Advances in Mathematics 458 (2024) 109960

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

From local nets to Euler elements

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ARTICLE INFO

Article history:
Received 19 December 2023
Received in revised form 6 September 2024
Accepted 16 September 2024
Available online xxxx
Communicated by Pramod Achar

MSC:
22D10
81T05

Keywords:
Lie theory
Euler element
Algebraic quantum field theory
Tomita-Takesaki theory
Modular Hamiltonian
Standard subspace

ABSTRACT

Various aspects of the geometric setting of Algebraic Quantum Field Theory (AQFT) models related to representations of the Poincaré group can be studied for general Lie groups, whose Lie algebra contains an Euler element, i.e., $\operatorname{ad} h$ is diagonalizable with eigenvalues in $\{-1, 0, 1\}$. This has been explored by the authors and their collaborators during recent years. A key property in this construction is the Bisognano–Wichmann property (thermal property for wedge region algebras) concerning the geometric implementation of modular groups of local algebras.

In the present paper we prove that under a natural regularity condition, geometrically implemented modular groups arising from the Bisognano–Wichmann property are always generated by Euler elements. We also show the converse, namely that in presence of Euler elements and the Bisognano–Wichmann property, regularity and localizability hold in a quite general setting. Lastly we show that, in this generalized AQFT, in the vacuum representation, under analogous assumptions (regularity and Bisognano–Wichmann), the von Neumann algebras associated to wedge regions are type III_1 factors, a property that is well-known in the AQFT context.

Contents

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https://doi.org/10.1016/j.aim.2024.109960

1. Introduction

This paper is part of a project by the authors and collaborators aiming to deepen the relations between geometric properties of Algebraic Quantum Field Theory (AQFT), Lie theory and unitary representation theory; see [51,53,54,64,67,26].

Starting from fundamental properties of a relativistic quantum theory, the Bisognano–Wichmann (BW) property and the PT symmetry, a generalized setting to study AQFT models has been developed, that starts from the geometry and representations of the symmetry group as fundamental input. Through this description, it was possible to present a new large set of mathematical models in an abstract way (nets on abstract wedge spaces) or a geometric way (nets on open subsets of homogeneous spaces). A key role is played by the Bisognano–Wichmann property which in AQFT models ensures that the vacuum state is thermal for any geodesic observer in a wedge region (see e.g. [38] and references therein). In our context the Bisognano-Wichmann property serves to provide a geometric implementation of modular groups of some local algebras. Along this analysis, a fundamental role has been played by Euler elements that also have been extensively studied in Lie theory (see e.g. [51] and [53]) and here creates a bridge between Lie theory, the AQFT localization properties, and the modular theory of operator algebras.

Nets of standard subspaces (in the one-particle representation) are fundamental objects to analyze properties of AQFT Models. In particular, they play a central role in the recent study of entropy and energy inequalities (see [56,41,16,17] and references therein), new constructions in AQFT ([52,35,43,50]), and in a very large family of models (see references in [24]). Due to the Bisognano–Wichmann property and the PCT symmetry, the language of standard subspaces deeply relates the geometry of the symmetry group with its representation theory and the algebraic objects related to the local von Neumann algebras.

To introduce the main ideas of this paper, we first recall the key steps to understand the setting we developed for this generalized AQFT.

Geometric setting: In the physics context, the underlying manifolds are relativistic spacetimes, i.e., time-oriented Lorentzian manifolds. In Minkowski or de Sitter spacetime localization regions are called wedges and they are specified by one-parameter groups of Lorentz boosts fixing them. On 2-dimensional Minkowski spacetime, the conformal chiral components yield fundamental localization regions, corresponding to circle intervals, which are also specified by one-parameter groups of dilations of the Möbius group. So one can describe fundamental localization regions in terms of generators of certain one-parameter groups in the Lie algebra of the symmetry group. This framework can be generalized to the context where G is a (connected) Lie group whose Lie algebra $\mathfrak{g}$ contains an Euler element h ($\mathrm{ad}\, h$ is diagonalizable with eigenvalues in $\{-1, 0, 1\}$) to construct an abstract version of the correspondence between wedge regions and boost generators. In particular, one can associate to every connected simple Lie group G and any Euler element $h \in \mathfrak{g}$ a non-compactly causal symmetric space $M = G/H$ (see Section 2.1.3 and [53] for details). For the Lorentz group $G = \mathrm{SO}_{1,d}(\mathbb{R})_e$, we thus obtain de Sitter space $M = \mathrm{dS}^d$. In this case we associate to every boost generator (=Euler element) the corresponding wedge region, and, in the general context, a wedge region in M associated to h is a connected component of the open subset on which the flow of h is "future directed" (timelike in the Lorentzian case). More generally, for an Euler element in a reductive Lie algebra $\mathfrak{g}$, there exists a non-compactly causal symmetric space G/H in which one can identify wedge regions W, but localization extends to general non-empty open subsets, see Section 2.1 for details.

AQFT setting: Models in AQFT are determined by nets of von Neumann algebras indexed by open regions (non-empty, connected, open subsets) of the spacetime satisfying fundamental quantum and relativistic assumptions, in particular isotony, locality, Poincaré covariance, positivity of the energy, and existence of the vacuum vector with Reeh–Schlieder property. Nets of standard subspaces arise at least in two natural ways: as the one-particle nets in irreducible Poincaré representations, from which the free fields are obtained by second quantization, and by acting with the self-adjoint part of the local von Neumann algebras on a cyclic separating vacuum vector. The Bisognano–Wichmann property and the anti-unitary PCT symmetry determine the wedge subspaces and the key role in this identification is played by Tomita–Takesaki theory. This technique has been established by Brunetti, Guido and Longo in [11] for cases of physical relevance.

This construction has been realized in a much wider generality by the authors in the current project (cf. the references above) with the following idea: given an involutive automorphism σ of a Lie group G, an (anti-)unitary representation U of the extended group $G_\sigma = G \rtimes \{\mathbf{1}, \sigma\}$ on a Hilbert space $\mathcal{H}$, an Euler element h in the Lie algebra $\mathfrak{g}$ of G, and a G-transitive family $\mathcal{W}_+$ of abstract wedges (fibered over the adjoint orbit of h), then one can associate an "abstract net" $(\mathsf{H}(W))_{W\in\mathcal{W}_+}$ of standard subspaces of $\mathcal{H}$, giving a net only depending on the symmetry group. This construction builds on the Brunetti–Guido–Longo (BGL) construction ([11] and [36]).

Often this net can be realized geometrically on a causal homogeneous space M, in such a way that the abstract wedges acquire a geometric interpretation as wedge regions in M. Here we call a G-space *causal* if its tangent bundle $T(M)$ contains a family $C_m \subseteq T_m(M)$ of pointed, generating, closed convex cones which is invariant under the G-action. Typical examples are time-oriented Lorentzian manifolds on which G acts by time-orientation preserving symmetries or conformal maps. Given a representation of G_σ, one can then try to extend the canonical net of standard subspaces from the set of wedge regions to arbitrary open subsets $\mathcal{O} \subseteq M$. A net of real subspaces associates to open subsets of a causal homogeneous space real subspaces of localized states satisfying properties that are analogous to those of nets of von Neumann algebras: For a unitary representation $(U, \mathcal{H})$ of a connected a Lie group G and a homogeneous space $M = G/H$, we consider families $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ of closed real subspaces of $\mathcal{H}$, indexed by open subsets $\mathcal{O} \subseteq M$, with the following properties:

- (Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$
- (Cov) **Covariance:** $U(g)\mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.
- (RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.
- (BW) **Bisognano–Wichmann property:** There exists an open subset $W \subseteq M$ (called a *wedge region*), such that $\mathsf{H}(W)$ is standard with modular group $\Delta_{\mathsf{H}(W)}^{-it/2\pi} = U(\exp th)$, $t \in \mathbb{R}$, for some $h \in \mathfrak{g}$, for which $\exp(\mathbb{R}h).W \subseteq W$.

So one has to specify the real subspaces associated to wedge regions and identify their properties. There are different possibilities to extend to larger classes of open subsets, that in general do not coincide. One is based on specifying certain generator spaces in which a linear basis may correspond to components of a field on M and then obtain local subspaces in terms of test functions, see [64,26] for irreducible representations and Theorem 4.23 for general representations of reductive groups). Alternatively, one can specify maximal covariant nets which are isotonic and have the (BW) property, here discussed in Section 2.2.4.

In this paper we discuss the necessity and the consequences of considering Euler elements as fundamental objects for our constructions. We will further see how this choice will be consistent with AQFT models. This will be done by facing the following three questions:

Question 1. Is it necessary to consider Euler elements determining fundamental localization regions for one particle nets? Yes, it is a consequence of the Bisognano–Wichmann property and a natural regularity property: Given a standard subspace H whose modular group corresponds to a one-parameter subgroup $(\exp th)_{t\in\mathbb{R}}$ of G (BW property), in Theorem 3.1 we show that h is an Euler element if there exists an e-neighborhood $N \subseteq G$ for which $\bigcap_{g\in N} U(g)\mathtt{V}$ is cyclic. This result is abstract and does not refer to any geometry of wedges or subregions but can be applied to any net of real subspaces satisfying a minimal set of axioms, such as (Iso), (Cov), (RS) and (BW). Our Euler Element Theorem (Theorem 3.1) has particularly striking consequences for such nets. In this setting, it implies in particular that all modular groups, that are geometrically implementable by one-parameter subgroups of finite-dimensional Lie groups in the sense of the (BW) property, are generated by Euler elements. Similar regularity conditions are satisfied in many AQFT models and an analogous property has been used also in [6, Def. 3.1] and [71, Sect. IV.B].

The second question concerns the converse implication:

Question 2: Are the nets of standard subspaces associated to Euler elements regular? More precisely, let $h \in \mathfrak{g}$ be an Euler element, $\tau_h = e^{\pi i \operatorname{ad} h}$ the corresponding involution on $\mathfrak{g}$, and suppose that this involution on $\mathfrak{g}$ integrates to an involution τ_h^G on G, so that we can form the group $G_{\tau_h} := G \rtimes \{\mathrm{id}_G, \tau_h^G\}$. Given an (anti-)unitary representation of this group G_{τ_h}, we consider the canonical standard subspace $\mathtt{V} = \mathtt{V}(h, U) \subseteq \mathcal{H}$, specified by

$$\Delta_{\mathtt{V}} = e^{2\pi i \partial U(h)} \quad \text{and} \quad J_{\mathtt{V}} = U(\tau_h^G)$$

(cf. [11]). A natural way to address such regularity questions is to associate to $\mathtt{V}$ a net $\mathsf{H}^{\max}$ defined on open subsets of a homogeneous space $M = G/H$ by

$$\mathsf{H}^{\max}(\mathcal{O}) := \bigcap_{\mathcal{O}\subseteq g.W} U(g)\mathtt{V}$$

(cf. (20)). If every $g \in G$ with $g.W \subseteq W$ satisfies $U(g)\mathtt{V} \subseteq \mathtt{V}$, this leads to a covariant, isotone net with $\mathsf{H}^{\max}(W) = \mathtt{V}$. Regularity now corresponds to the existence of open subsets $\mathcal{O} \subseteq W$ with $N.\mathcal{O} \subseteq W$ for which $\mathsf{H}^{\max}(\mathcal{O})$ is cyclic (Reeh–Schlieder property). We show that regularity follows if the representation satisfies certain positivity conditions, namely that the “positive energy” cones $C_\pm$ in the abelian Lie subalgebras $\mathfrak{g}_{\pm 1}(h) = \ker(\operatorname{ad} h \mp \mathbf{1})$ are generating; see Theorem 4.9. This requirement can be weakened as follows. If $G = N \rtimes L$ is a semidirect product and we know already that the restriction $U|_L$ satisfies the regularity condition, then it suffices that the intersections $C_\pm \cap \mathfrak{n}_{\pm 1}(h)$ generate $\mathfrak{n}_{\pm 1}(h)$ (Theorem 4.11). This is in particular the case for positive energy representations of the connected Poincaré group $G = \mathcal{P} = \mathbb{R}^{1,d} \rtimes \mathcal{L}_+^\uparrow$. That representations of connected reductive groups always satisfy the regularity condition can be derived from some localizability property asserting for every (anti-)unitary representation the existence of a net on the associated non-compactly causal symmetric space, satisfying (Iso), (Cov), (RS) and (BW) (Theorem 4.23). In particular, the maximal net $\mathsf{H}^{\max}$ has this property. As every algebraic linear Lie group is a semidirect product $G = N \rtimes L$, where N is unipotent and L is reductive [34, Thm. VIII.4.3], many questions related to regularity can be reduced to representations of nilpotent groups. These regularity results include all the physically relevant one-particle models; for instance the U(1)-current and its derivatives (covariant under the Möbius group) satisfy the hypotheses of Theorem 4.9 and so do the one-particle representations of the Poincaré group, to which Theorem 4.11 applies, but not Theorem 4.9.

Question 3: What can we say on nets of von Neumann algebras? Once fundamental localization regions are specified, it is natural to discuss nets of von Neumann algebras on causal homogeneous spaces as above. Such nets exist because second quantization of one-particle nets on causal homogeneous spaces produces nets of operator algebras. Here, second quantization nets correspond to bosonic second quantization in AQFT, in general a spin-statistics result is still to be obtained. For a recent systematic construction of twisted second quantization functors, we refer to [21]. The results on von Neumann algebras presented in this paper apply to general geometric relative position of von Neumann algebras, and second quantization provides examples of nets on causal homogeneous G-spaces. In Section 5, Theorem 5.15 implies that, given a connected Lie group G, when the BW property and a suitable regularity property hold, and there is a unique G-fixed state (the vacuum state), then the wedge algebras are factors of type III_1 with respect to Connes' classification of factors [18]. This extends the known results in AQFT dealing with more specific groups and spaces (see for instance [25,37,27,12,6] and references therein). Here the key property for an Euler element $h \in \mathfrak{g}$ implementing the modular group through the BW property is to be *anti-elliptic*, i.e., any quotient $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ ($\mathfrak{n} \trianglelefteq \mathfrak{g}$ an ideal), for which the image of h in $\mathfrak{q}$ is elliptic[1] is at most one-dimensional and linearly generated by the image of h. If $\mathfrak{g}$ is simple, then $\mathfrak{g}$ has no non-trivial quotients, so that any Euler element $h \in \mathfrak{g}$ is anti-elliptic, but Theorem 5.15 covers much more general situations. We actually do not need to start this discussion with a vacuum vector, but with a vector that is invariant under $U(\exp(\mathbb{R}h))$. The case of a non-unique invariant vector is discussed in Section 5.4 in terms of a direct integral decomposition taking all structures into account.

Along the paper, only very few comments on locality, or its twisted version, will come up. This is because the regularity property as well as the localization property merely refer to a subspace, resp., a subalgebra. To implement (twisted-) locality conditions,

[1] We call $x \in \mathfrak{g}$ *elliptic* if $\operatorname{ad} x$ is semisimple with purely imaginary spectrum, i.e., diagonalizable over $\mathbb{C}$ with purely imaginary eigenvalues.

suitable wedge complements have to be introduced (cf. [51]). In our general setting, some work still has to be done to adapt the second quantization procedure.

Recently, operator algebraic techniques have been very fruitful for the study of energy inequalities. In many of these results the modular Hamiltonian is instrumental. This object corresponds to the logarithm of the modular operator of a local algebra of a specific "wedge region", which in some cases can be identified with the generator of a one-parameter group of spacetime symmetries by the Bisognano–Wichmann property (see for instance [56,41,16,45,17,2,46,44]). In our setting, we start with a general Lie algebra element $h \in \mathfrak{g}$ specifying the flow implemented by the modular operator through the (BW) property. Then

$$\log \Delta_{\mathsf{H}(W)} = 2\pi i \cdot \partial U(h)$$

is the corresponding modular Hamiltonian. In this case, we know from Theorems 3.1 and 5.15 that h has to be an Euler element. *In particular we obtain an abstract algebraic characterization of those elements in the Lie algebra of the symmetry group that may correspond to modular Hamiltonians.* The study of the modular flow on the manifold is particularly relevant. In order to find regions where to prove energy inequalities, one may also need to deform the modular flow ([56,15]). Due to the recent characterization of modular flows on homogeneous space, a specific geometric analysis is expected to be possible.

This paper is structured as follows: In Section 2 we recall the fundamental geometry of Euler elements, both abstractly and on causal homogeneous spaces. In Section 2.1 we recall the geometry of standard subspaces, properties of nets of standard subspaces and the axioms (Iso), (Cov), (BW) and (RS). In particular, Section 2.2.4 introduces minimal and maximal nets on open subsets of a causal homogeneous space $M = G/H$ that are associated to an Euler element $h \in \mathfrak{g}$ and a corresponding wedge region $W \subseteq M$.

In Sections 3, 4 and 5 we discuss Questions 1,2 and 3, respectively. Our key result, the Euler Element Theorem (Theorem 3.1) is proved in Subsection 3.1. In Subsection 3.2 we describe its implications for operator algebras with cyclic separating vectors (Theorems 3.7 and 3.9). The main results of Subsection 4.1 are Theorems 4.9 and 4.11, deriving regularity from positive spectrum conditions. In Subsection 4.2, we turn to localizability aspects of nets of real subspaces. Here our main results are Theorem 4.23, asserting localizability for reductive groups in all representations in all non-empty open subsets of the associated non-compactly causal symmetric space for a suitably chosen wedge region. This allows us to derive that, for the Poincaré group, localizability in spacelike cones is equivalent to the positive energy condition (Theorem 4.25). In Section 5 we continue the discussion of applications of our results to standard subspaces and von Neumann algebras $\mathcal{M}$ by systematically using Moore's Eigenvector Theorem 5.1. The first main result in this section is Theorem 5.11, characterizing for (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h} the subspace $\mathtt{V}_G = \bigcap_{g \in G} U(g)\mathtt{V}$ as the set of fixed points of a certain normal subgroup specified by Moore's Theorem. The second one is Theorem 5.15 that combines Moore's Theorem with Theorem 3.7 to obtain a criterion for $\mathcal{M}$ to be a factor of type III_1. If $\mathcal{M}$ is not a factor, but $\mathcal{M}'$ and $\mathcal{M}$ are conjugate under G, we show that all the structure we discuss survives the central disintegration of $\mathcal{M}$.

We conclude this paper with an outlook section and four appendices, concerning background on operator algebras, unitary Lie group representations, direct integrals, and some more specific observations needed to discuss examples.

Notation

- Strips in $\mathbb{C}$: $\mathcal{S}_\beta = \{z \in \mathbb{C}\colon 0 < \operatorname{Im} z < \beta\}$ and $\mathcal{S}_{\pm\beta} = \{z \in \mathbb{C}\colon |\operatorname{Im} z| < \beta\}$.
- The neutral element of a group G is denoted e, and G_e is the identity component.
- The Lie algebra of a Lie group G is denoted $\mathbf{L}(G)$ or $\mathfrak{g}$.
- For an involutive automorphism σ of G, we write $G^\sigma = \{g \in G\colon \sigma(g) = g\}$ for the subgroup of fixed points and $G_\sigma := G \rtimes \{\mathrm{id}_G, \sigma\}$ for the corresponding group extension.
- $\mathrm{AU}(\mathcal{H})$ is the group of unitary or antiunitary operators on a complex Hilbert space.
- An (anti-)unitary representation of G_σ is a homomorphism $U\colon G_\sigma \to \mathrm{AU}(\mathcal{H})$ with $U(G) \subseteq \mathrm{U}(\mathcal{H})$ for which $J := U(\sigma)$ is antiunitary, i.e., a conjugation.
- Unitary or (anti-)unitary representations on the complex Hilbert space $\mathcal{H}$ are denoted as pairs $(U, \mathcal{H})$.
- $\overline{U}$ is the canonical unitary representation on the complex conjugate space $\overline{\mathcal{H}}$, where the operators $\overline{U}(g) = U(g)$ are the same, but the complex structure is given by $I\xi := -i\xi$.
- If $(U, \mathcal{H})$ is a unitary representation of G and J a conjugation with $JU(g)J = U(\sigma(g))$ for $g \in G$, the canonical extension $U^\sharp$ of U to G_σ is specified by $U^\sharp(\sigma) := J$ (cf. Definition 2.23).
- If G is a group acting on a set M and $W \subseteq M$ a subset, then the stabilizer subgroup of W in G is denoted $G_W := \{g \in G\colon g.W = W\}$, and $S_W := \{g \in G\colon g.W \subseteq W\}$.
- A closed real subspace $\mathtt{V}$ of a complex Hilbert space $\mathcal{H}$ is called *standard* if $\mathtt{V} \cap i\mathtt{V} = \{0\}$ and $\mathtt{V} + i\mathtt{V}$ is dense in $\mathcal{H}$.
- If $\mathfrak{g}$ is a Lie algebra and $h \in \mathfrak{g}$, then $\mathfrak{g}_\lambda(h) = \ker(\operatorname{ad} h - \lambda\mathbf{1})$ is the λ-eigenspace of $\operatorname{ad} h$ and $\mathfrak{g}^\lambda(h) = \bigcup_k \ker(\operatorname{ad} h - \lambda\mathbf{1})^k$ is the generalized λ-eigenspace.
- An element h of a Lie algebra $\mathfrak{g}$ is called
  - *hyperbolic* if $\operatorname{ad} h$ is diagonalizable over $\mathbb{R}$
  - *elliptic* or *compact* if $\operatorname{ad} h$ is semisimple with purely imaginary spectrum, i.e., $\overline{e^{\mathbb{R} \operatorname{ad} h}}$ is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$.
- A *causal G-space* is a smooth G-space M, endowed with a G-invariant *causal structure*, i.e., a field $(C_m)_{m \in M}$ of pointed generating closed convex cones $C_m \subseteq T_m(M)$.
- For a unitary representation $(U, \mathcal{H})$ of G we write:
  - $\partial U(x) = \frac{d}{dt}\big|_{t=0} U(\exp tx)$ for the infinitesimal generator of the unitary one-parameter group $(U(\exp tx))_{t \in \mathbb{R}}$ in the sense of Stone's Theorem.

– $\mathtt{d}U\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ for the representation of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{H}^\infty$ of smooth vectors. Then $\partial U(x) = \overline{\mathtt{d}U(x)}$ (operator closure) for $x \in \mathfrak{g}$.

Acknowledgment: We thank Roberto Longo and Detlev Buchholz for helpful discussions. We also thank the referee for reading the paper so carefully and for many suggestions, including a relaxation of the assumptions of Theorem 3.1. VM was partially supported by the MIUR Excellence Department Project 2023-2027 MatMod@TOV (CUP E83C23000330006) awarded to the Department of Mathematics of University of Rome Tor Vergata, by the Fondi di Ricerca Scientifica d'Ateneo 2021, OAQM, CUP E83C22001800005, and the European Research Council Advanced Grant 669240 QUEST. VM also thanks INdAM-GNAMPA. The research of K.-H. Neeb was partially supported by DFG-grant NE 413/10-2.

2. Preliminaries

In this section we recall fundamental geometric structures related to Euler elements of Lie algebras and corresponding symmetric spaces. Its main purpose is to introduce notation and some general techniques that will be used throughout the paper. Subsection 2.1 deals with abstract wedge spaces of graded Lie groups G_σ and how they can be related to sets of wedge regions in homogeneous causal G-spaces $M = G/H$. Subsection 2.2 then turns to nets of real subspaces $\mathsf{H}(\mathcal{O})$, associated to open subsets $\mathcal{O}$ of some homogeneous space of G. Here we introduce the basic axioms (Iso), (Cov), (RS) and (BW). We also show that, if (BW) holds for some $h \in \mathfrak{g}$ and some wedge region $W \subseteq M$, for which $g.W \subseteq W$ implies $g.\mathsf{H}(W) \subseteq \mathsf{H}(W)$, we obtain minimal and maximal isotone, covariant nets $\mathsf{H}^{\rm min}$ and $\mathsf{H}^{\rm max}$ satisfying (BW), such that any other net H with these properties satisfies

$$\mathsf{H}^{\rm min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}^{\rm max}(\mathcal{O})$$

on all open subsets $\mathcal{O} \subseteq M$. We also study basic properties of intersections of standard subspaces in G-orbits.

2.1. The geometry of Euler elements

In this subsection we recall some fundamental geometric structures related to Euler elements in the Lie algebra $\mathfrak{g}$ of a Lie group G. For more details and background, we refer to [51,53,54,65].

2.1.1. Euler elements

Let G be a connected Lie group, the Lie algebra of a Lie group G is denoted $\mathbf{L}(G)$ or $\mathfrak{g}$. For an involutive automorphism σ of G, we write $G^\sigma = \{g \in G\colon \sigma(g) = g\}$ for the subgroup of fixed points and $G_\sigma := G \rtimes \{\mathrm{id}_G, \sigma\}$ for the corresponding group extension. Then

$$\varepsilon\colon G_\sigma \to (\{\pm 1\}, \cdot), \quad (g, \mathrm{id}_G) \mapsto 1, \quad (g, \sigma) \mapsto -1$$

is a group homomorphism that defines on G_σ the structure of a $\mathbb{Z}_2$-*graded Lie group.*

Remark 2.1. (a) The group G_σ depends on σ, but two involutive automorphisms σ_1 and σ_2 lead to isomorphic extensions $G_{\sigma_1} \cong G_{\sigma_2}$ if and only if $\sigma_2\sigma_1^{-1}$ is an inner automorphism $c_y(x) = yxy^{-1}$ for some $y \in G$ with $\sigma_1(y) = y^{-1}$ (hence also $\sigma_2(y) = y^{-1}$). Specifically

$$\Phi\colon G_{\sigma_2} \to G_{\sigma_1}, \quad (g, \mathrm{id}_G) \mapsto (g, \mathrm{id}_G), \quad (e, \sigma_2) \mapsto (y, \sigma_1)$$

defines an isomorphism because

$$(y, \sigma_1)(g, \mathrm{id}_G)(y, \sigma_1)^{-1} = (y\sigma_1(g)y^{-1}, \mathrm{id}_G) = (\sigma_2(g), \mathrm{id}_G)$$

and

$$(y, \sigma_1)^2 = (y\sigma_1(y), \mathrm{id}_G) = (e, \mathrm{id}_G).$$

(b) If σ is inner, then the above argument shows that $G_\sigma \cong G \times \{\pm 1\}$ is a product group. Therefore (anti-)unitary representations $(U, \mathcal{H})$ of G_σ restrict to unitary representations U of G for which there exists a conjugation J commuting with $U(G)$. Then the real Hilbert space $\mathcal{H}^J$ is $U(G)$-invariant, and $(U, \mathcal{H})$ is simply the complexification of the so-obtained real orthogonal representation of G on which J acts by complex conjugation.

Definition 2.2. (a) We call an element h of the finite dimensional real Lie algebra $\mathfrak{g}$ a (non-central) *Euler element* if $\mathrm{ad}\, h$ is non-zero and diagonalizable with $\mathrm{Spec}(\mathrm{ad}\, h) \subseteq \{-1, 0, 1\}$. In particular the eigenspace decomposition with respect to $\mathrm{ad}\, h$ defines a 3-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h), \qquad \text{where} \qquad \mathfrak{g}_\nu(h) = \ker(\mathrm{ad}\, h - \nu\, \mathrm{id}_{\mathfrak{g}})$$

Then $\tau_h(y_j) = (-1)^j y_j$ for $y_j \in \mathfrak{g}_j(h)$ defines an involutive automorphism of $\mathfrak{g}$.

We write $\mathcal{E}(\mathfrak{g})$ for the set of Euler elements in $\mathfrak{g}$. The orbit of an Euler element h under the group $\mathrm{Inn}(\mathfrak{g}) = \langle e^{\mathrm{ad}\, \mathfrak{g}} \rangle$ of *inner automorphisms* is denoted with $\mathcal{O}_h = \mathrm{Inn}(\mathfrak{g})h \subseteq \mathfrak{g}$. We say that h is *symmetric* if $-h \in \mathcal{O}_h$.

(b) The set

$$\mathcal{G} := \mathcal{G}(G_\sigma) := \{(h, \tau) \in \mathfrak{g} \times G_\sigma \colon \tau^2 = e, \varepsilon(\tau) = -1, \mathrm{Ad}(\tau)h = h\}$$

is called the *abstract wedge space of* G_σ. An element $(h,\tau) \in \mathcal{G}$ is called an *Euler couple* or *Euler wedge* if $h \in \mathcal{E}(\mathfrak{g})$ and

$$\mathrm{Ad}(\tau) = \tau_h. \tag{1}$$

Then τ is called an *Euler involution*. We write $\mathcal{G}_E \subseteq \mathcal{G}$ for the subset of Euler couples.

(c) On $\mathfrak{g}$ we consider the *twisted adjoint action* of G_σ which changes the sign on odd group elements:

$$\mathrm{Ad}^\varepsilon \colon G_\sigma \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}^\varepsilon(g) := \varepsilon(g)\,\mathrm{Ad}(g). \tag{2}$$

It extends to an action of G_σ on $\mathcal{G}$ by

$$g.(h,\tau) := (\mathrm{Ad}^\varepsilon(g)h, g\tau g^{-1}). \tag{3}$$

(d) (Order structure on $\mathcal{G}$) For a given $\mathrm{Ad}^\varepsilon(G)$-invariant pointed closed convex cone $C_\mathfrak{g} \subseteq \mathfrak{g}$, we obtain an order structure on $\mathcal{G}$ as follows ([51, Def. 2.5]). We associate to $W = (h,\tau) \in \mathcal{G}$ a semigroup S_W whose unit group is $S_W \cap S_W^{-1} = G_W$, the stabilizer of W. It is specified by

$$S_W := \exp(C_+)G_W \exp(C_-) = G_W \exp\big(C_+ + C_-\big).$$

Here the convex cones $C_\pm$ are the intersections

$$C_\pm := \pm C_\mathfrak{g} \cap \mathfrak{g}^{-\tau} \cap \mathfrak{g}_{\pm 1}(h), \quad \text{where} \quad \mathfrak{g}^{\pm\tau} := \{y \in \mathfrak{g} \colon\ \mathrm{Ad}(\tau)(y) = \pm y\}. \tag{4}$$

That S_W is a semigroup follows from [61, Thm. 2.16], applied to the Lie subalgebra

$$L_W := (C_+ - C_+) + \mathfrak{g}_0(h)^\tau + (C_- - C_-),$$

in which h is an Euler element. That L_W is a Lie algebra follows from $[C_+, C_+] = [C_-, C_-] = \{0\}$. To see this, observe that $\mathfrak{g}_+ := \sum_{\lambda>0} \mathfrak{g}_\lambda(h)$ is a nilpotent Lie algebra, so that the subspace $\mathfrak{n} := (C_U \cap \mathfrak{g}_+) - (C_U \cap \mathfrak{g}_+)$ is a nilpotent Lie algebra generated by the pointed invariant cone $C_U \cap \mathfrak{g}_+$, hence abelian by [59, Ex. VII.3.21].

Then S_W defines a G-invariant partial order on the orbit $G.W \subseteq \mathcal{G}$ by

$$g_1.W \leq g_2.W \quad :\Longleftrightarrow \quad g_2^{-1}g_1 \in S_W. \tag{5}$$

In particular, $g.W \leq W$ is equivalent to $g \in S_W$.

(e) (Duality operation) The notion of a "causal complement" is defined on the abstract wedge space as follows: For $W = (h,\tau) \in \mathcal{G}$, we define the *dual wedge* by $W' := (-h,\tau) = \tau.W$. Note that $(W')' = W$ and $(gW)' = gW'$ for $g \in G$ by (3). This relation fits the geometric interpretation in the context of wedge domains in spacetime manifolds.

Remark 2.3. If $h \in \mathfrak{g}$ is an Euler element in a simple real Lie algebra, then the cases where the involution τ_h is inner are classified in [55].

Remark 2.4. Let $W = (h, \tau) \in \mathcal{G}$ and consider $y \in \mathfrak{g}$. Then $\exp(\mathbb{R}y)$ fixes W if and only if

$$[y, h] = 0 \quad \text{and} \quad y = \mathrm{Ad}(\tau)y.$$

If (h, τ) is an Euler couple, then $\mathrm{Ad}(\tau)y = \mathrm{Ad}(\tau_h)y = y$ follows from $y \in \mathfrak{g}_0(h)$, so that

$$\mathfrak{g}_W := \{y \in \mathfrak{g}\colon \exp(\mathbb{R}y) \subseteq G_W\} = \mathfrak{g}_0(h) = \ker(\mathrm{ad}\, h). \tag{6}$$

Definition 2.5. (The abstract wedge space) For a fixed couple $W_0 = (h, \tau) \in \mathcal{G}$, the orbits

$$\mathcal{W}_+(W_0) := G.W_0 \subseteq \mathcal{G} \quad \text{and} \quad \mathcal{W}(W_0) := G_\sigma.W_0 \subseteq \mathcal{G}$$

are called the *positive* and the *full abstract wedge space containing* W_0.

Here is a classification of real simple Lie algebras containing Euler elements. The families are determined by their root system:

Theorem 2.6. ([51, Thm. 3.10]) *Suppose that* $\mathfrak{g}$ *is a non-compact simple real Lie algebra and that* $\mathfrak{a} \subseteq \mathfrak{g}$ *is maximal* ad*-diagonalizable with restricted root system* $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ *of type* X_n*. We follow the conventions of the tables in* [7] *for the classification of irreducible root systems and the enumeration of the simple roots* $\alpha_1, \ldots, \alpha_n$*. For each* $j \in \{1, \ldots, n\}$*, we consider the uniquely determined element* $h_j \in \mathfrak{a}$ *satisfying* $\alpha_k(h_j) = \delta_{jk}$*. Then every Euler element in* $\mathfrak{g}$ *is conjugate under inner automorphism to exactly one* h_j*. For every irreducible root system, the Euler elements among the* h_j *are the following:*

$$A_n : h_1, \ldots, h_n, \qquad B_n : h_1, \qquad C_n : h_n, \qquad D_n : h_1, h_{n-1}, h_n, \quad E_6 : h_1, h_6, \quad E_7 : h_7. \tag{7}$$

For the root systems BC_n*,* E_8*,* F_4 *and* G_2 *no Euler element exists (they have no* 3*-grading). The symmetric Euler elements (see* Definition 2.2(a)*) are*

$$A_{2n-1} : h_n, \qquad B_n : h_1, \qquad C_n : h_n, \qquad D_n : h_1, \qquad D_{2n} : h_{2n-1}, h_{2n}, \qquad E_7 : h_7. \tag{8}$$

Example 2.7. (Wedge regions in Minkowski and de Sitter spacetimes) The Minkowski spacetime is the manifold $\mathbb{R}^{1,d}$ endowed with the Minkowski metric

$$ds^2 = dx_0^2 - dx_1^2 - \ldots - dx_d^2.$$

The de Sitter spacetime is the Minkowski submanifold $\mathrm{dS}^d = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} : \mathbf{x}^2 - x_0^2 = 1\}$, endowed with the metric obtained by restriction of the Minkowski metric to dS^d. In the literature the x_0-coordinate is often denoted t as it is interpreted as a time coordinate. The symmetry groups of isometries for these spaces are the (proper) Poincaré group $\mathcal{P}_+ = \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})$ on Minkowski space $\mathbb{R}^{1,d}$ and the (proper) Lorentz group $\mathcal{L}_+ = \mathrm{SO}_{1,d}(\mathbb{R})$ on dS^d.

The generator $h \in \mathfrak{so}_{1,d}(\mathbb{R})$ of the Lorentz boost on the (x_0, x_1)-plane

$$h(x_0, x_1, x_2, \ldots, x_d) = (x_1, x_0, 0, \ldots, 0)$$

is an Euler element. It combines with the spacetime reflection

$$j_h(x) = (-x_0, -x_1, x_2, \ldots, x_d)$$

to the Euler couple $(h, j_h) \in \mathcal{G}(\mathcal{L}_+) \subseteq \mathcal{G}(\mathcal{P}_+)$, for the graded Lie groups $\mathcal{L}_+ = \mathrm{SO}_{1,d}(\mathbb{R})$ and $\mathcal{P}_+$. The spacetime region

$$W_R = \{x \in \mathbb{R}^{1,d} : |x_0| < x_1\}$$

is called the *standard right wedge* in Minkowski space, and

$$W_R^{\mathrm{dS}} := W_R \cap \mathrm{dS}^d$$

is the corresponding wedge region in de Sitter space. Note that W_R and therefore W_R^{dS} are invariant under $\exp(\mathbb{R}h)$. Poincaré transformed regions $W = g.W_R, g \in \mathcal{P}_+$, are called *wedge regions in Minkowski space*; likewise the regions $W^{\mathrm{dS}} = g.W_R^{\mathrm{dS}}$, $g \in \mathcal{L}_+$, are called *wedge regions in de Sitter space*. To $W = g.W_R$ we associate the boost group $\Lambda_W(t) := \exp(t \operatorname{Ad}(g)h)$. They are in equivariant one-to-one correspondence with abstract Euler couples in $\mathcal{G}_E(\mathcal{P}_+)$ and $\mathcal{G}_E(\mathcal{L}_+)$, respectively. Here the couple (h, j_h) corresponds to W_R and W_R^{dS}, respectively (cf. [63, Lemma 4.13], [51, Rem. 2.9(e)] and [11, Sect. 5.2]).

2.1.2. Wedge domains in causal homogeneous spaces

In this subsection we recall how to specify suitable wedge regions $W \subseteq M$ in a causal homogeneous space $M = G/H$. Motivated by the Bisognano–Wichmann property (BW) in AQFT, the modular flow, namely the flow of the one-parameter group generated by an Euler element on a causal homogeneous space M should be timelike future-oriented. Indeed, the modular flow corresponds to the inner time evolution of Rindler wedges (see [19] and also [6,13,5], [16, §3]). In our context this means that the *modular vector field*

$$X_h^M(m) := \frac{d}{dt}\Big|_{t=0} \exp(th).m \tag{9}$$

should satisfy

$$X_h^M(m) \in C_m^\circ \quad \text{for all} \quad m \in W,$$

where the causal structure on M is specified by the G-invariant field $(C_m)_{m\in M}$ of closed convex cones $C_m \subseteq T_m(M)$. If this condition is satisfied in one $m \in M$, we may always replace h by a conjugate and thus assume that it holds in the base point $m = eH$. Then the connected component

$$W := W_M^+(h)_{eH} \tag{10}$$

of the base point $eH \in M$ in the *positivity region*

$$W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\} \tag{11}$$

is the natural candidate for a domain for which (BW) could be satisfied. Note that this domain depends on h and the causal structure on M and that W is invariant under the connected stabilizer G_e^h of h, hence in particular under $\exp(\mathbb{R}h)$. These "wedge regions" have been studied for compactly and non-compactly causal symmetric spaces in [66] and [65,54], respectively.

Remark 2.8. If $Z(G) = \{e\}$, then each Euler element $h \in \mathfrak{g}$ determines a pair $(h, \tau_h) \in \mathcal{G}_E$ uniquely. So the stabilizers $G^{(h,\tau_h)}$ and G^h coincide and we may identify $\mathcal{W}_+(h, \tau_h) \subseteq \mathcal{G}_E$ with the adjoint orbit $\mathcal{O}_h = \mathrm{Ad}(G)h$. We thus obtain a natural map from $\mathcal{W}_+(h, \tau_h) \cong \mathcal{O}_h$ to regions in M by $g.(h, \tau_h) \mapsto g.W_M^+(h)$. If, in addition, G^h preserves the connected component $W \subseteq W_M^+(h)$ (which is in particular the case if $W_M^+(h)$ is connected, hence equal to W), this leads to a map from the abstract wedge space $\mathcal{W}_+(h, \tau_h)$ to the geometric wedge space on M. Proposition 2.9 below implies that it is isotone if the order on $\mathcal{W}_+(h, \tau_h)$ is specified by the invariant cone C_M from (12).

The compression semigroup of a wedge region

Let $M = G/H$ be a causal homogeneous space and $(C_m)_{m\in M}$ its causal structure. Writing $G \times TM \to TM, (g, v) \mapsto g.v$ for the action of G on the tangent bundle, this means that $g.C_m = C_{g.m}$ for $g \in G, m \in M$. Identifying $T_{eH}(M)$ with $\mathfrak{g}/\mathfrak{h}$, we consider the projection $p \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ and the cone $C := C_{eH} \subseteq \mathfrak{g}/\mathfrak{h}$. For $y \in \mathfrak{g}$, the corresponding vector field on M is given by

$$X_y^M(gH) = \frac{d}{dt}\Big|_{t=0} \exp(ty).gH = g.\frac{d}{dt}\Big|_{t=0} \exp(t\,\mathrm{Ad}(g)^{-1}y).eH = g.p(\mathrm{Ad}(g)^{-1}y).$$

The set

$$C_M := \{y \in \mathfrak{g} \colon (\forall m \in M) X_y^M(m) \in C_m\} = \bigcap_{g\in G} \mathrm{Ad}(g)p^{-1}(C) \tag{12}$$

is a closed convex $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$. If G acts effectively on M, then it is also pointed because elements in $C_M \cap -C_M$ correspond to vanishing vector fields on M.

This cone is a geometric analog of the positive cone C_U corresponding to a unitary representation of G (see (19)). The following observation shows that it behaves in many respects similarly (cf. [61]).

As any connected component $W \subseteq W_M^+(h) \subseteq M$ is invariant under $\exp(\mathbb{R}h)$, the same holds for the closed convex cone

$$C_W := \{y \in \mathfrak{g} \colon (\forall m \in W)\ X_y^M(m) \in C_m\} \supseteq C_M.$$

Below we show that this cone determines the tangent wedge of the compression semigroup of W.

Proposition 2.9. *For a connected component $W \subseteq W_M^+(h)$, its compression semigroup*

$$S_W := \{g \in M \colon g.W \subseteq W\}$$

is a closed subsemigroup of $\mathfrak{g}$ with $G_W := S_W \cap S_W^{-1} \supseteq G_e^h$ and

$$\mathbf{L}(S_W) := \{x \in \mathfrak{g} \colon \exp(\mathbb{R}_+ x) \subseteq S_W\} = \mathfrak{g}_0(h) + C_{W,+} + C_{W,-}, \tag{13}$$

where the two convex cones $C_{W,\pm}$ are the intersections $\pm C_W \cap \mathfrak{g}_{\pm 1}(h)$. In particular, the convex cone $\mathbf{L}(S_W)$ has interior points if C_M does.

Proof. As $W \subseteq M$ is an open subset, its complement $W^c := M \setminus W$ is closed, and thus

$$S_W = \{g \in G \colon g^{-1}.W^c \subseteq W^c\}$$

is a closed subsemigroup of G, so that its tangent wedge $\mathbf{L}(S_W)$ is a closed convex cone in $\mathfrak{g}$ ([32]).

Let $m = gH \in W$, so that $p(\mathrm{Ad}(g)^{-1}h) \in C^\circ$. For $x \in \mathfrak{g}_1(h)$ we then derive from $\mathfrak{g}_2(h) = \{0\}$ that

$$e^{t\,\mathrm{ad}\,x} h = h + t[x,h] \quad \text{for} \quad t \in \mathbb{R}.$$

This leads to

$$\begin{aligned} p(\mathrm{Ad}(\exp(tx)g)^{-1}h) &= p(\mathrm{Ad}(g)^{-1}e^{-t\,\mathrm{ad}\,x}h) = p(\mathrm{Ad}(g)^{-1}(h - t[x,h])) \\ &= p(\mathrm{Ad}(g)^{-1}(h + tx)) = p(\mathrm{Ad}(g)^{-1}h) + tp(\mathrm{Ad}(g)^{-1}x). \end{aligned}$$

For $x \in C_{W,+}$, we have $p(\mathrm{Ad}(g)^{-1}x) \in C$, so that $p(\mathrm{Ad}(\exp(tx)g)^{-1}h) \in C^\circ$ for $t \geq 0$, which in turn implies that $\exp(tx).m \in W$ for $m \in W$ and $t \geq 0$. So $\exp(\mathbb{R}_+ x) \subseteq S_W$, and thus $x \in \mathbf{L}(S_W)$. It likewise follows that $C_{W,-} \subseteq \mathbf{L}(S_W)$. The invariance of W under the identify component G_e^h of the centralizer of h further entails $\mathfrak{g}_0(h) \subseteq \mathbf{L}(S_W)$, so that

$$C_{W,+} + \mathfrak{g}_0(h) + C_{W,-} \subseteq \mathbf{L}(S_W). \tag{14}$$

We now prove the converse inclusion. If $X^M_x(m) \not\in C_m$, i.e., $p(\mathrm{Ad}(g)^{-1}x) \not\in C$, then there exists a $t_0 > 0$ with

$$p(\mathrm{Ad}(g)^{-1}h) + t_0 \cdot p(\mathrm{Ad}(g)^{-1}x) \not\in C$$

([59, Prop. V.1.6]), so that $\exp(t_0 x).m \not\in W$. We conclude that

$$\mathbf{L}(S_W) \cap \mathfrak{g}_1(h) = C_{W,+}.$$

Further, the invariance of the closed convex cone $\mathbf{L}(S_W)$ under $e^{\mathbb{R}\,\mathrm{ad}\,h}$ implies that, for $x = x_{-1} + x_0 + x_1 \in \mathbf{L}(S_W)$ and $x_j \in \mathfrak{g}_j(h)$, we have

$$x_{\pm 1} = \lim_{t\to\infty} e^{\mp t} e^{\pm t\,\mathrm{ad}\,h} x \in \mathbf{L}(S_W) \cap \mathfrak{g}_{\pm 1}(h) = C_{W,\pm},$$

which implies the other inclusion $\mathbf{L}(S_W) \subseteq C_{W,+} + \mathfrak{g}_0(h) + C_{W,-}$, hence equality by (14).

Let $p_\pm \colon \mathfrak{g} \to \mathfrak{g}_{\pm 1}(h)$ denote the projection along the other eigenspaces of $\mathrm{ad}\,h$. Then

$$C_{W,\pm} \supseteq C_{M,\pm} := \pm C_M \cap \mathfrak{g}_{\pm 1}(h) = \pm p_\pm(C_M)$$

also follows from [67, Lemma 3.2]. Therefore $C_M^\circ \neq \emptyset$ implies $C_{W,\pm}^\circ \neq \emptyset$, and this is equivalent to $\mathbf{L}(S_W)^\circ \neq \emptyset$. □

Remark 2.10. In many situations, such as the action of $\mathrm{PSL}_2(\mathbb{R})$ on the circle $\mathbb{S}^1 \cong \mathbb{P}_1(\mathbb{R})$, the cones $C_{W,\pm} \supseteq C_{M,\pm}$ coincide, and we believe that this is probably always the case. It is easy to see that, if $x \in C_{W,+}$, then the positivity region

$$\Omega_x := \{m \in M \colon X^M_x(m) \in C_m\}$$

contains W (by definition), and it is also invariant under $\exp(\mathbb{R}h)$ and $\exp(\mathbb{R}x)$, to that

$$\Omega_x \supseteq \bigcup_{t>0} \exp(-tx).W. \tag{15}$$

Clearly, $\Omega_x = M$ follows if the right hand side of (15) is dense in M, but we now show that Minkowski space provides an example where $\Omega_x = M$ without the right hand side of (15) being dense in M.

If G is the connected Poincaré group acting on Minkowski space $M = \mathbb{R}^{1,d}$ and

$$W = W_R = \{(x_0, \mathbf{x}) \colon x_1 > |x_0|\},$$

then

$$S_W = \overline{W} \rtimes \big(\mathrm{SO}_{d-1}(\mathbb{R}) \times \mathrm{SO}_{1,1}(\mathbb{R})^{\uparrow}\big)$$

([63, Lemma 4.12]) implies that

$$C_{W,\pm} = \mathbf{L}(S_W) \cap \mathfrak{g}_1(h) = \mathbb{R}_+(\pm \mathbf{e}_0 + \mathbf{e}_1)$$

consists of constant vector fields, so that $C_{W,\pm} = C_{M,\pm}$ in this case. Here we see that, for $x = \mathbf{e}_0 + \mathbf{e}_1 \in C_{W,+}$, the domain $\Omega_x = W - \mathbb{R}_+ x$ is an open half space, hence in particular not dense in M. Therefore we cannot expect the domain Ω_x in (15) to be dense in M.

2.1.3. Non-compactly causal spaces

Let G be a connected simple Lie group and $h \in \mathfrak{g}$ be an Euler element. This implies in particular that G is not compact. The associated non-compactly causal symmetric spaces are obtained as follows (see [53, Thm. 4.21] for details). We choose a Cartan involution θ on $\mathfrak{g}$ with $\theta(h) = -h$, write $K := G^\theta$ for the corresponding group of fixed points, and consider the involution $\tau_{\rm nc} := \tau_h \theta \in \mathrm{Aut}(\mathfrak{g})$. Assuming that the involution $\tau_{\rm nc}$ integrates to an involution $\tau_{\rm nc}^G$ on G, we consider a subgroup $H \subseteq \mathrm{Fix}(\tau_{\rm nc}^G) = G^{\tau_{\rm nc}^G}$ that is open (hence has the same Lie algebra $\mathfrak{h} = \mathfrak{g}^{\tau_{\rm nc}}$) and for which $H \cap K$ fixes h. Then $M := G/H$ is the corresponding *non-compactly causal symmetric space*, where the invariant causal structure is determined by the maximal pointed closed convex cone $C \subseteq \mathfrak{g}^{-\tau_{\rm nc}} \cong T_{eH}(M)$ containing h. This construction ensures in particular that $eH \in W_M^+(h)$. Assume, in addition, that $G = \mathrm{Inn}(\mathfrak{g})$ is centerfree. Then [54, Cor. 7.2] identifies W from (11) with the "observer domain" $W(\gamma)$ associated to the geodesic $\gamma(t) = \mathrm{Exp}_{eH}(th)$ in M.[2] Further, [54, Prop. 7.3] thus implies that the stabilizer G_W of W coincides with the centralizer G^h of h:

$$G_W = G^h,$$

so that, for centerfree groups, we may identify the wedge space

$$\mathcal{W}(M,h) := G.W \cong G/G_W = G/G^h \cong \mathcal{O}_h$$

with the adjoint orbit $\mathcal{O}_h$ of h.

If, more generally, G is only assumed connected and $M = G/H$ is a corresponding non-compactly causal symmetric space, then the connected component $W := W_M^+(h)_{eH} \subseteq M$ containing eH is the natural wedge region and $G_{W_M} \subseteq G^h$ may be a proper subgroup. Typical examples arise for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ (see [26, Rem. 5.13]).

For non-compactly causal symmetric spaces, we typically have $G_{\tau_{\rm nc}} \not\cong G_{\tau_h}$ because the product $\tau_{\rm nc}\tau_h$ need not be inner (cf. Remark 2.1). If, for instance, $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ and $\tau_{\rm nc}$

[2] The symmetric space M carries a natural G-invariant affine connection ∇, and in this sense this curve is a geodesic. Concretely, the geodesics are the curves of the form $\eta(t) = g.\exp(ty)$, $y \in \mathfrak{g}$ with $\tau_{\rm nc}(y) = -y$.

is complex conjugation with respect to $\mathfrak{h}$ (non-compactly causal of complex type), then τ_h is complex linear and $\tau_{\rm nc}$ is antilinear, hence their product is antilinear and therefore not inner.

From $\tau_{\rm nc} = \theta\tau_h$ we derive $\tau_{\rm nc}\tau_h = \theta$, which leads to the question when θ is inner. For a characterization of these cases, we refer to [55].

2.1.4. Compactly causal spaces

Let G be a connected Lie group and $M = G/H$ be a compactly causal symmetric space, where $H \subseteq G^{\tau_{\rm cc}}$ is an open subgroup and $\tau_{\rm cc}$ is an involutive automorphism of G. We assume that there exists an Euler element $h \in \mathfrak{h} = \mathfrak{g}^\tau_{\rm cc}$, so that we obtain a so-called *modular compactly causal symmetric Lie algebra* $(\mathfrak{g}, \tau_{\rm cc}, C, h)$ (cf. [65]). Here $C \subseteq \mathfrak{q} := \mathfrak{g}^{-\tau_{\rm cc}}$ is a pointed generating closed convex cone, invariant under $\mathrm{Ad}(H)$, whose interior C° consists of elliptic elements. We further assume that the involution τ_h on $\mathfrak{g}$ integrates to an involutive automorphism τ_h^G of G such that $\tau_h^G(H) = H$ and the existence of a pointed generating $\mathrm{Ad}(G)$-invariant cone $C_\mathfrak{g} \subseteq \mathfrak{g}$ such that

$$-\tau(C_\mathfrak{g}) = C_\mathfrak{g} \quad \text{and} \quad C = C_\mathfrak{g} \cap \mathfrak{q}.$$

Then $eH \in M$ is a fixed point of the modular flow and there exists a unique connected component

$$W = W_M^+(h)_{eH}$$

of the positivity domain $W_M^+(h)$ that contains eH in its boundary. Then [65, Thm. 9.1] asserts that

$$S_W := \{g \in G \colon g.W \subseteq W\} = G_W \exp(C_\mathfrak{g}^c),$$

where $G_W = \{g \in G \colon g.W = W\}$ and

$$C_\mathfrak{g}^c := C_{\mathfrak{g},+} + C_{\mathfrak{g},-} \subseteq \mathfrak{g}^{-\tau_h} \quad \text{for} \quad C_{\mathfrak{g},\pm} := \pm C_\mathfrak{g} \cap \mathfrak{g}_{\pm 1}(h). \tag{16}$$

The cone $C_\mathfrak{g}^c$ is $-\tau_{\rm cc}$-invariant with

$$(C_\mathfrak{g}^c)^{-\tau_{\rm cc}} = C_\mathfrak{g}^c \cap \mathfrak{q} = C_+ + C_- \quad \text{for} \quad C_\pm := \pm C_\mathfrak{g} \cap \mathfrak{q}_{\pm 1}(h) = \pm C \cap \mathfrak{q}_{\pm 1}(h). \tag{17}$$

Here

$$G_W = G_e^h H^h \subseteq G^h$$

is an open subgroup with the Lie algebra $\mathfrak{g}_0(h)$ and the wedge space

$$\mathcal{W}(M, h) := G.W \cong G/G_W$$

carries the structure of a symmetric space ([66, Prop. 9.2]). Covering issues related to $\mathcal{W}(M,h)$ are discussed in [66, Rem. 9.4].

Remark 2.11. In general $\tau_{\rm cc} \neq \tau_h$ and also $\tau_{\rm cc} \neq \tau_h\theta$ for Cartan involutions θ with $\theta(h) = -h$. The latter products $\tau_h\theta$ are precisely the involutions $\tau_{\rm nc}$, corresponding to non-compactly causal symmetric spaces. In general we also have $G_{\tau_{\rm cc}} \not\cong G_{\tau_h}$ because the product $\tau_{\rm cc}\tau_h$ need not be inner (cf. Remark 2.1), as the following example shows. If $(\mathfrak{g}, \tau_{\rm cc})$ is compactly causal of group type, then $\mathfrak{g} \cong \mathfrak{h} \oplus \mathfrak{h}$ with $\tau_{\rm cc}(x,y) = (y,x)$, whereas τ_h preserves both ideals. Therefore $\tau_{\rm cc}\tau_h$ flips the ideals, hence cannot be inner. If $(\mathfrak{g}, \tau_{\rm cc})$ is of Cayley type, then (by definition) $\tau_{\rm cc} = \tau_h$ for an Euler element h.

If $\mathfrak{g}$ is simple, then it is of hermitian type, so that all Euler elements in $\mathfrak{g}$ are conjugate ([51, Prop. 3.11]). The relation

$$\tau_{\rm cc}\,\mathrm{Ad}(g)\tau_h\,\mathrm{Ad}(g)^{-1} = \tau_{\rm cc}\tau_h\,\mathrm{Ad}(\tau_h^G(g)g^{-1})$$

then shows that $\tau_{\rm cc}\tau_h$ is inner for one Euler element if and only if this is the case for all Euler elements. As we have seen above, this is true for Cayley type spaces.

2.2. The geometry of nets of real subspaces

In this section we recall some fundamental properties of the geometry of standard subspaces on generalized one-particle nets. We refer to [39,51,63] for more details. Sections 2.2.4 and 2.2.5 and contain some new observations that will become relevant below.

2.2.1. Standard subspaces

We call a closed real subspace H of the complex Hilbert space $\mathcal{H}$ *cyclic* if $\mathsf{H} + i\mathsf{H}$ is dense in $\mathcal{H}$, *separating* if $\mathsf{H} \cap i\mathsf{H} = \{0\}$, and *standard* if it is cyclic and separating. We write $\mathrm{Stand}(\mathcal{H})$ for the set of standard subspaces of $\mathcal{H}$. The symplectic orthogonal of a real subspace H is defined by the symplectic form $\mathrm{Im}\langle\cdot,\cdot\rangle$ on $\mathcal{H}$ via

$$\mathsf{H}' = \{\xi \in \mathcal{H} : (\forall \eta \in \mathsf{H})\ \ \mathrm{Im}\langle \xi, \eta\rangle = 0\}.$$

Then H is separating if and only if H' is cyclic, hence H is standard if and only if H' is standard. For a standard subspace H, we define the *Tomita operator* as the closed antilinear involution

$$\mathsf{H} + i\mathsf{H} \to \mathsf{H} + i\mathsf{H}, \quad \xi + i\eta \mapsto \xi - i\eta.$$

The polar decomposition $J_{\mathsf{H}}\Delta_{\mathsf{H}}^{\frac{1}{2}}$ of this operator defines an antiunitary involution J_{H} (a conjugation) and the modular operator Δ_{H}. For the modular group $(\Delta_{\mathsf{H}}^{it})_{t\in\mathbb{R}}$, we then have

$$J_{\mathsf{H}}\mathsf{H} = \mathsf{H}', \quad \Delta_{\mathsf{H}}^{it}\mathsf{H} = \mathsf{H} \qquad \text{for every} \quad t \in \mathbb{R}$$

and the modular relations

$$J_{\mathsf{H}}\Delta_{\mathsf{H}}^{it}J_{\mathsf{H}} = \Delta_{\mathsf{H}}^{it} \qquad \text{for every} \quad t \in \mathbb{R}.$$

One also has $\mathsf{H} = \mathrm{Fix}(J_{\mathsf{H}}\Delta_{\mathsf{H}}^{1/2})$ ([39, Thm. 3.4]). This construction leads to a one-to-one correspondence between couples (Δ, J) satisfying the modular relation and standard subspaces:

Proposition 2.12. ([39, Prop. 3.2]) *The map $\mathsf{H} \mapsto (\Delta_{\mathsf{H}}, J_{\mathsf{H}})$ is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of pairs (Δ, J), where J is a conjugation, $\Delta > 0$ selfadjoint with $J\Delta J = \Delta^{-1}$.*

From Proposition 2.12 we easily deduce:

Lemma 2.13. ([49, Lemma 2.2]) *Let $\mathsf{H} \subset \mathcal{H}$ be a standard subspace and $U \in \mathrm{AU}(\mathcal{H})$ be a unitary or anti-unitary operator. Then $U\mathsf{H}$ is also standard and $U\Delta_{\mathsf{H}}U^* = \Delta_{U\mathsf{H}}^{\varepsilon(U)}$ and $UJ_{\mathsf{H}}U^* = J_{U\mathsf{H}}$, where $\varepsilon(U) = 1$ if U is unitary and $\varepsilon(U) = -1$ if it is antiunitary.*

Proposition 2.14. ([39], [67, Prop. 2.1]) *Let $\mathtt{V} \subseteq \mathcal{H}$ be a standard subspace with modular objects (Δ, J). For $\xi \in \mathcal{H}$, we consider the orbit map $\alpha^\xi \colon \mathbb{R} \to \mathcal{H}, t \mapsto \Delta^{-it/2\pi}\xi$. Then the following are equivalent:*

(i) $\xi \in \mathtt{V}$.
(ii) $\xi \in \mathcal{D}(\Delta^{1/2})$ *with* $\Delta^{1/2}\xi = J\xi$.
(iii) *The orbit map $\alpha^\xi \colon \mathbb{R} \to \mathcal{H}$ extends to a continuous map*

$$\{z \in \mathbb{C} \colon 0 \le \mathrm{Im}\, z \le \pi\} \to \mathcal{H}$$

which is holomorphic on the interior and satisfies $\alpha^\xi(\pi i) = J\xi$.
(iv) *There exists $\eta \in \mathcal{H}^J$ whose orbit map α^η extends to a map*

$$\{z \in \mathbb{C} \colon |\,\mathrm{Im}\, z| \le \pi/2\} \to \mathcal{H}$$

which is continuous, holomorphic on the interior, and satisfies $\alpha^\eta(-\pi i/2) = \xi$.

2.2.2. The Brunetti–Guido–Longo (BGL) net

Here we recall a construction we introduced in [51] that generalizes the algebraic construction of free fields for AQFT models presented in [11].

If (U, G) is an (anti-)unitary representation of G_σ, then we obtain a standard subspace $\mathsf{H}_U(W)$ determined for $W = (h, \tau) \in \mathcal{G}$ by the couple of operators (cf. Proposition 2.12):

$$J_{\mathsf{H}_U(W)} = U(\tau) \quad \text{and} \quad \Delta_{\mathsf{H}_U(W)} = e^{2\pi i \partial U(h)}, \tag{18}$$

and thus a G-equivariant map $\mathsf{H}_U : \mathcal{G} \to \mathrm{Stand}(\mathcal{H})$. This is the so-called *BGL net*

$$\mathsf{H}_U^{\mathrm{BGL}} : \mathcal{G}(G_\sigma) \to \mathrm{Stand}(\mathcal{H}).$$

In the following theorem, we need the *positive cone*

$$C_U := \{x \in \mathfrak{g} \colon -i \cdot \partial U(x) \geq 0\}, \qquad \partial U(x) = \frac{d}{dt}\Big|_{t=0} U(\exp tx) \tag{19}$$

of a unitary representation U. It is a closed, convex, $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$.

Theorem 2.15. *Let $C_\mathfrak{g} \subseteq \mathfrak{g}$ be a pointed generating closed convex cone contained in the positive cone C_U of the (anti-)unitary representation $(U, \mathcal{H})$ of G_σ. Then the BGL net*

$$\mathsf{H}_U^{\mathrm{BGL}} : \mathcal{G}(G_\sigma) \to \mathrm{Stand}(\mathcal{H})$$

is G_σ-covariant and isotone with respect to the $C_\mathfrak{g}$-order on $\mathcal{G}(G_\sigma)$.

The BGL net also satisfies twisted locality conditions and PT symmetry. We refer to [51] for a detailed discussion. In this picture we have not required σ to be an Euler involution so $\mathcal{G}_E(G_\sigma)$ may in particular be trivial (see Example 2.16). This general presentation is motivated by the results in Section 3 that will exhibit the existence of an Euler element in $\mathfrak{g}$ and an involution τ_h^G, defining a graded group G_{τ_h}, as a consequence of a certain regularity condition for associated standard subspaces in unitary representations of G.

Example 2.16. It is easy to construct graded groups G_σ for which $\mathcal{G}_E(G_\sigma) = \emptyset$, i.e., no Euler couples exist. For example, we may consider $G = \mathrm{SL}_2(\mathbb{R})$ and the involutive automorphism $\theta(g) = (g^\top)^{-1}$ (Cartan involution). We claim that $G_\theta = G \rtimes \{\mathbf{1}, \theta\}$ contains no Euler couples. In fact, if (h, τ) is an Euler couple, then $\mathrm{Ad}(\tau) = \tau_h$. Identifying the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, endowed with its Cartan–Killing form, with 3-dimensional Minkowski space $\mathbb{R}^{1,2}$, we have $\mathrm{Ad}(G) = \mathrm{Ad}(G_\theta) = \mathrm{SO}_{1,2}(\mathbb{R})_e$, a connected group. But the automorphisms τ_h are contained in $\mathrm{SO}_{1,2}(\mathbb{R})^\downarrow$ because they reverse the causal orientation. Hence no involution $\tau = (g, \theta) \in G_\theta$ satisfies $\mathrm{Ad}(\tau) = \tau_h$. Clearly, the picture changes if we replace θ by an involution τ_h^G, where $h \in \mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{1,2}(\mathbb{R})$ is an Euler element.

2.2.3. Nets on homogeneous spaces

For a unitary representation $(U, \mathcal{H})$ of a connected a Lie group G and a homogeneous space $M = G/H$, we are interested in families $(\mathsf{H}(\mathcal{O}))_{\mathcal{O} \subseteq M}$ of closed real subspaces of $\mathcal{H}$, indexed by open subsets $\mathcal{O} \subseteq M$; so-called *nets of real subspaces on M*. Below we work in a more general context, where the connection between the abstract and the geometric wedges is less strict. For such nets, we consider the following properties:

(Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$.

(Cov) **Covariance:** $U(g)\mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.

(RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.

(BW) **Bisognano–Wichmann property:** There exists an open subset $W \subseteq M$ (called a *wedge region*), such that $\mathsf{H}(W)$ is standard with modular operator $\Delta_{\mathsf{H}(W)} = e^{2\pi i \partial U(h)}$ for some $h \in \mathfrak{g}$.

Nets satisfying (Iso), (Cov), (RS), (BW) have been constructed on non-compactly causal symmetric spaces in [26], and on compactly causal spaces in [66].

In some cases there is a one-to-one correspondence between the abstract wedge space $\mathcal{W}_+ \subseteq \mathcal{G}_E(G_\sigma)$ and the set $\mathcal{W}_M := \{g.W : g \in G\}$ of wedge regions in M, see Remark 2.8. In these cases, the BGL net on $\mathcal{W}_+$ can be considered as a net on concrete wedge regions in M, satisfying the previous assumptions, on the set $\mathcal{W}_M$ of wedge regions in M. A general correspondence theorem still has to be established. If $\mathtt{V}$ is a standard subspace with $\Delta_{\mathtt{V}} = e^{2\pi i \partial U(h)}$, then $\mathsf{H}(g.W) := U(g)\mathtt{V}$ yields a well-defined net on $\mathcal{W}_M$ if $g.W = W$ implies $U(g)\mathtt{V} = \mathtt{V}$. If $\ker U$ is discrete, the latter condition means that $\mathrm{Ad}(g)h = h$ and $U(g)J_{\mathtt{V}}U(g)^{-1} = J_{\mathtt{V}}$.

2.2.4. Minimal and maximal nets of real subspaces

To add a geometric context to the nets of standard subspaces that we have already encountered in terms of the BGL construction (cf. Theorem 2.15), we now fix an Euler element $h \in \mathfrak{g}$ and a homogeneous space $M = G/H$ of G, in which we consider an open subset W invariant under the one-parameter group $\exp(\mathbb{R}h)$. We call W and its translates gW, $g \in G$, "wedge regions". At the outset, we do not assume any specific properties of W, but Lemma 2.17 will indicate which properties good choices of W should have. Let $(U, \mathcal{H})$ be an (anti-)unitary representation of G_{τ_h} and $\mathtt{V} = \mathtt{V}(h, U)$ the corresponding standard subspace. For an open subset $\mathcal{O} \subseteq M$, we put

$$\mathsf{H}^{\max}(\mathcal{O}) := \bigcap_{g \in G, \mathcal{O} \subseteq gW} U(g)\mathtt{V} \quad \text{and} \quad \mathsf{H}^{\min}(\mathcal{O}) := \overline{\sum_{g \in G, gW \subseteq \mathcal{O}} U(g)\mathtt{V}}. \tag{20}$$

We call $\mathsf{H}^{\max}$ the *maximal net*, in accordance with [72]; and $\mathsf{H}^{\min}$ the *minimal net*.

This leads to $\mathsf{H}^{\max}(\mathcal{O}) = \mathcal{H}$ (the empty intersection) if there exists no $g \in G$ with $\mathcal{O} \subseteq gW$, i.e., $\mathcal{O}$ is not contained in any wedge region. We likewise get $\mathsf{H}^{\min}(\mathcal{O}) := \{0\}$ (the empty sum) if there exists no $g \in G$ with $gW \subseteq \mathcal{O}$, i.e., $\mathcal{O}$ contains no wedge region.

We also note that, if we write

$$\mathcal{O}^\wedge := \Big(\bigcap_{gW \supseteq \mathcal{O}} gW\Big)^\circ \supseteq \mathcal{O} \quad \text{and} \quad \mathcal{O}^\vee := \bigcup_{gW \subseteq \mathcal{O}} gW \subseteq \mathcal{O},$$

then $\mathcal{O}^\wedge$ and $\mathcal{O}^\vee$ are open subsets satisfying $(\mathcal{O}^\wedge)^\wedge = \mathcal{O}^\wedge$, $(\mathcal{O}^\vee)^\vee = \mathcal{O}^\vee$, and

$$\mathsf{H}^{\max}(\mathcal{O}^\wedge) = \mathsf{H}^{\max}(\mathcal{O}) \quad \text{and} \quad \mathsf{H}^{\min}(\mathcal{O}^\vee) = \mathsf{H}^{\min}(\mathcal{O}). \tag{21}$$

So, effectively, the maximal net "lives" on all open subsets $\mathcal{O}$ satisfying $\mathcal{O} = \mathcal{O}^\wedge$ (interiors of intersections of wedge regions) and the minimal net on those open subsets satisfying $\mathcal{O} = \mathcal{O}^\vee$ (unions of wedge regions).

Lemma 2.17. *The following assertions hold:*

(a) *The nets* $\mathsf{H}^{\max}$ *and* $\mathsf{H}^{\min}$ *on* M *satisfy* (Iso) *and* (Cov).
(b) *The set of all open subsets* $\mathcal{O} \subseteq M$ *for which* $\mathsf{H}^{\max}(\mathcal{O})$ *is cyclic is* G*-invariant.*
(c) *The following are equivalent:*
  (i) $S_W := \{g \in G \colon gW \subseteq W\} \subseteq S_{\mathtt{V}}$.
  (ii) $\mathsf{H}^{\max}(W) = \mathtt{V}$.
  (iii) $\mathsf{H}^{\max}(W)$ *is standard.*
  (iv) $\mathsf{H}^{\max}(W)$ *is cyclic.*
  (v) $\mathsf{H}^{\min}(W) = \mathtt{V}$.
  (vi) $\mathsf{H}^{\min}(W)$ *is standard.*
  (vii) $\mathsf{H}^{\min}(W)$ *is separating.*
(d) *The cyclicity of a subspace* $\mathsf{H}^{\max}(\mathcal{O})$ *is inherited by subrepresentations, direct sums, direct integrals and finite tensor products.*

Proof. (a) Isotony is clear and covariance of the maximal net follows from

$$\mathsf{H}^{\max}(g_0\mathcal{O}) = \bigcap_{g_0\mathcal{O}\subseteq gW} U(g)\mathtt{V} = U(g_0) \bigcap_{g_0\mathcal{O}\subseteq gW} U(g_0^{-1}g)\mathtt{V} = U(g_0)\mathsf{H}^{\max}(\mathcal{O}).$$

The argument for the minimal net is similar.

(b) follows from covariance.

(c) (i) $\Leftrightarrow$ (ii): Clearly, $\mathsf{H}^{\max}(W) \subseteq \mathtt{V}$, and equality holds if and only if $W \subseteq gW$ implies $U(g)\mathtt{V} \supseteq \mathtt{V}$, which is equivalent to $S_W^{-1} \subseteq S_{\mathtt{V}}^{-1}$, and this is equivalent to (i).

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (ii): By covariance and $\exp(\mathbb{R}h).W = W$, the subspace $\mathsf{H}^{\max}(W) \subseteq \mathtt{V}$ is invariant under the modular group $U(\exp \mathbb{R}h)$ of $\mathtt{V}$. If $\mathsf{H}^{\max}(W)$ is cyclic, then it is also standard, as a subspace of $\mathtt{V}$, so that [39, Prop. 3.10] implies $\mathsf{H}^{\max}(W) = \mathtt{V}$.

(i) $\Leftrightarrow$ (v) follows with a similar argument as the equivalence of (i) and (ii).

(v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) are trivial.

(vii) $\Rightarrow$ (v): By covariance and $\exp(\mathbb{R}h).W = W$, the subspace $\mathsf{H}^{\min}(W) \supseteq \mathtt{V}$ is invariant under the modular group $U(\exp \mathbb{R}h)$ of $\mathtt{V}$. If $\mathsf{H}^{\min}(W)$ is separating, then it is also standard, because it contains $\mathtt{V}$. Now [39, Prop. 3.10] implies $\mathsf{H}^{\min}(W) = \mathtt{V}$.

(d) We use that

$$\mathsf{H}^{\max}(\mathcal{O}) = \mathtt{V}_A := \bigcap_{g\in A} U(g)\mathtt{V} \quad \text{for} \quad A := \{g \in G \colon g^{-1}\mathcal{O} \subseteq W\}. \tag{22}$$

For a direct sum representation $U = U_1 \oplus U_2$ we have in particular $\mathtt{V} = \mathtt{V}_1 \oplus \mathtt{V}_2$, which leads to

$$\mathtt{V}_A = \mathtt{V}_{1,A} \oplus \mathtt{V}_{2,A} \tag{23}$$

because $U(g)^{-1}(v_1, v_2) \in \mathtt{V}$ is equivalent to $U_j(g)^{-1}v_j \in \mathtt{V}_j$ for $j = 1, 2$. We thus obtain

$$\mathsf{H}^{\max}(\mathcal{O}) = \mathsf{H}_1^{\max}(\mathcal{O}) \oplus \mathsf{H}_2^{\max}(\mathcal{O}).$$

This proves that cyclicity of $\mathsf{H}^{\max}(\mathcal{O})$ is inherited by subrepresentations and direct sums. For finite tensor products, the assertion follows from Lemma D.1. If $U = \int_X^\oplus U_m \, d\mu(m)$ is a direct integral, then (22) and Lemma C.3(a) imply that

$$\mathsf{H}^{\max}(\mathcal{O}) = \int_X^\oplus \mathsf{H}_m^{\max}(\mathcal{O}) \, d\mu(m) \tag{24}$$

for direct integrals. So Lemma C.1 implies that $\mathsf{H}^{\max}(\mathcal{O})$ is cyclic if every $\mathsf{H}_m^{\max}(\mathcal{O})$ is cyclic in $\mathcal{H}_m$. □

Lemma 2.17(d) implies in particular that a direct integral representation $(U, \mathcal{H})$ is (h, W)-localizable in a family of subsets of M in the sense of Definition 4.17 if μ-almost all representations $(U_m, \mathcal{H}_m)$ have this property. For the case where G is the Poincaré group and $M = \mathbb{R}^{1,d}$, a similar argument can be found in [11, Lemma 4.3].

Remark 2.18. (The case where S_W is a group) If the semigroup S_W is a group, i.e., $S_W = G_W = \{g \in G : g.W = W\}$ is a group and $\ker(U)$ is discrete, then the inclusion $S_W \subseteq S_{\mathtt{V}}$ is equivalent to

$$G_W \subseteq G_{\mathtt{V}} = G^{h,J} = \{g \in G^h : JU(g)J = U(g)\} \tag{25}$$

(cf. Lemma 2.13). In the context of causal homogeneous spaces, the definition of W as a connected component of $W_M^+(h)$ (see §2.1.2) implies that $\exp(\mathbb{R}h) \subseteq G_e^h \subseteq G_W$, and we have in many concrete examples that $G_W \subseteq G^h$ and $\mathbf{L}(G_W) = \mathfrak{g}^h$ (see [65,66,54] and §§2.1.3 and 2.1.4). However, $U(G_W)$ need not commute with J, so that (25) may fail. Examples arise already for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$; see [26, Rem. 5.13].

Lemma 2.19. *Let $(U, \mathcal{H})$ be an (anti-)unitary representation of G_{τ_h} and H a net of real subspaces on open subsets of M satisfying* (Iso), (Cov) *and $\mathsf{H}(W) = \mathtt{V}$ with respect to $h \in \mathfrak{g}$ and $W \subseteq M$. Then*

$$\mathsf{H}^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}^{\max}(\mathcal{O})$$

for each open subset $\mathcal{O} \subseteq M$ and equality holds for all domains of the form $\mathcal{O} = g.W$, $g \in G$ (wedge regions in M).

If $\emptyset \neq W \neq M$, then we have in particular

$$\mathsf{H}^{\min}(\emptyset) = \{0\} \subseteq \mathsf{H}^{\max}(\emptyset) = \bigcap_{g \in G} U(g)\mathtt{V} \quad \text{and} \quad \mathsf{H}^{\min}(M) = \overline{\sum_{g \in G} U(g)\mathtt{V}} \subseteq \mathsf{H}^{\max}(M) = \mathcal{H}.$$

Proof. First we show that the three properties (Iso), (Cov) and $\mathsf{H}(W) = \mathtt{V}$ of the net H imply that $S_W \subseteq S_{\mathtt{V}}$. In fact, $g.W \subseteq W$ implies

$$U(g)\mathtt{V} = U(g)\mathsf{H}(W) \overset{\text{(Cov)}}{=} \mathsf{H}(g.W) \overset{\text{(Iso)}}{\subseteq} \mathsf{H}(W) = \mathtt{V}.$$

From Lemma 2.17(c) we thus obtain $\mathsf{H}^{\max}(W) = \mathsf{H}^{\min}(W) = \mathtt{V}$. Hence $\mathsf{H}(gW) = U(g)\mathtt{V} = \mathsf{H}^{\max}(gW) = \mathsf{H}^{\min}(gW)$ by covariance for any $g \in G$ (Lemma 2.17(a)). Further, isotony shows that $\mathcal{O} \subseteq gW$ implies $\mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}(gW) = U(g)\mathtt{V}$, so that $\mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}^{\max}(\mathcal{O})$. Likewise, $gW \subseteq \mathcal{O}$ implies $U(g)\mathtt{V} = \mathsf{H}(gW) \subseteq \mathsf{H}(\mathcal{O})$, and thus $\mathsf{H}^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O})$. □

Definition 2.20. (a) (Causal complement) Let $M = \mathbb{R}^{1,d}$ be Minkowski space. Its causal structure allows us to define the *causal complement (or the spacelike complement)* of an open subset $\mathcal{O} \subset M$ by

$$\mathcal{O}' = \{x \in M : (\forall y \in \mathcal{O})\, (y - x)^2 < 0\}^\circ. \tag{26}$$

This is the interior of the set of all the points that cannot be reached from E with a timelike or lightlike curve.

(b) (Spacelike cones) In Minkowski space $\mathbb{R}^{1,d}$, we call an open subset $\mathcal{O}$ *spacelike* if $x_0^2 < \mathbf{x}^2$ holds for all $(x_0, \mathbf{x}) \in \mathcal{O}$. A spacelike open subset is called a *spacelike (convex) cone* if, in addition, it is a (convex) cone.

(c) (Double cone) A *double cone* is, up to Poincaré covariance, the causal closure

$$\mathbb{B}_r'' = (r\mathbf{e}_0 - V_+) \cap (-r\mathbf{e}_0 + V_+)$$

of an open ball of the time zero hyper-plane $\mathbb{B}_r = \{(0, \mathbf{x}) \in \mathbb{R}^{1,d} : \mathbf{x}^2 < r^2\}$.

Remark 2.21. We continue to use the notation from Example 2.7 and Definition 2.20. Let $d \geq 2$ and $M \supset \mathcal{D} \mapsto \mathsf{H}(\mathcal{D}) \subset \mathcal{H}$ be a net of standard subspaces on double cones (cf. Definition 2.20(c)), let U be a representation of the Poincaré group $\mathcal{P}_+^\uparrow$ satisfying (Iso), (Cov), (RS) and the following properties

1. *Positivity of the energy:* The support of the spectral measure of the space-time translation group is contained in

$$\overline{V_+} = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} : x_0^2 - \mathbf{x}^2 \geq 0, x_0 \geq 0\}.$$

2. Locality: $\mathcal{D}_1 \subset \mathcal{D}_2' \Rightarrow \mathsf{H}(\mathcal{D}_1) \subset \mathsf{H}(\mathcal{D}_2)'$.

3. Bisognano–Wichmann property: Let $W \subset M$ be a wedge region, as introduced in 2.7. Then
$$\mathsf{H}(W) = \overline{\sum_{\mathcal{D}\subset W} \mathsf{H}(\mathcal{D})}, \tag{27}$$
is standard with $\Delta_{\mathsf{H}(W)}^{-it/2\pi} = U(\Lambda_W(t))$, where $\Lambda_W(t)$ is the corresponding one-parameter group of boosts (cf. Example 2.7).

The Bisognano–Wichmann property implies *wedge duality (or essential duality)*:
$$\mathsf{H}(W') = \mathsf{H}(W)'.$$
Here W' is the causal complement of the wedge W, as in (26) (see [49, Prop. 2.7]).

For a double cone $\mathcal{D}$ we define
$$\mathsf{H}(\mathcal{D}') := \overline{\sum_{\mathcal{D}_1\subset \mathcal{D}'} \mathsf{H}(\mathcal{D}_1)} \tag{28}$$
and obtain the following net on double cones
$$M \supset \mathcal{D} \longmapsto \mathsf{H}^d(\mathcal{D}) := \mathsf{H}(\mathcal{D}')' = \bigcap_{\mathcal{D}_1\subset\mathcal{D}'} \mathsf{H}(\mathcal{D}_1)'.$$
By locality one has in general that $\mathsf{H}(\mathcal{D}) \subset \mathsf{H}^d(\mathcal{D})$. The net $\mathsf{H}^d(\mathcal{D})$ is called the *dual net* of H. If $\mathsf{H}(\mathcal{D}) = \mathsf{H}^d(\mathcal{D})$, then the net H is said to satisfy *Haag duality*. Given two relatively spacelike double cones $\mathcal{D}_1$ and $\mathcal{D}_2$, there always exists a wedge region W such that $\mathcal{D}_1 \subset W$ and $\mathcal{D}_2 \subset W'$ ([76, Prop. 3.1]). For every double cone $\mathcal{D}$, we further have $\mathcal{D} = \bigcap_{W\supset\mathcal{D}} W$. As a consequence $\mathsf{H}(\mathcal{D}') = \overline{\sum_{W\supset\mathcal{D}'} \mathsf{H}(W)}$ (with the definition of $\mathsf{H}(W)$ given in (27)). With respect to $\mathsf{V} = \mathsf{H}(W_R)$, this leads to
$$\mathsf{H}^{\min}(\mathcal{D}') = \mathsf{H}(\mathcal{D}') \quad \text{and} \quad \mathsf{H}^d(\mathcal{D}) = \mathsf{H}^{\min}(\mathcal{D}')'.$$
We further obtain
$$\mathsf{H}^d(\mathcal{D}) = \bigcap_{W\supset\mathcal{D}} \mathsf{H}(W) = \bigcap_{g\in\mathcal{P}_+^\uparrow, gW_R\supset\mathcal{D}} \mathsf{H}(gW_R) = \mathsf{H}^{\max}(\mathcal{D}).$$

For the case $d = 1$ one still has
$$\mathsf{H}^d(\mathcal{D}) = \bigcap_{W\supset\mathcal{D}} \mathsf{H}(W) = \bigcap_{g\in G, gW_R\supset\mathcal{D}} \mathsf{H}(gW_R) = \mathsf{H}^{\max}(\mathcal{D}),$$
but, to this end, one has to consider the maximal net with respect to a unitary representation $(U, \mathcal{H})$ of the group $G = \mathcal{P}^\uparrow = \langle \mathcal{P}_+^\uparrow, r\rangle$, where $r(x_0, x_1) = (x_0, -x_1)$ and H is also

covariant for $U(r)$. Indeed, every double cone is the intersection of $W_R + a$ and $W_R' + b$ for some $a, b \in \mathbb{R}^{1,d}$, but W_R and W_R' belong to disjoint orbits of wedges with respect to $\mathcal{P}_+^\uparrow$. However, they belong to the same orbit of $\mathcal{P}^\uparrow$ because $W_R' = rW_R$.

Alternatively, starting with a unitary representation $(U, \mathcal{H})$ of $\mathcal{P}_+^\uparrow$ for which H is covariant, we can use Theorem 3.4 to extend U to an (anti-)unitary representation of $\mathcal{P}_+$ by $U(\tau_h) := J_{\mathsf{H}(W_R)}$. Then $\mathcal{P}_+$ acts covariantly on the net on wedge regions.[3] Hence $\tau_h W_R = W_R'$ implies the equality

$$\bigcap_{W \supset \mathcal{D}} \mathsf{H}(W) = \bigcap_{g \in \mathcal{P}_+, gW_R \supset \mathcal{D}} U(g)\mathsf{H}(W_R) = \bigcap_{g \in \mathcal{P}_+, gW_R \supset \mathcal{D}} \mathsf{H}(gW_R) =: \widetilde{\mathsf{H}}^{\max}(\mathcal{D}),$$

where $\widetilde{\mathsf{H}}^{\max}(\mathcal{D})$ now is defined with respect to the (anti-)unitary representation of $\mathcal{P}_+$. If both constructions apply, then $\mathsf{H}^{\max}(\mathcal{D}) = \widetilde{\mathsf{H}}^{\max}(\mathcal{D})$.

We can conclude a correspondence between the maximal net construction and the dual net construction but, since we will not deal with locality in this paper, a more detailed analysis is postponed to future works.

2.2.5. Intersections of standard subspaces

Standing assumption in the remainder of this section: Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$ an Euler element. Assume that the involution τ_h integrates to an involution τ_h^G on G. For an (anti-)unitary representation $(U, \mathcal{H})$ of the group $G_{\tau_h} := G \rtimes \{\mathrm{id}_G, \tau_h^G\}$, we call

$$\mathsf{V} := \mathsf{V}(h, U) := \mathsf{H}_U^{\mathrm{BGL}}(h, \tau_h^G) \tag{29}$$

the *canonical standard subspace associated to* (h, U). Its modular objects are $J = U(\tau_h^G)$ and $\Delta = e^{2\pi i \partial U(h)}$.

For a subset $A \subseteq G$, we consider the closed real subspace

$$\mathsf{V}_A := \mathsf{V}_A(h, U) := \bigcap_{g \in A} U(g)\mathsf{V}. \tag{30}$$

We shall be interested in criteria for these real subspaces to be cyclic. An important property of these subspaces is that they are well adapted to direct sums and direct integrals (Lemma C.3 and (23)).

These concepts require (anti-)unitary representations of G_{τ_h}, but often unitary representations of G are easier to deal with. The following lemma translates between unitary and (anti-)unitary representations and their properties. It is our version of a closely related technique developed in [11, Props. 4.1, 4.2], which is based on density properties of intersections of dense complex subspaces of $\mathcal{H}$.

[3] One can also argue with Borchers' Theorem, positivity of the energy and the Bisognano–Wichmann property.

Lemma 2.22. (The (anti-)unitary extension) *Let $(U, \mathcal{H})$ be a unitary representation of G and write $\overline{\mathcal{H}}$ for the Hilbert space $\mathcal{H}$, endowed with the opposite complex structure. Then the following assertions hold:*

(a) *On $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \overline{\mathcal{H}}$ we obtain by $\widetilde{U}(g) := U(g) \oplus U(\tau_h^G(g))$ a unitary representation which extends by $\widetilde{U}(\tau_h)(v,w) := \widetilde{J}(v,w) := (w,v)$ to an (anti-)unitary representation of G_{τ_h}. The corresponding standard subspace $\widetilde{\mathtt{V}} := \mathtt{V}(h, \widetilde{U})$ coincides with the graph*

$$\widetilde{\mathtt{V}} = \Gamma(\Delta^{1/2}), \tag{31}$$

and its modular operator is $\widetilde{\Delta} := \Delta \oplus \Delta^{-1}$.

(b) *If U extends to an (anti-)unitary representation of G_{τ_h} by $J = U(\tau_h)$, then the following assertions hold:*
   1. *$\Phi \colon \mathcal{H}^{\oplus 2} \to \widetilde{\mathcal{H}}, \Phi(v,w) = (v, Jw)$ is a unitary intertwiner of $\widetilde{U}$ and the (anti-)unitary representation $U^\sharp$ of G_{τ_h} on $\mathcal{H}^{\oplus 2}$, given by*

$$U^\sharp|_G = U^{\oplus 2} \quad \text{and} \quad U^\sharp(\tau_h)(v,w) := J^\sharp(v,w) := (Jw, Jv).$$

   2. *The standard subspace $\mathtt{V}^\sharp := \mathtt{V}(h, U^\sharp)$ coincides with the graph $\Gamma(T_\mathtt{V})$ of the Tomita operator $T_\mathtt{V} = J\Delta^{1/2}$ of $\mathtt{V}$.*
   3. *The (anti-)unitary representation $\widetilde{U}$ is equivalent to the (anti-)unitary representation $U^{\oplus 2}$ of G_{τ_h} on $\mathcal{H}^{\oplus 2}$.*
   4. *If $A \subseteq G$ is a subset, then $\widetilde{\mathtt{V}}_A$ is cyclic in $\widetilde{\mathcal{H}}$ if and only if $\mathtt{V}_A$ is cyclic in $\mathcal{H}$.*

Proof. (a) The first assertion is a direct verification (cf. [63, Lemma 2.10]). Since

$$\widetilde{\Delta} = e^{2\pi i \partial \widetilde{U}(h)} = \Delta \oplus \Delta^{-1},$$

the description of the standard subspace $\widetilde{\mathtt{V}} = \mathrm{Fix}(\widetilde{J}\widetilde{\Delta}^{1/2})$ follows immediately.

(b) (1) Clearly, Φ is a complex linear isometry that intertwines the (anti-)unitary representation $\widetilde{U}$ with the (anti-)unitary representation $U^\sharp$.

(2) As $\Delta^\sharp = \Phi^{-1}\widetilde{\Delta}\Phi = \Delta \oplus \Delta$, the relation

$$(v,w) = J^\sharp(\Delta^\sharp)^{1/2}(v,w) = (J\Delta^{1/2}w, J\Delta^{1/2}v) = (T_\mathtt{V} w, T_\mathtt{V} v)$$

is equivalent to $w = T_\mathtt{V} v$. Hence $\mathtt{V}^\sharp = \Gamma(T_\mathtt{V})$.

(3) As the restrictions of $U^{\oplus 2}$ and $U^\sharp$ to G coincide, [63, Thm. 2.11] implies their equivalence as (anti-)unitary representations. However, in the present concrete case it is easy to see an intertwining operator. The matrix

$$A := \frac{1}{2}\begin{pmatrix} (1+i)\mathbf{1} & (1-i)\mathbf{1} \\ (1-i)\mathbf{1} & (1+i)\mathbf{1} \end{pmatrix} \quad \text{with} \quad A^2 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

defines a unitary operator on $\mathcal{H}^{\oplus 2}$ commuting with $U^\sharp(G)$. It satisfies $J^{\oplus 2}AJ^{\oplus 2} = A^* = A^{-1}$, so that

$$AJ^{\oplus 2}A^{-1} = A^2 J^{\oplus 2} = J^\sharp.$$

(4) If $U|_G$ extends to an (anti-)unitary representation U of G_{τ_h} on $\mathcal{H}$, then (3) implies that $\widetilde{U} \cong U^{\oplus 2}$, and any equivalence $\Psi\colon (\widetilde{U}, \widetilde{\mathcal{H}}) \to (U^{\oplus 2}, \mathcal{H}^{\oplus 2})$ maps $\widetilde{\mathtt{V}}_A$ to $(\mathtt{V} \oplus \mathtt{V})_A = \mathtt{V}_A \oplus \mathtt{V}_A$ (see (23)). Therefore $\widetilde{\mathtt{V}}_A$ is cyclic if and only if $\mathtt{V}_A$ is cyclic in $\mathcal{H}$. $\square$

The following definition extends the classical type of irreducible complex representations to the case where the involution on G is non-trivial. For a unitary representation $(U, \mathcal{H})$, we write $(\overline{U}, \overline{\mathcal{H}})$ for the canonical unitary representation on the complex conjugate space $\overline{\mathcal{H}}$ by $\overline{U}(g) = U(g)$. We observe that, for an (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h}, its *commutant*

$$\begin{aligned} U(G_{\tau_h})' &= \{A \in B(\mathcal{H})\colon (\forall g \in G_{\tau_h})\, AU(g) = U(g)A\} \\ &= \{A \in U(G)'\colon U(\tau_h^G)A = AU(\tau_h^G)\} \end{aligned}$$

is only a real subalgebra of $B(\mathcal{H})$ because some $U(g)$ are antilinear.

Definition 2.23. ([63, Def. 2.12]) Let $(U, \mathcal{H})$ be an irreducible unitary representation of G. We say that U is (with respect to τ_h), of

- *real type* if there exists an antiunitary involution J on $\mathcal{H}$ such that $U^\sharp(\tau_h) := J$ extends U to an (anti-)unitary representation $U^\sharp$ of G_{τ_h} on $\mathcal{H}$, i.e., $JU(g)J = U(\tau_h^G(g))$ for $g \in G$. Then the commutant of $U^\sharp(G_{\tau_h})$ is $\mathbb{R}$.
- *quaternionic type* if there exists an antiunitary complex structure I on $\mathcal{H}$ satisfying $IU(g)I^{-1} = U(\tau_h^G(g))$ for $g \in G$. Then $\overline{U} \circ \tau_h^G \cong U$, U has no extension on the same space, and the (anti-)unitary representation $(\widetilde{U}, \widetilde{\mathcal{H}})$ of G_{τ_h} with $\widetilde{U}|_G \cong U \oplus (\overline{U} \circ \tau_h^G)$ is irreducible with commutant $\mathbb{H}$.
- *complex type* if $\overline{U} \circ \tau_h^G \not\cong U$. This is equivalent to the non-existence of $V \in \mathrm{AU}(\mathcal{H})$ such that $U(\tau_h^G(g)) = VU(g)V^{-1}$ for all $g \in G$. Then $(\widetilde{U}, \widetilde{\mathcal{H}})$ is an irreducible (anti-)unitary representation of G_{τ_h} with commutant $\mathbb{C}$.

Example 2.24. (a) On the Poincaré group $\mathcal{P} = \mathbb{R}^{1,d} \rtimes \mathcal{L}_+^\uparrow$ we consider the involution $\tau_h^G(g) = j_h g j_h$, corresponding to conjugation with

$$j_h(x_0, x_1, \ldots, x_d) = (-x_0, -x_1, x_2, \ldots, x_d),$$

so that $\mathcal{P}_{\tau_h} \cong \mathcal{P}_+$. Then all irreducible positive energy representations of $\mathcal{P}$ are of real type except the massless finite helicity representations that are of complex type (see [58, App. A] for $m > 0$, and [77, Thm. 9.10] for the general case).

(b) (cf. [63, Ex. 2.16(c)]) Consider the irreducible unitary representation of $G = \mathrm{SU}_2(\mathbb{C}) \cong \mathrm{Spin}_3(\mathbb{R})$ on $\mathbb{C}^2 \cong \mathbb{H}$ (by left multiplication) where the complex structure on $\mathbb{H}$ is defined by the right multiplication with $\mathbb{C}$. This representation is of quaternionic type with respect to $\sigma = \mathrm{id}$, but of real type with respect to the involution $\sigma(g) = \overline{g}$.

Remark 2.25. (Antiunitary tensor products) Let $G = G_1 \times G_2$ be a product of type I groups and τ an involutive automorphism of G preserving both factors, i.e., $\tau = \tau_1 \times \tau_2$. We want to describe irreducible (anti-)unitary representations $(U, \mathcal{H})$ of the group $G_\tau = G \rtimes \{\mathrm{id}_G, \tau\}$ using [63, Thm. 2.11(d)].

(a) The first possibility is that $U|_G$ is irreducible, so that $U(G)' \cong \mathbb{R}$. Then

$$(U|_G, \mathcal{H}) \cong (U_1, \mathcal{H}_1) \otimes (U_2, \mathcal{H}_2)$$

with irreducible unitary representations $(U_j, \mathcal{H}_j)$ of G_j both extending to (anti-)unitary representations $U_j^\sharp$ of G_j. Hence both U_1 and U_2 are of real type.

(b) The second possibility is that $U|_G$ is reducible with $U(G)' \cong \mathbb{C}$ or $\mathbb{H}$, so that

$$U|_G \cong V \oplus (\overline{V} \circ \tau),$$

where $(V, \mathcal{K})$ is an irreducible unitary representation of G of complex or quaternionic type. Now $V = U_1 \otimes U_2$, and thus

$$\mathcal{H} \cong (\mathcal{H}_1 \otimes \mathcal{H}_2) \oplus (\overline{\mathcal{H}}_1 \otimes \overline{\mathcal{H}}_2), \quad U|_G \cong (U_1 \otimes U_2) \oplus (\overline{U_1} \circ \tau_1 \otimes \overline{U_2} \circ \tau_2).$$

If U_j is of complex type, then $\overline{U_j} \circ \tau_j \not\cong U_j$ implies that V is of complex type. If both U_1 and U_2 are of quaternionic type, then $\overline{U_j} \circ \tau_j \cong U_j$ for $j = 1, 2$ implies $\overline{V} \circ \tau \cong V$, so that V is of quaternionic type.

Proposition 2.26. *Assume that G has at most countably many connected components and that $A \subseteq G$ is a subset. Then the following are equivalent:*

(a) *For all (anti-)unitary representations $(U, \mathcal{H})$ of G_{τ_h}, the subspace $\mathtt{V}_A$ is cyclic.*
(b) *For all irreducible (anti-)unitary representations $(U, \mathcal{H})$ of G_{τ_h}, the subspace $\mathtt{V}_A$ is cyclic.*
(c) *For all irreducible unitary representations $(U, \mathcal{H})$ of G, the subspace $\widetilde{\mathtt{V}}_A$ is cyclic in $\widetilde{\mathcal{H}}$.*
(d) (Characterization in terms of unitary representations) *For all unitary representations $(U, \mathcal{H})$ of G, the subspace $\widetilde{\mathtt{V}}_A$ is cyclic in $\widetilde{\mathcal{H}}$.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): Let $(U, \mathcal{H})$ be an irreducible unitary representation and $(\widetilde{U}, \widetilde{\mathcal{H}})$ its natural (anti-)unitary extension. Then either $\widetilde{U}$ is an irreducible (anti-)unitary representations (if

U is of complex or quaternionic type) or a direct sum of two irreducible representations (if U is of real type) (cf. Definition 2.23). In view of (23), the cyclicity of $\mathtt{V}_A$ is inherited by direct sums, so that (c) follows from (b).

(c) ⇒ (d): Let $(U, \mathcal{H})$ be a unitary representation of G. Decomposing U into a direct sum of cyclic representations, we may assume that U is cyclic, hence that $\mathcal{H}$ is separable. Using [23, Thm. 8.5.2, §18.7], we can write U as a direct integral

$$U = \int_X^{\oplus} U_x \, d\mu(x)$$

of irreducible representations $(U_x)_{x \in X}$. Then

$$\widetilde{U} = \int_X^{\oplus} \widetilde{U}_x \, d\mu(x)$$

implies that $\widetilde{\mathtt{V}}_A = \int_X^{\oplus} \widetilde{\mathtt{V}}_{x,A} \, d\mu(x)$ by (78) and Lemma C.3(a). Further, Lemma C.3(b) implies that $\widetilde{\mathtt{V}}_A$ is cyclic because all subspaces $\widetilde{\mathtt{V}}_{x,A}$ are cyclic by (c).

(d) ⇒ (a): If $(U, \mathcal{H})$ is an (anti-)unitary representation of G_{τ_h}, then its restriction to G has an (anti-)unitary extension $(\widetilde{U}, \widetilde{\mathcal{H}})$ which by Lemma 2.22(b)(1) is equivalent to $U^{\oplus 2}$. Hence the cyclicity of $\widetilde{\mathtt{V}}_A \cong \mathtt{V}_A \oplus \mathtt{V}_A$ implies that $\mathtt{V}_A$ is cyclic. □

3. Modular groups are generated by Euler elements

In this section we show that, if the modular group of a standard subspace $\mathtt{V}$ is obtained from a unitary representation of a finite-dimensional Lie group G and a certain regularity condition is satisfied, then its infinitesimal generator is an Euler element $h \in \mathfrak{g}$ and the modular conjugation $J_{\mathtt{V}}$ induces on G the involution corresponding to $\tau_h = e^{\pi i \operatorname{ad} h}$ on $\mathfrak{g}$ (Theorem 3.1 in Section 3.1). In Subsection 3.2 we describe the implications of this result in the context of operator algebras with cyclic separating vectors (Theorem 3.7). In this context, we also obtain an explicit description of the identity component of the subsemigroup $S_{\mathcal{M}}$ of G leaving a von Neumann algebra $\mathcal{M}$ invariant.

3.1. The Euler element theorem

The following theorem is a key result of this paper on which all other discussions build. An important consequence is relation (33) which provides an extension of U to an (anti-)unitary representation of G_{τ_h} on the same space. Note that, besides connectedness, no assumptions are made on the structure of G, in particular G does not have to be semisimple.

Theorem 3.1. (Euler Element Theorem) *Let G be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$. Let $(U, \mathcal{H})$ be a unitary representation of G for which*

$$\ker(\mathtt{d}U) \cap [h, \mathfrak{g}] = \{0\}. \tag{32}$$

Suppose that $\mathtt{V}$ is a standard subspace and $N \subseteq G$ an identity neighborhood such that

(a) $U(\exp(th)) = \Delta_{\mathtt{V}}^{-it/2\pi}$ *for $t \in \mathbb{R}$, and*
(b) $\mathtt{V}_N := \bigcap_{g \in N} U(g)\mathtt{V}$ *is cyclic.*

Then h is an Euler element (or central) and the conjugation $J_{\mathtt{V}}$ satisfies

$$J_{\mathtt{V}} U(\exp x) J_{\mathtt{V}} = U(\exp \tau_h(x)) \quad \text{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}. \tag{33}$$

In Theorem D.2 we characterize those Euler elements for which a standard subspace satisfying (a) exists in every unitary representation of G.

Proof. First we argue that it suffices to consider representations with discrete kernel, i.e., for which the ideal $\mathfrak{n} := \ker(\mathtt{d}U)$ is trivial. Our assumption (32) implies that there exists a vector complement $E \subseteq \mathfrak{g}$ of the ideal $\mathfrak{n}$ containing $[h, \mathfrak{g}]$. Then $\mathfrak{g} = \mathfrak{n} \oplus E$ is an $\operatorname{ad} h$-invariant decomposition, so that h is an Euler element in $\mathfrak{g}$ (or central) if and only if its canonical image is an Euler element in the quotient $\mathfrak{g}/\mathfrak{n}$. We may therefore pass to the factorized representation of the quotient group $G/\ker U$, which has discrete (actually trivial) kernel. So it suffices to assume that $\ker(U)$ is discrete.

Part 1: $\operatorname{ad} h$ is diagonalizable with integral eigenvalues: For $x \in \mathfrak{g}$, we write

$$x(s) := e^{s \operatorname{ad} h} x \in \mathfrak{g}.$$

Pick $\xi \in \mathtt{V}_N$. Then we have for $\psi \in \mathcal{H}$

$$\begin{aligned} \langle \psi, U(\exp(sh)\exp(tx))\xi \rangle &= \langle \psi, U(\exp(tx(s))\exp(sh))\xi \rangle \\ &= \langle U(\exp(-tx(s)))\psi, U(\exp(sh))\xi \rangle. \end{aligned} \tag{34}$$

By Hypothesis (b), there exists a $\delta > 0$ such that $U(\exp tx)\xi \in \mathtt{V}$ for $|t| < \delta$, so that $U(\exp tx)\xi$ is contained in the domain of $\Delta_{\mathtt{V}}^{1/2} = e^{\pi i \cdot \partial U(h)}$. Therefore the left hand side of (34) can be continued analytically in s to a continuous function on the closure of the strip $\mathcal{S}_\pi$ which is holomorphic in the interior (Proposition 2.14).

To obtain an analytic extension of the right hand side, we assume that $\psi \in \mathcal{H}^\omega$ is an analytic vector for U. Then there exists an open convex 0-neighborhood $B \subseteq \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ (depending on ξ) and a holomorphic map

$$\eta_\psi \colon B \to \mathcal{H} \quad \text{with} \quad \eta_\psi(x) = U(\exp x)\psi \quad \text{for} \quad x \in B \cap \mathfrak{g}$$

and

$$\eta_\psi(z) = \sum_{n=0}^\infty \frac{1}{n!}(\mathtt{d}U(z))^n\psi \quad \text{for} \quad z \in B.$$

Writing $\mathcal{H}(B)$ for the set of all these vectors ψ, we know that $\bigcup_{n\in\mathbb{N}} \mathcal{H}(\frac{1}{n}B)$ is dense in $\mathcal{H}$ ([68]). Shrinking δ, we may assume that

$$e^{z\,\mathrm{ad}\,h}tx \subseteq B \quad \text{for} \quad |t| \leq \delta, |z| \leq 2\pi.$$

Then, for a fixed t with $|t| \leq \delta$, the function $s \mapsto U(\exp(-tx(s)))\psi$ can be continued analytically to the open disc $\mathcal{D} := \{z \in \mathbb{C} : |z| < 2\pi\}$. Further, $s \mapsto U(\exp sh)\xi$ has an analytic continuation to the strip $\mathcal{S}_\pi$. We conclude that both sides of (34) extend analytically to $\mathcal{D} \cap \mathcal{S}_\pi$ with continuous boundary values. We thus obtain for any fixed t with $|t| \leq \delta$ and $s = \pi i$ the equality

$$\langle \psi, e^{\pi i \cdot \partial U(h)} U(\exp tx)\xi\rangle = \langle \eta_\psi(-te^{-\pi i\,\mathrm{ad}\,h}x), e^{\pi i\cdot \partial U(h)}\xi\rangle.$$

As $U(\exp tx)\xi \in \mathtt{V}$ and $\Delta_{\mathtt{V}}^{1/2} = e^{\pi i\cdot\partial U(h)}$, this is equivalent to

$$\langle \psi, J_{\mathtt{V}} U(\exp tx)\xi\rangle = \langle \eta_\psi(-te^{-\pi i\,\mathrm{ad}\,h}x), J_{\mathtt{V}}\xi\rangle.$$

The real subspace $\mathtt{V}_N$ spans a dense subspace of $\mathcal{H}$, so that, for each analytic vector $\psi \in \mathcal{H}^\omega$, there exists a $\delta_\psi > 0$, such that

$$U(\exp -tx)J_{\mathtt{V}}\psi = J_{\mathtt{V}}\eta_\psi(-te^{-\pi i\,\mathrm{ad}\,h}x) \quad \text{for} \quad |t| \leq \delta_\psi. \tag{35}$$

Multiplication with $J_{\mathtt{V}}$ on the left yields

$$J_{\mathtt{V}}U(\exp -tx)J_{\mathtt{V}}\psi = \eta_\psi(-te^{-\pi i\,\mathrm{ad}\,h}x) \tag{36}$$

For a fixed $t_0 = \delta_\psi$, (35) shows in particular that the G-orbit map of $J_{\mathtt{V}}\psi$ is real analytic in an e-neighborhood because

$$z \mapsto \eta_\psi(-te^{-\pi i\,\mathrm{ad}\,h}z)$$

defines a holomorphic function on a 0-neighborhood of $\mathfrak{g}_{\mathbb{C}}$. We therefore have $J_{\mathtt{V}}\mathcal{H}^\omega \subseteq \mathcal{H}^\omega$. As both sides of (36) are differentiable in $t = 0$, we now obtain

$$J_{\mathtt{V}}\mathtt{d}U(x)J_{\mathtt{V}}\psi = \mathtt{d}U(e^{-\pi i\,\mathrm{ad}\,h}x)\psi \quad \text{for} \quad \psi \in \mathcal{H}^\omega.$$

The left hand side is a skew-symmetric operator on $\mathcal{H}^\omega$, so that $\mathtt{d}U(e^{-\pi i\,\mathrm{ad}\,h}x)$ is skew-symmetric on $\mathcal{H}^\omega$. As $\ker(\mathtt{d}U) = \mathbf{L}(\ker U) = \{0\}$, it follows that

$$\tau_h(x) := e^{-\pi i \operatorname{ad} h} x \in \mathfrak{g} \quad \text{for} \quad x \in \mathfrak{g}$$

because $\mathtt{d}U(z)$ is skew hermitian on $\mathcal{H}^\omega$ if and only if $z \in \mathfrak{g}$.

This means that the automorphism $\tau_h \in \operatorname{Aut}(\mathfrak{g}_\mathbb{C})$ preserves the real subspace $\mathfrak{g} \subseteq \mathfrak{g}_\mathbb{C}$ and that we have

$$J_\mathtt{V} \mathtt{d}U(x) J_\mathtt{V} = \mathtt{d}U(e^{-\pi i \operatorname{ad} h} x) \quad \text{on} \quad \mathcal{H}^\omega \quad \text{for every} \quad x \in \mathfrak{g}. \tag{37}$$

Applying this relation twice, we arrive at

$$\mathtt{d}U(x) = J_\mathtt{V}^2 \mathtt{d}U(x) J_\mathtt{V}^2 = \mathtt{d}U(\tau_h^2 x) \quad \text{on} \quad \mathcal{H}^\omega \quad \text{for every} \quad x \in \mathfrak{g}.$$

As $\mathtt{d}U$ is injective, this shows that $e^{-2\pi i \operatorname{ad} h} = \tau_h^2 = \operatorname{id}_\mathfrak{g}$. This in turn implies that $\operatorname{ad} h$ is diagonalizable with integral eigenvalues ([33, Exer. 3.2.12]). We also note that (37) entails

$$J_\mathtt{V} U(\exp x) J_\mathtt{V} = U(\exp \tau_h(x)) \quad \text{for} \quad x \in \mathfrak{g}$$

because any dense subspace consisting of analytic vectors is a core by Nelson's Theorem.

Part 2: h is an Euler element: Let $k \in \mathbb{Z}$ be an eigenvalue of $\operatorname{ad} h$. We have to show that $|k| \leq 1$. So let us assume that $|k| \geq 2$ and show that this leads to a contradiction. Let $x \in \mathfrak{g}$ be a corresponding eigenvector, so that $[h, x] = kx$. In view of (b), there exists a $\delta > 0$ such that

$$U(\exp tx) U(\exp sh) \mathtt{V}_N \subseteq \mathtt{V} \quad \text{for} \quad |t| + |s| < \delta.$$

Let

$$M := \partial U(h) \quad \text{and} \quad Q := \partial U(x)$$

denote the infinitesimal generators of the 1-parameter groups $U(\exp th)$ and $U(\exp tx)$, respectively. Suppose that $\xi = U(\exp rh)\eta = e^{rM}\eta$ for $\eta \in \mathtt{V}_N$ and $|r| < \delta$, so that $\xi \in \mathtt{V}$. As in Part 1, for $|t| + |r| < \delta$ and any entire vector $\psi \in \mathcal{H}$ of Q, both sides of

$$\langle \psi, U(\exp(sh) \exp(tx)) \xi \rangle = \langle \psi, U(\exp(te^{sk} x) \exp(sh)) \xi \rangle$$

extend analytically in s into $\mathcal{S}_\pi$. For $s := \frac{\pi i}{|k|}$ we have $\operatorname{Im} s < \pi$, so that we obtain for any $\eta \in \mathtt{V}_N$

$$\langle \psi, e^{\frac{\pi i}{|k|} M} e^{tQ} e^{rM} \eta \rangle = \langle \psi, e^{-tQ} e^{\frac{\pi i}{|k|} M} e^{rM} \eta \rangle \quad \text{for} \quad |t| + |r| < \delta.$$

As this holds for a dense set of vectors ψ, we derive that

$$e^{\frac{\pi i}{|k|} M} e^{tQ} e^{rM} \eta = e^{-tQ} e^{\frac{\pi i}{|k|} M} e^{rM} \eta \quad \text{for} \quad |t| + |r| < \delta. \tag{38}$$

Now let $E \subseteq \mathbb{R}$ be a bounded Borel subset and $P_{iM}(E)$ the corresponding spectral projection of the selfadjoint operator iM on $\mathcal{H}$. We multiply the relation (38) on the left with $P_{iM}(E)$ to obtain

$$e^{\frac{\pi i}{|k|}M} P_{iM}(E) e^{tQ} e^{rM} \eta = P_{iM}(E) e^{-tQ} e^{\frac{\pi i}{|k|}M} e^{rM} \eta. \tag{39}$$

Next we observe that $e^{\frac{\pi i}{|k|}M} P_{iM}(E)$ is a bounded operator and, as $\pi \geq \frac{2\pi}{|k|}$, the vector η is contained in the domain of $e^{\frac{2\pi i}{|k|}M}$, so that its orbit map $r \mapsto e^{rM}\eta$ extends analytically to the strip $\mathcal{S}_{\frac{2\pi}{k}}$. So both sides of (39) have analytic continuations in r to the strip $\mathcal{S}_{\frac{\pi}{|k|}}$. Hence by uniqueness of analytic continuation, (39) also holds for all real r and $|t| < \delta$. Let

$$\mathcal{H}_\eta := \overline{\operatorname{span}\{e^{rM}\eta \colon r \in \mathbb{R}\}}$$

denote the cyclic subspace generated by η under $e^{\mathbb{R}M} = U(\exp \mathbb{R}h)$. We then obtain from (39) that

$$e^{\frac{\pi i}{|k|}M} P_{iM}(E) e^{tQ} \zeta = P_{iM}(E) e^{-tQ} e^{\frac{\pi i}{|k|}M} \zeta \quad \text{for} \quad \zeta \in \mathcal{H}_\eta.$$

As $\mathcal{H}_\eta$ is invariant under the von Neumann algebra generated by $e^{\mathbb{R}M}$, it is invariant under all spectral projections, i.e. $P_{iM}(E)\mathcal{H}_\eta \subset \mathcal{H}_\eta$. This shows that

$$e^{\frac{\pi i}{|k|}M} P_{iM}(E) e^{tQ} P_{iM}(E) \eta = P_{iM}(E) e^{-tQ} e^{\frac{\pi i}{|k|}M} P_{iM}(E) \eta.$$

As all operators in this identity are bounded and $\mathtt{V}_N$ spans a dense subspace of $\mathcal{H}$, we arrive at the relation

$$e^{\frac{\pi i}{|k|}M} P_{iM}(E) e^{tQ} P_{iM}(E) = P_{iM}(E) e^{-tQ} P_{iM}(E) e^{\frac{\pi i}{|k|}M} \quad \text{for} \quad |t| < \delta. \tag{40}$$

Hence the operator

$$P_{iM}(E) e^{\frac{2\pi i}{|k|}M} P_{iM}(E) = \big(P_{iM}(E) e^{\frac{\pi i}{|k|}M} P_{iM}(E)\big)^2$$

commutes with $P_{iM}(E)e^{tQ}P_{iM}(E)$ for $|t| < \delta$, because applying (40) twice shows that

$$\begin{aligned} P_{iM}(E) e^{\frac{2\pi i}{|k|}M} P_{iM}(E) e^{tQ} P_{iM}(E) &= P_{iM}(E) e^{\frac{\pi i}{|k|}M} P_{iM}(E) e^{-tQ} P_{iM}(E) e^{\frac{\pi i}{|k|}M} P_{iM}(E) \\ &= P_{iM}(E) e^{tQ} P_{iM}(E) e^{\frac{2\pi i}{|k|}M} P_{iM}(E). \end{aligned}$$

As the von Neumann algebra on $P_{iM}(E)\mathcal{H}$ generated by $P_{iM}(E)e^{\frac{2\pi i}{|k|}M}P_{iM}(E)$ contains the unitary one-parameter group $P_{iM}(E)e^{\mathbb{R}M}P_{iM}(E)$, it follows that

$$\begin{aligned}P_{iM}(E)e^{sM}e^{tQ}P_{iM}(E) &= P_{iM}(E)e^{sM}P_{iM}(E)e^{tQ}P_{iM}(E)\\ &= P_{iM}(E)e^{tQ}P_{iM}(E)e^{sM}P_{iM}(E)\\ &= P_{iM}(E)e^{tQ}e^{sM}P_{iM}(E) \quad \text{for} \quad s\in\mathbb{R}, |t|<\delta.\end{aligned}$$

As E was arbitrary, this implies that $e^{\mathbb{R}M}$ commutes with $e^{\mathbb{R}Q}$, contradicting the assumption $|k| \geq 2$. We therefore have $|k| \leq 1$ and thus h is an Euler element. □

Remark 3.2. If N is an e-neighborhood in G, then so is N^{-1}. Therefore condition (b) in Theorem 3.1 is equivalent to the following:

(b') There exists a cyclic subspace $\mathsf{K} \subset \mathsf{H}$ such that $U(g)\mathsf{K} \subset \mathtt{V}$ for every $g \in N$.

Indeed, if (b) holds, then $\mathsf{K} := \mathtt{V}_N$ satisfies (b') for the e-neighborhood N^{-1}. If, conversely, (b') holds, then $\mathtt{V}_{N^{-1}} \supseteq \mathsf{K}$ is cyclic. When nets of standard subspaces are considered in the next sections, then Properties (b) and (b') will be related to regularity and localizability in a specific region, respectively (cf. Definition 4.1 and Lemma 4.20).

Starting points for the development of the proof of Theorem 3.1 were [6] for Part 1 and [71] for Part 2. Accordingly, we recover one of R. Strich's results as the following corollary.

Corollary 3.3. (Strich's Theorem for standard subspaces) *Let $\lambda \in \mathbb{R}^\times$ and consider a two-dimensional connected Lie group G whose Lie algebra is $\mathfrak{g} = \mathbb{R}x + \mathbb{R}h$ with $[h, x] = \lambda x$. Let $(U, \mathcal{H})$ be a unitary representation of G with $\partial U(x) \neq 0$. Suppose that $\mathsf{H} \subseteq \mathtt{V}$ are standard subspaces such that*

(a) $U(\exp(-\beta th)) = \Delta_{\mathtt{V}}^{it}$ *for* $t \in \mathbb{R}$.
(b) $U(\exp tx)U(\exp sh)\mathsf{H} \subseteq \mathtt{V}$ *for* $|s| + |t| < \delta$ *and some* $\delta > 0$.

Then $\beta = \frac{2\pi}{|\lambda|}$.

Proof. Since $\mathbb{R}x = [h, \mathfrak{g}]$ intersects $\ker(\mathsf{d}U)$ trivially because $\partial U(x) \neq 0$, Theorem 3.1 implies that $\frac{\beta}{2\pi}h$ is an Euler element in $\mathfrak{g}$, so that $\frac{\beta|\lambda|}{2\pi} = 1$. □

Theorem 3.4. *Let $(U, \mathcal{H})$ be a unitary representation of the connected Lie group G with* $\ker(U)$ *discrete. If $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ is a net of real subspaces on (the open subsets of) a G-manifold M that satisfies* (Iso), (Cov), (RS) *and* (BW), *then the Lie algebra element h satisfying*

$$\Delta_{\mathsf{H}(W)} = e^{2\pi i\, \partial U(h)}$$

is an Euler element, and the conjugation $J := J_{\mathsf{H}(W)}$ satisfies

$$JU(\exp x)J = U(\exp \tau_h(x)) \quad \textit{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}.$$

Proof. Let $\mathcal{O} \subseteq W$ be a non-empty open, relatively compact subset. Then $\overline{\mathcal{O}}$ is a compact subset of the open set W, so that

$$N := \{g \in G : g.\overline{\mathcal{O}} \subseteq W\}$$

is an open e-neighborhood in G. For every $g \in N$ we have by (Cov) and (Iso)

$$g.\mathsf{H}(\mathcal{O}) = \mathsf{H}(g.\mathcal{O}) \subseteq \mathsf{H}(W) \overset{\text{(BW)}}{=} \mathtt{V}.$$

Further (RS) implies that $\mathsf{H} := \mathsf{H}(\mathcal{O})$ is cyclic, hence standard because it is contained in $\mathtt{V}$. Now the assertion follows from Theorem 3.1. □

Theorem 6.2 in [6] can be rephrased for standard subspaces. Then it becomes a consequence of our Theorem 3.4. With the notation introduced in Example 2.7, we state the following corollary:

Corollary 3.5. (Borchers–Buchholz Theorem for standard subspaces) *Let U be a unitary representation of the Lorentz group $G = \mathrm{SO}_{1,d}(\mathbb{R})^{\uparrow}$ on a Hilbert space $\mathcal{H}$, acting covariantly on an isotone net $(\mathsf{H}(\mathcal{O}))_{\mathcal{O} \subseteq \mathrm{dS}^d}$ of standard subspace on open regions of de Sitter spacetime. If $\beta > 0$ is such that*

$$U(\exp(th)) = \Delta_{\mathsf{H}(W_R^{\mathrm{dS}})}^{-\frac{it}{\beta}} \quad \textit{for} \quad t \in \mathbb{R}, \tag{41}$$

then $\beta = 2\pi$.

Proof. The net of standard subspaces $(\mathsf{H}(\mathcal{O}))_{\mathcal{O} \subset \mathrm{dS}^d}$ with the Lorentz group representation $(U, \mathcal{H})$ fit the hypotheses of Theorem 3.4 with respect to the Lie algebra element $\widetilde{h} = \frac{\beta}{2\pi} h$, as

$$\Delta_{\mathsf{H}(W_R^{\mathrm{dS}})} = e^{2\pi i \partial U(\widetilde{h})}.$$

We conclude that $\widetilde{h}$ is an Euler element. Since h is also an Euler element in $\mathfrak{so}_{1,d}(\mathbb{R})$ and $\beta > 0$, we must have $\beta = 2\pi$. □

Remark 3.6. (a) An important consequence of Theorem 3.1 is that τ_h integrates to an involutive automorphism τ_h^G on the group $U(G) \cong G/\ker(U)$ that is uniquely determined by

$$\tau_h^G(\exp x) = \exp(\tau_h(x)) \quad \text{for} \quad x \in \mathfrak{g}.$$

To see this, let $q_G : \widetilde{G} \to G$ denote the universal covering of G and $\tau_h^{\widetilde{G}}$ the automorphism of $\widetilde{G}$ integrating $\tau_h \in \mathrm{Aut}(\mathfrak{g})$. Replacing G by $\widetilde{G}$ and U by $U \circ q_G$, we may assume that $G = \widetilde{G}$. Then (33) implies that

$$JU(g)J = U(\tau_h^G(g)) \quad \text{for} \quad g \in G. \tag{42}$$

It follows that $\tau_h^G(\ker U) = \ker U$, and hence that τ_h^G factors through an automorphism of the quotient group $G/\ker U \cong U(G)$.

Whenever τ_h^G exists (which by the preceding is the case if G is simply connected or if U is injective), U extends to an (anti-)unitary representation of the Lie group

$$G_{\tau_h} = G \rtimes \{\mathrm{id}_G, \tau_h^G\} \quad \text{by} \quad U(\tau_h^G) := J. \tag{43}$$

In the setting of Theorem 3.1, $(U, \mathcal{H})$ cannot be a multiple of an irreducible representation of complex type. Indeed, in this case there exists no anti-unitary operator J on $\mathcal{H}$ such that

$$U(\tau_h(g)) = JU(g)J^{-1} \quad \text{for} \quad g \in G. \tag{44}$$

So the conclusion of Theorem 3.1 fails, and therefore one of the two assumptions (a) and (b) must be violated. Given $h \in \mathfrak{g}$, it is easy to construct a standard subspace satisfying (a) by taking $\Delta_{\mathsf{V}} := e^{2\pi i \partial U(h)}$ as Tomita operator and any conjugation J commuting with $\partial U(h)$. The existence of such a conjugation only requires the unitary equivalence of the selfadjoint operators $i\partial U(h)$ and $-i\partial U(h)$ ([62, Prop. 3.1]). This is much weaker than (44) and satisfied in all unitary representations if $\mathfrak{g}$ is semisimple and h an Euler element (Theorem D.2). So Hypothesis (b) has to fail and thus regularity is lost. However, the doubling process from Lemma 2.22(a) leads to a context where (44) can be implemented.

This accords with the comment after Theorem 4.13 in [24], where a similar argument is used to show that factorial representations with finite non-zero helicity of the Poincaré group $\mathcal{P}_+^\uparrow$ of $\mathbb{R}^{1,3}$ cannot act on a net of standard subspaces on spacelike cones (Definition 2.20). We briefly recall the ideas here. Let $(U, \mathcal{H})$ be a factorial representation of finite non-zero helicity, acting covariantly on a net of standard subspaces on spacelike cones $\mathcal{C} \mapsto \mathsf{H}(\mathcal{C})$. By [24, Cor. 4.4], H has the (BW) property with respect to the pair (h, W_R) (see Example 2.7).[4] Following [30, Prop. 2.4] (or in our general setting [51, Thm. 4.28]), a representation of finite non-zero helicity acting on a net of standard subspaces on spacelike cones extends to a covariant (anti-)unitary representation of the proper Poincaré group $\mathcal{P}_+$ as in (44). As representations of finite non-zero helicity are of complex type ([77, Thm. 9.10]), we arrive at a contradiction.

Clearly, this example is compatible with the (BW) property in the form of condition (a) in Theorem 3.1. By continuity of the Poincaré action on $\mathbb{R}^{1,3}$, there always exists a spacelike cone $\mathcal{C} \subseteq \bigcap_{g \in N} gW$ if $N \subset \mathcal{P}_+^\uparrow$ is a sufficiently small neighborhood of the identity and W is a wedge region. For $\mathsf{V} = \mathsf{H}(W)$, we then obtain $\mathsf{H}(\mathcal{C}) \subset \mathsf{V}_N = \bigcap_{g \in N} g\mathsf{H}(W)$, and thus V_N is cyclic whenever $\mathsf{H}(\mathcal{C})$ is (which follows from (RS)). In particular, spacelike cone localization of standard subspaces ensures the regularity condition (b) in the setting

[4] It actually suffices to require the net to assign standard subspaces to wedge regions.

of Theorem 3.1 and this regularity condition for $\mathsf{H}(\mathcal{C})$ ensures the geometric property used in [30, Prop. 2.4] to obtain an extension to an (anti-)unitary representation of $\mathcal{P}^\uparrow$. As stressed for this specific case in [24], one needs to couple finite non-zero helicity representations with opposite helicities to provide an environment for non-trivial nets of standard subspaces.

(b) If $\mathtt{V}_N = \mathtt{V}$, then $\mathtt{V}$ is $U(G)$-invariant because the connected Lie group G is generated by the identity neighborhood N. In this case $h \in \mathfrak{g}$ is central, which follows from the discreteness of $\ker(U)$ because $U(G)$ commutes with $\Delta_{\mathtt{V}}$. Then we obtain on $\mathcal{H}^J$ a real representation of G.

(c) If $\mathfrak{g}$ is a compact Lie algebra, then every Euler element $h \in \mathfrak{g}$ is central, so that $\tau_h = \mathrm{id}_{\mathfrak{g}}$. Therefore the cyclicity of $\mathtt{V}_N$ as in Theorem 3.1 implies that $J_{\mathtt{V}}$ and $\Delta_{\mathtt{V}}$ commute with $U(G)$, and thus $U(g)\mathtt{V} = \mathtt{V}$ for $g \in G$. Therefore, a standard subspace $\mathtt{V}$ associated to a pair $(h, \tau) \in \mathcal{G}(G_\sigma)$ by the BGL construction can only satisfy the regularity condition in Theorem 3.1(b) if $\mathtt{V}$ and $\mathcal{H}^{J_{\mathtt{V}}}$ are $U(G)$-invariant. Therefore the representation $(U, \mathcal{H})$ is the complexification of the real representation of U on $\mathcal{H}^J = \mathtt{V}$. Conversely, for every real orthogonal representation $(U, \mathcal{E})$ of G, the real subspace $\mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$ is standard with $\Delta_{\mathcal{E}} = \mathbf{1}$ and $U_{\mathbb{C}}(G)$ leaves $\mathcal{E}$ invariant, so that the regularity condition is satisfied for trivial reasons.

3.2. An application to operator algebras

The following theorem is a version of the Euler Element Theorem 3.1 for operator algebras. We consider the following setup:

- (Uni) Let $(U, \mathcal{H})$ be a unitary representation of the **connected** Lie group G with discrete kernel, so that the derived representation $\mathtt{d}U$ is injective.
- (M) Let Ω be a unit vector and $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra for which Ω is cyclic and generating. We write $(\Delta_{\mathcal{M},\Omega}, J_{\mathcal{M},\Omega})$ for the corresponding modular objects.
- (Fix) $\Omega \in \mathcal{H}^G$, i.e., Ω is fixed by $U(G)$.
- (Mod) **Modularity:** There exists an element $h \in \mathfrak{g}$ for which $e^{2\pi i \partial U(h)} = \Delta_{\mathcal{M},\Omega}$. As $\ker(U)$ is discrete, h is uniquely determined.
- (Reg) **Regularity:** For some e-neighborhood $N \subseteq G$, the vector Ω is still cyclic (and obviously separating) for the von Neumann algebra

$$\mathcal{M}_N := \bigcap_{g \in N} \mathcal{M}_g, \qquad \text{where} \qquad \mathcal{M}_g = U(g)\mathcal{M}U(g)^{-1}.$$

  This implies that $(\mathcal{M}_N)'$ is a von Neumann algebra containing $\mathcal{M}'_g = U(g)\mathcal{M}'U(g)^{-1}$ for $g \in N$ and that Ω is cyclic and separating for $(\mathcal{M}_N)'$.

Theorem 3.7. *Assume* (Uni), (M), (Fix), (Reg) *and* (Mod). *Then h is an Euler element and the modular conjugation $J = J_{\mathcal{M},\Omega}$ of the pair $(\mathcal{M}, \Omega)$ satisfies*

$$JU(\exp x)J = U(\exp \tau_h(x)) \quad \textit{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}.$$

Proof. Clearly, Ω is also separating for $\mathcal{M}_N$. Let $\mathcal{M}_{\rm sa} := \{M \in \mathcal{M} \colon M^* = M\}$ be the subspace of hermitian elements in $\mathcal{M}$. Then we obtain the two standard subspaces

$$\mathtt{V} := \overline{\mathcal{M}_{\rm sa}\Omega} \supseteq \mathsf{H} := \overline{(\mathcal{M}_N)_{\rm sa}\Omega}. \tag{45}$$

Further $U(g)^{-1}\mathcal{M}_N U(g) \subseteq \mathcal{M}$ for $g \in N$ implies $U(g)^{-1}\mathsf{H} \subseteq \mathtt{V}$. Hence $\mathsf{H} \subseteq \mathtt{V}_N$, and the assertion follows from Theorem 3.1. □

Example 3.8. (The minimal group) For $G = \mathbb{R}$, $\mathfrak{g} = \mathbb{R}h$, and the unitary one-parameter group $U(t) := \Delta_{\mathcal{M},\Omega}^{-it/2\pi}$, the conditions (Uni), (M), (Fix), (Mod) and (Reg) are satisfied because the Tomita–Takesaki Theorem ensures that $\mathcal{M}_g = \mathcal{M}$ for every $g \in G$. The conclusion of Theorem 3.7 then reduces to the fact that $J_{\mathcal{M},\Omega}$ commutes with the modular group.

Endomorphism semigroups

We consider the context from Theorem 3.7, where G is a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$, $h \in \mathfrak{g}$ is an Euler element, $(U, \mathcal{H})$ is an (anti-)unitary representation of G_{τ_h} with discrete kernel, $J = U(\tau_h^G)$, and $\mathtt{V} = \mathtt{V}(h, U) \subseteq \mathcal{H}$ is the associated standard subspace. We also have a von Neumann algebra $\mathcal{M}$ with cyclic separating vector Ω for which

$$\mathtt{V} = \mathtt{V}_{\mathcal{M}} := \overline{\mathcal{M}_{\rm sa}\Omega}.$$

Here the equality of $\mathtt{V}$ and $\mathtt{V}_{\mathcal{M}}$ follows from the equality of their modular objects and Proposition 2.12.

We consider *the endomorphism semigroup of $\mathcal{M}$ in G* by

$$S_{\mathcal{M}} := \{g \in G \colon U(g)\mathcal{M}U(g)^{-1} \subseteq \mathcal{M}\}.$$

Typically it is hard to get fine information on the semigroup $S_{\mathcal{M}}$, but combining results from [61] with Theorem 3.7, we actually get a full description of its identity component by comparing it with the endomorphism semigroup

$$S_{\mathtt{V}} := \{g \in G \colon U(g)\mathtt{V} \subseteq \mathtt{V}\}.$$

Theorem 3.9. (The endomorphism semigroup) *Suppose that* (Uni), (M), (Fix), (Reg) *and* (Mod) *are satisfied. With the pointed cones $C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$, we have the following description of the identity component of the semigroup $S_{\mathcal{M}}$:*

$$(S_{\mathcal{M}})_e = (G_{\mathcal{M}})_e \exp(C_+ + C_-) = \exp(C_+)(G_{\mathcal{M}})_e \exp(C_-) \quad \textit{and} \quad \mathbf{L}(G_{\mathcal{M}}) = \mathfrak{g}_0(h).$$

In particular $(G_{\mathcal{M}})_e = \langle \exp \mathfrak{g}_0(h) \rangle$.

Proof. As U has discrete kernel, h is an Euler element and $\mathtt{V} = \mathtt{V}_{\mathcal{M}}$, [61, Thms. 2.16, 3.4] imply that

$$S_{\mathtt{V}_{\mathcal{M}}} = G_{\mathtt{V}_{\mathcal{M}}} \exp(C_+ + C_-) = \exp(C_+) G_{\mathtt{V}_{\mathcal{M}}} \exp(C_-). \tag{46}$$

Further, $g \in S_{\mathcal{M}}$ yields $U(g)\mathtt{V}_{\mathcal{M}} = \mathtt{V}_{\mathcal{M}_g} \subseteq \mathtt{V}_{\mathcal{M}}$ because $U(g)$ fixes Ω, and therefore

$$S_{\mathcal{M}} \subseteq S_{\mathtt{V}_{\mathcal{M}}}. \tag{47}$$

Let N be an e-neighborhood as in (Reg) and $g \in S_{\mathtt{V}_{\mathcal{M}}} \cap N$. Then $\mathcal{M}'_N$ contains both algebras $\mathcal{M}'$ and $\mathcal{M}'_g = U(g)\mathcal{M}'U(g)^{-1}$, and $U(g)\mathtt{V}_{\mathcal{M}} \subseteq \mathtt{V}_{\mathcal{M}}$ implies $U(g)\mathtt{V}'_{\mathcal{M}} \supseteq \mathtt{V}'_{\mathcal{M}}$. Further, Ω is cyclic and separating for $\mathcal{M}'_N$ and

$$\mathtt{V}_{\mathcal{M}'_g} = U(g)\mathtt{V}_{\mathcal{M}'} = U(g)\mathtt{V}'_{\mathcal{M}} \supseteq \mathtt{V}'_{\mathcal{M}} = \mathtt{V}_{\mathcal{M}'}.$$

As Ω is cyclic and separating for $\mathcal{M}'_g$ and $\mathcal{M}'$, [39, Prop. 3.24] implies that $\mathcal{M}'_g \supseteq \mathcal{M}'$, which leads to $\mathcal{M}_g \subseteq \mathcal{M}$, i.e., $g \in S_{\mathcal{M}}$. This proves that

$$S_{\mathcal{M}} \cap N = S_{\mathtt{V}_{\mathcal{M}}} \cap N.$$

Since the semigroups $\exp(C_\pm)$ and $(G_{V_{\mathcal{M}}})_e$ are generated by their intersections with N, it follows that $(S_{\mathtt{V}_{\mathcal{M}}})_e = \exp(C_+)(G_{V_{\mathcal{M}}})_e \exp(C_-) \subseteq S_{\mathcal{M}}$. Now the assertion follows from the fact that the connected components of $S_{\mathtt{V}_{\mathcal{M}}}$ are products of connected components of the group $G_{\mathtt{V}_{\mathcal{M}}}$ and $\exp(C_+ + C_-)$ (polar decomposition of $S_{\mathtt{V}_{\mathcal{M}}}$). □

Remark 3.10. Davidson's paper [22] contains interesting results on the relation between the stabilizer groups $G_{\mathcal{M}}$ and $G_{\mathtt{V}_{\mathcal{M}}}$, also on the level of endomorphism semigroups.

(a) [22, Thm. 4] considers a unitary one-parameter group $U_t = e^{itH}$ that fixes Ω and leaves the standard subspace $\mathtt{V}_{\mathcal{M}}$ invariant. It asserts that, if the set

$$\mathcal{D}(\delta) := \{X \in \mathcal{M} \colon [H, X] \in \mathcal{M}\}$$

is such that $\mathcal{D}(\delta)\Omega$ is a core for H in $\mathcal{H}$, then $\mathrm{Ad}(U_t)\mathcal{M} = \mathcal{M}$ for all $t \in \mathbb{R}$.

(b) Davidson considered in [22, Thm. 5] a unitary one-parameter group $U_t = e^{itH}$ fixing Ω such that $U_t\mathtt{V}_{\mathcal{M}} \subseteq \mathtt{V}_{\mathcal{M}}$ for $t \geq 0$. He shows that, if

$$\mathtt{V}_\varepsilon := \bigcap_{0 \leq t \leq \varepsilon} U_t \mathtt{V}_{\mathcal{M}}$$

is cyclic for some $\varepsilon > 0$, then $\mathrm{Ad}(U_t)\mathcal{M} \subseteq \mathcal{M}$ for $t \geq 0$. This condition is quite close to the assumption in our Theorem 3.1 and the regularity conditions discussed in the following section.

4. Regularity and localizability

If $(U, \mathcal{H})$ is a unitary representation of the Lie group G and $\mathtt{V} \subseteq \mathcal{H}$ a standard subspace with $\Delta_{\mathtt{V}} = e^{2\pi i \partial U(h)}$ for some $h \in \mathfrak{g}$, then the Euler Element Theorem (Theorem 3.1) describes a sufficient condition for h to be an Euler element, and in this case it even implies the extension of U to an (anti-)unitary extension of G_{τ_h} by $J_{\mathtt{V}}$. In this section we study the converse problem: Assuming that h is an Euler element and $(U, \mathcal{H})$ an (anti-)unitary representation of G_{τ_h}, when is $\mathtt{V}_N$ cyclic for some e-neighborhood $N \subseteq G$? We then call U regular with respect to h. In Subsection 4.1 we discuss various permanence properties of regularity and also sufficient conditions, such as Theorems 4.9 and 4.11, deriving regularity from positive spectrum conditions.

In Subsection 4.2, we turn to localizability aspects of nets of real subspaces. Starting with an (anti-)unitary representation of G_{τ_h} and the corresponding standard subspace $\mathtt{V} = \mathtt{V}(h, U)$, we consider a maximal net $\mathsf{H}^{\max}$ associated to some wedge region $W \subseteq M = G/H$. We then say that $(U, \mathcal{H})$ is (h, W)-localizable in those subsets $\mathcal{O} \subseteq M$ for which the real subspace $\mathsf{H}^{\max}$ is cyclic. Here the starting point is to assume this for W, which by Lemma 2.17 implies that $\mathsf{H}^{\max}(W) = \mathtt{V}$, so that the net $\mathsf{H}^{\max}$ satisfies (Iso), (Cov) and (BW), but not necessarily the Reeh–Schlieder condition. In this context our main results are Theorem 4.23, asserting localizability for reductive groups in all representations in all non-empty open subsets of the associated non-compactly causal symmetric space for a suitably chosen wedge region. For the Lorentz group $\mathrm{SO}_{1,d}(\mathbb{R})_e$ and its simply connected covering $\mathrm{Spin}_{1,d}(\mathbb{R})$, this leads to localization in open subsets of de Sitter space dS^d. Relating open subsets of dS^d with open spacelike cones in Minkowski space $\mathbb{R}^{1,d}$, this allows us to derive that, for the Poincaré group, localizability in spacelike cones is equivalent to the positive energy condition (Theorem 4.25).

4.1. Regularity

Definition 4.1. We call an (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h} *regular with respect to* h, or h*-regular*, if there exists an e-neighborhood $N \subseteq G$ such that $\mathtt{V}_N = \bigcap_{g \in N} U(g)\mathtt{V}$ is cyclic. Replacing N by its interior, we may always assume that N is open.

Remark 4.2. In these terms, Theorem 3.1 asserts that, if U is a unitary representation with discrete kernel, $\mathtt{V}$ is a standard subspace and $h \in \mathfrak{g}$ with $\Delta_{\mathtt{V}} = e^{2\pi i\, \partial U(h)}$, then h-regularity implies that h is an Euler element and that the prescription $U(\tau_h) := J$ extends U to an (anti-)unitary representation of G_{τ_h}.

This leads us to the problem to determine which (anti-)unitary representations $(U, \mathcal{H})$ of G_{τ_h} are h-regular. We start with a few general observations.

Examples 4.3. (a) If G is abelian, then $\tau_h = \mathrm{id}_{\mathfrak{g}}$ and J commutes with $U(G)$. Therefore $U(g)\mathtt{V} = \mathtt{V}$ for all $g \in G$ and thus all representations are regular.

(b) From [26], combined with [70], it follows that all irreducible (anti-)unitary representations are regular for any Euler element if G is a simple connected Lie group. In Corollary 4.24 below, this is extended to all connected real reductive Lie groups.

(c) Let $L = \mathrm{SO}_{1,d}(\mathbb{R})_e$ be the connected Lorentz group and $h \in \mathfrak{so}_{1,d}(\mathbb{R})$ be a boost generator. Then all (anti-)unitary representations of the proper Lorentz group $L_+ \cong L_{\tau_h}$ are h-regular. This follows for $d = 1$ from (a) and, for $d \geq 2$, from (b).

Lemma 4.4. *For an (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h}, the following assertions hold:*

(a) *If $U = U_1 \oplus U_2$ is a direct sum, then U is h-regular if and only if U_1 and U_2 are h-regular.*
(b) *If U is h-regular, then every subrepresentation is h-regular.*
(c) *Assume that G has at most countably many connected components and let $U = \int_X^{\oplus} U_m \, d\mu(m)$ be an (anti-)unitary direct integral representation of G_{τ_h}, then U is regular if and only if there exists an e-neighborhood $N \subseteq G$ such that, for μ-almost every $m \in X$, the subspace $\mathtt{V}_{m,N}$ is cyclic.*

Proof. (a) If $U \cong U_1 \oplus U_2$, then (23) implies that $\mathtt{V}_N = \mathtt{V}_{1,N} \oplus \mathtt{V}_{2,N}$ for every identity neighborhood $N \subseteq G$. In particular, $\mathtt{V}_N$ is cyclic if and only if $\mathtt{V}_{1,N}$ and $\mathtt{V}_{2,N}$ are.

(b) follows immediately from (a).

(c) Applying Lemma C.3(b) to $A := N$, we obtain (c). □

To deal with tensor products, we need the following observations from [42]:

Lemma 4.5. *Let $\mathtt{V}_j \subseteq \mathcal{H}_j$, $j = 1, \ldots, n$, be standard subspaces with the modular data (Δ_j, J_j). Then the closed real span*

$$\mathtt{V} := \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_n$$

of the elements $v_1 \otimes \cdots \otimes v_n$, $v_j \in \mathtt{V}_j$, is a standard subspace of

$$\mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$$

with modular data

$$\Delta = \Delta_1 \otimes \cdots \otimes \Delta_n \quad \text{and} \quad J = J_1 \otimes \cdots \otimes J_n.$$

Moreover,

$$\mathtt{V}' = \mathtt{V}'_1 \otimes \cdots \otimes \mathtt{V}'_n.$$

Proof. The first assertion follows easily by induction from the case $n = 2$ ([42, Prop. 2.6]). The second assertion follows by induction from [42, Prop. 2.5]. □

Example 4.6. Consider the group $G = \widetilde{\mathrm{SL}}_2(\mathbb{R})$, an Euler element $h \in \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ (they are all conjugate) and an irreducible (anti-)unitary representation $(U_1, \mathcal{H}_1)$ of G_{τ_h} for which $U_1(Z(G)) \not\subseteq \{\pm\mathbf{1}\}$. We then consider the (anti-)unitary representation

$$U := U_1 \otimes \overline{U_1} \quad \text{of } G_{\tau_h} \text{ on } \quad \mathcal{H}_1 \otimes \overline{\mathcal{H}_1}$$

and observe that $U_1(Z(G)) \subseteq \mathbb{T}\mathbf{1}$ implies that U factors through the group $G/Z(G) \cong \mathrm{PSL}_2(\mathbb{R})$. For $\mathtt{V}_1 := \mathtt{V}(h, U_1)$, $\mathtt{V}'_1 = \mathtt{V}(h, \overline{U}_1)$, and $\mathtt{V} := \mathtt{V}(h, U) = \mathtt{V}_1 \otimes \mathtt{V}'_1$, by Lemma 4.5 and Proposition 2.12, we have

$$\mathtt{V}_{Z(G)} = \mathtt{V} = \mathtt{V}_1 \otimes \mathtt{V}'_1 \subseteq \mathcal{H} = \mathcal{H}_1 \otimes \overline{\mathcal{H}_1}$$

by Lemma 2.13.

However, $U_1(Z(G)) \subseteq \mathbb{T}\mathbf{1}$ is a subgroup containing non-real numbers, so that

$$\mathtt{V}_{1,Z(G)} = \bigcap_{z \in Z(G)} U_1(z)\mathtt{V}_1 = \{0\}.$$

We therefore have

$$\mathtt{V}_{Z(G)} = \mathtt{V} \neq \mathtt{V}_{1,Z(G)} \otimes \mathtt{V}'_{1,Z(G)} = \{0\}.$$

Example 4.7. Another example from AQFT, where strict inclusions of the type (80) arise, is contained in [57, Sect. 4.2.2]. We present the example in a slightly different way from [57] in order to fit it with the language introduced in this paper. It is obtained by second quantization of the tensor product of U(1)-current chiral one-particle nets. Consider the 1+1-dimensional Minkowski spacetime $\mathbb{R}^{1,1}$ with the quadratic form $x^2 = x_0^2 - x_1^2$, where spacetime events are denoted $x = (x_0, x_1)$. One can now pass to chiral coordinates:

$$(x_+, x_-) = \Big(\frac{x_0 + x_1}{\sqrt{2}}, \frac{x_0 - x_1}{\sqrt{2}}\Big) \tag{48}$$

In these coordinates, the right and left wedge in $\mathbb{R}^{1,1}$ are given by

$$W_R = \mathbb{R}_+ \times \mathbb{R}_- \quad \text{and} \quad W_L = \mathbb{R}_- \times \mathbb{R}_+.$$

Consider the BGL net $(\mathsf{H}(I))_{I \subseteq \mathbb{R}_\infty}$ indexed by intervals on the compactified real line $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, associated with the (anti-)unitary lowest weight 1 representation $(U, \mathcal{H})$

of the Möbius group $\mathsf{Möb}_{\tau_h}$ with respect to the Euler element $h \in \mathfrak{sl}_2(\mathbb{R})$, the generator of the dilations, acting by $\exp(th)x = e^t x$. We form the tensor product net

$$\mathbb{R}^{1,1} \supset I_1 \times I_2 \mapsto \widetilde{\mathsf{H}}(I_1 \times I_2) := \mathsf{H}(I_1) \otimes \mathsf{H}(I_2) \subset \mathcal{H} \otimes \mathcal{H},$$

where I_1 and I_2 are intervals in $\mathbb{R}_\infty$. A pair of intervals specifies a region

$$\mathcal{D}_{I_1,I_2} := \{(x_+, x_-) \in \mathbb{R}^{1,1} : x_+ \in I_1, x_- \in I_2\}.$$

Here we only consider intervals $I_1, I_2 \subseteq \mathbb{R}$, so that the product set $I_1 \times I_2 \subseteq \mathbb{R}^2_\infty$ can be identified with $\mathcal{D}_{I_1,I_2}$, and this set is connected.

The net $\widetilde{\mathsf{H}}$ on “rectangles” in $\mathbb{R}^2_\infty$ is covariant, it is actually the BGL-net on rectangles by Lemma 4.5 and Proposition 2.12, for the representation $U \otimes U$ of the group

$$\mathsf{Möb}^2_{\tau_h} := (\mathsf{Möb} \times \mathsf{Möb})_{(\tau_h,\tau_h)}.$$

Note that the identity component of the Poincaré group $\mathcal{P}_+^\uparrow$ and the dilation group $(D(t))_{t\in\mathbb{R}^+}$ are contained in the group $\mathsf{Möb}^2$. Let r be the space reflection $r(x_0, x_1) = (x_0, -x_1)$, resp., by $r(x_+, x_-) = (x_-, x_+)$. We consider the group $\mathsf{Möb}^2_{r,\tau_h}$, generated by $\mathsf{Möb}^2_{\tau_h}$ and r. We implement the reflection r unitarily on $\mathcal{H} \otimes \mathcal{H}$ as the flip, acting on simple tensors by $U(r)(\xi \otimes \eta) = \eta \otimes \xi$. This extends $U \otimes U$ to an (anti-)unitary representation $U^{(2)}$ of $\mathsf{Möb}^2_{r,\tau_h}$ for which the net $\widetilde{\mathsf{H}}$ is covariant. Now let

$$G \cong \mathbb{R}^{1,1} \rtimes (\mathbb{R}^+ \times \mathrm{O}_{1,1}(\mathbb{R}))^\uparrow \cong \mathcal{P}^\uparrow \rtimes \mathbb{R}^+$$

be the subgroup of $\mathsf{Möb}^2_r$ generated by $\mathcal{P}^\uparrow = \mathbb{R}^{1,1} \rtimes \mathrm{O}_{1,1}(\mathbb{R})^\uparrow$ and positive dilations. Clearly,

$$\widetilde{\mathsf{H}}(W_R) = \mathsf{H}(\mathbb{R}^+) \otimes \mathsf{H}(\mathbb{R}^-) \quad \text{and} \quad \widetilde{\mathsf{H}}(W_L) = \mathsf{H}(\mathbb{R}^-) \otimes \mathsf{H}(\mathbb{R}^+).$$

Let $I_1 = (a, b)$ and $I_2 = (c, d)$ be bounded real intervals. Then

$$I_1 \times I_2 = W^R_{a,c} \cap W^L_{b,d},$$

where

$$W^R_{a,c} = (\mathbb{R}^+ + a) \times (\mathbb{R}^- + c) \quad \text{and} \quad W^L_{b,d} = (\mathbb{R}^- + b) \times (\mathbb{R}^+ + d).$$

Let $A = \{g_1, g_2\} \subseteq \mathsf{Möb} \times \mathsf{Möb}$, where $g_1 W_R = W^R_{a,c}$ and $g_2 W_R = W^L_{b,d}$. For

$$\mathsf{V} := \widetilde{\mathsf{H}}(W_R),$$

we now derive from isotony

$$\mathsf{V}_A = \widetilde{\mathsf{H}}(W^R_{a,c}) \cap \widetilde{\mathsf{H}}(W^L_{b,d}) \supset \widetilde{\mathsf{H}}(I_1 \times I_2) = \mathsf{H}(I_1) \otimes \mathsf{H}(I_2) = \widetilde{\mathsf{H}}(W^R_{a,c} \cap W^L_{b,d}). \tag{49}$$

We now consider $\widetilde{\mathsf{H}}^{\max}$, the maximal net with respect to G. In [57, Sect. 4.4.2] it is proved that $\widetilde{\mathsf{H}}^{\max}(I_1 \times I_2) = \mathsf{V}_A$ properly contains $\widetilde{\mathsf{H}}(I_1 \times I_2) = \mathsf{H}(I_1) \otimes \mathsf{H}(I_2)$. The idea of the proof is that the net $\widetilde{\mathsf{H}}$ is $\mathsf{M\ddot{o}b} \times \mathsf{M\ddot{o}b}$-covariant by construction, but the net on Minkowski space

$$\mathbb{R}^{1+1} \supset I_1 \times I_2 \longmapsto \widetilde{\mathsf{H}}^{\max}(I_1 \times I_2) \subset \mathcal{H}, \qquad I_1, I_2 \subset \mathbb{R}$$

is only G-covariant and, consequently, they have to be different. It is easy to see (again by construction) that the net $\widetilde{\mathsf{H}}^{\max}$ is G-covariant with respect to $U^{(2)}|_G$. In order to prove that it is not $\mathsf{M\ddot{o}b} \times \mathsf{M\ddot{o}b}$-covariant, one can argue as follows: The representation

$$(U \otimes U)|_{\mathcal{P}^\uparrow} = \int_{\mathbb{R}_+}^{\oplus} U_m d\nu(m)$$

disintegrates to a direct integral of all positive mass representations $(U_m, \mathcal{H}_m), m > 0$, of $\mathcal{P}^\uparrow$. On wedge regions, the net is the BGL net, hence disintegrates into the BGL nets H_m over $\mathbb{R}_+ = (0, \infty)$

$$\widetilde{\mathsf{H}}(W) = \int_{\mathbb{R}_+}^{\oplus} \mathsf{H}_m(W)\, d\nu(m) \subset \int_{\mathbb{R}_+}^{\oplus} \mathcal{H}_m d\nu(m).$$

By (DI2) from Appendix C, we also have

$$\widetilde{\mathsf{H}}^{\max}(\mathcal{D}) = \int_{\mathbb{R}_+}^{\oplus} \mathsf{H}_m^{\max}(\mathcal{D})\, d\nu(m) \subset \int_{\mathbb{R}_+}^{\oplus} \mathcal{H}_m d\nu(m)$$

for all open doublecones $\mathcal{D} = I_1 \times I_2$. We associate the following subspace to the forward light cone:

$$\mathsf{K}(V_+) := \overline{\sum_{\mathcal{D} \subset V_+} \widetilde{\mathsf{H}}^{\max}(\mathcal{D})},$$

where the union is extended over all double cones $\mathcal{D}$ contained in V_+.

Following [57, Prop. 4.3], we have $\overline{\sum_{\mathcal{D} \subset V_+} \widetilde{\mathsf{H}}_m(\mathcal{D})} = \mathcal{H}_m$, so that $\mathsf{K}^{\max}(V_+)$ is not separating because

$$\mathsf{K}(V_+) = \overline{\sum_{\mathcal{D} \subset V_+} \widetilde{\mathsf{H}}^{\max}(\mathcal{D}))} = \overline{\int_{\mathbb{R}_+}^{\oplus} \sum_{\mathcal{D} \subset V_+} \mathsf{H}_m(\mathcal{D}) d\nu(m)} = \int_{\mathbb{R}_+}^{\oplus} \mathcal{H}_m d\nu(m) = \mathcal{H}.$$

Let $g \in \mathsf{M\ddot{o}b} \times \mathsf{M\ddot{o}b}$ be such that $g\mathcal{D} = V_+$ for some bounded interval $\mathcal{D}$. We conclude that there is no unitary operator $Q \in \mathrm{U}(\mathcal{H})$, implementing g in the sense that $Q\widetilde{\mathsf{H}}^{\max}(\mathcal{D}) \supseteq \widetilde{\mathsf{H}}^{\max}(\widetilde{\mathcal{D}})$ holds for all double cones $\widetilde{\mathcal{D}} \subseteq V_+$. In fact, the former is a standard subspace and sum of the spaces on the right is not separating.

Lemma 4.8. *Let $(U, \mathcal{H})$ be an (anti-)unitary representation of G_{τ_h} for which the cones*

$$C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$$

have interior points in $\mathfrak{g}_{\pm 1}(h)$ with respect to the subspace topology. Then, for $\mathtt{V} = \mathtt{V}(h, U)$, the semigroup $S_{\mathtt{V}} = \{g \in G \colon U(g)\mathtt{V} \subseteq \mathtt{V}\}$ has dense interior, i.e., $S_{\mathtt{V}} = \overline{S_{\mathtt{V}}^\circ}$.

Note that, if C_U has interior points, then so do the cones $C_\pm$, because they are the projections of $\pm C_U$ onto $\mathfrak{g}_{\pm 1}(h)$.

Proof. Let $G^r := G/\ker(U)$ and $\mathfrak{n} := \mathbf{L}(\ker U) = \ker(\mathtt{d}U)$. We write $U^r \colon G^r \to \mathrm{U}(\mathcal{H})$ for the unitary representation of G^r defined by U. Then

$$C_U = C_U + \mathfrak{n} \quad \text{and} \quad C_U/\mathfrak{n} = C_{U^r}.$$

Moreover, for $\mathfrak{n}_\lambda(h) = \mathfrak{n} \cap \mathfrak{g}_{\pm\lambda}(h)$ we have

$$\mathfrak{g}^r_\lambda(h) \cong \mathfrak{g}_\lambda(h)/\mathfrak{n}_\lambda(h) \quad \text{for} \quad \lambda = 1, 0, -1.$$

Therefore the cones

$$C^r_\pm := \pm C_{U^r} \cap \mathfrak{g}^r_{\pm 1}(h) = C_\pm/\mathfrak{n}_{\pm 1}(h)$$

are generating and

$$S^r_{\mathtt{V}} := \{g \in G^r \colon U^r(g)\mathtt{V} \subseteq \mathtt{V}\} = G^r_{\mathtt{V}} \exp(C^r_+ + C^r_-)$$

by [61, Thms. 3.4 and 2.16]. To see that this semigroup has dense interior, it suffices to show that e can be approximated by interior points. Since both cones $C^r_\pm$ have dense interior and the map

$$\mathfrak{g}_0(h) \times \mathfrak{g}_1(h) \times \mathfrak{g}_{-1}(h) \to G, \quad (x_0, x_1, x_{-1}) \mapsto \exp(x_0)\exp(x_1 + x_{-1})$$

is a local diffeomorphism around $(0, 0, 0)$, the semigroup $S^r_{\mathtt{V}}$ has dense interior. As $S_{\mathtt{V}} \subseteq G$ is the full inverse image of $S^r_{\mathtt{V}}$ under the quotient map $G \to G^r$, which has continuous local sections, it has dense interior as well. □

Theorem 4.9. (Regularity via positive energy) *If $(U, \mathcal{H})$ is an (anti-)unitary representation of G_{τ_h} for which the cones*

$$C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$$

are generating in $\mathfrak{g}_{\pm 1}(h)$*, then* $(U, \mathcal{H})$ *is regular.*

Proof. For a subset $N \subseteq G$ and $g_0 \in G$, we note that

$$U(g_0)\mathtt{V} \subseteq \mathtt{V}_N \qquad \Leftrightarrow \qquad N^{-1}g_0 \subseteq S_{\mathtt{V}}. \tag{50}$$

From Lemma 4.8 we infer that $S_{\mathtt{V}}$ has an interior point g_0, so that the above condition is satisfied for some e-neighborhood N. As $U(g)\mathtt{V}$ is cyclic, it follows in particular that $\mathtt{V}_N$ is cyclic. □

Remark 4.10. (a) The condition on the cone $C_\pm$ to be generating holds for positive energy representations of the Möbius group. Up to sign, the only pointed, generating (in the sense of having interior points) closed convex Ad-invariant cone is

$$C := \{X \in \mathfrak{g} \colon V_X \geq 0\} = \left\{X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : b \geq 0, c \leq 0, a^2 \leq -bc\right\}.$$

For the Euler element $h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we have

$$C_\pm = \pm C \cap \mathfrak{g}_{\pm 1}(h), \quad C_+ = \mathbb{R}_+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_- = \mathbb{R}_+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and the half lines $C_\pm$ in $\mathfrak{g}_{\pm 1}(h)$ also have interior points. In general the generating property of the cones $C_\pm$ in $\mathfrak{g}_{\pm 1}(h)$ is rather strong. For instance it is not satisfied by positive energy representations of the Poincaré group on $\mathbb{R}^{1,3}$. Theorem 4.11 will show how to derive regularity if the cones $C_\pm$ are not generating; see Remark 4.12.

(b) From the proof of Theorem 4.9 one can derive some more specific quantitative information. If N is an e-neighborhood contained in $g_0^{-1}S_{\mathtt{V}}$ for some $g_0 \in S_{\mathtt{V}}$, then the argument implies that $\mathtt{V}_N$ is cyclic.

(c) If $[\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)] = \{0\}$, then $B := \exp(\mathfrak{g}_1(h) + \mathfrak{g}_{-1}(h))$ is an abelian subgroup of G and $S_{\mathtt{V}} \supseteq G_e^h \exp(C_+ + C_-)$. If $C \subseteq B$ is any compact e-neighborhood, then there exists a $b_0 \in S_{\mathtt{V}}$ with $C^{-1}b_0 \subseteq S_{\mathtt{V}}$, so that $G_e^h C^{-1} b_0 \subseteq S_{\mathtt{V}}$ and thus $\mathtt{V}_{CG_e^h} = \mathtt{V}_C \supseteq U(b_0)\mathtt{V}$ is cyclic. It follows that N can be chosen arbitrarily large, whenever the cones $C_\pm$ are generating. A typical example is given by the 3-dimensional Poincaré algebra in dimension $1 + 1$.

Note that the subgroups $G_{\pm 1}(h) := \exp(\mathfrak{g}_{\pm 1}(h)) \subseteq G$ are abelian.

Theorem 4.11. *Suppose that* $G = R \rtimes L$ *is a semidirect product. Let* $(U, \mathcal{H})$ *be an antiunitary representation such that*

- $(U|_L, \mathcal{H})$ *is regular, and*

- *the cones* $C_\pm := \pm C_U \cap \mathfrak{r}_{\pm 1}(h)$ *generate* $\mathfrak{r}_{\pm 1}(h)$.

Then $(U, \mathcal{H})$ *is regular.*

Proof. First, let $N_L \subseteq L$ be an e-neighborhood for which $\mathsf{H} := \mathsf{V}_{N_L}$ is cyclic. Our assumption implies that $S_\mathsf{V} \cap R$ has interior points in R (Lemma 4.8). Hence there exists $r_0 \in (S_\mathsf{V} \cap R)^\circ$ and an e-neighborhood $N_R \subseteq R$ with $r_0 N_R^{-1} \subseteq S_\mathsf{V}$. Then

$$U(\ell)U(r)U(r_0)^{-1}\mathsf{V} \supseteq U(\ell)\mathsf{V} \supseteq \mathsf{H} \quad \text{for} \quad \ell \in N_L,\ r \in N_R,$$

and so regularity follows. □

Remark 4.12. The condition on the cones $C_\pm$ in Theorem 4.9 is stronger than the positive energy condition $C_U \neq \{0\}$. The latter assumes the existence of a positive cone C in the Lie algebra that $-i\partial U(x) \geq 0$ for every $x \in C$ but does not require the generating property. Theorem 4.11 shows that, in order to recover the regularity of the net on Minkowski spacetime, one has to look at the representation of the Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \mathcal{L}_+^\uparrow$ and to check the non-triviality of the one-dimensional cones $C_\pm$ in the eigenspaces $\mathfrak{r}_{\pm 1}(h) = \mathbb{R}(\mathbf{e}_0 \pm \mathbf{e}_1)$ (light rays) in the subalgebra $\mathfrak{r} \cong \mathbb{R}^{1,3}$ corresponding to translations, and the regularity property for the restriction of the representation to the identity component $\mathcal{L}_+^\uparrow$ of the Lorentz group. The first property is equivalent to the usual positive energy condition on Poincaré representations, namely the joint spectrum of the translations is contained in $\{x \in \mathbb{R}^{1,3} : x^2 \geq 0, x_0 \geq 0\}$. The second one holds for every representation of the Lorentz group, see Example 4.3 and Theorem 4.23 below.

Remark 4.13. (a) If G is simply connected, then $G \cong R \rtimes S$, where S is semisimple and R is the solvable radical. In view of Theorem 4.23, which guarantees localizability for representations of S, Theorem 4.11 applies whenever the cones $C_U \cap \mathfrak{r}_{\pm 1}(h)$ are generating, i.e., the restriction of the representation to the abelian subgroups $R_\pm := \exp(\mathfrak{r}_{\pm 1}(h))$ have a generating positive cone.

(b) A similar remark applies to (coverings of) identity components of real algebraic groups. They are semidirect products $G = N \rtimes L$, where N is unipotent and L is reductive ([34, Thm. VIII.4.3]). For these groups Theorem 4.11 applies whenever the cones $C_U \cap \mathfrak{n}_{\pm 1}(h)$ are generating.

(c) Presently we do not know if all (anti-)unitary representations of Lie groups of the form G_{τ_h}, $h \in \mathfrak{g}$ an Euler element, are regular. The preceding discussion shows that, to answer this question, a more detailed analysis of the case of solvable groups has to be undertaken.

Proposition 4.14. *Let* $h \in \mathfrak{g}$ *be an Euler element and* G_{τ_h} *as above. An (anti-)unitary representation* $(U, \mathcal{H})$ *of* G_{τ_h} *is regular if and only if its restriction to the connected normal subgroup* $N_h^\natural$ *with Lie algebra*

$$\mathfrak{n}_h^\natural := \mathfrak{g}_1(h) + (\mathbb{R}h + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)]) + \mathfrak{g}_{-1}(h)$$

is regular.

Note that the equality of $\mathfrak{g} = \mathfrak{n}_h^\natural$ is equivalent to the Euler element h being anti-elliptic in $\mathfrak{g}$ (cf. Definition 5.3 below).

Proof. Since $\mathfrak{g} = \mathfrak{n}_h^\sharp + \mathfrak{g}_0(h)$ on the Lie algebra level, we obtain $G = N_h^\natural G_e^h$ for the corresponding integral subgroups, where $N_h^\sharp$ is a normal subgroup with Lie algebra $\mathfrak{n}_h^\sharp$ and $\mathbf{L}(G_e^h) = \mathfrak{g}_0(h)$. Then $G_e^h \subseteq G^{h,\tau_h} \subseteq G_{\mathtt{V}}$. For any e-neighborhood $N \subseteq N_h^\natural$, we therefore have

$$\bigcap_{g \in NG_e^h} U(g)\mathtt{V} = \bigcap_{g \in N} U(g)\mathtt{V}.$$

Therefore U is regular if and only if $U|_{N_h^\natural}$ is regular. □

Proposition 4.15. *We consider a group $G = E \rtimes \mathbb{R}$, where E is a finite-dimensional vector space with Lie algebra of the form*

$$\mathfrak{g} = E \rtimes \mathbb{R}h,$$

where h is an Euler element. Then all (anti-)unitary representations of G_{τ_h} are regular.

Proof. Let $E_j := \{v \in E \colon [h,v] = jv\}$ be the h-eigenspaces in E. By Proposition 4.14, it suffices to verify regularity on the subgroup $N_h^\sharp = (E_1 \oplus E_{-1}) \rtimes \mathbb{R}$. Using systems of imprimitivity, it follows that all irreducible unitary representations of such groups factor through representations of groups for which $\dim E_{\pm 1} \leq 1$. In fact, all orbits of $e^{\mathbb{R}\,\mathrm{ad}\,h}$ in $E^* = E_{-1}^* \oplus E_1^*$ are contained in an at most 2-dimensional subspace because, for $\alpha = \alpha_{-1} + \alpha_1$, we have

$$e^{\mathrm{ad}\,h}.\alpha = e^{-t}\alpha_{-1} + e^t\alpha_1 \in \mathbb{R}\alpha_{-1} + \mathbb{R}\alpha_1.$$

As irreducible unitary representations of G are built from $\exp(\mathbb{R}h)$-ergodic covariant projection-valued measures on E^*, we can mod out $\ker\alpha_{\pm j}$ to reduce to the situation where $\dim E_{\pm 1} \leq 1$.

This reduces the problem to the cases where $\mathfrak{g}$ is abelian, $\mathfrak{aff}(\mathbb{R})$ or the 2d-Poincaré Lie algebra $\mathfrak{p}(2) = \mathbb{R}^{1,1} \rtimes \mathfrak{so}_{1,1}(\mathbb{R})$. The simple orbit structure for $\mathbb{R}$ on the dual space E^* implies that in these cases the cones

$$C_\pm := \pm C_U \cap E_{\pm 1}$$

are always non-trivial, hence generating. Now regularity of all irreducible (anti-)unitary representations follows from Theorem 4.9.

Moreover, Remark 4.10 implies that, for all compact e-neighborhoods $N \subseteq G$ (which project to compact identity neighborhoods in the three types of quotient groups), the subspaces $\mathtt{V}_N$ are cyclic. As N is independent of the representation, we can use Lemma 4.4 (c) to obtain the result in general. □

Remark 4.16. Let $(U, \mathcal{H})$ be an irreducible (anti-)unitary representation of the connected Lie group G and $0 \neq v \in \mathcal{D}(\Delta^{1/2})$ be an analytic vector. If $\xi \in \mathtt{V}_A$, then $U(g)^{-1}\xi \in \mathtt{V}$ holds for all g in A and, if $A^\circ \neq \emptyset$, then the analyticity of the map $G \to \mathtt{V}, g \mapsto U(g)^{-1}v$ and the closedness of $\mathtt{V}$ imply that $U(G)v \subseteq \mathtt{V}$, so that

$$\mathtt{V}_A \cap \mathcal{H}^\omega \subseteq \mathtt{V}_G. \tag{51}$$

If $\mathtt{V}_A \cap \mathcal{H}^\omega$ is dense in $\mathtt{V}_A$ and $\mathtt{V}_A$ is cyclic, it follows that $\mathtt{V}_G$ is cyclic. Its invariance under the modular group of $\mathtt{V}$ then implies that $\mathtt{V} = \mathtt{V}_G$ ([39, Prop. 3.10]). Therefore $\mathtt{V}$ is G-invariant and thus h is central in $\mathfrak{g}$ if $\ker(U)$ is discrete. In view of [3, Thm. 7.12], one should not expect that $\mathtt{V}$ contains non-zero analytic vectors if $\mathtt{V}_G = \{0\}$. For more details on the subspace $\mathtt{V}_G$, we refer to Section 5.2 below.

4.2. Localizability

In this section we study localizability properties of unitary representations of a connected Lie group G.

Definition 4.17. We say that the (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h} is (h, W)-*localizable* in those open subsets $\mathcal{O} \subseteq M$ for which $\mathsf{H}^{\max}(\mathcal{O})$ is cyclic.

The following remark shows that already the localizability condition in the wedge region W has consequences for the representation.

Remark 4.18. By Lemma 2.17(c) the property of (h, W)-localizability implies $S_W \subseteq S_{\mathtt{V}}$. From [61, Thm. 3.4] we recall that

$$S_{\mathtt{V}} := \{g \in G \colon U(g)\mathtt{V} \subseteq \mathtt{V}\} = G_{\mathtt{V}} \exp(C_+ + C_-) \quad \text{with} \quad C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h) \tag{52}$$

if $\ker(U)$ is discrete. If the Lie wedge

$$\mathbf{L}(S_W) = \{x \in \mathfrak{g} \colon \exp(\mathbb{R}_+ x) \subseteq S_W\}$$

is not contained in $\mathfrak{g}_0(h)$ (see Proposition 2.9 for a description of this cone for positivity domains), this implies that one of the two cones

$$\mathbf{L}(S_{\mathtt{V}}) \cap \mathfrak{g}_{\pm 1}(h) = C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$$

is non-zero and thus $C_U \neq \{0\}$. If $S_W = G_W$ is a group, this conclusion does not work, so that localizability does not require any spectral condition, in particular $C_U = \{0\}$ is possible.

Remark 4.19. For the canonical nets obtained from pairs (h, W) on a homogeneous space $M = G/H$ through two (anti-)unitary representations U_1, U_2 of G_{τ_h}, as in (20), Lemma D.1 shows that, for a tensor product representation $U = U_1 \otimes U_2$, we have

$$\mathsf{H}^{\max}(\mathcal{O}) \supseteq \mathsf{H}_1^{\max}(\mathcal{O}) \otimes \mathsf{H}_2^{\max}(\mathcal{O}),$$

and in general equality does not hold (Example 4.6).

Lemma 4.20. (Localizability implies regularity) *Let $\emptyset \neq \mathcal{O} \subseteq W \subseteq M$ be open subsets such that $N := \{g \in G \colon g^{-1}\mathcal{O} \subseteq W\}$ is an e-neighborhood. If $(U, \mathcal{H})$ is an (anti-)unitary representation for which $\mathsf{H}^{\max}(W) = \mathtt{V}$ and $\mathsf{H}^{\max}(\mathcal{O})$ is cyclic, then it is regular.*

Proof. By assumption $\mathsf{H}^{\max}(\mathcal{O})$ is cyclic, and

$$\mathsf{H}^{\max}(\mathcal{O}) \subseteq \bigcap_{g \in N} \mathsf{H}^{\max}(gW) = \bigcap_{g \in N} U(g)\mathsf{H}^{\max}(W) = \bigcap_{g \in N} U(g)\mathtt{V} = \mathtt{V}_N.$$

It follows that $\mathtt{V}_N$ is cyclic. □

Nets satisfying (Iso) and (Cov) can easily be constructed as follows. Given an (anti-)unitary representation $(U, \mathcal{H})$ of G_τ, the subspace $\mathcal{H}^\infty \subseteq \mathcal{H}$ of vectors $v \in \mathcal{H}$ for which the orbit map $U^v \colon G \to \mathcal{H}, g \mapsto U(g)v$, is smooth (*smooth vectors*) is dense and carries a natural Fréchet topology for which the action of G on this space is smooth ([29,60], [64, App. A], and Appendix B). The space $\mathcal{H}^{-\infty}$ of continuous antilinear functionals $\eta \colon \mathcal{H}^\infty \to \mathbb{C}$ (*distribution vectors*) contains in particular Dirac's kets $\langle \cdot, v\rangle$, $v \in \mathcal{H}$, so that we obtain complex linear embeddings

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\infty},$$

where G acts on all three spaces by representations denoted U^∞, U and $U^{-\infty}$, respectively.

All of the three above representations can be integrated to the convolution algebra $C_c^\infty(G, \mathbb{C})$ of test functions, for instance $U^{-\infty}(\varphi) := \int_G \varphi(g) U^{-\infty}(g)\, dg$, where dg stands for a left Haar measure on G. The operators $U(\varphi)$ are continuous maps $\mathcal{H} \to \mathcal{H}^\infty$, so that their adjoints $U^{-\infty}(\varphi)$ define maps $\mathcal{H}^{-\infty} \to \mathcal{H}$. For any real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$, we can therefore associate to every open subset $\mathcal{O} \subseteq G$, the closed real subspace

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{\mathrm{span}_{\mathbb{R}} U^{-\infty}(C_c^\infty(\mathcal{O}, \mathbb{R}))\mathtt{E}}. \tag{53}$$

On a homogeneous space $M = G/H$ with the projection map $q\colon G \to M$, we now obtain a "push-forward net"

$$\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) := \mathsf{H}^G_{\mathtt{E}}(q^{-1}(\mathcal{O})). \tag{54}$$

This assignment satisfies (Iso) and (Cov), so that a key problem is to specify subspaces $\mathtt{E}$ of distribution vectors for which (RS) and (BW) hold as well.

Suppose that $\mathfrak{g}$ is simple and $h \in \mathfrak{g}$ an Euler element, and that $M = G/H$ is the corresponding non-compactly causal symmetric space (cf. Subsection 2.1.3). In [26] a net of standard subspaces H^M_E has been constructed on open regions of M, satisfying (Iso), (Cov), (RS), (BW), where $W = W^+_M(h)_{eH}$. The following lemma applies in particular to these nets:

Lemma 4.21. *Let* $(U, \mathcal{H})$ *be an (anti-)unitary representation and* $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ *be a real subspace with* $\mathtt{V} = \mathsf{H}^M_{\mathtt{E}}(W)$. *If the net* $\mathsf{H}^M_{\mathtt{E}}$ *has the Reeh–Schlieder property* (RS)*, then* $\mathsf{H}^{\max}(\mathcal{O})$ *is cyclic for any non-empty open subset* $\mathcal{O} \subseteq M$.

Proof. Since $\mathsf{H}^M_{\mathtt{E}}(\mathcal{O})$ is cyclic for each non-empty open subset $\mathcal{O} \subseteq M$ by (RS), it suffices to verify that $\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) \subseteq \mathsf{H}^{\max}(\mathcal{O})$. As the net $\mathsf{H}^M_{\mathtt{E}}$ is covariant, isotone and has the BW property with respect to h and W, this follows from Lemma 2.19. □

Example 4.22. We now describe an example of a net $\mathsf{H}^M_{\mathtt{E}}$ constructed from a standard subspace $\mathtt{V} = \mathtt{V}(h, U)$ for which the corresponding maximal net $\mathsf{H}^{\max}$ is strictly larger on some open subsets. Here $M = \mathbb{R}$, with its natural causal structure, on which we consider the group $\mathrm{Aff}(\mathbb{R})_e$, acting by affine maps.

On the space $C^\infty_c(\mathbb{R}, \mathbb{R})$ of real-valued test functions on $\mathbb{R}$, we consider the positive definite hermitian form, given by

$$\langle f, g \rangle_1 := \int_{\mathbb{R}_+} p\overline{\widehat{f}(p)}\widehat{g}(p)\, dp = \int_{\mathbb{R}_+} p\widehat{f}(-p)\widehat{g}(p)\, dp$$

where the Fourier transform is defined $\widehat{f}(p) = \int_{\mathbb{R}} e^{ipx} f(x)\, dx$. We write $\mathcal{H}^{(1)}$ for the Hilbert space obtained by completion with respect to this scalar product and $\eta\colon C^\infty_c(\mathbb{R}, \mathbb{R}) \to \mathcal{H}^{(1)}$ for the canonical inclusion. The symplectic form corresponding to its imaginary part is

$$\sigma_1(f, g) = \mathrm{Im} \int_{\mathbb{R}_+} p\widehat{f}(-p)\widehat{g}(p)\, dp = \frac{1}{2i} \int_{\mathbb{R}} p\widehat{f}(-p)\widehat{g}(p)\, dp = \pi \int_{\mathbb{R}} f(x)g'(x)\, dx. \tag{55}$$

Let $G := \mathrm{Aff}(\mathbb{R})_e$ be the connected affine group. Then the canonical action of G on $C^\infty_c(\mathbb{R}, \mathbb{R})$ by $(g.f)(x) := f(g^{-1}x)$ preserves the hermitian form and the Fourier transform intertwines it with the unitary representation on $L^2(\mathbb{R}_+, pdp)$ by

$$(\widetilde{U}(b,a)F)(p) = e^{ibp} aF(ap), \quad b \in \mathbb{R}, a, p \in \mathbb{R}_+.$$

As $\widetilde{U}$ extends to an irreducible unitary representation $\widetilde{U}$ of $\mathrm{PSL}_2(\mathbb{R})$ (cf. [26, §5.4]), Corollary D.7 implies that $\widetilde{U}$ is irreducible over $\mathbb{R}$. It follows in particular that the Fourier transform $C_c^\infty(\mathbb{R},\mathbb{R}) \to L^2(\mathbb{R}_+, p\,dp)$ has dense range. We thus obtain a real linear isometric bijection $\mathcal{H}^{(1)} \to L^2(\mathbb{R}_+, p\,dp)$. Bypassing the Fourier transform, we can also write the scalar product, as

$$\langle f, g\rangle_1 = \int_{\mathbb{R}_+} \overline{\widehat{f}(p)}\widehat{g}(p)\, pdp = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f(x)} g(y) \frac{(-1)}{(y-x+i0)^2}\, dx\, dy.$$

We consider the unitary representation $U^{(1)}$ of G on $\mathcal{H}^{(1)}$, for which the Fourier transform is an intertwining operator onto $L^1(\mathbb{R}_+, p\,dp)$. Note that $\mathcal{H}^{(1)}$ may also be considered as a Hilbert subspace of $\mathcal{S}'(\mathbb{R})$ via the map $\iota(g)(f) = \langle f, g\rangle_1$ for $f, g \in \mathcal{S}(\mathbb{R})$, i.e.,

$$\iota(g) = g * D \quad \text{with} \quad D(x) = \frac{(-1)}{(-x+i0)^2}.$$

The antilinear involution $JF = -\overline{F}$ on $L^2(\mathbb{R}_+, p\,dp)$ defines a conjugation on $\mathcal{H}^{(1)}$ that extends $U^{(1)}$ to an (anti-)unitary representation $G_{\tau_h} \cong \mathbb{R} \rtimes \mathbb{R}^\times = \mathrm{Aff}(\mathbb{R})$ for the Euler element $h = (0,1) \in \mathfrak{g}$. Here $(h,-1) \in \mathcal{G}_E(\mathrm{Aff}(\mathbb{R}))$ and $\mathcal{W}_+ = G.(h,-1)$ can be identified with the set of open real half-lines, bounded from below.

Clearly,

$$\mathsf{H}^{(1)}(\mathcal{O}) := \overline{\eta(C_c^\infty(\mathcal{O},\mathbb{R}))}$$

defines a net of real subspaces in $\mathcal{H}^{(1)}$ that is isotone and G-covariant. Furthermore (55) implies that this net is local in the sense that disjoint open intervals map to symplectically orthogonal real subspaces. It also satisfies the Reeh–Schlieder property and also the BW property in the sense that

$$\mathsf{V} = \mathsf{V}(h, U) = \mathsf{H}^{(1)}(\mathbb{R}_+)$$

(cf. [39,67]). Here the main point is to verify that the constant function 1, a distribution vector for the representation on $L^2(\mathbb{R}_+, p\,dp)$ satisfies the abstract KMS condition

$$J1 = -1 = \Delta^{1/2} 1 \quad \text{for} \quad \Delta = e^{2\pi i \partial U(0,1)} \tag{56}$$

(cf. [3]). As $\widetilde{U}(0, e^t)1 = e^t$, the relation (56) follows immediately. For $k \geq 2$, we also have the following subnets, generated by the derivatives of test functions via

$$\mathsf{H}^{(k)}(\mathcal{O}) = \overline{\{\eta(f^{(k-1)}) \colon f \in C_c^\infty(\mathcal{O},\mathbb{R})\}} \subseteq \mathsf{H}^{(1)}(\mathcal{O}).$$

These nets are also isotone and G-covariant. It is known from [39, Prop. 4.2.3] and [31] that, for every bounded interval $I \subseteq \mathbb{R}$ and $k < \ell$, the subspace $\mathsf{H}^{(\ell)}(I) \subseteq \mathsf{H}^{(k)}(I)$ is proper with

$$\dim\left(\mathsf{H}^{(k)}(I)/\mathsf{H}^{(\ell)}(I)\right) = \ell - k.$$

On the other hand, when $I = (a, \infty)$ is an unbounded interval, then $\mathsf{H}^{(k)}(I) = \mathsf{H}^{(1)}(I)$ for every $k \in \mathbb{N}$. Furthermore, on intervals, $\mathsf{H}^{(k)}$ is a restriction of the BGL net associated to the unitary positive energy representation $\widetilde{U}^{(k)}$ of $\mathrm{PSL}_2(\mathbb{R})$ of lowest weight k ([39, Thm. 3.6.7]).

Finally, we explain how to write these nets in the form $\mathsf{H}^{\mathbb{R}}_{\mathtt{E}_k}$ for suitable one-dimensional subspaces $\mathtt{E}_k = \mathbb{R}\alpha_k \subseteq \mathcal{H}^{-\infty}$ of distribution vectors of the representation $(U^{(1)}, \mathcal{H}^{(1)})$. To this end, we consider the Fourier transform $L^2(\mathbb{R}_+, p\,dp) \to \mathcal{O}(\mathbb{C}_+), \mathcal{F}_1(F)(z) = \int_{\mathbb{R}_+} e^{ipz} F(p)\, p dp$, which maps unitarily onto the reproducing kernel Hilbert space $\mathcal{H}_1 \subseteq \mathcal{O}(\mathbb{C}_+)$ with reproducing kernel

$$Q(z,w) = \frac{-1}{(z - \overline{w})^2} \quad \text{for} \quad z, w \in \mathbb{C}_+ = \mathbb{R} + i\mathbb{R}_+$$

([67]). Here J acts by $(JF)(z) := -\overline{F(-\overline{z})}$ and the affine group by

$$(U_1(b,a)F)(z) = a^{-1}F(a^{-1}(z+b)).$$

The discussion in [26, §5.4] shows that

$$\alpha_1(x) := (x + i0)^{-2}, \quad \text{resp.} \quad \alpha_1(z) = \frac{1}{z^2},$$

is a distribution vector that is an eigenvector for the dilation group, satisfying $U_1^{-\infty}(a)\alpha_1 = a\alpha_1$ and $J\alpha_1 = -\alpha_1$. For $\mathtt{E}_1 := \mathbb{R}\alpha_1$, the corresponding standard subspace $\mathsf{H}^{\mathbb{R}}_{\mathtt{E}_1}(\mathcal{O})$ is therefore generated by the elements $U_1^{-\infty}(\varphi)\alpha_1 = \varphi^\vee * \alpha_1$, $\varphi \in C^\infty_c(\mathcal{O}, \mathbb{R})$, so that we obtain for each open subset $\mathcal{O} \subseteq \mathbb{R}$:

$$\mathsf{H}^{\mathbb{R}}_{\mathtt{E}_1}(\mathcal{O}) = \mathsf{H}^{(1)}(-\mathcal{O}).$$

We also note that

$$U^{(1)}(\varphi^{(k)})\alpha_1 = \varphi^{(k),\vee} * \alpha_1 = (-1)^k \varphi^\vee * \alpha_1^{(k)},$$

so that we obtain

$$\mathsf{H}^{(k)}(-\mathcal{O}) = \mathsf{H}^{\mathbb{R}}_{\mathtt{E}_k}(\mathcal{O}) \quad \text{for} \quad \mathtt{E}_k = \mathbb{R}\alpha_k, \quad \alpha_k := \alpha_1^{(k-1)}.$$

An example on 1+1-dimensional Minkowski spacetime is described in Remark 4.7.

Localizability for reductive groups

In this section we assume that $\mathfrak{g}$ is reductive and that G is a corresponding connected Lie group. We choose an involution θ on $\mathfrak{g}$ in such a way that it fixes the center pointwise and restricts to a Cartan involution on the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Then the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ satisfies $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{k}$. We write $K := G^\theta$ for the subgroup of θ-fixed points in G.

We write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma,$$

where $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g})$ is the center and each ideal $\mathfrak{g}_\gamma$ is simple. Accordingly, we have

$$h = h_0 + \sum_\gamma h_\gamma,$$

where $h_\gamma \in \mathfrak{g}_\gamma$ either vanishes or is an Euler element in $\mathfrak{g}_\gamma$. We assume that $\theta(h_\gamma) = -h_\gamma$ for each $\gamma \in \Gamma$. We decompose Γ as

$$\Gamma = \Gamma_0 \dot\cup \Gamma_1 \quad \text{with} \quad \Gamma_0 := \{\gamma \in \Gamma \colon h_\gamma = 0\}, \tag{57}$$

so that $h = h_0 + \sum_{\gamma\in\Gamma_1} h_\gamma$. Then we obtain an involutive automorphism τ on $\mathfrak{g}$ by

$$\tau(x) = \begin{cases} x & \text{for} \quad x \in \mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}), \\ x & \text{for} \quad x \in \mathfrak{g}_\gamma, \gamma \in \Gamma_0, \\ \tau_h\theta(x) & \text{for} \quad x \in \mathfrak{g}_\gamma, \gamma \in \Gamma_1, \end{cases}$$

and we assume that τ integrates to an involutive automorphism τ^G of G. We write $\mathfrak{h} := \mathfrak{g}^\tau$ and $\mathfrak{q} := \mathfrak{g}^{-\tau}$ for the τ-eigenspaces in $\mathfrak{g}$. Then there exists in $\mathfrak{q}$ a unique maximal pointed generating $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant cone C containing $h' := \sum_{\gamma\in\Gamma_1} h_\gamma$ in its interior ([53]) We choose an open θ-invariant subgroup $H \subseteq G^\tau$ satisfying $\mathrm{Ad}(H)C = C$. By [53, Cor. 4.6], this is equivalent to $H_K = H \cap K$ fixing h. Here we use that H has a polar decomposition $H = H_K \exp(\mathfrak{h}_\mathfrak{p})$, so that the above condition implies that $\mathrm{Ad}(H)h = e^{\mathrm{ad}\,\mathfrak{h}_\mathfrak{p}} h$. Then

$$M = G/H \tag{58}$$

is called the corresponding *non-compactly causal symmetric space*. The normal subgroups $G_0 = Z(G)_e$ and G_j for $h_j = 0$, are contained in H, hence act trivially on M. The homogeneous space M carries a G-invariant causal structure, represented by a field $(C_m)_{m\in M}$ of closed convex cones $C_m \subseteq T_m(M)$, which is uniquely determined by $C_{eH} = C \subseteq \mathfrak{q} \cong T_{eH}(M)$.

The modular vector field

$$X_h^M(m) = \frac{d}{dt}\Big|_{t=0} \exp(th).m \tag{59}$$

on M determines a positivity region

$$W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\} \tag{60}$$

and the connected component $W := W_M^+(h)_{eH}$ of the base point $eH \in M$ is called the *wedge region in* M.

Note that the following theorem does not require any assumption concerning the irreducibility of the representation. Although its proof draws heavily from [26], which deals with irreducible representations, Proposition 2.26 is a convenient tool to reduce to this situation.

Theorem 4.23. (Localization for reductive groups) *If* $\mathfrak{g}$ *is reductive and* $(U, \mathcal{H})$ *is an (anti-)unitary representation of* G_{τ_h}, *then the canonical net* $\mathsf{H}^{\max}$ *on the non-compactly causal symmetric space* $M = G/H$ *associated to* h *as in* (58) *satisfies*

(a) $\mathtt{V} = \mathsf{H}^{\max}(W)$, *i.e.*, $S_W \subseteq S_{\mathtt{V}}$, *and*
(b) $\mathsf{H}^{\max}(\mathcal{O})$ *is cyclic for every non-empty open subset* $\mathcal{O} \subseteq M$.

Proof. In view of Lemma 2.17(c), assertion (a) follows from (b), applied to $\mathcal{O} = W$. So it suffices to verify (b). By Proposition 2.26 we may further assume that $(U, \mathcal{H})$ is irreducible. Replacing G by its simply connected covering leads to a product structure

$$G \cong G_0 \times \prod_{\gamma\in\Gamma} G_\gamma.$$

Moreover,

$$\mathfrak{q} = \bigoplus_{\gamma\in\Gamma_1} \mathfrak{q}_\gamma \quad \text{and} \quad C = \sum_{\gamma\in\Gamma_1} C_\gamma \quad \text{with} \quad C_\gamma = C \cap \mathfrak{q}_\gamma$$

(cf. (57)).

We first consider irreducible representations of the factor groups G_{γ,τ_h}. If $h_\gamma \in \mathfrak{g}_\gamma$ is trivial or central, then the standard subspace $\mathtt{V}$ is G_j-invariant, so that $\mathtt{V} = \mathtt{V}_{G_j}$. For all other simple factors (h_γ, W_γ)-localizability in the family of all non-empty open subsets of the associated non-compactly causal symmetric space follows from [26, Thm. 4.10] for linear groups and by combining it with [70] for the general case. This implies the assertion for all irreducible (anti-)unitary representations of the factor groups G_{γ,τ_h} and G_{0,τ_h}.

Let $U_0 \otimes \bigotimes_{\gamma\in\Gamma} U_\gamma$ be an irreducible unitary representation of G and extend it by some conjugation of the form $J = J_0 \otimes \bigotimes_{\gamma\in\Gamma} J_\gamma$ to an irreducible (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h} on a Hilbert space that is a subspace of the tensor product of the spaces

$$\widetilde{\mathcal{H}}_\gamma = \mathcal{H}_\gamma \oplus \overline{\mathcal{H}_\gamma}.$$

By Remark 2.25, all irreducible (anti-)unitary representations of G_{τ_h} are subrepresentations of tensor products of irreducible (anti-)unitary representations of the factor groups. We thus obtain all irreducible (anti-)unitary representations of G_{τ_h}. Therefore the assertion follows from the fact that (b) is inherited by subrepresentations, direct sums, and finite tensor products (Lemma 2.17(d)). □

Corollary 4.24. (Regularity for reductive groups) *Let G be a connected reductive Lie group. Then there exists an e-neighborhood $N \subseteq G$ such that for every separable (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h}, the real subspace*

$$\mathtt{V}(h, U)_N = \bigcap_{g \in N} U(g)\mathtt{V}(h, U)$$

is cyclic. In particular, $(U, \mathcal{H})$ is h-regular.

Proof. Let $\mathcal{O} \subseteq W \subseteq M = G/H$ be an open subset whose closure $\overline{\mathcal{O}}$ is relatively compact. In Theorem 4.23 we have seen that $\mathsf{H}^{\max}(\mathcal{O})$ is cyclic. Further

$$N := \{g \in G : g\mathcal{O} \subseteq W\} \supseteq \{g \in G : g\overline{\mathcal{O}} \subseteq W\}$$

is an e-neighborhood because $\overline{\mathcal{O}} \subseteq W$ is compact. Therefore the h-regularity of $(U, \mathcal{H})$ follows from Lemma 4.20. □

Localizability for the Poincaré group

The following result is well-known ([11, Thm. 4.7]). Here we derive it naturally in the context of our theory for general Lie groups. It connects regularity, resp., localizability with the positive energy condition.

Theorem 4.25. (Localization for the Poincaré group) *Let $(U, \mathcal{H})$ be an (anti-)unitary representation of the proper Poincaré group $\mathcal{P}_+ = \mathbb{R}^{1,d} \rtimes \mathcal{L}_+$ (identified with $\mathcal{P}_{\tau_h}$) and consider the standard boost h and the corresponding Rindler wedge $W_R \subseteq \mathbb{R}^{1,d}$. Then $(U, \mathcal{H})$ is (h, W_R)-localizable in the set of all spacelike open cones if and only if it is a positive energy representation, i.e.,*

$$C_U \supseteq \overline{V_+} := \{(x_0, \mathbf{x}) : x_0 \geq 0, x_0^2 \geq \mathbf{x}^2\}. \tag{61}$$

These representations are regular.

Note that $\mathrm{Ad}(\mathcal{P}_+^\uparrow)$ acts transitively on the set $\mathcal{E}(\mathfrak{p})$ of Euler elements, so that the choice of a specific Euler element h is inessential.

Proof. First we show that the positive energy condition is necessary for localizability in spacelike cones. In fact, the localizability condition implies in particular that $\mathsf{H}(W_R)$ is cyclic, so that Lemma 2.17 implies $S_{W_R} \subseteq S_{\mathtt{V}}$. As a consequence, $\mathbf{e}_1 + \mathbf{e}_0 \in C_U$, and thus $\overline{V_+} \subseteq C_U$ by Lorentz invariance of C_U. Therefore $(U, \mathcal{H})$ is a positive energy representation.

Now we assume that $(U, \mathcal{H})$ is a positive energy representation. For the standard boost we have $h \in \mathfrak{l} \cong \mathfrak{so}_{1,d}(\mathbb{R})$, and the restriction $(U|_{L_+}, \mathcal{H})$ is (h, W)-localizable in the family of all non-empty open subsets of dS^d, where $W = W_R \cap \mathrm{dS}^d$ is the canonical wedge region (Theorem 4.23).

Next we recall from [63, Lemma 4.12] that

$$S_{W_R} = \{g \in \mathcal{P}_+^\uparrow : gW_R \subseteq W_R\} = \overline{W_R} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})^\uparrow_{W_R},$$

where

$$\mathrm{SO}_{1,d}(\mathbb{R})^\uparrow_{W_R} = \mathrm{SO}_{1,1}(\mathbb{R})^\uparrow \times \mathrm{SO}_{d-2}(\mathbb{R})$$

is connected, hence coincides with L^h_e. It follows that

$$S_{W_R} = G^h_e \exp([0, \infty)(\mathbf{e}_0 + \mathbf{e}_1)) \exp([0, \infty)(-\mathbf{e}_0 + \mathbf{e}_1)).$$

Let us assume that $(U, \mathcal{H})$ is a positive energy representation, i.e., that $C_U \supseteq \overline{V_+}$ (cf. (61)). Then

$$C_\pm = [0, \infty)(\mathbf{e}_1 \pm \mathbf{e}_0) \subseteq \overline{W_R}, \quad \text{so that} \quad S_{W_R} \subseteq S_{\mathtt{V}}.$$

By Lemma 2.17(c), the net $\mathsf{H}^{\max}$ satisfies $\mathsf{H}^{\max}(W_R) = \mathtt{V}$.

Now suppose that $\mathcal{C} \subseteq W_R$ is a spacelike cone, so that

$$\mathcal{C} = \mathbb{R}_+(\mathcal{C} \cap \mathrm{dS}^d),$$

where $\mathcal{C} \cap \mathrm{dS}^d$ is an open subset of the wedge region $W = W_R \cap \mathrm{dS}^d$ in de Sitter space. For $g^{-1} = (v, \ell) \in \mathcal{P}_+^\uparrow$, the condition $\mathcal{C} \subseteq g.W_R$ is equivalent to

$$g^{-1}.\mathcal{C} = v + \ell.\mathcal{C} \subseteq W_R,$$

which in turn means that $v \in \overline{W_R}$ and $\ell.\mathcal{C} \subseteq W_R$. Then

$$U(g)\mathtt{V} = U(\ell)^{-1}U(v)^{-1}\mathtt{V} \supseteq U(\ell)^{-1}\mathtt{V}$$

follows from $\overline{W_R} \subseteq S_{\mathtt{V}}$, and therefore

$$\begin{aligned}\mathsf{H}^{\max}(\mathcal{C}) &= \bigcap_{\mathcal{C}\subseteq g.W_R} U(g)\mathtt{V} \supseteq \bigcap_{\mathcal{C}\subseteq \ell^{-1}.W_R} U(\ell)^{-1}\mathtt{V} \\ &= \bigcap_{\mathcal{C}\cap \mathrm{dS}^d\subseteq \ell^{-1}.(W_R\cap \mathrm{dS}^d)} U(\ell)^{-1}\mathtt{V} = \mathsf{H}^{\max}_{U|_L}(\mathcal{C}\cap \mathrm{dS}^d).\end{aligned}$$

We conclude that, on spacelike cones with vertex in 0, the net $\mathsf{H}^{\max}$ dominates the net $\mathsf{H}^{\max}_{U|_L}$ on de Sitter space. As the latter net has the Reeh–Schlieder property by Theorem 4.23, and all spacelike cones can be translated to one with vertex 0, localization in spacelike cones follows.

Finally we show that $(U, \mathcal{H})$ is regular. For $v \in W_R$ and a pointed spacelike cone C with $v + C \subseteq W$, there exists an e-neighborhood $N \subseteq G$ with $v + C \subseteq g.W$ for all $g \in N$. This implies that $\mathsf{H}^{\max}(v + C) \subseteq \mathtt{V}_N$, so that $(U, \mathcal{H})$ is regular. □

Remark 4.26. Infinite helicity representations $(U, \mathcal{H})$ of $\mathcal{P}_+$ in $\mathbb{R}^{1,d}$ are **not** localizable in double cones (Definition 2.20). Let $\mathtt{V} = \mathsf{H}^{BGL}_U(W)$ for $W = (h, j_h)$ be as in Example 2.7. In [42, Thm. 6.1] it is proved that, if $\mathcal{O} \subseteq \mathbb{R}^{1,d}$ is a double cone, then

$$\mathsf{H}^{\max}(\mathcal{O}) = \bigcap_{\mathcal{O}\subseteq g.W_R} U(g)\mathtt{V} = \{0\}. \tag{62}$$

The argument to conclude (62) can be sketched as follows. Infinite spin representations are massless representations, i.e., the support of the spectral measure of the space-time translation group is

$$\partial V_+ = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} : x_0^2 - \mathbf{x}^2 = 0, x_0 \geq 0\}.$$

Covariant nets of standard subspaces on double cones in massless representations are also dilation covariant in the sense that the representation of $\mathcal{P}_+$ extends to the Poincaré and dilation group $\mathbb{R}^{1,d} \rtimes (\mathbb{R}^+ \times \mathcal{L})$, and the net is also covariant under this larger group, cf. [42, Prop. 5.4]. When $d = 3$, this follows from the fact that, due to the Huygens' Principle, one can associate by additivity a standard subspace to the forward lightcone $\mathsf{H}(V_+) = \overline{\sum_{\mathcal{O}\subset V_+} \mathsf{H}(\mathcal{O})}$ (sum over all double cones in V_+) and the modular group of $\mathsf{H}(V_+)$ is geometrically implemented by the dilation group. As massless infinite helicity representations are not dilation covariant, it follows that they do not permit localization in double cones. Properties of the free wave equation permit to extend this argument to any space dimension $d \geq 2$ including even space dimensions, and the Huygens Principle fails ([42, Sect. 8.2]). However, in Theorem 4.25, we recover in our general setting the result contained in [11, Thm. 4.7] that all positive energy representations of $\mathcal{P}_+$ are localizable in spacelike cones.

5. Moore's theorem and its consequences

In this section we continue the discussion of applications of our results to von Neumann algebras $\mathcal{M}$ with cyclic separating vector Ω, started in Subsection 3.2. First we explain the consequences of Moore's Eigenvector Theorem 5.1 (cf. [48, Thm. 1.1]). Here the main point is that the properties (Mod) and (M) (from Subsection 3.2) imply that Ω is fixed by the one-parameter group $U(\exp(\mathbb{R}h))$ and Moore's Theorem allows us to find conditions for G under which this implies that Ω is fixed under G. Note that, for semisimple Lie groups, Moore's Theorem also follows from the Howe–Moore Theorem on the vanishing of matrix coefficients at infinity for all unitary representations non containing non-zero fixed vectors (cf. [78, Thm. 2.2.20]).

The first main result in this section is Theorem 5.11, characterizing for (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h} the subspace $\mathsf{V}_G = \bigcap_{g\in G} U(g)\mathsf{V}$ as the set of fixed points of a certain normal subgroup specified by Moore's Theorem. The second one is Theorem 5.15 that combines Moore's Theorem with Theorem 3.7 to obtain a criterion for $\mathcal{M}$ to be a factor of type III_1. The third one is Proposition 5.22 which shows that all the structure we discuss survives the central disintegration of $\mathcal{M}$, provided $\mathcal{M}'$ and $\mathcal{M}$ are conjugate under $U(G)$.

5.1. Moore's theorem

Theorem 5.1. (Moore's Eigenvector Theorem) *Let G be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$. Further, let $\mathfrak{n}_h \trianglelefteq \mathfrak{g}$ be the smallest ideal of $\mathfrak{g}$ such that the image of h in the quotient Lie algebra $\mathfrak{g}/\mathfrak{n}_h$ is elliptic.*

Suppose that $(U, \mathcal{H})$ is a continuous unitary representation of G and $\Omega \in \mathcal{H}$ an eigenvector for the one-parameter group $U(\exp \mathbb{R}h)$. Then

(a) *Ω is fixed by the normal subgroup $N_h := \langle \exp \mathfrak{n}_h \rangle \trianglelefteq G$, and*
(b) *the restriction of $i\cdot\partial U(h)$ to the orthogonal complement of the space $\mathcal{H}^{N_h}$ of N_h-fixed vectors has absolutely continuous spectrum.*

The ideal $\mathfrak{n}_h \trianglelefteq \mathfrak{g}$ has the property that the corresponding closed normal subgroup $N_h \trianglelefteq G_e$ generated by $\exp(\mathfrak{n}_h)$ fixes Ω, hence acts trivially on the projective orbit $G.[\Omega] \subseteq \mathbb{P}(\mathcal{H})$. As $\mathrm{ad}\, h$ induces an elliptic element on $\mathfrak{g}/\mathfrak{n}_h$, the group G/N_h has a basis of e-neighborhoods invariant under $\exp(\mathbb{R}h)$.

Corollary 5.2. *Let G be a connected finite-dimensional Lie group. Suppose that $(U, \mathcal{H})$ is a unitary representation of G with discrete kernel and that $h \in \mathfrak{g}$ is such that $\partial U(h)$ has a G-cyclic eigenvector in $\mathcal{H}$. Then* $\mathrm{ad}(h)$ *is elliptic.*

Proof. It suffices to show that $\mathfrak{n}_h = \{0\}$. As the subgroup $N_h \trianglelefteq G$ is normal, the subspace $\mathcal{H}^{N_h}$ of N_h-fixed vectors is G-invariant: For $\xi \in \mathcal{H}^{N_h}$, $g \in G$ and $n \in N_h$, we have

$$U(n)U(g)\xi = U(g)U(g^{-1}ng)\xi = U(g)\xi.$$

The G-cyclic eigenvector Ω of $\partial U(h)$ is contained in $\mathcal{H}^{N_h}$ by Moore's Theorem, so that $\mathcal{H} = \mathcal{H}^{N_h}$. Therefore $\mathfrak{n}_h \subseteq \ker(\mathsf{d}U) = \{0\}$. $\square$

In many situations, Moore's Theorem implies that eigenvectors of one-parameter subgroups are actually fixed by G. These cases are easily detected with the following concept:

Definition 5.3. We call $h \in \mathfrak{g}$ *anti-elliptic* if $\mathfrak{n}_h + \mathbb{R}h = \mathfrak{g}$.

Remark 5.4. In [71] a closely related property is introduced for Lie algebra elements: An element $x \in \mathfrak{g}$ for which $\operatorname{ad} x$ is diagonalizable is said to be *essential* if

$$\mathfrak{g} = \mathbb{R}x + [x, \mathfrak{g}] + \operatorname{span}[[x, \mathfrak{g}], [x, \mathfrak{g}]].$$

As $\mathfrak{g} = \sum_{\lambda \in \mathbb{R}} \mathfrak{g}_\lambda(x)$ and $[x, \mathfrak{g}] = \sum_{\lambda \neq 0} \mathfrak{g}_\lambda(x)$, this is equivalent to

$$\mathfrak{g}_0(x) = \mathbb{R}x + \sum_{\lambda \neq 0} [\mathfrak{g}_\lambda(x), \mathfrak{g}_{-\lambda}(x)].$$

In this case the ideal $\mathfrak{n}_x$ contains all eigenspaces $\mathfrak{g}_\lambda(x)$ for $\lambda \neq 0$, hence also the brackets $[\mathfrak{g}_\lambda(x), \mathfrak{g}_{-\lambda}(x)]$. As

$$\mathfrak{i} := \sum_{\lambda \neq 0} \mathfrak{g}_\lambda(x) + \sum_{\lambda \neq 0} [\mathfrak{g}_\lambda(x), \mathfrak{g}_{-\lambda}(x)]$$

is an ideal of $\mathfrak{g}$ for which the image of x in $\mathfrak{g}/\mathfrak{i}$ is central, it follows that $\mathfrak{i} = \mathfrak{n}_x$. Therefore an ad-diagonalizable element is essential if and only if it is anti-elliptic. In this sense our concept of anti-ellipticity extends Strich's concept of essentiality to general Lie algebra elements.

Remark 5.5. Any Euler element h in a simple Lie algebra is anti-elliptic. But $h = \frac{1}{2}\operatorname{diag}(1, -1)$ is an Euler element in the reductive Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ with $\mathfrak{n}_h = \mathfrak{sl}_2(\mathbb{R}) \ni h$. So it is not anti-elliptic.

Moore's Theorem immediately yields:

Corollary 5.6. *If $h \in \mathfrak{g}$ is anti-elliptic and $(U, \mathcal{H})$ is a unitary representation of a connected Lie group G with Lie algebra $\mathfrak{g}$, then* $\ker(\partial U(h)) = \mathcal{H}^G$.

Proof. As $\ker(\partial U(h))$ consists of eigenvectors for $U(\exp \mathbb{R}h)$, Moore's Theorem implies that they are fixed by $U(N_h)$. Anti-ellipticity of h further implies that $G = N_h \exp(\mathbb{R}h)$, so that they are fixed by G. $\square$

Examples 5.7. (a) If $\mathfrak{g}$ is simple and $h \in \mathfrak{g}$ is not elliptic, then $\mathfrak{n}_h \neq \{0\}$ implies $\mathfrak{n}_h = \mathfrak{g}$, so that h is anti-elliptic. If, more generally, $\mathfrak{g}$ is reductive such that $\mathfrak{g} = \mathbb{R}h + [\mathfrak{g}, \mathfrak{g}]$ and no restriction of $\operatorname{ad} h$ to a simple ideal of $\mathfrak{g}$ is elliptic, then h is anti-elliptic.

(b) Consider a semidirect sum of Lie algebras $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{l}$ and an element $h \in \mathfrak{l}$ such that

$$\mathrm{Spec}(\mathrm{ad}\, h|_{\mathfrak{r}}) \cap i\mathbb{R} = \emptyset \tag{63}$$

and h is anti-elliptic in $\mathfrak{l}$. Then h is anti-elliptic in $\mathfrak{g}$. In fact, our assumption implies that $\mathfrak{r} \subseteq \mathfrak{n}_h$, so that $\mathfrak{g}/\mathfrak{n}_h \cong \mathfrak{l}/(\mathfrak{l} \cap \mathfrak{n}_h) \cong \mathfrak{l}/\mathfrak{l}_h$ is linearly generated by the image of h. This implies that $\mathfrak{g} = \mathfrak{n}_h + \mathbb{R}h$.

(c) If $\mathfrak{g} = \mathbb{R}x + \mathbb{R}h$ with $[h, x] = \lambda x$ and $\lambda \neq 0$, then $\mathfrak{n}_h = \mathbb{R}x$, so that h is anti-elliptic (cf. [71]).

(d) Consider the boost generator $h \in \mathfrak{so}_{1,1}(\mathbb{R}) \subseteq \mathfrak{p}(2) = \mathbb{R}^{1,1} \rtimes \mathfrak{so}_{1,1}(\mathbb{R})$, the $2d$-Poincaré–Lie algebra. Then $\mathfrak{n}_h = \mathbb{R}^{1,1}$ and $\mathfrak{g} = \mathfrak{n}_h + \mathbb{R}h$, so that h is anti-elliptic.

(e) From (a) and (b) it follows immediately that, for $d \geq 3$, any boost generator $h \in \mathfrak{so}_{1,d-1}(\mathbb{R}) \subseteq \mathfrak{p}(d) = \mathbb{R}^{1,d-1} \rtimes \mathfrak{so}_{1,d-1}(\mathbb{R})$ is anti-elliptic. Here we use that the representation of $\mathfrak{so}_{1,d-1}(\mathbb{R})$ on $\mathbb{R}^{1,d-1}$ is irreducible.

(f) Suppose that $\mathfrak{g}$ is reductive and $h \in \mathfrak{g}$ is an Euler element. Since every ideal of a reductive Lie algebra possesses a complementary ideal ([33, Def. 5.7.1]), we can write $\mathfrak{g} = \mathfrak{n}_h \oplus \mathfrak{b}$. We write accordingly $h = h_0 + h_1$ with $h_0 \in \mathfrak{n}_h$ and $h_1 \in \mathfrak{b}$. If $\mathfrak{n}_h$ is not central, then h_0 is an Euler element of $\mathfrak{n}_h$. Further, h_1 is elliptic in $\mathfrak{b} \cong \mathfrak{g}/\mathfrak{n}_h$. From the direct sum decomposition we thus infer that h_0 is an Euler element of $\mathfrak{g}$ and that h_1 is elliptic.

Lemma 5.8. *If $h \in \mathfrak{g}$ is an Euler element, then*

$$\mathfrak{n}_h = \mathfrak{g}_1(h) + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)] + \mathfrak{g}_{-1}(h).$$

In particular, h is anti-elliptic if and only if

$$\mathfrak{g}_0(h) \subseteq \mathbb{R}h + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)].$$

Proof. Clearly, $\mathfrak{g}_{\pm 1}(h) \subseteq \mathfrak{n}_h$ implies that $\mathfrak{n}_h$ contains the ideal

$$\mathfrak{n} := \mathfrak{g}_1(h) + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)] + \mathfrak{g}_{-1}(h).$$

As the image of h in $\mathfrak{g}/\mathfrak{n}$ is central, we have $\mathfrak{n}_h = \mathfrak{n}$. Hence h is anti-elliptic if and only if $\mathfrak{g}_0(h) \subseteq \mathbb{R}h + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)]$. □

Remark 5.9. If h is an Euler element, then Lemma 5.8 shows that

$$\mathfrak{g} = \mathfrak{n}_h + \mathfrak{g}_0(h),$$

so that the summation map is a surjective homomorphism $\mathfrak{n}_h \rtimes \mathfrak{g}_0(h) \twoheadrightarrow \mathfrak{g}$. Hence $\mathfrak{g}$ is a quotient of $\mathfrak{n}_h \rtimes \mathfrak{g}_0(h)$, where $h \in \mathfrak{g}_0(h)$ is central.

Remark 5.10. If h is an Euler element, then

$$\mathfrak{n}_h^\natural := \mathfrak{n}_h + \mathbb{R}h = \mathfrak{g}_1(h) + (\mathbb{R}h + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)]) + \mathfrak{g}_{-1}(h)$$

is an ideal of $\mathfrak{g}$. It is the minimal ideal containing h, and therefore the corresponding integral subgroup of G is generated by $\exp(\mathrm{Ad}(G)\mathbb{R}h)$. Therefore h is anti-elliptic if and only if the modular groups $\exp(\mathrm{Ad}(g)\mathbb{R}h)$ generate G.

5.2. Non-degeneracy

Let $(U, \mathcal{H})$ be an (anti-)unitary representation of G_{τ_h}, where $h \in \mathfrak{g}$ is an Euler element and $\mathtt{V} = \mathtt{V}(h, U)$ is the canonical standard subspace.

We consider the G-invariant closed real subspace

$$\mathtt{V}_G = \bigcap_{g \in G} U(g)\mathtt{V}. \tag{64}$$

We call the couple $(U, \mathtt{V})$ *non-degenerate* if $\mathtt{V}_G = \{0\}$. We now explain how this property is related to the structure introduced in the previous section.

Theorem 5.11. *Suppose that G is connected, $h \in \mathfrak{g}$ is an Euler element, $(U, \mathcal{H})$ an (anti-)unitary representation of G_{τ_h}, and $\mathtt{V} = \mathtt{V}(h, U)$ the corresponding standard subspace. Then $\mathtt{V}_G = \mathtt{V} \cap \mathcal{H}^{N_h}$, where N_h is the normal subgroup from* Moore's Theorem 5.1.

Proof. Let $\mathcal{H}_1 := \mathcal{H}^{N_h}$ and $\mathcal{H}_2 := \mathcal{H}_1^\perp$. As $N_h \trianglelefteq G$ is a normal subgroup of G_{τ_h}, the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is $U(G_{\tau_h})$-invariant, so that $U = U_1 \oplus U_2$, accordingly. Since this group contains $J_{\mathtt{V}}$ and the modular group, it follows that

$$\mathtt{V} = \mathtt{V}_1 \oplus \mathtt{V}_2 \quad \text{with} \quad \mathtt{V}_1 = \mathtt{V} \cap \mathcal{H}^{N_h} \quad \text{and} \quad \mathtt{V}_2 = \mathtt{V} \cap (\mathcal{H}^{N_h})^\perp,$$

where $\mathtt{V}_1 = \mathtt{V}(h, U_1)$.

"$\supseteq$": On $\mathcal{H}_1$ the group N_h acts trivially, so that $\mathfrak{g} = \mathfrak{n}_h + \mathfrak{g}_0(h)$ (Lemma 5.8) implies that $U_1(G) = U_1(\langle \exp \mathfrak{g}_0(h) \rangle)$ commutes with the modular group $U_1(\exp \mathbb{R}h)$ of $\mathtt{V}_1$. Further $\mathfrak{g}_0(h) = \mathfrak{g}^{\tau_h}$ shows that $U_1(G)$ also commutes with $J_1 = U_1(\tau_h^G)$, and therefore $\mathtt{V}_1$ is $U_1(G)$-invariant. This proves that $\mathtt{V}_1 \subseteq \mathtt{V}_G$.

"$\subseteq$": We consider the closed $U(G)$-invariant subspace $\mathcal{H}_0 := \overline{\mathtt{V}_G + i\mathtt{V}_G}$ and note that $\mathtt{V}_G$ is a standard subspace of $\mathcal{H}_0$. As $\mathtt{V}_G$ is invariant under $U(\exp \mathbb{R}h) = \Delta_{\mathtt{V}}^{i\mathbb{R}}$, the modular group of $\mathtt{V}$, it follows from [39, Cor. 2.1.8] that

$$\Delta_{\mathtt{V}_G} = e^{2\pi i\, \partial U_0(h)} \quad \text{for} \quad U_0(g) := U(g)|_{\mathcal{H}_0}.$$

The $U_0(G)$-invariance of the standard subspace $\mathtt{V}_G$ implies that $U_0(G)$ commutes with its modular operator, hence with $\partial U_0(h)$, and thus $\partial U([h, x]) = 0$ for $x \in \mathfrak{g}$. This implies that $[h, \mathfrak{g}] \subseteq \ker \mathtt{d}U_0$, so that the ideal $\ker(\mathtt{d}U_0) \trianglelefteq \mathfrak{g}$ contains $\mathfrak{g}_{\pm 1}(h)$, hence also

$$\mathfrak{n}_h = \mathfrak{g}_1(h) + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)] + \mathfrak{g}_{-1}(h)$$

(cf. Lemma 5.8). This is turn shows that $\mathcal{H}_0 \subseteq \mathcal{H}^{N_h}$, hence $\mathtt{V}_G \subseteq \mathtt{V} \cap \mathcal{H}^{N_h}$. □

Corollary 5.12. *If G is connected and $h \in \mathfrak{n}_h$, then*

$$\mathtt{V}_G = \mathtt{V} \cap \mathtt{V}'.$$

Proof. Theorem 5.11 shows that $\mathtt{V}_G \subseteq \mathcal{H}^{N_h}$, and since $h \in \mathfrak{n}_h$ by assumption, $\mathtt{V}_G$ is fixed by its modular group, hence contained in $\mathrm{Fix}(\Delta_{\mathtt{V}}) \cap \mathtt{V} = \mathtt{V} \cap \mathtt{V}'$.

If, conversely, $v \in \mathtt{V} \cap \mathtt{V}'$, then v is fixed by $U(\exp \mathbb{R}h) = \Delta_{\mathtt{V}}^{i\mathbb{R}}$, hence by definition of N_h also by N_h, so that $v \in \mathtt{V} \cap \mathcal{H}^{N_h} = \mathtt{V}_G$ (Theorem 5.11). □

With the standard subspace $\mathtt{V}_G \subseteq \mathcal{H}^{N_h}$, the preceding corollary yields an orthogonal decomposition

$$\mathtt{V} = \mathtt{V}_G \oplus \mathtt{V}_{\mathrm{symp}},$$

where $\mathtt{V}_{\mathrm{symp}} \subseteq (\mathcal{H}^{\mathbb{R}}, \omega)$ is a symplectic subspace for $\omega = \mathrm{Im}\langle \cdot, \cdot \rangle$, and $\mathtt{V}_{\mathrm{symp}} = \mathtt{V}(h, U_s)$ for the (anti-)unitary representation U_s of G_{τ_h} on $(\mathcal{H}^{N_h})^{\perp}$.

Corollary 5.13. *If G is connected and $\mathfrak{n}_h = \mathfrak{g}$, then the following are equivalent:*

(a) $\mathtt{V}_G = \{0\}$, *i.e.* $(U, \mathtt{V})$ *is non-degenerate.*
(b) $\mathcal{H}^G = \{0\}$.
(c) $\mathtt{V} \cap \mathtt{V}' = \{0\}$.
(d) *The closed real subspace $\widetilde{\mathtt{V}}$ generated by $U(G)\mathtt{V}$ coincides with $\mathcal{H}$.*

Proof. Theorem 5.11 implies that $\mathtt{V}_G = \mathtt{V} \cap \mathcal{H}^G$, which is a standard subspace of the G_{τ_h}-invariant subspace $\mathcal{H}^G$. This implies the equivalence of (a) and (b). The equivalence of (a) and (c) follows from Corollary 5.12. To connect with (d), we note that

$$J_{\mathtt{V}}\mathtt{V}_G = \bigcap_{g \in G} J_{\mathtt{V}} U(g)\mathtt{V} = \bigcap_{g \in G} U(\tau(g)) J_{\mathtt{V}}\mathtt{V} = \bigcap_{g \in G} U(\tau(g))\mathtt{V}' = \bigcap_{g \in G} U(g)\mathtt{V}' = (U(G)\mathtt{V})'$$

shows that (d) is equivalent to (a). □

Remark 5.14. (a) Let $h \in \mathfrak{g}$ be an Euler element, if h is symmetric, then $h \in \mathfrak{n}_h$. Indeed in this case there exists a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \simeq \mathfrak{sl}_2(\mathbb{R})$ and h is an Euler element of $\mathfrak{h}$ [51, Corollary 3.14]. Then $h \in [\mathfrak{h}_1, \mathfrak{h}_{-1}] \subset \mathfrak{n}_h$.

(b) If h is not symmetric, then Corollary 5.12 does not apply. Indeed let $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}}$ be the one-particle net associated to the free field in dimension $1+1$ with mass $m > 0$ and let U be the mass m representation of the identity component $\mathcal{P}_+^{\uparrow} = \mathbb{R}^{1,1} \rtimes \mathcal{L}_+^{\uparrow}$ of the Poincaré group. The wedge subspaces $\mathtt{V} := \mathsf{H}(W_R)$ and $\mathsf{H}(W_L)$ are symplectic factor subspaces satisfying

$$\mathsf{H}(W_R)' = \mathsf{H}(W_L) \quad \text{and} \quad \mathsf{H}(W_R) \cap \mathsf{H}(W_L) = \{0\}.$$

Here the wedge W_R is associated to an Euler couple (h, τ_h) (cf. Example 2.7), and since h is neither symmetric in $\mathcal{P}_+^\uparrow$ nor in $\mathcal{L}_+^\uparrow$ (note that $\mathfrak{so}_{1,1}(\mathbb{R}) \cong \mathbb{R}$ is abelian), there is no g such that $gW_R = W_L$. One can restrict the symmetry group to $H := \mathcal{L}_e$ as well as the representation $U|_H$, acting as automorphisms of $\mathsf{H}(W_R)$. Then $\mathtt{V}_H = \mathtt{V} \neq \mathtt{V} \cap \mathtt{V}' = \{0\}$ since the subspace $\mathtt{V} = \mathsf{H}(W_R)$ is symplectic.

(c) The containment $h \in \mathfrak{n}_h$ does not imply that h is symmetric: For instance no Euler element $h \in \mathfrak{sl}_3(\mathbb{R})$ is symmetric, but $h \in \mathfrak{g} = \mathfrak{n}_h$ follows from the simplicity of $\mathfrak{sl}_3(\mathbb{R})$.

5.3. Consequences of Moore's theorem for operator algebras

For the discussion in this section, we recall the conditions (Uni), (M), (Fix), (Mod) and (Reg) from Section 3.2.

Theorem 5.15. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$ anti-elliptic. Let $(U, \mathcal{H})$ be a unitary representation of G with discrete kernel, $\mathcal{N} \subset \mathcal{M} \subseteq B(\mathcal{H})$ an inclusion of von Neumann algebras, and $\Omega \in \mathcal{H}$ a unit vector which is cyclic and separating for $\mathcal{N}$ and $\mathcal{M}$. Assume that*

(Mod) $e^{2\pi i \partial U(h)} = \Delta_{\mathcal{M},\Omega}$, *and*
(Reg') $\{g \in G\colon\ \mathrm{Ad}(U(g))\mathcal{N} \subseteq \mathcal{M}\}$ *is an e-neighborhood in G.*

Then the following assertions hold:

(a) *h is an Euler element.*
(b) *The conjugation $J := J_{\mathcal{M},\Omega}$ satisfies*

$$JU(\exp x)J = U(\exp \tau_h(x)) \quad \textit{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}. \tag{65}$$

(c) $\mathcal{H}^G = \ker(\partial U(h))$.
(d) *The restriction of $i\partial U(h)$ to the orthogonal complement of the subspace $\mathcal{H}^{N_h}$, of fixed vectors of the codimension-one normal subgroup N_h, has absolutely continuous spectrum.*

If, in addition, $\mathcal{H}^G = \mathbb{C}\Omega \neq \mathcal{H}$, then $\mathcal{M}$ is factor of type III_1.

Proof. Our assumptions clearly imply (Uni), (M) and (Mod). Let $N \subseteq G$ be the e-neighborhood specified by (Reg'). Then $\mathcal{M}_N \supseteq \mathcal{N}$, so that (Reg) is also satisfied. As h is anti-elliptic and $\Omega \in \ker(\partial U(h))$ by (Mod), Corollary 5.6 implies that

$$\Omega \in \mathcal{H}^G = \ker(\partial U(h)),$$

which is (c). Now Theorem 3.7 implies (a) and (b). Further, (d) follows from Moore's Theorem.

If, in addition, $\mathcal{H}^G = \mathbb{C}\Omega \neq \mathcal{H}$, then

$$\mathbb{C}\Omega = \ker(\partial U(h)) = \ker(\Delta_{\mathcal{M},\Omega} - \mathbf{1}),$$

so that $\mathcal{M}$ is a factor of type III_1 by Proposition A.1(e) because $\mathcal{H} = \overline{\mathcal{M}\Omega}$ implies $\mathcal{M} \neq \mathbb{C}\mathbf{1}$ and $\mathbb{C}\Omega = \ker(\Delta_{\mathcal{M},\Omega} - \mathbf{1})$ implies $\Delta_{\mathcal{M},\Omega} \neq \mathbf{1}$. □

In our context, [6, Thm. 6.2] becomes the following corollary. We use the notation from 2.7.

Corollary 5.16. (Borchers–Buchholz Theorem) *Let $(U, \mathcal{H})$ be a unitary representation of the Lorentz group $G = \mathrm{SO}_{1,d}(\mathbb{R})^\uparrow$ acting covariantly on an isotone net $(\mathcal{A}(\mathcal{O}))_{\mathcal{O} \subseteq \mathrm{dS}^d}$ of von Neumann algebras on open subsets of de Sitter spacetime, i.e., $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ (isotony) and $\mathrm{Ad}(U(g))(\mathcal{A}(\mathcal{O})) = \mathcal{A}(g\mathcal{O})$ with $g \in G$ (G-covariance). Let $\Omega \in \mathcal{H}$ be a fixed vector of $U(G)$ that is cyclic and separating for any $\mathcal{A}(\mathcal{O})$. Assume that the vacuum state $\omega(\cdot) = \langle \Omega, \cdot\, \Omega\rangle$ is a KMS state for $\mathcal{A}(W_R)$ with inverse temperature $\beta > 0$ with respect to the one-parameter group $(U(\exp th))_{t\in\mathbb{R}}$, namely for every pair $X, Y \in \mathcal{A}(W_R)$, there exists an analytic function $F_{X,Y}$ on the strip*

$$\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z < \beta\}$$

with continuous boundary values satisfying

$$F(t) = \omega(X\, \mathrm{Ad}(U(\exp th))(Y)), \quad F(t + i\beta) = \omega(\mathrm{Ad}(U(\exp th))(Y)X), \quad t \in \mathbb{R}.$$

Then $\beta = 2\pi$.

Proof. For $\mathcal{O} \Subset W_R$, there exists an open neighborhood of the identity $N \subset \mathrm{SO}_{1,d}(\mathbb{R})^\uparrow$ such that $\mathcal{O} \subset gW_R^{\mathrm{dS}}$ for all $g \in N$. Let $\mathcal{M} := \mathcal{A}(W_R^{\mathrm{dS}})$. By covariance, $\mathcal{N} := \mathcal{A}(\mathcal{O})$ satisfies (Reg') in Theorem 5.15. The KMS property implies that $\mathrm{Ad}(U(\exp th)) = \mathrm{Ad}(\Delta_{\mathcal{A},\Omega}^{-it/\beta})$ (cf. [4, Thm. III.4.7.2]) and, since the representation of $\mathcal{A}(W_R^{\mathrm{dS}})$ on $\mathcal{H}$ is the GNS representation for ω, we have that $U\left(\exp\left(\frac{\beta t}{2\pi} h\right)\right) = \Delta^{-\frac{it}{2\pi}}_{\mathcal{A}(W_R^{\mathrm{dS}}),\Omega}$, and Theorem 5.15 applies. We conclude that $\frac{\beta}{2\pi} h$ is an Euler element, but since h is also an Euler element in $\mathfrak{so}_{1,d}(\mathbb{R})$, it follows that $\beta = 2\pi$. □

Definition 5.17. We write $\mathcal{A} := \big(\bigcup_{g\in G} \mathcal{M}_g\big)'' \subseteq B(\mathcal{H})$ for the von Neumann algebra generated by all algebras $\mathcal{M}_g = U(g)\mathcal{M}U(g)^{-1}$. Let $(\mathcal{M}')_G := \bigcap_{g\in G} \mathcal{M}'_g$ and note that

$$\mathcal{A}' = \bigcap_{g\in G} \mathcal{M}'_g = (\mathcal{M}')_G. \tag{66}$$

We also write $\widetilde{\mathcal{A}}$ for the von Neumann algebra generated by $\mathcal{A}$ and $J\mathcal{A}J$ with $J = J_{\mathcal{M},\Omega}$, i.e., by all algebras $\mathcal{M}_g$ and $(\mathcal{M}')_g$, $g \in G$. Then $\widetilde{\mathcal{A}}' \subseteq \mathcal{M} \cap \mathcal{M}' = \mathcal{Z}(\mathcal{M})$ and, more precisely,

$$\widetilde{\mathcal{A}}' = \mathcal{Z}(\mathcal{M})_G = \bigcap_{g\in G} \mathcal{Z}(\mathcal{M})_g \tag{67}$$

is the maximal G-invariant subalgebra of $\mathcal{Z}(\mathcal{M})$.

Lemma 5.18. *Let $\alpha_t := \mathrm{Ad}(\Delta^{it}) \in \mathrm{Aut}(\mathcal{M})$ be the modular automorphisms of the von Neumann algebra $\mathcal{M}$ corresponding to the cyclic separating vector Ω. If* (Uni), (M), (Fix), (Reg) *and* (Mod) *are satisfied and h is anti-elliptic, then*

(a) *$\mathcal{A}' \subseteq \mathcal{M}'$ is invariant under $\mathrm{Ad}(U(G))$.*
(b) *$(\mathcal{M}')^G = (\mathcal{M}')^\alpha = (\mathcal{A}')^G$.*
(c) *$\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M}^G = \mathcal{M}^\alpha$.*

Proof. (a) $\mathcal{A}' \subseteq \mathcal{M}'$ holds by definition, and $\mathcal{A}'$ is $U(G)$-invariant.
(b) By (Mod), we have $(\mathcal{M}')^G \subseteq (\mathcal{M}')^\alpha$. To show the converse, suppose that $A \in \mathcal{M}'$ is fixed by α. As h is anti-elliptic, $A\Omega \in \mathcal{H}^\Delta = \mathcal{H}^G$ (Corollary 5.6), which implies that

$$U(g)AU(g)^{-1}\Omega = U(g)A\Omega = A\Omega.$$

If $g \in N$, with N as in (Reg), then $\mathcal{M}' \cup \mathcal{M}'_g \subseteq \mathcal{M}'_N$ and Ω is separating for $\mathcal{M}'_N$, so that we obtain

$$U(g)AU(g)^{-1} = A.$$

We conclude that A commutes with $U(N)$, and since the connected group G is generated by the identity neighborhood N, it follows that A commutes with $U(G)$. This shows that $(\mathcal{M}')^G = (\mathcal{M}')^\alpha$.

As $\mathcal{A}$ is G-invariant, so it holds $\mathcal{A}' \subseteq \mathcal{M}'$. Further,

$$(\mathcal{A}')^G \subseteq (\mathcal{M}')^G \subseteq (\mathcal{M}')_G = \mathcal{A}'$$

by (66). This implies that $(\mathcal{A}')^G = (\mathcal{M}')^G$.
(c) Using the relation $\mathcal{M} = J\mathcal{M}'J$ and the fact that J normalizes $U(G)$ (Theorem 3.7) and commutes with $U(\exp \mathbb{R}h)$, the equality $\mathcal{M}^G = \mathcal{M}^\alpha$ follows from (b) by conjugating with J. Further $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M}^\alpha$ follows from the fact that modular automorphisms fix the center pointwise ([9, Prop. 5.3.28]). □

Proposition 5.19. *Suppose that* (Uni), (M), (Fix), (Mod) *and* (Reg) *are satisfied, that h is anti-elliptic, and that $\Delta \neq \mathbf{1}$. For the assertions*

(a) *The net $(\mathcal{M}_g)_{g\in G}$ is irreducible, i.e., $\mathcal{A} = B(\mathcal{H})$.*
(b) $\mathcal{A}' = (\mathcal{M}')_G = \bigcap_{g\in G} \mathcal{M}'_g = \mathbb{C}\mathbf{1}$.
(c) $\mathcal{M}_G = \bigcap_{g\in G} \mathcal{M}_g = \mathbb{C}\mathbf{1}$.
(d) $\mathcal{H}^G = \mathbb{C}\Omega$.
(e) *$\mathcal{M}$ is a type III_1 factor.*

we have the implications:

$$\text{(a)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(c)} \Rightarrow \text{(d)} \Rightarrow \text{(e)}.$$

Note that (d) is stronger than $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbf{1}$.

Proof. (a) $\Leftrightarrow$ (b) follows from $\mathcal{A}' = \bigcap_{g\in G}(\mathcal{M}_g)' = (\mathcal{M}')_G$.
(b) $\Leftrightarrow$ (c): As $JU(G)J = U(G)$ by Theorem 3.7 and $J\mathcal{M}J = \mathcal{M}'$, we have $J\mathcal{M}_G J = (\mathcal{M}')_G$. Therefore (b) and (c) are equivalent.
(c) $\Rightarrow$ (d): From Proposition A.1(a) and Lemma 5.18(c), we know that

$$\mathcal{H}^G = \mathcal{H}^\Delta \overset{\text{A.1}}{=} \overline{\mathcal{M}^\alpha\Omega} = \overline{\mathcal{M}^G\Omega}. \tag{68}$$

Therefore $\mathcal{M}^G \subseteq \mathcal{M}_G = \mathbb{C}\mathbf{1}$ implies that $\mathcal{H}^G = \mathbb{C}\Omega$.
(d) $\Rightarrow$ (e): As h is anti-elliptic, we have $\mathcal{H}^G = \mathcal{H}^\Delta$ (Corollary 5.6), so that Proposition A.1(e) implies that $\mathcal{M}$ is a factor of type III_1. □

Remark 5.20. If $G = \mathbb{R}$ acts as the modular group of $(\mathcal{M}, \Omega)$, then $\mathcal{A} = \mathcal{M}$, $\widetilde{\mathcal{A}} = (\mathcal{M} \cup \mathcal{M}')''$, and $\widetilde{\mathcal{A}}' = \mathcal{Z}(\mathcal{M})$. So $\widetilde{\mathcal{A}}' = \mathbb{C}\mathbf{1}$ is equivalent to $\mathcal{M}$ being a factor, but, in general, this does not imply that $\mathcal{H}^G = \mathcal{H}^\Delta = \mathbb{C}\Omega$ because we may have $\mathcal{M}^\alpha \neq \mathbb{C}\mathbf{1}$ (cf. Remark 5.21(b)).

Remark 5.21. (a) The implication "(e) $\Rightarrow$ (c)" holds if there exists a $g \in G$ such that $\mathcal{M}_g = U(g)\mathcal{M}U(g)^{-1} \subseteq \mathcal{M}'$. Then $\mathcal{M}_G \subseteq \mathcal{Z}(\mathcal{M})$, and if $\mathcal{M}$ is a factor, it follows that $\mathcal{M}_G = \mathbb{C}\mathbf{1}$, so that (e) implies (c).

If the Euler element h is not symmetric, i.e., there exists no $g \in G$ such that $\mathrm{Ad}(g)h = -h$, then (e) does not always imply (a). For instance, let $\mathbb{R}^{1,1} \supset \mathcal{O} \to \mathcal{M}(\mathcal{O})$ be the free field of mass $m > 0$ in $1 + 1$ dimensions and let U be the mass m representation of the identity component of the Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{1,1} \rtimes \mathcal{L}_+^\uparrow$. The algebras $\mathcal{M}(W_R)$ and $\mathcal{M}(W_L)$ corresponding to the right and left wedges are invariant under the Lorentz action and of type III_1. This follows from uniqueness of the vacuum state and Proposition 5.19. In particular, the "one wedge net" $W_R \to \mathcal{M}(W_R)$ together with the representation $U|_{\mathcal{L}_+^\uparrow}$ satisfies (Uni), (M), (Fix), (Mod) and (Reg) but the algebra generated by $\mathrm{Ad}(U(\mathcal{L}_+^\uparrow))\mathcal{M}(W_R) = \mathcal{M}(W_R)$ is properly contained in $\mathcal{B}(\mathcal{H})$ (see also Example 3.8).

(b) The implication "(e) $\Rightarrow$ (d)" is related to the ergodicity of the state on the type III_1-factor $\mathcal{M}$ specified by Ω: By (68), ergodicity of the state defined by Ω is equivalent to $\mathcal{H}^G = \mathbb{C}\Omega$. This does in general not follow from (e) because non-ergodic states always exist for a type III_1-factor (Remark A.2). Concretely, such states can be obtained as follows: Consider a type III_1 factor $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and the algebra $M_2(\mathbb{C})$ of complex 2×2-matrices. Then $\widetilde{\mathcal{M}} = \mathcal{M} \otimes M_2(\mathbb{C})$ is a type III_1 factor ([74, Thm. V.2.30]). For a faithful normal state ω on $\mathcal{M}$, we consider the state on $\widetilde{\mathcal{M}}$ specified by

$$(\omega \otimes \varphi_{11})(m \otimes x) = \omega(m)x_{11}.$$

This is a non-ergodic (non-faithful) state on the type III_1 factor $\widetilde{\mathcal{M}}$.

(c) Suppose that $\mathcal{M} = \mathcal{M}_G$, i.e., that $\mathcal{M}$ is normalized by $U(G)$. Then $G = G_{\mathcal{M}}$ and $\Omega \in \mathcal{H}^G$ imply $G = G_{\mathtt{V}_{\mathcal{M}}}$, so that h is central in $\mathfrak{g}$ and therefore $\tau_h = \mathrm{id}_G$. The example described in point (a) with $G = \mathcal{L}_e$ is of this type.

5.4. The degenerate case

Proposition 5.19 describes the non-degenerate case, where $\mathcal{H}^G = \mathbb{C}\Omega$. If $\mathcal{H}^G$ is not one-dimensional, we now obtain a direct integral decomposition, in accordance with the AQFT literature, see [40, Cor. 6.2.10], [2, Sect. 4.4], [6, Sect. 5].

The following proposition extends 5.19 to the case where the vacuum Ω is not cyclic. We will comment on conditions (a) and (b) in Remark 5.23 below.

Proposition 5.22. *Suppose that $\mathcal{H}$ is separable. Let $(\alpha_t)_{t\in\mathbb{R}}$ be the modular automorphisms of $\mathcal{M}$ with respect to the cyclic separating vector Ω and $(U, \mathcal{H})$ a unitary representation of G, such that:*

(a) (Uni), (M), (Fix), (Reg) *and* (Mod) *and h is anti-elliptic in $\mathfrak{g}$.*
(b) *$\mathcal{M}' = \mathcal{M}_{g_0}$ for some $g_0 \in G$.*

Then we have direct integral decompositions

$$\mathcal{M} = \int_X^\oplus \mathcal{M}_x \, d\mu(x), \qquad U = \int_X^\oplus U_x \, d\mu(x), \qquad \text{and} \qquad \mathcal{A} = \int_X^\oplus B(\mathcal{H}_x) d\mu(x).$$

We have a measurable decomposition $X = X_0 \dot\cup X_1$, where $\dim \mathcal{H}_x = 1$ for $x \in X_0$ and the representations $(U_x)_{x\in X_0}$ are trivial. For $x \in X_1$, the algebras $\mathcal{M}_x$ are factors of type III_1 and $(\mathcal{M}_x, \Omega_x, \underline{U}_x)$ satisfies (Uni), (M), (Fix), (Reg) *and* (Mod), *where $\underline{U}_x$ is the representation of $G/\ker(U_x)$ induced by U_x.*

Proof. From $\mathcal{M}' = \mathcal{M}_{g_0}$ for some $g_0 \in G$, we derive that $\mathcal{A}' \subseteq \mathcal{Z} := \mathcal{M} \cap \mathcal{M}'$. Using Lemma 5.18(b),(c), we obtain

$$\mathcal{A}' = (\mathcal{A}')^G \subseteq \mathcal{Z} = \mathcal{Z}^G \subseteq (\mathcal{M}')^G = (\mathcal{M}')^\alpha = (\mathcal{A}')^G = \mathcal{A}', \tag{69}$$

so that

$$\mathcal{Z}^G = \mathcal{Z} = (\mathcal{M}')^\alpha = \mathcal{A}'. \tag{70}$$

By [8, Thm. 4.4.3], there exists a finite standard measure space (X, μ), a unitary Φ such that

$$\Phi\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x)$$

and $U\mathcal{Z}U^*$ acts on the direct integral as the algebra $L^\infty(X, \mu)$ of diagonal operator. From [8, Thm. 4.4.6(a)], passing to the commutant one can easily see that $\mathcal{A}= \mathcal{Z}'$ can be represented as the direct integral von Neumann algebra xof decomposable operators:

$$\Phi\mathcal{A}\Phi^* = \int_X^\oplus B(\mathcal{H}_x)\, d\mu(x).$$

If $\mathcal{C}$ is a von Neumann subalgebra of $\mathcal{A}$, then $\Phi\mathcal{C}\Phi^* \subset \Phi\mathcal{A}\Phi^*$ and there exists a measurable family of von Neumann algebras $X \ni x \mapsto \mathcal{C}_x \subset B(\mathcal{H}_x)$ for almost every $x \in X$ [74, Thms. 8.21, 8.23]. In particular $U\mathcal{C}U^* = \int_X^\oplus \mathcal{C}_x\, d\mu(x)$. Since U does not depend on the subalgebra, hereafter in the proof we will work on the direct integral Hilbert space, i.e. we will assume $\mathcal{H} = \int_X^\oplus \mathcal{H}_x\, d\mu(x)$.

With this argument we can also assume that, on the same standard finite measure space (X, μ), we have

$$(\mathcal{M}, \mathcal{H}) = \int_X^\oplus (\mathcal{M}_x, \mathcal{H}_x)\, d\mu(x), \tag{71}$$

for which $\mathcal{Z} \cong L^\infty(X, \mu)$ is the diagonal algebra and almost every $\mathcal{M}_x$ is a factor ([74, Cor. 8.20]).

As $\mathcal{Z}$ commutes with $U(G)$, we have

$$U(G) \subseteq \mathcal{Z}' = \mathcal{A}'' = \mathcal{A}. \tag{72}$$

Hence the separable C^*-algebra $C^*(U(G))$ is contained in $\mathcal{Z}' = \mathcal{A}$, so that [8, Cor. 4.4.8] yields a direct integral decomposition of the unitary representation

$$(U, \mathcal{H}) = \int_X^{\oplus} (U_x, \mathcal{H}_x)\, d\mu(x).$$

For $x \in X$, the kernel $\ker U_x$ may not be discrete, so that (Uni) holds for $(U_x, \mathcal{H}_x)$ only as a representation $\underline{U}_x$ of $G/\ker(U_x)$.

Since U is a direct integral representation, we have

$$(\mathcal{M}_g, \mathcal{H}) = \int_X^{\oplus} ((\mathcal{M}_g)_x, \mathcal{H}_x)\, d\mu(x). \tag{73}$$

By Proposition A.1(a), $\Omega \in \mathcal{H}^G \subseteq \mathcal{H}^\Delta$ is a cyclic separating vector for $\mathcal{Z} = (\mathcal{M}')^\alpha$. Writing $\Omega = (\Omega_x)_{x\in X}$, it follows that almost no Ω_x vanishes, and thus

$$\mathcal{H}^G = \int_X^{\oplus} \mathbb{C}\Omega_x\, d\mu(x) \cong L^2(X, \mu).$$

Replacing $\mathcal{N}$ in (Reg) by the von Neumann algebra $\mathcal{M}_N = \bigcap_{g\in N} \mathcal{M}_g$, where $N \subseteq G$ is an e-neighborhood satisfying (Reg), we see that $\mathcal{M}_N \subseteq \mathcal{Z}'$ also decomposes according to the direct integral. We also obtain

$$\mathcal{M}_N = \int_X^{\oplus} (\mathcal{M}_x)_N\, d\mu(x),$$

from Lemma C.4. Theorem 5.15 now shows that $\partial U(h)$ also decomposes in such a way that

$$\ker(\partial U_x(h)) = \mathbb{C}\Omega_x \tag{74}$$

for almost every $x \in X$.

Since Ω is cyclic and separating for $\mathcal{M}$, the vectors $\Omega_x \in \mathcal{H}_x$ must be cyclic separating for the von Neumann algebras $\mathcal{M}_x$ for almost every $x \in X$ (easy argument by contradiction, we also refer to [75, Thm. VIII.4.8] for a more general case). We therefore obtain (Uni), (M), (Fix), (Mod) and (Reg) for the algebras $\mathcal{M}_x \subseteq B(\mathcal{H}_x)$ and the representations $\underline{U}_x$ of $G/\ker(U_x)$ on $\mathcal{H}_x$. Finally, since $\mathcal{A}'$ is the diagonal algebra,

$$\mathbb{C}\mathbf{1} = (\mathcal{A}_x)' = \bigcap_{g\in G} (\mathcal{M}'_x)_g$$

holds for almost every $x \in X$ (Lemma C.4 and [8, Thm. 4.4.5]).

The condition $\Delta_x \neq \mathbf{1}$ is by (74) equivalent to $\dim \mathcal{H}_x > 1$, and in this case Proposition 5.19 applies to the configuration in the Hilbert space $\mathcal{H}_x$ and shows that $\mathcal{M}_x$ is a type III_1-factor. If $\dim \mathcal{H}_x = 1$, then $\mathcal{M}_x = \mathbb{C}\mathbf{1}$ and $\partial U_x(h) = 0$ implies the triviality of the representation U_x because

$$\mathcal{H}_x^G = \ker(\partial U(h))_x = \mathbb{C}\Omega_x = \mathcal{H}_x$$

(Theorem 5.15(c)).

We now define $X_1 := \{x \in X\colon \dim \mathcal{H}_x > 1\}$ and $X_0 := \{x \in X\colon \dim \mathcal{H}_x = 1\}$. Then the triples $(\mathcal{M}_x, \mathcal{H}_x, U_x)$ satisfy (M), (Fix), (Reg), (Mod), and (Uni) for the representation $\underline{U}_x$ of $G/\ker(U_x)$. $\square$

Remark 5.23. (a) If h is not a symmetric Euler element, the condition $\mathcal{M}' \subset \mathcal{M}_{g_0}$ may not hold (Remark 5.21(a)).

(b) In Proposition 5.22 it was crucial that $\mathcal{M}' = \mathcal{M}_{g_0}$ for some $g_0 \in G$, in order to obtain the disintegration. Furthermore, $\mathcal{A}' = \mathcal{Z} = \mathcal{Z}^G$ implies $U(G) \subset \mathcal{A}$. In the general case it is not clear when the group $U(G)$ is contained in $\mathcal{A}$. In [6, Prop. 4.1], this follows from the KMS property of the wedge modular groups together with their geometric action, where it is used that boosts generate the Lorentz group to see that $U(G) \subseteq \mathcal{A}'' = \mathcal{A}$. In our argument $U(G) \subseteq \mathcal{A}'' = \mathcal{A}$ does not require that G is generated by an orbit of Euler elements.

(c) In the proof of Proposition 5.22, we disintegrated $\mathcal{M} = \int_X^\oplus \mathcal{M}_x \, d\mu(x)$ and $U = \int_X^\oplus U_x \, d\mu(x)$ in order to apply Proposition 5.19 fiberwise and conclude that, for almost every $x \in X_1$, the algebra $\mathcal{M}_x$ is a type III_1 factor. We actually have deduced (M), (Fix), (Reg), (Mod) for almost every the triple $(\mathcal{M}_x, U_x, \Omega_x)$ and (Uni) for $(\mathcal{M}_x, \underline{U}_x, \Omega_x)$. In particular we could apply Proposition 5.19 for almost every triple $(\mathcal{M}_x, \underline{U}_x, \Omega_x)$, where all the properties (M), (Fix), (Reg), (Mod) and (Uni) hold. Actually, it is not needed to assume (Uni) on U_x to conclude the type III_1 property of $\mathcal{M}_x$. Along this paper, (Uni) is necessary to ensure that $\mathtt{d}U$ is injective and in particular that $\mathtt{d}U(h)$ determines h uniquely. In the proof of Proposition 5.22 we only need that

$$(\mathcal{Z})_x = (\mathcal{M}^\alpha)_x = \mathbb{C} \cdot \mathbf{1}_{\mathcal{H}_x} \tag{75}$$

to apply Proposition A.1(e). We obtain (75) as follows: Let $g_0 \in G$, such that $\mathcal{M}' = \mathcal{M}_{g_0} \in \mathcal{A}$. Then we have $\mathcal{M}_x' = (\mathcal{M}_{g_0})_x$, hence $\mathcal{Z}(\mathcal{M}_x) = \mathcal{Z}(\mathcal{M})_x = \mathcal{Z}_x$ for a.e. $x \in X$. Furthermore, $\mathcal{M}^\alpha = \int_X^\oplus (\mathcal{M}^\alpha)_x \, d\mu(x)$, and since $\mathcal{Z} = \mathcal{M}^\alpha = \mathbb{C} \cdot \mathbf{1}$, then $\mathcal{Z}_x = (\mathcal{M}^\alpha)_x = \mathbb{C} \cdot \mathbf{1}_{\mathcal{H}_x}$ for almost every $x \in X$.

(d) Condition (b) in Proposition 5.22 implies that $\mathcal{M}' \subset \mathcal{A}$. If $\mathcal{M}' \not\subset \mathcal{A}$ then Proposition 5.22 does not apply in the present form. One may consider the larger von Neumann algebra $\widetilde{\mathcal{A}}$ generated by the G-transforms of $\mathcal{M}$ and $\mathcal{M}'$. Lemma 5.18(c) then implies that G acts trivially on $\mathcal{Z}(\mathcal{M})$, so that (67) entails $\widetilde{\mathcal{A}}' = \mathcal{Z}(\mathcal{M})$. Then $\widetilde{\mathcal{A}}$ contains $U(G)$, and one can repeat large portions of the proof of Proposition 5.22 to disintegrate the triple $(\mathcal{M}, U, \widetilde{\mathcal{A}})$. However, in this situation, the conclusion one can draw from $\mathcal{Z}(\mathcal{M}_x) = \mathbb{C}\mathbf{1}$, i.e., if $\mathcal{M}_x$ is a factor, is weaker. In particular, $\mathcal{M}_x^\alpha$ can be larger than $\mathbb{C}\mathbf{1}$, so that $\mathcal{M}_x$ need not be of type III_1 (cf. Remark 5.20).

6. Outlook

This paper develops a language concerning properties of nets of standard subspaces that provides descriptions on several levels of abstraction. It also incorporates a series of recent results from a new point of view. [6,10] aim to deduce properties of QFT on de Sitter/anti-de Sitter spacetime from the thermal property of the vacuum state for a geodesic observer. In [14], the authors deduce AQFT properties from the assumption on the state on the quasi-local algebra to be passive for a uniformly accelerated observer in n-dimensional anti-de Sitter spacetime for $n \geq 2$. [71] aims to unify the previous approaches by considering passive states for an observer traveling along worldlines in order to prove the thermal property of the vacuum and the Reeh–Schlieder property. His purpose was also to look for an abstract setting that, in the end, was lacking concrete examples. Our context may provide the proper setting in which such questions can be investigated and where one has a large zoo of diverse examples.

If one starts with a standard subspace $\mathtt{V}$ and a unitary representation $(U, \mathcal{H})$ of G, then there are many ways to formulate conditions on a net of standard subspaces containing $\mathtt{V}$ that ensure the Bisognano–Wichmann property, or at least modular covariance, in the sense that the modular groups associated to wedge regions act geometrically; see [49,51]. Results in these directions have recently been established in [52], and our Euler Element Theorem (Theorem 3.1) can also be considered as a tool to investigate the Bisognano–Wichmann property. However, a satisfying answer to the long-standing questions related to modular covariance for nets of standard subspaces and the Bisognano–Wichmann property in free and interacting nets of von Neumann algebras requires further research. For a recent approach to the situation for Minkowski spacetime through scattering theory, we refer to [24] and references therein.

In this paper, we do not analyze locality properties. Indeed, in our AQFT context it may happen that, on the same symmetric space M, there are no causally complementary wedge regions. This happens if the Euler element corresponding to the wedge W is not symmetric, so that there exists no $g \in G$ with $gW = W'$ (cf. [54]). If h is a symmetric Euler element and the center of G is non-trivial, many complementary wedges appear. This has been studied in [51] at the abstract level, but an analysis on symmetric spaces is still missing. Once a one-particle net is established one would aim at a second quantization procedure which takes care of a one-particle Spin-Statistics Theorem anticipated in [51]. Interesting new possibilities for twisted second quantization procedures may be derived from the recent paper [21].

Wedges on causal homogeneous space have been described in [66,53,54]. The construction of covariant local nets of standard subspaces on open regions have been described in [26,66]. Having now understood that Euler elements are the natural generators of the geometric flows of modular Hamiltonians on a causal homogeneous space (see Theorem 3.1

and Theorem 5.15), one is interested in a general geometric description of entropy and energy inequalities on these spaces and their relation with the representation theory of Lie groups ([56,15,16]).

Appendix A. Factor types and modular groups

We assume that $\Omega \in \mathcal{H}$ is a cyclic and separating unit vector for the von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$. We consider the automorphism group $(\alpha_t)_{t\in\mathbb{R}}$ of $\mathcal{M}$ defined by the modular group via

$$\alpha_t(M) = \Delta^{it} M \Delta^{-it}, \quad t \in \mathbb{R}, M \in \mathcal{M}.$$

We write $\mathcal{M}^\alpha$ for the subalgebra of α-fixed elements and $\mathcal{H}^\Delta := \ker(\Delta - \mathbf{1})$ for the subspace of fixed vectors of the modular group.

Proposition A.1. *The following assertions hold:*

(a) $\mathcal{M}^\alpha\Omega \subseteq \mathcal{H}^\Delta$ *is a dense subspace.*
(b) $\mathcal{H}^\Delta = \mathbb{C}\Omega$ *if and only if* $\mathcal{M}^\alpha = \mathbb{C}\mathbf{1}$*, i.e., that* $(\mathcal{M}, \mathbb{R}, \alpha)$ *is ergodic.*
(c) $\mathcal{M}^\alpha \supseteq \mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$*. In particular,* $\mathcal{M}$ *is a factor if* $(\mathcal{M}, \mathbb{R}, \alpha)$ *is ergodic.*
(d) *The von Neumann algebra* $\mathcal{M}$ *is semi-finite if and only if the modular automorphisms* $(\alpha_t)_{t\in\mathbb{R}}$ *are inner, i.e., can be implemented by a unitary one-parameter group of* $\mathcal{M}$*. If* Δ *is non-trivial and inner, then* $\mathcal{M}^\alpha \neq \mathbb{C}\mathbf{1}$*.*
(e) *If* $\mathcal{H}^\Delta = \mathbb{C}\Omega$ *and* $\Delta \neq \mathbf{1}$*, then* $\mathcal{M}$ *is a factor of type* III_1*.*

Proof. (a) The inclusion $\mathcal{M}^\alpha\Omega \subseteq \mathcal{H}^\Delta$ is clear. That $\mathcal{M}^\alpha\Omega$ is dense in $\mathcal{H}^\Delta$ follows from [40, Prop. 6.6.4], applied with $G = \mathbb{R}$ and $U_t = \Delta^{it}$.

(b) This follows from (a) and the fact that Ω is a separating vector.

(c) Here we use that modular groups fix the center pointwise; see [9, Prop. 5.3.28].

(d) The first assertion follows from [73, Thm. 3.1.6]. If $(\alpha_t)_{t\in\mathbb{R}}$ is inner and non-trivial, then the spectral projections of the corresponding infinitesimal generator are contained in $\mathcal{M}^\alpha$, showing that $\mathcal{M}^\alpha \neq \mathbb{C}\mathbf{1}$.

(e) From (b) we infer that $\mathcal{M}^\alpha = \mathbb{C}\mathbf{1}$, so that (c) implies that $\mathcal{M}$ is a factor. By (d) it is of type III because Δ is non-trivial (here we use $\mathcal{M} \neq \mathbb{C}\mathbf{1}$), but cannot be inner by ergodicity.[5] We have to exclude the types III_0 and III_λ for $\lambda \in (0, 1)$. By [75, Prop. XII.3.15], if $\mathcal{M}$ is of type III_0, then the center of $\mathcal{M}^\alpha$ is non-atomic. As this is not the case for $\mathcal{M}^\alpha = \mathbb{C}$, this case is excluded.

[5] At this point [40, Prop. 6.6.5] implies that $\mathcal{M}$ is of type III_1, but as Longo's argument is very condensed, we provide some more details.

Let $\Gamma(\mathcal{M}) \subseteq \mathbb{R}_+^\times \cong \widehat{\mathbb{R}}$ denote the Connes spectrum of α on $\mathcal{M}$, which by [73, Prop. 3.3.3] coincides with the spectrum of α on $\mathcal{M}$. Now [73, Prop. 3.4.7] asserts that, if $\mathcal{M}$ and $\mathcal{M}^\alpha$ are factors, then

$$\Gamma(\mathcal{M}) = S(\mathcal{M}) \cap \mathbb{R}_+^\times = \mathbb{R}_+^\times \cap \sigma(\Delta_\omega)$$

for any faithful separating normal state ω. If $\mathcal{M}$ is of type III_λ, then $\Gamma(\mathcal{M}) = \lambda^{\mathbb{Z}}$ (cf. [73, Def. 3.3.10]), so that the modular group α is periodic. By [75, Exer. XII.2], this implies that $\mathcal{M}^\alpha$ is a factor of type II_1, contradicting $\mathcal{M}^\alpha = \mathbb{C}\mathbf{1}$. So type III_λ is also ruled out. Alternatively, one can use [18, Lemma 4.2.3], asserting that, if $\mathcal{M}$ is a factor and 1 is isolated in $\sigma(\Delta_\omega)$, then $\mathcal{M}^\alpha$ contains a maximal abelian subalgebra of $\mathcal{M}$. In our context this contradicts $\mathcal{M}^\alpha = \mathbb{C}\mathbf{1}$. $\square$

Remark A.2. We have seen above that $\mathcal{M}$ is a type III_1-factor if $(\mathcal{M}, \mathbb{R}, \alpha)$ is ergodic. According to [47], the converse also holds in the sense that, if $\mathcal{M}$ is a type III_1-factor, then the set of ergodic states is a dense G_δ in the set of all faithful normal states. That there are also faithful normal states that are not ergodic follows from [20, Cor. 8], that asserts for each hyperfinite factor $\mathcal{R}$ the existence of faithful normal states of $\mathcal{M}$ with $\mathcal{M}^\alpha \supseteq \mathcal{R}$.

Remark A.3. From Proposition A.1(a) it follows that the J-fixed vector Ω is cyclic and separating in $\mathcal{H}^\Delta$ for the subalgebra $\mathcal{M}^\alpha$. Hence $J\mathcal{M}J = \mathcal{M}'$ implies that the same holds of $(\mathcal{M}')^\alpha$ because $J\mathcal{H}^\Delta = \mathcal{H}^\Delta$. We therefore have a standard form representation of $\mathcal{M}^\alpha$ on $\mathcal{H}^\Delta$. Note that the standard subspace $\mathtt{V} = \mathtt{V}_{M,\Omega}$ satisfies

$$\mathtt{V} \cap \mathcal{H}^\Delta = \mathtt{V}^\Delta = \mathtt{V} \cap \mathtt{V}' = \mathtt{V} \cap \mathcal{H}^J$$

and contains the standard subspace $\overline{\mathcal{M}_h^\alpha.\Omega}$ of $\mathcal{H}^\Delta$. This implies that the corresponding modular operator is trivial, so that $\omega_\Omega(A) := \langle \Omega, A\Omega \rangle$ is a trace on $\mathcal{M}^\alpha$ ([9, Prop. 5.3.3]).

Remark A.4. Suppose that $\mathcal{M} = \mathcal{R}(\mathtt{V})$ is a second quantization algebra. Then $\mathcal{R}(\mathtt{V} \cap \mathtt{V}') = \mathcal{R}(\mathtt{V}) \cap \mathcal{R}(\mathtt{V})'$ by the Duality Theorem ([1]), so that $\mathcal{R}(\mathtt{V})$ is a factor if and only if $\mathtt{V}$ is symplectic, which is equivalent to

$$\ker(\Delta_{\mathtt{V}} - \mathbf{1}) = \{0\}.$$

We also have $\Delta = \Gamma(\Delta_{\mathtt{V}})$ for the corresponding standard subspace $\mathtt{V}$. Therefore $\mathcal{F}(\mathcal{H})^\Delta = \mathbb{C}\Omega$ implies that $\mathcal{H}^{\Delta_{\mathtt{V}}} = \{0\}$, which is equivalent to $\mathcal{R}(\mathtt{V})$ being a factor, but we have seen in Proposition A.1(a) that $\mathcal{F}(\mathcal{H})^\Delta = \mathbb{C}\Omega$ even implies that $\mathcal{M}$ is a factor of type III_1.

If $\mathcal{R}(\mathtt{V})$ is a factor of type I, then the modular group is inner and, if $\mathtt{V} \neq \{0\}$, it follows that $\mathcal{R}(\mathtt{V})^\alpha \neq \mathbb{C}\mathbf{1}$. In view of Proposition A.1(a), this implies that $\mathcal{F}(\mathcal{H})^\Delta \neq \mathbb{C}\Omega$.

Appendix B. Smooth and analytic vectors

For a unitary representation $(U, \mathcal{H})$ of a Lie group G, we write $\mathcal{H}^\infty \subseteq \mathcal{H}$ for the subspace of *smooth vectors*, i.e., elements $\xi \in \mathcal{H}$ whose orbit map

$$U^\xi \colon G \to \mathcal{H}, g \mapsto U(g)\xi$$

is smooth. For $x \in \mathfrak{g}$, we write $\partial U(x)$ for the infinitesimal generator of the one-parameter group $U(\exp tx)$, so that $U(\exp tx) = e^{t\partial U(x)}$. On this dense subspace we have the *derived representation*

$$\mathtt{d}U \colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(\mathcal{H}^\infty), \quad \mathtt{d}U(x+iy)\xi := \partial U(x)\xi + i\partial U(y)\xi \quad \text{for} \quad x, y \in \mathfrak{g}, \xi \in \mathcal{H}^\infty$$

for the derived representation of $\mathfrak{g}_{\mathbb{C}}$ on this dense subspace. We also write $\mathcal{H}^\omega \subseteq \mathcal{H}^\infty$ for the subspace of analytic vectors which is dense in $\mathcal{H}$ ([68, Thm. 4], [28]). As $\mathcal{H}^\infty$ is dense and $U(G)$-invariant, $\partial U(x)$ is the closure of $\mathtt{d}U(x)$ ([69, Thm. VIII.10]).

For an analytic vector $\xi \in \mathcal{H}^\omega$, we then have

$$U^\xi(\exp x) = U(\exp x)\xi = \sum_{n=0}^\infty \frac{1}{n!}(\mathtt{d}U(x))^n \xi$$

for every x in a sufficiently small 0-neighborhood $U^\xi_{\mathfrak{g}} \subseteq \mathfrak{g}$. Analytic continuation implies that, after possibly shrinking $U^\xi_{\mathfrak{g}}$, the power series on the right converges on the 0-neighborhood $U^\xi_{\mathfrak{g}_{\mathbb{C}}} := U^\xi_{\mathfrak{g}} + iU^\xi_{\mathfrak{g}} \subseteq \mathfrak{g}_{\mathbb{C}}$ and defines a holomorphic function

$$\eta_\xi \colon U^\xi_{\mathfrak{g}_{\mathbb{C}}} \to \mathcal{H}, \quad \eta_\xi(z) := \sum_{n=0}^\infty \frac{1}{n!}(\mathtt{d}U(z))^n \xi. \tag{76}$$

If $\ker(U)$ is discrete, then $\mathtt{d}U$ is injective on $\mathfrak{g}$. But for $z \in \mathfrak{g}_{\mathbb{C}}$ the adjoint $\mathtt{d}U(z)^\dagger$ on $\mathtt{d}U(z)$ on the pre-Hilbert space $\mathcal{H}^\infty$ satisfies

$$\mathtt{d}U(x+iy)^\dagger = -\mathtt{d}U(x) + i\mathtt{d}U(y) = \mathtt{d}U(-x+iy) \quad \text{for} \quad x, y \in \mathfrak{g}.$$

This implies that $\mathtt{d}U \colon \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(\mathcal{H}^\infty)$ is also injective because $0 = \mathtt{d}U(x+iy) = \mathtt{d}U(x) + i\mathtt{d}U(y)$ implies that the hermitian and the skew-hermitian part of this operator on $\mathcal{H}^\infty$ vanish, and thus $\mathtt{d}U(x) = \mathtt{d}U(y) = 0$.

Lemma B.1. *For $z \in \mathfrak{g}_{\mathbb{C}}$, let $\mathtt{d}U^\omega(z)$ denote the restriction of $\mathtt{d}U(z)$ to $\mathcal{H}^\omega$. Then*

$$\mathtt{d}U(z) \subseteq \mathtt{d}U^\omega(-\overline{z})^* \tag{77}$$

In particular, the representation $\mathtt{d}U^\omega$ of $\mathfrak{g}_{\mathbb{C}}$ is injective if $\ker(U)$ is discrete. If this is the case, then $\mathtt{d}U^\omega(z)$ is skew-symmetric if and only if $z \in \mathfrak{g}$.

Proof. We have

$$\langle \xi, \mathtt{d}U^\omega(z)\eta\rangle = \langle \mathtt{d}U(-\overline{z})\xi, \eta\rangle \quad \text{for all} \quad \xi \in \mathcal{H}^\infty, \eta \in \mathcal{H}^\omega,$$

which is (77). In particular, we see that $\mathtt{d}U^\omega(z) = 0$ implies $\mathtt{d}U(z) = 0$, so that $\ker(\mathtt{d}U) = \ker(\mathtt{d}U^\omega)$. Suppose that $\ker(U)$ is discrete, so that $\mathtt{d}U$ and $\mathtt{d}U^\omega$ are injective. Then $\mathtt{d}U^\omega(z)$ is skew-symmetric if and only if $z - \overline{z} \in \ker(\mathtt{d}U^\omega) = \{0\}$, which is equivalent to $z \in \mathfrak{g}$. □

Appendix C. Some facts on direct integrals

Let $\mathcal{H} = \int_X^\oplus \mathcal{H}_m \, d\mu(m)$ be a direct integral of Hilbert spaces on a standard measure space (X, μ). We call a closed real subspace $\mathsf{H} \subseteq \mathcal{H}$ *decomposable* if it is of the form

$$\mathsf{H} = \int_X^\oplus \mathsf{H}_m \, d\mu(m),$$

where $(\mathsf{H}_m)_{m\in X}$ is a measurable field of closed real subspaces. Now let $(\mathsf{H}^k)_{k\in K}$ be an at most countable family of decomposable real subspaces. Then we have ([57, Lemma B.3]):

(DI1) $\mathsf{H}' = \int_X^\oplus \mathsf{H}_m' \, d\mu(m)$.
(DI2) $\bigcap_{k\in K} \mathsf{H}^k = \int_X^\oplus \bigcap_{k\in K} \mathsf{H}_m^k \, d\mu(m)$.
(DI3) $\overline{\sum_k \mathsf{H}^k} = \int_X^\oplus \overline{\sum_k \mathsf{H}_m^k} \, d\mu(m)$.

Lemma C.1. *The subspace* H *is cyclic/separating/standard if and only if* μ*-almost all* H_m *have this property.*

Proof. (a) First we deal with the separating property. By (DI2) we have

$$\mathsf{H} \cap i\mathsf{H} = \int_X^\oplus (\mathsf{H}_m \cap i\mathsf{H}_m) \, d\mu(m),$$

and this space is trivial if and only if μ-almost all spaces $\mathsf{H}_m \cap i\mathsf{H}_m$ are trivial, which means that H_m is separating.

(b) The subspace H is cyclic if and only if H' is separating. By (DI1) and (a) this means that μ-almost all H_m' are separating, i.e., that H_m is cyclic.

(c) By (a) and (b) H is standard if and only if μ-almost all H_m are cyclic and separating, i.e., standard. □

Lemma C.2. *For a countable family* $(\mathsf{H}^k)_{k\in K}$ *of decomposable cyclic closed real subspaces, the intersection* $\mathtt{V} := \bigcap_{k\in K} \mathsf{H}^k$ *is cyclic if and only if, for* μ*-almost every* $m \in X$*, the subspace* $\mathtt{V}_m := \bigcap_{k\in K} \mathsf{H}_m^k$ *is cyclic.*

Proof. By (DI2), we have $\mathtt{V} = \int_X^\oplus \mathtt{V}_m \, d\mu(m)$, so that the assertion follows from Lemma C.1. □

For a direct integral

$$(U, \mathcal{H}) = \int_X^\oplus (U_m, \mathcal{H}_m) \, d\mu(m)$$

of (anti-)unitary representations of G_{τ_h}, the canonical standard subspace $\mathtt{V} = \mathtt{V}(h, U) \subseteq \mathcal{H}$ from (29) is specified by the decomposable operator $J\Delta^{1/2} = U(\tau_h)e^{\pi i \, \partial U(h)}$, hence decomposable:

$$\mathtt{V} = \int_X^\oplus \mathtt{V}_m \, d\mu(m). \tag{78}$$

Lemma C.3. *Assume that G has at most countably many components. For any subset $A \subseteq G$ and a real subspace $\mathsf{H} \subseteq \mathcal{H}$, we put*

$$\mathsf{H}_A := \bigcap_{g \in A} U(g)\mathsf{H}. \tag{79}$$

Then the following assertions hold:

(a) *If H is decomposable, then $\mathsf{H}_A = \int_X^\oplus \mathsf{H}_{m,A} \, d\mu(m)$.*
(b) *H_A is cyclic if and only if μ-almost all $\mathsf{H}_{m,A}$ are cyclic.*

Proof. (a) As G has at most countably many components, it carries a separable metric. Hence there exists a countable subset $B \subseteq A$ which is dense in A. For $\xi \in \mathcal{H}$, we have

$$\xi \in \mathsf{H}_A \quad \text{if and only if} \quad U(A)^{-1}\xi \subseteq \mathsf{H}.$$

Now the closedness of H and the density of B in A show that this is equivalent to $U(B)^{-1}\xi \subseteq \mathsf{H}$, i.e., to $\xi \in \mathsf{H}_B$. This shows that $\mathsf{H}_A = \mathsf{H}_B$. We likewise obtain $\mathsf{H}_{m,A} = \mathsf{H}_{m,B}$ for every $m \in X$. Hence the assertion follows by applying (DI2) to $\mathsf{H}_B = \mathsf{H}_A$.

(b) follows from (a) and Lemma C.1. □

We refer to [8] for the basics on direct integrals.

Lemma C.4. *Let $\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x)$, a direct integral von Neumann algebra $\mathcal{A} = \int_X^\oplus \mathcal{A}_x d\mu(x)$ and a strongly continuous, unitary, direct integral representation of a connected Lie group G, $(U, \mathcal{H}) = \int_X^\oplus (U_x, \mathcal{H}_x) d\mu(x)$. Let $N \subset G$ a subset, then*

$$\bigcap_{g\in N}\mathcal{A}_g=\int_X^{\oplus}\bigcap_{g\in N}(\mathcal{A}_g)_x\,d\mu(x)$$

where $\mathcal{A}_g = U(g)\mathcal{A}U(g)^*$.

Proof. As G has at most countably many components, it carries a separable metric. Hence there exists a countable subset $N_0 \subseteq N$ which is dense in N. For $A \in B(\mathcal{H})$, the map

$$F\colon G\to B(\mathcal{H}),\quad F(g)=U(g)AU(g)^*,$$

is weak operator continuous, so that the set of all $g \in G$ with $F(g) \in \bigcap_{g\in N_0}\mathcal{A}_g$ is a closed subset, hence contains N. We conclude that

$$\bigcap_{g\in N_0}\mathcal{A}_g=\bigcap_{g\in N}\mathcal{A}_g.$$

We likewise obtain for every $x \in X$ the relation

$$\bigcap_{g\in N_0}\mathcal{A}_{x,g}=\bigcap_{g\in N}\mathcal{A}_{x,g}\quad\text{for}\quad\mathcal{A}_{x,g}=U_x(g)\mathcal{A}_xU_x(g)^*.$$

From [8, Prop. 4.4.6(b)] we thus obtain

$$\bigcap_{g\in N}\mathcal{A}_g=\bigcap_{g\in N_0}\mathcal{A}_g=\int_X^{\oplus}\bigcap_{g\in N_0}\mathcal{A}_{x,g}\,d\mu(x)=\int_X^{\oplus}\bigcap_{g\in N}\mathcal{A}_{x,g}\,d\mu(x).$$

Finally, we observe that, for every $g \in G$

$$\mathcal{A}_g=\int_X^{\oplus}(\mathcal{A}_g)_x\,d\mu(x)=\int_X^{\oplus}\mathcal{A}_{x,g}\,d\mu(x)$$

follows by the uniqueness of the direct integral decomposition. □

Appendix D. Some facts on (anti-)unitary representations

D.1. Standard subspaces in tensor products

Lemma D.1. *Suppose that* $(U, \mathcal{H}) = \otimes_{j=1}^n (U_j, \mathcal{H}_j)$ *is a tensor product of (anti-)unitary representations of* G_{τ_h}. *Then the standard subspace* $\mathtt{V} = \mathtt{V}(h, U)$ *is a tensor product*

$$\mathtt{V}=\mathtt{V}_1\otimes\cdots\otimes\mathtt{V}_n,$$

and for every non-empty subset $A \subseteq G$ *the subset* $\mathtt{V}_A := \bigcap_{g\in A} U(g)\mathtt{V}$ *satisfies*

$$\mathtt{V}_A \supseteq \mathtt{V}_{1,A} \otimes \cdots \otimes \mathtt{V}_{n,A}. \tag{80}$$

Proof. We have $\xi \in \mathtt{V}_A$ if and only if $U(A)^{-1}\xi \subseteq \mathtt{V}$. This shows that any $\xi = \xi_1 \otimes \cdots \otimes \xi_n$ with $\xi_j \in \mathtt{V}_{j,A}$ is contained in $\mathtt{V}_A$, which is (80). □

D.2. Existence of standard subspaces for unitary representations

The following theorem characterizes those Euler elements which, in every unitary representation, generate a modular group of some standard subspace.

Theorem D.2. (Euler elements generating modular groups) *Let* G *be a connected Lie group and* $h \in \mathfrak{g}$ *an Euler element. We consider the following assertions:*

(a) $h \in [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)]$.
(b) *For all quotients* $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$, $\mathfrak{n} \trianglelefteq \mathfrak{g}$, *in which the image of* h *is central, we have* $h \in \mathfrak{n}$, *so that its image in* $\mathfrak{q}$ *vanishes.*
(c) *For all unitary representation* $(U, \mathcal{H})$ *of* G, *the selfadjoint operator* $i\partial U(h)$ *is unitarily equivalent to* $-i\partial U(h)$.
(d) *For all unitary representation* $(U, \mathcal{H})$ *of* G, *there exists a standard subspace* $\mathtt{V}$ *such that* $\Delta_{\mathtt{V}} = e^{2\pi i\partial U(h)}$.

Then we have the implications

$$(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d),$$

and if G *is simply connected, then all assertions are equivalent.*

Proof. (a) $\Leftrightarrow$ (b): The ± 1-eigenspaces for the image of h in $\mathfrak{q}$ are the spaces $\mathfrak{q}_{\pm 1} = \mathfrak{g}_{\pm 1}(h)/\mathfrak{n}_{\pm 1}(h)$. That the image of h is central in $\mathfrak{q}$ means that both these spaces are trivial, i.e., that $\mathfrak{g}_{\pm 1}(h) \subseteq \mathfrak{n}$. As $\mathfrak{n}$ is a subalgebra, this means that

$$\mathfrak{i} := \mathfrak{g}_1(h) + \mathfrak{g}_{-1}(h) + [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)] \subseteq \mathfrak{n}.$$

As $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, condition (b) means that $h \in \mathfrak{i}$, but as $h \in \mathfrak{g}_0(h)$, this is equivalent to (a).

(b) $\Rightarrow$ (c): We argue by induction on $\dim G$. Passing to the quotient group $G/\ker(U)$, we may w.l.o.g. assume that U has discrete kernel. If h is central, then $h = 0$ by (b), so that (c) holds trivially because $\pm i\partial U(h) = 0$.

So we may assume that h is not central. Hence there exists a non-zero $x \in \mathfrak{g}_{\pm 1}(h)$. We consider the 2-dimensional subalgebra $\mathfrak{b} := \mathbb{R}h + \mathbb{R}x \cong \mathfrak{aff}(\mathbb{R})$ and the corresponding integral subgroup $B := \exp(\mathbb{R}x)\exp(\mathbb{R}h)$, which is isomorphic to $\mathrm{Aff}(\mathbb{R})_e$.

We may w.l.o.g. assume that $\mathcal{H}^G = \{0\}$ because (c) obviously holds for trivial representations. Then Moore's Theorem 5.1 implies that

$$\ker(\partial U(x)) \subseteq \mathcal{H}^{N_x}, \tag{81}$$

where $N_x \trianglelefteq G$ is a normal integral subgroup whose Lie algebra $\mathfrak{n}_x$ is the smallest ideal of $\mathfrak{g}$ such that the image $\overline{x}$ of x in the quotient Lie algebra $\mathfrak{g}/\mathfrak{n}_x$ is elliptic. As $x = \pm[h, x]$ is ad-nilpotent (the h-eigenspace decomposition implies that $(\mathrm{ad}\, x)^3 = 0$), its image $\overline{x}$ in $\mathfrak{g}/\mathfrak{n}_x$ must be central. So $\overline{x} = \pm[\overline{h}, \overline{x}] = 0$ implies $x \in \mathfrak{n}_x$. Using that N_x is a normal subgroup, we see that $\mathcal{H}^{N_x}$ is G-invariant, and the representation of G on this space factors through a representation of the quotient group $G/\overline{N_x}$ of strictly smaller dimension. By the induction hypothesis, our assertion holds for this representation.

We may therefore consider the representation of G on the orthogonal complement $(\mathcal{H}^{N_x})^\perp$. In view of (81), we may assume that $\ker(\partial U(x)) = \{0\}$. Then the restriction of U to the 2-dimensional subgroup B is a direct sum or irreducible representations of B in which x acts non-trivially, and every such representation is equivalent to one of the representations $(U_\pm, L^2(\mathbb{R}))$, where

$$(U_\pm(\exp(sx)\exp(th))f)(p) = e^{\pm ise^p} f(p+t) \quad \text{for} \quad s, t, p \in \mathbb{R} \tag{82}$$

(cf. [63, Prop. 2.38]). For both these representations, the operator $i\partial U_\pm(h)$ is equivalent to the selfadjoint operator $i\frac{d}{dp}$ on $L^2(\mathbb{R}, dp)$. This implies that $i\partial U(h)$ is unitarily equivalent to $-i\partial U(h)$.

(c) $\Leftrightarrow$ (d): The existence of a standard subspace $\mathtt{V}$ with $\Delta_{\mathtt{V}} = e^{2\pi i\partial U(h)}$ is equivalent to the existence of a conjugation J commuting with $\partial U(h)$. In view of [62, Prop. 3.1], this is equivalent to the existence of a unitary operator S with $Si\partial U(h)S^{-1} = -i\partial U(h)$. Therefore (c) and (d) are equivalent.

(c) $\Rightarrow$ (b): We assume that G is simply connected. If (b) is not satisfied, then there exists a quotient $\mathfrak{q} = \mathfrak{g}/\mathfrak{n}$ in which the image $\overline{h}$ of h is central but non-zero. Hence the corresponding quotient group $Q := G/N$ (as G is simply connected, N is closed and Q exists [33, Thms. 11.1.5, 11.1.21]) has a non-trivial irreducible unitary representation $(U, \mathcal{H})$ with $\partial U(\overline{h}) \neq 0$. The irreducibility of U implies that $\partial U(\overline{h}) = i\lambda\mathbf{1}$ for some $\lambda \in \mathbb{R}^\times$. Then $-i\partial U(\overline{h}) = \lambda\mathbf{1}$ is not unitarily equivalent to $-\lambda\mathbf{1} = i\partial U(\overline{h})$. Composing U with the quotient map $G \to Q$, we see that (c) cannot be satisfied. This shows that (c) implies (b). □

Corollary D.3. *If $\mathfrak{g}$ is semisimple and $h \in \mathfrak{g}$ is an Euler element, then there exists for every unitary representation $(U, \mathcal{H})$ of G a standard subspace $\mathtt{V}$ with $\Delta_{\mathtt{V}} = e^{2\pi i\partial U(h)}$.*

Proof. As all quotients of the semisimple Lie algebra $\mathfrak{g}$ are semisimple, hence have trivial center, condition (b) in Theorem D.2 is satisfied. □

Example D.4. (An example where $(c) \Rightarrow (b)$ fails) Consider the group $G_1 := \mathbb{T}^2 \times \widetilde{\mathrm{SL}}_2(\mathbb{R})$. Then $Z := Z(\widetilde{\mathrm{SL}}_2(\mathbb{R})) \cong \mathbb{Z}$, and there exists a homomorphism $\gamma\colon Z \to \mathbb{T}^2$ with dense range because the element $(e^{\pi i\sqrt{2}}, e^{\pi i\sqrt{3}})$ generates a dense subgroup of $\mathbb{T}^2$. Now

$$D := \{(\gamma(z), z)\colon z \in Z\}$$

is a discrete central subgroup in G_1, so that $G := G_1/D$ is a connected reductive Lie group with Lie algebra $\mathfrak{g} = \mathbb{R}^2 \oplus \mathfrak{sl}_2(\mathbb{R})$. Its commutator group (G, G) is the integral subgroup corresponding to $\mathfrak{sl}_2(\mathbb{R})$. As it contains a dense subgroup of the torus $\mathbb{T}^2$, it is dense in G.

Let $h = h_z + h_s \in \mathfrak{g}$ be an Euler element with $h_z \neq 0$ and $h_s \neq 0$. Then $\mathfrak{g}_{\pm 1}(h) = \mathfrak{g}_{\pm 1}(h_s) \subseteq \mathfrak{sl}_2(\mathbb{R})$ shows that (b) fails. We now verify (c), so that (c) does not imply (b) for all connected Lie groups.

Pick a non-zero $x \in \mathfrak{g}$ with $[h, x] = x$. As in the proof of "(b) $\Rightarrow$ (c)" above, we see that $x \in \mathfrak{n}_x$, so that $\mathfrak{sl}_2(\mathbb{R}) = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}_x$. Hence $(G, G) \subseteq N_x$, and the density of (G, G) implies $\overline{N_x} = G$. We conclude that, for every unitary representation $(U, \mathcal{H})$ of G, we have $\ker(\partial U(x)) = \mathcal{H}^G$. Clearly, (c) holds for the trivial representation of G on $\mathcal{H}^G$, and by the argument under "(b) $\Rightarrow$ (c)" it also holds for the representation on $\ker(\partial U(x))^\perp$. Therefore (c) holds for G.

Remark D.5. (a) If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then its simply connected covering $q_G\colon \widetilde{G} \to G$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$. All unitary representations of G yield by composition with q_G unitary representations of $\widetilde{G}$, but not all representations of $\widetilde{G}$ are obtained this way. If (c) holds for G, it may still fail for $\widetilde{G}$ (Example D.4).

(b) For a semidirect product $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ with $\mathfrak{r}$ solvable and $\mathfrak{s}$ semisimple, where h is an Euler element contained in $\mathfrak{s}$, the equivalence of (a) and (b) in Theorem D.2 implies that $h \in [\mathfrak{s}_1(h), \mathfrak{s}_{-1}(h)] \subseteq [\mathfrak{g}_1(h), \mathfrak{g}_{-1}(h)]$, so that Theorem D.2 applies to any simply connected Lie group G with Lie algebra $\mathfrak{g}$.

This argument applies in particular to the Poincaré Lie algebra $\mathfrak{g} = \mathbb{R}^{1,d} \rtimes \mathfrak{so}_{1,d}(\mathbb{R})$ and the Euler element $h \in \mathfrak{so}_{1,d}(\mathbb{R})$ generating a boost.

D.3. A criterion for real irreducibility

The following lemma is needed in the discussion of Example 4.22 below.

Proposition D.6. *Any irreducible unitary representation $(U, \mathcal{H})$ of G for which*

$$C_U \neq -C_U$$

is also irreducible as a real representation.

Proof. Let $(U^{\mathbb{R}}, \mathcal{H}^{\mathbb{R}})$ be the underlying real representation. Then its complexification is of the form $U^{\mathbb{R}}_{\mathbb{C}} \cong U \oplus \overline{U}$, as complex representations, where $C_{\overline{U}} = -C_U$. As $C_U \neq -C_U$, the representations U and $\overline{U}$ are not equivalent. Therefore the commutant of $U^{\mathbb{R}}_{\mathbb{C}}$ is isomorphic to $\mathbb{C}^2$, and this implies that the commutant of $U^{\mathbb{R}}(G)$ in $B(\mathcal{H}^{\mathbb{R}})$ cannot be larger than $\mathbb{C}\mathbf{1}$. Hence it contains no non-trivial projections, and thus $(U^{\mathbb{R}}, \mathcal{H}^{\mathbb{R}})$ is irreducible. □

Corollary D.7. *For any irreducible unitary positive energy representation* $(U, \mathcal{H})$ *of* $\widetilde{\mathrm{SL}}_2(\mathbb{R})$*, and any Euler element* $h \in \mathfrak{sl}_2(\mathbb{R})$*, the restriction to the subgroup* $P = \exp(\mathbb{R}h)\exp(\mathfrak{g}_1(h))$ *is irreducible as a real orthogonal representation.*

Proof. We know that, in all cases, the representation $U_P := U|_P$ of $P \cong \mathrm{Aff}(\mathbb{R})_e = \mathbb{R} \times \mathbb{R}_+$ is equivalent to the representation on $L^2(\mathbb{R}_+, \mathbb{C})$, given by

$$(U_P(b,a)f)(p) = a^{1/2} e^{ibp} f(ap).$$

Hence $(U_P, \mathcal{H})$ is the unique irreducible positive energy representation of P. Now the assertion follows from Proposition D.6. □

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