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Following Schubert varieties under Feigin's degeneration of the flag variety

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Abstract
We study the effect of Feigin's flat degeneration of the type A flag variety on the defining ideals of its Schubert varieties. In particular, we describe two classes of Schubert varieties which stay irreducible under the degenerations and in several cases we are able to encode reducibility of the degenerations in terms of symmetric group combinatorics. As a side result, we obtain an identification of some *degenerate Schubert varieties* (i.e. the vanishing sets of initial ideals of the ideals of Schubert varieties with respect to Feigin's Gröbner degeneration) with Richardson varieties in higher rank partial flag varieties.

Keywords Schubert varieties · Gröbner degenerations · Feigin's degeneration

1 Introduction

Let G be a complex simple Lie group and let $P \subset G$ be a parabolic subgroup. In [10], Feigin introduced a flat degeneration of the flag variety G/P, which is equipped with an action of the M-fold product of the additive group of the field (M being the dimension of a maximal unipotent subgroup of G).[1]

[1] Feigin's degeneration should not be confused with the toric degeneration of the flag variety that is associated to the Feigin–Fourier–Littelmann–Vinberg polytope. In fact, the degenerate flag variety studied in this paper is not toric, but does admit a degeneration to the toric variety associated with the FFLV polytope.

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These degenerations of flag varieties (and some generalizations in type A) have been intensively studied in the past years from many different perspectives (see, for example, [4–6, 9, 11, 18]).

In this paper, we deal with the effect of Feigin's degeneration on the Schubert varieties inside $\mathcal{F}\ell_n := SL_n/B$, for B the Borel subgroup of upper triangular matrices. In [10] it is shown that in type A the degeneration $\mathcal{F}\ell_n^a$ of $\mathcal{F}\ell_n$ can be embedded into a product of projective spaces, exactly as $\mathcal{F}\ell_n$, and that the defining ideal is generated by degenerate Plücker relations. More precisely, the defining ideal $\mathcal{I}_{\mathcal{F}\ell_n}$ of $\mathcal{F}\ell_n$ is generated by Plücker relations and the defining ideal $\mathcal{I}^a_{\mathcal{F}\ell_n}$ is obtained as the initial ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n})$ with respect to a weight vector $\mathbf{w}$ (whose components are indexed by Plücker coordinates), as described in Sect. 2.2.1. Moreover, if $v \in S_n$ is a permutation, it is well-known that the ideal $\mathcal{I}_v$ of the Schubert variety $X_v = \overline{BvB/B} \subseteq \mathcal{F}\ell_n$ is generated by the Plücker relations together with vanishing of certain Plücker coordinates (see Sect. 2.3 for a more precise formulation). Thus it is natural to ask what happens to $\mathcal{I}_v$ under Feigin's degeneration, that is to investigate $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$.

From the first non-trivial example, it is already clear that not all Schubert varieties under Feigin's degeneration will stay irreducible: for $n = 3$, indeed, one of the six Schubert varieties degenerates to a reducible variety. Therefore, a considerable part of this paper is directed towards understanding this reducibility phenomenon.

We recall that the cohomology ring of $\mathcal{F}\ell_n$ can be identified (after doubling the degree) with its Chow ring, and the latter is generated by Schubert classes. Moreover, it is shown in [5] that $\mathcal{F}\ell_n^a$ admits an affine paving and hence its cohomology ring can be identified (up to doubling the degree) with its Chow ring. Therefore we expect the above mentioned reducibility phenomenon to be related to the surjectivity of the cohomology ring map $\psi : H^*(\mathcal{F}\ell_n^a, \mathbb{Z}) \to H^*(\mathcal{F}\ell_n, \mathbb{Z})$ proven in [18]. It would be interesting to investigate this relationship, and in particular deduce a description of the kernel of ψ in terms of Schubert classes.

We should mention here that what we refer to as *Feigin's degeneration* is in fact a modified version of his original construction, which was coming from Lie theory. The version we deal with in this paper is the one which has been studied in [6]. The variety one obtains in this way is isomorphic to Feigin's original degeneration, but in some sense it behaves better with respect to Schubert varieties. In fact, Caldero noticed in [2] that there does not exist a (flat) toric degeneration of the flag variety under which all Schubert varieties degenerate to toric varieties. For $n = 3$ (which is the only case, apart from $n = 2$, in which $\mathcal{F}\ell_n^a$ is toric) our version of the degeneration preserves irreducibility of all but one Schubert variety, while two Schubert varieties would become reducible under Feigin's original definition. This is why we feel that in this setting the definition we use is sort of optimal.

Before focusing on Schubert varieties which become reducible after degenerating, we first describe some cases in which they stay irreducible (see Sect. 3). In particular, we prove that there is a class of Schubert varieties (indexed by permutations which are less or equal than a distinguished Coxeter element) whose defining ideals are not affected by the degeneration (see Proposition 2).

Section 4 is devoted to sufficient conditions on the permutation v such that the initial ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$ is not prime. The strategy is as follows: we look for Plücker relations whose initial term is a (degree 2) monomial when considered modulo the Plücker coordinates vanishing on $X_v^a := V(\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v))$, or, equivalently, vanishing on X_v. The efficiency of some of the conditions we give is then tested by looking at the $n = 4$ and $n = 5$ examples, for which we can detect all initial ideals containing monomials (see Tables 1 and 2).

In previous joint work with Cerulli Irelli [6], the second author proved that the degenerate flag variety $\mathcal{F}\ell_n^a$ can be embedded in the flag variety SL_{2n-2}/P of partial flags in $\mathbb{C}^{2n-2}$ consisting of odd dimensional spaces (that is, $P = P_{\omega_1+\omega_3+\ldots+\omega_{2n-3}}$). Under this embedding, it was shown in [6] that $\mathcal{F}\ell_n^a$ is isomorphic to a Schubert variety. From this fact (together with classical results) one could obtain a new proof of projective normality, Frobenius splitting, and rationality of the singularities of $\mathcal{F}\ell_n^a$. In Sect. 5 we further exploit such an isomorphism and study the effect of Feigin's degeneration on Schubert varieties inside SL_{2n-2}/P. The idea is to show irreducibility of the degeneration of some Schubert variety by proving that the above-mentioned embedding sends it to a Richardson variety. Although our main focus is the analysis of Plücker relations (cf. Sects. 4 and 3), for which there is no need to move to a higher rank (partial) flag variety, we decided to dedicate a section to the connection with Richardson varieties. By comparing Proposition 2 with Lemma 7 we obtain a realization of some Richardson varieties inside SL_{2n-2}/P as Schubert varieties in a lower rank (complete) flag variety.

The last section of the paper deals with Schubert divisors, that is Schubert varieties of codimension one in $\mathcal{F}\ell_n$. By applying our reducibility criteria from Sect. 4, we are able to prove that if n is even all Schubert divisors become reducible, while for n odd this happens for all but one. In this case, the remaining divisor is shown to be isomorphic to a Richardson variety inside SL_{2n-2}/P, and hence irreducible.

We want to point out that our paper is very different in spirit from [11], where irreducible flat degenerations of Schubert varieties corresponding to some special Weyl group elements (*triangular elements*) are produced by considering PBW-degenerations of Demazure modules $V_w(\lambda)$ and then realizing the desired degeneration as the closure of an appropriate $\mathbb{G}_a^M$-orbit inside $\mathbb{P}(V_w(\lambda))$. So for any Schubert variety which is indexed by a triangular element (see [11, Definition 1]) one can construct a flat irreducible degeneration via Fourier's procedure, while in this article we fix the degeneration (Feigin's) of the whole flag variety and study its effect on Schubert varieties (which are hence simultaneously degenerated).

Since a first draft of this paper appeared on the arxiv more research has been done regarding degenerations of Schubert varieties. Among them [7], where similar methods are employed to study Gröbner degenerations of Schubert varieties, and [3, 15] which are very different in flavour and closely related to [11].

2 Preliminaries and notation

2.1 Symmetric group combinatorics

The combinatorics of the symmetric group control many geometric properties of $\mathcal{F}\ell_n$ and its Schubert varieties, therefore we spend a little time here introducing the notation we will need later on.

For any two positive integers $i, j \in \mathbb{Z}_{\geq 1}$, with $i \leq j$ we denote by $[i, j] := \{a \in \mathbb{Z} \mid i \leq a \leq j\}$. Moreover, we use the short hand notation $[j] := [1, j]$. We write $\binom{[n]}{k}$ for the set of subsets of cardinality k inside $[n]$.

Let $n \geq 2$ and denote by S_n the symmetric group. Recall that the symmetric group S_n admits a presentation as a Coxeter group, with set of simple reflections $\{s_i \mid i = 1, \ldots, n-1\}$, where s_i denotes the transposition $(i, i+1)$. We will use the standard terminology and say that a product $s_{i_1} \ldots s_{i_r}$ is a reduced expression for $w \in S_n$ if $w = s_{i_1} \ldots s_{i_r}$ and all other expressions of w as a product of simple reflections $w = s_{j_1} \ldots s_{j_t}$ are such that $t \geq r$. In this case $r = \ell(w)$ is called the *length* of w. We denote by $\leq$ the Bruhat order on S_n and recall the following equivalent characterization (see, for example, [1, Theorem 2.1.5]): For $v \in S_n$ and $i, j \in [n]$ set

$$w^{i,j} = \#\{a \in [i] \mid w(a) \geq j\}. \tag{2.1}$$

Then

$$v \leq u \quad \Leftrightarrow \quad v^{i,j} \leq u^{i,j} \text{ for all } i, j. \tag{2.2}$$

Below we will also need that if $v \in S_n$ and $i \in [n-1]$, then

$$vs_i < v \quad \Leftrightarrow \quad v(i) > v(i+1),$$

or, equivalently,

$$s_i v < v \quad \Leftrightarrow \quad v^{-1}(i) > v^{-1}(i+1).$$

The symmetric group S_n acts on $\binom{[n]}{k}$ for any k: if $I = \{i_1, \ldots, i_k\} \in \binom{[n]}{k}$ then

$$v(I) := \{v(i_1), \ldots v(i_k)\}.$$

This action is transitive, so that $\binom{[n]}{k}$ is identified with the S_n-orbit of $[k]$ and hence with the set of minimal length coset representatives in $S_n/\langle s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_{n-1}\rangle$. In this way the Bruhat order on S_n induces a partial order on $\binom{[n]}{k}$ (see, for instance, [1, Proposition 2.5.1]) that we also denote by $\leq$. Such an order has an explicit description if we arrange the elements of the subsets in increasing order: let $I = \{i_1 < i_2 < \ldots < i_k\}$, $J = \{j_1 < j_2 < \ldots < j_k\} \in \binom{[n]}{k}$, then

$$I \leq J \quad \Leftrightarrow \quad i_t \leq j_t \text{ for all } t \in [k].$$

We will sometimes write elements $v \in S_n$ as $[v(1), v(2), \ldots, v(n)]$. This is referred to as the *one-line* notation.

2.1.1 Sequences

In the following sections, we will often need to deal with sequences $(i_1, \ldots, i_k)$ rather than sets $\{i_1, \ldots, i_k\}$. We denote by $\mathcal{S}(n, k)$ the set of sequences of k pairwise distinct numbers between 1 and n.

Given two sequences $I_1 = (i_1^{(1)}, \dots, i_k^{(1)}) \in \mathcal{S}(n,k)$, $I_2 = (i_1^{(2)}, \dots, i_l^{(2)}) \in \mathcal{S}(n,l)$ such that $I_1 \cap I_2 = \emptyset$, we denote by $(I_1, I_2) := (i_1^{(1)}, \dots, i_k^{(1)}, i_1^{(2)}, \dots i_l^{(2)}) \in \mathcal{S}(n, k+l)$ the sequence obtained by concatenation.

If $d \geq k$, $L \in \mathcal{S}(n,d)$ and $J = (j_1, \dots, j_k) \in \mathcal{S}(n,k)$, then the sequence $L' = (L \backslash (l_{r_1}, \dots, l_{r_k})) \cup (j_1, \dots j_k) \in \mathcal{S}(n,d)$ is obtained from L by replacing the subsequence $(l_{r_1}, \dots, l_{r_k})$ with $(j_1, \dots, j_k)$, that is $l'_a = l_a$ if $a \notin \{r_1, \dots, r_k\}$ and $l'_a = j_b$ if $a = r_b$. There is a forgetful map

$$F : \mathcal{S}(n,k) \to \binom{[n]}{k}, \quad (i_1, \dots, i_k) \mapsto \{i_1, \dots i_k\}.$$

By abuse of notation, if $I \in \mathcal{S}(n,k)$ and $v \in S_n$, we will write $I \leq v([k])$ instead of $F(I) \leq v([k])$ (and $I \geq v([k])$, $I \nleq v([k])$, etc., will have an analogous meaning).

2.1.2 A special Coxeter element

The Coxeter element $c = s_{n-1}s_{n-2}\cdots s_2 s_1 \in S_n$ will play an important role later on. Observe that in the one-line notation

$$c = [n, 1, 2, 3 \dots, n-1].$$

Thus, by [1, Proposition 2.4.8], for a subset $I \in \binom{[n]}{d}$ the following holds

$$I \leq c([d]) \Leftrightarrow I = [d-1] \cup \{b\} \text{ for } d \leq b \leq n. \tag{2.3}$$

2.2 Basics on the flag variety

Let $n \geq 2$. We denote by $\mathcal{F}\ell_n$ the variety of complete flags in $\mathbb{C}^n$. Let $(e_i)_{1 \leq i \leq n}$ be the standard basis of $\mathbb{C}^n$. Let $B \subset SL_n$ be the Borel subgroup of upper triangular matrices. The group SL_n acts transitively on $\mathcal{F}\ell_n$ and we can identify the flag variety with the quotient SL_n/B by looking at the SL_n-orbit of the standard flag $E_\bullet = (\{0\} \subset E_1 \subset \cdots \subset E_{n-1} \subset \mathbb{C}^n) \in \mathcal{F}\ell_n$ with

$$E_i := \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_i\} \quad (i = 1, \dots n-1).$$

Recall that under the left action of B, the flag variety decomposes as a union of Schubert cells indexed by the elements of the symmetric group S_n:

$$SL_n/B = \bigsqcup_{v \in S_n} BvB/B$$

where, by abuse of notation, v in BvB/B denotes the corresponding permutation matrix in SL_n. Finally, let X_v be the Schubert variety, that is the closure $\overline{BvB/B}$ of a Schubert cell.

Analogously, also B_-, the Borel subgroup of lower triangular matrices, acts by left multiplication on SL_n/B, providing the decomposition:

$$SL_n/B = \bigsqcup_{u \in S_n} B_- uB/B.$$

We denote by X^u the opposite Schubert variety $\overline{B_- uB/B}$. In Sect. 5, we will also consider Richardson varieties $X_v^u := X_v \cap X^u$.

2.2.1 Plücker relations

Our main reference for Plücker coordinates and relations is [12], while we refer to [10] for the degenerate Plücker relations.

We start by recalling the Plücker embedding of a Grassmannian. Recall that $(e_i)_{1 \leq i \leq n}$ is the standard basis of $\mathbb{C}^n$, so that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$$

is a basis of $\wedge^k \mathbb{C}^n$. Let $(\wedge^k \mathbb{C}^n)^*$ be the dual vector space, then the Plücker coordinate $p_{i_1,\ldots,i_k} \in (\wedge^k \mathbb{C}^n)^*$ for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is defined to be the basis element dual to $e_{i_1} \wedge \ldots \wedge e_{i_k}$. For $i_1, \ldots, i_k \in [n]$ pairwise distinct, but not necessarily increasing, the Plücker coordinate $p_{i_1,\ldots,i_k}$ has the following property

$$p_{\sigma(i_1),\ldots,\sigma(i_k)} = (-1)^{\ell(\sigma)} p_{i_1,\ldots,i_k} \quad \text{for all } \sigma \in S_n.$$

Denote by p_I the Plücker coordinate corresponding to a sequence $I = (i_1, \ldots, i_k) \in \mathcal{S}(n,k)$. In the following sections it will be sometimes convenient to simplify notation and index some Plücker coordinates by a set instead of a sequence. This has to be interpreted as being indexed by the sequence obtained by arranging the elements of the set in an increasing order.

The Plücker embedding is the map

$$\mathrm{Gr}(k, \mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^n) \tag{2.4}$$

which sends a k-dimensional subspace V of $\mathbb{C}^n$ to the collection of the images of V under the Plücker coordinates.

The flag variety is embedded into the product of Grassmannians

$$\mathcal{F}\ell_n \hookrightarrow \mathrm{Gr}(1, \mathbb{C}^n) \times \mathrm{Gr}(2, \mathbb{C}^n) \times \cdots \times \mathrm{Gr}(n-1, \mathbb{C}^n).$$

By composing the latter embedding with the embedding (2.4) for each Grassmannian in the product, we get

$$\mathcal{F}\ell_n \hookrightarrow \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2 \mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1} \mathbb{C}^n).$$

Denote by $\mathcal{I}_{\mathcal{F}\ell_n}$ the (homogeneous) ideal of $\mathcal{F}\ell_n$ in $\mathbb{C}[p_{i_1,\dots,i_k} \mid 1 \le i_1 < i_2 < \dots < i_k \le n,\ k \in [n-1]]$ with respect to this embedding. Then $\mathcal{I}_{\mathcal{F}\ell_n}$ is generated by elements in

$$\{R^k_{(j_1,\dots,j_e),(l_1,\dots,l_d)} \mid e \le d,\ k \in [e]\}$$

given by

$$R^k_{J,L} = p_J p_L - \sum_{1\le r_1<\dots<r_k\le d} p_{J'} p_{L'}, \tag{2.5}$$

where $L = (l_1,\dots,l_d) \in \mathcal{S}(n,d)$, $J = (j_1,\dots,j_e) \in \mathcal{S}(n,e)$, $L' = (L\backslash(l_{r_1},\dots,l_{r_k})) \cup (j_1,\dots,j_k)$ and $J' = (J\backslash(j_1,\dots,j_k)) \cup (l_{r_1},\dots,l_{r_k})$. The elements $R^k_{J,L}$ will be referred to as Plücker relations. To simplify notation we set

$$\mathcal{L}^k_{J,L} = \left\{ (J',L') \;\middle|\; \begin{smallmatrix} \exists 1\le r_1<\dots<r_k\le \#L, \\ J'=(J\backslash(j_1,\dots,j_k))\cup(l_{r_1},\dots,l_{r_k}), \\ L'=(L\backslash(l_{r_1},\dots,l_{r_k}))\cup(j_1,\dots,j_k) \end{smallmatrix} \right\}. \tag{2.6}$$

The weight vector $\mathbf{w} \in \mathbb{R}^{\binom{n}{1}+\dots+\binom{n}{n-1}}$ is defined componentwise by setting for $I = \{i_1,\dots,i_k\} \in \binom{[n]}{k}$

$$\mathbf{w}_I = \#\{r \mid k \le i_r \le n-1\}.$$

If $I_1,\dots,I_r \in \binom{[n]}{k}$, the $\mathbf{w}$-weight of the monomial $\prod_{t=1}^r p_{I_t}$ is $\sum_{t=1}^r \mathbf{w}_{I_t}$, while the initial form of a polynomial f consists of the sum of those monomials whose $\mathbf{w}$-weight is *minimal* among the weights of all monomials in f. Given an ideal $\mathcal{I} \subset \mathbb{C}[p_{i_1,\dots,i_d} \mid 1 \le i_1 < i_2 < \dots < i_d \le n, d \in [n-1]]$ its *initial ideal* is $\mathrm{in}_{\mathbf{w}}(\mathcal{I}) = (\mathrm{in}_{\mathbf{w}}(f) \mid f \in \mathcal{I})$. A (finite) set of elements in $\mathcal{I}$ whose initial forms generate $\mathrm{in}_{\mathbf{w}}(\mathcal{I})$ is called a *Gröbner basis*. A Gröbner basis for $\mathcal{I}_{\mathcal{F}\ell_n}$ whose elements are the Plücker relations (2.5) is computed in [10, Theorem 3.13]. The initial forms of its elements are given by

$$\mathrm{in}_{\mathbf{w}}(R^k_{J,L}) = p_J p_L - \sum_{\substack{(J',L')\in\mathcal{L}^k_{J,L} \\ \{l_{r_1},\dots,l_{r_k}\}\cap[e,d-1]=\emptyset}} p_{J'} p_{L'},$$

where the leading term is non-zero, only if

$$\{j_1,\dots,j_k\} \cap [e,d-1] = \emptyset. \tag{2.7}$$

We can choose J, L in such a way that (2.7) holds. Observe that for $e = d$, we always have $\mathrm{in}_{\mathbf{w}}(R^k_{J,L}) = R^k_{J,L}$ since the condition (2.7) is empty.

Definition 1 ([10]) The *degenerate flag variety* is the vanishing of the ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n})$, that is

$$\mathcal{F}\ell_n^a := V(\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{\mathcal{F}\ell_n})) \subset \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2\mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1}\mathbb{C}^n).$$

Remark 1 Feigin's original definition, valid for any simple Lie group, was different from the one we have just given, which is an equivalent characterization of the type A degenerate flag variety by [10, Theorem 3.13]. Explicitly, to obtain our degeneration from Feigin's original one, a global shift by -1 (modulo n) to all indices is needed. As already mentioned in the introduction, we modify Feigin's definition to match the one considered in [6], since we believe that it exhibits a better behavior with respect to Schubert varieties. Indeed, by [2] in any (flat) toric degeneration of the flag variety, under which all Schubert varieties degenerate to toric varieties, at least one of them becomes reducible. For $n = 3$ (which, together with $n = 2$, is the only case in which $\mathcal{F}\ell_n^a$ is toric) the version from [6] of the degeneration preserves irreducibility of all but one Schubert variety. Certainly, by this choice we loose some symmetry, as with Feigin's original definition we would have two Schubert varieties becoming reducible (thus a symmetry exchanging 1 and 2).

2.3 Ideals for Schubert varieties and their degeneration

For $v \in S_n$ the defining ideal of the Schubert variety $X_v \subset \mathcal{F}\ell_n$ is given by the vanishing of $(p_I)_{I \not\leq v([\#I])}$. It is shown in [17, §10.12] (see also [16, Theorem 3]) that by embedding $X_v \hookrightarrow \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2\mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1}\mathbb{C}^n)$, we obtain the ideal

$$\mathcal{I}_v := \mathcal{I}_{\mathcal{F}\ell_n} + (p_I)_{I \not\leq v([\#I])} \tag{2.8}$$

of $\mathbb{C}[p_{i_1,\ldots,i_d} \mid 1 \leq i_1 < i_2 < \ldots < i_d \leq n,\ d \in [n-1]]$. Feigin's degeneration of the flag variety induces a degeneration $X_v^a \subset \mathcal{F}\ell_n^a$ of any Schubert variety $X_v \subset \mathcal{F}\ell_n$:

$$X_v^a := V(\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)) \subset \mathbb{P}\mathbb{C}^n \times \mathbb{P}(\wedge^2\mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n-1}\mathbb{C}^n). \tag{2.9}$$

In what follows we study the initial ideals $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$ in detail. Note that $\mathrm{in}_{\mathbf{w}}(p_I) = p_I$ for all $I \in \mathcal{S}(n, d)$, for all $d \in [n-1]$. Moreover, we have an inclusion:

$$\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v) \supseteq (\mathrm{in}_{\mathbf{w}}(R_{J,L}^k))_{k,J,L} + (p_I)_{I \not\leq v([\#I])}. \tag{2.10}$$

The following example shows that this inclusion may be strict. In the proof of Theorem 1 instead we will encounter examples of (2.10) being an equality.

Example 1 Consider the ideal $\mathcal{I}_{\mathcal{F}\ell_4}$. Among its Plücker relations we have

$$p_4 p_{123} - p_3 p_{124} + p_2 p_{134} - p_1 p_{234}.$$

The first two terms have $\mathbf{w}$-weight 1 while the last two have $\mathbf{w}$-weight 2, so its initial form is $p_4 p_{123} - p_3 p_{124}$. Now consider $v = s_1 s_2 s_3 \in S_4$, which in the one-line notation is [2, 3, 4, 1]. Hence, $\{p_I\}_{I \not\leq v([\#I])} = \{p_3, p_4, p_{14}, p_{24}, p_{34}\}$. In particular, this implies that $f := p_2 p_{134} - p_1 p_{234} \in \mathcal{I}_v$ and by definition its initial form lies in $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$. As both monomials have the same $\mathbf{w}$-weight, f is equal to its initial form. Notice however that f does not lie in $(\mathrm{in}_{\mathbf{w}}(R^k_{J,L}))_{k,J,L} + (p_I)_{I \not\leq v([\#I])}$. This demonstrates that the containment in (2.10) is strict in general.

3 Two classes of irreducible X_v^a

We investigate two classes of Schubert varieties which degenerate to irreducible varieties. In this section we use some basics on Gröbner bases which we summarize for completeness. For more details we refer to [14, 20].

A *term order* on $\mathbb{C}[x_1, \dots, x_n]$ is a total order $<$ on the set of monic monomials in $\mathbb{C}[x_1, \dots, x_n]$ such that for every α, β, γ in $\mathbb{Z}^n_{\geq 0}$ we have that

$$\text{(i) } 1 \leq \mathbf{x}^\alpha, \text{ and (ii) if } \mathbf{x}^\alpha < \mathbf{x}^\beta, \text{ then } \mathbf{x}^{\alpha+\gamma} < \mathbf{x}^{\beta+\gamma}.$$

The *initial monomial* of an element $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ with respect to $<$ is $\mathrm{in}_<(f) := \max_<\{\mathbf{x}^\alpha \mid c_\alpha \neq 0\}$. The *initial ideal* of an ideal $J \subseteq \mathbb{C}[x_1, \dots, x_n]$ with respect to $<$ is defined as $\mathrm{in}_<(J) := (\mathrm{in}_<(f) \mid f \in J)$.

Let $\mathrm{in}_<(J)$ be a monomial initial ideal of the ideal J for some term order $<$ on $\mathbb{C}[x_1, \dots, x_n]$. Then the set $\mathbb{B}_< := \{\bar{\mathbf{x}}^\alpha \mid \mathbf{x}^\alpha \notin \mathrm{in}_<(J)\}$ is a vector space basis of $\mathbb{C}[x_1, \dots, x_n]/J$ (and $\mathbb{C}[x_1, \dots, x_n]/\mathrm{in}_<(J)$) called *standard monomial basis*, see e.g. [20, Proposition 1.1].

Let $<$ be a term order on $\mathbb{C}[x_1, \dots, x_n]$ and $\mathcal{G} = \{g_1, \dots, g_s\}$ a finite generating set of an ideal J. Then the *S-polynomial* of g_i and g_j is defined as

$$S(g_i, g_j) := \frac{\mathrm{lcm}(\mathrm{in}_<(g_i), \mathrm{in}_<(g_j))}{\mathrm{in}_<(g_i)} g_i - \frac{\mathrm{lcm}(\mathrm{in}_<(g_i), \mathrm{in}_<(g_j))}{\mathrm{in}_<(g_j)} g_j.$$

Buchberger's criterion says that $\mathcal{G}$ is a Gröbner basis if and only if for all $1 \leq i < j \leq s$ the S-polynomial $S(g_i, g_j)$ reduces to zero with respect to $\{g_1, \dots, g_s\}$, see e.g [14, Theorem 2.3.2].

3.1 Small flag varieties

The main result of this section is the following.

Theorem 1 *Let $v \in S_n$ be the minimal representative of the longest word in $S_n/\langle s_1, \dots, s_i, s_{i+r}, \dots, s_{n-1}\rangle$ for suitable i and r. Then*

$$X_v^a \cong \mathcal{F}\ell_r^a.$$

In particular, X_v^a is irreducible.

Before we prove the result, let us establish some useful lemmata. Note that written in one-line notation v is of form

$$v = [1, 2, \dots, i, i + r, i + r - 1, \dots, i + 1, i + r + 1, \dots, n].$$

So $v(j) = j$ for $j \in [i] \cup [i + r + 1, n]$ and $v(i + k) = i + r - k + 1$ for $k \in [r]$.

Lemma 1 *For the Schubert variety we have $X_v \cong \mathcal{F}\ell_r$. In particular, the only non-vanishing Plücker coordinates besides $p_{[s]}$ for $s \leq n-1$ are associated with the index sets in*

$$\mathcal{J}_v = \{I \mid I = [i] \cup \{l_1, \dots, l_s\}, s \in [r-1],\ l_j \in [i+1, i+r]\ \forall j\}. \tag{3.1}$$

Proof There is a bijection

$$\rho : \mathcal{J}_v \to \bigcup_{s=1}^{r-1} \binom{[r]}{s}, \quad I \mapsto \tilde{I},$$

where if $I = [i] \cup \{l_1, l_2 \dots, l_s\}$, we set $\tilde{I} = \{l_1 - i, l_2 - i, \dots, l_s - i\}$. This induces a bijection between the set of Plücker coordinates $\neq p_{[s]}, s \in [n-1] \backslash [i+1, i+r]$, which are non-vanishing on X_v (that is, the ones involved in the relevant Plücker relations) and Plücker coordinates $(\tilde{p}_K)$ which generate the coordinate ring of $\mathcal{F}\ell_r$. Notice that for J, L with $F(J), F(L) \in \mathcal{J}_v$, the Plücker relation $R^k_{J,L}$ is not identically 0 if and only if $R^k_{\tilde{J},\tilde{L}}$ is not identically 0 (since this happens for $k \in [\#(L \backslash (L \cap J))] = [\#(\tilde{L} \backslash (\tilde{L} \cap \tilde{J}))]$). In particular, $\mathcal{I}_v$ is generated by $\{R^k_{J,L}\}_{k,J,L \in \mathcal{J}_v} \cup \{p_I\}_{I \notin \mathcal{J}_v \cup \{[s] \mid s \in [n-1]\}}$. We extend the bijection to a map

$$\rho : \mathbb{C}[p_I \mid I \subset [n]] \to \mathbb{C}[p_{\tilde{I}} \mid \tilde{I} \subset [r]], \quad p_I \mapsto \begin{cases} p_{\tilde{I}} & \text{if } I \in \mathcal{J}_v \\ 0 & \text{otherwise} \end{cases}. \tag{3.2}$$

Then $\rho((R^k_{J,L})_{k;J,L \in \mathcal{J}_v}) = \mathcal{I}_{\mathcal{F}\ell_r}$. □

Remark 2 Note that one could also prove the previous lemma geometrically, since the Schubert variety is the closure of the Borel orbit.

Next, we establish a connection between the defining ideal of the degenerate flag variety $\mathcal{F}\ell^a_r$ and the initial ideal defining X^a_v. We keep the notation introduced in (3.2) and (3.1).

Lemma 2 *Let $\tilde{\mathbf{w}}$ be the weight for $\mathcal{F}\ell_r$, then $\rho(\mathrm{in}_{\mathbf{w}}((R^k_{J,L})_{k,J,L \in \mathcal{J}_v})) = \mathrm{in}_{\tilde{\mathbf{w}}}(\mathcal{I}_{\mathcal{F}\ell_r})$.*

Proof Let $L = ((1, \dots, i), (l_1, \dots, l_d)) > J = ((j_1, \dots, j_e), (1, \dots, i))$. Consider the relation $R^k_{\tilde{J},\tilde{L}}$. Without loss of generality we can assume that J and L are chosen in such a way that $\mathrm{in}_{\mathbf{w}}(R^k_{J,L})$ contains the monomial $p_J p_L$. All other monomials $p_{J'} p_{L'}$ in $\mathrm{in}_{\mathbf{w}}(R^k_{J,L})$ are obtained from $p_J p_L$ by choosing $1 \leq r_1 < \dots < r_k \leq i + d$, such that $\{l_{r_1}, \dots, l_{r_k}\} \cap [i + e, i + d - 1] = \emptyset$. This is the case if and only if $\{\tilde{l}_{r_1}, \dots, \tilde{l}_{r_k}\} \cap [e, d - 1] = \emptyset$. □

In what follows we use Feigin's standard monomial basis given by semistandard PBW-tableaux. As we work throughout the paper with conventions for the weight vector $\mathbf{w}$ as in [6] a global shift in the indices of all Plücker variables is needed before we can use Feigin's basis in our setting. Whenever we use the combinatorics from [10] in this section we assume we have applied the global shift to our index sets.

Recall that by [10, Theorem 4.10] there exists a standard monomial basis (indexed by *semistandard PBW-tableaux*) for $\mathbb{C}[p_{\tilde{I}} \mid \tilde{I} \subset [r]]/\operatorname{in}_{\tilde{\mathbf{w}}}(\mathcal{I}_{\mathcal{F}\ell_r})$ (and $\mathbb{C}[p_{\tilde{I}} \mid \tilde{I} \subset [r]]/\mathcal{I}_{\mathcal{F}\ell_r}$), denote it by $\mathbb{B}_{\mathrm{PBW}}$.

Lemma 3 *There exists a term order $\prec$ on $\mathbb{C}[p_{\tilde{I}} \mid \tilde{I} \subset [r]]$ such that $\mathbb{B}_{\mathrm{PBW}}$ equals the standard monomial basis given by monomials not contained in $\operatorname{in}_\prec(\mathcal{I}_{\mathcal{F}\ell_r})$. Moreover, the set $\{R^k_{\tilde{J},\tilde{L}}\}$ is a Gröbner basis for $\mathcal{I}_{\mathcal{F}\ell_r}$ with respect to $\prec$.*

Proof In [10, Lemma 4.9] Feigin introduces a partial order $\leq$ on $\mathbb{C}[p_{\tilde{I}} \mid \tilde{I} \subset [r]]$ such that for every monomial $\mathbf{p}^{a_T} \in \mathbb{C}[p_{\tilde{I}} \mid \tilde{I} \subset [r]]$ corresponding to a non-semistandard PBW-tableau T there exists an element $f \in \operatorname{in}_{\tilde{\mathbf{w}}}(\mathcal{I}_{\mathcal{F}\ell_r})$ that contains $\mathbf{p}^{a_T}$ in its support and further satisfies

$$\mathbf{p}^{a_T} \geq \mathbf{p}^a \text{ for all } \mathbf{p}^a \text{ non zero monomial in } f.$$

Moreover, $f = \mathbf{p}^v R^k_{\tilde{J},\tilde{L}}$ for a fixed monomial $\mathbf{p}^v$ that divides $\mathbf{p}^{a_T}$ and certain $k, \tilde{J}, \tilde{L}$. Given $\tilde{\mathbf{w}}$ and the partial order $\leq$ we define a term order on $\mathbb{C}[p_{\tilde{I}} : \tilde{I} \subset [r]]$ as follows: $\mathbf{p}^u \prec \mathbf{p}^v$ if and only if

(1) $\tilde{\mathbf{w}} \cdot u > \tilde{\mathbf{w}} \cdot v$, or [2]
(2) $\tilde{\mathbf{w}} \cdot u = \tilde{\mathbf{w}} \cdot v$, and $\mathbf{p}^u \leq \mathbf{p}^v$, or
(3) $\tilde{\mathbf{w}} \cdot u = \tilde{\mathbf{w}} \cdot v$, $\mathbf{p}^u$ and $\mathbf{p}^v$ are not comparable with respect to $\leq$, and $\mathbf{p}^u <_{\mathrm{lex}} \mathbf{p}^v$.

Here $<_{\mathrm{lex}}$ denotes the lexicographic order on $\mathbb{C}[p_{\tilde{I}} : \tilde{I} \subset [r]]$ with underlying lexicographic order on the variables corresponding to their index sets. Our term order $\prec$ is a refined version of a term order induced by a weight (see, for example the order $\prec_w$ in [20, page 4]). In particular, [20, Proposition 1.8] holds also in our case and we have

$$\operatorname{in}_\prec(\operatorname{in}_{\tilde{\mathbf{w}}}(\mathcal{I}_{\mathcal{F}\ell_r})) = \operatorname{in}_\prec(\mathcal{I}_{\mathcal{F}\ell_r}). \tag{3.3}$$

From [10, Proof of Lemma 4.9] it follows that for $\mathbf{p}^{a_T}$ and f as above we have

$$\operatorname{in}_\prec(f) = \mathbf{p}^{a_T} \in \operatorname{in}_\prec(\mathcal{I}_{\mathcal{F}\ell_r}). \tag{3.4}$$

In particular, the cosets of the *standard monomials*, i.e. $\mathbf{p}^u \notin \operatorname{in}_\prec(\mathcal{I}_{\mathcal{F}\ell_r})$, form a (standard monomial) basis for $\mathbb{C}[p_{\tilde{I}} : \tilde{I} \subset [r]]/\mathcal{I}_{\mathcal{F}\ell_r}$. By (3.3) they also form a basis for $\mathbb{C}[p_{\tilde{I}} : \tilde{I} \subset [r]]/\operatorname{in}_{\tilde{\mathbf{w}}}(\mathcal{I}_{\mathcal{F}\ell_r}))$, denote it by $\mathbb{B}_\prec$. In particular, we deduce from (3.4) that every standard monomial corresponds to a semistandard PBW-tableaux. Hence,

[2] Note the switch here: $\mathbf{p}^u \prec \mathbf{p}^v$ if $\tilde{\mathbf{w}} \cdot u > \tilde{\mathbf{w}} \cdot v$. This is because we have chosen to use the minimum convention for initial ideals with respect to weight vectors while for initial ideals with respect to term orders the maximum is considered.

$\mathbb{B}_{\prec} \subset \mathbb{B}_{\mathrm{PBW}}$. But as both are bases for the same algebra they have to be equal. This implies the first claim. The second follows as $f = \mathbf{p}^v R^k_{\tilde{J},\tilde{L}}$, and so in particular $\mathbf{p}^{a_T} \in (\mathrm{in}_{\prec}(R^k_{\tilde{J},\tilde{L}}))_{k,\tilde{J},\tilde{L}}$. □

Proposition 1 *The set* $\{R^k_{J,L}\}_{k,J,L\in\mathcal{J}_v} \cup \{p_I\}_{I\notin\mathcal{J}_v}$ *is a Gröbner basis for* $\mathcal{I}_v$ *and* $\mathbf{w}$*, denoted by* $\mathcal{G}_{v;\mathbf{w}}$.

Proof We use $\prec$ as defined in the proof of Lemma 3 and the map ρ from (3.2) to define a term order on $\mathbb{C}[p_I : I \subset [n]]$:

$$\mathbf{p}^u < \mathbf{p}^t \quad \Leftrightarrow \quad \mathbf{p}^u \notin (p_I)_{I\notin\mathcal{J}_v} \ni \mathbf{p}^t, \;\text{ or }\; \mathbf{p}^t, \mathbf{p}^u \notin (p_I)_{I\notin\mathcal{J}_v} \;\text{ and }\; \rho(\mathbf{p}^u) \prec \rho(\mathbf{p}^t).$$

By definition of $<$ and Lemma 2 we have $\mathrm{in}_<(\mathrm{in}_{\mathbf{w}}(R^k_{J,L})_{k;J,L\in\mathcal{J}_v}) = \mathrm{in}_<((R^k_{J,L})_{k;J,L\in\mathcal{J}_v})$. Moreover, as the $R^k_{\tilde{J},\tilde{L}}$ constitute a Gröbner basis for $\mathcal{I}_{\mathcal{F}\ell_r}$ and $\prec$ by Lemma 3, it follows from Buchberger's criterion that the S-polynomials of pairs of these elements reduce to zero. Given the map ρ, the same must be true for S-polynomials of elements $R^k_{J,L}$ with $J, L \in \mathcal{J}_v$ with respect to the term order $<$. Hence, in order to verify the claim we only need to compute S-polynomials of the relevant Plücker relations and the vanishing Plücker variables. Consider $R^k_{J,L}$ with $J, L \in \mathcal{J}_v$ and p_I with $I \notin \mathcal{J}_v$. Then $\mathrm{in}_<(R^k_{J,L})$ and $\mathrm{in}_<(p_I)$ are relatively prime. So by [14, Lemma 2.3.1] their S-polynomials reduces to zero over $R^k_{J,L}$ and p_I. As the same is true for the S-polynomials of variables $p_I, p_{I'}$ with $I, I' \notin \mathcal{J}_v$, the claim follows by Buchberger's criterion. □

Proof of Theorem 1 We need to show that the isomorphism of X_v and $\mathcal{F}\ell_r$ induced by ρ is induces an isomorphism between the corresponding degenerations. This is true as by Lemma 2 and Proposition 1 ρ maps the initial ideal defining X^a_v to the ideal defining $\mathcal{F}\ell^a_r$. Lastly, by [10, §5.1] the degenerate flag variety is the closure of a homogeneous space and therefore irreducible. As $X^a_v \cong \mathcal{F}\ell^a_r$ by the above, the claim follows. □

Let $\underline{i} = \{i_1, \ldots, i_r\} \subsetneq [n-1]$. We set $m := \min\{\underline{i}\}$, $M := \max\{\underline{i}\}$, and $r := M - m + 1$. Let $v \in \langle s_{i_1}, \cdots, s_{i_r}\rangle \subset S_n$ and denote by $\widetilde{v}$ the element $\widetilde{s}_{i_1-m+1} \cdots \widetilde{s}_{i_r-m+1} \in S_r$. In this notation, from the proof of Theorem 1 we can deduce the following result, which in this case allows one to reduce to smaller rank flag varieties.

Corollary 1 *Let* $\underline{i} = \{i_1, \ldots, i_r\} \subsetneq [n]$ *and* $v \in \langle s_{i_1}, \cdots, s_{i_r}\rangle \subset S_n$. *Then for* $X^a_v \subset \mathcal{F}\ell^a_n$ *we have*

$$X^a_v \cong X^a_{\widetilde{v}} \subset \mathcal{F}\ell^a_r .$$

3.2 Isomorphic degenerate and original Schubert varieties

In the following we present another instance in which a Schubert variety stays irreducible under Feigin's degeneration of $\mathcal{F}\ell_n$. In fact, for the class of varieties we deal with in this section a stronger property holds: the degeneration process does not deform them, that is X^a_v is isomorphic to the original Schubert variety X_v.

Recall that we denote by $c \in S_n$ the special Coxeter element $c = s_{n-1}s_{n-2}\cdots s_2 s_1$.

Proposition 2 *Let $v \leq c$. Then $\mathcal{I}_v = \mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$.*

Proof Recall that $\mathcal{I}_v = (\{p_I\}_{I \not\leq v([\#I])} \cup \{R^k_{J,L}\}_{k,J,L})$ We will show that $R^k_{J,L} - \mathrm{in}_{\mathbf{w}}(R^k_{J,L}) \in (p_I)_{I \not\leq v([\#I])}$ for all k, J, L. If $R^k_{J,L} = \mathrm{in}_{\mathbf{w}}(R^k_{J,L})$ we are done. Otherwise we have

$$R^k_{J,L} - \mathrm{in}_{\mathbf{w}}(R^k_{J,L}) = \sum_{\substack{(J',L') \in \mathcal{L}^k_{J,L} \\ \{l_{r_1},\ldots,l_{r_k}\} \cap [e,d-1] \neq \emptyset}} p_{J'} p_{L'} \neq 0.$$

We claim that in this case $L' \not\leq v([d])$ holds. Note that $\{l_{r_1}, \ldots, l_{r_k}\} \cap [e, d-1] \neq \emptyset$ implies in particular that there exists $x \in [e, d-1]$ with $x \notin L' = (L \backslash (l_{r_1}, \ldots, l_{r_k})) \cup (j_1, \ldots, j_k)$. By (2.3),

$$v \leq c \Leftrightarrow v([d]) = [d-1] \cup \{b\} \text{ with } d \leq b \leq n$$

it follows that $p_{L'} \in (p_I)_{I \not\leq v([\#I])}$. And further, $R^k_{J,L} - \mathrm{in}_{\mathbf{w}}(R^k_{J,L}) \in (p_I)_{I \not\leq v([\#I])}$. Hence,

$$\mathcal{I}_v = (R^k_{J,L})_{k,J,L} + (p_I)_{I \not\leq v([\#I])} = (\mathrm{in}_{\mathbf{w}}(R^k_{J,L}))_{k,J,L} + (p_I)_{I \not\leq v([\#I])} \subseteq \mathrm{in}_{\mathbf{w}}(\mathcal{I}_v).$$

Consider any term order $<$ so that $\mathrm{in}_<(\mathcal{I}_v) = \mathrm{in}_<(\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v))$. Such a term order exists as $\mathcal{I}_v$ is homogeneous. Then the reduced Gröbner basis for $\mathcal{I}_v$ with respect to $<$ is also a reduced Gröbner basis for $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$ with respect to $<$. As reduced Gröbner bases are unique the claim follows. □

4 Criteria for reducibility

In this section we examine when Schubert varieties become reducible after being degenerated. We give a number of sufficient conditions for certain monomials of degree two to be contained in the initial ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ for $w \in S_n$ by repeated applications of (2.10).

Definition 2 Let $w \in S_n$. A monomial $f = \prod_{J \subset [n]} p_J^{\epsilon_J} \in \mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$, where $\epsilon_J \in \{0, 1\}$, is called an *honest monomial* if f has degree at least 2 and $f \notin (p_I)_{I \not\leq w([\#I])}$.

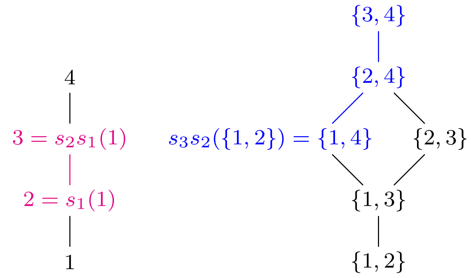
The following Lemma is straightforward:

Lemma 4 *Let $w \in S_n$. If $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains an honest monomial then it fails to be prime.*

4.1 Relations between Gr(1, $\mathbb{C}^n$) and Gr(2, $\mathbb{C}^n$)

We start the discussion by focusing on very special Plücker relations, namely those between Plücker coordinates on $\mathrm{Gr}(1, \mathbb{C}^n)$ and on $\mathrm{Gr}(2, \mathbb{C}^n)$. In this case, we can classify the $w \in S_n$ for which $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains an honest monomial of type $p_{\{i\}} p_{\{j,k\}}$.

Fig. 1 The Bruhat posets of $\mathrm{Gr}(1, \mathbb{C}^4)$ and $\mathrm{Gr}(2, \mathbb{C}^4)$ with intervals given by $s_1 \leq \bar{v} \leq s_2 s_1$ and $s_3 s_2 \leq \bar{\bar{v}}$ as in Theorem 2 for $j = 2, k = 4$



For $v \in S_n$ denote by $\overline{v}$ the minimal length representative of the coset of v in $S_n/\langle s_2, s_3 \ldots . s_{n-1}\rangle$ and $\overline{\overline{v}}$ the minimal length representative of the coset of v in $S_n/\langle s_1, s_3, s_4, \ldots, s_{n-1}\rangle$.

Theorem 2 *Let $v \in S_n$ and $1 < j < k \leq n$. Then $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$ contains the honest monomial $p_{\{j\}}p_{\{1,k\}}$ if and only if v satisfies*

$$s_{j-1}s_{j-2}\cdots s_2 s_1 \leq \overline{v} \leq s_{k-2}s_{k-3}\cdots s_2 s_1 \ \textit{and}\ s_{k-1}s_{k-2}\cdots s_3 s_2 \leq \overline{\overline{v}}.$$

The conditions on $\overline{v}$ and $\overline{\overline{v}}$ in Theorem 2 are depicted for S_4 with $j = 2, k = 4$ in Fig. 1.

Proof To simplify notation, for $a \in [n]$ we denote $p_a := p_{(a)}$, and for $a, b \in [n]$ we write $p_{a,b}$ instead of $p_{(a,b)}$. We will only consider Plücker coordinates corresponding to increasing sequences in this proof and hence adapt the signs.

Consider for $1 \leq i < j < k \leq n$ the Plücker relation $R^1_{(i),(j,k)} = p_i p_{j,k} - p_j p_{i,k} + p_k p_{i,j}$. Note that if $\mathrm{in}_{\mathbf{w}}(R^1_{(i),(j,k)}) = R^1_{(i),(j,k)}$ the relation will not produce an honest monomial in $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ for any $w \in S_n$ as $\mathcal{I}_w$ is prime. Note that $R^1_{(i),(j,k)} \neq \mathrm{in}_{\mathbf{w}}(R^1_{(i),(j,k)})$ only if $i = 1$. In this case

$$\mathrm{in}_{\mathbf{w}}(p_1 p_{j,k} - p_j p_{1,k} + p_k p_{1,j}) = -p_j p_{1,k} + p_k p_{1,j}.$$

As $j < k$, if p_j vanishes on the Schubert variety X_v, then so does p_k. Hence, both monomials are zero on X_v. Similarly, if $p_{1,j}$ vanishes on X_v, then so does $p_{1,k}$. Our aim is to determine $v \in S_n$ such that one of the two terms of $\mathrm{in}_{\mathbf{w}}(R^1_{(i),(j,k)})$ lies in $(p_I)_{I \not\leq v([\#I])}$ but the other does not as in this case, the ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$ contains an honest monomial. A priori, there are two cases for the restriction of p_k and $p_{1,k}$ to X_v:

(1) $p_{1,k} \neq 0$ and $p_k = 0$,
(2) $p_{1,k} = 0$ and $p_k \neq 0$.

We will show that in fact the second case can never happen. Both cases yield conditions on $\overline{v}$ and $\overline{\overline{v}}$ (keeping also in mind that we do not want p_j and $p_{1,j}$ to vanish). In the first case we have the following conditions

$$s_{j-1}s_{j-2}\cdots s_2 s_1 \leq \overline{v} \leq s_{k-2}s_{k-3}\cdots s_2 s_1 \ \text{and}\ s_{k-1}s_{k-2}\cdots s_3 s_2 \leq \overline{\overline{v}}, \quad (4.1)$$

respectively, in the second case we have

$$s_{k-1}s_{k-2}\cdots s_2s_1 \leq \overline{v} \text{ and } s_{j-1}s_{j-2}\cdots s_3s_2 \leq \overline{\overline{v}} \leq s_{k-2}s_{k-3}\cdots s_3s_2. \tag{4.2}$$

Assume $v \in S_n$ is chosen such that the minimal length representatives of the cosets fulfill the inequalities in (4.2). Then

$$s_{k-1}s_{k-2}\cdots s_2s_1 \leq v \leq s_{k-2}\cdots s_2x$$

for some $x \in \langle s_1, s_3, \ldots, s_{n-1}\rangle$. Observe that $s_{k-1}\cdots s_1(1) = k$ and

$$s_{k-2}\cdots s_2x(1) = \begin{cases} 1 & \text{if } s_1x > x \\ k-1 & \text{if } s_1x < x. \end{cases}$$

With the notation as in (2.1) this implies $(s_{k-1}\cdots s_1)^{1,k} = 1 > (s_{k-2}\cdots s_2x)^{1,k} = 0$. But $s_{k-1}\cdots s_1 \leq s_{k-2}\cdots s_2x$, contradicting (2.2). Hence, case (4.2) never applies. □

Remark 3 Theorem 2 is enough to detect all Schubert varieties in $\mathcal{F}\ell_3 \hookrightarrow \mathrm{Gr}(1, \mathbb{C}^3) \times \mathrm{Gr}(2, \mathbb{C}^3)$ which become reducible under Feigin's degeneration. In fact, the only Schubert variety having this property is the one indexed by s_1s_2. All the other permutations but the longest element (which indexes the Schubert variety corresponding to the irreducible variety $\mathcal{F}\ell_n^a$) are $\leq c = s_2s_1$ and hence, by Proposition 2, are irreducible.

4.2 Monomials from other relations

Items (1) to (5) of Theorem 3, formulated below, provide sufficient conditions on $w \in S_n$ for the initial ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ to contain a degree two honest monomial originating from a Plücker relation between Plücker coordinates on adjacent Grassmannians, that is $\mathrm{Gr}(k, \mathbb{C}^n)$ and on $\mathrm{Gr}(k+1, \mathbb{C}^n)$ for suitable k. Notice that here we are only producing sufficient conditions, so that for $k = 1$ we clearly obtain a weaker result than Theorem 2. Theorem 3 (6) and (7) deal with Plücker relations between not necessarily adjacent Grassmannians.

Table 1 (resp. Table 2 in the appendix) show to which permutations $w \in S_4$ (resp. S_5) each one of the points of Theorem 3 applies. The computations were performed in *Sage* [8] and *Macaulay2* [13].

Let $w \in S_n$. In the following, it will be convenient to set $w([0]) := \emptyset$. Moreover, since $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_e) = \mathcal{I}_e$, we can exclude the case $w = e$ right away in the following theorem.

Theorem 3 *Let $w \in S_n \setminus \{e\}$. If one of the following conditions holds for w, then $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains an honest monomial of degree 2:*

(1) *there exist $i \in [n-1]$ with $ws_i > w$ and $j \in [n]$ such that*

$$i, j \leq w(i), i \neq j \text{ and } i, j \notin w([i-1]) \cup \{w(i+1)\};$$

Table 1 Applying our results to the case $n = 4$

w		mono	(1)	(2)	(3)	(4)	(5)	(6)	(7)	Thm1	Prop2	Thm5
[1, 2, 3, 4]	1	–	–	–	–	–	–	–	–	–	×	–
[1, 2, 4, 3]	s_3	–	–	–	–	–	–	–	–	×	×	–
[1, 3, 2, 4]	s_2	–	–	–	–	–	–	–	–	×	×	–
[1, 3, 4, 2]	s_2s_3	×	×	–	×	–	–	×	–	–	–	–
[1, 4, 2, 3]	s_3s_2	–	–	–	–	–	–	–	–	–	×	–
[1, 4, 3, 2]	$s_2s_3s_2$	–	–	–	–	–	–	–	–	×	–	–
[2, 1, 3, 4]	s_1	–	–	–	–	–	–	–	–	×	×	–
[2, 1, 4, 3]	s_3s_1	–	–	–	–	–	–	–	–	–	–	–
[2, 3, 1, 4]	s_1s_2	×	×	–	×	–	–	×	–	–	–	–
[2, 3, 4, 1]	$s_1s_2s_3$	×	×	–	×	×	–	×	–	–	–	–
[2, 4, 1, 3]	$s_3s_1s_2$	×	×	–	×	–	–	×	–	–	–	–
[2, 4, 3, 1]	$s_1s_2s_3s_2$	×	×	–	×	×	–	×	–	–	–	–
[3, 1, 2, 4]	s_2s_1	–	–	–	–	–	–	–	–	–	×	–
[3, 1, 4, 2]	$s_2s_3s_1$	×	–	–	×	–	–	×	–	–	–	–
[3, 2, 1, 4]	$s_1s_2s_1$	–	–	–	–	–	–	–	–	×	–	–
[3, 2, 4, 1]	$s_1s_2s_3s_1$	×	×	–	–	×	–	×	–	–	–	–
[3, 4, 1, 2]	$s_2s_3s_1s_2$	×	×	×	×	–	×	×	–	–	–	–
[3, 4, 2, 1]	$s_1s_2s_3s_1s_2$	×	×	–	×	–	–	×	–	–	–	×
[4, 1, 2, 3]	$s_3s_2s_1$	–	–	–	–	–	–	–	–	–	×	–
[4, 1, 3, 2]	$s_2s_3s_2s_1$	–	–	–	–	–	–	–	–	–	–	–
[4, 2, 1, 3]	$s_3s_1s_2s_1$	–	–	–	–	–	–	–	–	–	–	–
[4, 2, 3, 1]	$s_1s_2s_3s_2s_1$	×	×	–	–	×	–	–	×	–	–	×
[4, 3, 1, 2]	$s_2s_3s_1s_2s_1$	×	–	×	–	–	×	–	–	–	–	×
[4, 3, 2, 1]	$s_1s_2s_3s_1s_2s_1$	–	–	–	–	–	–	–	–	×	–	–
24		11	9	2	8	4	2	9	1	6	7	3

The first two columns give w in one-line notation and as reduced expression; the column *mono* indicates whether the initial ideal contains a monomial; the columns (1) to (7) indicate whether or not the criteria (1) to (7) of Theorem 3 apply; the last three columns indicate if Theorem 1, Proposition 2 and Theorem 5, respectively, apply

(2) *there exist* $i \in [3, n-1]$ *with* $ws_i > w$ *and* $l, x \in [n]$ *with* $x \neq i-1, l \leq w(i)$ *and* $w(i+1) \leq x, i-1$*, such that*

$$i-1, x \in w([i-1]) \cup \{w(i+1)\} \text{ and } l \notin w([i-1]) \cup \{w(i+1)\};$$

(3) *there exist* $j \in [2, n-1]$ *with* $s_j w > w$ *and* $i \in [n-1], i < j$ *such that*

$$j \in w([i]), i \notin w([i]), \text{ and } j+1 \leq w(i+1);$$

(4) *there exists* $i \in [n-2]$ *with* $s_i w < w$ *and* $j \in [n]$ *such that*

$$i, j \notin w([i+1]), j \leq w(i+2), i+1 \in w([i+1]) \text{ and } i+1 < j;$$

(5) *there exist* $i \in [2, n-1]$ *and* $l \in [2, n], l > i$ *with*

$$i \notin w([i+1]), l \in w([i]), l > w(i+1) \text{ and } i > w(i+1);$$

(6) *for* $i \in [n]$*, minimal with* $w(i) \neq i$*, it holds* $w(i) < n$ *and, for the minimal* $j \in [i+1, n-1]$ *such that* $w(j) > w(i)$*, it holds* $w(i) \notin [j-1]$*;*
(7) *for* $i \in [n]$*, minimal with* $w(i) \neq i$*, it holds* $w(i) = n$ *and, for the minimal* $j \in [i+2, n-1]$*, such that* $w(j) > w(i+1)$*, it holds* $i \notin w([i+1, j-1])$*.*

Proof (1) Assume there exist i, j fulfilling the conditions above. Let J be any sequence such that $F(J) = w([i-1]) \cup \{j\}$ and $j_1 = j$, and let L be any sequence such that $F(L) = w([i-1]) \cup \{i, w(i+1)\}$. Then the Plücker relation $R^1_{J,L}$ equals

$$p_J p_L - p_{(J\setminus(j))\cup(i)} p_{(L\setminus(i))\cup(j)} - p_{(J\setminus(j))\cup(w(i+1))} p_{(L\setminus(w(i+1)))\cup(j)}.$$

Taking the initial form with respect to $\mathbf{w}$ we obtain

$$\text{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(j))\cup(w(i+1))} p_{(L\setminus(w(i+1)))\cup(j)}.$$

Restricting to X_w, we have $p_{(J\setminus(j))\cup(w(i+1))} = p_{(w([i-1]),w(i+1))} = 0$ as $ws_i > w$ and so $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

(2) Assume such i, l, x exist. Let J be any sequence such that $F(J) = (w([i-1]) \cup \{w(i+1)\})\setminus\{i-1\}$ and $j_1 = x$, and let L be any sequence such that $F(L) = (w([i-1]) \cup \{w(i+1), l\}) \setminus \{x\}$ the Plücker relation $R^1_{J,L}$, i.e.

$$p_J p_L - p_{(J\setminus(x))\cup(i-1)} p_{(L\setminus(i-1))\cup(x)} - p_{(J\setminus(x))\cup(l)} p_{(L\setminus(l))\cup(x)}.$$

Taking the initial form with respect to $\mathbf{w}$ we obtain

$$\text{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(x))\cup(l)} p_{(L\setminus(l))\cup(x)}.$$

Note that $(F(L) \setminus \{l\}) \cup \{x\} = w([i-1]) \cup \{w(i+1)\}$ and so restricting to X_w we have $p_{(L\setminus(l))\cup(x)} = 0$ as $ws_i > w$. So $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

(3) Assume such i and j exist and take J any sequence such that $F(J) = w([i])$ and $j_1 = j$, and L any sequence such that $F(L) = (w([i]) \cup \{i, j+1\}) \setminus \{j\}$. Note that $j \in w([i])$ and $s_j w > w$ imply $j+1 \notin w([i+1])$. Then

$$R^1_{J,L} = p_J p_L - p_{(J\setminus(j))\cup(i)} p_{(L\setminus(i))\cup(j)} - p_{(J\setminus(j))\cup(j+)} p_{(L\setminus(j+1))\cup(j)}.$$

Taking the initial form with respect to $\mathbf{w}$ we obtain

$$\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(j))\cup(j+1)} p_{(L\setminus(j+1))\cup(j)}.$$

As $(J\setminus(j))\cup(j+1) \not\leq w([\#J])$ restricting to X_w we have $p_{(w([i])\setminus(j))\cup(j+1)} = 0$. Hence, $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

(4) Assume such i and j exist and consider L any sequence such that $F(L) = w([i+1]) \cup \{j\}$, and J any sequence such that $F(J) = s_i w([i+1]) = (w([i+1]) \setminus \{i+1\}) \cup \{i\}$ and $j_1 = i$. Then

$$R^1_{J,L} = p_J p_L - p_{(J\setminus(i))\cup(i+1)} p_{(L\setminus(i+1))\cup(i)} - p_{(J\setminus(i))\cup(j)} p_{(L\setminus(j))\cup(i)}$$

Taking the initial form with respect to $\mathbf{w}$ yields

$$\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(i))\cup(j)} p_{(L\setminus(j))\cup(i)}$$

Now $(J \setminus (i)) \cup (j) = (w([i+1]) \setminus (i+1)) \cup (j)$, but restricting to X_w we have $p_{(J\setminus(i))\cup(j)} = 0$ as $j > i+1$. Hence, $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

(5) Assume such i, l exist, take $J = w([i])$ and $L = (w([i+1]) \setminus \{l\}) \cup \{i\}$. Consider the relation $R^1_{J,L}$:

$$p_J p_L - p_{(J\setminus(l))\cup(i)} p_{(L\setminus(i))\cup(l)} - p_{(J\setminus(l))\cup(w(i+1))} p_{(L\setminus(w(i+1)))\cup(l)}.$$

Taking the initial form with respect to $\mathbf{w}$ yields

$$\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(l))\cup(w(i+1))} p_{(L\setminus(w(i+1)))\cup(l)}.$$

Restricting to X_w we have $(F(L) \setminus \{w(i+1)\}) \cup \{l\} = (F(w([i+1]) \setminus \{w(i+1)\}) \cup \{i\}$ and $p_{(w([i+1])\setminus(w(i+1)))\cup(i)} = 0$ as $i > w(i+1)$. So $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains the monomial $p_J p_L$.

(6) First note that as $w \neq e$ we have that $w(i) \neq i$ in particular implies $i < n$. Consider J any sequence such that $F(J) = w([i]) = [i-1] \cup \{w(i)\}$ with $j_1 = w(i)$. Let L be any sequence such that $F(L) = [j-1] \cup \{w(j)\}$. As $w(i) \notin [j-1]$ implies $w(i) > j-1$ and so $w(j) > w(i) > j-1$, then the set $[j-1] \cup \{w(j)\}$ has cardinality j. So,

$$R^1_{J,L} = p_J p_L - p_{(w(j),[i-1])} p_{(L\setminus(w(j)))\cup(w(i))} - \sum_{r\in[i,j-1]} p_{(r,[i-1])} p_{(L\setminus(r))\cup(w(i))}.$$

Taking the initial form with respect to $\mathbf{w}$ yields

$$\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(w(j),[i-1])} p_{(L \setminus (w(j))) \cup (w(i))}.$$

Since $w(j) > w(i)$, the coordinate $p_{(w(j),[i-1])}$ vanishes in the coordinate ring of X_w, so that $\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) \in \mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ is a monomial.

(7) Consider J any sequence such that $F(J) = [i] \cup \{n\} = w([i]) \cup \{i\}$ such that $j_1 = i$, and let L be any sequence such that $F(L) = [i-1] \cup [i+1, j-1] \cup \{w(j), n\}$. Note that $L \leq w([j])$ as $i \notin w([i+1, j-1])$, and hence we get

$$R^1_{J,L} = p_J p_L - p_{(w(j),w([i]))} p_{(L \setminus (w(j))) \cup (i)} - \sum_{r \in [i+1, j-1]} p_{(r,w([i]))} p_{(L \setminus (r)) \cup (i)}$$

with initial term $\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(w(j),w([i]))} p_{(L \setminus (w(j))) \cup (i)}$. Further observe that $w(j) > w(i+1) \geq i$, which implies that $p_{(w(j),w([i]))}$ vanishes in the coordinate ring of X_w. Then $R^1_{J,L}$ produces a monomial.

□

Remark 4 In principle, we could have assumed $i \in \{2, 3, \ldots, n-1\}$ in Theorem 3 (2). Instead, we exclude the case $i = 2$, since it never happens under the other assumptions, for which we would have $w(3) \leq 1$ and $ws_2(2) = w(3) > w(2)$ contradicting each other.

Remark 5 In points (6) and (7) of Theorem 3, such a j need not exist, in which case the criterion would simply not apply.

4.2.1 Efficiency of the various criteria from Theorem 3

We want to comment here on how efficient the various criteria of Theorem 3 are, based on the data we have collected for S_4 (see Table 1) and S_5 (see Table 2). The data can be found at the homepage: https://www.matem.unam.mx/~lara/schubert/.

For $n = 4$, there are 11 permutations w such that at least one Plücker relation degenerates to a monomial. In the S_5-case, this happens for 85 permutations.

Among the criteria collected in Theorem 3, point (6) seems to be the most powerful: it detects 9 out of 11 permutations for S_4, and 65 out of 85 for S_5. To cover the missing two permutations for S_4 it is enough to combine Theorem 3 (6) with one of the points (1),(4),(7) and one between (2) and (5). So that it is enough to apply three of our criteria to find all $w \in S_4$ such that $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains a Plücker relation which degenerates to a monomial.

Theorem 3 (1) picks 9 out of 11 permutations in S_4, and 64 out of 85 for S_5.

Theorem 3 (3) covers 8 out of 11 permutations yielding monomial initial ideals for S_4 and 57 out of 85 for S_5.

Theorem 3 (4) detects 4 permutations for S_4 and 36 permutations for S_5.

Theorem 3 (2) and (5) both finds 2 permutations for $n = 4$ and 22 for $n = 5$, but the elements they see are different.

Finally, Theorem 3 (7) applies to only one permutation, resp. 8 permutations, in the $n = 4$, resp. $n = 5$, case, but it is necessary to cover all the permutations in S_5 containing monomial degenerate Plücker relations. For example, it is the only one among our criteria which can be applied to $s_1s_2s_3s_4s_3s_1s_2s_1$.

4.3 Plücker relations not degenerating to monomials

In this section we study some cases in which none of the Plücker relations produces a monomial in the defining ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$. Clearly, this does not have to be equivalent to the irreducibility of the degeneration, but it turns out to be the case for $n = 3$ (by Remark 3) and $n \in \{4, 5\}$ (by Macaulay2 computations). We do not know whether such an equivalence holds in general.

We have seen in Sect. 3.2 that if $w \leq c = s_{n-1}s_{n-2}\cdots s_2s_1$, then the initial ideal $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ coincides with $\mathcal{I}_w$. In the following proposition we will show that if we multiply c on the right by simple reflections $s_{k_1}, \ldots, s_{k_r}$ which commute pairwise and each appear at most once, then none of the Plücker relations degenerates to a monomial in $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_{k_1}\cdots s_{k_r}})$.

Table 1 (resp. Table 2 in the appendix) show which statements apply to which elements of S_4 (resp. S_5).

Proposition 3 *For any $h \in [n-1]$, none of the Plücker relations degenerates to a monomial in* $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_h})$.

Proof First of all notice that if $h = 1$, then $cs_1 < c$ and the claim follows from Proposition 2, which says that $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_c) = \mathcal{I}_c$.

If $h \in [2, n-1]$, then $cs_h > c$. In this case, if $J \leq c([\#J])$ and $L \leq c([\#L])$, then $\mathrm{in}_{\mathbf{w}}(R^m_{J,L})$ being a monomial on $X^a_{cs_h}$ implies that it is a monomial on X^a_c too. But this is not possible, again by Proposition 2. Therefore we can assume that $L \not\leq c([\#L])$ or $J \not\leq c([\#J])$. We set $k := h - 1 \in [n-2]$ for convenience.

Recall that for any $i \in [k] \cup [k+2, n-1]$

$$\begin{aligned} cs_{k+1}/\langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n-1}\rangle &= s_r \cdots s_i/\langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n-1}\rangle \\ &= c/\langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n-1}\rangle. \end{aligned}$$

In one-line notation $cs_{k+1} = [n, 1, \ldots, k-1, k+1, k, k+2, \ldots, n-1]$. Hence, if $I \leq cs_{k+1}([\#I])$, but $I \not\leq c([\#I])$, then $\#I = k+1$ and it must hold

$$F(I) = [k-1] \cup \{k+1, i\} \text{ with } i \in [k+2, n]. \tag{4.3}$$

Therefore a Plücker $R^m_{J,L}$ can produce a monomial in $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_h})$ only if J is a sequence such that $F(J) = [k-1] \cup \{k+1, j\}$ with $j_1 = j$ or $F(L) = [k-1] \cup \{k+1, l\}$ for $j, l \in [k+2, n]$. If $\#J = \#L$, then $\mathrm{in}_{\mathbf{w}}(R^m_{J,L}) = R^m_{J,L}$, hence we only have to consider the case $\#J < \#L$.

Let $\#L = p > k+1$, then by (4.3) we have $F(J) = [k-1] \cup \{k+1, j\}$ and $F(L) = [p-1] \cup \{l\}$ for $j_1 = j \in [k+2, n]$ and $l \in [p, n]$. Note that $j \in J$ is the only possible element to swap for elements in L non-trivially, so that we impose

$j \notin L$ (otherwise $R^m_{J,L} = 0$ for any m). Remember that we may assume $j \in [p, n]$. Then

$$\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(j))\cup(l)} p_{(L\setminus(l))\cup(j)} - p_{(J\setminus(j))\cup(k)} p_{(L\setminus(k))\cup(j)}. \quad (4.4)$$

As $[k-1] \cup \{k+1, l\} \leq cs_{k+1}([k+1])$ and $[k-1, p-1] \cup \{j\} \leq cs_{k+1}([p])$ at least two terms are non-zero on $X_{cs_{k+1}}$.

Now, assume $\#L = k+1$ and $\#J = q < k+1$. Then we have

$$F(L) = [k-1] \cup \{k+1, l\} \text{ and } F(J) = [q-1] \cup \{j\},$$

for $j = j_1, l \in [k+2, n]$ and $j \notin L$ in order for the relation to be non-trivial. We obtain

$$\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(j))\cup(k+1)} p_{(L\setminus(k+1))\cup(j)} - p_{(J\setminus(j))\cup(l)} p_{(L\setminus(l))\cup(j)}. \quad (4.5)$$

As $[q-1] \cup \{l\} \leq cs_{k+1}([q])$ and $[k-1] \cup \{k+1, j\} \leq cs_{k+1}([k+1])$, the relation $R^1_{J,L}$ does not degenerate to a monomial. □

Corollary 2 *Let $h \in [n-1]$. Then $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_h})$, as an ideal in the quotient $\mathbb{C}[p_I]/(p_I \mid I \not\leq cs_h([\#I]))$, admits a set of generators of type $p_I p_J - p_{I'} p'_J$ for appropriate $I, I', J, J' \subset [n]$.*

Proof First note that if $h = 1$, then by Proposition 2 $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_1}) = \mathcal{I}_{cs_1}$. The Plücker relations involving non-vanishing Plücker coordinates on X_{cs_1} are for $q < p \leq j < l \leq n$ the following *pure differences* (i.e. a sums of two monomials, one of which with coefficient 1 the other with coefficient −1)

$$p_{[q-1]\cup\{j\}} p_{[p-1]\cup\{l\}} - p_{[q-1]\cup\{l\}} p_{[p-1]\cup\{j\}}.$$

Notice that the index sets of the Plücker coordinates in the above equation (as well as in the rest of this proof) are sets, and hence by convention, as sequences they are arranged in an increasing order, while in the proof of the previous result we always had $j = j_1$. This only affect the relation by a global sign.

If $h \in [2, n-1]$, we can set again $k := h - 1$. In the proof of Proposition 3 we have seen in equations (4.4) and (4.5) the form of the additional relations for cs_{k+1}. Note that in (4.4) we have $[k-1] \cup [k+1, p-1] \cup \{j, l\} \not\leq cs_{k+1}([p])$ and hence, the middle term vanishes on $X_{cs_{k+1}}$. Similarly observe for (4.5) that $[k-1] \cup \{j, l\} \not\leq cs_{k+1}([k+1])$ as $j, l \geq k+2$. So all generators of $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_{k+1}})$ are pure differences in $\mathbb{C}[p_I|]/(p_I \mid I \not\leq cs_{k+1}(\#I))$. □

Remark 6 Note that while $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ and $\mathcal{I}_w$ have the same generators for $w \leq c$, this is not true for cs_{k+1} with $k \geq 1$. Here taking the initial ideal with respect to $\mathbf{w}$ modifies the generators.

The following proposition generalizes Proposition 3 to a product of pairwise distinct commuting simple reflections.

Proposition 4 *Take $k_1, \ldots, k_r \in [n-1]$ with $|k_i - k_j| > 1$ for all $i \neq j$, then none of the Plücker relations degenerates to a monomial in* $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_{k_1}\cdots s_{k_r}})$.

Proof We may assume $k_1 < k_2 < \ldots < k_r$ without loss of generality. Moreover, since we are multiplying by pairwise distinct commuting reflections, and as Plücker relations only involve pairs of Grassmannians, it is enough to consider the cases $r = 1,\ 2$. The case $r = 1$ was dealt with in Proposition 3, so we are left with $r = 2$.

We consider two cases: firstly, we deal with the case $k_1 = 1$, and then we suppose $k_1 \neq 1$.

If $k_1 = 1$, $cs_1 < c$ can be identified with the Coxeter element $\tilde{c} = \tilde{s}_{n-2} \ldots \tilde{s}_1$ in S_{n-1} (via $s_i \mapsto \tilde{s}_{i-1}$ for $i \in [2, n-1]$). In this case, $cs_1 s_{k_2} \in \langle s_2, \ldots, s_{n-1}\rangle$ and, by Corollary 1, we have $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_{cs_1 s_{k_2}}) = \mathrm{in}_{\mathbf{w}}(\mathcal{I}_{\tilde{c}\tilde{s}_{k_2}})$. We then apply Proposition 3 to obtain the claim.

Now denote $k_1 := k+1$ and $k_2 := g+1$ and recall, that by assumption $k < g+1$. As in the proof of Proposition 3, we only have to deal with Plücker relations $R^m_{J,L}$ with $\#J \neq \#L$, where $J \not\leq cs_{k+1}s_{g+1}([\#J])$ or $L \not\leq cs_{k+1}s_{g+1}([\#L])$. We can further reduce to the case $\#J = k+1$, $j_1 = j$, and $\#L = g+1$, otherwise the Plücker relations are the same as the ones considered in Proposition 3, and the result has been proven above.

Consider relations $R^m_{J,L}$ with $\#J = k+1, \#L = g+1$ and $J \leq cs_{k+1}s_{g+1}([k+1])$, $J \not\leq c([k+1])$ and $L \leq cs_{k+1}s_{g+1}([g+1])$, $L \not\leq c([g+1])$. We have shown in Proposition 3 that in this case it must hold

$$F(J) = [k-1] \cup \{k+1, j\}, \quad F(L) = [g-1] \cup \{g+1, l\}$$

with $j \in [k+2, n]$ and $l \in [g+2, n]$. In order for the relation to be non-trivial we may assume $j \notin L$. Since $k+1 \in [g-1]$, the only relation to be considered is

$$\begin{aligned} R^1_{J,L} = p_J p_L - p_{(J\setminus(j))\cup(l)} p_{(L\setminus(l))\cup(j)} - p_{(J\setminus(j))\cup(g+1)} p_{(L\setminus(g+1))\cup(j)} \\ - \sum_{r \in [k+1, g-1]} p_{(J\setminus(j))\cup(r)} p_{(L\setminus(r))\cup(j)}. \end{aligned}$$

It degenerates to

$$\mathrm{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(j))\cup(l)} p_{(L\setminus(l))\cup(j)} - p_{(J\setminus(j))\cup(g+1)} p_{(L\setminus(g+1))\cup(j)}.$$

The monomial $p_{(J\setminus(j))\cup(l)} p_{(L\setminus(l))\cup(j)}$ does not vanish on the coordinate ring of $X_{cs_{k+1}s_{l+1}}$ (and thus of $X_{cs_{k_1}\ldots s_{k_r}}$). Hence, $\mathrm{in}_{\mathbf{w}}(R^1_{J,L})$ is not monomial and this finishes the proof. □

Lemma 5 below shows that the Coxeter word $c = s_{n-1} \cdots s_2 s_1$ is in fact special among all Coxeter words regarding the degeneration.

Lemma 5 *Let $w \in S_n$ have a reduced expresion $\underline{w} = s_{i_r} \cdots s_{i_1}$ with $i_k \neq i_l$ for all $k \neq l$. Then none of the Plücker relations degenerates to a monomial in* $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ *if and only if $w \leq c$.*

Proof "$\Leftarrow$" by Proposition 2.
"$\Rightarrow$" Assume $w = s_{i_r} \dots s_{i_1}$ is a product of pairwise distinct simple reflections. First note that $w \not\leq c$ implies there exists an $i_k \in \{i_1, \dots, i_r\}$ such that $i_k + 1 = i_l$ for $l < k$. We choose $i = i_k$, such that k is minimal with this property. In particular, if there exists t with $i_t + 1 = i$ then $t < k$. Since s_i commutes with all reflections s_{i_m} with $m > k$, as in this case $i_m \neq i \pm 1$ by minimality of k, we observe

$$w = s_i s_{i_r} \dots s_{i_{k+1}} s_{i_{k-1}} \dots s_{i_1} \in s_i \langle s_1, s_2 \dots, s_{i-1}, s_{i+1}, \dots s_{n-1} \rangle.$$

We deduce that $w([i]) = [i-1] \cup \{i+1\}$. Moreover, notice $w(i+1) \geq i+2$, since $i+1$ is moved only by s_i and s_{i+1}, but we apply s_{i+1} first and by hypothesis there are no other occurrences of s_{i+1}. We can now produce the degree two monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ by choosing as J any sequence such that $F(J) = w([i])$ and $j_1 = i+1$, and as L any sequence with $F(L) = [i] \cup \{i+2\}$, so that

$$R^1_{J,L} = p_J p_L - p_{(J\setminus(i+1))\cup(i+2)} p_{(L\setminus(i+2))\cup(i+1)} - p_{(J\setminus(i+1))\cup(i)} p_{(L\setminus(i))\cup(i+1)},$$
$$\text{in}_{\mathbf{w}}(R^1_{J,L}) = p_J p_L - p_{(J\setminus(i+1))\cup(i+2)} p_{(L\setminus(i+2))\cup(i+1)}.$$

As $[i-1] \cup \{i+2\} \not\leq w([i])$ the second term vanishes on X_w. □

4.4 More and more monomials

If we can write a permutation $u \in S_n$ as a product of two permutations v, w belonging to two distinct parabolic subgroups which centralize each other, then we can check how a Plücker relation degenerates on $\mathcal{I}_u$ by looking at the ideals $\mathcal{I}_v$ and $\mathcal{I}_w$. Lemma 6 concerns defining ideals for Schubert varieties and allows us to deduce Corollary 3, which suggests an inductive procedure on n to find Schubert varieties that become reducible under Feigin's degeneration.

Lemma 6 *For $v, w \in S_n$ assume there exist two sets of simple reflections $\mathcal{S}_v = \{s_{i_1}, \dots, s_{i_r}\}$ and $\mathcal{S}_w = \{s_{j_1}, \dots, s_{j_s}\}$ such that $|i_h - j_l| > 1$ for all $h \in [r]$, $l \in [s]$ with $v \in \langle \mathcal{S}_v \rangle$ and $w \in \langle \mathcal{S}_w \rangle$. Then for all sequences J, L with $k \leq \#J$ we have*

$$R^k_{J,L}|_{X_{vw}} = R^k_{J,L}|_{X_v} \text{ or } R^k_{J,L}|_{X_{vw}} = R^k_{J,L}|_{X_w}.$$

Corollary 3 *Let $v, w \in S_n$ assume there exist two sets of simple reflections $\mathcal{S}_v = \{s_{i_1}, \dots, s_{i_r}\}$ and $\mathcal{S}_w = \{s_{j_1}, \dots, s_{j_s}\}$ such that $|i_h - j_l| > 1$ for all $h \in [r]$, $l \in [s]$ with $v \in \langle \mathcal{S}_v \rangle$ and $w \in \langle \mathcal{S}_w \rangle$. Then*

(1) *None of the $R^k_{J,L}$ degenerates to a monomial nor in $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ neither in $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$, if and only if none of the $R^k_{J,L}$ degenerates to a monomial in $\text{in}_{\mathbf{w}}(\mathcal{I}_{vw})$.*
(2) *If $\text{in}_{\mathbf{w}}(\mathcal{I}_w)$ or $\text{in}_{\mathbf{w}}(\mathcal{I}_v)$ contains a monomial degenerate Plücker relation, then so does $\text{in}_{\mathbf{w}}(\mathcal{I}_{vw})$.*

Remark 7 From the previous corollary we see that the bigger n is, the more Schubert varieties become reducible after degenerating them à la Feigin, since there are several ways of embedding S_m into S_n for $m < n$ as a parabolic subgroup. Indeed, the number of permutations $v \in S_n$ such that at least one Plücker relation degenerates to a monomial in $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_v)$ is 0,1,11,85 for $n = 2, 3, 4, 5$, respectively. We would like to mention here the observation that there is exactly one sequence in the On-Line Encyclopedia of Integer Sequences [19, Sequence A129180] whose first four terms are 0, 1, 11, 85, namely the *Total area below all Schroeder paths of semilength n*.

Example 2 From Table 1 we see that there are three permutation to which none of our results apply, namely s_3s_1, $s_2s_3s_2s_1$ and $s_3s_1s_2s_1$. In the first case the initial ideal coincides with the ideal defining the Schubert variety. The latter two though are examples of permutations w where $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w) \neq \mathcal{I}_w$ but nonetheless $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ is prime. In these cases we don't know if the degenerate Schubert varieties are (up to isomorphism) Schubert varieties in some other flag variety or if they are genuinely *different* varieties.

5 Degenerate Schubert and Richardson varieties

In this section we explore how degenerate Schubert varieties behave under the embedding of the degenerate flag variety $\mathcal{F}\ell_n^a$ into a larger partial flag variety given by Cerulli Irelli and the second author in [6].

5.1 Degenerate flag varieties and flag varieties of higher rank

We start by introducing some notation and recalling the main result of [6].

Let ω_i denote the i-th fundamental weight for SL_{2n-2} and consider the parabolic subgroup $P := P_{\omega_1+\omega_3+\cdots+\omega_{2n-3}}$ of SL_{2n-2}. Then, SL_{2n-2}/P is the variety of (partial) flags in $\mathbb{C}^{2n-2}$ whose points are flags of vector spaces of odd dimensions. Its Schubert varieties $\widetilde{X}_w$ are indexed by minimal length coset representatives $w \in S_{2n-2}/W_P$, where W_P is the Weyl group of the Levi of P. More precisely, if $\widetilde{s_i} \in S_{2n-2}$ denotes the simple transposition $(i, i+1)$, then $W_P = \langle \widetilde{s}_2, \widetilde{s}_4, \ldots \widetilde{s}_{2n-4}\rangle$. Let $w_n \in S_{2n-2}$ be defined by

$$w_n(i) = \begin{cases} r & \text{if } i = 2r, r \geq 1, \\ n + r - 1 & \text{if } i = 2r - 1, r \in [n-1]. \end{cases}$$

The following Theorem can be found in [6].

Theorem 4 ([6]) *The degenerate flag variety $\mathcal{F}\ell_n^a$ is isomorphic to the Schubert variety $\widetilde{X}_{w_n} \subset SL_{2n-2}/P$.*

5.1.1 Translation into Plücker coordinates

We describe here the isomorphism of Theorem 4 in terms of Plücker coordinates. Recall that whenever we index Plücker coordinates by a set, we really mean the associated sequence obtained by increasingly ordering the elements of the given set.

Let $J \in \binom{[2n-2]}{2k-1}$, with $k \in [n-1]$, then $J \leq w_n([2k-1]) = [k-1] \cup [n, n+k-1]$ if and only if

$$[k-1] \subset J \subset [k+n-1]. \tag{5.1}$$

In order to give the translation of the isomorphism in terms of coordinate rings, we need to set some notation. Let $k \in [n-1]$, we denote by $\{\leq w_n\}^{(2k-1)}$ the set of $J \in \binom{[2n-2]}{2k-1}$, with $J \leq w_n([2k-1])$. There is hence a bijection

$$\{\leq w_n\}^{(2k-1)} \to \binom{[n]}{k}, \quad J \mapsto \tau_k(J \setminus [k-1]) \tag{5.2}$$

where $\tau_k : [n+k-1] \to [n]$ is given by

$$\tau_k(j) \mapsto \begin{cases} j & \text{if } j \in [k,n], \\ j-n & \text{if } j \in [n+1, n+k-1]. \end{cases}$$

For a sequence $I = (i_1, \ldots, i_k) \in \mathcal{S}(n,k)$ we set $\tau_k(I) := (\tau_k(i_1), \ldots \tau_k(i_k)) \in \mathcal{S}(n,k)$. If $\rho_k : [n] \to [k, n+k-1]$ is given by

$$\rho_k(j) \mapsto \begin{cases} j & \text{if } j \in [k,n], \\ j+n & \text{if } j \in [k-1], \end{cases}$$

then the inverse map to (5.2) is given by

$$\binom{[n]}{k} \to \{\leq w_n\}^{(2k-1)}, \quad I \mapsto [k-1] \cup \rho_k(I).$$

On the level of sequences, this lifts to a map

$$\begin{array}{rcl} \mathcal{S}(n,k) & \stackrel{\widetilde{\rho}_k}{\to} & \left\{J \in \mathcal{S}(2n-2, 2k-1) \mid F(J) \in \{\leq w_n\}^{(2k-1)}\right\}, \\ (i_1, \ldots, i_k) & \mapsto & (1, 2, \ldots, k-1, \rho_k(i_1), \ldots, \rho_k(i_k)) \end{array}$$

Fix an ordered basis $(\tilde{e}_j)_{j \in [2n-2]}$ of $\mathbb{C}^{2n-2}$, then the linear algebraic description of $\widetilde{X}_{w_n}$ is

$$\widetilde{X}_{w_n} = \left\{ \{0\} \subset W_1 \subset W_3 \subset \ldots \subset W_{2n-3} \;\middle|\; \begin{array}{c} W_{2k-1} \in \mathrm{Gr}(2k-1, \mathbb{C}^{2n-2}) \\ \mathrm{span}_{\mathbb{C}}\{\widetilde{e}_j \mid j \in [k-1]\} \subset W_{2k-1}, \\ W_{2k-1} \subset \mathrm{span}_{\mathbb{C}}\{\widetilde{e}_j \mid j \in [n+k-1]\}. \end{array} \right\}$$

Denote by $(e_i)_{i \in [n]}$ an ordered basis for $\mathbb{C}^n$. For $k \in [n-1]$ define the projection operator (which we also denote by π_k as in [6])

$$\begin{array}{rrl} \pi_k : \mathrm{span}_{\mathbb{C}}\{\widetilde{e}_j \mid j \in [n+k-1]\} \to & \mathbb{C}^n = & \mathrm{span}_{\mathbb{C}}\{e_i \mid i \in [n]\}, \\ & \widetilde{e}_j \mapsto & \begin{cases} e_{\tau_k(j)} & \text{if } j \in [k, n+k-1], \\ 0 & \text{otherwise} \end{cases} \end{array}$$

Then there is an isomorphism, which we denote by the same symbol, of algebraic varieties

$$\widetilde{X}_{w_n}^{(2k-1)} := \left\{ U \,\middle|\, \begin{array}{c} U \in \mathrm{Gr}(2k-1, \mathbb{C}^{2n-2}) \\ \mathrm{span}_{\mathbb{C}}\{\widetilde{e}_j \mid j \in [2i-2]\} \subset U, \\ U \subset \mathrm{span}_{\mathbb{C}}\{\widetilde{e}_j \mid j \in [n+2k-2]\}. \end{array} \right\} \xrightarrow{\pi_k} \mathrm{Gr}(k, \mathbb{C}^n),$$
$$U \mapsto \pi_k(U)$$

and the desired isomorphism (cf. [6]) is given by

$$\xi : \widetilde{X}_{w_n} \to \mathcal{F}\ell_n^a, \quad (W_{2k-1})_{k\in[n-1]} \mapsto (\pi_k(W_{2k-1}))_{k\in[n-1]}. \tag{5.3}$$

Remark 8 In [6], an embedding of $\zeta : \mathcal{F}\ell_n^a \hookrightarrow SL_{2n-2}/P$ is given, and hence the isomorphism from Theorem 4 is rather the inverse of the isomorphism ξ we consider here. We prefer to work with ξ instead of ζ since in this way we obtain an induced map from the coordinate ring of $\mathcal{F}\ell_n^a$ to the coordinate ring of $\widetilde{X}_{w_n}$, which we make explicit in the following.

For SL_{2n-2}/P we also have an embedding into the product of Grassmannians

$$SL_{2n-2}/P \hookrightarrow \mathrm{Gr}(1, \mathbb{C}^{2n-2}) \times Gr(3, \mathbb{C}^{2n-2}) \times \cdots \times \mathrm{Gr}(2n-3, \mathbb{C}^{2n-2}),$$

and hence a Plücker embedding. Plücker coordinates for $\mathrm{Gr}(2k-1, \mathbb{C}^{2n-2})$ with $k \in [n-1]$ are denoted by $\tilde{p}_J$, $J \in \mathcal{S}(2n-2, 2k-1)$. Let $I = (i_1, \ldots, i_k)$ then

$$\pi_k^* : \mathbb{C}[\mathrm{Gr}(k,n)] \to \mathbb{C}[\widetilde{X}_w^{(2k-1)}], \quad p_I \mapsto \widetilde{p}_{\widetilde{\rho}_k(I)}.$$

As π_k^* is compatible with Plücker relations, we have an isomorphism

$$\xi^* : \mathbb{C}[\mathcal{F}\ell_n^a] \to \mathbb{C}[\widetilde{X}_{w_n}], \quad p_I \mapsto \pi_{\#I}^*(p_I).$$

Notice that even if I is ordered increasingly, $\widetilde{\rho}_k(I)$ needs not be ordered increasingly. To get an increasing sequence we have to multiply by some sign. While keeping track of the sign is fundamental to check that Plücker relations are satisfied, it is not relevant to us, as we only deal with vanishing of certain Plücker coordinates, which of course vanish independently of their sign.

5.2 Richardson varieties in SL_{2n-2}/P

Let $u, v \in S_{2n-2}$ be minimal length coset representatives of S_{2n-2}/W_P and assume that $u \leq v$. We denote by $\widetilde{X}_v^u := \widetilde{X}_v \cap \widetilde{X}^u \subseteq SL_{2n-2}/P$ the corresponding Richardson variety. Recall that its defining ideal in $\mathbb{C}[p_I \mid \#I \equiv 1 (\mathrm{mod}\ 2),\ I \subset [2n-2]]$ is

$$\mathcal{I}_v^u = (R_{J,L}^k) + (p_I)_{I \not\leq v([\#I])} + (p_I)_{I \not\geq u([\#I])}. \tag{5.4}$$

In the following we will show that for appropriate permutations $x \in S_n$, $u, v \in S_{2n-2}$ with $u \leq v \leq w_n$, the isomorphism ξ^* induces an isomorphism between the coordinate rings

$$\mathbb{C}[X_x^a] \to \mathbb{C}[\widetilde{X}_v^u].$$

To stress out the fact that such an isomorphism really comes from the embedding ζ, we will express it as $\zeta(X_x^a) = \widetilde{X}_v^u$.

Since $\mathbb{C}[X_x^a] = \mathbb{C}[\mathcal{F}\ell_n^a]/(p_I \mid I \nleq x([\#I]))$ and $\mathbb{C}[\widetilde{X}_v^u] = \mathbb{C}[SL_{2n-2}/P]/(p_K \mid K \nleq v([\#K]),\ K \ngeq u([\#K]))$, the claim will be proven by verifying that

$$((K \leq v([\#K]) \text{ and } K \geq u([\#K])) \quad \Rightarrow \quad \tau_k(K \setminus [k-1]) \leq x([k]), \tag{5.5}$$

where $k := \frac{\#K+1}{2}$, and the opposite direction

$$I \leq x(\#I) \quad \Rightarrow \quad \begin{pmatrix} [k-1] \cup \rho_{\#I}(I) \leq v([n-1+\#I]) \\ [k-1] \cup \rho_{\#I}(I) \geq u([n-1+\#I]) \end{pmatrix}. \tag{5.6}$$

An important role will be played by the following permutation $y_n \in S_{2n-2}$:

$$y_n(i) = \begin{cases} 1 & \text{if } i = 1, \\ r+1 & \text{if } i = 2r, r \in [n-1], \\ n+r-1 & \text{if } i = 2r-1, r \in [n-1]. \end{cases}$$

Notice that for any $m \in [n-1]$

$$\tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n(i) = \begin{cases} m+1 & \text{if } i = 1, \\ r & \text{if } i = 2r,\ r \in [m], \\ r+1 & \text{if } i = 2r,\ r \in [m+1, n-1], \\ n+r-1 & \text{if } i = 2r-1, r \in [n-1], \end{cases}$$

and, by (2.2), $y_n < \tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n \leq w_n$.

Lemma 7 *Let $m \in [n-1]$ and $x := s_m s_{m-1} \dots s_1 \in S_n$. Then,*

$$\zeta(X_x^a) = \widetilde{X}^{y_n}_{\tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n}.$$

Proof Let $I \in \binom{[n]}{k}$. Then, by (2.3), $I \leq x([k])$ if and only if

$$I = \begin{cases} [k-1] \cup \{i\}, i \in [k, m+1] & \text{if } k \leq m, \\ [k] & \text{if } k > m. \end{cases}$$

On the other hand, let $K \in \binom{2n-2}{2k-1}$, then both $K \leq \tilde{s}_m \tilde{s}_{m-1} \dots \tilde{s}_1 y_n([2k-1])$ and $K \geq y_n([2k-1])$ hold if and only if

$$K = \begin{cases} [k-1] \cup [n+1, n+k-1] \cup \{i\}, i \in [k, m+1] & \text{if } k \leq m, \\ [k] \cup [n+1, n+k-1] & \text{if } k > m. \end{cases}$$

These two facts imply (5.5) and (5.6). □

Combining Lemma 7 with Proposition 2 we obtain the following corollary.

Corollary 4 *Let $x = s_m s_{m-1} \cdots s_1 \leq c$ and consider the Schubert variety $X_x \subset \mathcal{F}\ell_n$. Then there is an isomorphism*

$$X_x \cong \widetilde{X}^{y_n}_{\tilde{s}_m \tilde{s}_{m-1} \ldots \tilde{s}_1 y_n} \subset SL_{2n-2}/P.$$

6 Schubert divisors

In this section we focus on Schubert divisors and apply the results from previous sections to them. In this case we can completely answer the question whether or not they stay irreducible under the degeneration.

Let $w_0 \in S_n$ be the longest element, then all Schubert divisors are indexed by permutations of the form $w = w_0 s_i$ for $i \in [n-1]$. Note that

$$w(k) = \begin{cases} n-k+1 & \text{if } k \neq i, i+1, \\ n-i & \text{if } k = i, \\ n-i+1 & \text{if } k = i+1. \end{cases}$$

The following Theorem 5 is an application of Theorem 3 (1) and (2).

Theorem 5 *Let $n > 2$ and $w \in S_n$ be such that $ws_i = w_0$. If n is odd assume $i \neq \frac{n+1}{2}$, for even n there is no additional assumption. Then X^a_w is reducible.*

Proof We consider four cases separately: $i < \frac{n}{2}$, $i = \frac{n}{2}$, $i \geq \frac{n+3}{2}$, and $i = \frac{n+2}{2}$. Notice that they cover all possiblities, since $i > \frac{n}{2}$ together with the assumption $i \neq \frac{n+1}{2}$ implies $i > \frac{n+1}{2}$, hence $i \geq \frac{n+2}{2}$. We will deal with the first two cases by applying Theorem 3 (1), while we will use Theorem 3 (2) for the remaining two.

First of all, notice that $w_0 = ws_i > w$.

Case 1: If $i < \frac{n}{2}$, then

$$w(k) = n-k+1 \geq n-i+2 > \frac{n}{2}+2 > i, \quad \text{for any } k \leq i-1, \tag{6.1}$$
$$w(i) = n-i > \frac{n}{2} > i,$$

and

$$w(i+1) = n-i+1 > \frac{n}{2}+1 > i. \tag{6.2}$$

We conclude that $i \notin w([i+1])$ and we can hence apply Theorem 3 (1) with $j = w(i)$.

Case 2: If $i = \frac{n}{2}$, then (6.1) and (6.2) still hold, but $w(i) = i$, so that $i \notin w([i-1]) \cup \{w(i+1)\}$, but we cannot choose $j = w(i)$. Nevertheless, (6.1) and (6.2) imply that any j with $j \leq i-1 < i = w(i)$ (which exists, since $n > 2$) fulfills the hypotheses of Theorem 3 (1).

Case 3: Let $i \geq \frac{n+3}{2}$, so that $n \leq 2i-3$ and $n-i+2 \leq 2i-3-i+2 = i-1$. Note further that $w(i+1) = n+i-1 \leq \frac{n+3}{2} - 1 \leq i-1$. Thus $w(n-i+2) = i-1 \in w([i-1])$ and we can apply Theorem 3 (2) with $l = w(i)$ and $x = w(i+1)$.
Case 4: Consider $i = \frac{n}{2} + 1$. In this case, $w(i+1) = n-i+1 = \frac{n}{2} = i-1 \in w([i-1]) \cup \{w(i+1)\}$ and we can apply Theorem 3 (2) with x any element in $w([i-1])$ and $l = w(i)$. □

For flag varieties $\mathcal{F}\ell_n$ with n odd, the next proposition explains why the case of $w_0 s_i$ for $i = \frac{n+1}{2}$ is special. This is another instance, of a degenerate Schubert variety being isomorphic to a Richardson variety in SL_{2n-2}/P. However, unlike the degenerate Schubert varieties of form X_v^a, for $v \leq c$, this one is not isomorphic to the original Schubert variety.

Proposition 5 *Let $i \geq 2$ and $n = 2i-1$. Then $\zeta(X^a_{w_0 s_i}) = \widetilde{X}^{\tilde{s}_{2i-1}}_{w_n}$.*

Proof First note that $w_0 s_i([i]) = \{n-i\} \cup [n-i+2, n] = \{i-1\} \cup [i+1, n]$ and $w_0([i]) = [n-i+1, n] = [i, n]$. Let $J \in \binom{n}{k}$, then $J \not\leq w_0 s_i([k]) = [n-k+1, n]$ if and only if $k = i$ and $J = [i, n]$.

On the other hand, recall that $w_n([2k-1]) = [k-1] \cup [n+k-1, n]$ and

$$\tilde{s}_{2i-1}([2k-1]) = \begin{cases} [2k-1] & \text{if } k \neq i, \\ [2i-2] \cup \{2i\} & \text{if } k = i. \end{cases}$$

If $K \in \binom{2n-2}{2k-1}$ is such that $K \leq w_n([2k-1])$, then $K \not\geq \tilde{s}_{2i-1}([2k-1])$ if and only if $k = i$ and $K = [2i-1] = [n]$.

At this point the claim follows from $\pi_i^*(p_{[i,n]}) = \tilde{p}_{[i-1]\cup\rho_i([i,n])} = \tilde{p}_{[n]}$. □

Corollary 5 (1) *If n is even, then all Schubert divisors $X_{w_0 s_i} \subset \mathcal{F}\ell_n$ become reducible under Feigin's degeneration.*

(2) *If n is odd, then the Schubert divisor $X_{w_0 s_{\frac{n+1}{2}}} \subset \mathcal{F}\ell_n$ stays irreducible under Feigin's degeneration, while all the others become reducible.*

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Data Availability All data mentiones throughout the article is available online at the homepage https://www.matem.unam.mx/\protect\unhbox\voidb@x\penalty\@M\lara/schubert/.

Table 2 Initial ideals $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ (see Sect. 2.3) for $w \in S_5$ and which criteria to detect initial monomials from Theorem 3 apply

w one-line	w reduced word	mono	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[1, 2, 3, 4, 5]	1	–	–	–	–	–	–	–	–
[1, 2, 3, 5, 4]	s_4	–	–	–	–	–	–	–	–
[1, 2, 4, 3, 5]	s_3	–	–	–	–	–	–	–	–
[1, 2, 4, 5, 3]	s_3s_4	×	×	–	×	–	–	×	–
[1, 2, 5, 3, 4]	s_4s_3	–	–	–	–	–	–	–	–
[1, 2, 5, 4, 3]	$s_3s_4s_3$	–	–	–	–	–	–	–	–
[1, 3, 2, 4, 5]	s_2	–	–	–	–	–	–	–	–
[1, 3, 2, 5, 4]	s_4s_2	–	–	–	–	–	–	–	–
[1, 3, 4, 2, 5]	s_2s_3	×	×	–	×	–	–	×	–
[1, 3, 4, 5, 2]	$s_2s_3s_4$	×	×	–	×	×	–	×	–
[1, 3, 5, 2, 4]	$s_4s_2s_3$	×	×	–	×	–	–	×	–
[1, 3, 5, 4, 2]	$s_2s_3s_4s_3$	×	×	–	×	×	–	×	–
[1, 4, 2, 3, 5]	s_3s_2	–	–	–	–	–	–	–	–
[1, 4, 2, 5, 3]	$s_3s_4s_2$	×	–	–	×	–	–	×	–
[1, 4, 3, 2, 5]	$s_2s_3s_2$	–	–	–	–	–	–	–	–
[1, 4, 3, 5, 2]	$s_2s_3s_4s_2$	×	×	–	–	×	–	×	–
[1, 4, 5, 2, 3]	$s_3s_4s_2s_3$	×	×	×	×	–	×	×	–
[1, 4, 5, 3, 2]	$s_2s_3s_4s_2s_3$	×	×	–	×	–	–	×	–
[1, 5, 2, 3, 4]	$s_4s_3s_2$	–	–	–	–	–	–	–	–
[1, 5, 2, 4, 3]	$s_3s_4s_3s_2$	–	–	–	–	–	–	–	–
[1, 5, 3, 2, 4]	$s_4s_2s_3s_2$	–	–	–	–	–	–	–	–
[1, 5, 3, 4, 2]	$s_2s_3s_4s_3s_2$	×	×	–	–	×	–	–	×
[1, 5, 4, 2, 3]	$s_3s_4s_2s_3s_2$	×	–	×	–	–	×	–	–
[1, 5, 4, 3, 2]	$s_2s_3s_4s_2s_3s_2$	–	–	–	–	–	–	–	–
[2, 1, 3, 4, 5]	s_1	–	–	–	–	–	–	–	–
[2, 1, 3, 5, 4]	s_4s_1	–	–	–	–	–	–	–	–
[2, 1, 4, 3, 5]	s_3s_1	–	–	–	–	–	–	–	–
[2, 1, 4, 5, 3]	$s_3s_4s_1$	×	×	–	×	–	–	–	–
[2, 1, 5, 3, 4]	$s_4s_3s_1$	–	–	–	–	–	–	–	–
[2, 1, 5, 4, 3]	$s_3s_4s_3s_1$	–	–	–	–	–	–	–	–
[2, 3, 1, 4, 5]	s_1s_2	×	×	–	×	–	–	×	–
[2, 3, 1, 5, 4]	$s_4s_1s_2$	×	×	–	×	–	–	×	–
[2, 3, 4, 1, 5]	$s_1s_2s_3$	×	×	–	×	×	–	×	–
[2, 3, 4, 5, 1]	$s_1s_2s_3s_4$	×	×	–	×	×	–	×	–
[2, 3, 5, 1, 4]	$s_4s_1s_2s_3$	×	×	–	×	×	–	×	–
[2, 3, 5, 4, 1]	$s_1s_2s_3s_4s_3$	×	×	–	×	×	–	×	–
[2, 4, 1, 3, 5]	$s_3s_1s_2$	×	×	–	×	–	–	×	–
[2, 4, 1, 5, 3]	$s_3s_4s_1s_2$	×	×	–	×	–	–	×	–
[2, 4, 3, 1, 5]	$s_1s_2s_3s_2$	×	×	–	×	×	–	×	–

Table 2 continued

w one-line	w reduced word	mono	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[2, 4, 3, 5, 1]	$s_1s_2s_3s_4s_2$	×	×	–	×	×	–	×	–
[2, 4, 5, 1, 3]	$s_3s_4s_1s_2s_3$	×	×	×	×	×	×	×	–
[2, 4, 5, 3, 1]	$s_1s_2s_3s_4s_2s_3$	×	×	–	×	×	–	×	–
[2, 5, 1, 3, 4]	$s_4s_3s_1s_2$	×	×	–	×	–	–	×	–
[2, 5, 1, 4, 3]	$s_3s_4s_3s_1s_2$	×	×	–	×	–	–	×	–
[2, 5, 3, 1, 4]	$s_4s_1s_2s_3s_2$	×	×	–	×	×	–	×	–
[2, 5, 3, 4, 1]	$s_1s_2s_3s_4s_3s_2$	×	×	–	×	×	–	×	–
[2, 5, 4, 1, 3]	$s_3s_4s_1s_2s_3s_2$	×	×	×	×	×	×	×	–
[2, 5, 4, 3, 1]	$s_1s_2s_3s_4s_2s_3s_2$	×	×	–	×	×	–	×	–
[3, 1, 2, 4, 5]	s_2s_1	–	–	–	–	–	–	–	–
[3, 1, 2, 5, 4]	$s_4s_2s_1$	–	–	–	–	–	–	–	–
[3, 1, 4, 2, 5]	$s_2s_3s_1$	×	–	–	×	–	–	×	–
[3, 1, 4, 5, 2]	$s_2s_3s_4s_1$	×	–	–	×	×	–	×	–
[3, 1, 5, 2, 4]	$s_4s_2s_3s_1$	×	–	–	×	–	–	×	–
[3, 1, 5, 4, 2]	$s_2s_3s_4s_3s_1$	×	–	–	×	×	–	×	–
[3, 2, 1, 4, 5]	$s_1s_2s_1$	–	–	–	–	–	–	–	–
[3, 2, 1, 5, 4]	$s_4s_1s_2s_1$	–	–	–	–	–	–	–	–
[3, 2, 4, 1, 5]	$s_1s_2s_3s_1$	×	×	–	–	×	–	×	–
[3, 2, 4, 5, 1]	$s_1s_2s_3s_4s_1$	×	×	–	–	×	–	×	–
[3, 2, 5, 1, 4]	$s_4s_1s_2s_3s_1$	×	×	–	–	×	–	×	–
[3, 2, 5, 4, 1]	$s_1s_2s_3s_4s_3s_1$	×	×	–	–	×	–	×	–
[3, 4, 1, 2, 5]	$s_2s_3s_1s_2$	×	×	×	×	–	×	×	–
[3, 4, 1, 5, 2]	$s_2s_3s_4s_1s_2$	×	×	–	×	×	×	×	–
[3, 4, 2, 1, 5]	$s_1s_2s_3s_1s_2$	×	×	–	×	–	–	×	–
[3, 4, 2, 5, 1]	$s_1s_2s_3s_4s_1s_2$	×	×	–	×	–	–	×	–
[3, 4, 5, 1, 2]	$s_2s_3s_4s_1s_2s_3$	×	×	×	×	–	–	×	–
[3, 4, 5, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3$	×	×	–	×	–	–	×	–
[3, 5, 1, 2, 4]	$s_4s_2s_3s_1s_2$	×	×	×	×	–	×	×	–
[3, 5, 1, 4, 2]	$s_2s_3s_4s_3s_1s_2$	×	×	–	×	×	×	×	–
[3, 5, 2, 1, 4]	$s_4s_1s_2s_3s_1s_2$	×	×	–	×	–	–	×	–
[3, 5, 2, 4, 1]	$s_1s_2s_3s_4s_3s_1s_2$	×	×	–	×	–	–	×	–
[3, 5, 4, 1, 2]	$s_2s_3s_4s_1s_2s_3s_2$	×	×	×	×	–	–	×	–
[3, 5, 4, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3s_2$	×	×	–	×	–	–	×	–
[4, 1, 2, 3, 5]	$s_3s_2s_1$	–	–	–	–	–	–	–	–
[4, 1, 2, 5, 3]	$s_3s_4s_2s_1$	×	–	–	×	–	–	×	–
[4, 1, 3, 2, 5]	$s_2s_3s_2s_1$	–	–	–	–	–	–	–	–
[4, 1, 3, 5, 2]	$s_2s_3s_4s_2s_1$	×	×	–	–	×	–	×	–
[4, 1, 5, 2, 3]	$s_3s_4s_2s_3s_1$	×	–	×	×	–	×	×	–
[4, 1, 5, 3, 2]	$s_2s_3s_4s_2s_3s_1$	×	–	–	×	–	–	×	–
[4, 2, 1, 3, 5]	$s_3s_1s_2s_1$	–	–	–	–	–	–	–	–

Table 2 continued

w one-line	w reduced word	mono	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[4, 2, 1, 5, 3]	$s_3s_4s_1s_2s_1$	×	–	–	×	–	–	×	–
[4, 2, 3, 1, 5]	$s_1s_2s_3s_2s_1$	×	×	–	–	×	–	–	–
[4, 2, 3, 5, 1]	$s_1s_2s_3s_4s_2s_1$	×	×	–	–	×	–	×	–
[4, 2, 5, 1, 3]	$s_3s_4s_1s_2s_3s_1$	×	×	×	–	×	×	×	–
[4, 2, 5, 3, 1]	$s_1s_2s_3s_4s_2s_3s_1$	×	×	–	–	×	–	×	–
[4, 3, 1, 2, 5]	$s_2s_3s_1s_2s_1$	×	–	×	–	–	×	–	–
[4, 3, 1, 5, 2]	$s_2s_3s_4s_1s_2s_1$	×	–	–	–	×	×	×	–
[4, 3, 2, 1, 5]	$s_1s_2s_3s_1s_2s_1$	–	–	–	–	–	–	–	–
[4, 3, 2, 5, 1]	$s_1s_2s_3s_4s_1s_2s_1$	×	–	–	–	–	–	×	–
[4, 3, 5, 1, 2]	$s_2s_3s_4s_1s_2s_3s_1$	×	×	×	×	–	–	×	–
[4, 3, 5, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3s_1$	×	×	–	×	–	–	×	–
[4, 5, 1, 2, 3]	$s_3s_4s_2s_3s_1s_2$	×	×	×	×	–	×	×	–
[4, 5, 1, 3, 2]	$s_2s_3s_4s_2s_3s_1s_2$	×	×	–	×	–	×	×	–
[4, 5, 2, 1, 3]	$s_3s_4s_1s_2s_3s_1s_2$	×	×	×	×	–	×	×	–
[4, 5, 2, 3, 1]	$s_1s_2s_3s_4s_2s_3s_1s_2$	×	×	–	×	–	–	×	–
[4, 5, 3, 1, 2]	$s_2s_3s_4s_1s_2s_3s_1s_2$	×	×	×	×	–	–	×	–
[4, 5, 3, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3s_1s_2$	×	×	–	×	–	–	×	–
[5, 1, 2, 3, 4]	$s_4s_3s_2s_1$	–	–	–	–	–	–	–	–
[5, 1, 2, 4, 3]	$s_3s_4s_3s_2s_1$	–	–	–	–	–	–	–	–
[5, 1, 3, 2, 4]	$s_4s_2s_3s_2s_1$	–	–	–	–	–	–	–	–
[5, 1, 3, 4, 2]	$s_2s_3s_4s_3s_2s_1$	×	×	–	–	×	–	–	–
[5, 1, 4, 2, 3]	$s_3s_4s_2s_3s_2s_1$	×	–	×	–	–	×	–	–
[5, 1, 4, 3, 2]	$s_2s_3s_4s_2s_3s_2s_1$	–	–	–	–	–	–	–	–
[5, 2, 1, 3, 4]	$s_4s_3s_1s_2s_1$	–	–	–	–	–	–	–	–
[5, 2, 1, 4, 3]	$s_3s_4s_3s_1s_2s_1$	–	–	–	–	–	–	–	–
[5, 2, 3, 1, 4]	$s_4s_1s_2s_3s_2s_1$	×	×	–	–	×	–	–	×
[5, 2, 3, 4, 1]	$s_1s_2s_3s_4s_3s_2s_1$	×	×	–	–	×	–	–	×
[5, 2, 4, 1, 3]	$s_3s_4s_1s_2s_3s_2s_1$	×	×	×	–	×	×	–	×
[5, 2, 4, 3, 1]	$s_1s_2s_3s_4s_2s_3s_2s_1$	×	×	–	–	×	–	–	×
[5, 3, 1, 2, 4]	$s_4s_2s_3s_1s_2s_1$	×	–	×	–	–	×	–	–
[5, 3, 1, 4, 2]	$s_2s_3s_4s_3s_1s_2s_1$	×	–	–	–	×	×	–	–
[5, 3, 2, 1, 4]	$s_4s_1s_2s_3s_1s_2s_1$	–	–	–	–	–	–	–	–
[5, 3, 2, 4, 1]	$s_1s_2s_3s_4s_3s_1s_2s_1$	×	–	–	–	–	–	–	×
[5, 3, 4, 1, 2]	$s_2s_3s_4s_1s_2s_3s_2s_1$	×	×	×	×	–	–	–	×
[5, 3, 4, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3s_2s_1$	×	×	–	×	–	–	–	×
[5, 4, 1, 2, 3]	$s_3s_4s_2s_3s_1s_2s_1$	×	–	×	–	–	×	–	–
[5, 4, 1, 3, 2]	$s_2s_3s_4s_2s_3s_1s_2s_1$	×	–	–	–	–	×	–	–
[5, 4, 2, 1, 3]	$s_3s_4s_1s_2s_3s_1s_2s_1$	×	–	×	–	–	×	–	–

Table 2 continued

w one-line	w reduced word	mono	(1)	(2)	(3)	(4)	(5)	(6)	(7)
[5, 4, 2, 3, 1]	$s_1s_2s_3s_4s_2s_3s_1s_2s_1$	–	–	–	–	–	–	–	–
[5, 4, 3, 1, 2]	$s_2s_3s_4s_1s_2s_3s_1s_2s_1$	×	–	×	–	–	–	–	–
[5, 4, 3, 2, 1]	$s_1s_2s_3s_4s_1s_2s_3s_1s_2s_1$	–	–	–	–	–	–	–	–
120		85	64	22	57	36	22	65	8

Appendix

Table 2 shows which of the criteria for $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ to contain a monomial apply to which elements $w \in S_5$. It has to be read as follows: the first column contains $w \in S_5$ written in one-line notation, the second as a reduced word. In the third column "×" indicates that $\mathrm{in}_{\mathbf{w}}(\mathcal{I}_w)$ contains a monomial, resp. "–" that it does not. The last columns labeled (1) to (7) indicate which of the points of Theorem 3 apply to w. The last row indicates how often × appears in the corresponding column.

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