

# Monge-Ampère measures on contact sets

Eleonora Di Nezza & Stefano Trapani

## Abstract

Let  $(X, \omega)$  be a compact Kähler manifold of complex dimension  $n$  and  $\theta$  be a smooth closed real  $(1, 1)$ -form on  $X$  such that its cohomology class  $\{\theta\} \in H^{1,1}(X, \mathbf{R})$  is pseudoeffective. Let  $\varphi$  be a  $\theta$ -psh function, and let  $f$  be a continuous function on  $X$  with bounded distributional laplacian with respect to  $\omega$  such that  $\varphi \leq f$ . Then the non-pluripolar measure  $\theta_\varphi^n := (\theta + dd^c \varphi)^n$  satisfies the equality:

$$\mathbf{1}_{\{\varphi=f\}} \theta_\varphi^n = \mathbf{1}_{\{\varphi=f\}} \theta_f^n,$$

where, for a subset  $T \subseteq X$ ,  $\mathbf{1}_T$  is the characteristic function. In particular we prove that

$$\theta_{P_\theta(f)}^n = \mathbf{1}_{\{P_\theta(f)=f\}} \theta_f^n \quad \text{and} \quad \theta_{P_\theta[\varphi](f)}^n = \mathbf{1}_{\{P_\theta[\varphi](f)=f\}} \theta_f^n.$$

## 1 Introduction

Starting from the works of Zaharjuta [Z76] and Siciak [S77], that years later have been take over by Bedford and Taylor [BT, BT82, BT86], *envelopes of plurisubharmonic functions* started to be of interest and to play an important role in the development of the pluripotential theory on domains of  $\mathbb{C}^n$ .

When, relying on the Bedford and Taylor theory in the local case, the foundations of a pluripotential theory on compact Kähler manifolds has been developed [GZ05, GZ07], *envelopes of quasi-plurisubharmonic functions* started to be intensively studied.

As geometric motivations we can mention, among others, the study of geodesics in the space of Kähler metrics [Chen00, Dar17, Ber17, RWN17, CTW18, DDL1, CMc19] and the transcendental holomorphic Morse inequalities on projective manifolds [WN19].

The two basic (and related) questions are about the regularity of envelopes and the behaviour of their Monge-Ampère measures. To fix notations, let  $(X, \omega)$  be a compact Kähler manifold of complex dimension  $n$ ,  $\theta$  be a smooth closed real  $(1, 1)$ -form and let  $f$  be a function on  $X$  bounded from above. We are going to refer to  $f$  as “barrier function”. Then the “prototype” of an envelope construction is

$$P_\theta(f) := (\{u \in \text{PSH}(X, \theta), u \leq f\})^*.$$

Such a function is either a genuine  $\theta$ -plurisubharmonic function or identically  $-\infty$ . When  $f = -\mathbf{1}_T$  is the negative characteristic function of a subset  $T$ , then  $P_\theta(f) = f_T^*$  is the so called relative extremal function of  $T$  [GZ05]. When  $f = 0$  then  $P_\theta(0) = V_\theta$  is a distinguished potential with minimal singularities.

The study of such envelopes has lead to several works. We start summarizing them in the case of a smooth barrier function  $f$ .

The first result to mention is [Ber09] where the author proves that in the case  $\theta \in c_1(L)$  where  $L$  is a big line bundle over  $X$ , the envelope  $P_\theta(f)$  is  $C^{1,1}$  on  $\text{Amp}(\{\theta\})$  and moreover

$$\theta_{P_\theta(f)}^n = \mathbf{1}_{\{P_\theta(f)=f\}} \theta_f^n. \quad (1)$$

After [BD12], people started to work on possible generalisations of the above results in the case of a pseudoeffective class  $\{\theta\}$ , that does not necessarily represents the first Chern class of a line bundle. If we assume  $\{\theta\}$  big and nef, Berman [Ber19], using PDE methods, proved that the envelope  $P_\theta(f)$  is in  $C^{1,\alpha}$  on  $\text{Amp}(\{\theta\})$  for any  $\alpha \in (0, 1)$  and the identity in (1) holds. The optimal regularity  $C^{1,1}$  in the Kähler case was then proved independently by [T18] and [CZ19] while the big and nef case was settled in [CTW18].

For general pseudoeffective classes only the following inequality of measures is known [DDL1]:

$$\theta_{P_\theta(f)}^n \leq \mathbf{1}_{\{P_\theta(f)=f\}} \theta_f^n. \quad (2)$$

In the study of geodesics, [RWN17] introduced another type of envelope: given a  $\theta$ -plurisubharmonic function  $\varphi$ , the so called *maximal envelope* is defined as

$$P_\theta[\varphi](f) := \left( \lim_{C \rightarrow +\infty} P_\theta(\min(\varphi + C, f)) \right)^*.$$

In the same paper, under the assumption  $\{\theta\} = c_1(L)$ , they proved the equality

$$\theta_{P_\theta[\varphi](f)}^n = \mathbf{1}_{\{P_\theta[\varphi](f)=f\}} \theta_f^n. \quad (3)$$

In the case of general pseudoeffective classes, the inequality  $\leq$  in (3) was proved in [DDL2], whereas the equality is derived in [Mc19] when  $\varphi$  has analytic singularities.

In the literature, regularity questions about envelopes for functions  $f$  that are less regular have been also addressed: in the case  $\theta$  is a Kähler form, Darvas and Rubinstein [DR16] proved that if  $f$  is  $C^1/C^{1,\bar{1}}$ , its envelope is  $C^1/C^{1,\bar{1}}$  as well; while Guedj, Lu and Zeriahi [GLZ19] proved that if  $f$  is a continuous function,  $P_\theta(f)$  is also continuous. They also proved that its Monge-Ampère measure (w.r.t. any big class  $\{\theta\}$ ) is supported on the contact set  $\{P_\theta(f) = f\}$ .

In the present paper we prove the following theorem:

**Theorem 1.** *Let  $\theta$  be smooth closed real  $(1, 1)$ -form on  $X$  such that the cohomology class  $\{\theta\}$  is pseudoeffective. Let  $\varphi$  be a  $\theta$ -plurisubharmonic function and  $f \in C^{1,\bar{1}}(X)$ . Then the non-pluripolar product  $\theta_\varphi^n$  satisfies the equality*

$$\mathbf{1}_{\{\varphi=f\}} \theta_\varphi^n = \mathbf{1}_{\{\varphi=f\}} \theta_f^n.$$

In particular, when the barrier function  $f$  is in  $C^{1,\bar{1}}(X)$  we obtain the equality in (1) and (3) in the general pseudoeffective case. Note that, at the best of our knowledge, the equality in (3) is new even in the case of a Kähler class.

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## 2 Preliminaries

Let  $(X, \omega)$  be a compact Kähler manifold of dimension  $n$  and fix  $\theta$  a smooth closed real  $(1, 1)$ -form. A function  $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$  is called *quasi-plurisubharmonic* (qpsH for short) if locally  $\varphi = \rho + u$ , where  $\rho$  is smooth and  $u$  is a plurisubharmonic function. We say that  $\varphi$  is  $\theta$ -plurisubharmonic ( $\theta$ -psh for short) if it is quasi-plurisubharmonic and  $\theta_\varphi := \theta + i\partial\bar{\partial}\varphi \geq 0$  in the weak sense of currents on  $X$ . We let  $\text{PSH}(X, \theta)$  denote the space of all  $\theta$ -psh functions on  $X$ . The class  $\{\theta\}$  is *pseudoeffective* if  $\text{PSH}(X, \theta) \neq \emptyset$  and it is *big* if there exists  $\psi \in \text{PSH}(X, \theta)$  such that  $\theta + i\partial\bar{\partial}\psi \geq \varepsilon\omega$  for some  $\varepsilon > 0$ , or equivalently if  $\text{PSH}(X, \theta - \varepsilon\omega) \neq \emptyset$ .

When  $\theta$  is non-Kähler, elements of  $\text{PSH}(X, \theta)$  can be quite singular, and we distinguish the potential with the smallest singularity type in the following manner:

$$V_\theta := \sup\{u \in \text{PSH}(X, \theta) \text{ such that } u \leq 0\}.$$

A function  $\varphi \in \text{PSH}(X, \theta)$  is said to have minimal singularities if it has the same singularity type as  $V_\theta$ , i.e.,  $|\varphi - V_\theta| \leq C$  for some  $C > 0$ . Note that, given any  $\theta$ -psh function  $\varphi$  we have  $\varphi - \sup_X \varphi \leq V_\theta$ .

Given  $\theta^1, \dots, \theta^n$  closed smooth real  $(1, 1)$ -forms representing pseudoeffective cohomology classes and  $\varphi_j \in \text{PSH}(X, \theta^j)$ ,  $j = 1, \dots, n$ , following the construction of Bedford-Taylor [BT, BT82] in the local setting, it has been shown in [BEGZ10] that the sequence of positive measures

$$\mathbf{1}_{\cap_j \{\varphi_j > V_{\theta^j} - k\}} \theta_{\max(\varphi_1, V_{\theta^1} - k)}^1 \wedge \dots \wedge \theta_{\max(\varphi_n, V_{\theta^n} - k)}^n \quad (4)$$

has total mass (uniformly) bounded from above and is non-decreasing in  $k \in \mathbb{R}$ , hence converges weakly to the so called *non-pluripolar product*

$$\theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_n}^n.$$

The resulting positive measure does not charge pluripolar sets. In the particular case when  $\varphi_1 = \varphi_2 = \dots = \varphi_n = \varphi$  and  $\theta^1 = \dots = \theta^n = \theta$  we will denote by  $\theta_\varphi^n$  the non-pluripolar Monge-Ampère measure of  $\varphi$ . As a consequence of Bedford-Taylor theory it can be seen that the measures in (4) all have total mass less than  $\int_X \theta_{V_{\theta^1}}^1 \wedge \dots \wedge \theta_{V_{\theta^n}}^n$ , in particular, after letting  $k \rightarrow \infty$

$$\int_X \theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_n}^n \leq \int_X \theta_{V_{\theta^1}}^1 \wedge \dots \wedge \theta_{V_{\theta^n}}^n.$$

We recall that the *plurifine topology* is the weakest topology for which qpsH functions are continuous and that the non-pluripolar Monge-Ampère measure satisfies a locality condition with respect to the plurifine topology [BEGZ10, Section 1.2], i.e. if  $\varphi_j, \psi_j, j = 1, \dots, n$ , are  $\theta^j$ -psh functions such that  $\varphi_j = \psi_j$  on  $U$  an open set in the plurifine topology, then

$$\mathbf{1}_U \theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_n}^n = \mathbf{1}_U \theta_{\psi_1}^1 \wedge \dots \wedge \theta_{\psi_n}^n. \quad (5)$$

We also note that sets of the form  $\{\varphi < \psi\}$ , where  $\varphi, \psi$  are qpsH functions, are open in the plurifine topology.

In the following we are going to work with some well known envelope constructions:

$$P_\theta(f), P_\theta(f_1, \dots, f_k), P_\theta[\varphi](f), P_\theta[\varphi].$$

Given  $f, f_1, \dots, f_k$  functions on  $X$  bounded from above, we consider the “rooftop envelopes”

$$P_\theta(f) := (\sup\{v \in \text{PSH}(X, \theta), v \leq f\})^*$$

and

$$P_\theta(f_1, \dots, f_k) := P_\theta(\min(f_1, \dots, f_k)) = (\sup\{v \in \text{PSH}(X, \theta), v \leq \min(f_1, \dots, f_k)\})^*.$$

Then, given a  $\theta$ -psh function  $\varphi$ , the above procedure allows us to introduce

$$P_\theta[\varphi](f) := \left( \lim_{C \rightarrow +\infty} P_\theta(\varphi + C, f) \right)^*.$$

Note that by definition we have  $P_\theta[\varphi](f) = P_\theta[\varphi](P_\theta(f))$ . When  $f = 0$ , we will simply write  $P_\theta[\varphi] := P_\theta[\varphi](0)$ . We emphasize that the functions  $P_\theta(f)$ ,  $P_\theta(f_1, \dots, f_k)$  and  $P_\theta[\varphi](f)$  are either  $\theta$ -psh or identically equal to  $-\infty$ . Moreover, observe that if  $-C \leq f \leq C$ , then  $V_\theta - C \leq P_\theta(f) \leq V_\theta + C$ ; hence  $P_\theta(f)$  is a well defined  $\theta$ -psh function. Moreover, if we assume  $\varphi \leq f$  the following sequence of inequalities holds:

$$\varphi \leq P_\theta[\varphi](f) \leq P_\theta(f) \leq f. \quad (6)$$

As in [DR16, Section 2] we denote by  $C^{1,\bar{1}}(X)$  the space of continuous function with bounded distributional laplacian w.r.t.  $\omega$ . Elliptic regularity and Sobolev’s embedding theorem imply that  $C^{1,\bar{1}}(X) \subset W^{2,p} \subset C^{1,\alpha}$  for any  $p \geq 1$  and  $\alpha \in (0, 1)$ . Here  $W^{2,p}$  denotes the Sobolev space of functions with all derivatives up to second order in  $L^p$ . By Hölder inequality, any polynomial having as coefficients the second derivatives of  $f$  is in any  $L^q$ ,  $q \geq 1$ . In particular  $(\theta + dd^c f)^n = h \omega^n$  where  $h \in L^q$ .

### 3 Monge-Ampère measures

The starting point is given by the following Lemma that deeply relies on [Ber09, Theorem 3.4] and on [DR16, Theorem 2.5]:

**Lemma 3.1.** *Let  $f_1, f_2 \in C^{1,\bar{1}}(X)$ . Then  $P_\omega(f_1, f_2)$  is also  $C^{1,\bar{1}}$ , and for  $i = 1, 2$  the functions  $f_i$  and  $P_\omega(f_1, f_2)$  are equal up to second order at almost every point on the set  $\{P_\omega(f_1, f_2) = f_i\}$ . In particular, the functions  $f_1, f_2, P_\omega(f_1, f_2)$  are equal up to second order at almost every point on the set  $\{P_\omega(f_1, f_2) = f_1 = f_2\}$ . In particular*

$$\mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_1}^n = \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_2}^n.$$

Moreover the measures

$$\mathbf{1}_{\{P_\omega(f_1, f_2) = f_1\}} \omega_{f_1}^n, \quad \mathbf{1}_{\{P_\omega(f_1, f_2) = f_2\}} \omega_{f_2}^n, \quad \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_j}^n \quad (j = 1, 2)$$

are positive and

$$\begin{aligned} \omega_{P_\omega(f_1, f_2)}^n &= \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1\}} \omega_{f_1}^n + \mathbf{1}_{\{P_\omega(f_1, f_2) = f_2\}} \omega_{f_2}^n - \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_1}^n \\ &= \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1\}} \omega_{f_1}^n + \mathbf{1}_{\{P_\omega(f_1, f_2) = f_2\}} \omega_{f_2}^n - \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_2}^n \end{aligned}$$

*Proof.* Observe that  $P_\omega(f_1, f_2)$  is a genuine  $\omega$ -psh function. Moreover, by [DR16, Theorem 2.5] the function  $P_\omega(f_1, f_2) \in C^{1,1}(X)$ , hence the measures  $\omega_{f_1}^n$ ,  $\omega_{f_2}^n$  and  $\omega_{P_\omega(f_1, f_2)}^n$  are absolutely continuous with respect to the Lebesgue measure. We set  $\Psi_1 = P_\omega(f_1, f_2) - f_1$ ,  $\Psi_2 = P_\omega(f_1, f_2) - f_2$  and  $\Psi_3 = f_1 - f_2$ . Since the functions  $\Psi_i$  are  $C^1$  the sets  $A_i = \{\Psi_i = 0, d\Psi_i \neq 0\}$  are real hypersurfaces and they have Lebesgue measure zero. Since  $\Psi_i \in W^{2,p}$ , the set  $B_i$  where the real hessian matrix of  $\Psi_i$  does not exist also have Lebesgue measure zero. Finally by [KS, page 53, Lemma A4], the set  $C_i$  where  $\Psi_i = 0, d\Psi_i = 0$  and the real hessian matrix of  $\Psi_i$  exists but it is non zero, has Lebesgue measure zero as well. Then for  $1 \leq i \leq 3$ , the function  $\Psi_i$  is zero up to order two almost everywhere on the set  $\{\Psi_i = 0\}$ . Furthermore by [BT82, Corollary 9.2] the measure  $\omega_{P_\omega(f_1, f_2)}^n$  is supported on the set  $\{\Psi_1 = 0\} \cup \{\Psi_2 = 0\}$ . Set  $U_1 = \{f_1 < f_2\}, U_2 = \{f_1 > f_2\}$ , and note that  $U_1$  and  $U_2$  are open sets such that  $U_1 \cap \{\Psi_2 = 0\} = \emptyset$  and  $U_2 \cap \{\Psi_1 = 0\} = \emptyset$ . Then we can argue that

$$\mathbf{1}_{U_1 \cap \{\Psi_1 = 0\}} \omega_{P_\omega(f_1, f_2)}^n = \mathbf{1}_{U_1 \cap \{\Psi_1 = 0\}} \omega_{f_1}^n, \quad \mathbf{1}_{U_2 \cap \{\Psi_2 = 0\}} \omega_{P_\omega(f_1, f_2)}^n = \mathbf{1}_{U_2 \cap \{\Psi_2 = 0\}} \omega_{f_2}^n.$$

This implies that the measures  $\mathbf{1}_{U_1 \cap \{\Psi_1 = 0\}} \omega_{f_1}^n$  and  $\mathbf{1}_{U_2 \cap \{\Psi_2 = 0\}} \omega_{f_2}^n$  are positive. By the above argument, we can also guarantee that at almost every point of the set  $(\{\Psi_1 = 0\} \cup \{\Psi_2 = 0\}) \cap \{\Psi_3 = 0\} = \{P_\omega(f_1, f_2) = f_1 = f_2\}$  the functions  $f_1, f_2$  and  $P_\omega(f_1, f_2)$  coincide up to order two. Therefore we also have

$$\mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{P_\omega(f_1, f_2)}^n = \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_1}^n = \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_2}^n,$$

and in particular it follows that, for any  $j = 1, 2$ , the measure  $\mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_j}^n$  is positive. Combining all the above equalities we get that for any  $j = 1, 2$ ,

$$\begin{aligned} & \omega_{P_\omega(f_1, f_2)}^n \\ &= \mathbf{1}_{U_1 \cap \{\Psi_1 = 0\}} \omega_{f_1}^n + \mathbf{1}_{U_2 \cap \{\Psi_2 = 0\}} \omega_{f_2}^n + \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_j}^n \\ &= \mathbf{1}_{\{\Psi_1 = 0\}} \omega_{f_1}^n - \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_1}^n + \mathbf{1}_{\{\Psi_2 = 0\}} \omega_{f_2}^n - \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_2}^n + \mathbf{1}_{\{P_\omega(f_1, f_2) = f_1 = f_2\}} \omega_{f_j}^n. \end{aligned}$$

□

**Theorem 3.2.** *Let  $\theta^1, \dots, \theta^n$  be smooth closed real  $(1, 1)$ -forms on  $X$  such that the cohomology classes  $\{\theta^1\}, \dots, \{\theta^n\}$  are pseudoeffective. For  $1 \leq i \leq n$ , let  $\varphi_i$  be a  $\theta^i$ -psh function and  $f_i$  be a  $C^{1,1}$  function on  $X$  such that  $\varphi_i \leq f_i$ . Then the non-pluripolar product  $\theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_n}^n$  satisfies the equality:*

$$\mathbf{1}_{\bigcap_j \{\varphi_j = f_j\}} \theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_n}^n = \mathbf{1}_{\bigcap_j \{\varphi_j = f_j\}} \theta_{f_1}^1 \wedge \dots \wedge \theta_{f_n}^n \quad (7)$$

The proof below is inspired by [WN19].

*Proof. Step 1.* We start proving the case when  $\theta := \theta^1 = \theta^2 = \dots = \theta^n$  is a Kähler form,  $\varphi := \varphi_1 = \varphi_2 = \dots = \varphi_n$  and  $f := f_1 = f_2 = \dots = f_n$  are both  $\theta$ -psh functions. Let  $\phi_j$  be a sequence of smooth functions on  $X$  decreasing to  $\varphi$  and define  $\psi_j := P_\theta(\phi_j, f)$ . Note that, since  $\varphi$  is  $\theta$ -psh and  $\varphi \leq \phi_j, f$ , we have

$$\varphi \leq \psi_j \leq \min(\phi_j, f) \leq f. \quad (8)$$

In particular,

$$\{\varphi = f\} \subseteq \{\psi_j = \min(\phi_j, f)\} \cap \{\min(\phi_j, f) = f\} = \{\psi_j = f\} \cap \{\phi_j \geq f\}. \quad (9)$$

Moreover, thanks to (8) we can infer that the functions  $\psi_j$  are decreasing to  $\varphi$  as  $j$  goes to  $+\infty$ . From Lemma 3.1 we then get

$$\theta_{\psi_j}^n = \mathbf{1}_{\{\psi_j=f\}}\theta_f^n + \mathbf{1}_{\{\psi_j=\phi_j\}}\theta_{\phi_j}^n - \mathbf{1}_{\{\psi_j=\phi_j=f\}}\theta_{\phi_j}^n \geq \mathbf{1}_{\{\psi_j=f\}}\theta_f^n \geq \mathbf{1}_{\{\varphi=f\}}\theta_f^n, \quad (10)$$

where the last inequality follows from (9).

Fix  $C > 0$  such that  $\min f > -C$  and  $g$  a non negative continuous function on  $X$ . By [GZ17, Theorem 4.26] we know that for any sequence of uniformly bounded quasi-continuous functions  $\chi_j$  converging in capacity to a bounded quasi-continuous function  $\chi$ , the measure  $\chi_j \theta_{\max(\psi_j, -C)}^n$  weakly converges to  $\chi \theta_{\max(\varphi, -C)}^n$  as  $j$  goes to  $+\infty$ . Fix  $\varepsilon > 0$ . We set

$$h_j^{C,\varepsilon} := \frac{\max(\psi_j + C, 0)}{\max(\psi_j + C, 0) + \varepsilon}, \quad h^{C,\varepsilon} := \frac{\max(\varphi + C, 0)}{\max(\varphi + C, 0) + \varepsilon}$$

and we observe that  $h_j^{C,\varepsilon}, h^{C,\varepsilon}$  are quasi-continuous (uniformly) bounded functions (with values in  $[0, 1]$ ) and that  $h_j^{C,\varepsilon}$  converges in capacities to  $h^{C,\varepsilon}$ . The last statement follows from the fact that

$$\tilde{h}_j^{C,\varepsilon} := \frac{\max(\varphi + C, 0)}{\max(\psi_j + C, 0) + \varepsilon} \leq h_j^{C,\varepsilon} \leq \frac{\max(\psi_j + C, 0)}{\max(\varphi + C, 0) + \varepsilon} := \hat{h}_j^{C,\varepsilon}$$

and  $\tilde{h}_j^{C,\varepsilon}, \hat{h}_j^{C,\varepsilon}$  are monotone sequences (increasing and decreasing, respectively) converging to  $h^{C,\varepsilon}$ .

Moreover, since  $h_j^{C,\varepsilon} = 0$  if  $\psi_j \leq -C$  and  $h^{C,\varepsilon} = 0$  if  $\varphi \leq -C$ , by the locality of the Monge-Ampère measure with respect to the plurifine topology [BEGZ10, Section 1.2], we have

$$h_j^{C,\varepsilon} g \theta_{\psi_j}^n = h_j^{C,\varepsilon} g \theta_{\max(\psi_j, -C)}^n, \quad \text{and} \quad h^{C,\varepsilon} g \theta_{\varphi}^n = h^{C,\varepsilon} g \theta_{\max(\varphi, -C)}^n.$$

Combining all the above we get

$$\begin{aligned} \int_X g \theta_{\varphi}^n &\geq \int_X h^{C,\varepsilon} g \theta_{\varphi}^n \\ &= \int_X h^{C,\varepsilon} g \theta_{\max(\varphi, -C)}^n \\ &= \lim_{j \rightarrow +\infty} \int_X h_j^{C,\varepsilon} g \theta_{\max(\psi_j, -C)}^n \\ &= \lim_{j \rightarrow +\infty} \int_X h_j^{C,\varepsilon} g \theta_{\psi_j}^n \\ &\geq \lim_{j \rightarrow +\infty} \int_{\{\varphi=f\}} h_j^{C,\varepsilon} g \theta_f^n \\ &= \int_{\{\varphi=f\}} h^{C,\varepsilon} g \theta_f^n \end{aligned}$$

where the last inequality follows from (10) while the last identity follows from the fact that convergence in capacity implies  $L^1$ -convergence [GZ17, Lemma 4.24]. Observe that since we choose  $\min f > -C$ , the functions  $h^{C,\varepsilon} > 0$  on  $\{\varphi = f\}$ . Letting  $\varepsilon \rightarrow 0$ , the dominated convergence theorem gives that

$$\int_X g \theta_{\varphi}^n \geq \int_{\{\varphi=f\}} g \theta_f^n.$$

Since the above inequality holds for any non negative continuous function  $g$ , by Riesz' representation theorem [Cohn, Theorem 7.2.8] we then we derive the inequality between measures

$$\theta_\varphi^n \geq \mathbf{1}_{\{\varphi=f\}} \theta_f^n. \quad (11)$$

Then, by [DDL4, Lemma 4.5] we get the equality

$$\mathbf{1}_{\{\varphi=f\}} \theta_\varphi^n = \mathbf{1}_{\{\varphi=f\}} \theta_f^n. \quad (12)$$

**Step 2.** The next step is to prove the equality in (12) when  $\theta$  is merely pseudoeffective and not necessarily Kähler. Also, we assume that there exists  $A > 0$  such that  $\theta + A\omega$  is a Kähler form and  $f$  is  $(\theta + A\omega)$ -psh. Observe that, since  $\varphi$  is  $\theta$ -psh function, then  $\varphi$  is also  $\theta + t\omega$ -psh, for  $t \geq 0$ . Let  $g \in C^0(X, \mathbf{R})$  and consider the function

$$Q(t) := \int_{\{\varphi=f\}} g(\theta + t\omega + dd^c \varphi)^n - \int_{\{\varphi=f\}} g(\theta + t\omega + dd^c f)^n$$

defined for  $t \geq 0$ . Then by multilinearity of the non-pluripolar product and the multilinearity of the product of forms, it is clear that  $Q(t)$  is a polynomial in  $t$  of the form:

$$Q(t) = \sum_{j=0}^n \left( \binom{n}{j} \int_{\{\varphi=f\}} g \left( (\theta + dd^c \varphi)^j - (\theta + dd^c f)^j \right) \wedge \omega^{n-j} \right) t^{n-j}.$$

Thanks to (7) we can infer that for any  $t > A$

$$\mathbf{1}_{\{\varphi=f\}} (\theta + t\omega + dd^c \varphi)^n = \mathbf{1}_{\{\varphi=f\}} (\theta + t\omega + dd^c f)^n.$$

This implies that the polynomial  $Q(t)$  is identically zero for  $t > A$ , hence  $Q(t) \equiv 0$ . It follows  $Q(0) = 0$ . Since  $g \in C^0(X, \mathbf{R})$  is arbitrary we have the desired equality between measures.

**Step 3.** We now prove equality (12) when  $f \in C^{1,\bar{1}}$  and not necessarily qps. Choose  $A > 0$  such that  $\theta + A\omega$  is a Kähler form. Since the function  $\varphi$  is  $(\theta + A\omega)$ -psh, then

$$\varphi \leq P_{\theta+A\omega}(f) \leq f. \quad (13)$$

In particular

$$\{\varphi = f\} = \{\varphi = P_{\theta+A\omega}(f)\} \cap \{P_{\theta+A\omega}(f) = f\}. \quad (14)$$

Observe that, even if  $P_{\theta+A\omega}(f)$  and  $f$  are not  $\theta$ -psh they are both  $C^{1,\bar{1}}$  [DR16, Theorem 2.5]. Lemma 3.1 applied to the functions  $f_1 = f_2 = f$  ensures that the functions  $P_{\theta+A\omega}(f)$  and  $f$  are equal up to second order at almost every point on the set  $\{P_{\theta+A\omega}(f) = f\}$ . Hence

$$\mathbf{1}_{\{P_{\theta+A\omega}(f)=f\}} \theta_{P_{\theta+A\omega}(f)}^n = \mathbf{1}_{\{P_{\theta+A\omega}(f)=f\}} \theta_f^n. \quad (15)$$

Moreover, since  $P_{\theta+A\omega}(f)$  is  $(\theta + A\omega)$ -psh, Step 2 ensures that

$$\mathbf{1}_{\{\varphi=P_{\theta+A\omega}(f)\}} \theta_\varphi^n = \mathbf{1}_{\{\varphi=P_{\theta+A\omega}(f)\}} \theta_{P_{\theta+A\omega}(f)}^n. \quad (16)$$

If we multiply (16) by  $\mathbf{1}_{\{P_{\theta+A\omega}(f)=f\}}$  and use (15) and (14) we derive (12).

**Step 4.** We now prove the general case. Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in (\mathbf{R}^+)^n$ . We set

$\theta^\lambda := \sum_j \lambda_j \theta^j$ ,  $\varphi_\lambda := \sum_j \lambda_j \varphi_j$ ,  $f_\lambda := \sum_j \lambda_j f_j$ . Observe that  $\varphi_\lambda$  is  $\theta^\lambda$ -psh and  $f_\lambda$  is still  $C^{1,\bar{1}}$  and that

$$\bigcap_{j=1}^n \{\varphi_j = f_j\} \subseteq \{\varphi_\lambda = f_\lambda\}. \quad (17)$$

By Step 3 and (17) we get

$$\mathbf{1}_{\bigcap_j \{\varphi_j = f_j\}} (\theta_{\varphi_\lambda}^\lambda)^n = \mathbf{1}_{\bigcap_j \{\varphi_j = f_j\}} (\theta_{f_\lambda}^\lambda)^n.$$

Using the multilinearity of the non-pluripolar product we get an identity between two polynomials in the variables  $\lambda_1, \dots, \lambda_n$ . Comparing the coefficients of  $\lambda_1 \cdots \lambda_n$  we obtain

$$\mathbf{1}_{\bigcap_j \{\varphi_j = f_j\}} \theta_{\varphi_1}^1 \wedge \dots \wedge \theta_{\varphi_n}^n = \mathbf{1}_{\bigcap_j \{\varphi_j = f_j\}} \theta_{f_1}^1 \wedge \dots \wedge \theta_{f_n}^n.$$

□

**Remark 3.3.** One can not expect Theorem 3.2 to hold when the barrier function  $f$  is singular. The following counterexample shows indeed that (7) does not hold when  $f$  is merely continuous.

Let  $\mathbb{B} \subset X$  be a small open ball and let  $\rho$  be a smooth potential such that  $\omega = dd^c \rho$  in a neighbourhood of  $\bar{\mathbb{B}}$ . We solve the Dirichlet problem

$$(dd^c(\rho + v))^n = 0 \quad \text{in } \mathbb{B}, \quad v|_{\partial \mathbb{B}} = 0.$$

Since the boundary data is continuous, [GZ17, Proposition 1.6 and Corollary 1.17] guarantees the existence of a continuous solution  $v \geq 0$  which is  $\omega$ -psh in  $\mathbb{B}$ . We then define

$$f := \begin{cases} v & \text{in } \mathbb{B} \\ 0 & \text{in } X \setminus \mathbb{B}. \end{cases}$$

By construction  $f$  is a continuous  $\omega$ -psh function and  $f \geq 0$ . On the other hand we observe that

$$\int_{X \setminus \mathbb{B}} \omega_f^n = \int_X \omega_f^n = \int_X \omega^n > \int_{X \setminus \mathbb{B}} \omega^n.$$

Since  $\{f = 0\} \subseteq X \setminus \mathbb{B}$ , we then deduce that the two measures  $\mathbf{1}_{\{f=0\}} \omega^n$  and  $\mathbf{1}_{\{f=0\}} \omega_f^n$  can not coincide.

**Corollary 3.4.** *Let  $\varphi \in \text{PSH}(X, \theta)$  and  $f \in C^{1,\bar{1}}(X)$  be such that  $\varphi \leq f$ . We have:*

- i)  $\theta_{P_\theta(f)}^n = \mathbf{1}_{\{P_\theta(f)=f\}} \theta_f^n$ .
- ii)  $\theta_{P[\varphi](f)}^n = \mathbf{1}_{\{P[\varphi](f)=f\}} \theta_f^n$ .
- iii)  $\mathbf{1}_{\{\varphi=P_\theta(f)\}} \theta_{P_\theta(f)}^n = \mathbf{1}_{\{\varphi=P_\theta(f)\}} \theta_\varphi^n$  and  $\mathbf{1}_{\{\varphi=P_\theta[\varphi](f)\}} \theta_{P_\theta[\varphi](f)}^n = \mathbf{1}_{\{\varphi=P_\theta[\varphi](f)\}} \theta_\varphi^n$ . Moreover, the measure  $\theta_\varphi^n$  is supported on the set  $\{\varphi = f\} \cup \{\varphi < P_\theta[\varphi](f)\}$ .
- iv) *The following conditions are equivalent:*
  - 1)  $\mathbf{1}_{\{\varphi < P_\theta[\varphi](f)\}} \theta_\varphi^n = 0$
  - 2) either  $\theta_\varphi^n = 0$ , or  $\varphi = P_\theta[\varphi](f)$ .



v) Assume  $\theta_\varphi^n > 0$ . The set  $\{P_\theta[\varphi](f) = f, \varphi < f\}$  has measure zero w.r.t.  $\theta_f^n$  if and only if  $\varphi = P[\varphi](f)$ .

*Proof.* Then statement in (i) immediately follows from Theorem 3.2 and [DDL1, Proposition 2.16].

Since  $P_\theta[\varphi](f) = P_\theta[\varphi](P_\theta(f))$  and  $\varphi \leq P_\theta(f)$ , by [DDL2, Theorem 3.8] we have

$$\theta_{P_\theta[\varphi](f)}^n \leq \mathbf{1}_{\{P_\theta[\varphi](f)=P_\theta(f)\}} \theta_{P_\theta(f)}^n = \mathbf{1}_{\{P_\theta[\varphi](f)=f\}} \theta_f^n,$$

where in the last equality we used (i). The opposite inequality follows from Theorem 3.2. This proves (ii).

Let's now prove (iii). By (i) and by Theorem 3.2,

$$\mathbf{1}_{\{\varphi=P_\theta(f)\}} \theta_{P_\theta(f)}^n = \mathbf{1}_{\{\varphi=P_\theta(f)=f\}} \theta_f^n = \mathbf{1}_{\{\varphi=P_\theta(f)=f\}} \theta_\varphi^n \leq \mathbf{1}_{\{\varphi=P_\theta(f)\}} \theta_\varphi^n. \quad (18)$$

The other inequality is given by [DDL4, Lemma 4.5]. In particular the inequality in (18) is in fact an equality, hence

$$\mathbf{1}_{\{P_\theta(f)=\varphi<f\}} \theta_\varphi^n = 0.$$

Using (ii) and Theorem 3.2, the same arguments of above give

$$\mathbf{1}_{\{\varphi=P_\theta[\varphi](f)\}} \theta_{P_\theta[\varphi](f)}^n = \mathbf{1}_{\{\varphi=P_\theta[\varphi](f)\}} \theta_\varphi^n \quad \text{and} \quad \mathbf{1}_{\{P_\theta[\varphi](f)=\varphi<f\}} \theta_\varphi^n = 0.$$

We now prove (iv). If  $\theta_\varphi^n = 0$  or  $\varphi = P_\theta[\varphi](f)$  then clearly  $\mathbf{1}_{\{\varphi < P_\theta[\varphi](f)\}} \theta_\varphi^n = 0$ . This proves that 2) implies 1). Viceversa we assume 1) and that  $\int_X \theta_\varphi^n > 0$ . By [DDL2, Remark 2.5]

$$\int_X \theta_\varphi^n = \int_X \theta_{P_\theta[\varphi](f)}^n = \int_X \theta_{P_\theta[\varphi]}^n. \quad (19)$$

The domination principle [DDL2, Proposition 3.11] gives the conclusion.

We finally prove (v). By assumption and by (ii) we have

$$\int_{\{\varphi < P_\theta[\varphi](f)\}} \theta_{P_\theta[\varphi](f)}^n = \int_{\{\varphi < P_\theta[\varphi](f)\} \cap \{P_\theta[\varphi](f)=f\}} \theta_f^n = 0.$$

Using (19), (iii) and the above we get that

$$\int_{\{\varphi < P_\theta[\varphi](f)\}} \theta_\varphi^n = \int_X \theta_\varphi^n - \int_{\{\varphi=P_\theta[\varphi](f)\}} \theta_\varphi^n = \int_{\{\varphi < P_\theta[\varphi](f)\}} \theta_{P_\theta[\varphi](f)}^n = 0.$$

Once again the conclusion follows from the domination principle [DDL2, Proposition 3.11].  $\square$

Next we give a formula relating the Monge-Ampère measure of  $P_\theta(f_1, \dots, f_k)$  to the  $(n, n)$ -forms  $\theta_{f_j}^n$ ,  $1 \leq j \leq k$ . Set  $R := \{P_\theta(f_1, \dots, f_k) = \min\{f_1, \dots, f_k\}\}$ , and let  $\mathcal{I}$  be the family of all non empty subsets of  $\{1, \dots, k\}$ . For  $I \in \mathcal{I}$ , we let

$$R_I = \{x \in X : f_i(x) = \min(f_1, \dots, f_k)(x) = P_\theta(f_1, \dots, f_k)(x) \text{ iff } i \in I\}.$$

Then  $\{R_I\}_{I \in \mathcal{I}}$  gives a partition of  $R$  by Borel sets.

**Proposition 3.5.** *Let  $f_1, \dots, f_k \in C^{1,\bar{1}}(X)$ . For  $I \in \mathcal{I}, I = \{i_1, i_2, \dots, i_r\}$  we have:*

$$\mathbf{1}_{R_I} \theta_{f_{i_1}}^n = \dots = \mathbf{1}_{R_I} \theta_{f_{i_r}}^n = \mathbf{1}_{R_I} \theta_{P_\theta(f_1, \dots, f_k)}^n := \mu_I.$$

Moreover,  $\theta_{P_\theta(f_1, \dots, f_k)}^n = \sum_{I \in \mathcal{I}} \mu_I$ .

*Proof.* By [DDL1, Proposition 2.16] the measure  $\theta_{P_\theta(f_1, \dots, f_k)}^n$  is supported on the contact set  $R$ . Moreover, for any  $h = 1, \dots, r$ , we have  $P_\theta(f_1, \dots, f_k) \leq f_{i_h}$  with equality on the set  $R_I$ . The conclusion follows from Theorem 3.2.  $\square$

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SORBONNE UNIVERSITÉ  
 eleonora.dinezza@imj-prg.fr

UNIVERSITÁ DI ROMA TORVERGATA  
 trapani@axp.mat.uniroma2.it