a fixed Cartan super-datum (A, p) ; furthermore, $P_s := 2^{-1}(P + P^T) = DA =: P_A$, and finally $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of P .

By Lemma 2.1.8, there is a *straight* realization $\tilde{\mathcal{R}} := (\tilde{\mathfrak{h}}, \tilde{\Pi}, \tilde{\Pi}^\vee)$ of P and an epimorphism of realizations $\pi : \tilde{\mathcal{R}} \twoheadrightarrow \mathcal{R}$; moreover, up to possibly enlarging $\tilde{\mathfrak{h}}$, we can assume that $t := \text{rk}(\tilde{\mathfrak{h}}) \geq 3n - \text{rk}(P_A)$. Then, by Proposition 2.2.6(a) — with $P' = P_A := DA$ — there exists some $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that $P_A = P_\Phi$ while $\mathcal{R}_A := \tilde{\mathcal{R}}_\Phi$ is a straight realization of $P_A = P_\Phi$. Hence, conversely, we have also $P = (P_A)_{-\Phi}$ and $\mathcal{R} = (\mathcal{R}_A)_{-\Phi}$. Then there exists an isomorphism of

Lie superbialgebras $\phi_P^{-\Phi} : \mathfrak{g}_P^{\tilde{\mathcal{R}},p} = \mathfrak{g}_{(P_A)_{-\Phi}}^{(\mathcal{R}_A)_{-\Phi},p} \xrightarrow{\cong} (\mathfrak{g}_{P_A}^{\mathcal{R}_A,p})^{j-\Phi}$, explicitly described as in Proposition 3.2.3, up to switching Φ to $-\Phi$; moreover, the Lie cobracket in $\mathfrak{g}_P^{\tilde{\mathcal{R}},p} = \mathfrak{g}_{(P_A)_{-\Phi}}^{(\mathcal{R}_A)_{-\Phi},p}$ is described as in Proposition 3.2.3(c), which in terms of the defining generators of $\mathfrak{g}_P^{\tilde{\mathcal{R}},p}$ reads exactly like (3.9) in the claim.

Now, the epimorphism of realizations $\pi : \tilde{\mathcal{R}} \twoheadrightarrow \mathcal{R}$ “include” an epimorphism $\pi : \tilde{\mathfrak{h}} \twoheadrightarrow \mathfrak{h}$ of (Abelian) Lie $\mathbb{k}[[\hbar]]$ -superalgebras; moreover, it follows at once from the presentation of both $\mathfrak{g}_P^{\tilde{\mathcal{R}},p}$ and $\mathfrak{g}_P^{\mathcal{R},p}$ that π extends to an epimorphism of Lie superalgebras $\mathcal{L}_\pi : \mathfrak{g}_P^{\tilde{\mathcal{R}},p} \twoheadrightarrow \mathfrak{g}_P^{\mathcal{R},p}$, and that $\text{Ker}(\mathcal{L}_\pi)$ is generated by $\text{Ker}(\pi)$. Furthermore, $\text{Ker}(\pi)$ lies in the center of $\mathfrak{g}_P^{\tilde{\mathcal{R}},p}$, by definitions and by Lemma 2.1.9; moreover, the Lie cobracket of $\mathfrak{g}_P^{\tilde{\mathcal{R}},p}$ kills $\text{Ker}(\pi)$, as it is trivial on $\tilde{\mathfrak{h}}$, so the latter is a Lie superbideal in the Lie superbialgebra $\mathfrak{g}_P^{\tilde{\mathcal{R}},p}$. Thus $\mathfrak{g}_P^{\mathcal{R},p}$ inherits via $\mathcal{L}_\pi$ a quotient Lie superbialgebra structure from $\mathfrak{g}_P^{\tilde{\mathcal{R}},p}$, again described by (3.9), q.e.d. $\square$

Definition 3.2.5. Every such Lie superbialgebra $\mathfrak{g}_P^{\mathcal{R},p}$ considered above will be called *multiparameter Lie superbialgebra*, in short *MpLSbA*. In addition, we say that the MpLSbA $\mathfrak{g}_P^{\mathcal{R},p}$ is *straight*, or *small*, or *minimal*, or *split*, if such is $\mathcal{R}$.

Finally, we define the *rank* of $\mathfrak{g}_P^{\mathcal{R},p}$ as $\text{rk}(\mathfrak{g}_P^{\mathcal{R},p}) := \text{rk}(\mathcal{R}) = \text{rk}_{\mathbb{k}}(\mathfrak{h})$. $\diamond$

Remarks 3.2.6. Let $\mathfrak{n}_+^{\mathcal{R}}$, resp. $\mathfrak{n}_-^{\mathcal{R}}$, be the Lie sub-superalgebra of $\mathfrak{g}_P^{\mathcal{R},p}$ generated by all the E_i 's, resp. all the F_i 's; and let $\mathfrak{b}_{P,\pm}^{\mathcal{R},p} := \mathfrak{h} \oplus \mathfrak{n}_{\pm}^{\mathcal{R}} = \mathfrak{n}_{\pm}^{\mathcal{R}} \oplus \mathfrak{h}$. Then:

- (a) $\mathfrak{b}_{P,\pm}^{\mathcal{R},p}$ is a Lie sub-superbialgebra of $\mathfrak{g}_P^{\mathcal{R},p}$,
- (b) there exist obvious *triangular decompositions*

$$\mathfrak{g}_P^{\mathcal{R},p} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \quad , \quad \mathfrak{g}_P^{\mathcal{R},p} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad (3.10)$$

The following, last two result stress the fact that the dependence of MpLSbA's on realizations — for a common, fixed multiparameter matrix P — is *functorial*:

Proposition 3.2.7. *Let (A, p) be a fixed Cartan super-datum and let $P \in M_n(\mathbb{k})$ be of Cartan type A . If both $\mathcal{R}'$ and $\mathcal{R}''$ are realizations of P and $\phi : \mathcal{R}' \longrightarrow \mathcal{R}''$ is a morphism between them, then there exists a unique morphism of Lie superbialgebras $\mathcal{L}_{\phi} : \mathfrak{g}_P^{\mathcal{R}',p} \longrightarrow \mathfrak{g}_P^{\mathcal{R}'',p}$ that extends the morphism $\phi : \mathfrak{h}' \longrightarrow \mathfrak{h}''$ given by ϕ ; moreover, $\mathcal{L}_{\text{id}_{\mathcal{R}}} = \text{id}_{\mathfrak{g}_P^{\mathcal{R},p}}$ and $\mathcal{L}_{\phi' \circ \phi} = \mathcal{L}_{\phi'} \circ \mathcal{L}_{\phi}$ (whenever $\phi' \circ \phi$ is defined). In a nutshell, the construction $\mathcal{R} \mapsto \mathfrak{g}_P^{\mathcal{R},p}$ (for fixed P) is functorial in $\mathcal{R}$.*

Further, if ϕ is an epimorphism, resp. a monomorphism, then $\mathcal{L}_{\phi}$ is an epimorphism, resp. a monomorphism, as well. Finally, for any morphism $\phi : \mathcal{R}' \longrightarrow \mathcal{R}''$, the kernel $\text{Ker}(\mathcal{L}_{\phi})$ of $\mathcal{L}_{\phi}$ coincides with $\text{Ker}(\phi)$, and the latter is central in $\mathfrak{g}_P^{\mathcal{R}',p}$.

In particular, when ϕ , and hence also $\mathcal{L}_{\phi}$, is an epimorphism, we have — setting $\mathfrak{k} := \text{Ker}(\phi)$ — a central exact sequence of Lie superbialgebras

$$0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g}_P^{\mathcal{R}',p} \xrightarrow{\mathcal{L}_{\phi}} \mathfrak{g}_P^{\mathcal{R}'',p} \longrightarrow 0$$

Proof. The existence of $\mathcal{L}_{\phi}$ is obvious, as well as all the other claims; we only spend a moment on $\text{Ker}(\phi)$ being central. Lemma 2.1.9 implies $\text{Ker}(\phi) \subseteq \bigcap_{j \in I} \text{Ker}(\alpha'_j)$;

then from the relations among the generators of $\mathfrak{g}_P^{\mathcal{R}'}$ we get that each element in $\text{Ker}(\phi)$ commutes with all generators of $\mathfrak{g}_P^{\mathcal{R}'}$, thus $\text{Ker}(\phi)$ is central. $\square$

Corollary 3.2.8. *With notation as above, if $\mathcal{R}' \cong \mathcal{R}''$ then $\mathfrak{g}_P^{\mathcal{R}',p} \cong \mathfrak{g}_P^{\mathcal{R}'',p}$.*

In particular, all MpLSbA's built upon split realizations, respectively small realizations, of the same matrix P and sharing the same rank of $\mathfrak{h}$, are isomorphic to each other, hence they are independent (up to isomorphisms) of the specific realization, but only depend on P and on the rank of $\mathfrak{h}$.

Proof. This follows at once from Proposition 3.2.7 together with the uniqueness result in Proposition 2.1.5 and Proposition 2.1.7. $\square$

3.3. Deformations of MpLSbA's by toral twists.

We will now go and study deformations of MpLSbA's, following §3.1.3. We begin with deformations by twist, choosing the latter to be of a special type.

Let $\mathfrak{g}_P^{\mathcal{R},p}$ be a MpLSbA as in §3.2 above; then $\mathfrak{h}$ is a free $\mathbb{k}$ -module of finite rank $t := \text{rk}(\mathfrak{h})$: we fix in it a $\mathbb{k}$ -basis $\{H_g\}_{g \in \mathcal{G}}$, where $\mathcal{G}$ is an index set with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) =: t$. We begin introducing the so-called “toral” twists for $\mathfrak{g}_P^{\mathcal{R},p}$.

Definition 3.3.1. For any antisymmetric matrix $\Theta = (\theta_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k})$, we set

$$j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k \in \mathfrak{h} \otimes \mathfrak{h} \subseteq \mathfrak{g} \otimes \mathfrak{g} \quad (3.11)$$

and we call this j_Θ the toral twist (or “twist of toral type”) associated with Θ . $\diamond$

Next result — which explains our use of terminology — follows by construction; in particular, it makes use of the antisymmetry condition on Θ .

Lemma 3.3.2. For any $\Theta = (\theta_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k})$, the element j_Θ in Definition 3.3.1 is a twist element for the Lie superbialgebra $\mathfrak{g}_P^{\mathcal{R},p}$, in the sense of (3.3). $\square$

Concerning deformations of MpLSbA’s by toral twists, our main result is the next one. To settle its content, let $P \in M_t(\mathbb{k})$ be a multiparameter matrix of Cartan type with associated Cartan super-datum (A, p) , let $\mathcal{R}$ be a realization of it, and let $\mathfrak{g}_P^{\mathcal{R},p}$ be the associated multiparameter Lie bialgebra; then, for any given antisymmetric matrix $\Theta = (\theta_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k})$, let $j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k$ be the associated twist as in (3.11). Moreover, we consider the “deformed” multiparameter matrix $P_\Theta = (p_{i,j}^\Theta)_{i,j \in I} := P - \mathfrak{A} \Theta \mathfrak{A}^T$ as in (2.2) — again of Cartan type, the same as P — and its corresponding “deformed” realization $\mathcal{R}_\Theta := (\mathfrak{h}, \Pi, \Pi_\Theta^\vee := \{T_{\Theta,i}^+, T_{\Theta,i}^-\}_{i \in I})$.

Theorem 3.3.3. There is a Lie superbialgebra isomorphism $f_P^\Theta : \mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p} \xrightarrow{\cong} (\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$ (notation as above, with in right-hand side the twist deformation $(\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$ of $\mathfrak{g}_P^{\mathcal{R},p}$ by j_Θ occurs) given by $E_i \mapsto E_i$, $T \mapsto T$ and $F_i \mapsto F_i$ for all $i \in I$, $T \in \mathfrak{h}$.

In particular, the class of all MpLSbA’s of any fixed Cartan type and of fixed rank is stable by toral twist deformations. Moreover, inside it the subclass of all such MpLSbA’s associated with straight, resp. small, realizations is stable as well.

Proof. By definition — cf. §3.1.3 — the Lie superalgebra structure in $(\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$ is the same as in $\mathfrak{g}_P^{\mathcal{R},p}$, and the latter only depends on the α_j ’s and the sums $S_j := 2^{-1}(T_j^+ + T_j^-)$ ($j \in I$). Now, both the α_j ’s and the S_j ’s do not change (again by construction) when we pass from $\mathfrak{g}_P^{\mathcal{R},p}$ to $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ or viceversa; therefore, the formulas in the claim (mapping each generator of $\mathfrak{g}_{P_\Theta}^{\mathcal{R}_\Theta,p}$ onto the same name generator of $\mathfrak{g}_P^{\mathcal{R},p} = (\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$) provide an isomorphism of Lie superalgebras.

Now consider the toral twist $j_\Theta := \sum_{g,k=1}^t \theta_{gk} H_g \otimes H_k$ given in (3.11). By (3.4)

$$\delta^{j_\Theta}(x) := \delta(x) - x \cdot j_\Theta = \delta(x) - \sum_{g,k=1}^t \theta_{gk} ([x, H_g] \otimes H_k + H_g \otimes [x, H_k])$$

for all $x \in \mathfrak{g}$. Now take $x := E_\ell$ ($\ell \in I$): then the previous formula reads

$$\begin{aligned} \delta^{j_\Theta}(E_\ell) &:= \delta(E_\ell) - \sum_{g,k=1}^t \theta_{gk} ([E_\ell, H_g] \otimes H_k + H_g \otimes [E_\ell, H_k]) = \\ &= \delta(E_\ell) - \sum_{g,k=1}^t \theta_{gk} (-\alpha_\ell(H_g) E_\ell \otimes H_k - H_g \otimes \alpha_\ell(H_k) E_\ell) = \\ &= T_\ell^+ \otimes E_\ell - E_\ell \otimes T_\ell^+ + \sum_{g,k=1}^t \theta_{gk} (\alpha_\ell(H_g) E_\ell \otimes H_k + \alpha_\ell(H_k) H_g \otimes E_\ell) = \end{aligned}$$

$$\begin{aligned}
&= \left(T_\ell^+ + \sum_{g,k=1}^t \theta_{kg} \alpha_\ell(H_g) H_k \right) \otimes E_\ell - E_\ell \otimes \left(T_\ell^+ + \sum_{g,k=1}^t \theta_{kg} \alpha_\ell(H_g) H_k \right) = \\
&= T_{\Theta,\ell}^+ \otimes E_\ell - E_\ell \otimes T_{\Theta,\ell}^+ = 2 T_{\Theta,\ell}^+ \wedge E_\ell
\end{aligned}$$

hence in short we get $\delta^{j_\Theta}(E_\ell) = 2 T_{\Theta,\ell}^+ \wedge E_\ell$. Similar computations give

$$\delta^{j_\Theta}(T) = 0, \quad \delta^{j_\Theta}(F_\ell) = 2 T_{\Theta,\ell}^- \wedge F_\ell, \quad \forall \ell \in I, T \in \mathfrak{h}$$

This means that, through the Lie superalgebra isomorphism f_P^Θ , the Lie coalgebra structure of $(\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$ is described on generators exactly like that of $\mathfrak{g}_{P_\Theta}^{\mathcal{R},p}$, with the new “coroots” $T_{\Theta,\ell}^\pm$ ($\ell \in I$) in $(\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$ playing the role of the coroots $T_\ell^\pm$ ($\ell \in I$) in $\mathfrak{g}_{P_\Theta}^{\mathcal{R},p}$. Thus $f_P^\Theta : \mathfrak{g}_P^{\mathcal{R},p} \longrightarrow (\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$ is an isomorphism of Lie superbialgebras. $\square$

In fact, the previous result can be reversed, somehow. Namely, our next result shows, in particular, that every straight small MpLSbA can be realized as a toral twist deformation of the Yamane’s MpLSbA $\mathfrak{g}_A^{\mathcal{R},p}$ (as in §3.2.1), cf. claim (c) below.

Theorem 3.3.4. *With assumptions as above, let P and P' be two matrices of Cartan type with the same associated Cartan super-datum (A, p) , i.e. $P_s = P'_s$.*

(a) *Let $\mathcal{R}$ be a straight realization of P and let $\mathfrak{g}_P^{\mathcal{R},p}$ be the associated MpLSbA. Then there exists a matrix $\Theta \in \mathfrak{so}_t(\mathbb{k})$ such that $P' = P_\Theta$, the corresponding $\mathcal{R}_\Theta$ is a straight realization of $P' = P_\Theta$, and for the twist element j_Θ as in (3.11) we have*

$$\mathfrak{g}_{P'}^{\mathcal{R}_\Theta,p} \cong (\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$$

In a nutshell, if $P'_s = P_s$ then from any straight MpLSbA over P we can obtain by toral twist deformation a straight MpLSbA (of the same rank) over P' .

(b) *Let $\mathcal{R}$ and $\mathcal{R}'$ be straight small realizations of P and P' respectively, with $\text{rk}(\mathcal{R}) = \text{rk}(\mathcal{R}') =: t$, and let $\mathfrak{g}_P^{\mathcal{R},p}$ and $\mathfrak{g}_{P'}^{\mathcal{R}',p}$ be the associated MpLSbA’s. Then there exists a matrix $\Theta \in \mathfrak{so}_t(\mathbb{k})$ such that for the twist element j_Θ as in (3.11) we have*

$$\mathfrak{g}_{P'}^{\mathcal{R}',p} \cong (\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$$

In a nutshell, if $P'_s = P_s$ then any straight small MpLSbA over P' is isomorphic to a toral twist deformation of any straight small MpLSbA over P of the same rank.

(c) *Every straight small MpLSbA is isomorphic to some toral twist deformation of Yamane’s MpLSbA $\mathfrak{g}_A^{\mathcal{R},p}$ (over $P_A = DA = P_s$, cf. §3.2.1) of the same rank.*

Proof. (a) By Theorem 3.3.3 it is enough to find $\Theta \in \mathfrak{so}_t(\mathbb{k})$ such that $P' = P_\Theta$, that is $P' = P - \mathfrak{A} \Theta \mathfrak{A}^T$; but this is guaranteed by Lemma 2.2.5, so we are done.

(b) This follows from claim (a), along with the uniqueness of straight small realizations — cf. Proposition 2.1.5(b) — and Proposition 3.2.7.

(c) This follows from (b), once we take as $\mathfrak{g}_{P'}^{\mathcal{R}',p}$ the given straight small MpLSbA and as $\mathfrak{g}_P^{\mathcal{R},p}$ Yamane’s MpLSbA $\mathfrak{g}_{P'}^{\mathcal{R}',p}$ over $P := P_A = DA$. $\square$

Observations 3.3.5.

(a) Theorems 3.3.3 and 3.3.4 allow the following interpretation. Our MpLSbA’s $\mathfrak{g}_P^{\mathcal{R},p}$ depend on the multiparameter P ; at a further level, once we perform onto $\mathfrak{g}_P^{\mathcal{R},p}$ a deformation by toral twist, the outcome $\mathfrak{g}_{P,\Theta}^{\mathcal{R},p} := (\mathfrak{g}_P^{\mathcal{R},p})^{j_\Theta}$ depends on two multiparameters, namely P and Θ . So all the $\mathfrak{g}_{P,\Theta}^{\mathcal{R},p}$ ’s form a (seemingly) richer family of “twice-multiparametric” Lie superbialgebras. In spite of this, Theorem 3.3.3 proves that this family still coincides with our initial family of MpLSbA’s.

In short, Theorems 3.3.3 and 3.3.4 show the following. The dependence of the Lie superbialgebra structure of $\mathfrak{g}_{P,\Theta}^{\mathcal{R},p}$ on the “double parameter” (P, Θ) is “split” in the algebraic structure (ruled by P) and in the coalgebraic structure (ruled by Θ). Now, Theorems 3.3.3 and 3.3.4 enable us to “polarize” this dependence so to codify it either entirely within the algebraic structure (while the coalgebraic one is reduced to a “canonical form”) — so the single multiparameter P_Θ is enough to describe it — or entirely within the coalgebraic structure (with the algebraic one being reduced to the “standard” one) — so that one multiparameter Θ_P is enough to describe it.

(b) As we saw at the end of §2.2.7, the (sub)class of *split* realizations is *not closed* under twist deformation; as a consequence, the subclass of all MpLSbA’s that are “split” is not closed either under twist deformation.

3.4. Deformations of MpLSbA’s by toral 2-cocycles.

Let $\mathfrak{g}_P^{\mathcal{R},p}$ be a MpLSbA as in §3.2, and keep notation as above. Dually to what we did before in §3.3, we consider now the so-called “toral” 2-cocycles for $\mathfrak{g}_P^{\mathcal{R},p}$ and the corresponding deformations by them.

Definition 3.4.1. Fix an antisymmetric $\mathbb{k}$ -linear map $\chi : \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{k}$ such that

$$\chi(S_i, -) = 0 = \chi(-, S_i) \quad \forall i \in I \quad (3.12)$$

where $S_i := 2^{-1}(T_i^+ + T_i^-)$ for all $i \in I$ (cf. Definition 2.1.3); in other words, we have $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$, cf. (2.5). Moreover, let $\pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R},p}} : \mathfrak{g}_P^{\mathcal{R},p} \longrightarrow \mathfrak{h}$ be the canonical linear projection induced by any one of the triangular decompositions in (3.10). We define

$$\chi_{\mathfrak{g}} := \chi \circ \left(\pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R},p}} \otimes \pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R},p}} \right) : \mathfrak{g}_P^{\mathcal{R},p} \otimes \mathfrak{g}_P^{\mathcal{R},p} \longrightarrow \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{k}$$

and we call it *the toral 2-cocycle* (or “2-cocycle of toral type”) associated with χ . $\diamond$

The following lemma explains our use of terminology:

Lemma 3.4.2. *For any antisymmetric $\mathbb{k}$ -linear map $\chi : \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{k}$ obeying (3.12),*

$$\chi_{\mathfrak{g}} := \chi \circ \left(\pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R},p}} \otimes \pi_{\mathfrak{h}}^{\mathfrak{g}_P^{\mathcal{R},p}} \right) : \mathfrak{g}_P^{\mathcal{R},p} \otimes \mathfrak{g}_P^{\mathcal{R},p} \longrightarrow \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathbb{k}$$

is a 2-cocycle map for the Lie bialgebra $\mathfrak{g}_P^{\mathcal{R},p}$, in the sense of (3.5).

Proof. Follows from [GG4, Lemma 3.4.2] *mutatis mutandis*. $\square$

A dual analog of Theorems 3.3.3 and 3.3.4 asserts that one may obtain any (straight, minimal, split) MpLSbA as a 2-cocycle deformation of some Yamane’s MpLSbA. To state properly the result, we need some notation. Let (A, p) be a Cartan super-datum, $P \in M_t(\mathbb{k})$ a multiparameter matrix of Cartan type with Cartan matrix A , $\mathcal{R}$ be a realization of it, and $\mathfrak{g}_P^{\mathcal{R},p}$ the associated multiparameter Lie superbialgebra. For any $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ as in (2.5), let $\chi_{\mathfrak{g}} : \mathfrak{g}_P^{\mathcal{R},p} \otimes \mathfrak{g}_P^{\mathcal{R},p} \longrightarrow \mathbb{k}$ be the 2-cocycle as in Lemma 3.4.2.

Consider the antisymmetric matrix $\mathring{X} := \left(\mathring{\chi}_{ij} = \chi(T_i^+, T_j^+) \right)_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k})$. By Proposition 2.3.2, we have a matrix $P_{(\chi)}$ and a realization $\mathcal{R}_{(\chi)}$ of it, given by

$$P_{(\chi)} := P + \mathring{X} = \left(p_{ij}^{(\chi)} := p_{ij} + \mathring{\chi}_{ij} \right)_{i,j \in I}, \quad \Pi_{(\chi)} := \left\{ \alpha_i^{(\chi)} := \alpha_i \pm \chi(-, T_i^{\pm}) \right\}_{i \in I}$$

Note that $P_{(\chi)}$ is also a multiparameter matrix of Cartan type with Cartan matrix A , for we are adding just an antisymmetric part.

Theorem 3.4.3. *Keep notation as above.*

(a) *There exists a Lie superbialgebra isomorphism $\mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)},p} \simeq (\mathfrak{g}_P^{\mathcal{R},p})_{\chi_{\mathfrak{g}}}$.*

(b) *Let P and P' be two matrices of Cartan type with the same associated Cartan matrix A . Then the following holds:*

(b.1) *Let $\mathcal{R}$ be a split realization of P and let $\mathfrak{g}_P^{\mathcal{R},p}$ be the associated MpLSbA. Then there is $\chi \in \text{Alt}_{\mathbb{k}}^S(\mathfrak{h})$ such that $P' = P_{(\chi)}$, the corresponding $\mathcal{R}_{(\chi)}$ is a split realization of P' , and for the 2-cocycle $\chi_{\mathfrak{g}}$ as in Definition 3.4.1 we have*

$$\mathfrak{g}_{P'}^{\mathcal{R}_{(\chi)},p} \cong (\mathfrak{g}_P^{\mathcal{R}})_{\chi_{\mathfrak{g}}}$$

(b.2) *Let $\mathcal{R}$ and $\mathcal{R}'$ be split minimal realizations of P and P' respectively, and let $\mathfrak{g}_P^{\mathcal{R},p}$ and $\mathfrak{g}_{P'}^{\mathcal{R}',p}$ be the associated MpLSbA's. Then there exists $\chi \in \text{Alt}_{\mathbb{k}}^S(\mathfrak{h})$ such that for the 2-cocycle $\chi_{\mathfrak{g}}$ as in Definition 3.4.1 we have*

$$\mathfrak{g}_{P'}^{\mathcal{R}',p} \cong (\mathfrak{g}_P^{\mathcal{R},p})_{\chi_{\mathfrak{g}}}$$

(b.3) *Every split minimal MpLSbA is isomorphic to some toral 2-cocycle deformation of Yamane's Lie superbialgebra.*

Proof. (a) Since the Lie coalgebra structure in $(\mathfrak{g}_P^{\mathcal{R},p})_{\chi_{\mathfrak{g}}}$ is the same as in $\mathfrak{g}_P^{\mathcal{R},p}$, and the latter coincides with the one in $\mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)},p}$, we have a Lie coalgebra isomorphism among them. Let us show that it is a Lie algebra isomorphism as well. The Lie bracket in $(\mathfrak{g}_P^{\mathcal{R},p})_{\chi_{\mathfrak{g}}}$ is a deformation of that of $\mathfrak{g}_P^{\mathcal{R},p}$ by the formula (3.6):

$$[x, y]_{\chi_{\mathfrak{g}}} := [x, y] - (\partial_* \chi_{\mathfrak{g}})(x \otimes y) = [x, y] - x_{[1]} \chi_{\mathfrak{g}}(x_{[2]}, y) - (-1)^{|x||y_{[1]}|} y_{[1]} \chi_{\mathfrak{g}}(x, y_{[2]})$$

Let us check that the modified defining relations in $(\mathfrak{g}_P^{\mathcal{R},p})_{\chi_{\mathfrak{g}}}$ coincide with the ones in $\mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)},p}$. First note that by the formula above, $[x, y]_{\chi_{\mathfrak{g}}} = [x, y]$ whenever x, y are elements in the radical of $\chi_{\mathfrak{g}}$; in particular for all elements E_i, F_i for all $i \in I$. Hence, we need to check only those relations involving elements in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{P_{(\chi)}}^{\mathcal{R}_{(\chi)},p}$. As the Lie cobracket on this subalgebra is trivial, we get that

$$[T', T'']_{\chi_{\mathfrak{g}}} = [T', T''] - T'_{[1]} \chi_{\mathfrak{g}}(T'_{[2]}, T'') + T''_{[1]} \chi_{\mathfrak{g}}(T'_{[2]}, T') = [T', T''] = 0$$

for all $T', T'' \in \mathfrak{h}$. Take now $i \in I$: since $\delta(T) = 0$, $\delta(E_i) = 2T_i^+ \wedge E_i$ and $\delta(F_i) = 2T_i^- \wedge F_i$, we get that

$$\begin{aligned} [T, E_i]_{\chi_{\mathfrak{g}}} &= [T, E_i] - E_i \chi_{\mathfrak{g}}(T_i^+, T) = \alpha_i(T) E_i + \chi(T, T_i^+) E_i = \alpha_i^{(\chi)}(T) E_i \\ [T, F_i]_{\chi_{\mathfrak{g}}} &= [T, F_i] - F_i \chi_{\mathfrak{g}}(T_i^-, T) = -\alpha_i(T) F_i + \chi(T, T_i^-) F_i = \\ &= -\left(\alpha_i(T) F_i - \chi(T, T_i^-) F_i\right) = -\alpha_i^{(\chi)}(T) F_i \end{aligned}$$

(b.1) By (a), it is enough to find an antisymmetric $\mathbb{k}$ -linear $\chi : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathbb{k}$ obeying (3.12) such that $P' = P_{(\chi)}$; this is guaranteed by Proposition 2.3.5. Now (b.2) follows from (b.1), and the uniqueness of split realizations — cf. Proposition 2.1.5(b), and (b.3) follows from (b.2), taking as $\mathfrak{g}_{P'}^{\mathcal{R}',p}$ the given split minimal MpLSbA and for $\mathfrak{g}_P^{\mathcal{R},p}$ Yamane's MpLSbA which by definition is straight and split minimal. $\square$

Remark 3.4.4. The theorem above tell us that the class of all MpLSbA's of any fixed Cartan type and of fixed rank is stable by toral 2-cocycle deformations, and inside it, the subclass of all such MpLSbA's associated with *split*, resp. *minimal*, realizations is stable as well. Also, for P and P' two matrices of Cartan type with the same associated Cartan matrix A , one can construct by a toral 2-cocycle deformation a split MpLSbA over one of them from any split MpLSbA over the other. Besides, any split minimal MpLSbA over P' is isomorphic to a toral 2-cocycle deformation of any split minimal MpLSbA over P .

3.4.5. Generalizations. The entire analysis and constructions devised in this section can be applied *verbatim* to the setup of *affine* Lie superalgebras, starting from their presentation as provided in [Ya2] and [Ya3]. One has to take care of even more additional relations and their transformations, but this is indeed a matter of bookkeeping, nothing more. What is more relevant, though, is that the key ideas underlying the whole recipe can indeed be applied to a much larger framework (and specific examples): we expand on this in [GG5].

4. FORMAL MULTIPARAMETER QUE SUPERALGEBRAS (=FOMPQUESA's)

This section is devoted to introduce formal multiparameter quantized universal enveloping superalgebras (in short, FoMpQUESA's) and to study their deformations.

4.1. The Hopf (super)algebra setup and QUESA's.

We will deal in the following with Hopf superalgebras and Hopf algebras alike, for which we will use standard notation. Most of the cases, the ground ring will be the ring $\mathbb{k}[[\hbar]]$ of formal power series in the indeterminate $\hbar$ with coefficients in the field $\mathbb{k}$ (which is always assumed to have characteristic zero). For this ring, we will consider $\hbar$ -adically complete $\mathbb{k}[[\hbar]]$ -modules: these form a tensor category, whose tensor product is nothing but the $\hbar$ -adic completion of the standard (algebraic) tensor product. When we deal with “super” objects (i.e., they are $\mathbb{Z}_2$ -graded), the tensor product is graded as well, so that the relevant category is indeed a (non-trivially) braided symmetric category. In particular, in this category a “Hopf (super)algebra” will be a Hopf algebra in the generalized, categorical sense (i.e., with respect to the tensor category) — in particular, it will be topological and $\hbar$ -adically complete: for further details, see for instance [CP], [KS], and [Ge] for the super case.

4.1.1. Topological issues and topological Hopf (super)algebras. The ring $\mathbb{k}[[\hbar]]$ mentioned above carries a natural topology, called the *$\hbar$ -adic topology*, coming from the so-called *$\hbar$ -adic norm* with respect to which it is complete, namely

$$\|a_n \hbar^n + a_{n+1} \hbar^{n+1} + \dots\| := C^{-n} \quad (a_n \neq 0)$$

where $C > 1$ is any fixed constant in $\mathbb{R}$. In this sense, we shall consider *topological* $\mathbb{k}[[\hbar]]$ -modules and the *completed* tensor products among them, which we denote by $\widehat{\otimes}_{\mathbb{k}[[\hbar]]}$ or simply by $\widehat{\otimes}$. For any $\mathbb{k}$ -vector space V , set

$$V[[\hbar]] := \left\{ \sum_{n \geq 0} v_n \hbar^n \mid v_n \in V, \forall n \geq 0 \right\}$$

then $V[[\hbar]]$ is a complete $\mathbb{k}[[\hbar]]$ -module. We call a topological $\mathbb{k}[[\hbar]]$ -module *topologically free* if it is isomorphic to $V[[\hbar]]$ for some $\mathbb{k}$ -vector space V . For two topologically free modules $V[[\hbar]]$ and $W[[\hbar]]$ one has that $V[[\hbar]] \widehat{\otimes} W[[\hbar]] \cong (V \otimes W)[[\hbar]]$.

All completed tensor products between $\mathbb{k}[[\hbar]]$ -modules will be denoted simply by $\otimes$, unless we intend to stress the topological aspect. In particular, we make no distinction on the notation between Hopf superalgebras and topological Hopf superalgebras; we assume it is well-understood from the context.

4.1.2. Quantized Universal Enveloping Superalgebras. Recall that a *deformation superalgebra* is a topological, unital, associative $\mathbb{k}[[\hbar]]$ -superalgebra A which is topologically free as a $\mathbb{k}[[\hbar]]$ -module. Conversely, a deformation of a (unital, associative) $\mathbb{k}$ -superalgebra A_0 is by definition a deformation algebra A such that $A_0 \cong A/\hbar A$. The same criteria apply to the notion of “deformation Hopf superalgebra”, just replacing “topological, unital, associative $\mathbb{k}[[\hbar]]$ -superalgebra” with “topological Hopf $\mathbb{k}[[\hbar]]$ -superalgebra”. Following Geer (cf. [Ge], §2.3, and references therein), we say that a deformation Hopf superalgebra H is a *quantized universal enveloping superalgebra* (or *QUESA* in short) if $H/\hbar H \cong U(\mathfrak{g})$ for some Lie superalgebra $\mathfrak{g}$. In particular, the Lie superbracket in $\mathfrak{g}$ comes from the multiplication in $U(\mathfrak{g}) \cong H/\hbar H$. Moreover, this $\mathfrak{g}$ inherits a Lie supercoalgebra structure from the QUESA, making it into a Lie *superbialgebra*, thanks to the following result:

Theorem 4.1.3. (cf. [Ge, §2.3]) *Let H be a quantized universal enveloping superalgebra with $H/\hbar H \cong U(\mathfrak{g})$. Then the Lie superalgebra $\mathfrak{g}$ is naturally equipped with a Lie superbialgebra structure, whose Lie cobracket is defined by*

$$\delta(x) := \frac{\Delta(x) - \Delta^{\text{op}}(x)}{\hbar} \pmod{\hbar} \quad (4.1)$$

where $x \in H$ is any lifting of $x \in \mathfrak{g} \subseteq U(\mathfrak{g}) \cong H/\hbar H$. $\square$

Definition 4.1.4. [CP, Ge] The semiclassical limit of a quantized universal enveloping superalgebra H is the Lie superbialgebra $(\mathfrak{g}, [-, -], \delta)$ where $\mathfrak{g}$ is the Lie superalgebra such that $H/\hbar H \cong U(\mathfrak{g})$ and δ is defined as above. Conversely, we say that H is a quantization of the Lie superbialgebra $(\mathfrak{g}, [-, -], \delta)$. $\diamond$

4.1.5. Hopf superalgebra deformations. There exist two standard methods to deform Hopf algebras in braided (symmetric) categories, thus in particular Hopf superalgebras (and even quasi-Hopf superalgebras, more in general), leading to so-called “twist deformations” and to “2-cocycle deformations”: hereafter we recall both procedures — which are naturally dual to each other, in a very precise sense — adapting them to the setup of *topological* Hopf superalgebras, then later on we apply them to formal quantum supergroups. We will be somewhat sketchy, since these objects for a Hopf algebra in any symmetric tensor category are defined in the same way as in the category of vector spaces; further details can be found in literature.

Twist deformations: Let H be a Hopf superalgebra (possibly in topological sense). We call “*twist element*”, or just a “*twist*”, of H any even (hence homogeneous), invertible element $\mathcal{F} \in H \otimes H$ such that

$$\mathcal{F}_{1,2}(\Delta \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{2,3}(\text{id} \otimes \Delta)(\mathcal{F}) \quad , \quad (\epsilon \otimes \text{id})(\mathcal{F}) = 1 = (\text{id} \otimes \epsilon)(\mathcal{F})$$

Then H bears a second Hopf superalgebra structure, denoted $H^{\mathcal{F}}$ and called *twist deformation* of the old one, with the old product, unit and counit, but with new

“twisted” coproduct $\Delta^{\mathcal{F}}$ and antipode $\mathcal{S}^{\mathcal{F}}$ given by

$$\Delta^{\mathcal{F}}(x) := \mathcal{F} \Delta(x) \mathcal{F}^{-1} \quad , \quad \mathcal{S}^{\mathcal{F}}(x) := v \mathcal{S}(x) v^{-1} \quad \forall x \in H$$

where $v := \sum_{\mathcal{F}} \mathcal{S}(f'_1) f'_2$ — with $\sum_{\mathcal{F}} f'_1 \otimes f'_2 = \mathcal{F}^{-1}$ — is invertible in H . When H is in fact a topological Hopf superalgebra — meaning that, in particular, its coproduct Δ takes values into $H \otimes H$ where now “ $\otimes$ ” stands for a suitable topological tensor product — then the same notions still make sense, and the related results apply again, up to properly reading them. Moreover, if H is only a superbialgebra (possibly topological), then the same recipe makes $H^{\mathcal{F}}$ into a superbialgebra again.

Cocycle deformations: Let $(H, m, 1, \Delta, \epsilon)$ be a superbialgebra over a ring $\mathbb{k}$. We call “(normalized) Hopf 2-cocycle” (or just a “2-cocycle” if no confusion arises) any convolution invertible, parity-preserving map σ in $\text{Hom}_{\mathbb{k}}(H \otimes H, \mathbb{k})$ such that

$$(-1)^{|b_{(2)}||c_{(1)}|} \sigma(b_{(1)}, c_{(1)}) \sigma(a, b_{(2)} c_{(2)}) = (-1)^{|a_{(2)}||b_{(1)}|} \sigma(a_{(1)}, b_{(1)}) \sigma(a_{(2)} b_{(2)}, c)$$

and $\sigma(a, 1) = \epsilon(a) = \sigma(1, a)$ for all $a, b, c \in H$.

Using a 2-cocycle σ it is possible to define a new superalgebra structure on H by deforming the multiplication. Indeed, define $m_{\sigma} = \sigma * m * \sigma^{-1} : H \otimes H \rightarrow H$ by $m_{\sigma}(a, b) = a \cdot_{\sigma} b = (-1)^{|a_{(2)}||b_{(1)}| + |a_{(3)}||b_{(1)}| + |a_{(3)}||b_{(2)}|} \sigma(a_{(1)}, b_{(1)}) a_{(2)} b_{(2)} \sigma^{-1}(a_{(3)}, b_{(3)})$ for all $a, b \in H$. If in addition H is a Hopf superalgebra with antipode $\mathcal{S}$, then define also the “deformed antipode” $\mathcal{S}_{\sigma} : H \rightarrow H$ by

$$\mathcal{S}_{\sigma}(a) = \sigma(a_{(1)}, \mathcal{S}(a_{(2)})) \mathcal{S}(a_{(3)}) \sigma^{-1}(\mathcal{S}(a_{(4)}), a_{(5)}) \quad \forall a \in H$$

It is known that $(H, m_{\sigma}, 1, \Delta, \epsilon)$ is in turn a superbialgebra, and $(H, m_{\sigma}, 1, \Delta, \epsilon, \mathcal{S}_{\sigma})$ is a Hopf superalgebra: we shall call such a new structure on H a *cocycle deformation* of the old one, and we shall graphically denote it by H_{σ} .

N.B.: in §5.2 later on, we will actually apply a suitable generalization of this procedure, namely the process of “deformation by polar-2-cocycle”: this is introduced in [GG4] for Hopf algebras that are quantum groups, we shall presently extend it (in a specific example) to the context of quantum supergroups.

4.1.6. Some q -numbers. Let $\mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials with integral coefficients in the indeterminate q . For every $n \in \mathbb{N}_+$ we define

$$\begin{aligned} (0)_q &:= 1 \quad , \quad (n)_q := \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1} = \sum_{s=0}^{n-1} q^s \quad (\in \mathbb{Z}[q]) \\ (n)_q! &:= (0)_q(1)_q \cdots (n)_q := \prod_{s=0}^n (s)_q \quad , \quad \binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!} \quad (\in \mathbb{Z}[q]) \\ [0]_q &:= 1 \quad , \quad [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-(n-1)} + \cdots + q^{n-1} = \sum_{s=0}^{n-1} q^{2s-n+1} \quad (\in \mathbb{Z}[q, q^{-1}]) \\ [n]_q! &:= [0]_q[1]_q \cdots [n]_q = \prod_{s=0}^n [s]_q \quad , \quad \left[\begin{matrix} n \\ k \end{matrix} \right]_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (\in \mathbb{Z}[q, q^{-1}]) \end{aligned}$$

In particular, we have the identities

$$(n)_{q^2} = q^{n-1} [n]_q \quad , \quad (n)_{q^2}! = q^{\frac{n(n-1)}{2}} [n]_q! \quad , \quad \binom{n}{k}_{q^2} = q^{k(n-k)} \left[\begin{matrix} n \\ k \end{matrix} \right]_q .$$

Moreover, for any field $\mathbb{F}$ we can think of Laurent polynomials as functions on $\mathbb{F}^\times$, hence for any $q \in \mathbb{F}^\times$ we shall read every symbol above as a suitable element in $\mathbb{F}$.

4.2. Formal multiparameter QUESA's (=FoMpQUESA's).

We introduce now the notion of *formal multiparameter quantum universal enveloping superalgebra* — or just “FoMpQUESA”, in short — in particular proving, somewhat indirectly, that it carries a structure of Hopf superalgebra.

Hereafter, $\mathbb{k}$ is again a field of characteristic zero, $\mathbb{k}[[\hbar]]$ the ring of formal power series in $\hbar$ with coefficients in $\mathbb{k}$. In any topological, $\mathbb{k}[[\hbar]]$ -adically complete $\mathbb{k}[[\hbar]]$ -algebra $\mathcal{A}$, if $X \in \mathcal{A}$ we set $e^{\hbar X} := \exp(\hbar X) = \sum_{n=0}^{+\infty} \hbar^n X^n / n! \in \mathcal{A}$.

Definition 4.2.1. For later use, we fix some more notation:

(a) Let $A := (a_{i,j})_{i,j \in I}$ be a fixed Cartan matrix as in Definition 2.1.2, with $I := \{1, \dots, n\}$, and let $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ be a square matrix of Cartan type associated with A (cf. Definition 2.1.3(d)), so $P + P^T = 2DA$, i.e. $p_{ij} + p_{ji} = 2d_i a_{ij}$ for all $i, j \in I$, which implies $p_{ii} = 2d_i$ for all $i \in I$ (notation of §2.1).

We define the following elements in $\mathbb{k}[[\hbar]]$, for all $i, j \in I$:

$$\begin{aligned} q &:= e^{\hbar} = \exp(\hbar) , \quad q_i := e^{\hbar d_i} = q^{d_i} , \quad q_{ij} := e^{\hbar p_{ij}} = q^{p_{ij}} , \quad q_{ij}^{1/2} := e^{\hbar p_{ij}/2} \\ k_{ij} &:= e^{\hbar p_{i,j}/2} e^{-\hbar p_{j,i}/2} = q^{+p_{ij}/2} q^{-p_{ji}/2} = q_{ij}^{1/2} q_{ji}^{-1/2} \end{aligned}$$

In particular we have $q_{ii}^{1/2} = e^{\hbar d_i} = q_i$ and $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ for all $i, j \in I$.

For any $i \in I$, we set $\nu_i := q^{\varepsilon_i}$, where the ε_i 's are given as follows:

$$\begin{aligned} (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) &:= (6, -2, -2, -2) \quad \text{if } A = F_4 , \\ (\varepsilon_1, \varepsilon_2, \varepsilon_3) &:= (-2, 2, 6) \quad \text{if } A = G_3 , \\ \varepsilon_i &:= \pm 1 \quad \forall i \quad \text{otherwise, but for the conditions} \\ \varepsilon_n &= +\varepsilon_{n-1} \text{ if } A = D_n(I) , \quad \varepsilon_n = -\varepsilon_{n-1} \text{ if } A = D_n(II) \end{aligned} \tag{4.2}$$

In fact this corresponds to choosing an orthogonal bilinear form — in some vector space containing the roots of the Lie superalgebra built out of A and some parity function — so that $\varepsilon_i = (\epsilon_i, \epsilon_i)$, cf. [Ya2].

(b) Given any superalgebra A over some ring R , we define the “supercommutator” $[X, Y]_c := XY - c(-1)^{p(X)p(Y)} YX$ for any $c \in R$ and for any homogeneous $X, Y \in A$, and then extend by linearity as usual. Moreover, for $c = 1$ we adopt the simpler notation $[X, Y] := [X, Y]_1$. $\diamond$

We can now define our FoMpQUESA's, using notation as in Definition 4.2.1 above.

Definition 4.2.2. Let $(A := (a_{i,j})_{i,j \in I}, p)$ be a Cartan super-datum, as in Definition 2.1.2, and $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ a matrix of Cartan type associated with A . We fix a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P as in Definition 2.1.3.

(a) We define the *formal multiparameter quantum universal super enveloping algebra* — in short *formal MpQUESA*, or simply *FoMpQUESA* — with multiparameter P and realization $\mathcal{R}$ as follows. It is the unital, associative, topological, $\hbar$ -adically complete $\mathbb{k}[[\hbar]]$ -superalgebra $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ generated (in topological sense)

by the $\mathbb{k}[[\hbar]]$ -submodule $\mathfrak{h}$ together with elements E_i, F_i (for all $i \in I$), with the $\mathbb{Z}_2$ -grading given by setting parity on generators

$$|T| := \bar{0} \quad \forall \quad T \in \mathfrak{h} \quad , \quad |E_i| := p(i) =: |F_i| \quad \forall \quad i \in I$$

and with relations (for all $T, T', T'' \in \mathfrak{h}, i, j \in I$)

$$T E_j - E_j T = +\alpha_j(T) E_j \quad , \quad T' T'' = T'' T' \quad , \quad T F_j - F_j T = -\alpha_j(T) F_j \quad (4.3)$$

$$[E_i, F_j] = \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} \quad (4.4)$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} k_{ij}^s E_i^{1-a_{ij}-s} E_j E_i^s = 0 \quad (i \neq j, p(i) = \bar{0}) \quad (4.5)$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} k_{ji}^s F_i^{1-a_{ij}-s} F_j F_i^s = 0 \quad (i \neq j, p(i) = \bar{0}) \quad (4.6)$$

$$[E_i, E_j]_{k_{ij}} = 0 \quad , \quad [F_i, F_j]_{k_{ji}} = 0 \quad \text{if} \quad a_{i,j} = 0 \quad (4.7)$$

$$\left[[[E_i, E_j]_{\nu_j k_{ij}}, E_k]_{\nu_{j+1} k_{ik} k_{jk}}, E_j \right]_{k_{ij} k_{kj}} = 0 \quad \text{in any of the following cases:} \quad (4.8)$$

$$\begin{array}{ccc} i & j & k \\ \times & \text{---} \otimes \text{---} & \times \end{array} \quad \begin{array}{ccc} i & j & k \\ \times & \text{---} \otimes \Rightarrow & \bullet \end{array} \quad \begin{array}{ccc} i & j & k \\ \times & \text{---} \otimes \Rightarrow \Rightarrow & \times \end{array}$$

$$\left[[[E_{n-1}, E_n]_{\nu_n k_{n-1,n}}, E_n]_{k_{n-1,n}}, E_n \right]_{\nu_n^{-1} k_{n-1,n}} = 0 \quad (4.9)$$

$$\text{with} \quad \begin{array}{ccc} n-1 & & n \\ \times & \Rightarrow \Rightarrow & \bullet \end{array} \quad \text{and } A \text{ of type } B_n$$

$$\begin{aligned} & \left[[E_{n-2}, E_{n-1}]_{\nu_{n-1} k_{n-2,n-1}}, E_n \right]_{\nu_n k_{n-2,n} k_{n-1,n}} - \\ & - \left[[E_{n-2}, E_n]_{\nu_{n-1} k_{n-2,n}}, E_{n-1} \right]_{\nu_n k_{n-2,n-1} k_{n,n-1}} = 0 \end{aligned} \quad (4.10)$$

$$\text{with} \quad \begin{array}{ccc} & n \\ & \otimes \\ n-2 & \diagup & \\ \times & & \\ & \parallel & \\ & \otimes & \\ & n-1 \end{array} \quad \text{and } A \text{ of type } D_n(II)$$

$$\begin{aligned} & \left[[[[E_{n-2}, E_{n-1}]_{\nu_{n-1} k_{n-2,n-1}}, E_n]_{\nu_n k_{n-2,n-1} k_{n-1,n}}, \right. \\ & \left. [E_{n-2}, E_{n-1}]_{\nu_{n-1} k_{n-2,n-1}}, E_{n-1} \right]_{\nu_{n-1} k_{n-2,n-1}} = 0 \end{aligned} \quad (4.11)$$

$$\text{for } \begin{array}{c} n-2 \quad n-1 \quad n \\ \otimes \text{---} \otimes \text{---} \leftarrow \circ \end{array} \quad \text{and } A \text{ of type } C_n$$

$$\begin{aligned} & \left[\left[\left[\left[[E_{n-3}, E_{n-2}]_{\nu_{n-2} k_{n-3, n-2}}, E_{n-1} \right]_{\nu_{n-1} k_{n-3, n-1} k_{n-2, n-1}}, \right. \right. \\ & \left. \left. E_n \right]_{2 \nu_n k_{n-3, n} k_{n-2, n} k_{n-1, n}}, E_{n-1} \right]_{\nu_n k_{n-3, n-1} k_{n-2, n-1} k_{n, n-1}}, \\ & \left. E_{n-2} \right]_{\nu_{n-1} k_{n-3, n-2} k_{n-1, n-2}^2 k_{n, n-2}}, E_{n-1} \right]_{k_{n-3, n-1} k_{n-2, n-1}^2 k_{n, n-1}} = 0 \end{aligned} \quad (4.12)$$

$$\text{for } \begin{array}{c} n-3 \quad n-2 \quad n-1 \quad n \\ \times \text{---} \otimes \text{---} \otimes \text{---} \leftarrow \circ \end{array} \quad \text{and } A \text{ of type } C_n$$

and also the like of relations (4.8), (4.9), (4.10), (4.11), (4.12) with the E_i 's replaced by F_i 's, the q_{ij} 's by q_{ji} 's and the k_{ij} 's by k_{ji} 's.

N.B.: In (4.12) we have removed a subscript “2” which appears in Yamane's original formula in [Ya1], seemingly as a misprint. Also, the like of formulas (4.5) and (4.6) in [loc. cit.] have “ $+|a_{ij}|$ ” instead of “ $-a_{ij}$ ”, yet the two do coincide because of the assumption $p(i) = \bar{0}$.

(b) We say that $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ is *straight*, or *small*, or *minimal*, or *split*, if such is $\mathcal{R}$; also, we define the *rank* of $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ as $\text{rk}(U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})) := \text{rk}(\mathcal{R}) = \text{rk}_{\mathbb{k}[[\hbar]]}(\mathfrak{h})$.

(c) We define the *Cartan subalgebra* $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{h})$, or just $U_{\hbar}(\mathfrak{h})$, of a FoMpQUESA $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ as being the unital, $\hbar$ -adically complete topological $\mathbb{k}[[\hbar]]$ -subsuperalgebra of $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ generated by the $\mathbb{k}[[\hbar]]$ -submodule $\mathfrak{h}$.

(d) We define the *positive, resp. negative, Borel subsuperalgebra* $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{b}_+)$, resp. $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{b}_-)$, of $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ to be the unital, $\hbar$ -adically complete topological $\mathbb{k}[[\hbar]]$ -subsuperalgebra of $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ generated by $\mathfrak{h}$ and the E_i 's, resp. by $\mathfrak{h}$ and the F_i 's ($i \in I$).

(e) We define the *positive, resp. negative, nilpotent subsuperalgebra* $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{n}_+)$, resp. $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{n}_-)$, of a FoMpQUESA $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ to be the unital, $\hbar$ -adically complete topological $\mathbb{k}[[\hbar]]$ -subsuperalgebra of $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ generated by all the E_i 's, resp. all the F_i 's ($i \in I$). $\diamond$

Remark 4.2.3. In Definition 4.2.2(a), all relations (4.7) through (4.12), as well as their counterparts with the F_ℓ 's replacing the E_ℓ 's, are all given in terms of “iterated q -supercommutators” (either identifying such a q -supercommutator to zero, or identifying two of them). Similarly, it is important to stress that also *the quantum Serre's relations, namely (4.5) and (4.6), can be expressed in that way.* We prove this for (4.5), the case of (4.6) being entirely similar, up to switching every q_{ij} , resp. every k_{ij} , to q_{ji} , resp. to k_{ji} .

Indeed, let $E_{i,j}$ denote the left-hand side of (4.5). Then with notation as in Definition 4.2.1(b), straightforward computations yield

$$a_{i,j} = 0 \implies E_{i,j} = [E_i, E_j]_{q_{ij}} \quad (4.13)$$

$$a_{i,j} = -1 \implies E_{i,j} = [E_i, [E_i, E_j]_{q_{ij}}]_{q_{ii} q_{ij}} \quad (4.14)$$

$$a_{i,j} = -2 \implies E_{i,j} = \left[E_i, [E_i, [E_i, E_j]_{q_{ij}}]_{q_{ii} q_{ij}} \right]_{q_{ii}^2 q_{ij}} \quad (4.15)$$

$$a_{i,j} = -3 \implies E_{i,j} = \left[E_i, \left[E_i, [E_i, [E_i, E_j]_{q_{ij}}]_{q_{ii} q_{ij}} \right]_{q_{ii}^2 q_{ij}} \right]_{q_{ii}^3 q_{ij}} \quad (4.16)$$

In more general terms, the iterated q -supercommutators arise from a *braided* adjoint action. This braiding is the canonical one given in the category of super Yetter-Drinfeld modules over the Cartan subalgebra.

The following two results underscore that the dependence of FoMpQUSEA's on realizations (which includes that on the multiparameter matrix) is functorial:

Proposition 4.2.4. *Let $P \in M_n(\mathbb{k}[[\hbar]])$, and $p : I \rightarrow \mathbb{Z}_2$ a parity function. If both $\mathcal{R}'$ and $\mathcal{R}''$ are realizations of P and $\underline{\phi} : \mathcal{R}' \rightarrow \mathcal{R}''$ is a morphism between them, then there exists a unique morphism $U_{\underline{\phi}} : U_{P,\hbar}^{\mathcal{R}',p}(\mathfrak{g}) \rightarrow U_{P,\hbar}^{\mathcal{R}'',p}(\mathfrak{g})$ of unital topological $\mathbb{k}[[\hbar]]$ -algebras that extends the morphism $\phi : \mathfrak{h}' \rightarrow \mathfrak{h}''$ of $\mathbb{k}[[\hbar]]$ -modules given by $\underline{\phi}$; moreover, $U_{\text{id}_{\mathcal{R}}} = \text{id}_{U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})}$ and $U_{\phi' \circ \phi} = U_{\phi'} \circ U_{\phi}$ (whenever $\phi' \circ \phi$ is defined). Therefore, the construction $\mathcal{R} \mapsto U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ — for any fixed P and p — is functorial in $\mathcal{R}$.*

Moreover, if $\underline{\phi}$ is an epimorphism, resp. a monomorphism, then $U_{\underline{\phi}}$ is an epimorphism, resp. a monomorphism, as well.

Finally, for any morphism $\underline{\phi} : \mathcal{R}' \rightarrow \mathcal{R}''$, the kernel $\text{Ker}(U_{\underline{\phi}})$ of $U_{\underline{\phi}}$ is the two-sided ideal in $U_{P,\hbar}^{\mathcal{R}',p}(\mathfrak{g})$ generated by $\text{Ker}(\phi)$, and the latter is central in $U_{P,\hbar}^{\mathcal{R}',p}(\mathfrak{g})$.

Proof. This is just Proposition 4.2.3 in [GG3], but for the addition of the parity function, which nevertheless is harmless. $\square$

Corollary 4.2.5. *With notation as above, if $\mathcal{R}' \cong \mathcal{R}''$ then $U_{P,\hbar}^{\mathcal{R}',p}(\mathfrak{g}) \cong U_{P,\hbar}^{\mathcal{R}'',p}(\mathfrak{g})$.*

In particular, all FoMpQUSEA's built upon split realizations, respectively small realizations, of the same matrix P and sharing the same rank of $\mathfrak{h}$ and the same parity function p , are isomorphic to each other — hence they are independent (up to isomorphisms) of the specific realization, but only depend on P , on the rank of $\mathfrak{h}$, and on the parity function.

Proof. Again, this is essentially Corollary 4.2.4 in [GG3]. $\square$

Examples 4.2.6. (Yamane's QUESA's as examples of FoMpQUESA's)

Fix a Cartan super-datum (A, p) as in Definition 2.1.2, with D the diagonal matrix such that DA is symmetric, and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of $P := DA$.

(a) Assume that $\mathcal{R}$ is straight and such that $T_i^+ = T_i^-$ for all $i \in I$. Then $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ is nothing but the quantized universal enveloping superalgebra (=QUESA) introduced by Yamane in [Ya1], and denoted by $U_{\hbar}(\mathcal{G}) = U_{\hbar}(\Pi, p)$. See also [Ge] for a further study of these QUESA's.

(b) Assume that $\mathcal{R}$ is straight and split. Then $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ is nothing but the “quantum double version” of Yamane’s $U_h(\mathcal{G}) = U_h(\Pi, p)$, again introduced in [Ya1] where it is denoted by $\mathcal{D}'$.

(c) Assume that $\mathcal{R}$ is just straight. Then $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ is a suitable quotient (by an ideal generated by a submodule of the “Cartan subalgebra”) of the “quantum double version” of Yamane’s QUESA, i.e. the $\mathcal{D}'$ mentioned in (b) above. We refer to such an example as a “Yamane’s QUESA”.

4.2.7. Hopf superalgebra structure on FoMpQUESA’s. The examples of FoMpQUESA’s from Yamane’s work can be modified so to yield more examples.

Let us begin with (A, p) , $P_A := DA$ and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P_A , as in Definition 4.2.2; we assume in addition that $\mathcal{R}$ be straight, so that $U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g})$ is just a “Yamane’s QUESA”, as in Example 4.2.6(c) above. From Yamane’s work [Ya1] we know that $U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g})$ bears indeed a *Hopf structure* which is given on (super)algebra generators by the formulas

$$\begin{aligned}\Delta(E_\ell) &= E_\ell \otimes 1 + e^{+\hbar T_\ell^+} \otimes E_\ell \\ \Delta(T) &= T \otimes 1 + 1 \otimes T \\ \Delta(F_\ell) &= F_\ell \otimes e^{-\hbar T_\ell^-} + 1 \otimes F_\ell\end{aligned}\tag{4.17}$$

(for all $T \in \mathfrak{h}$, $\ell \in I$) for the coproduct, and for the counit and antipode by

$$\epsilon(E_\ell) = 0, \quad \epsilon(T) = 0, \quad \epsilon(F_\ell) = 0\tag{4.18}$$

$$\mathcal{S}(E_\ell) = -e^{-\hbar T_\ell^+} E_\ell, \quad \mathcal{S}(T) = -T, \quad \mathcal{S}(F_\ell) = -F_\ell e^{+\hbar T_\ell^-}\tag{4.19}$$

Now we fix in $\mathfrak{h}$ any $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, where $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$. Pick an antisymmetric $(t \times t)$ -matrix $\Phi = (\phi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$, and set

$$\mathfrak{J}_\Phi := \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k \in \mathfrak{h} \otimes \mathfrak{h} \subseteq U_{P_A,h}^{\mathcal{R}}(\mathfrak{h}) \otimes U_{P_A,h}^{\mathcal{R}}(\mathfrak{h})$$

By direct check, we see that the element

$$\mathcal{F}_\Phi := e^{\hbar 2^{-1} \mathfrak{J}_\Phi} = \exp\left(\hbar 2^{-1} \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k\right)\tag{4.20}$$

in $U_{P_A,h}^{\mathcal{R}}(\mathfrak{h}) \widehat{\otimes} U_{P_A,h}^{\mathcal{R}}(\mathfrak{h})$ is actually a *twist* for $U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g})$, in the sense of §4.1.5. Using it, we construct a new (topological) Hopf algebra $(U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$, isomorphic to $U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g})$ as an algebra but with a new, twisted coalgebra structure, as in §4.1.5. A direct calculation yields formulas for the new coproduct on generators, namely

$$\begin{aligned}\Delta^{\mathcal{F}_\Phi}(E_\ell) &= E_\ell \otimes \mathcal{L}_{\Phi,\ell}^{+1} + e^{+\hbar T_\ell^+} \mathcal{K}_{\Phi,\ell}^{+1} \otimes E_\ell & (\forall \ell \in I) \\ \Delta^{\mathcal{F}_\Phi}(T) &= T \otimes 1 + 1 \otimes T & (\forall T \in \mathfrak{h}) \\ \Delta^{\mathcal{F}_\Phi}(F_\ell) &= F_\ell \otimes \mathcal{L}_{\Phi,\ell}^{-1} e^{-\hbar T_\ell^-} + \mathcal{K}_{\Phi,\ell}^{-1} \otimes F_\ell & (\forall \ell \in I)\end{aligned}$$

with $\mathcal{L}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{gk} H_k}$, $\mathcal{K}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k} = \mathcal{L}_{\Phi,\ell}^{-1}$ for $\ell \in I$. Similarly, the “twisted” antipode $\mathcal{S}^{\mathcal{F}_\Phi}$ and the counit $\epsilon^{\mathcal{F}_\Phi} := \epsilon$ are given by

$$\begin{aligned} \mathcal{S}^{\mathcal{F}_\Phi}(E_\ell) &= -e^{-\hbar T_\ell^+} \mathcal{K}_{\Phi,\ell}^{-1} E_\ell \mathcal{L}_{\Phi,\ell}^{-1}, & \epsilon^{\mathcal{F}_\Phi}(E_\ell) &= 0 & (\forall \ell \in I) \\ \mathcal{S}^{\mathcal{F}_\Phi}(T) &= -T, & \epsilon^{\mathcal{F}_\Phi}(T) &= 0 & (\forall T \in \mathfrak{h}) \\ \mathcal{S}^{\mathcal{F}_\Phi}(F_\ell) &= -\mathcal{K}_{\Phi,\ell}^{+1} F_\ell \mathcal{L}_{\Phi,\ell}^{+1} e^{+\hbar T_\ell^-}, & \epsilon^{\mathcal{F}_\Phi}(F_\ell) &= 0 & (\forall \ell \in I) \end{aligned}$$

In short, starting from Yamane’s QUESA $U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ we get, via a twist deformation process, a new Hopf superalgebra $(U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$; moreover, one immediately see that, by construction, this $(U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ is yet another QUESA (in the sense of §4.1.2). Again by construction, this QUESA is of “multiparametric nature”, in that it depends on the entries of Φ as “parameters”. Indeed, by means of a suitable change of generators (hence a change of presentation), the following result shows that $(U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ is in fact a FoMpQUESA in the sense of Definition 4.2.2:

Proposition 4.2.8. *Let (A, p) be a Cartan super-datum, and $\mathcal{R}$ a realization of $P_A := DA$; we assume that $\mathcal{R}$ be straight, so $U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ is just a “Yamane’s QUESA”, as in Example 4.2.6(c). We fix a twist $\mathcal{F}_\Phi := \exp\left(\hbar 2^{-1} \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k\right)$ as above, and we let $(U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ be the corresponding deformed Hopf superalgebra.*

Then there exists an isomorphism of topological superalgebras

$$f_P^\Phi : U_{(P_A)_\Phi, \hbar}^{\mathcal{R}_\Phi, p}(\mathfrak{g}) \xrightarrow{\cong} (U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$$

given by $E_i \mapsto E_i^\Phi := \mathcal{L}_{\Phi,i}^{-1} E_i$, $T \mapsto T$ and $F_i \mapsto F_i^\Phi := F_i \mathcal{K}_{\Phi,i}^{+1}$ with $\mathcal{L}_{\Phi,i} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_i(H_g) \phi_{gk} H_k}$ and $\mathcal{K}_{\Phi,i} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_i(H_g) \phi_{kg} H_k} (= \mathcal{L}_{\Phi,i}^{-1})$ for all $i \in I$, $T \in \mathfrak{h}$, where $\mathcal{R}_\Phi$ and P_Φ are the twist deformations (via Φ) of $\mathcal{R}$ and P , as in Definition 2.2.3.

Proof. It is clear that the E_i^Φ ’s, the T ’s and the F_i^Φ ’s (for $i \in I$ and $T \in \mathfrak{h}$) altogether generate $(U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$, as the latter as an algebra is just $U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$. So we find a complete presentation of this algebra just by rewriting with respect to these new generators the old relations among the old ones, i.e. the E_i ’s, the T ’s and the F_i ’s. Hereafter we present the explicit computations for some of these relations, the other ones can be easily filled in by the interested reader.

First of all, from the relations in (4.3) one gets at once identical relations with the E_j^Φ ’s, resp. the F_j^Φ ’s, replacing the E_j ’s, resp. the F_j ’s.

Second, direct computations give

$$\begin{aligned} [E_i, F_j] - \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} &= [\mathcal{L}_{\Phi,i}^{+1} E_i^\Phi, F_j^\Phi \mathcal{K}_{\Phi,j}^{-1}] - \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} = \\ &= \mathcal{L}_{\Phi,i}^{+1} E_i^\Phi F_j^\Phi \mathcal{K}_{\Phi,j}^{-1} - (-1)^{p(i)p(j)} F_j^\Phi \mathcal{K}_{\Phi,j}^{-1} \mathcal{L}_{\Phi,i}^{+1} E_i^\Phi - \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} = \\ &= \mathcal{L}_{\Phi,i}^{+1} E_i^\Phi F_j^\Phi \mathcal{K}_{\Phi,j}^{-1} - (-1)^{p(i)p(j)} \mathcal{L}_{\Phi,i}^{+1} F_j^\Phi E_i^\Phi \mathcal{K}_{\Phi,j}^{-1} - \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} = \end{aligned}$$

$$\begin{aligned}
&= \mathcal{L}_{\Phi,i}^{+1} (E_i^\Phi F_j^\Phi - (-1)^{p(i)p(j)} F_j^\Phi E_i^\Phi) \mathcal{K}_{\Phi,j}^{-1} - \delta_{i,j} \mathcal{L}_{\Phi,i}^{+1} \frac{e^{+\hbar T_{\Phi,i}^+} - e^{-\hbar T_{\Phi,i}^-}}{q_i^{+1} - q_i^{-1}} \mathcal{K}_{\Phi,j}^{-1} = \\
&= \mathcal{L}_{\Phi,i}^{+1} \left(E_i^\Phi F_j^\Phi - (-1)^{p(i)p(j)} F_j^\Phi E_i^\Phi - \delta_{i,j} \frac{e^{+\hbar T_{\Phi,i}^+} - e^{-\hbar T_{\Phi,i}^-}}{q_i^{+1} - q_i^{-1}} \right) \mathcal{K}_{\Phi,j}^{-1} = \\
&= \mathcal{L}_{\Phi,i}^{+1} \left([E_i^\Phi, F_j^\Phi] - \delta_{i,j} \frac{e^{+\hbar T_{\Phi,i}^+} - e^{-\hbar T_{\Phi,i}^-}}{q_i^{+1} - q_i^{-1}} \right) \mathcal{K}_{\Phi,j}^{-1}
\end{aligned}$$

from which we see that the “old” relation $[E_i, F_j] - \delta_{i,j} \frac{e^{+\hbar T_i^+} - e^{-\hbar T_i^-}}{q_i^{+1} - q_i^{-1}} = 0$ is equivalent to the new one, that is $[E_i^\Phi, F_j^\Phi] - \delta_{i,j} \frac{e^{+\hbar T_{\Phi,i}^+} - e^{-\hbar T_{\Phi,i}^-}}{q_i^{+1} - q_i^{-1}} = 0$.

Third, recall that all relations from (4.5) through (4.12) can be written in terms of iterated q -supercommutators (cf. §4.2.3). Now, for a single such q -supercommutator we have, for all $i, j \in I$ and $\nu \in \mathbb{K}[[\hbar]]$,

$$\begin{aligned}
[E_i, E_j]_\nu &= [\mathcal{L}_{\Phi,i}^{+1} E_i^\Phi, \mathcal{L}_{\Phi,j}^{+1} E_j^\Phi]_\nu = \mathcal{L}_{\Phi,i}^{+1} E_i^\Phi \mathcal{L}_{\Phi,j}^{+1} E_j^\Phi - (-1)^{p(i)p(j)} \nu \mathcal{L}_{\Phi,j}^{+1} E_j^\Phi \mathcal{L}_{\Phi,i}^{+1} E_i^\Phi = \\
&= e^{-2^{-1}\hbar \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} E_i^\Phi E_j^\Phi - \\
&\quad - (-1)^{p(i)p(j)} \nu e^{-2^{-1}\hbar \sum_{g,k=1}^t \alpha_i(H_g) \phi_{gk} \alpha_j(H_k)} \mathcal{L}_{\Phi,j}^{+1} \mathcal{L}_{\Phi,i}^{+1} E_j^\Phi E_i^\Phi = \\
&= e^{-2^{-1}\hbar \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} \times \\
&\quad \times \left(E_i^\Phi E_j^\Phi - (-1)^{p(i)p(j)} \nu e^{+\hbar \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)} E_j^\Phi E_i^\Phi \right) = \\
&= e^{-2^{-1}\hbar \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} [E_i^\Phi, E_j^\Phi]_{\nu e^{+\hbar \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)}}
\end{aligned}$$

i.e., using notation $k_{ij}^\Phi := e^{+\hbar \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)}$, we find

$$[E_i, E_j]_\nu = (k_{ij}^\Phi)^{-1/2} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} [E_i^\Phi, E_j^\Phi]_{\nu k_{ij}^\Phi} \quad (4.21)$$

Similarly, a parallel calculation gives

$$[F_i, F_j]_\nu = [F_i^\Phi, F_j^\Phi]_{\nu k_{ji}^\Phi} \mathcal{K}_{\Phi,i}^{-1} \mathcal{K}_{\Phi,j}^{-1} (k_{ji}^\Phi)^{-1/2} \quad (4.22)$$

More in general, a similar computation yields

$$E_{i_1}^\Phi \cdots E_{i_\ell}^\Phi = \left(\prod_{1 \leq r \leq s \leq \ell} q^{-2^{-1} \sum_{g,k=1}^t \alpha_{i_s}(H_g) \phi_{gk} \alpha_{i_r}(H_k)} \right) \mathcal{L}_{\Phi,i_1}^{+1} \cdots \mathcal{L}_{\Phi,i_\ell}^{+1} E_{i_1} \cdots E_{i_\ell} \quad (4.23)$$

and then from this one can deduce formulas generalizing (4.21) that express any q -supercommutator of products of the E_i^Φ 's in terms of a similar supercommutator of products of the E_i 's. A similar outcome holds with “ F ” replacing “ E ” throughout.

Applying these arguments — in particular, applying the like of (4.21) — one can deduce all relations from (4.5) through (4.12) for the E_j^Φ 's, resp. the F_j^Φ 's, from the parallel relations for the E_j 's, resp. the F_j 's. Hereafter we show that in a few cases.

For instance, the original relation (4.7) in $U_{P_A, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ reads $[E_i, E_j] = 0$ because $k_{i,j} = 1$ when $a_{i,j} = 0$ for the matrix P_A (as it is symmetric). But then this relation is *equivalent*, by (4.21), to $[E_i^\Phi, E_j^\Phi]_{k_{ij}^\Phi} = 0$, q.e.d.

Similarly, consider the case of the quantum Serre relation (4.5) for $a_{i,j} = -2$. Direct computations — taking into account (4.15), (4.21) and (2.2), that in particular

imply $q_{i,j} k_{i,j}^\Phi = e^{+\hbar p_{i,j}^\Phi} =: q_{i,j}^\Phi$ — give us

$$\begin{aligned}
 E_{i,j} &= \left[E_i, [E_i, [E_i, E_j]_{q_{ij}}]_{q_{ii} q_{ij}} \right]_{q_{ii}^2 q_{ij}} = \\
 &= (k_{i,j}^\Phi)^{-1/2} \left[E_i, [E_i, \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} [E_i^\Phi, E_j^\Phi]_{q_{i,j}^\Phi}]_{q_{ii} q_{ij}} \right]_{q_{ii}^2 q_{ij}} = \\
 &= (k_{i,j}^\Phi)^{-1} \left[E_i, \mathcal{L}_{\Phi,i}^{+2} \mathcal{L}_{\Phi,j}^{+1} [E_i^\Phi, [E_i^\Phi, E_j^\Phi]_{q_{i,j}^\Phi}]_{q_{ii}^\Phi q_{ij}^\Phi} \right]_{q_{ii}^2 q_{ij}} = \\
 &= (k_{i,j}^\Phi)^{-3/2} \mathcal{L}_{\Phi,i}^{+3} \mathcal{L}_{\Phi,j}^{+1} \left[E_i^\Phi, [E_i^\Phi, [E_i^\Phi, E_j^\Phi]_{q_{i,j}^\Phi}]_{q_{ii}^\Phi q_{ij}^\Phi} \right]_{(q_{ii}^\Phi)^2 q_{ij}^\Phi} = \\
 &= (k_{i,j}^\Phi)^{-3/2} \mathcal{L}_{\Phi,i}^{+3} \mathcal{L}_{\Phi,j}^{+1} E_{i,j}^\Phi
 \end{aligned}$$

from which we infer that the quantum Serre relation $E_{i,j} = 0$ for the E_ℓ 's is equivalent to the similar quantum Serre relation $E_{i,j}^\Phi = 0$ for the E_ℓ^Φ 's.

The other quantum Serre relations are disposed of in a similar way.

Finally, as to the other relations we consider for instance the case of (4.8). In case $P = P_A$ (which is symmetric, hence the $k_{i,j}$'s are one) relation (4.8) reads just

$$\begin{aligned}
 &\left[[[E_i, E_j]_{\nu_j}, E_k]_{\nu_{j+1}}, E_j \right] = 0. \text{ Now, direct computations give} \\
 &\left[[[E_i, E_j]_{\nu_j}, E_k]_{\nu_{j+1}}, E_j \right] = q_{ij}^{-1} \left[[\mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} [E_i^\Phi, E_j^\Phi]_{\nu_i k_{ij}}, E_k]_{\nu_{j+1}}, E_j \right] = \\
 &= q_{ij}^{-1} q_{ik}^{-1} q_{jk}^{-1} \left[\mathcal{L}_{\Phi,k}^{+1} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} [[E_i^\Phi, E_j^\Phi]_{\nu_j k_{ij}}, E_k]_{\nu_{j+1} k_{ik} k_{jk}}, E_j \right] = \\
 &= q_{ij}^{-2} q_{ik}^{-1} q_{jk}^{-1} q_{kj}^{-1} q_{jj}^{-1} \mathcal{L}_{\Phi,k}^{+1} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+2} \left[[[E_i^\Phi, E_j^\Phi]_{\nu_j k_{ij}}, E_k]_{\nu_{j+1} k_{ik} k_{jk}}, E_j^\Phi \right]_{k_{ij} k_{jj} k_{kj}} = \\
 &= q_{ij}^{-2} q_{ik}^{-1} q_{jk}^{-1} q_{kj}^{-1} q_{jj}^{-1} \mathcal{L}_{\Phi,k}^{+1} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+2} \left[[[E_i^\Phi, E_j^\Phi]_{\nu_j k_{ij}}, E_k]_{\nu_{j+1} k_{ik} k_{jk}}, E_j^\Phi \right]_{k_{ij} k_{kj}}
 \end{aligned}$$

from which we eventually see that relation (4.8) for the E_ℓ 's is equivalent to relation (4.8) for the E_ℓ^Φ 's. Similarly one deals with relations (4.9) through (4.12).

As a different method, one can adopt the following. For any $b, d \in \mathbb{N}$ a special instance of (4.23) reads just $(E_i^\Phi)^b E_j^\Phi (E_i^\Phi)^d = q^{\epsilon(i,j)} \mathcal{L}_{\Phi,i}^{d+b} \mathcal{L}_{\Phi,j}^{+1} E_i^\Phi E_j E_i^d$ with

$$\begin{aligned}
 \epsilon(i, j) &:= -2^{-1} \binom{d+b}{2} \sum_{g,k=1}^t \alpha_i(H_g) \phi_{gk} \alpha_i(H_k) - 2^{-1} d \sum_{g,k=1}^t \alpha_i(H_g) \phi_{gk} \alpha_j(H_k) - \\
 &\quad - 2^{-1} b \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k) = 2^{-1} (d-b) \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)
 \end{aligned}$$

so that $q^{\epsilon(i,j)} = e^{\hbar 2^{-1} (d-b) \sum_{g,k=1}^t \alpha_j(H_g) \phi_{gk} \alpha_i(H_k)} = (k_{i,j}^\Phi)^{(d-b)/2}$. Reversing that identity, we get $E_i^b E_j E_i^d = (k_{i,j}^\Phi)^{(d-b)/2} \mathcal{L}_{\Phi,i}^{-(d+b)} \mathcal{L}_{\Phi,j}^{-1} (E_i^\Phi)^b E_j^\Phi (E_i^\Phi)^d$, which for the pair $(b, d) = (1-a_{ij}-s, s)$ reads

$$E_i^{1-a_{ij}-s} E_j E_i^s = (k_{i,j}^\Phi)^{(a_{ij}-1)/2} (k_{i,j}^\Phi)^s \mathcal{L}_{\Phi,i}^{a_{ij}-1} \mathcal{L}_{\Phi,j}^{-1} (E_i^\Phi)^{1-a_{ij}-s} E_j^\Phi (E_i^\Phi)^s$$

Finally, replacing this last identity in the left-hand side of (4.5) we get

$$\begin{aligned}
 \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} k_{ij}^s E_i^{1-a_{ij}-s} E_j E_i^s &= (k_{i,j}^\Phi)^{(a_{ij}-1)/2} \mathcal{L}_{\Phi,i}^{a_{ij}-1} \mathcal{L}_{\Phi,j}^{-1} \times \\
 &\times \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} q_{ij}^{+s/2} q_{ji}^{-s/2} (k_{i,j}^\Phi)^s (E_i^\Phi)^{1-a_{ij}-s} E_j^\Phi (E_i^\Phi)^s =
 \end{aligned}$$

$$\begin{aligned}
&= (k_{ij}^\Phi)^{(a_{ij}-1)/2} \mathcal{L}_{\Phi,i}^{a_{ij}-1} \mathcal{L}_{\Phi,j}^{-1} \times \\
&\quad \times \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} (q_{ij}^\Phi)^{+s/2} (q_{ji}^\Phi)^{-s/2} (E_i^\Phi)^{1-a_{ij}-s} E_j^\Phi (E_i^\Phi)^s
\end{aligned}$$

— since $k_{ij}^\Phi = (k_{ji}^\Phi)^{-1}$ and $q_{ij} k_{ij}^\Phi = q_{i,j}^\Phi$ — and then we notice that the last factor (i.e., the last line above) is just the left-hand side of the quantum Serre relation (4.5) for the E_ℓ^Φ 's: so we deduce that (4.5) for the E_ℓ 's — inside $U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g})$ — is indeed equivalent to (4.5) for the E_ℓ^Φ 's inside $(U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$, and we are done.

In the same way, replacing the E_k 's with the F_k 's, one disposes of all relations involving the F_k 's (alone) turning into relations involving the F_k^Φ 's (again alone).

To sum up, the previous analysis prove that the E_i^Φ 's, the T 's and the F_i^Φ 's altogether do generate the algebra $(U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ and the original ideal of defining relations of $U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g})$ — described in terms of the original generators — is also generated by the same relations *but written in terms of the new generators and of the deformed multiparameter matrix P_Φ* . In a nutshell, using $\{E_i^\Phi, T, F_i^\Phi \mid i \in I, T \in \mathfrak{h}\}$ as set of generators, we get for $(U_{P_A,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ nothing but the defining presentation of $U_{P_\Phi,h}^{\mathcal{R},p}(\mathfrak{g})$. In a different wording, this means exactly that there exists a well-defined isomorphism f_P^Φ of topological superalgebras as given in the claim. $\square$

A direct consequence of Proposition 4.2.8 is that the FoMpQUESA $U_{P_\Phi,h}^{\mathcal{R},p}(\mathfrak{g})$ inherits, via pull-back through the isomorphism f_P^Φ , a full structure of Hopf superalgebra, more precisely one of QUESA. In fact, by a “reverse-engineering argument”, we see that this result extends to *all* FoMpQUESA's (hence the very use of the terminology “FoMpQUESA” is indeed correct!). This is the content of our next result:

Theorem 4.2.9. *Every FoMpQUESA $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ is indeed a (topological) Hopf superalgebra, whose Hopf structure is given on (super)algebra generators by the formulas*

$$\begin{aligned}
\Delta(E_\ell) &= E_\ell \otimes 1 + e^{+\hbar T_\ell^+} \otimes E_\ell, \\
\Delta(T) &= T \otimes 1 + 1 \otimes T, \\
\Delta(F_\ell) &= F_\ell \otimes e^{-\hbar T_\ell^-} + 1 \otimes F_\ell,
\end{aligned} \tag{4.24}$$

(for all $T \in \mathfrak{h}$, $\ell \in I$) for the coproduct, and for the counit and antipode by

$$\epsilon(E_\ell) = 0, \quad \epsilon(T) = 0, \quad \epsilon(F_\ell) = 0 \tag{4.25}$$

$$\mathcal{S}(E_\ell) = -e^{-\hbar T_\ell^+} E_\ell, \quad \mathcal{S}(T) = -T, \quad \mathcal{S}(F_\ell) = -F_\ell e^{+\hbar T_\ell^-} \tag{4.26}$$

Proof. By assumption, we have a fixed Cartan super-datum (A, p) , as in Definition 2.1.2, such that the symmetric part of P is $P_s := 2^{-1}(P + P^T) = D A =: P_A$, with $P = (p_{i,j})_{i,j \in I}$ and $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of P .

By Lemma 2.1.8, there is a *straight* realization $\tilde{\mathcal{R}} := (\tilde{\mathfrak{h}}, \tilde{\Pi}, \tilde{\Pi}^\vee)$ of the matrix P and an epimorphism of realizations $\pi : \tilde{\mathcal{R}} \twoheadrightarrow \mathcal{R}$. Up to possibly enlarging $\tilde{\mathfrak{h}}$, we can assume that $t := \text{rk}(\tilde{\mathfrak{h}}) \geq 3n - \text{rk}(P_A)$. Then, by Proposition 2.2.6(a) — with $P' = P_A := DA$ — we get that there exists some $\Phi \in \mathfrak{so}_t(\mathbb{K}[[\hbar]])$ such that $P_A = P_\Phi$ while $\mathcal{R}_A := \tilde{\mathcal{R}}_\Phi$ is a straight realization of $P_A = P_\Phi$. Hence, conversely,

we have also $P = (P_A)_{-\Phi}$ and $\tilde{\mathcal{R}} = (\mathcal{R}_A)_{-\Phi}$. Then by Proposition 4.2.8 there exists an isomorphism of topological superalgebras

$$f_{P_A}^{-\Phi} : U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g}) = U_{(P_A)_{-\Phi},h}^{(\mathcal{R}_A)_{-\Phi},p}(\mathfrak{g}) \xrightarrow{\cong} (U_{P_A,h}^{\mathcal{R}_A,p}(\mathfrak{g}))^{\mathcal{F}_{-\Phi}}$$

which is described by explicit formulas as in Proposition 4.2.8 — up to paying attention to switching Φ to $-\Phi$. Eventually, we can pull back onto $U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g})$ — via $f_{P_A}^{-\Phi}$ — the Hopf structure of $(U_{P_A,h}^{\mathcal{R}_A,p}(\mathfrak{g}))^{\mathcal{F}_{-\Phi}}$: then, when tracking the whole construction, the explicit description of this Hopf structure on the generators of $U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g})$ is given exactly by the formulas in the claim.

Finally, Proposition 4.2.4 gives an epimorphism $U_{\pi} : U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g}) \longrightarrow U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ of unital topological $\mathbb{k}[[\hbar]]$ -algebras that extends the epimorphism $\pi : \tilde{\mathfrak{h}} \longrightarrow \mathfrak{h}$ of $\mathbb{k}[[\hbar]]$ -modules given by π ; moreover, $\text{Ker}(\pi)$ is the two-sided ideal in $U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g})$ generated by $\text{Ker}(\pi)$, and the latter is central in $U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g})$. But then, since $\text{Ker}(\pi)$ is made of primitive elements, $\text{Ker}(\pi)$ is automatically a Hopf ideal: therefore, the quotient algebra $U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) = U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g}) / \text{Ker}(\pi)$ inherits via the algebra epimorphism U_{π} the whole Hopf structure of $U_{P,h}^{\tilde{\mathcal{R}},p}(\mathfrak{g})$, hence we are done. $\square$

Corollary 4.2.10. *The Cartan subalgebra $U_{\hbar}(\mathfrak{h})$ and the Borel subsuperalgebras $U_{P,h}^{\mathcal{R},p}(\mathfrak{b}_{\pm})$ are actually (topological) Hopf subsuperalgebras of $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$, their Hopf structure being described again via formulas (4.24), (4.25) and (4.26). $\square$*

The next result is a straightforward “upgrading” Proposition 4.2.4:

Proposition 4.2.11. *Let $\phi : \mathcal{R}' \longrightarrow \mathcal{R}''$ be a morphism between realizations of a same matrix $P \in M_n(\mathbb{k}[[\hbar]])$. Then the morphism of unital topological $\mathbb{k}[[\hbar]]$ -superalgebras $U_{\phi} : U_{P,h}^{\mathcal{R}',p}(\mathfrak{g}) \longrightarrow U_{P,h}^{\mathcal{R}'',p}(\mathfrak{g})$ granted by Proposition 4.2.4 is indeed a morphism of (unital topological) Hopf $\mathbb{k}[[\hbar]]$ -superalgebras.*

If we set $\mathfrak{k} := \text{Ker}(\phi)$, then $U_{\hbar}(\mathfrak{k})$ is a unital, $\hbar$ -adically complete $\mathbb{k}[[\hbar]]$ -subsuperalgebra of $U_{P,h}^{\mathcal{R}',p}(\mathfrak{g})$ which is a central Hopf subsuperalgebra, isomorphic to a quantum Cartan (in the sense of Definition 4.2.2(c)), and $\text{Ker}(U_{\phi}) = U_{P,h}^{\mathcal{R}',p}(\mathfrak{g}) U_{\hbar}(\mathfrak{k})^+$ — where $U_{\hbar}(\mathfrak{k})^+$ is the augmentation ideal of $U_{\hbar}(\mathfrak{k})$. In particular, if U_{ϕ} is an epimorphism, then $U_{P,h}^{\mathcal{R}''}(\mathfrak{g}) \cong U_{P,h}^{\mathcal{R}',p}(\mathfrak{g}) / U_{P,h}^{\mathcal{R}',p}(\mathfrak{g}) U_{\hbar}(\mathfrak{k})^+$. $\square$

We conclude this subsection proving that every FoMpQUESA is indeed a QUESA in the sense of §4.1.2: later on, in §6, we will study in detail their semiclassical limit.

Theorem 4.2.12. *Every FoMpQUESA is a QUESA.*

Proof. Let $U_{\hbar} := U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ be any given FoMpQUESA, and let $U_0 := U_{\hbar} / \hbar U_{\hbar}$ denote its semiclassical limit; also, for any $X \in U_{\hbar}$ we set $\overline{X} := X \pmod{\hbar U_{\hbar}}$ for its coset modulo $\hbar U_{\hbar}$ in U_0 .

First of all, $U_{\hbar}$ is topologically free (as a $\mathbb{k}[[\hbar]]$ -module). Indeed, assume first that $\mathcal{R}$ is straight and $\mathfrak{h}$ is “large enough”, namely $\text{rk}(\mathfrak{h}) \geq 3n - \text{rk}(DA)$: then — see the proof of Theorem 4.2.9 — $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ is isomorphic (as a topological algebra) to

some Yamane’s QUESAs, and the latter is known to be topologically free, by [Ya1], hence the same holds true for $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$. The general case (for any $\mathfrak{h}$ and any $\mathcal{R}$) then is obtained like in the proof of Theorem 4.2.9 by reducing to that of some straight realization with large enough Cartan (sub)algebra (for more details, see the proof of the non-super case in [GG3], Theorem 6.1.4 — the present case is similar).

Second, as U_h is generated by the E_i ’s, F_i ’s and T ’s ($i \in I$, $T \in \mathfrak{h}$), it follows that U_0 is generated by the $\overline{E}_i$ ’s, $\overline{F}_i$ ’s and T ’s. Moreover, the formulas for the coproduct in U_h imply that the latter generators are *primitive* in U_0 . Therefore, the Hopf algebra U_0 is generated by its subspace $P(U_0)$ of primitive elements, hence — by Milnor–Moore Theorem for Hopf superalgebras cf. [Ko, Proposition 3.2] — we conclude that $P(U_0)$ is a Lie superalgebra and U_0 is its universal enveloping superalgebra, hence we are done. $\square$

We finish with a typical “triangular decomposition result” for our FoMpQUESA’s:

Theorem 4.2.13.

Every FoMpQUESA $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ admits triangular decompositions such as

$$U_{P,h}^{\mathcal{R},p}(\mathfrak{n}_+) \otimes U_{P,h}^{\mathcal{R},p}(\mathfrak{h}) \otimes U_{P,h}^{\mathcal{R},p}(\mathfrak{n}_-) \cong U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) \cong U_{P,h}^{\mathcal{R},p}(\mathfrak{n}_-) \otimes U_{P,h}^{\mathcal{R},p}(\mathfrak{h}) \otimes U_{P,h}^{\mathcal{R},p}(\mathfrak{n}_+)$$

and similarly for Borel sub-superalgebras we have

$$U_{P,h}^{\mathcal{R},p}(\mathfrak{n}_{\pm}) \otimes U_{P,h}^{\mathcal{R},p}(\mathfrak{h}) \cong U_{P,h}^{\mathcal{R},p}(\mathfrak{b}_{\pm}) \cong U_{P,h}^{\mathcal{R},p}(\mathfrak{h}) \otimes U_{P,h}^{\mathcal{R},p}(\mathfrak{n}_{\pm})$$

Proof. The claim follows from a standard, fairly “classical” argument, which is detailed in the *Third Proof* in [GG3, §4.4.9]; that one is written there for the non-super setup, but it applies *verbatim* to the present context as well. $\square$

5. DEFORMATIONS OF FOMPQUESA’S

After introducing formal MpQUEAs in the previous section, now we go and study their deformations, either by twist or by polar–2–cocycle (both of “toral type”, say).

5.1. Deformations of FoMpQUESAs by toral twists.

We discuss now suitable twist deformations (of “toral type”) of FoMpQUESAs, proving that they are again FoMpQUESAs. In fact, this is but a generalization of the argument that we already used in §4.2 to *define* our FoMpQUESAs, as they are indeed introduced as deformations by toral twists of Yamane’s QUESAs.

5.1.1. Toral twist deformations of $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$. We fix a Cartan super-datum (A, p) as in Definition 2.1.2, a multiparameter matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ associated with A , and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it, as in §2. In particular, we have $d_i := p_{ii}/2$ ($i \in I$), and $\mathfrak{h}$ is a free $\mathbb{k}[[\hbar]]$ –module of finite rank $t := \text{rk}(\mathfrak{h})$. We fix in $\mathfrak{h}$ any $\mathbb{k}[[\hbar]]$ –basis $\{H_g\}_{g \in \mathcal{G}}$, where $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Now let $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ be the FoMpQUESA given in Definition §4.2.2. We pick $\Phi = (\phi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$, and set

$$\mathfrak{J}_\Phi := \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k \in \mathfrak{h} \otimes \mathfrak{h} \subseteq U_{P,h}^{\mathcal{R}}(\mathfrak{h}) \otimes U_{P,h}^{\mathcal{R}}(\mathfrak{h}).$$

By direct check, we see that the element

$$\mathcal{F}_\Phi := e^{\hbar 2^{-1} \mathfrak{J}_\Phi} = \exp \left(\hbar 2^{-1} \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k \right) \quad (5.1)$$

in $U_{P,h}^{\mathcal{R},p}(\mathfrak{h}) \hat{\otimes} U_{P,h}^{\mathcal{R},p}(\mathfrak{h})$ is actually a *twist* for $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ in the sense of §4.1.5. Using it, we construct a new (topological) Hopf algebra $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$, isomorphic to $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ as an algebra but with a new, twisted coalgebra structure, as in §4.1.5.

Our key result now is that this new Hopf algebra $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ is yet another FoMpQUESA, and even more we know exactly which one it is:

Theorem 5.1.2. *There exists an isomomorphism of topological Hopf algebras*

$$f_P^\Phi : U_{P_\Phi,h}^{\mathcal{R},p}(\mathfrak{g}) \xrightarrow{\cong} (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$$

given by $E_\ell \mapsto E_\ell^\Phi := \mathcal{L}_{\Phi,\ell}^{-1} E_\ell$, $T \mapsto T$, $F_\ell \mapsto F_\ell^\Phi := F_\ell \mathcal{K}_{\Phi,\ell}^{+1}$ ($\forall \ell \in I$, $T \in \mathfrak{h}$), with $\mathcal{L}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{gk} H_k}$, $\mathcal{K}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k}$ ($= \mathcal{L}_{\Phi,\ell}^{-1}$).

In particular, the class of all FoMpQUESAs of any fixed Cartan type and of fixed rank is stable by toral twist deformations. Moreover, inside it the subclass of all such FoMpQUESAs associated with straight, resp. small, realizations is stable as well.

Similar, parallel statements hold true for the Borel FoMpQUEAs, i.e. there exist isomorphisms $f_{P,\pm}^\Phi : U_{P_\Phi,h}^{\mathcal{R},p}(\mathfrak{b}_\pm) \xrightarrow{\cong} (U_{P,h}^{\mathcal{R},p}(\mathfrak{b}_\pm))^{\mathcal{F}_\Phi}$ given by formulas as above.

Proof. The proof follows closely in the footprints of Proposition 4.2.8, so we can be somewhat quicker now. First of all, $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ coincides with $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ as an algebra, so we can take the generators of the latter as generators of the former too.

Direct calculation yields explicit formulas for the new coproduct $\Delta^{\mathcal{F}_\Phi}$, namely

$$\begin{aligned} \Delta^{\mathcal{F}_\Phi}(E_\ell) &= E_\ell \otimes \mathcal{L}_{\Phi,\ell}^{+1} + e^{+\hbar T_\ell^+} \mathcal{K}_{\Phi,\ell}^{+1} \otimes E_\ell & (\forall \ell \in I) \\ \Delta^{\mathcal{F}_\Phi}(T) &= T \otimes 1 + 1 \otimes T & (\forall T \in \mathfrak{h}) \\ \Delta^{\mathcal{F}_\Phi}(F_\ell) &= F_\ell \otimes \mathcal{L}_{\Phi,\ell}^{-1} e^{-\hbar T_\ell^-} + \mathcal{K}_{\Phi,\ell}^{-1} \otimes F_\ell & (\forall \ell \in I) \end{aligned}$$

with $\mathcal{L}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{gk} H_k}$, $\mathcal{K}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k}$ ($= \mathcal{L}_{\Phi,\ell}^{-1}$) for $\ell \in I$. Similarly, the “twisted” antipode $\mathcal{S}^{\mathcal{F}_\Phi}$ and the counit $\epsilon^{\mathcal{F}_\Phi} := \epsilon$ are given by

$$\begin{aligned} \mathcal{S}^{\mathcal{F}_\Phi}(E_\ell) &= -e^{-\hbar T_\ell^+} \mathcal{K}_{\Phi,\ell}^{-1} E_\ell \mathcal{L}_{\Phi,\ell}^{-1}, & \epsilon^{\mathcal{F}_\Phi}(E_\ell) &= 0 & (\forall \ell \in I), \\ \mathcal{S}^{\mathcal{F}_\Phi}(T) &= -T, & \epsilon^{\mathcal{F}_\Phi}(T) &= 0 & (\forall T \in \mathfrak{h}), \\ \mathcal{S}^{\mathcal{F}_\Phi}(F_\ell) &= -\mathcal{K}_{\Phi,\ell}^{+1} F_\ell \mathcal{L}_{\Phi,\ell}^{+1} e^{+\hbar T_\ell^-}, & \epsilon^{\mathcal{F}_\Phi}(F_\ell) &= 0 & (\forall \ell \in I). \end{aligned}$$

Tiding up, from the explicit description of the coproduct $\Delta^{\mathcal{F}_\Phi}$, it follows that $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ is generated by group-likes and skew-primitive elements; hence, it is a *pointed* Hopf superalgebra. Moreover, both Hopf superalgebras $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ and $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ have the same coradical and the same space of skew-primitive elements. As the coproduct is changed by twist, one sees that the skew-primitive generators of $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$, which are $(1, g)$ -primitive or $(g, 1)$ -primitive with respect to Δ , for some $g \in G(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))$, become (h, k) -primitive for $\Delta^{\mathcal{F}_\Phi}$. Looking at the coradical filtration, and the associated graded Hopf superalgebra, one finds from that set of generators some new $(1, \ell)$ -primitive or $(\ell, 1)$ -primitives for $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$. This leads to devise (new) *twisted generators* and a corresponding presentation for

$(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$, which yields a Hopf superalgebra isomorphism between $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ and a new FoMpQUESA with suitable multiparameter matrix and realization.

Motivated by the above analysis, we introduce now in $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ the “twisted” generators $E_\ell^\Phi := \mathcal{L}_{\Phi,\ell}^{-1} E_\ell$, $F_\ell^\Phi := F_\ell \mathcal{K}_{\Phi,\ell}^{+1}$ (for all $\ell \in I$); clearly, these along with all the T ’s in $\mathfrak{h}$ still generate $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi} = U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$. Moreover, we introduce the twisted “distinguished toral elements” (or “coroots”) that were already defined in (2.1), i.e. $T_{\Phi,\ell}^\pm := T_\ell^\pm \pm \sum_{g,k=1}^t \alpha_\ell(H_g) \phi_{kg} H_k$.

Finally, again from §2.2.1, we recall also $P_\Phi := (p_{i,j}^\Phi)_{i,j \in I}$ and $\mathcal{R}_\Phi := (\mathfrak{h}, \Pi, \Pi_\Phi^\vee)$, with the latter being a realization of the former indeed.

As a second step, the defining relations in the superalgebra $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ give new relations between twisted generators. Namely, by straightforward computations — for instance using that $\mathcal{K}_{\Phi,j} E_i = e^{\hbar 2^{-1} \sum_{g,k=1}^t \alpha_j(H_g) \phi_{kg} \alpha_i(H_k)} E_i \mathcal{K}_{\Phi,j}$ and that $e^{+\hbar T_{\Phi,i}^\pm} = e^{+\hbar T_i^\pm} (\mathcal{K}_{\Phi,i} \mathcal{L}_{\Phi,i}^{-1})^{\pm 1}$ — one proves that inside $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ the following identities hold true (for all $T, T', T'' \in \mathfrak{h}$, $i, j \in I$):

$$\begin{aligned} T E_j^\Phi - E_j^\Phi T &= +\alpha_j(T) E_j^\Phi, & T F_j^\Phi - F_j^\Phi T &= -\alpha_j(T) F_j^\Phi \\ T' T'' &= T'' T', & [E_i^\Phi, F_j^\Phi] &= \delta_{i,j} \frac{e^{+\hbar T_{\Phi,i}^+} - e^{-\hbar T_{\Phi,i}^-}}{q_i^{+1} - q_i^{-1}}, \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} (k_{ij}^\Phi)^s (E_i^\Phi)^{1-a_{ij}-s} E_j^\Phi (E_i^\Phi)^s &= 0 & \left(\begin{array}{l} i \neq j \\ p(i)=\bar{0} \end{array} \right) \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} (k_{ji}^\Phi)^s (F_i^\Phi)^{1-a_{ij}-s} F_j^\Phi (F_i^\Phi)^s &= 0 & \left(\begin{array}{l} i \neq j \\ p(i)=\bar{0} \end{array} \right) \end{aligned}$$

— with $q_{i,j}^\Phi := e^{\hbar p_{i,j}^\Phi}$ ($i, j \in I$), so that $q_{i,i}^\Phi = e^{\hbar p_{i,i}^\Phi} = e^{\hbar p_{i,i}} = e^{\hbar 2d_i} = q_i^2$, as well as $k_{i,j}^\Phi := q_{i,j}^\Phi q_{j,i}^\Phi = e^{\hbar (p_{i,j}^\Phi - p_{j,i}^\Phi)}$ ($i, j \in I$) — and also

$$[E_i^\Phi, E_j^\Phi]_{k_{ij}^\Phi} = 0, \quad [F_i^\Phi, F_j^\Phi]_{k_{ji}^\Phi} = 0 \quad \text{if } a_{i,j} = 0$$

as well as all other relations listed in Definition 4.2.2, namely (4.8) through (4.12) — including their “counterparts” with the F_ℓ ’s replacing the E_ℓ ’s throughout. Indeed, apart from the first two lines (which follow from direct approach), all the other formulas follows from iterated applications of (4.21) — that is $[E_i, E_j]_\nu = (k_{i,j}^\Phi)^{-1/2} \mathcal{L}_{\Phi,i}^{+1} \mathcal{L}_{\Phi,j}^{+1} [E_i^\Phi, E_j^\Phi]_{\nu k_{ij}^\Phi}$ where the left-hand side now is to be read inside $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ — and (4.22) alike, just sticking to the very same argument that was followed in the proof of Proposition 4.2.8.

Third, the Hopf operators on the “twisted” generators read (for $\ell \in I$, $T \in \mathfrak{h}$)

$$\begin{aligned} \Delta^{\mathcal{F}_\Phi}(E_\ell^\Phi) &= E_\ell^\Phi \otimes 1 + e^{+\hbar T_{\Phi,\ell}^+} \otimes E_\ell^\Phi, & \mathcal{S}^{\mathcal{F}_\Phi}(E_\ell^\Phi) &= -e^{-\hbar T_{\Phi,\ell}^+} E_\ell^\Phi, & \epsilon^{\mathcal{F}_\Phi}(E_\ell^\Phi) &= 0 \\ \Delta^{\mathcal{F}_\Phi}(T) &= T \otimes 1 + 1 \otimes T, & \mathcal{S}^{\mathcal{F}_\Phi}(T) &= -T, & \epsilon^{\mathcal{F}_\Phi}(T) &= 0 \\ \Delta^{\mathcal{F}_\Phi}(F_\ell^\Phi) &= F_\ell^\Phi \otimes e^{-\hbar T_{\Phi,\ell}^-} + 1 \otimes F_\ell^\Phi, & \mathcal{S}^{\mathcal{F}_\Phi}(F_\ell^\Phi) &= -F_\ell^\Phi e^{+\hbar T_{\Phi,\ell}^-}, & \epsilon^{\mathcal{F}_\Phi}(F_\ell^\Phi) &= 0 \end{aligned}$$

At last, it is now immediate to check that mapping

$$E_\ell \mapsto E_\ell^\Phi := \mathcal{L}_{\Phi,\ell}^{-1} E_\ell, \quad T \mapsto T, \quad F_\ell \mapsto F_\ell^\Phi := F_\ell \mathcal{K}_{\Phi,\ell}^{+1} \quad (\forall \ell \in I, T \in \mathfrak{h})$$

does indeed provide a well-defined isomorphism of (topological) superalgebras that is in fact also one of Hopf superalgebras, q.e.d. $\square$

As a matter of fact, *the previous result can be somehow reversed*, as the following shows: in particular, loosely speaking, we end up finding that *every straight small FoMpQUESA can be realized as a toral twist deformation of the “standard” Yamane’s QUESA* — cf. claim (c) in Theorem 5.1.3 here below.

Theorem 5.1.3. *With assumptions as above, let P and P' be two matrices of Cartan type with the same associated Cartan matrix A , namely $P_s = P'_s$.*

(a) *Let $\mathcal{R}$ be a straight realization of P and let $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ be the associated FoMpQUESA. Then there is a straight realization $\tilde{\mathcal{R}}'$ of P' and a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that for the associated twist element $\mathcal{F}_\Phi$ as in (5.1) we have*

$$U_{P',h}^{\tilde{\mathcal{R}}',p}(\mathfrak{g}) \cong (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$$

In a nutshell, if $P'_s = P_s$ then from any straight FoMpQUESA over P we can obtain by toral twist deformation a straight FoMpQUESA (of the same rank) over P' .

Conversely, if $\mathcal{R}'$ is any straight realization of P' and $U_{P',h}^{\mathcal{R}',p}(\mathfrak{g})$ is the associated FoMpQUESA, then there exists a straight realization $\hat{\mathcal{R}}$ of P and a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that for the associated twist element $\mathcal{F}_\Phi$ as in (5.1) we have

$$U_{P',h}^{\mathcal{R}',p}(\mathfrak{g}) \cong (U_{P,h}^{\hat{\mathcal{R}},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$$

(b) *Let $\mathcal{R}$ and $\mathcal{R}'$ be straight small realizations of P and P' respectively, with $\text{rk}(\mathcal{R}) = \text{rk}(\mathcal{R}') = t$, and let $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ and $U_{P',h}^{\mathcal{R}',p}(\mathfrak{g})$ be the associated FoMpQUESAs. Then there exists a matrix $\Phi \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ such that for $\mathcal{F}_\Phi$ as in (5.1) we have*

$$U_{P',h}^{\mathcal{R}',p}(\mathfrak{g}) \cong (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$$

In short, if $P'_s = P_s$, any straight small FoMpQUESA over P' is isomorphic to a toral twist deformation of any straight small FoMpQUESA over P of same rank.

(c) *Every straight small FoMpQUESA is isomorphic to some toral twist deformation of Yamane’s standard FoMpQUESA (over $DA = P_s$) of the same rank.*

(d) *Similar, parallel statements hold true for the Borel FoMpQUESAs.*

Proof. (a) By Theorem 5.1.2 it is enough to find $\Phi \in \mathfrak{so}_t(\mathbb{k})$ such that $P' = P_\Phi$, that is $P' = P - \mathfrak{A}\Phi\mathfrak{A}^T$; but this is guaranteed by Proposition 2.2.6(a).

(b) This follows directly from claim (a) and Proposition 2.2.6(b).

(c) This follows from (b), with $U_{P',h}^{\mathcal{R}',p}(\mathfrak{g})$ the given straight small FoMpQUESA and $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ the standard FoMpQUESA over $P := DA$ as in Yamane’s definition.

(d) This is clear from definitions. $\square$

Observation 5.1.4. Theorems 5.1.2 and 5.1.3 have the following interpretation. Our FoMpQUESAs $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ are quantum objects depending on the multiparameter P ; but when we perform onto $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ a deformation by twist as in §5.1.1, the output $U_{P,h,\Phi}^{\mathcal{R},p}(\mathfrak{g}) := (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_\Phi}$ seemingly depends on two multiparameters, namely P and Φ . Thus all these $U_{P,h,\Phi}^{\mathcal{R},p}(\mathfrak{g})$ ’s form a seemingly richer family of “twice-multiparametric” formal QUESAs. Nevertheless, Theorem 5.1.2 above proves that

this family actually *coincides* with the family of all FoMpQUESAs, although the latter seems a priori smaller.

In short, Theorems 5.1.2 and 5.1.3 show the following. The dependence of the Hopf structure of $U_{P,\hbar}^{\mathcal{R},\Phi}(\mathfrak{g})$ on the “double parameter” (P, Φ) is “split” in the algebraic structure (ruled by P) and in the coalgebraic structure (ruled by Φ); now Theorems 5.1.2 and 5.1.3 enable us to “polarize” this dependence so to codify it either entirely within the algebraic structure (while the coalgebraic one is cast into a “canonical form”), so that the single multiparameter P_Φ is enough to describe it, or entirely within the coalgebraic structure (with the algebraic one being reduced to the “standard” Yamane’s one), hence one multiparameter alone, say Φ_P , is enough.

Remark 5.1.5. Since the subclass of *split* realizations is *not closed* under twist (cf. the end of §2.2.7), the subclass of all “split” FoMpQUESAs is not closed too under twist deformation; this is a quantum analogue of Observations 3.3.5(b).

5.2. Deformations of FoMpQUESAs by toral polar–2–cocycles.

We discuss now suitable deformations of FoMpQUESAs by polar–2–cocycles (of “toral type”), proving that they are again FoMpQUESAs.

5.2.1. Deformations of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ by toral polar–2–cocycles. In §4.1.5 we introduced the notion of “2–cocycle” and the procedure of deformation by a 2–cocycle for a given Hopf superalgebra. In the setup of QUESAs, the construction has an important variation, leading to the notion of “polar–2–cocycle” for which the procedure of deformation mentioned above still applies. This theory is developed in [GG4] for the case of quantized universal enveloping algebras (=QUEAs), i.e. those QUESAs whose super-structure is trivial (namely, their odd part is zero). The task of extending the results in [GG4] — stated and proved for QUEAs: see, in particular, §3.3 therein — to the setup of QUESAs is harmless, though technical. However, we do not need that: instead, we will introduce special “polar–2–cocycles” for our FoMpQUESAs (a special class of QUESAs, indeed) and then we will show by direct check that the procedure of deformation by 2–cocycles still work if we use instead these *polar–2–cocycles*. Thus we will be applying a general fact, but proving its validity “on the fly” instead of relying on a general, theoretical proof.

5.2.2. Polar–2–cocycles of $U_{P,\hbar}(\mathfrak{h})$. Let (A, p) be a Cartan super-datum. Fix again $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type with associated Cartan matrix A , a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it and the (topological) Hopf algebra $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, as in §2 and §4.2. We consider special 2–cocycles of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, called “toral” as they are induced from the quantum torus. To this end, like in §5.1.1, we fix in $\mathfrak{h}$ a $\mathbb{k}[[\hbar]]$ –basis $\{H_g\}_{g \in \mathcal{G}}$, where $\mathcal{G}$ is an index set with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Like in §2.3.1, we fix an antisymmetric, $\mathbb{k}[[\hbar]]$ –bilinear map $\chi : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{k}[[\hbar]]$, that corresponds to some $X = (\chi_{g\gamma})_{g,\gamma \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$ via $\chi_{g\gamma} = \chi(H_g, H_\gamma)$. We also consider the antisymmetric matrix $\mathring{X} := \left(\mathring{\chi}_{ij} = \chi(T_i^+, T_j^+) \right)_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$. Any such map χ induces uniquely an antisymmetric, $\mathbb{k}[[\hbar]]$ –bilinear map

$$\tilde{\chi}_U : U_{\hbar}(\mathfrak{h}) \times U_{\hbar}(\mathfrak{h}) \longrightarrow \mathbb{k}[[\hbar]]$$

as follows. By definition, $U_{\hbar}(\mathfrak{h}) := U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{h})$ is an $\hbar$ -adically complete topologically free Hopf algebra isomorphic to $\widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h}) := \bigoplus_{n \in \mathbb{N}} \widehat{S_{\mathbb{k}[[\hbar]]}^n}(\mathfrak{h})$ — the $\hbar$ -adic completion of the symmetric algebra $S_{\mathbb{k}[[\hbar]]}(\mathfrak{h}) = \bigoplus_{n \in \mathbb{N}} S_{\mathbb{k}[[\hbar]]}^n(\mathfrak{h})$ — hence the following makes sense:

Definition 5.2.3. We define $\tilde{\chi}_U$ as the unique $\mathbb{k}[[\hbar]]$ -linear (hence $\hbar$ -adically continuous) map $U_{\hbar}(\mathfrak{h}) \otimes U_{\hbar}(\mathfrak{h}) \xrightarrow{\tilde{\chi}_U} \mathbb{k}[[\hbar]]$ such that (with identifications as above)

$$\begin{aligned} \tilde{\chi}_U(z, 1) &:= \epsilon(z) =: \tilde{\chi}_U(1, z) \quad , \quad \tilde{\chi}_U(x, y) := \chi(x, y) \quad \forall \quad z \in \widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h}) \quad , \quad x, y \in S_{\mathbb{k}[[\hbar]]}^1(\mathfrak{h}) \\ \tilde{\chi}_U(x, y) &:= 0 \quad \forall \quad x \in S_{\mathbb{k}}^r(\mathfrak{h}) \quad , \quad y \in S_{\mathbb{k}}^s(\mathfrak{h}) : r, s \geq 1, r + s > 2 \quad \diamond \end{aligned}$$

By construction, $\tilde{\chi}_U$ is a normalized Hochschild 2-cocycle on $U_{\hbar}(\mathfrak{h})$, that is

$$\epsilon(x) \tilde{\chi}_U(y, z) - \tilde{\chi}_U(xy, z) + \tilde{\chi}_U(x, yz) - \tilde{\chi}_U(x, y) \epsilon(z) = 0 \quad \forall \quad x, y, z \in U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$$

Given two linear maps $\eta, \vartheta : U_{\hbar}(\mathfrak{h}) \otimes U_{\hbar}(\mathfrak{h}) \longrightarrow \mathbb{k}[[\hbar]]$, one defines the *convolution product* $\eta * \vartheta : U_{\hbar}(\mathfrak{h})^{\otimes 2} \longrightarrow \mathbb{k}[[\hbar]]$ by $(\eta * \vartheta)(x \otimes y) := \eta(x_{(1)}, y_{(1)}) \vartheta(x_{(2)}, y_{(2)})$ for all $x, y \in U_{\hbar}(\mathfrak{h})$ — using the coalgebra structure of $U_{\hbar}(\mathfrak{h})^{\otimes 2} \cong \widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h})^{\otimes 2} \cong \widehat{S}_{\mathbb{k}[[\hbar]]}(\mathfrak{h} \oplus \mathfrak{h})$. Then by η^{*m} we denote the m -th power with respect to the convolution product of any map η as above; in particular, we set $\eta^{*0} := \epsilon \otimes \epsilon$.

The following result, proved by direct computation, describes the powers of $\tilde{\chi}_U$:

Lemma 5.2.4. For all $H_+, H_- \in \mathfrak{h}$ and $k, \ell, m \in \mathbb{N}_+$, we have

$$\tilde{\chi}_U^{*m}(H_+^k, H_-^\ell) = \begin{cases} \delta_{k,m} \delta_{\ell,m} (m!)^2 \chi(H_+, H_-)^m & \text{for } m \geq 1, \\ \delta_{k,0} \delta_{\ell,0} & \text{for } m = 0. \end{cases} \quad \square$$

Definition 5.2.5. Keep notation as above. We define χ_U as the unique map from $U_{\hbar}(\mathfrak{h}) \otimes_{\mathbb{k}[[\hbar]]} U_{\hbar}(\mathfrak{h})$ to $\mathbb{k}((\hbar))$ given by the exponentiation of $\hbar^{-1} 2^{-1} \tilde{\chi}_U$, i.e.

$$\chi_U := e^{\hbar^{-1} 2^{-1} \tilde{\chi}_U} = \sum_{m \geq 0} \hbar^{-m} \tilde{\chi}_U^{*m} / 2^m m! \quad \diamond$$

Lemma 5.2.6. The map χ_U is a well-defined, normalized, $\mathbb{k}((\hbar))$ -valued Hopf 2-cocycle for $U_{\hbar}(\mathfrak{h}) := U_{P,\hbar}^{\mathcal{R}}(\mathfrak{h})$, such that, for all $H_+, H_- \in \mathfrak{h}$ and $K_{\pm} := e^{\hbar H_{\pm}}$,

$$\chi_U^{\pm 1}(H_+, H_-) = \pm \hbar^{-1} 2^{-1} \chi(H_+, H_-) \quad , \quad \chi_U(K_+, K_-) = e^{\hbar 2^{-1} \chi(H_+, H_-)}$$

Proof. This is just Lemma 5.2.5 in [GG3]. $\square$

5.2.7. Toral polar-2-cocycles of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$. The previous construction provides, starting from χ , a normalized Hopf 2-cocycle $\chi_U : U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{h}) \times U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{h}) \longrightarrow \mathbb{k}((\hbar))$.

We assume now that the map χ satisfies the additional requirement (2.3), in other words we require that $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ — notation of §2.3. The latter map canonically induces a $\mathbb{k}[[\hbar]]$ -bilinear map $\bar{\chi} : \bar{\mathfrak{h}} \times \bar{\mathfrak{h}} \longrightarrow \mathbb{k}[[\hbar]]$, where $\bar{\mathfrak{h}} := \mathfrak{h}/\mathfrak{s}$ with $\mathfrak{s} := \text{Span}_{\mathbb{k}[[\hbar]]}(\{S_i\}_{i \in I})$, given by

$$\bar{\chi}(T' + \mathfrak{s}, T'' + \mathfrak{s}) := \chi(T', T'') \quad \forall \quad T', T'' \in \mathfrak{h}$$

Now, replaying the construction above but with $\bar{\mathfrak{h}}$ and $\bar{\chi}$ replacing $\mathfrak{h}$ and χ , we construct a normalized Hopf 2-cocycle $\bar{\chi}_U : U_{P,\hbar}^{\mathcal{R},p}(\bar{\mathfrak{h}}) \times U_{P,\hbar}^{\mathcal{R},p}(\bar{\mathfrak{h}}) \longrightarrow \mathbb{k}((\hbar))$;

for the latter, the analogue of Lemma 5.2.6 holds true again. Moreover, note that $U_{P,\hbar}^{\mathcal{R},p}(\bar{\mathfrak{h}}) \cong \widehat{S}_{\mathbb{k}[[\hbar]]}(\bar{\mathfrak{h}})$, and, thanks to (2.4), there exists a Hopf superalgebra epimorphism $\pi : U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \longrightarrow U_{P,\hbar}^{\mathcal{R},p}(\bar{\mathfrak{h}})$ given by $\pi(E_i) := 0$, $\pi(F_i) := 0$ — for $i \in I$ — and $\pi(T) := (T + \mathfrak{s}) \in \bar{\mathfrak{h}} \subseteq U_{P,\hbar}^{\mathcal{R}}(\bar{\mathfrak{h}})$ — for $T \in \mathfrak{h}$. Then we consider

$$\sigma_\chi := \bar{\chi}_U \circ (\pi \times \pi) : U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \times U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \longrightarrow \mathbb{k}((\hbar))$$

which is *automatically* a normalized, $\mathbb{k}((\hbar))$ -valued Hopf 2-cocycle on $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$.

Definition 5.2.8. We call “*polar-2-cocycles of toral type*”, or “*toral polar-2-cocycles*”, of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ all normalized Hopf 2-cocycles σ_χ obtained — from all $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ — via the above construction. We denote by $\mathcal{Z}_2(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))$ the set of all of them — which is actually independent of the multiparameter P , indeed.

N.B.: in fact, the σ_χ ’s above are “*polar-2-cocycles of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$* ” in the sense of the definition given in [GG4], once the latter is adapted to QUESAs. $\diamond$

5.2.9. Formulas for the σ_χ -deformed product. Let $\sigma_\chi \in \mathcal{Z}_2(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))$ be a toral 2-cocycle as above. Following §4.1.5, using σ_χ we introduce in $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ a “deformed product”, hereafter denoted by $\dot{\sigma}_\chi$; then $X^{(n)\sigma_\chi} = X \dot{\sigma}_\chi \cdots \dot{\sigma}_\chi X$ will denote the n -th power of any $X \in U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ with respect to this deformed product.

Directly from definitions, sheer computation yields the following formulas, relating the deformed product with the old one (for all $T', T'', T \in \mathfrak{h}$, $i, j \in I$, $m, n \in \mathbb{N}$):

$$\begin{aligned} T' \dot{\sigma}_\chi T'' &= T' T'' \quad , \quad T^{(n)\sigma_\chi} = T^n \quad , \quad E_i \dot{\sigma}_\chi F_j = E_i F_j \quad , \quad F_j \dot{\sigma}_\chi E_i = F_j E_i \\ T \dot{\sigma}_\chi E_j &= T E_j + 2^{-1} \chi(T, T_j^+) E_j \quad , \quad E_j \dot{\sigma}_\chi T = E_j T + 2^{-1} \chi(T_j^+, T) E_j \\ T \dot{\sigma}_\chi F_j &= T F_j + 2^{-1} \chi(T, T_j^-) F_j \quad , \quad F_j \dot{\sigma}_\chi T = F_j T + 2^{-1} \chi(T_j^-, T) F_j \\ E_{i_1} \dot{\sigma}_\chi E_{i_2} \dot{\sigma}_\chi \cdots \dot{\sigma}_\chi E_{i_{t-1}} \dot{\sigma}_\chi E_{i_t} &= e^{+\hbar 2^{-1} \sum_{c < e} \hat{\chi}_{ce}} E_{i_1} E_{i_2} \cdots E_{i_{t-1}} E_{i_t} \\ E_i^{(m)\sigma_\chi} &= \prod_{\ell=1}^{m-1} \sigma_\chi(e^{+\hbar \ell T_i^+}, e^{+\hbar T_i^+}) E_i^m = E_i^m \\ E_i^m \dot{\sigma}_\chi E_j^n &= \sigma_\chi(e^{+\hbar m T_i^+}, e^{+\hbar n T_j^+}) E_i^m E_j^n = e^{+\hbar m n 2^{-1} \hat{\chi}_{ij}} E_i^m E_j^n \\ E_i^{(m)\sigma_\chi} \dot{\sigma}_\chi E_j \dot{\sigma}_\chi E_k^{(n)\sigma_\chi} &= \prod_{\ell=1}^{m-1} \sigma_\chi(e^{+\hbar \ell T_i^+}, e^{+\hbar T_i^+}) \cdot \prod_{t=1}^{n-1} \sigma_\chi(e^{+\hbar t T_k^+}, e^{+\hbar T_k^+}) \cdot \\ &\quad \cdot \sigma_\chi(e^{+\hbar m T_i^+}, e^{+\hbar T_j^+}) \sigma_\chi(e^{+\hbar(m T_i^+ + T_j^+)}, e^{+\hbar n T_k^+}) E_i^m E_j E_k^n \\ F_{i_1} \dot{\sigma}_\chi F_{i_2} \dot{\sigma}_\chi \cdots \dot{\sigma}_\chi F_{i_{t-1}} \dot{\sigma}_\chi F_{i_t} &= e^{-\hbar 2^{-1} \sum_{c < e} \hat{\chi}_{ce}} F_{i_1} F_{i_2} \cdots F_{i_{t-1}} F_{i_t} \\ F_i^{(m)\sigma_\chi} &= \prod_{\ell=1}^{m-1} \sigma_\chi^{-1}(e^{-\hbar \ell T_i^-}, e^{-\hbar T_i^-}) F_i^m = F_i^m \\ F_i^m \dot{\sigma}_\chi F_j^n &= \sigma_\chi^{-1}(e^{-\hbar m T_i^-}, e^{-\hbar n T_j^-}) F_i^m F_j^n = e^{-\hbar m n 2^{-1} \hat{\chi}_{ij}} F_i^m F_j^n \\ F_i^{(m)\sigma_\chi} \dot{\sigma}_\chi F_j \dot{\sigma}_\chi F_k^{(n)\sigma_\chi} &= \prod_{\ell=1}^{m-1} \sigma_\chi^{-1}(e^{-\hbar \ell T_i^-}, e^{-\hbar T_i^-}) \cdot \prod_{t=1}^{n-1} \sigma_\chi^{-1}(e^{-\hbar t T_k^-}, e^{-\hbar T_k^-}) \cdot \\ &\quad \cdot \sigma_\chi^{-1}(e^{-\hbar m T_i^-}, e^{-\hbar T_j^-}) \sigma_\chi^{-1}(e^{-\hbar(m T_i^- + T_j^-)}, e^{-\hbar n T_k^-}) F_i^m F_j F_k^n \end{aligned}$$

Note also that from the identities $T^{(n)\sigma_\chi} = T^n$ (for $T \in \mathfrak{h}$ and $n \in \mathbb{N}$) it follows that the exponential of any toral element with respect to the deformed product $\dot{\sigma}_\chi$ is the same as with respect to the old one.

Observation 5.2.10. The whole procedure of 2-cocycle deformation by σ_χ should apply to the scalar extension $\mathbb{k}((\hbar)) \otimes_{\mathbb{k}[[\hbar]]} U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, as *a priori* σ_χ takes values in $\mathbb{k}((\hbar))$ rather than in $\mathbb{k}[[\hbar]]$. Nevertheless, the formulas above show that $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ is actually *closed* for the deformed product “ $\dot{\sigma}_\chi$ ” provided by this procedure, hence the deformation does “restrict” to $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ itself, so that we eventually end up with a well-defined 2-cocycle deformation $(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$.

A first, direct consequence of these formulas is the following:

Proposition 5.2.11. *The deformed algebra $(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$ is still generated by the elements E_i , F_i and T (with $i \in I$ and $T \in \mathfrak{h}$) of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$.*

5.2.12. Deformations of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ by toral polar-2-cocycle. Our key result concerns polar-2-cocycle deformations by means of toral polar-2-cocycles. In order to state it, we need some more notation, which we now settle.

Let (A, p) be a Cartan super-datum, $P := (p_{ij})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ a multiparameter matrix of Cartan type with associated Cartan matrix A — cf. Definition 2.1.3 — and fix a realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of P . Fix an antisymmetric $\mathbb{k}[[\hbar]]$ -bilinear map $\chi : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{k}[[\hbar]]$ enjoying (2.3) — that is, $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$; then we associate with it the matrix $\dot{X} := (\dot{\chi}_{ij} = \chi(T_i^+, T_j^+))_{i,j \in I}$ as above. Note that $+\chi(-, T_j^+) = -\chi(-, T_j^-)$ for all $j \in I$, as direct consequence of (2.3). Basing on the above, like in §2.3 we define

$$P_{(\chi)} := P + \dot{X} = \left(p_{ij}^{(\chi)} := p_{ij} + \dot{\chi}_{ij} \right)_{i,j \in I}, \quad \Pi_{(\chi)} := \left\{ \alpha_i^{(\chi)} := \alpha_i \pm \chi(-, T_i^\pm) \right\}_{i \in I}$$

Then, still from §2.3 we know that $P_{(\chi)}$ is a matrix of Cartan type — the same of P indeed — and $\mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ is a realization of it.

We are now ready for our result on deformations by toral polar-2-cocycle.

Theorem 5.2.13. *There exists an isomorphism of topological Hopf superalgebras*

$$(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi} \cong U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g})$$

which is the identity on generators. In short, every toral polar-2-cocycle deformation of a FoMpQUESA is another FoMpQUESA, whose multiparameter $P_{(\chi)}$ and realization $\mathcal{R}_{(\chi)}$ depend on the original P and $\mathcal{R}$, as well as on χ , as explained in §2.3.

Similar statements hold true for the Borel FoMpQUESAs and their deformations by σ_χ , namely there exist isomorphisms $(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{b}_\pm))_{\sigma_\chi} \cong U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)},p}(\mathfrak{b}_\pm)$.

Proof. We begin by noting the following key

Fact: The generators of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, when thought of as elements of the deformed algebra $(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$, obey the defining relations of the (same name) generators of $U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g})$ — with respect to the deformed product $\dot{\sigma}_\chi$.

Let us assume for a moment the above Fact does hold true. Then it implies that there exists a well-defined homomorphism of topological Hopf superalgebras, say $\ell : U_{P_{(\chi)},\hbar}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}) \longrightarrow (U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$ given on generators by $\ell(E_i) := E_i$, $\ell(F_i) := F_i$, $\ell(T) := T$ ($i \in I$, $T \in \mathfrak{h}$) — in short, it is the identity on

generators. Moreover, thanks to Proposition 5.2.11 this is in fact an epimorphism. As an application of this result, there exists also an epimorphism of topological Hopf superalgebras $\ell' : U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) \longrightarrow (U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}))_{\sigma_{-\chi}}$ which again is the identity on generators — just replace χ with $-\chi$ and P with $P_{(\chi)}$.

Mimicking what we did for $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$, we can construct, out of χ , a normalized Hopf 2-cocycle $\dot{\sigma}_\chi$ for $(U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}))_{\sigma_{-\chi}}$; then we also have a similar 2-cocycle σ'_χ on $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ defined as the pull-back of $\dot{\sigma}_\chi$ via ℓ' , and a unique, induced Hopf algebra homomorphism $\ell'_{\dot{\sigma}_\chi} : (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma'_\chi} \longrightarrow \left((U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}))_{\sigma_{-\chi}} \right)_{\dot{\sigma}_\chi}$ between deformed Hopf algebras, which is once more the identity on generators. Now, tracking the whole construction we see that $\sigma'_\chi = \sigma_\chi$, so that $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma'_\chi} = (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$, and $\left((U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}))_{\sigma_{-\chi}} \right)_{\dot{\sigma}_\chi} = U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g})$. Then composition gives homomorphisms

$$\begin{aligned} \ell'_{\dot{\sigma}_\chi} \circ \ell &: U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}) \longrightarrow (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi} \longrightarrow U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}) \\ \ell \circ \ell'_{\dot{\sigma}_\chi} &: (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi} \longrightarrow U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}) \longrightarrow (U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi} \end{aligned}$$

which (both) are the identity on generators: hence in the end we get $\ell'_{\dot{\sigma}_\chi} \circ \ell = id$ and $\ell \circ \ell'_{\dot{\sigma}_\chi} = id$, thus in particular ℓ is an isomorphism, q.e.d.

We are now left with the task of proving the Fact above, i.e. a bunch of relations.

Indeed, most relations follow at once from the formulas in §5.2.9. Namely, the identities $T' \dot{\sigma}_\chi T'' = T' T''$ imply $T' \dot{\sigma}_\chi T'' = T'' \dot{\sigma}_\chi T'$ — for all $T', T'' \in \mathfrak{h}$. Also, from $T \dot{\sigma}_\chi E_j = T E_j + 2^{-1} \chi(T, T_j^+) E_j$ and $E_j \dot{\sigma}_\chi T = E_j T + 2^{-1} \chi(T_j^+, T) E_j$ — for all $T \in \mathfrak{h}$ and $j \in I$ — we get (since χ is antisymmetric)

$$T \dot{\sigma}_\chi E_j - E_j \dot{\sigma}_\chi T = \left(\alpha_j(T) + 2^{-1} (\chi - \chi^T)(T, T_j^+) \right) E_j = +\alpha_j^{(\chi)}(T) E_j$$

A similar, straightforward analysis also yields $T \dot{\sigma}_\chi F_j - F_j \dot{\sigma}_\chi T = -\alpha_j^{(\chi)}(T) F_j$.

The identities $E_i \dot{\sigma}_\chi F_j = E_i F_j$ and $F_j \dot{\sigma}_\chi E_i = F_j E_i$ in turn imply

$$E_i \dot{\sigma}_\chi F_j - (-1)^{p(i)p(j)} F_j \dot{\sigma}_\chi E_i = E_i F_j - (-1)^{p(i)p(j)} F_j E_i = \frac{\delta_{i,j} (e^{+\hbar T_i^+} - e^{-\hbar T_i^-})}{e^{+\hbar p_{ii}/2} - e^{-\hbar p_{ii}/2}}$$

Eventually, since $p_{ii} = p_{ii}^{(\chi)}$ by definition, and the exponential of toral elements with respect to $\dot{\sigma}_\chi$ is the same as with respect to the old product, we conclude that

$$E_i \dot{\sigma}_\chi F_j - (-1)^{p(i)p(j)} F_j \dot{\sigma}_\chi E_i = \delta_{i,j} \frac{e_{\sigma_\chi}^{+\hbar T_i^+} - e_{\sigma_\chi}^{-\hbar T_i^-}}{e^{+\hbar p_{ii}^{(\chi)}/2} - e^{-\hbar p_{ii}^{(\chi)}/2}}$$

where $e_{\sigma_\chi}^X$ denotes the exponential of any X with respect to $\dot{\sigma}_\chi$ — which we already noticed that is the same as the exponential with respect to the original product.

What is less trivial is proving the other relations (for the deformed product); we do this only for the relations involving the E_i 's, those for the F_i 's being similar.

We begin with “quantum Serre relations”, i.e. those of the form

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} (k_{ij}^{(\chi)})^s E_i^{(1-a_{ij}-s)\sigma_\chi} \dot{\sigma}_\chi E_j \dot{\sigma}_\chi E_i^{(s)\sigma_\chi} = 0$$

for all $i \neq j$ in I , with $p(i) = \bar{0}$: hereafter we set $q_{ij}^{(\chi)} := e^{\hbar p_{ij}^{(\chi)}} = e^{\hbar(p_{ij} + \tilde{\chi}_{ij})}$ and $k_{ij}^{(\chi)} := (q_{ij}^{(\chi)})^{+1/2} (q_{ji}^{(\chi)})^{-1/2}$ ($i, j \in I$) noting that $q_{ii}^{(\chi)} = q_{ii}$ and $q_i^{(\chi)} = e^{+\hbar p_{ii}^{(\chi)}/2} = q_i$ ($i \in I$), since $(P_{(\chi)})_s = P_s = DA$; moreover, we set also $\kappa_{ij} := e^{\hbar \tilde{\chi}_{ij}}$, noting then that $q_{ij}^{(\chi)} = q_{ij} \kappa_{ij}$ ($i, j \in I$).

To prove the above identity, we analyze all factors in the summands separately.

Claim 1: For $i \neq j \in I$ we have $k_{ij}^{(\chi)} = k_{ij} \kappa_{ij}$, hence $(k_{ij}^{(\chi)})^s = k_{ij}^s \kappa_{ij}^s$ for $s \in \mathbb{N}$.

This follows by direct computation. Next claim instead follows from §5.2.9:

Claim 2: Fix $i \neq j$ in I , $p(i) = \bar{0}$, and write $m := 1 - a_{ij}$. Then

$$E_i^{(m-s)\sigma_\chi} \dot{\sigma}_\chi E_j \dot{\sigma}_\chi E_i^{(s)\sigma_\chi} = \sigma_\chi(K_i^{m-s}, K_j) \sigma_\chi(K_i^{m-s} K_j, K_i^s) E_i^{m-s} E_j E_i^s$$

Now we evaluate the value of the toral 2-cocycle using the exponentials.

Claim 3: For all $i, j \in I$ and $m, s, \ell \in \mathbb{N}$, we have

$$\begin{aligned} (a) \quad & \sigma_\chi(K_i^\ell, K_j) = \kappa_{ij}^{\ell/2} \\ (b) \quad & \sigma_\chi(K_i^{m-s} K_j, K_i^s) = \kappa_{ij}^{s/2} \\ (c) \quad & \sigma_\chi(K_i^{m-s}, K_j) \sigma_\chi(K_i^{m-s} K_j, K_i^s) = \kappa_{ij}^{(m-2s)/2} \end{aligned}$$

All assertions follow by computation using Lemma 5.2.6. Indeed, for (a) we have

$$\sigma_\chi(K_i^\ell, K_j) = \bar{\chi}_U(K_i^\ell, K_j) = e^{\hbar 2^{-1} \chi(\ell T_i^+, T_j^+)} = \kappa_{ij}^{\ell/2}$$

For item (b), Lemma 5.2.6 yields

$$\begin{aligned} \sigma_\chi(K_i^{m-s} K_j, K_i^s) &= \bar{\chi}_U(K_i^{m-s} K_j, K_i^s) = \bar{\chi}_U(e^{\hbar((m-s)T_i^+ + T_j^+)}, e^{\hbar s T_i^+}) = \\ &= e^{\hbar 2^{-1} \chi((m-s)T_i^+ + T_j^+, s T_i^+)} = \kappa_{ij}^{s/2} \end{aligned}$$

Now, putting altogether (a) and (b) we eventually get (c).

Finally, Claims 1, 2 and 3 altogether yield, for $m := 1 - a_{ij}$,

$$\begin{aligned} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} (k_{ij}^{(\chi)})^s E_i^{(m-s)\sigma_\chi} \dot{\sigma}_\chi E_j \dot{\sigma}_\chi E_i^{(s)\sigma_\chi} &= \\ &= \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} k_{ij}^s \kappa_{ij}^s \kappa_{ij}^{(m-2s)/2} E_i^{m-s} E_j E_i^s = \\ &= \kappa_{ij}^{m/2} \sum_{s=0}^m (-1)^s \begin{bmatrix} m \\ s \end{bmatrix}_{q_i} k_{ij}^s E_i^{m-s} E_j E_i^s = 0 \end{aligned}$$

where the last equality comes from the quantum Serre relation w.r.t. the old product.

Finally, we have to check relations (4.7) through (4.12). To this end, like for the proof of Proposition 4.2.8 – where one also had to check relations, in a very similar manner — we observe that the relations under scrutiny involve iterated q -supercommutators, so the key step is to check how a single q -supercommutator change under polar-2-cocycle deformation. In this respect, we fix notation $[X, Y]_\mu^{\dot{\sigma}_\chi}$ to denote the μ -supercommutator between X and Y with respect to the deformed product “ $\dot{\sigma}_\chi$ ”. Then direct computation, using formulas in §5.2.9, gives (for all

$i, j, k \in I$ and for all $\nu, \mu \in \mathbb{k}[[\hbar]]$)

$$\begin{aligned}
[E_i, E_j]_\nu &= E_i E_j - \nu (-1)^{p(i)p(j)} E_j E_i = \\
&= \kappa_{ji}^{+1/2} E_i \dot{\sigma}_\chi E_j - \nu (-1)^{p(i)p(j)} \kappa_{ij}^{+1/2} E_j \dot{\sigma}_\chi E_i = \\
&= \kappa_{ij}^{-1/2} E_i \dot{\sigma}_\chi E_j - \nu (-1)^{p(i)p(j)} \kappa_{ij}^{+1/2} E_j \dot{\sigma}_\chi E_i = \\
&= \kappa_{ij}^{-1/2} (E_i \dot{\sigma}_\chi E_j - \nu (-1)^{p(i)p(j)} \kappa_{ij} E_j \dot{\sigma}_\chi E_i) = \kappa_{ij}^{-1/2} [E_i, E_j]_{\nu \kappa_{ij}}^{\dot{\sigma}_\chi}
\end{aligned}$$

thus in short

$$[E_i, E_j]_\nu = \kappa_{ij}^{-1/2} [E_i, E_j]_{\nu \kappa_{ij}}^{\dot{\sigma}_\chi} \quad (5.2)$$

Similarly, iterated q -supercommutators in $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ can be rewritten as suitable multiples of similar, iterated q -supercommutators in $\left(U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})\right)_{\dot{\sigma}_\chi}$, giving rise to “higher order analogues” of (5.2). Therefore, all relations from (4.7) to (4.12) for $\left(U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})\right)_{\dot{\sigma}_\chi}$ are deduced from the equivalent relations for $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ via repeated applications of (5.2) or of its higher order generalizations; indeed, the same applies to the quantum Serre relations as well, that we already disposed of independently.

For instance, the first of these higher order analogues comes from

$$\begin{aligned}
[[E_i, E_j]_\nu, E_k]_\mu &= (E_i E_j - \nu (-1)^{p(i)p(j)} E_j E_i) E_k - \\
&- (-1)^{(p(i)+p(j))p(k)} \mu E_k (E_i E_j - \nu (-1)^{p(i)p(j)} E_j E_i) = \\
&= \left(\kappa_{ji}^{+1/2} \kappa_{kj}^{+1/2} \kappa_{ki}^{+1/2} E_i \dot{\sigma}_\chi E_j - \nu (-1)^{p(i)p(j)} \kappa_{ij}^{+1/2} \kappa_{kj}^{+1/2} \kappa_{ki}^{+1/2} E_j \dot{\sigma}_\chi E_i \right) \dot{\sigma}_\chi E_k - \\
&- (-1)^{(p(i)+p(j))p(k)} \mu E_k \dot{\sigma}_\chi \left(\kappa_{ik}^{+1/2} \kappa_{jk}^{+1/2} \kappa_{ji}^{+1/2} E_i \dot{\sigma}_\chi E_j - \right. \\
&\quad \left. - \nu (-1)^{p(i)p(j)} \kappa_{jk}^{+1/2} \kappa_{ik}^{+1/2} \kappa_{ij}^{+1/2} E_j \dot{\sigma}_\chi E_i \right) = \\
&= \kappa_{ji}^{+1/2} \kappa_{kj}^{+1/2} \kappa_{ki}^{+1/2} [[E_i, E_j]_{\nu \kappa_{ij}}^{\dot{\sigma}_\chi}, E_k]_{\mu \kappa_{jk} \kappa_{ik}}^{\dot{\sigma}_\chi}
\end{aligned} \quad (5.3)$$

which eventually yields the “third order” identity

$$[[E_i, E_j]_\nu, E_k]_\mu = \kappa_{ji}^{+1/2} \kappa_{kj}^{+1/2} \kappa_{ki}^{+1/2} [[E_i, E_j]_{\nu \kappa_{ij}}^{\dot{\sigma}_\chi}, E_k]_{\mu \kappa_{jk} \kappa_{ik}}^{\dot{\sigma}_\chi} \quad (5.4)$$

To see how these apply, the easiest case is relation (4.7) for $\left(U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})\right)_{\dot{\sigma}_\chi}$, which is deduced from (4.7) for $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ applying (5.2) as follows

$$0 = [E_i, E_j]_{k_{ij}} = \kappa_{ij}^{-1/2} [E_i, E_j]_{k_{ij} \kappa_{ij}}^{\dot{\sigma}_\chi} \implies [E_i, E_j]_{k_{ij}^{(\chi)}}^{\dot{\sigma}_\chi} = 0$$

where we noticed that $k_{ij}^{(\chi)} := (q_{ij}^{(\chi)})^{+1/2} (q_{ji}^{(\chi)})^{-1/2} = k_{ij} \kappa_{ij}$ (cf. Definition 4.2.1).

Another, less trivial example is that of (4.10) for $\left(U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})\right)_{\dot{\sigma}_\chi}$: we deduce it from the parallel relation (4.10) for $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ by direct application of (5.4) in which we fix $(i, j, k) := (N-2, N-1, N)$, $(\nu, \mu) := (\nu_{N-1}, \nu_N)$ in the first addendum, and $(i, j, k) := (N-2, N, N-1)$, $(\nu, \mu) := (\nu_{N-1}, \nu_N)$ in the second addendum.

All other cases are entirely similar, so we leave them to the reader. $\square$

Remark 5.2.14. With notation of Theorem 5.2.13 above, we have $P_{(\chi)} - P = \Lambda$ for some *antisymmetric* matrix $\Lambda \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$. Conversely, under mild assumptions on P , this result can be “reversed” as it is shown below.

Theorem 5.2.15. *Let $P, P' \in M_n(\mathbb{k}[[\hbar]])$ be two matrices of Cartan type with the same associated Cartan matrix A .*

(a) *Let $\mathcal{R}$ be a split realization of P and $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ be the associated FoMpQUESA. Then there exists a split realization $\tilde{\mathcal{R}}'$ of P' , a matrix $\tilde{X} = (\tilde{\chi}_{ij})_{i,j \in I} \in \mathfrak{so}_n(\mathbb{k}[[\hbar]])$ and a toral polar-2-cocycle σ_χ such that*

$$U_{P',\hbar}^{\mathcal{R}',p}(\mathfrak{g}) \cong (U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$$

In short, if $P'_s = P_s$ then from any split FoMpQUESA over P we can obtain a split FoMpQUESA (of the same rank) over P' by a toral polar-2-cocycle deformation.

(b) *Let $\mathcal{R}$ be a split minimal realization of P . Then the FoMpQUESA $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ is isomorphic to a toral polar-2-cocycle deformation of Yamane’s standard double QUESA, that is there exists some bilinear map $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that*

$$U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \cong (U_{P_A,\hbar}^{\mathcal{R}_{st},p}(\mathfrak{g}))_{\sigma_\chi}$$

where $U_{P_A,\hbar}^{\mathcal{R}_{st},p}(\mathfrak{g})$ is the “quantum double” Yamane’s QUESA of Example 4.2.6(b).

(c) *Similar, parallel statements hold true for the Borel FoMpQUESAs.*

Proof. (a) By Proposition 2.3.5(a), there exists $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that $P' = P_{(\chi)}$ and the triple — constructed as in §2.3 — $\mathcal{R}' := \mathcal{R}_{(\chi)} = (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$ is a split realization of $P' = P_{(\chi)}$. Then $U_{P',\hbar}^{\mathcal{R}',p}(\mathfrak{g}) \cong (U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$ by Theorem 5.2.13.

(b) Yamane’s $U_{P_A,\hbar}^{\mathcal{R}_{st},p}(\mathfrak{g})$ is, in our language, nothing but the FoMpQUESA built upon a straight split realization $\mathcal{R}_{st} = (\mathfrak{h}, \Pi_{st}, \Pi_{st}^\vee)$ of DA , for which we write $\Pi_{st}^\vee = \{T_i^\pm\}_{i \in I}$ and $\Pi_{st} = \{\alpha_i^{(st)}\}_{i \in I}$. From Proposition 2.3.5(b) we have a suitable $\chi \in \text{Alt}_{\mathbb{k}[[\hbar]]}^S(\mathfrak{h})$ such that the realization $(\mathcal{R}_{st})_{(\chi)}$ obtained as deformation of $\mathcal{R}_{st}$ via χ coincides with $\mathcal{R}$; then Theorem 5.2.13 gives $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \cong (U_{P_A,\hbar}^{\mathcal{R}_{st},p}(\mathfrak{g}))_{\sigma_\chi}$. $\square$

5.2.16. Generalizations. The whole analysis and construction devised in this section can be also applied, *verbatim*, to the setup of QUESAs over *affine* Lie superalgebras, namely those introduced in [Ya2] and [Ya3]. Much like in the semiclassical case (cf. §3.4.5) one has to handle many more defining relations, keep track of their “mutations” under deformation, etc.: nevertheless, all this is a standard task, while the strategy stands exactly the same. On the other hand, it is worth stressing that the key ideas underlying our recipe also apply (when suitably adapted) to a much larger framework, including concrete examples: we work on this in [GG6].

6. QUANTIZATION AND SPECIALIZATION: FOMPQUESAS VS. MPLSBAS

In this section we consider the the interplay of specialization — performed onto quantum objects, namely our FoMpQUESAs — and, conversely, of quantization — applied to semiclassical objects, namely our MpLSbA’s.

To begin with, we shall see that every FoMpQUESA yields by specialization a suitable MpLSbA; conversely, for any MpLSbA there exists at least one quantization, in the form of a well-defined FoMpQUESA. Second, we shall study the interaction between the process of specialization (at $\hbar = 0$) of any FoMpQUESA and the process of deformation — either by (toral) twist or by (toral) 2-cocycle — of that same FoMpQUESA or of the MpLSbA which is its semiclassical limit. In particular, we will find that, in a natural sense, *the two processes commute with each other*.

6.1. Deformation vs. specialization for FoMpQUESA's and MpLSbA's.

In §4.1 we recalled that any QUESA defines, by specialization at $\hbar = 0$, a well-defined “semiclassical limit”, given by a specific Lie superbialgebra — or more precisely by the universal enveloping superalgebra of the latter. In addition, by Theorem 4.2.12 we know that our FoMpQUESA's are indeed QUESA's, so they do have semiclassical limits in the previous sense. We shall now go and see what these limits look like, i.e. what Lie superalgebras we do find in this way.

6.1.1. Formal MpQUESA's vs. MpLSbA's. Let us fix a Cartan super-datum (A, p) , a matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{K}[[\hbar]])$ of Cartan type with associated Cartan matrix A , and a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P . Out of these data, we associate the FoMpQUESA $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, as in §4.2, and also the MpLSbA $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p}$ introduced in §3.2, where we use the somewhat loose (yet obvious) notation such $\bar{\mathcal{R}} := \mathcal{R} \pmod{\hbar}$ and $\bar{P} := P \pmod{\hbar}$.

We begin by pointing out that, in a nutshell, FoMpQUESA's and MpLSbA's are in bijection through the specialization/quantization process, as one might expect:

Theorem 6.1.2. *With assumptions as above, $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ is a QUESA in the sense of §4.1.2, whose semiclassical limit is $U(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p})$. In other words, $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p}$ is the specialization of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, or — equivalently — $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ is a quantization of $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p}$.*

Proof. By Theorem 4.2.12 we know that $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ is a QUESA, so only have to describe its semiclassical limit. In fact, we aim to prove that $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, as a co-Poisson Hopf superalgebra, is isomorphic to $U(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p})$.

First of all, the presentation of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ implies that $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ is generated by the cosets (modulo $\hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$) of the F_i 's, T 's and E_i 's ($i \in I$, $T \in \mathfrak{h}$); moreover, these cosets $\bar{X} := X \pmod{\hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})}$ obey all relations induced modulo $\hbar$ by the defining relations among the original generators X of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$. On the other hand, the Lie superbialgebra $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p}$ is endowed, by construction, with a presentation (as a Lie superalgebra) by generators (the F_i 's, T 's and E_i 's) and relations, and explicit formulas for the value of the Lie supercobracket δ on the given generators. From this one gets an obvious presentation of $U(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p})$, where the generators are again the F_i 's, T 's and E_i 's and the Poisson cobracket δ on them is given by explicit formulas.

Comparing the presentation of $U(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p})$ with that of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})/\hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ we find that all the given relations among generators of the latter superalgebra do correspond to identical relations among the corresponding generators in the former: indeed, mapping X to $\bar{X}$ (for all $X \in \{E_i, F_i \mid i \in I\} \cup \mathfrak{h}$) turns every given relation

among the X 's into a same-look relation among the $\overline{X}$'s. Eventually, this provides a well-defined epimorphism of Hopf superalgebras

$$\phi : U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p}) \longrightarrow U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) / \hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) , \quad E_i \mapsto \overline{E}_i, T \mapsto \overline{T}, F_i \mapsto \overline{F}_i \quad (6.1)$$

($i \in I, T \in \mathfrak{h}$); furthermore, comparing the formulas on both sides for the co-Poisson superbracket we see that this is also a *co-Poisson* super Hopf epimorphism.

On the other hand, scalar restriction — via $\mathbb{k}[[\hbar]] \longrightarrow \mathbb{k}[[\hbar]]/\hbar \mathbb{k}[[\hbar]] \cong \mathbb{k}$ — makes $U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p})$ into a $\mathbb{k}[[\hbar]]$ -superalgebra. Then the same argument about relations yields a well-defined $\mathbb{k}[[\hbar]]$ -superalgebra epimorphism

$$\psi' : U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p}) , \quad E_i \mapsto E_i, T \mapsto T, F_i \mapsto F_i \quad (i \in I, T \in \mathfrak{h})$$

whose kernel contains $\hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$; therefore, a $\mathbb{k}$ -superalgebra epimorphism

$$\psi : U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) / \hbar U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p}) , \quad \overline{E}_i \mapsto E_i, \overline{T} \mapsto T, \overline{F}_i \mapsto F_i \quad (6.2)$$

($i \in I, T \in \mathfrak{h}$) is induced too. Now comparing (6.1) and (6.2) shows that ϕ and ψ are inverse to each other, so ψ is a super *Hopf* morphism too and we are done. $\square$

6.2. Intertwining deformation and specialization.

We shall now compare the process of deformation at the quantum level and at the semiclassical level, finding a very neat outcome: in a nutshell, *deformation (by twist or by 2-cocycle) commutes with specialization*.

6.2.1. Intertwining twist deformation and specialization. Fix again a Cartan super-datum (A, p) , a matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type, a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of it, and the associated FoMpQUESA $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ and MpLSbA $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p}$. In the free $\mathbb{k}[[\hbar]]$ -module $\mathfrak{h}$ of finite rank t , we fix a $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$, with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$.

Choose any antisymmetric matrix $\Psi = (\psi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$. Out of it, set

$$j_{\overline{\Psi}} := \sum_{g,k=1}^t \overline{\psi_{gk}} \overline{H_g} \otimes \overline{H_k} \in \overline{\mathfrak{h}} \otimes \overline{\mathfrak{h}} \quad \text{as in (3.11)}$$

$$\mathcal{F}_{\Psi} := \exp \left(\hbar 2^{-1} \sum_{g,k=1}^t \psi_{gk} H_g \otimes H_k \right) \in U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \hat{\otimes} U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}) \quad \text{as in (5.1)}$$

so $j_{\overline{\Psi}}$ is a (toral) *twist* for the Lie superbialgebra $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p}$ and $\mathcal{F}_{\Psi}$ is a (toral) *twist* for the Hopf superalgebra $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, we have the deformation $(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p})^{j_{\overline{\Psi}}}$ of $\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p}$ by the (Lie) twist $j_{\overline{\Psi}}$ and the deformation $(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_{\Psi}}$ of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$ by the (Hopf) twist $\mathcal{F}_{\Psi}$.

Again out of Ψ , we define the matrix P_{Ψ} and its realization $\mathcal{R}_{\Psi} := (\mathfrak{h}, \Pi, \Pi_{\Psi}^\vee)$, as in Proposition 2.2.2; then we have $U_{P_{\Psi},\hbar}^{\mathcal{R}_{\Psi},p}(\mathfrak{g})$ and $\mathfrak{g}_{\overline{P}_{\Psi}}^{\overline{\mathcal{R}}_{\Psi},p}$, again mutually linked by a quantization/specialization relationship.

We can now state our result, which in particular claims (in a nutshell) that “*deformation by twist commutes with specialization*”.

Theorem 6.2.2. $(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_{\Psi}}$ is a QUESA, whose semiclassical limit is isomorphic to $U((\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p})^{j_{\overline{\Psi}}})$. Indeed, $(U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_{\Psi}} \cong U_{P_{\Psi},\hbar}^{\mathcal{R}_{\Psi},p}(\mathfrak{g})$ and $(\mathfrak{g}_{\overline{P}}^{\overline{\mathcal{R}},p})^{j_{\overline{\Psi}}} \cong \mathfrak{g}_{\overline{P}_{\Psi}}^{\overline{\mathcal{R}}_{\Psi},p}$.

Proof. The claim follows, as direct application, from the isomorphisms

$$\left(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) \right)^{\mathcal{F}_\Psi} \cong U_{P_\Psi,h}^{\mathcal{R}_\Psi}(\mathfrak{g}) , \quad U_{P_\Psi,h}^{\mathcal{R}_\Psi}(\mathfrak{g}) / \hbar U_{P_\Psi,h}^{\mathcal{R}_\Psi}(\mathfrak{g}) \cong U_h(\mathfrak{g}_{\bar{P}_\Psi}^{\bar{\mathcal{R}}_\Psi}) , \quad \mathfrak{g}_{\bar{P}_\Psi}^{\bar{\mathcal{R}}_\Psi} \cong (\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}}})^{j_\Psi}$$

which come from Theorem 5.1.2, Theorem 6.1.2 and Theorem 3.3.3. $\square$

Remark 6.2.3. The above result is proved by direct inspection; it can also be deduced as an application of a general result about the “semiclassical limit” of a twist of a QUESA, namely [GG4, Theorem 3.1.2] (the result there is stated for the non-super case, but it extends straightforwardly to the super setup).

6.2.4. Intertwining 2-cocycle deformation and specialization. Now we analyse what happens when one combines *deformations by 2-cocycle* — for both FoMpQUESAs and MpLSbA’s — with the specialization process.

We begin with a Cartan super-datum (A, p) , a matrix $P := (p_{i,j})_{i,j \in I} \in M_n(\mathbb{k}[[\hbar]])$ of Cartan type, a realization $\mathcal{R} := (\mathfrak{h}, \Pi, \Pi^\vee)$ of P , and a fixed $\mathbb{k}[[\hbar]]$ -basis $\{H_g\}_{g \in \mathcal{G}}$ of $\mathfrak{h}$, with $|\mathcal{G}| = \text{rk}(\mathfrak{h}) = t$. Then $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ and $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p}$ are defined, interlocked through quantization/specialization.

Like in §5.2.2, we fix a $\mathbb{k}[[\hbar]]$ -bilinear map $\chi : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{k}[[\hbar]]$ which is antisymmetric and obeys (3.12). Taking everything modulo $\hbar$, this χ defines a similar antisymmetric, $\mathbb{k}$ -bilinear map $\gamma := (\chi \bmod \hbar) : \mathfrak{h}_0 \times \mathfrak{h}_0 \longrightarrow \mathbb{k}$ — with $\mathfrak{h}_0 := \mathfrak{h} / \hbar \mathfrak{h} = \bar{\mathfrak{h}}$ — which obeys (3.12) again, up to replacing “ χ ” with “ γ ”. Following §5.2.7, we construct out of χ a $\mathbb{k}((\hbar))$ -valued toral polar-2-cocycle $\sigma_\chi : U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) \otimes U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) \longrightarrow \mathbb{k}((\hbar))$, and then the deformed Hopf superalgebra $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$. Similarly, we construct as in §3.4 (but with “ γ ” replacing “ χ ”) a toral 2-cocycle $\gamma_{\bar{\mathfrak{g}}}$ for the Lie bialgebra $\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p}$, and out of it the 2-cocycle deformed Lie superbialgebra $(\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p})_{\gamma_{\bar{\mathfrak{g}}}}$.

Still out of χ , we define the matrix $P_{(\chi)}$ and its realization $\mathcal{R}_{(\chi)} := (\mathfrak{h}, \Pi_{(\chi)}, \Pi^\vee)$, as in Proposition 2.3.2; similarly, out of γ we get the matrix $P_{(\gamma)}$ and its realization $\mathcal{R}_{(\gamma)}$: then $P_{(\gamma)} = \bar{P}_{(\chi)}$ and $\mathcal{R}_{(\gamma)} = \bar{\mathcal{R}}_{(\chi)}$. Finally, attached to the latter we have $U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g})$ and $\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)},p} = \mathfrak{g}_{\bar{P}_{(\chi)}}^{\bar{\mathcal{R}}_{(\chi)},p}$, again connected via quantization/specialization.

Next result claims that “*deformation by 2-cocycle commutes with specialization*”.

Theorem 6.2.5. $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi}$ is a QUESA, with semiclassical limit $U((\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p})_{\gamma_{\bar{\mathfrak{g}}}})$.

Proof. The claim follows, as direct application, from the two isomorphisms

$$(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))_{\sigma_\chi} \cong U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}) , \quad U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}) / \hbar U_{P_{(\chi)},h}^{\mathcal{R}_{(\chi)},p}(\mathfrak{g}) \cong U_h(\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)},p})$$

of co-Poisson Hopf superalgebras (cf. Theorem 5.2.13 and Theorem 6.1.2) and the isomorphism of Lie superbialgebras $\mathfrak{g}_{P_{(\gamma)}}^{\mathcal{R}_{(\gamma)},p} \cong (\mathfrak{g}_{\bar{P}}^{\bar{\mathcal{R}},p})_{\gamma_{\bar{\mathfrak{g}}}}$ — cf. Theorem 3.4.3. $\square$

Remark 6.2.6. The previous result is proved by direct inspection; nevertheless, we can deduce it also as an application of a general result on the “semiclassical limit” of a polar-2-cocycle of a QUESA, namely [GG4, Theorem 3.3.72] (although the result there is stated for the non-super case, it does extend easily to the super setup).

6.3. A global synopsis.

In this paper we studied multiparametric versions of formal QUESA's, as well as their semiclassical limits. All these are presented by generators and relations, so their very definition highlights the relation between the multiparameters and the action of a fixed commutative subsuperalgebra: this is encoded in the notion of *realization* of a multiparameter matrix P , much like for Kac-Moody algebras. This allows us to relate the quantum objects with their semiclassical limit, and also multiparameter objects with standard ones: the latter step goes through deformation(s).

In conclusion we can say, a bit loosely, that:

- (a) multiparameters are encoded in realizations;
- (b) FoMpQUESA's are quantizations of MpLSbA's;
- (c) multiparameter objects are given by deformation of either the (super)algebra or the (super)coalgebra structure of (Yamane's) *uniparameter* objects.

Finally, we display a “pictorial sketch” of the global picture, as follows:

$$\begin{array}{ccc}
 U_{P_\Psi, \hbar}^{\mathcal{R}_\Psi, p}(\mathfrak{g}) \cong \left(U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g}) \right)^{\mathcal{F}_\Psi} & \xleftarrow[\text{Theorem 6.1.2}]{} & U\left(\mathfrak{g}_{P_\Psi}^{\bar{\mathcal{R}}_\Psi, p}\right) \cong U\left(\left(\mathfrak{g}_P^{\bar{\mathcal{R}}, p}\right)^{j_\Psi}\right) \\
 \uparrow \text{Theorem 5.1.3} & & \uparrow \text{Theorem 3.3.4} \\
 U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g}) & \xleftarrow[\text{Theorem 6.1.2}]{} & U\left(\mathfrak{g}_P^{\bar{\mathcal{R}}, p}\right) \\
 \uparrow \text{Theorem 5.2.15} & & \uparrow \text{Theorem 3.4.3} \\
 \left(U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g}) \right)_{\sigma_\chi} \cong U_{P_{(\chi)}, \hbar}^{\mathcal{R}_{(\chi)}, p}(\mathfrak{g}) & \xleftarrow[\text{Theorem 6.1.2}]{} & U\left(\left(\mathfrak{g}_P^{\bar{\mathcal{R}}, p}\right)_{\bar{\chi}}\right) \cong U\left(\mathfrak{g}_{P_{(\chi)}}^{\bar{\mathcal{R}}_{(\chi)}, p}\right)
 \end{array}$$

In the above drawing, each horizontal arrow (with dotted shaft) denotes a “quantization/specialization (upwards/downwards) relationship” — which involves the “continuous parameter” $\hbar$ — whereas each vertical arrow (having a waving shaft) stands for a relationship “via deformation” — that involves “discrete parameters”.

7. POLYNOMIAL MULTIPARAMETER QUESA'S AND THEIR DEFORMATIONS

Our FoMpQUESA's are a close analogue, for multiparameter depending supergroups, of the well-known (uniparameter) QUEA's $U_\hbar(\mathfrak{g})$ by Drinfeld, attached with finite-dimensional semisimple Lie algebras $\mathfrak{g}$. Now, a *polynomial* version of Drinfeld's $U_\hbar(\mathfrak{g})$ is also available, originally due to Jimbo and Lusztig — and a multiparameter version of that is known as well, cf. [GG2]. Similarly, we will now introduce a “polynomial version” of MpQUESA's, and we will shortly present its main features.

7.1. Multiparameters of polynomial type.

Given $n \in \mathbb{N}_+$ and $I := \{1, \dots, n\}$, let $\mathbb{Z}[\mathbf{x}^{\pm 1}] := \mathbb{Z}[\{x_{ij}^{\pm 1}\}_{i,j \in I}]$ be the ring of Laurent polynomials with coefficients in $\mathbb{Z}$ in the indeterminates x_{ij} ($i, j \in I$), and let $A := (a_{ij})_{i,j \in I}$ be an indecomposable, symmetrizable Cartan matrix as in

§4.2. Consider the quotient ring $\mathbb{Z}_A[\mathbf{q}^{\pm 1}] := \mathbb{Z}[\mathbf{x}^{\pm 1}] / \left(\{x_{ij} x_{ji} - x_{ii}^{a_{ij}}\}_{i,j \in I} \right)$ and denote by q_{ij} the coset of every x_{ij} (for $i, j \in I$) in $\mathbb{Z}_A[\mathbf{q}^{\pm 1}]$. This is the ring of global sections of an affine scheme over $\mathbb{Z}$, call it $\mathfrak{C}_A$.

From all the identities $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ in $\mathbb{Z}_A[\mathbf{q}^{\pm 1}]$, from the indecomposability of A and from the symmetry of DA — i.e., $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$ — one finds that there exists $j_0 \in I$ such that $q_{ii} = q_{j_0 j_0}^{d_i}$. From this and the relations between the q_{ij} 's, it is easy to argue that $\mathfrak{C}_A$ is a torus, of dimension $\binom{n}{2} + 1$: in particular, it is irreducible, hence $\mathbb{Z}_A[\mathbf{q}^{\pm 1}]$ is a domain.

Finally, we extend the ring $\mathbb{Z}_A[\mathbf{q}^{\pm 1}]$ using formal square roots, as follows. First, we consider the ring extension $\dot{R}_{\mathbf{q}}$ of $\mathbb{Z}_A[\mathbf{q}^{\pm 1}]$ generated by a square root of $q_{j_0 j_0}$, namely $\dot{R}_{\mathbf{q}} := (\mathbb{Z}_A[\mathbf{q}^{\pm 1}])[x] / (x^2 - q_{j_0 j_0})$ so that $q := q_{j_0 j_0}^{1/2} := \bar{x} \in \dot{R}_{\mathbf{q}}$. We still denote by q_{ij} the images in $\dot{R}_{\mathbf{q}}$ of the “old” elements with same name in $\mathbb{Z}_A[\mathbf{q}^{\pm 1}]$. Note that $\dot{R}_{\mathbf{q}} \cong (\mathbb{Z}[\{x_{ij}^{\pm 1}\}_{i,j \in I}])[x] / \left(\{x_{ij} x_{ji} - x_{ii}^{a_{ij}}\}_{i,j \in I} \cup \{x^2 - q_{j_0 j_0}\} \right)$. In turn, we define also

$$\dot{R}_{\mathbf{q}}^{\vee} := \mathbb{Z}[\{\xi_{ij}^{\pm 1/2}\}_{i,j \in I}][\xi^{\pm 1/2}] / \left(\{\xi_{ij}^{1/2} \xi_{ji}^{1/2} - (\xi_{ii}^{1/2})^{a_{ij}}\}_{i,j \in I} \cup \{(\xi^{1/2})^2 - \xi_{j_0 j_0}^{1/2}\} \right)$$

which is again a domain, and we denote by $q_{ij}^{\pm 1/2}$ and by $q^{\pm 1/2}$ the coset of $\xi_{ij}^{\pm 1/2}$ and $\xi^{\pm 1/2}$, respectively, in $\dot{R}_{\mathbf{q}}^{\vee}$. Then set also $q_i^{\pm 1} := q^{\pm d_i}$ for all $i \in I$; in particular, $q_{j_0 j_0}^{1/2} := q = q_{j_0}$. Finally, inside the field of fractions of $\dot{R}_{\mathbf{q}}^{\vee}$ we consider the subring $R_{\mathbf{q}}^{\vee}$ generated by $\dot{R}_{\mathbf{q}}^{\vee}$ and the elements $(q_i^{+1} - q_i^{-1})^{-1}$, for all $i \in I$.

At last, for later use we introduce notation $k_{ij} := q_{ij}^{1/2} q_{ji}^{-1/2}$ for all $i, j \in I$.

7.2. Polynomial multiparameter QUESA's.

We resume all assumptions and notation as in §4.2 and §7.1 above. In particular, we fix the multiparameter matrix $\mathbf{q} := (q_{ij})_{i,j \in I}$ and the ring $R_{\mathbf{q}}^{\vee}$ associated with some fixed symmetrizable, indecomposable Cartan matrix $A = (a_{ij})_{i,j \in I}$ as in §4.2: we shall then say that the multiparameter $\mathbf{q} := (q_{ij})_{i,j \in I}$ is of *Cartan type* A .

Definition 7.2.1. Let $(A := (a_{i,j})_{i,j \in I}, p)$ be a fixed Cartan super-datum, and $\mathbf{q} := (q_{ij})_{i,j \in I}$ a multiparameter matrix of Cartan type A ,

(a) We define the *polynomial multiparameter quantum universal super enveloping algebra* — in short *polynomial MpQUESA*, or simply *PolMpQUESA* — with multiparameter $\mathbf{q}$ as follows. It is the unital, associative $R_{\mathbf{q}}^{\vee}$ -superalgebra $U_{\mathbf{q}}(\mathfrak{g})$ generated by the elements $K_i^{\pm 1}$, $L_i^{\pm 1}$, E_i , F_i (for all $i \in I$), with $\mathbb{Z}_2$ -grading given

by setting $|K_i^{\pm 1}| := \bar{0} =: |L_i^{\pm 1}|$, $|E_i| := p(i) =: |F_i|$ ($\forall i \in I$), and with relations

$$\begin{aligned} K_i^{\pm 1} L_j^{\pm 1} &= L_j^{\pm 1} K_i^{\pm 1}, & K_i^{\pm 1} K_i^{\mp 1} &= 1 = L_i^{\pm 1} L_i^{\mp 1} \\ K_i^{\pm 1} K_j^{\pm 1} &= K_j^{\pm 1} K_i^{\pm 1}, & L_i^{\pm 1} L_j^{\pm 1} &= L_j^{\pm 1} L_i^{\pm 1} \\ K_i E_j K_i^{-1} &= q_{ij} E_j, & L_i E_j L_i^{-1} &= q_{ji}^{-1} E_j \\ K_i F_j K_i^{-1} &= q_{ij}^{-1} F_j, & L_i F_j L_i^{-1} &= q_{ji} F_j \\ [E_i, F_j] &= \delta_{i,j} \frac{K_i - L_i}{q_i^{+1} - q_i^{-1}} \end{aligned}$$

as well as relations (4.5), (4.6), (4.7) and all of relations (4.8) through (4.12) both for the E_i 's and for the F_i 's alike.

(b) We define the Cartan subalgebra $U_{\mathbf{q}}(\mathfrak{h})$ of a PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ as being the unital $R_{\mathbf{q}}^{\vee}$ -subsuperalgebra of $U_{\mathbf{q}}(\mathfrak{g})$ generated by the subset $\{K_i^{\pm 1}, L_i^{\pm 1} \mid i \in I\}$.

(c) We define the positive, resp. negative, nilpotent subsuperalgebra $U_{\mathbf{q}}(\mathfrak{n}_+)$, resp. $U_{\mathbf{q}}(\mathfrak{n}_-)$, of a PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ to be the unital $R_{\mathbf{q}}^{\vee}$ -subsuperalgebra of $U_{\mathbf{q}}(\mathfrak{g})$ generated by the subset $\{E_i \mid i \in I\}$, resp. the subset $\{F_i \mid i \in I\}$.

(d) We define the positive, resp. negative, Borel subsuperalgebra $U_{\mathbf{q}}(\mathfrak{b}_+)$, resp. $U_{\mathbf{q}}(\mathfrak{b}_-)$, of a PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ to be the unital $R_{\mathbf{q}}^{\vee}$ -subsuperalgebra of $U_{\mathbf{q}}(\mathfrak{g})$ generated by the subset $\{E_i, K_i^{\pm 1} \mid i \in I\}$, resp. the subset $\{F_i, L_i^{\pm 1} \mid i \in I\}$.

(e) The above are definitions for PolMpQUESA's (or their subalgebras) over the “generic multiparameter” $\mathbf{q}$ with entries in $R_{\mathbf{q}}^{\vee}$. If $\bar{\mathbf{q}} = (\bar{q}_{ij})_{i,j \in I}$ is any square matrix of size n with entries in $\mathbb{k}^{\times}$ which are not roots of 1, we can define the “specialized” PolMpQUESA $U_{\bar{\mathbf{q}}}(\mathfrak{g})$ as a $\mathbb{k}$ -algebra which has exactly the same presentation of $U_{\mathbf{q}}(\mathfrak{g})$ but for the $\bar{q}_{ij}$'s replacing the q_{ij} 's. Similarly, we define “specialized” Cartan, nilpotent and Borel subsuperalgebras $U_{\bar{\mathbf{q}}}(\mathfrak{h})$, $U_{\bar{\mathbf{q}}}(\mathfrak{n}_{\pm})$ and $U_{\bar{\mathbf{q}}}(\mathfrak{b}_{\pm})$.

Remark: The definitions in (a) through (d) are given over the ground ring $R_{\mathbf{q}}^{\vee}$ that is the least one for which the definitions make sense. Another natural choice would be to pick as ground ring the field of fractions of $R_{\mathbf{q}}^{\vee}$. The further results hereafter then still hold true, up to minimal adjustments. $\diamond$

The key point about PolMpQUESA's is the following:

Theorem 7.2.2. *Every PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ is a Hopf superalgebra over $R_{\mathbf{q}}^{\vee}$, with coproduct, counit and antipode given by (for all $i \in I$)*

$$\begin{aligned} \Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1}, & \epsilon(K_i^{\pm 1}) &= 1, & \mathcal{S}(K_i^{\pm 1}) &= K_i^{\mp 1} \\ \Delta(L_i^{\pm 1}) &= L_i^{\pm 1} \otimes L_i^{\pm 1}, & \epsilon(L_i^{\pm 1}) &= 1, & \mathcal{S}(L_i^{\pm 1}) &= L_i^{\mp 1} \\ \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \epsilon(E_i) &= 0, & \mathcal{S}(E_i) &= -K_i^{-1} E_i \\ \Delta(F_i) &= F_i \otimes L_i + 1 \otimes F_i, & \epsilon(F_i) &= 0, & \mathcal{S}(F_i) &= -F_i L_i^{-1} \end{aligned}$$

and Borel subsuperalgebras $U_{\mathbf{q}}(\mathfrak{b}_+)$ and $U_{\mathbf{q}}(\mathfrak{b}_-)$ are Hopf sub-superalgebras of $U_{\mathbf{q}}(\mathfrak{g})$.

Proof. Let DA the symmetrized Cartan matrix associated with the multiparameter $\mathbf{q}$ (see above). Let $\mathbb{K}_n := \mathbb{k}(\{x_{ij} \mid i, j \in I, i < j\})$ be the field of rational functions in $\binom{n}{2}$ indeterminates x_{ij} — with $i < j$ in $I := \{1, \dots, n\}$ — let $\mathbb{K}_n[[\hbar]]$ the associated ring of formal power series in $\hbar$, and let $\mathbb{K}_n((\hbar))$ be the field of formal Laurent series in $\hbar$. Now pick the matrix $P_{\underline{x}} := (p_{ij})_{i,j \in I} \in M_n(\mathbb{K}_n[[\hbar]])$ given by

$$p_{ij} := d_i a_{ij} + x_{ij} \quad \forall i < j, \quad p_{ii} := d_i a_{ii}, \quad p_{ij} := d_i a_{ij} - x_{ji} \quad \forall i > j$$

so that $P_{\underline{x}} + P_{\underline{x}}^T = 2D A$. Then, there exists a unique $\mathbb{k}$ -algebra morphism, say $\varphi : R_{\mathbf{q}}^{\vee} \longrightarrow \mathbb{K}_n((\hbar))$, given by $q_{ij} \mapsto \varphi(q_{ij}) = \exp(\hbar p_{ij})$, which is clearly injective since the q_{hk} 's only obey the relations $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ and the p_{hk} 's the relations $p_{ij} + p_{ji} = 2d_i a_{ij}$.

Choose any realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ of $P_{\underline{x}}$ over the ground ring $\mathbb{K}_n[[\hbar]]$, which is *straight* and *split* — its existence is granted by Proposition 2.1.5(a). Out of these data, we define the FoMpQUESA $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ still over the ground ring $\mathbb{K}_n[[\hbar]]$, and also its scalar extension $\mathbb{K}_n((\hbar)) \otimes_{\mathbb{K}_n[[\hbar]]} U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$. Using the monomorphism $\varphi : R_{\mathbf{q}}^{\vee} \longrightarrow \mathbb{K}_n((\hbar))$ mentioned above, by scalar restriction through it we consider $\mathbb{K}_n((\hbar)) \otimes_{\mathbb{K}_n[[\hbar]]} U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ as an $R_{\mathbf{q}}^{\vee}$ -superalgebra too.

Inside $\mathbb{K}_n((\hbar)) \otimes_{\mathbb{K}_n[[\hbar]]} U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ we look at the $R_{\mathbf{q}}^{\vee}$ -subsuperalgebra $\dot{U}_{\mathbf{q}}(\mathfrak{g})$ generated by $\left\{ \dot{E}_i, \dot{K}_i^{\pm 1} := \exp(\pm \hbar T_i^+), \dot{L}_i^{\pm 1} := \exp(\mp \hbar T_i^-), \dot{F}_i \mid i \in I \right\}$. An easy calculation, along with the fact that the set $\Pi^\vee = \{T_i^+, T_i^- \mid i \in I\}$ is linearly independent, proves that $U_{\mathbf{q}}(\mathfrak{g})$ is actually contained in $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$; moreover, the defining presentation for $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ induces for $\dot{U}_{\mathbf{q}}(\mathfrak{g})$ the same presentation as for $U_{\mathbf{q}}(\mathfrak{g})$. In short, $\dot{U}_{\mathbf{q}}(\mathfrak{g})$ is an isomorphic copy of $U_{\mathbf{q}}(\mathfrak{g})$ embedded inside $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$, via $\Phi : U_{\mathbf{q}}(\mathfrak{g}) \xrightarrow{\cong} \dot{U}_{\mathbf{q}}(\mathfrak{g}) \left(E_i \mapsto \dot{E}_i, K_i^{\pm 1} \mapsto \dot{K}_i^{\pm 1}, L_i^{\pm 1} \mapsto \dot{L}_i^{\pm 1}, F_i \mapsto \dot{F}_i \right)$.

Yet another computation yields $\Delta(\dot{K}_i^{\pm 1}) = \dot{K}_i^{\pm 1} \otimes \dot{K}_i^{\pm 1}$ and $\Delta(\dot{L}_i^{\pm 1}) = \dot{L}_i^{\pm 1} \otimes \dot{L}_i^{\pm 1}$ inside $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$: together with similar formulas for the antipode (and for the E_i 's and the F_i 's as well), this implies that the topological Hopf structure of $U_{P, \hbar}^{\mathcal{R}, p}(\mathfrak{g})$ does restrict to a (non-topological) Hopf structure on $\dot{U}_{\mathbf{q}}(\mathfrak{g})$. The outcome is that $\dot{U}_{\mathbf{q}}(\mathfrak{g})$ is indeed a (non-topological!) Hopf superalgebra over $R_{\mathbf{q}}^{\vee}$, hence the same holds true for $U_{\mathbf{q}}(\mathfrak{g})$ via pull-back of the structure through the superalgebra isomorphism Φ mentioned above. Finally, tracking the whole construction we see that this Hopf structure onto $U_{\mathbf{q}}(\mathfrak{g})$ is indeed described by the formulas in the main claim.

The last claim about Borel subsuperalgebras then is clear. $\square$

The proof of next claim is contained in that of Theorem 7.2.2 right above:

Corollary 7.2.3. *Every polynomial MpQUESA embeds as a Hopf sub-superalgebra inside (the suitable scalar extension of) any split straight formal MpQUESA defined by the same Cartan super-datum.* $\square$

Specializing the “generic parameters” q_{ij} to specific values in $\mathbb{k}$ we get at once

Corollary 7.2.4. *Every specialized PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ is a (non-topological) Hopf superalgebra over $\mathbb{k}$, with Hopf structure given as in Theorem 7.2.2.* $\square$

A special case in particular stands out:

Definition 7.2.5. We define the *standard* PolMpQUESA $U_{\check{\mathbf{q}}}(\mathfrak{g})$ as the specialized PolMpQUESA for the “standard multiparameter” $\check{\mathbf{q}} := (\check{q}_{ij} = q^{d_i a_{ij}})_{i, j \in I}$.

N.B.: this choice corresponds to specializing the parameters $x_{ij} = 0$ in the proof above, whereas the parameter q stands untouched.

Finally, from Theorem 4.2.13 we easily see that the triangular decompositions for formal MpQUESA's imply parallel decompositions for their polynomial counterparts

(just because the latter embed into the former, as shown in the proof of Theorem 7.2.2 above). The precise statement, whose proof is immediate, reads as follows:

Theorem 7.2.6. *Every PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ admits triangular decompositions*

$$\begin{aligned} U_{\mathbf{q}}(\mathfrak{n}_+) \otimes U_{\mathbf{q}}(\mathfrak{h}) \otimes U_{\mathbf{q}}(\mathfrak{n}_-) &\cong U_{\mathbf{q}}(\mathfrak{g}) \cong U_{\mathbf{q}}(\mathfrak{n}_-) \otimes U_{\mathbf{q}}(\mathfrak{h}) \otimes U_{\mathbf{q}}(\mathfrak{n}_+) \\ U_{\mathbf{q}}(\mathfrak{b}_+) \otimes U_{\mathbf{q}}(\mathfrak{b}_-) &\cong U_{\mathbf{q}}(\mathfrak{g}) \cong U_{\mathbf{q}}(\mathfrak{b}_-) \otimes U_{\mathbf{q}}(\mathfrak{b}_+) \end{aligned}$$

and similar decompositions also occur for Borel subsuperalgebras, namely

$$U_{\mathbf{q}}(\mathfrak{n}_{\pm}) \otimes U_{\mathbf{q}}(\mathfrak{h}) \cong U_{\mathbf{q}}(\mathfrak{b}_{\pm}) \cong U_{\mathbf{q}}(\mathfrak{h}) \otimes U_{\mathbf{q}}(\mathfrak{n}_{\pm})$$

A similar statement holds true for all specialized PolMpQUESA's as well. $\square$

7.3. Deformations of polynomial MpQUESA's.

We discuss now how the construction of deformations by (toral) twist or by (toral) polar-2-cocycles somehow “restrict” from formal MpQUESA's to *polynomial* ones — bearing in mind that the latter embed as (Hopf) sub-superalgebras of the former.

In short, stepping from the “formal” setup to the “polynomial” one, things get more tricky for deformations by (toral) twist, whereas they get instead simpler for deformations by (toral) polar-2-cocycles. We present these results a bit quickly, yet the interested reader will certainly be able to fill in further details as needed.

7.3.1. Deformations by toral twist for PolMpQUESA's. Let $U_{\mathbf{q}}(\mathfrak{g})$ be some PolMpQUESA, and $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ some split, straight FoMpQUESA such that the former embeds inside the latter, as in Corollary 7.2.3. Let us consider a toral twist for $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ as in §5.1.1, say $\mathcal{F}_{\Phi} := \exp\left(\hbar 2^{-1} \sum_{g,k=1}^t \phi_{gk} H_g \otimes H_k\right)$. This belongs to $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})^{\otimes 2}$, but not to $U_{\mathbf{q}}(\mathfrak{g})^{\otimes 2}$, unless in the trivial case $\Phi = (\phi_{gk})_{g,k \in \mathcal{G}} = 0$; hence in no way any such a toral twist for $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ can be seen as a toral twist for $U_{\mathbf{q}}(\mathfrak{g})$. Nevertheless, for suitable choices of $\Phi = (\phi_{gk})_{g,k \in \mathcal{G}} \in \mathfrak{so}_t(\mathbb{k}[[\hbar]])$, hence of $\mathcal{F}_{\Phi}$, the deformed coproduct Δ^{Φ} of $(U_{P,h}^{\mathcal{R},p}(\mathfrak{g}))^{\mathcal{F}_{\Phi}}$ maps $U_{\mathbf{q}}(\mathfrak{g})$ inside $U_{\mathbf{q}}(\mathfrak{g})^{\otimes 2}$, hence it is indeed a new, “twist-deformed” coproduct for the Hopf algebra $U_{\mathbf{q}}(\mathfrak{g})$.

In order to prove the above statement clearly, we note that, directly from the proof of Theorem 5.1.2, we see that the property that $\Delta^{\Phi}(U_{\mathbf{q}}(\mathfrak{g})) \subseteq U_{\mathbf{q}}(\mathfrak{g})^{\otimes 2}$ does hold true if and only if $\mathcal{L}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_{\ell}(H_g) \phi_{gk} H_k} \in U_{\mathbf{q}}(\mathfrak{h})$ and similarly $\mathcal{K}_{\Phi,\ell} := e^{+\hbar 2^{-1} \sum_{g,k=1}^t \alpha_{\ell}(H_g) \phi_{kg} H_k} \in U_{\mathbf{q}}(\mathfrak{h})$ — for all $\ell \in I$ — hence if and only if

$$2^{-1} \sum_{g,k=1}^t \alpha_{\ell}(H_g) \phi_{gk} H_k \in \text{Span}_{\mathbb{Z}}(\Pi^{\vee}) \ni 2^{-1} \sum_{g,k=1}^t \alpha_{\ell}(H_g) \phi_{kg} H_k \quad (\forall \ell \in I) \quad (7.1)$$

but since the matrix $\Phi = (\phi_{ij})_{i,j \in I}$ is antisymmetric, the double set of conditions above is equivalent to half of them, say the left-hand side conditions in (7.1) above.

Now, $\Pi^{\vee}$ can be completed to a $\mathbb{k}[[\hbar]]$ -basis of $\mathfrak{h}$ — by the splitness assumption — so let us assume from scratch that the basis $\{H_k\}_{k=1,\dots,t}$ actually include $\Pi^{\vee}$; even more, we assume that it coincide with it (i.e., the realization $\mathcal{R}$ is also *minimal*);

then the matrix $\Phi = (\phi_{ij})_{i,j \in I}$ can be written in blocks, say $\Phi = \begin{pmatrix} \Phi_+^+ & \Phi_+^- \\ \Phi_-^+ & \Phi_-^- \end{pmatrix}$ where $\Phi_{\varepsilon}^{\eta} = (\phi_{ij}^{\varepsilon,\eta})_{i,j \in I}$ with $\mathcal{F}_{\Phi} := \exp\left(\hbar 2^{-1} \sum_{\varepsilon,\eta \in \{+,-\}} \sum_{i,j=1}^n \phi_{ij}^{\varepsilon,\eta} T_i^{\varepsilon} \otimes T_j^{\eta}\right)$.

Then the necessary and sufficient conditions for having $\Delta^\Phi(U_{\mathbf{q}}(\mathfrak{g})) \subseteq U_{\mathbf{q}}(\mathfrak{g})^{\otimes 2}$ are

$$\sum_{i=1}^n 2^{-1} \phi_{ij}^{\varepsilon, \eta} \alpha_\ell(T_i^\varepsilon) \in \mathbb{Z} \quad \forall \ell, j \in I, \varepsilon, \eta \in \{+, -\} \quad (7.2)$$

Finally, recalling that $\alpha_\ell(T_i^+) = p_{i\ell}$ and $\alpha_\ell(T_i^-) = p_{\ell i}$, conditions (7.2) eventually read, in matrix form, as “integrality conditions” of the following form:

$$P^T \cdot \Phi_+^\eta \in 2M_n(\mathbb{Z}) \quad , \quad P \cdot \Phi_-^\eta \in 2M_n(\mathbb{Z}) \quad \forall \eta \in \{+, -\}$$

When these conditions are fulfilled (and only then), the deformation of $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ by the “compatible” toral twist $\mathcal{F}_\Phi$ does “restrict” to a deformation of the polynomial MpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ as well, that we denote by $U_{\mathbf{q}}(\mathfrak{g})^{(\mathcal{F}_\Phi)}$. The new coproduct, counit and antipode in $U_{\mathbf{q}}(\mathfrak{g})^{(\mathcal{F}_\Phi)}$ are described by formulas which are (a) unchanged for the $K_i^{\pm 1}$ ’s and $L_i^{\pm 1}$ ’s, and (b) the very same formulas as in the first part of the proof of Theorem 5.1.2 for the E_i ’s and F_i ’s — as those formulas now make sense as formulas inside $U_{\mathbf{q}}(\mathfrak{g})^{\otimes 2}$ or $U_{\mathbf{q}}(\mathfrak{g})$. Then the “polynomial counterpart” of Theorem 5.1.2 holds true, stating in particular that every “twist deformed” object $U_{\mathbf{q}}(\mathfrak{g})^{(\mathcal{F}_\Phi)}$ issued by this procedure — out of a given polynomial MpQUESA $U_{\mathbf{q}}(\mathfrak{g})$, through a toral twist $\mathcal{F}_\Phi$ which is “compatible” in the sense explained above — is yet another *polynomial* MpQUESA. And similar statements hold true for Borel subalgebras too.

7.3.2. Deformations by toral 2-cocycle for PolMpQUESA’s. Let again $U_{\mathbf{q}}(\mathfrak{g})$ and $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ be a PolMpQUESA and a split, straight and minimal FoMpQUESA, respectively, the former embedding into the latter as in Corollary 7.2.3.

Let $\sigma_\chi : U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) \times U_{P,h}^{\mathcal{R},p}(\mathfrak{g}) \longrightarrow \mathbb{k}(\hbar)$ be a toral polar-2-cocycle of $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ — cf. §5.2.7 — which by construction is defined starting from a $\mathbb{k}[[\hbar]]$ -bilinear map $\chi : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{k}[[\hbar]]$ that vanishes (both on the left and the right) on the S_i ’s ($i \in I$). Let us restrict this σ_χ to $U_{\mathbf{q}}(\mathfrak{g}) \times U_{\mathbf{q}}(\mathfrak{g})$: by construction, $\sigma_\chi|_{U_{\mathbf{q}}(\mathfrak{g}) \times U_{\mathbf{q}}(\mathfrak{g})}$ is uniquely determined by $\chi|_{U_{\mathbf{q}}(\mathfrak{h}) \times U_{\mathbf{q}}(\mathfrak{h})}$. Now, $U_{\mathbf{q}}(\mathfrak{h})$ is nothing but the group algebra over $R_{\mathbf{q}}$ of the free Abelian group $\Gamma \cong \mathbb{Z}^{2n}$ (generated by the K_i ’s and the L_i ’s); therefore $\chi|_{U_{\mathbf{q}}(\mathfrak{h}) \times U_{\mathbf{q}}(\mathfrak{h})}$ is uniquely determined by its restriction $\eta(\chi) := \chi|_{\Gamma \times \Gamma}$. A straightforward check shows that $\eta(\chi)$ is just a standard group 2-cocycle with values in $R_{\mathbf{q}}$, and likewise $\chi|_{U_{\mathbf{q}}(\mathfrak{h}) \times U_{\mathbf{q}}(\mathfrak{h})}$ is a *standard* Hopf 2-cocycle for the Hopf algebra $U_{\mathbf{q}}(\mathfrak{h})$, with values in ground ring $R_{\mathbf{q}}$, and the same holds for $\sigma_{\hat{\mathbf{q}}_\chi} := \sigma_\chi|_{U_{\mathbf{q}}(\mathfrak{g}) \times U_{\mathbf{q}}(\mathfrak{g})}$ — in particular, no more “polar” is involved, nor there is any need to resort to any larger ring than the fixed ground one. The outcome is that restricting to the PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ the deformation process onto the larger FoMpQUESA $U_{P,h}^{\mathcal{R},p}(\mathfrak{g})$ via the (toral) polar-2-cocycle σ_χ yields the same outcome as deforming $U_{\mathbf{q}}(\mathfrak{g})$ by its Hopf 2-cocycle $\sigma_{\hat{\mathbf{q}}_\chi}$ via the standard procedure in Hopf algebra theory.

By the way, note that (by construction) every such $\sigma_{\hat{\mathbf{q}}_\chi}$ vanishes on the subgroup Γ_0 of Γ generated by the products $K_i^{+1} L_i^{-1}$ ($i \in I$).

Conversely, starting from any standard group 2-cocycle $\hat{\sigma}_{\hat{\mathbf{q}}}$ on Γ with values in $R_{\mathbf{q}}$ which vanishes on the subgroup Γ_0 , we can uniquely extend it to a Hopf 2-cocycle of $U_{\mathbf{q}}(\mathfrak{h})$ and then even to a Hopf 2-cocycle $\sigma_{\hat{\mathbf{q}}}$ of $U_{\mathbf{q}}(\mathfrak{g})$ — adapting the recipe in §5.2.7. These Hopf 2-cocycles (in a standard sense: no “polar” is involved!) of $U_{\mathbf{q}}(\mathfrak{g})$

will in fact coincide with all those coming from restriction of toral polar-2-cocycles of $U_{P,\hbar}^{\mathcal{R},p}(\mathfrak{g})$, hence we will call these $\sigma_{\hat{\mathbf{q}}}$ “toral” as well.

To finish with, whenever we perform a deformation of $U_{\mathbf{q}}(\mathfrak{g})$ by any toral 2-cocycle $\sigma_{\hat{\mathbf{q}}}$ we get a genuine deformation of the polynomial MpQUESA $U_{\mathbf{q}}(\mathfrak{g})$, that we denote by $(U_{\mathbf{q}}(\mathfrak{g}))_{\sigma_{\hat{\mathbf{q}}}}$: its coalgebra structure is unchanged, whilst the new product is described by a straightforward (and easy), partial adaptation of formulas given in §5.2.9. Then the “polynomial counterpart” of Theorem 5.2.13 holds true, namely there exists an isomorphism $(U_{\mathbf{q}}(\mathfrak{g}))_{\sigma_{\hat{\mathbf{q}}}} \cong U_{\mathbf{q}*\hat{\mathbf{q}}}(\mathfrak{g})$ which is the identity on generators. In short, every toral 2-cocycle deformation of a PolMpQUESA is yet another PolMpQUESA, whose new multiparameter $\mathbf{q} * \hat{\mathbf{q}}$ is a suitable combination of the original one $\mathbf{q}$ and of the further parameters encoded within $\sigma_{\hat{\mathbf{q}}}$ — details can be easily deduced again from §5.2.9. In particular, one can prove that any PolMpQUESA $U_{\mathbf{q}}(\mathfrak{g})$ can be obtained from the standard PolMpQUESA $U_{\hat{\mathbf{q}}}(\mathfrak{g})$ by a 2-cocycle deformation, by choosing the parameter $\hat{\mathbf{q}}$ such that $\hat{\mathbf{q}} * \hat{\mathbf{q}} = \mathbf{q}$. Similar statements also hold true for the Borel PolMpQUESAs and their deformations by $\sigma_{\hat{\mathbf{q}}}$.

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^b DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS
UNIVERSIDAD NACIONAL DE LA PLATA — CMALP-CIC-CONICET
1900 LA PLATA, ARGENTINA — ggarcia@mate.unlp.edu.ar

AND

^b GUANGDONG-TECHNION ISRAEL INSTITUTE OF TECHNOLOGY,
NO. 241, DAXUE ROAD, SHANTOU, GUANGDONG PROVINCE, CHINA

[#] DIPARTIMENTO DI MATEMATICA,
UNIVERSITÀ DEGLI STUDI DI ROMA “TOR VERGATA” — INDAM / GNSAGA
VIA DELLA RICERCA SCIENTIFICA 1, I-00133 ROMA, ITALY — gavarini@mat.uniroma2.it

[‡] INDEPENDENT RESEARCHER,
ROMA, ITALY — margherita.paolini.mp@gmail.com