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On a Heisenberg-Type Uncertainty Principle in von Neumann Algebras

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Abstract

A refinement of the Heisenberg uncertainty principle has been proved by Luo using Wigner–Yanase information. Generalizations of this result have been proved by Yanagi and by other scholars for regular Quantum Fisher Information in the matrix case. In this paper, we prove these results in the von Neumann algebra setting.

Keywords: uncertainty principle; Wigner–Yanase–Dyson information; operator monotone functions

MSC: Primary 62B10, 94A17; Secondary 46L30, 46L60

1. Introduction

Starting from the Wigner–Yanase information $I_\rho(A) := \frac{1}{2}\mathrm{Tr}\big((i[\rho, A])^2\big)$ and the variance $V_\rho(A) := \mathrm{Tr}(\rho A^2) - \big(\mathrm{Tr}(\rho A)\big)^2$, Luo introduced in [1] a new measure of quantum uncertainty as

$$U_\rho(A) := \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho(A))^2}.$$

and proved an uncertainty principle for U in the form

$$U_\rho(A) \cdot U_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2. \tag{1}$$

The inequality can be seen as a refinement of the Heisenberg uncertainty principle

$$V_\rho(A) \cdot V_\rho(B) \geq \frac{1}{4}|\mathrm{Tr}(\rho[A, B])|^2.$$

Wigner–Yanase information is an example of Quantum Fisher Information. The family of all Quantum Fisher Information is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ (see below). To each function f of the class $\mathcal{F}_{op}$ one may associate the metric adjusted skew information $I^f_\rho(A)$, so one can define also a generalized quantum uncertainty U^f by the formula

$$U^f_\rho(A) := \sqrt{V_\rho(A)^2 - (V_\rho(A) - I^f_\rho(A))^2}.$$

It is natural to conjecture that inequality (1) is a particular case of a general inequality

$$U^f_\rho(A) \cdot U^f_\rho(B) \geq f(0)|\mathrm{Tr}(\rho[A, B])|^2, \qquad f \in \mathcal{F}^r_{op}. \tag{2}$$

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Actually, this is not the case and the inequality is true only for a proper subset of $\mathcal{F}_{op}^r$ [2,3]. To have a general uncertainty relation for U^f one has to use the smaller constant $f(0)^2$. Indeed the main result in [4] was the following inequality

$$U_\rho^f(A) \cdot U_\rho^f(B) \geq f(0)^2 |\mathrm{Tr}(\rho[A,B])|^2, \qquad f \in \mathcal{F}_{op}^r. \tag{3}$$

As for this article, we want to extend the validity of inequalities (3) and (2) to unbounded operators affiliated to a von Neumann algebra.

The use of von Neumann algebras in Quantum Information is not new, starting from the work of Umegaki in the 1960's, and it has been popular ever since, as it is witnessed, for example, by the books [5–10] and the papers [11–15].

In Section 2 we recall some notions on operator monotone functions. In Section 3 we prove some auxiliary lemmas, revolving around sesquilinear forms on the Hilbert space of the standard representation of a von Neumann algebra. In Section 4 we prove inequalities (3) and (2) for unbounded operators affiliated to a von Neumann algebra, respectively, in Theorem 2 and in Theorem 3.

2. Operator Monotone Functions

In quantum probability, operator monotone functions are extremely relevant because they parametrize Quantum Fisher Information(s) and Quantum Covariances.

Let $M_n := M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices. A function $f : (0, +\infty) \to \mathbb{R}$ is an *operator monotone (increasing)* if, for any $n \in \mathbb{N}$, and $A, B \in M_n$ such that $0 \leq A \leq B$, the inequalities $0 \leq f(A) \leq f(B)$ hold. An operator monotone function is *symmetric* if $f(x) = xf(x^{-1})$ and *normalized* if $f(1) = 1$.

Definition 1. *$\mathcal{F}_{op}$ is the class of functions $f : (0, +\infty) \to (0, +\infty)$ such that*

(i) f is operator monotone.

(ii) $f(1) = 1$.

(iii) $tf(t^{-1}) = f(t)$.

Example 1. *Examples of elements of $\mathcal{F}_{op}$ are given by the following list:*

$$f_{RLD}(x) := \frac{2x}{x+1}, \qquad f_{WY}(x) := \left(\frac{1+\sqrt{x}}{2}\right)^2,$$

$$f_{SLD}(x) := \frac{1+x}{2}, \qquad f_\alpha(x) := \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \qquad \alpha \in \left(0, \frac{1}{2}\right].$$

To each function $f \in \mathcal{F}_{op}$ one may associate an operator mean (Kubo-Ando [16]), a Quantum Fisher Information (Petz [17]); and a Quantum Covariance (Petz [18]) according to the following formulas:

$$m_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$
$$\langle A, B \rangle_{\rho,f} = \mathrm{Tr}(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)),$$
$$\mathrm{Cov}_\rho^f(A,B) = \mathrm{Tr}(A_0 \cdot m_f(L_\rho, R_\rho)(B_0)).$$

A number of applications in quantum information derive from the above Kubo–Ando–Petz results.

Remark 1. *Any $f \in \mathcal{F}_{op}$ satisfies*

$$\frac{2x}{1+x} \leq f(x) \leq \frac{1+x}{2}, \qquad \forall x > 0.$$

For $f \in \mathcal{F}_{op}$ define $f(0) := \lim_{x \to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{op}^{r} := \{f \in \mathcal{F}_{op} \mid f(0) \neq 0\}, \quad \mathcal{F}_{op}^{n} := \{f \in \mathcal{F}_{op} \mid f(0) = 0\}$$

and notice that trivially $\mathcal{F}_{op} = \mathcal{F}_{op}^{r} \dot{\cup} \mathcal{F}_{op}^{n}$.

Definition 2. *For $f \in \mathcal{F}_{op}^{r}$ we set*

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2 \frac{f(0)}{f(x)}\right] \qquad x > 0.$$

Theorem 1 ([19])**.** *The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_{op}^{r}$ and $\mathcal{F}_{op}^{n}$.*

Recently this result has proven to be useful in different parts of Quantum Information. The first example is the Dynamical Uncertainty Principle, which is the first inequality setting a quantum bound for the generalized variance of an odd number of observables, contrary to the Schrödinger–Robertson uncertainty principle [15,20].

The second example deals with the construction of quantifiers of quantum discord. Two of the most used ones are Local Quantum Uncertainty (LQU) and Interferometric Power (IP) [21,22]. Using the above theorem it has recently been proven that LQU and IP are just examples of a class of quantifiers parametrized by the functions $f \in \mathcal{F}_{op}^{r}$ [23].

See [24] for a recent survey of the applications of this correspondence.

We recall two inequalities that we use in the sequel.

Proposition 1. *Any $f \in \mathcal{F}_{op}$ satisfies*

$$f(0)^2(x-1)^2 \leq \frac{1}{4}(x+1)^2 - \tilde{f}(x)^2, \qquad x > 0.$$

If, moreover, $\frac{1}{2}(x+1) + \tilde{f}(x) \geq 2f(x)$, $\forall x > 0$, then

$$f(0)(x-1)^2 \leq \frac{1}{4}(x+1)^2 - \tilde{f}(x)^2, \qquad x > 0.$$

Proof. The first inequality is proved in [4]. The second inequality comes from ([3], Lemma 4.1). □

3. Auxiliary Lemmas

Let $\mathcal{M}$ be a von Neumann algebra, and ω a normal faithful state on $\mathcal{M}$. Associated with ω are a Hilbert space $\mathcal{H}_\omega$, a normal faithful representation π_ω of $\mathcal{M}$ on $\mathcal{B}(\mathcal{H}_\omega)$, and a cyclic and separating vector $\xi_\omega \in \mathcal{H}_\omega$, such that $\omega(a) = \langle \xi_\omega, \pi_\omega(a)\xi_\omega \rangle$, $a \in \mathcal{M}$. These are called the GNS (for Gelfang–Naimark–Segal) Hilbert space, representation, and vector, respectively. Consider now the antilinear densely defined operator $S_\omega a \xi_\omega := a^* \xi_\omega$, $a \in \mathcal{M}$. Its polar decomposition is $S_\omega = J_\omega \Delta_\omega^{1/2}$, where J_ω is an antilinear unitary, called modular conjugation, and Δ_ω is a positive unbounded self-adjoint operator, called modular operator. See [25,26] for more information on the general theory of von Neumann algebras. In the sequel, we denote by $T \widehat{\in} \mathcal{M}$ the fact that T is a closed, densely defined linear operator on $\mathcal{H}_\omega$, and is affiliated with $\mathcal{M}$.

For the reader's convenience, we recall the following folklore result.

Lemma 1. $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T \widehat{\in} \mathcal{M}, \xi_\omega \in \mathcal{D}(T) \cap \mathcal{D}(T^*)\}$.

Proof. See, e.g., ([27], Lemma 3.6). $\square$

To deal with unbounded operators, we introduce some sesquilinear forms on $\mathcal{H}$, and take [28] as our standard reference.

Definition 3. *Let $f \in \mathcal{F}_{op}$, and define the following sesquilinear forms*

$$\begin{aligned}
\mathscr{E}(\xi,\eta) &:= \langle \Delta_\omega^{1/2}\xi\,,\,\Delta_\omega^{1/2}\eta\rangle, \quad \xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2}),\\
\mathscr{E}_1(\xi,\eta) &:= \mathscr{E}(\xi,\eta) + \langle \xi\,,\,\eta\rangle, \quad \xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2}),\\
\mathscr{F}^f(\xi,\eta) &:= \langle f(\Delta_\omega)^{1/2}\xi\,,\,f(\Delta_\omega)^{1/2}\eta\rangle, \quad \xi,\eta \in \mathcal{D}(f(\Delta_\omega)^{1/2}),\\
\mathscr{G}^f(\xi,\eta) &:= \mathscr{F}^{f_{SLD}}(\xi,\eta) - \mathscr{F}^{\tilde f}(\xi,\eta), \quad \xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2}).
\end{aligned}$$

Remark 2. *The form $\mathscr{G}^f$ is used to define the f-correlation, while the others are introduced for technical purposes.*

Proposition 2. *Let $f \in \mathcal{F}_{op}$, and $\Delta_\omega = \int_0^\infty t\,dE(t)$ its spectral decomposition. Then, for any $\xi,\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, one has*

$$\begin{aligned}
\mathscr{F}_\omega^f(\xi,\eta) &= \int_0^\infty f(t)\,d\langle \xi\,,\,E(t)\eta\rangle,\\
\mathscr{G}_\omega^f(\xi,\eta) &= \int_0^\infty \left(\frac{1}{2}(1+t) - \tilde f(t)\right) d\langle \xi\,,\,E(t)\eta\rangle\,.
\end{aligned}$$

Proof. It follows from ([29], Prop. 4.15) that

$$\mathscr{F}_\omega^f(\xi,\eta) = \langle f(\Delta_\omega^{1/2})\xi\,,\,f(\Delta_\omega^{1/2})\eta\rangle = \int_0^{+\infty} \overline{f(t)^{1/2}} f(t)^{1/2}\,d\langle \xi\,,\,E(t)\eta\rangle = \int_0^{+\infty} f(t)\,d\langle \xi\,,\,E(t)\eta\rangle.$$

Then

$$\mathscr{G}_\omega^f(\xi,\eta) = \mathscr{F}_\omega^{f_{SLD}}(\xi,\eta) - \mathscr{F}_o^{\tilde f}(\xi,\eta) = \int_0^{+\infty} \frac{1}{2}(1+t)\,d\langle \xi\,,\,E(t)\eta\rangle - \int_0^{+\infty} \tilde f(t)\,d\langle \xi\,,\,E(t)\eta\rangle\,.$$

$\square$

Remark 3. (1) *It follows from ([28], example VI.1.13) that $\mathscr{E}$ and $\mathscr{F}^f$ [so also $\mathscr{E}_1$ and $\mathscr{G}^f$] are closed, positive, and symmetric sesquilinear forms.*
(2) *Observe that $\mathscr{F}^{f_{SLD}}(\xi,\eta) = \frac{1}{2}\mathscr{E}(\xi,\eta) + \frac{1}{2}\langle \xi\,,\,\eta\rangle = \frac{1}{2}\mathscr{E}_1(\xi,\eta)$. Indeed*

$$\begin{aligned}
\mathscr{F}^{f_{SLD}}(\xi,\eta) &= \langle f_{SLD}(\Delta_\omega)^{1/2}\xi\,,\,f_{SLD}(\Delta_\omega)^{1/2}\eta\rangle\\
&= \frac{1}{2}\langle (1+\Delta_\omega)^{1/2}\xi\,,\,(1+\Delta_\omega)^{1/2}\eta\rangle\\
&= \frac{1}{2}\langle \Delta_\omega^{1/2}\xi\,,\,\Delta_\omega^{1/2}\eta\rangle + \frac{1}{2}\langle \xi\,,\,\eta\rangle.
\end{aligned}$$

Some properties of the quadratic forms $\mathscr{F}^f$ are contained in the following Lemma.

Lemma 2. (1) $\mathcal{D}(\mathscr{F}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$,
(2) $f, g \in \mathcal{F}_{op}$, $f \le g \implies \mathscr{F}^f(\xi,\xi) \le \mathscr{F}^g(\xi,\xi)$*, for any* $\xi \in \mathcal{D}(\Delta_\omega^{1/2})$.
(3) $\mathscr{G}^f$ *is a symmetric sesquilinear form on* $\mathcal{D}(\mathscr{G}^f) \supset \mathcal{D}(\Delta_\omega^{1/2})$*, which is positive on* $\mathcal{D}(\Delta_\omega^{1/2})$.

Proof. (1) Since $f \in \mathcal{F}_{op} \implies 0 \le f(x) \le \frac{1}{2}(x+1)$, $\forall x > 0$, one has $\mathcal{D}(\Delta_\omega^{1/2}) \subset \mathcal{D}(f(\Delta_\omega)^{1/2})$.
(2) It is obvious.
(3) It follows from Lemma 2 and Remark 1. □

Since we consider only self-adjoint operators (i.e., observables, in physics parlance) we introduce the following definition.

Definition 4. *Set* $\mathcal{X}_\omega := \{A \widehat{\in} \mathcal{M} : A = A^*, \xi_\omega \in \mathcal{D}(A)\}$.

The following Proposition is used to prove that the f-correlation is symmetric.

Proposition 3. *Let* $A, B \in \mathcal{X}_\omega$, *and* $f \in \mathcal{F}_{op}$. *Then* $\mathscr{F}^f(B_0\xi_\omega, A_0\xi_\omega) = \mathscr{F}^f(A_0\xi_\omega, B_0\xi_\omega) \in \mathbb{R}$.

Proof. Indeed

$$\begin{aligned}
\mathscr{F}^f(B_0\xi_\omega, A_0\xi_\omega) &= \langle f(\Delta_\omega)^{1/2}B_0\xi_\omega\,,\, f(\Delta_\omega)^{1/2}A_0\xi_\omega\rangle \\
&= \langle f(\Delta_\omega)^{1/2}J_\omega\Delta_\omega^{1/2}B_0\xi_\omega\,,\, f(\Delta_\omega)^{1/2}J_\omega\Delta_\omega^{1/2}A_0\xi_\omega\rangle \\
&= \langle J_\omega f(\Delta_\omega)^{1/2}J_\omega\Delta_\omega^{1/2}A_0\xi_\omega\,,\, J_\omega f(\Delta_\omega)^{1/2}J_\omega\Delta_\omega^{1/2}B_0\xi\rangle \\
&= \langle f(\Delta_\omega^{-1})^{1/2}\Delta_\omega^{1/2}A_0\xi_\omega\,,\, f(\Delta^{-1})^{1/2}\Delta_\omega^{1/2}B_0\xi_\omega\rangle \\
&\overset{(a)}{=} \langle f(\Delta_\omega)^{1/2}A_0\xi_\omega\,,\, f(\Delta_\omega)^{1/2}B_0\xi_\omega\rangle = \mathscr{F}^f(A_0\xi_\omega, B_0\xi_\omega),
\end{aligned}$$

where in (a) we used $\tilde{f}(x^{-1}) = x^{-1}\tilde{f}(x)$, for $x > 0$.
Therefore

$$\begin{aligned}
\overline{\mathscr{F}^f(A_0\xi_\omega, B_0\xi_\omega)} &= \langle f(\Delta_\omega)^{1/2}A_0\xi_\omega\,,\, f(\Delta_\omega)^{1/2}B_0\xi_\omega\rangle^- = \langle f(\Delta_\omega)^{1/2}B_0\xi_\omega\,,\, f(\Delta_\omega)^{1/2}A_0\xi_\omega\rangle \\
&= \mathscr{F}^f(B_0\xi_\omega, A_0\xi_\omega) = \mathscr{F}^f(A_0\xi_\omega, B_0\xi_\omega)\,.
\end{aligned}$$

□

We can now introduce the main objects of study.

Definition 5. *For any* $f \in \mathfrak{F}_{op}$, *and any* $A, B \in \mathcal{X}_\omega$, *we set* $A_0 := A - \langle\xi_\omega, A\xi_\omega\rangle I$, $B_0 := B - \langle\xi_\omega, B\xi_\omega\rangle I$, *and define the bilinear forms*

$$\begin{aligned}
\mathrm{Cov}_\omega(A,B) &:= \mathrm{Re}\langle A_0\xi_\omega\,,\, B_0\xi_\omega\rangle, \\
\mathrm{Var}_\omega(A) &:= \mathrm{Cov}_\omega(A,A), \\
\mathrm{Corr}^f_\omega(A,B) &:= \mathrm{Re}\langle A_0\xi_\omega\,,\, B_0\xi_\omega\rangle - \langle\tilde{f}(\Delta_\omega)^{1/2}A_0\xi_\omega\,,\, \tilde{f}(\Delta_\omega)^{1/2}B_0\xi_\omega\rangle \\
&= \mathrm{Cov}_\omega(A,B) - \mathscr{F}^{\tilde{f}}(A_0\xi_\omega, B_0\xi_\omega), \\
I^f_\omega(A) &:= \mathrm{Corr}^f_\omega(A,A).
\end{aligned}$$

We refer to Corr^f *as the* f*-correlation and to* I^f *as the metric adjusted skew information associated to* f.

Remark 4. *Observe that in the matrix case* $\omega = \mathrm{Tr}(\rho\,\cdot)$, *for some density matrix* ρ, *and* $\Delta_\omega = L_\rho R_\rho^{-1}$, *so that the previous Definition is a true generalization of covariance and* f*-correlation in the matrix case.*

What follows provides the link between $\mathscr{G}^f$ and f-correlation.

Lemma 3. *Let* $A, B \in \mathcal{X}_\omega$, *and* $f \in \mathfrak{F}_{op}$. *Then*

(i) $\mathrm{Cov}_\omega(A,B) = \frac{1}{2}\mathscr{E}_1(A_0\xi_\omega, B_0\xi_\omega)$ *is a positive symmetric bilinear form;*

(ii) $\mathrm{Corr}^f_\omega(A,B) = \mathscr{G}^f_\omega(A_0\xi_\omega, B_0\xi_\omega) = \frac{1}{2}\mathscr{E}_1(A_0\xi_\omega, B_0\xi_\omega) - \mathscr{F}^{\tilde{f}}(A_0\xi_\omega, B_0\xi_\omega)$ *is a positive symmetric bilinear form.*

Proof. (i) Observe that

$$\begin{aligned}\langle B_0\xi_\omega\,,\, A_0\xi_\omega\rangle &= \langle B_0^*\xi_\omega\,,\, A_0^*\xi_\omega\rangle = \langle J_\omega\Delta_\omega^{1/2}B_0\xi_\omega\,,\, J_\omega\Delta_\omega^{1/2}A_0\xi_\omega\rangle \\ &= \langle \Delta_\omega^{1/2}A_0\xi_\omega\,,\, \Delta_\omega^{1/2}B_0\xi_\omega\rangle = \mathscr{E}(A_0\xi_\omega, B_0\xi_\omega).\end{aligned}$$

Then

$$\begin{aligned}2\,\mathrm{Cov}_\omega(A,B) &= \langle A_0\xi_\omega\,,\, B_0\xi_\omega\rangle + \langle B_0\xi_\omega\,,\, A_0\xi_\omega\rangle \\ &= \langle A_0\xi_\omega\,,\, B_0\xi_\omega\rangle + \mathscr{E}(A_0\xi_\omega, B_0\xi_\omega) = \mathscr{E}_1(A_0\xi_\omega, B_0\xi_\omega).\end{aligned}$$

The thesis follows from this and the fact that $\mathcal{D}(\Delta_\omega^{1/2}) = \{T\xi_\omega : T\widehat{\in}\mathcal{M}, \xi_\omega \in \mathcal{D}(T)\cap\mathcal{D}(T^*)\}$.
(ii) It follows from (i) and Lemma 2 (ii). $\square$

4. A Refinement of Heisenberg Uncertainty Relation

To each function f of the class $\mathcal{F}_{op}$ one may associate the metric adjusted skew information $I^f_\rho(A)$, a generalization of WYD information, and therefore one can define also a generalized quantum variance U^f by the formula

$$\begin{aligned}U^f_\omega(A) &:= \sqrt{\mathrm{Var}_\omega(A)^2 - (\mathrm{Var}_\omega(A) - I^f_\omega(A))^2}, \\ J^f_\omega(A) &:= 2\,\mathrm{Var}_\omega(A) - I^f_\omega(A).\end{aligned}$$

Proposition 4. *Set,* $\forall\xi \in \mathcal{D}(\Delta_\omega^{1/2})$,

$$\begin{aligned}\mathscr{U}^f(\xi) &:= \sqrt{\mathscr{F}^{sld}(\xi,\xi)^2 - \mathscr{F}^{\tilde{f}}(\xi,\xi)^2}, \\ \mathscr{I}^f(\xi) &:= \mathscr{F}^{sld}(\xi,\xi) - \mathscr{F}^{\tilde{f}}(\xi,\xi) = \mathscr{G}^f(\xi,\xi), \\ \mathscr{J}^f(\xi) &:= \mathscr{F}^{sld}(\xi,\xi) + \mathscr{F}^{\tilde{f}}(\xi,\xi).\end{aligned}$$

Then, for any $A \in \mathcal{X}_\omega$*, one has*
(1) $I^f_\omega(A) = \mathscr{I}^f(A_0\xi_\omega)$;
(2) $J^f_\omega(A) = \mathscr{J}^f(A_0\xi_\omega)$;
(3) $U^f_\omega(A) = \sqrt{I^f_\omega(A)\cdot J^f_\omega(A)}$;
(4) $U^f_\omega(A) = \mathscr{U}^f(A_0\xi_\omega)$;
(5) $0 \le I^f_\omega(A) \le U^f_\omega(A) \le \mathrm{Var}_\omega(A)$.

Proof. (1) From Remark 3, $\mathscr{F}^{f_{SLD}} = \frac{1}{2}\mathscr{E}_1$, and Definition 5 it follows that

$$\begin{aligned}I^f_\omega(A) &= \mathrm{Corr}^f_\omega(A,A) = \mathrm{Cov}_\omega(A,A) - \mathscr{F}^{\tilde{f}}(A_0\xi_\omega, A_0\xi_\omega) \\ &= \mathscr{F}^{f_{SLD}}(A_0\xi_\omega, A_0\xi_\omega) - \mathscr{F}^{\tilde{f}}(A_0\xi_\omega, A_0\xi_\omega) = \mathscr{I}^f(A_0\xi_\omega).\end{aligned}$$

(2) One has

$$\begin{aligned} J^f_\omega(A) &= 2\,\mathrm{Var}_\omega(A) - I^f_\omega(A) \\ &= 2\mathscr{F}^{f_{SLD}}(A_0\xi_\omega, A_0\xi_\omega) - (\mathscr{F}^{f_{SLD}}(A_0\xi_\omega, A_0\xi_\omega) - \mathscr{F}^{\tilde{f}}(A_0\xi_\omega, A_0\xi_\omega)) \\ &= \mathscr{F}^{f_{SLD}}(A_0\xi_\omega, A_0\xi_\omega) + \mathscr{F}^{\tilde{f}}(A_0\xi_\omega, A_0\xi_\omega) = \mathscr{J}^f(A_0\xi_\omega). \end{aligned}$$

(3) One has

$$\begin{aligned} U^f_\omega(A)^2 &= \mathrm{Var}_\omega(A)^2 - (\mathrm{Var}_\omega(A) - I^f_\omega(A))^2 \\ &= 2\,\mathrm{Var}_\omega(A) I^f_\omega(A) - I^f_\omega(A)^2 = I^f_\omega(A)\big(2\,\mathrm{Var}_\omega(A) - I^f_\omega(A)\big) = I^f_\omega(A)\cdot J^f_\omega(A). \end{aligned}$$

(4) It follows from (1)-(3).
(5) Since $I^f_\omega(A) = \mathrm{Var}_\omega(A) - \mathscr{F}^{\tilde{f}}_\omega(A_0\xi_\omega, A_0\xi_\omega) \le \mathrm{Var}_\omega(A)$, one has

$$U^f_\omega(A)^2 = 2\,\mathrm{Var}_\omega(A) I^f_\omega(A) - I^f_\omega(A)^2 \ge I^f_\omega(A)^2.$$

The last inequality is obvious. □

Before proving the main result of this paper, let us recall a version of Heisenberg uncertainty relation.

Proposition 5. *Let $\mathcal{H}$ be a Hilbert space, $\xi \in \mathcal{H}$, $\|\xi\| = 1$, $A, B \in \mathcal{L}(\mathcal{H})$ self-adjoint, and such that $\xi \in \mathcal{D}(A) \cap \mathcal{D}(B)$, $A_0 := A - \langle \xi\,,\, A\xi\rangle I$, $B_0 := B - \langle \xi\,,\, B\xi\rangle I$. Then*
(1) $|\mathrm{Im}\,\langle A_0\xi\,,\, B_0\xi\rangle| \le \|A_0\xi\|\,\|B_0\xi\|$;
(2) *if* $\xi \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$, $\frac{1}{2}|\langle \xi\,,\, i[A,B]\xi\rangle| \le \|A_0\xi\|\,\|B_0\xi\|$.

Proof. See [30]. □

We are ready for the main result of the paper.

Theorem 2. *Let $A, B \widehat{\in} \mathcal{M}$ self-adjoint, $\xi_\omega \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$, $A_0\xi_\omega, B_0\xi_\omega \in \mathcal{D}(\Delta_\omega)$. Then*

$$f(0)^2|\langle \xi_\omega\,,\, [A,B]\xi_\omega\rangle|^2 \le U^f_\omega(A) U^f_\omega(B).$$

Proof. One has

$$\begin{aligned} |\langle \xi_\omega\,,\, [A,B]\xi_\omega\rangle| &\overset{(a)}{=} |2i\,\mathrm{Im}\langle A_0\xi_\omega\,,\, B_0\xi_\omega\rangle| = |\langle A_0\xi_\omega\,,\, B_0\xi_\omega\rangle - \langle B_0\xi_\omega\,,\, A_0\xi_\omega\rangle| \\ &\overset{(b)}{=} |\langle A_0\xi_\omega\,,\, B_0\xi_\omega\rangle - \langle A_0\xi_\omega\,,\, \Delta_\omega B_0\xi_\omega\rangle| = |\langle A_0\xi_\omega\,,\, (I - \Delta_\omega)B_0\xi_\omega\rangle|, \end{aligned}$$

where in (a) we used Proposition 5, and in (b) a formula proved in Proposition 3. If $I - \Delta_\omega = V|I - \Delta_\omega|$ is its polar decomposition, one has

$$\begin{aligned} |\langle \xi_\omega\,,\, [A,B]\xi_\omega\rangle|^2 &= |\langle A_0\xi_\omega\,,\, V|I - \Delta_\omega|B_0\xi_\omega\rangle|^2 \\ &= |\langle |I - \Delta_\omega|^{1/2} V^* A_0\xi_\omega\,,\, |I - \Delta_\omega|^{1/2} B_0\xi_\omega\rangle|^2 \\ &\le \||I - \Delta_\omega|^{1/2} V^* A_0\xi_\omega\|^2\,\||I - \Delta_\omega|^{1/2} B_0\xi_\omega\|^2 \\ &= \langle |I - \Delta_\omega|^{1/2} V^* A_0\xi_\omega\,,\, |I - \Delta_\omega|^{1/2} V^* A_0\xi_\omega\rangle\langle |I - \Delta_\omega|^{1/2} B_0\xi_\omega\,,\, |I - \Delta_\omega|^{1/2} B_0\xi_\omega\rangle \\ &= \langle A_0\xi_\omega\,,\, V|I - \Delta_\omega|V^* A_0\xi_\omega\rangle\langle B_0\xi_\omega\,,\, |I - \Delta_\omega|B_0\xi_\omega\rangle \\ &\overset{(c)}{=} \langle A_0\xi_\omega\,,\, |I - \Delta_\omega|A_0\xi_\omega\rangle\langle B_0\xi_\omega\,,\, |I - \Delta_\omega|B_0\xi_\omega\rangle, \end{aligned}$$

where in (c) we used a property of the polar decomposition (see exercise 7.26 in [31], page 199). Let $\Delta_\omega = \int_0^{+\infty} \lambda dE(\lambda)$ be the spectral decomposition of Δ_ω. Then, for any $\xi \in \mathcal{D}(\Delta_\omega) \subset \mathcal{D}(\Delta_\omega^{1/2})$, one has

$$f(0)\langle \xi\,,\, |I-\Delta_\omega|\xi\rangle = \int_0^{+\infty} f(0)|1-\lambda| d\langle \xi\,,\, E(\lambda)\xi\rangle \overset{(d)}{\leq} \int_0^{+\infty} \left(\frac{1}{4}(\lambda+1)^2 - \widetilde{f}(\lambda)^2\right)^{1/2} d\langle \xi\,,\, E(\lambda)\xi\rangle$$
$$\leq \left(\int_0^{+\infty} \left(\frac{1}{2}(\lambda+1) - \widetilde{f}(\lambda)\right) d\langle \xi\,,\, E(\lambda)\xi\rangle\right)^{1/2} \left(\int_0^{+\infty} \left(\frac{1}{2}(\lambda+1) + \widetilde{f}(\lambda)\right) d\langle \xi\,,\, E(\lambda)\xi\rangle\right)^{1/2}, \tag{4}$$

where in (d) we used Proposition 1. Observe that, in analogy to Proposition 2, for any $\eta \in \mathcal{D}(\Delta_\omega^{1/2})$, one has

$$\mathscr{I}^f(\eta) = \int_0^{+\infty} \left(\frac{1}{2}(1+t) - \tilde{f}(t)\right) d\langle \eta\,,\, E(t)\eta\rangle,$$
$$\mathscr{J}^f(\eta) = \int_0^{+\infty} \left(\frac{1}{2}(1+t) + \tilde{f}(t)\right) d\langle \eta\,,\, E(t)\eta\rangle,$$

so that, if $\xi \in \mathcal{D}(\Delta_\omega)$, one has

$$f(0)\langle \xi\,,\, |I-\Delta_\omega|\xi\rangle \leq \sqrt{\mathscr{I}^f(\xi)\mathscr{J}^f(\xi)} = \mathscr{U}^f(\xi).$$

Therefore

$$f(0)^2 |\langle \xi_\omega\,,\, [A,B]\xi_\omega\rangle|^2 \leq \mathscr{U}^f(A_0\xi_\omega)\mathscr{U}^f(B_0\xi_\omega) = U^f_\omega(A)U^f_\omega(B).$$

□

Theorem 3. *Let $f \in \mathcal{F}_{op}$ be such that $\frac{1}{2}(x+1) + \tilde{f}(x) \geq 2f(x)$, $\forall x \in (0,+\infty)$, and $A, B \widehat{\in} \mathcal{M}$ self-adjoint, $\xi_\omega \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$, $A_0\xi_\omega, B_0\xi_\omega \in \mathcal{D}(\Delta_\omega)$. Then*

$$f(0)|\langle \xi_\omega\,,\, [A,B]\xi_\omega\rangle|^2 \leq U^f_\omega(A)U^f_\omega(B).$$

Proof. It is analogous to that of Proposition 2, but for inequality (4), which must be substituted with

$$\sqrt{f(0)}\langle \xi\,,\, |I-\Delta_\omega|\xi\rangle = \int_0^{+\infty} \sqrt{f(0)}|1-\lambda| d\langle \xi\,,\, E(\lambda)\xi\rangle \overset{(a)}{\leq} \int_0^{+\infty} \left(\frac{1}{4}(\lambda+1)^2 - \widetilde{f}(\lambda)^2\right)^{1/2} d\langle \xi\,,\, E(\lambda)\xi\rangle$$
$$\leq \left(\int_0^{+\infty} \left(\frac{1}{2}(\lambda+1) - \widetilde{f}(\lambda)\right) d\langle \xi\,,\, E(\lambda)\xi\rangle\right)^{1/2} \left(\int_0^{+\infty} \left(\frac{1}{2}(\lambda+1) + \widetilde{f}(\lambda)\right) d\langle \xi\,,\, E(\lambda)\xi\rangle\right)^{1/2},$$

where in (a) we used Proposition 1. □

Remark 5. *The condition in Theorem 3 allows us to verify that inequality (2) is not only satisfied by the functions f_α, but also by some other $f \in \mathcal{F}_{op}$. For example, the function $f(t) = f_{1/2}(t^{2/3})^{3/2} = \left(\frac{1+\sqrt[3]{t}}{2}\right)^3$ is in $\mathcal{F}_{op}$, as a consequence of [32], Corollary 4.3 (i). Moreover, f satisfies the condition in Theorem 3, since, setting $g(t) := \frac{t+1}{2} + \widetilde{f}(t) - 2f(t) = t + 1 - \frac{(t-1)^2}{2(\sqrt[3]{t}+1)^3} - \frac{(\sqrt[3]{t}+1)^3}{8}$, it suffices to prove $g(t) \geq 0$, $\forall t > 0$. Indeed, $g(0) = \frac{3}{8}$, and $g'(t) = \frac{3-6t^{1/3}+t^{2/3}+12t+33t^{4/3}+18t^{5/3}+3t^2}{8t^{2/3}(1+t^{1/3})^4} \geq 0$, for all $t > 0$, since it is easy to prove that $h(t) := 3 - 6t^{1/3} + 12t \geq 0$, $\forall t > 0$.*

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