Algebra & Number Theory

Volume 7

2013

No. 8

The geometry and combinatorics of cographic toric face rings

Sebastian Casalaina-Martin, Jesse Leo Kass and Filippo Viviani

msp

ALGEBRA AND NUMBER THEORY 7:8 (2013)
dx.doi.org/10.2140/ant.2013.7.1781

The geometry and combinatorics of cographic toric face rings

Sebastian Casalaina-Martin, Jesse Leo Kass and Filippo Viviani

In this paper, we define and study a ring associated to a graph that we call the cographic toric face ring or simply the cographic ring. The cographic ring is the toric face ring defined by the following equivalent combinatorial structures of a graph: the cographic arrangement of hyperplanes, the Voronoi polytope, and the poset of totally cyclic orientations. We describe the properties of the cographic ring and, in particular, relate the invariants of the ring to the invariants of the corresponding graph.

Our study of the cographic ring fits into a body of work on describing rings constructed from graphs. Among the rings that can be constructed from a graph, cographic rings are particularly interesting because they appear in the study of compactified Jacobians of nodal curves.

Introduction

In this paper, we define and study a ring $R(\Gamma)$ associated to a graph Γ that we call the cographic toric face ring or simply the cographic ring. The cographic ring $R(\Gamma)$ is the toric face ring defined by the following equivalent combinatorial structures of Γ: the cographic arrangement of hyperplanes $\mathscr{C}_\Gamma^\perp$, the Voronoi polytope Vor_Γ, and the poset of totally cyclic orientations $\mathbb{O}\mathscr{P}_\Gamma$. We describe the properties of the cographic ring and, in particular, relate the invariants of the ring to the invariants of the corresponding graph.

Our study of the cographic ring fits into a body of work on describing rings constructed from graphs. Among the rings that can be constructed from a graph, cographic rings are particularly interesting because they appear in the study of compactified Jacobians.

The authors establish the connection between $R(\Gamma)$ and the local geometry of compactified Jacobians in [Casalaina-Martin et al. 2011]. The compactified Jacobian $\bar{J}_X^d$ of a nodal curve X is the coarse moduli space parametrizing sheaves

MSC2010: primary 14H40; secondary 13F55, 05E40, 14K30, 05B35, 52C40.
Keywords: toric face rings, graphs, totally cyclic orientations, Voronoi polytopes, cographic arrangement of hyperplanes, cographic fans, compactified Jacobians, nodal curves.

on X that are rank-1, semistable, and of fixed degree d. These moduli spaces have been constructed by Oda and Seshadri [1979], Caporaso [1994], Simpson [1994], and Pandharipande [1996], and the different constructions are reviewed in Section 2 of [Casalaina-Martin et al. 2011]. In Theorem A of the same work, it is proved that the completed local ring of $\bar{J}^d_X$ at a point is isomorphic to a power series ring over the completion of $R(\Gamma)$ for a graph Γ constructed from the dual graph of X.

Also in [Casalaina-Martin et al. 2011], we studied the local structure of the universal compactified Jacobian, which is a family of varieties over the moduli space of stable curves whose fibers are closely related to the compactified Jacobians just discussed. (See Section 2 of [loc. cit.] for a discussion of the relation between the compactified Jacobians from the previous paragraph and the fibers of the universal Jacobian). Caporaso [1994] first constructed the universal compactified Jacobian, and Pandharipande [1996] gave an alternative construction. In [Casalaina-Martin et al. 2011, Theorem A] we gave a presentation of the completed local ring of the universal compactified Jacobian at a point, and we will explore the relation between that ring and the affine semigroup ring defined in Section 5A in the upcoming paper [Casalaina-Martin et al. 2012].

Cographic toric face rings are examples of toric face rings. Recall that a toric face ring is constructed from the same combinatorial data that is used to construct a toric variety: a fan. Let $H_{\mathbb{Z}}$ be a free, finite-rank $\mathbb{Z}$-module and $\mathscr{F}$ be a fan that decomposes $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ into (strongly convex rational polyhedral) cones. Consider the free k-vector space with basis given by monomials X^c indexed by elements $c \in H_{\mathbb{Z}}$. If we define a multiplication law on this vector space by setting

$$X^c \cdot X^{c'} = \begin{cases} X^{c+c'} & \text{if } c, c' \in \sigma \text{ for some } \sigma \in \mathscr{F}, \\ 0 & \text{otherwise} \end{cases}$$

and extending by linearity, then the resulting ring $R(\mathscr{F})$ is the toric face ring (over k) that is associated to $\mathscr{F}$.

We define the cographic toric face ring $R(\Gamma)$ of a graph Γ to be toric face ring associated to the fan that is defined by the cographic arrangement $\mathscr{C}^{\perp}_{\Gamma}$. The cographic arrangement is an arrangement of hyperplanes in the real vector space $H_{\mathbb{R}}$ associated to the homology group $H_{\mathbb{Z}} := H_1(\Gamma, \mathbb{Z})$ of the graph. Every edge of Γ naturally induces a functional on $H_{\mathbb{R}}$, and the zero locus of this functional is a hyperplane in $H_{\mathbb{R}}$, provided the functional is nonzero. The cographic arrangement is defined to be the collection of all hyperplanes constructed in this manner. The intersections of these hyperplanes define a fan $\mathscr{F}^{\perp}_{\Gamma}$, the cographic fan. The toric face ring associated to this fan is $R(\Gamma)$.

We study the fan $\mathscr{F}^{\perp}_{\Gamma}$ in Section 3. The main result of that section is Corollary 3.9, which provides two alternative descriptions of $\mathscr{F}^{\perp}_{\Gamma}$. First, using a theorem of Amini, we prove that $\mathscr{F}^{\perp}_{\Gamma}$ is equal to the normal fan of the Voronoi polytope Vor_{Γ}. As a

consequence, we can conclude that $\mathscr{F}_\Gamma^\perp$, considered as a poset, is isomorphic to the poset of faces of Vor_Γ ordered by reverse inclusion. Using work of Greene and Zaslavsky, we show that this common poset is also isomorphic to the poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations.

The combinatorial definition of $R(\Gamma)$ does not appear in [Casalaina-Martin et al. 2011]. Rather, the rings in that paper appear as invariants under a torus action. The following theorem, proven in Section 6 (Theorem 6.1), shows that the rings in [Casalaina-Martin et al. 2011] are (completed) cographic rings:

Theorem A. *Let Γ be a finite graph with vertices $V(\Gamma)$, oriented edges $\vec{E}(\Gamma)$, and source and target maps $s, t : \vec{E}(\Gamma) \to V(\Gamma)$. Let*

$$T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m \quad \text{and} \quad A(\Gamma) := \frac{k[U_{\vec{e}}, U_{\vec{e}} : e \in E(\Gamma)]}{(U_{\vec{e}} U_{\vec{e}} : e \in E(\Gamma))}.$$

If we make T_Γ act on $A(\Gamma)$ by

$$\lambda \cdot U_{\vec{e}} = \lambda_{s(\vec{e})} U_{\vec{e}} \lambda_{t(\vec{e})}^{-1},$$

then the invariant subring $A(\Gamma)^{T_\Gamma}$ is isomorphic to the cographic ring $R(\Gamma)$.

The cographic ring $R(\Gamma)$ has reasonable geometric properties. Specifically, in Theorem 5.7, we prove that $R(\Gamma)$ is

- of pure dimension $b_1(\Gamma) = \dim_{\mathbb{R}} H_1(\Gamma, \mathbb{R})$,
- Gorenstein,
- seminormal, and
- semi log canonical.

We also compute invariants of $R(\Gamma)$ in terms of the combinatorics of Γ. The invariants we compute are

- a description of $R(\Gamma)$ in terms of oriented subgraphs (Section 5B),
- the number of minimal primes in terms of orientations (Theorem 5.7(i)),
- the embedded dimension of $R(\Gamma)$ in terms of circuits (Theorem 5.7(vi)), and
- the multiplicity of $R(\Gamma)$ (Theorem 5.7(vii)).

Finally, it is natural to ask what information is lost in passing from Γ to $R(\Gamma)$. An answer to this question is given by Theorem 7.1, which states that $R(\Gamma)$ determines Γ up to three-edge connectivization.

Combinatorially defined rings, such as the cographic toric face ring, have long been used in the study of compactified Jacobians and, more generally, degenerate abelian varieties (see, e.g., [Mumford 1972; Oda and Seshadri 1979; Faltings and Chai 1990; Namikawa 1980; Alexeev and Nakamura 1999; Alexeev 2004]).

In particular, the ring $R(\Gamma)$ we study here is a special case of the rings $R_0(c)$ studied by Alexeev and Nakamura [1999, Theorem 3.17]. There the rings appear naturally as a by-product of Mumford's technique for degenerating an abelian variety. Alexeev and Nakamura [1999, Lemma 4.1] proved that $R_0(c)$ satisfies the Gorenstein condition, and the seminormality was established by Alexeev [2002]. In personal correspondence, Alexeev informed the authors that the techniques of those papers can also be used to establish other results in this paper such as the fact that $R(\Gamma)$ is semi log canonical.

In a different direction, the cographic ring is defined by the cographic fan $\mathscr{F}_\Gamma^\perp$, which is the normal fan to the Voronoi polytope Vor_Γ. There is a body of work studying similar polytopes and the algebra-geometric objects defined by these polytopes. Altmann and Hille [1999] define the polytope of flows associated to an oriented graph (or quiver). Associated to this polytope is a toric variety that they relate to a moduli space. There are also a number of recent papers that study the modular/integral flow polytope in $H_1(\Gamma, \mathbb{R})$. This study is motivated by the work of Beck and Zaslavsky [2006] on interpreting graph polynomials in terms of lattice points. Some recent papers on this topic are [Beck and Zaslavsky 2006; Breuer and Dall 2010; Breuer and Sanyal 2012; Chen 2010]. The paper [Breuer and Dall 2010], in particular, studies graph polynomials using tools from commutative algebra. The Voronoi polytope does not equal the modular/integral flow polytope or the polytope of flows of an oriented graph. It would, however, be interesting to further explore the relation between these polytopes. (We thank the anonymous referee for pointing out this literature.)

This paper suggests several other questions for further study. First, in Section 5A, we exhibit a collection of generators V_γ, indexed by oriented circuits γ, for $R(\Gamma \setminus T, \phi)$. What is an explicit set of generators for the ideal of relations between the variables V_γ? This problem is posed as Problem 5.5. Second, in Theorem 5.7, we give a formula for the multiplicity of $R(\Gamma)$ in terms of the subdiagram volume of certain semigroups associated to Γ. Problem 5.8 is to find an expression for this multiplicity in terms of well-known graph theory invariants. Third, we also prove in Theorem 5.7 that $\mathrm{Spec}(R(\Gamma))$ is semi log canonical. In Problem 5.9, we ask: which graphs Γ have the stronger property that $R(\Gamma)$ is semi divisorial log canonical?

1. Preliminaries

In this section, we review the definitions of the graph-theoretic objects considered in this paper. This will provide the reader with enough background to follow the main ideas of the proof of Theorem A (proven in Section 6) as well as the proofs of many of the geometric properties of cographic rings (proven in Section 4).

1A. ***Notation.*** Following notation of Serre [1980, §2.1], a *graph* Γ will consist of the data $(\vec{E} \underset{t}{\overset{s}{\rightrightarrows}} V, \vec{E} \xrightarrow{\iota} \vec{E})$, where V and $\vec{E}$ are sets, ι is a fixed-point free

involution, and s and t are maps satisfying $s(\vec{e}) = t(\iota(\vec{e}))$ for all $\vec{e} \in \vec{E}$. The maps s and t are called the *source* and *target* maps, respectively. We call $V =: V(\Gamma)$ the set of *vertices*. We call $\vec{E} =: \vec{E}(\Gamma)$ the set of *oriented edges*. We define the set of (*unoriented*) *edges* to be $E(\Gamma) = E := \vec{E}/\iota$. An *orientation of an edge* $e \in E$ is a representative for e in $\vec{E}$; we use the notation $\vec{e}$ and $\overleftarrow{e}$ for the two possible orientations of e. An *orientation of a graph* Γ is a section $\phi : E \to \vec{E}$ of the quotient map. An *oriented graph* consists of a pair (Γ, ϕ) where Γ is a graph and ϕ is an orientation. Given an oriented graph, we say that $\phi(e)$ is the *positive orientation* of the edge e. Given a subset $S \subseteq E$, we define $\vec{S} \subseteq \vec{E}$ to be the set of all orientations of the edges in S.

1B. ***Homology of a graph.*** Given a ring A, let $C_0(\Gamma, A) = \vec{C}_0(\Gamma, A)$ be the free A-module with basis $V(\Gamma)$ and $\vec{C}_1(\Gamma, A)$ be the A-module generated by $\vec{E}(\Gamma)$ with the relations $\overleftarrow{e} = -\vec{e}$ for every $e \in E(\Gamma)$. If we fix an orientation, then a basis for $\vec{C}_1(\Gamma, A)$ is given by the positively oriented edges; this induces an isomorphism with the usual group of 1-chains of the simplicial complex associated to Γ. These modules may be put into a chain complex. Define a boundary map ∂ by

$$\partial : \vec{C}_1(\Gamma, A) \to \vec{C}_0(\Gamma, A) = C_0(\Gamma, A), \quad \vec{e} \mapsto t(\vec{e}) - s(\vec{e}).$$

We will denote by $H_\bullet(\Gamma, A)$ the groups obtained from the homology of $\vec{C}_\bullet(\Gamma, A)$. The homology groups $H_\bullet(\Gamma, A)$ coincide with the homology groups of the topological space associated to Γ.

1C. ***The bilinear form.*** The vector space $\vec{C}_1(\Gamma, \mathbb{R})$ is endowed with a positive definite bilinear form

$$(\,\cdot\,,\,\cdot\,) : \vec{C}_1(\Gamma, \mathbb{R}) \otimes \vec{C}_1(\Gamma, \mathbb{R}) \to \mathbb{R}$$

that is uniquely determined by $(\vec{e}, \vec{e}) = 1$, $(\vec{e}, \overleftarrow{e}) = -1$, and $(\vec{e}, \vec{f}) = 0$ if $\vec{f} \neq \vec{e}, \overleftarrow{e}$. As above, fixing an orientation induces a basis for $\vec{C}_1(\Gamma, \mathbb{R})$, and in terms of such a basis, this is the standard inner product. By restriction, we get a positive definite bilinear form on $H_1(\Gamma, \mathbb{R}) \subseteq \vec{C}_1(\Gamma, \mathbb{R})$. The pairing $(\,\cdot\,,\,\cdot\,)$ allows us to form the product $(\vec{e}, v)$ of an oriented edge $\vec{e}$ with a vector $v \in \vec{C}_1(\Gamma, \mathbb{R})$ but not the product (e, v) of v with an unoriented vector. However, we will write $(e, v) = 0$ to mean $(\vec{e}, v) = 0$ for one (equivalently all) orientations of e.

1D. ***Cographic arrangement.*** We review the definition of the *cographic arrangement* $\mathscr{C}_\Gamma^\perp$ of Γ [Greene and Zaslavsky 1983, §8; Novik et al. 2002, §5].[1] To begin,

[1] The name "cographic arrangement" suggests the fact that $\mathscr{C}_\Gamma^\perp$ depends on the cographic matroid associated to Γ. The notation $\mathscr{C}_\Gamma^\perp$ is used in [Novik et al. 2002] while in [Greene and Zaslavsky 1983] the cographic arrangement is denoted by $\mathscr{H}^\perp[\Gamma]$. There is a dual notion, namely that of the graphic arrangement, which depends only on the graphic matroid associated to Γ and is denoted by

let $\mathscr{H}$ be the coordinate hyperplane arrangement in $\vec{C}_1(\Gamma, \mathbb{R})$. More precisely,

$$\mathscr{H} = \bigcup_{e \in E} \{v \in \vec{C}_1(\Gamma, \mathbb{R}) : (v, e) = 0\}.$$

The restriction of this hyperplane arrangement to $H_1(\Gamma, \mathbb{R})$ is called the cographic arrangement $\mathscr{C}_\Gamma^\perp$. More precisely,

$$\mathscr{C}_\Gamma^\perp = \bigcup_{\substack{e \in E \\ H_1(\Gamma,\mathbb{R}) \not\subseteq \ker(\,\cdot\,, e)}} \{v \in \vec{C}_1(\Gamma, \mathbb{R}) : (v, e) = 0\}.$$

The cographic arrangement partitions $H_1(\Gamma, \mathbb{R})$ into a finite collection of strongly convex rational polyhedral cones. These cones, together with their faces, form a (complete) fan that is defined to be the *cographic fan* and is denoted $\mathscr{F}_\Gamma^\perp$.[2] We give a more detailed enumeration of the cones of this fan in Section 3, where we discuss the poset of totally cyclic orientations.

Remark 1.1. The following observation used in the proof of Theorem A is proven in Corollary 3.4. We emphasize it here so that the reader may follow the proof of Theorem A having read just Section 1. *Let* $c = \sum_{e \in E} a_e \vec{e}$ *and* $c' = \sum_{e \in E} a'_e \vec{e}$ *be cycles in* $H_1(\Gamma, \mathbb{Z})$. *Then* c *and* c' *lie in a common cone of* $\mathscr{F}_\Gamma^\perp$ *if and only if, for all* $e \in E$, $a_e a'_e \geq 0$. In words, two cycles lie in a common cone if and only if every common edge is oriented in the same direction.

1E. ***Toric face rings.*** We recall the definition of a toric face ring associated to a fan. In [Ichim and Römer 2007, §2; Bruns et al. 2008, §2], the authors define more generally the toric face ring associated to a monoidal complex. The following definition is a special case:

Definition 1.2. Let $H_\mathbb{Z}$ be a free $\mathbb{Z}$-module of finite rank, and let $\mathscr{F}$ be a fan of (strongly convex rational polyhedral) cones in $H_\mathbb{R} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$ with support $\operatorname{Supp} \mathscr{F}$. The *toric face ring* $R_k(\mathscr{F})$ is the k-algebra whose underlying k-vector space has basis $\{X^c : c \in H_\mathbb{Z} \cap \operatorname{Supp} \mathscr{F}\}$ and whose multiplication is defined by

$$X^c \cdot X^{c'} = \begin{cases} X^{c+c'} & \text{if } c, c' \in \sigma \text{ for some } \sigma \in \mathscr{F}, \\ 0 & \text{otherwise.} \end{cases} \tag{1-1}$$

We will write $R(\mathscr{F})$ if we do not need to specify the base field k.

Remark 1.3. It follows from the definition that $R(\mathscr{F})$ is a reduced ring finitely generated over k. See also Section 5, especially (5-4), for more on generators and relations.

$\mathscr{C}_\Gamma$ in [Novik et al. 2002] and $\mathscr{H}[\Gamma]$ in [Greene and Zaslavsky 1983, §7]. The graphic arrangement of hyperplanes is also studied in [Orlik and Terao 1992, §2.4], where it is denoted by $\mathscr{A}(\Gamma)$.

[2] We use the notation $\mathscr{F}_\Gamma^\perp$ and the name "cographic fan" in order to be consistent with the notation $\mathscr{C}_\Gamma^\perp$ used in [Novik et al. 2002] for the cographic arrangement of hyperplanes.

A cographic toric face ring is the toric face ring associated to a cographic fan.

Definition 1.4. Let Γ be a finite graph. The *cographic toric face ring* $R_k(\Gamma)$ is the toric face k-ring $R(\mathscr{F}_\Gamma^\perp)$ associated to the cographic fan $\mathscr{F}_\Gamma^\perp$. We will write $R(\Gamma)$ if we do not need to specify the base field k.

1F. ***The Voronoi polytope.*** Following [Bacher et al. 1997], we define the *Voronoi polytope* of Γ by

$$\mathrm{Vor}_\Gamma := \{v \in H_1(\Gamma, \mathbb{R}) : (v, v) \le (v - \lambda, v - \lambda) \text{ for all } \lambda \in H_1(\Gamma, \mathbb{Z})\}.$$

The reader familiar with the Voronoi decomposition of $\mathbb{R}^n$ will recognize this polytope as the unique cell containing the origin in the Voronoi decomposition associated with the lattice $H_1(\Gamma, \mathbb{Z})$ endowed with the scalar product defined in Section 1C (see [Erdahl 1999; Alexeev 2004, §2.5] for more details).

To the Voronoi polytope, we can associate its *normal fan* $\mathscr{N}(\mathrm{Vor}_\Gamma)$, which is defined as follows. Given a face δ of Vor_Γ, we define the (strongly convex rational polyhedral) cone C_δ by

$$C_\delta = \{\alpha \in H_1(\Gamma, \mathbb{R}) : (\alpha, r) \ge (\alpha, r') \text{ for all } r \in \delta \text{ and } r' \in \mathrm{Vor}_\Gamma\}.$$

The normal fan $\mathscr{N}(\mathrm{Vor}_\Gamma)$ of Vor_Γ is the fan whose cones are the cones C_δ.

Remark 1.5. In Proposition 3.8, we will prove that the cographic fan $\mathscr{F}_\Gamma^\perp$ is equal to the normal fan of the Voronoi polytope $\mathscr{N}(\mathrm{Vor}_\Gamma)$.

2. Totally cyclic orientations

Here we define and study totally cyclic orientations of a graph. We also define an oriented circuit on a graph and describe the relation between these circuits and totally cyclic orientations.

2A. ***Subgraphs.*** In this subsection, we introduce some special subgraphs that will play an important role throughout the paper.

Given a graph Γ and a collection $S \subset E(\Gamma)$ of edges, we define $\Gamma \setminus S$ to be the graph, called a *spanning subgraph* (see, e.g., [Oda and Seshadri 1979, §4]), obtained from Γ by removing the edges in S and leaving the vertices unmodified. In other words, $\Gamma \setminus S$ consists of the data

$$(\overrightarrow{E(\Gamma) \setminus S} \underset{t}{\overset{s}{\rightrightarrows}} V, \overrightarrow{E(\Gamma) \setminus S} \xrightarrow{\iota} \overrightarrow{E(\Gamma) \setminus S}).$$

Of particular significance is the special case where $S = \{e\}$ consists of a single edge. If $\Gamma \setminus \{e\}$ has more connected components than Γ, then we say that e is a *separating edge*. The *set of all separating edges* is written $E(\Gamma)_{\mathrm{sep}}$.

Given a chain $c \in \vec{C}_1(\Gamma, \mathbb{R})$, we would like to refer to the underlying graph having only those edges in the support of c. More precisely, given $c \in \vec{C}_1(\Gamma, \mathbb{R})$, let $\mathrm{Supp}(c)$ denote the set of all edges e with the property that $(e, c) \neq 0$. We define Γ_c to be the subgraph of Γ with $V(\Gamma_c) := V(\Gamma)$ and $E(\Gamma_c) := \mathrm{Supp}(c)$. There is a distinguished orientation ϕ_c of Γ_c given by setting $\phi_c(e)$ equal to $\vec{e}$ if $(\vec{e}, c) > 0$ and to $\bar{e}$ otherwise. Using this subgraph, we can write c as

$$c = \sum_{e \in \mathrm{Supp}(c)} m_c(e)\phi_c(e) \tag{2-1}$$

with all $m_c(e) > 0$. Indeed, we have $m_c(e) = (\phi_c(e), c)$.

2B. ***Totally cyclic orientations and oriented circuits.*** Totally cyclic orientations will play a dominant role in what follows. We are going to review their definition and their basic properties.

Definition 2.1. If Γ is connected, then we say that an orientation ϕ of Γ is *totally cyclic* if there does not exist a nonempty proper subset $W \subset V(\Gamma)$ such that every edge e between a vertex in W and a vertex in the complement $V(\Gamma) \setminus W$ is oriented from W to $V \setminus W$ (i.e., the source of $\phi(e)$ lies in W and the target of $\phi(e)$ lies in $V(\Gamma) \setminus W$). If Γ is disconnected, then we say that an orientation of Γ is totally cyclic if the orientation induced on each connected component of Γ is totally cyclic.

Observe that if Γ is a graph with no edges, then the empty orientation of Γ is a totally cyclic orientation. Totally cyclic orientations are closely related to oriented circuits. Recall that a graph Δ is called *cyclic* if it is connected, free from separating edges, and satisfies $b_1(\Delta) = 1$. A cyclic graph together with a totally cyclic orientation is called an *oriented circuit*. A cyclic graph admits exactly two totally cyclic orientations.

Let $\overrightarrow{\mathrm{Cir}}(\Gamma)$ denote the set of all oriented circuits on Γ; that is, $\gamma = (\Delta, \phi_\Delta)$ is an element of $\overrightarrow{\mathrm{Cir}}(\Gamma)$ if Δ is a cyclic subgraph of Γ and ϕ_Δ is a totally cyclic orientation of Δ. We call $E(\Delta)$ the *support* of $\gamma = (\Delta, \phi_\Delta) \in \overrightarrow{\mathrm{Cir}}(\Gamma)$. There is a natural map

$$\overrightarrow{\mathrm{Cir}}(\Gamma) \to H_1(\Gamma, A),$$
$$\gamma = (\Delta, \phi_\Delta) \mapsto [\gamma] = \sum_{e \in E(\Delta)} \phi_\Delta(e).$$

With respect to the orientation ϕ of Γ, we can consider $\mathrm{Cir}_\phi(\Gamma) \subset \overrightarrow{\mathrm{Cir}}(\Gamma)$, the subset that consists of oriented circuits on Γ of the form $(\Delta, \phi|_\Delta)$ (i.e., oriented circuits whose orientation is compatible with ϕ).

Remark 2.2. The oriented circuits on Γ, i.e., the elements of $\overrightarrow{\mathrm{Cir}}(\Gamma)$, are the (signed) cocircuits of the cographic oriented matroid $M^*(\Gamma)$ of Γ or, equivalently, the (signed) circuits of the oriented graphic matroid $M(\Gamma)$ of Γ [Björner et al. 1999,

§1.1]. Many of the combinatorial results that follow can be naturally stated using this language. We will limit ourselves to pointing out the connection with the theory when relevant.

The next lemma clarifies the relationship between totally cyclic orientations and compatibly oriented circuits. Recall that an oriented path from $w \in V(\Gamma)$ to $v \in V(\Gamma)$ is a collection of oriented edges $\{\vec{e}_1, \ldots, \vec{e}_r\} \subset \vec{E}(\Gamma)$ such that $s(\vec{e}_1) = w$, $t(\vec{e}_i) = s(\vec{e}_{i+1})$ for any $i = 1, \ldots, r-1$, and $t(\vec{e}_r) = v$. If ϕ is an orientation of Γ, a path compatibly oriented with respect to ϕ is an oriented path as before of the form $\{\phi(e_1), \ldots, \phi(e_r)\}$.

Lemma 2.3. *Let Γ be a graph.*

(1) *The graph Γ admits a totally cyclic orientation if and only if $E(\Gamma)_{\text{sep}} = \varnothing$.*

(2) *Fix an orientation ϕ on Γ. The following conditions are equivalent*:

(a) *The orientation is totally cyclic.*

(b) *For any distinct $v, w \in V(\Gamma)$ belonging to the same connected component of Γ, there exists a path compatibly oriented with respect to ϕ from w to v.*

(c) *The cycles $[\gamma]$ associated to the $\gamma \in \text{Cir}_\phi(\Gamma)$ generate $H_1(\Gamma, \mathbb{Z})$, and $E(\Gamma)_{\text{sep}} = \varnothing$.*

(d) *Every edge $e \in E$ is contained in the support of a compatibly oriented circuit $\gamma \in \text{Cir}_\phi(\Gamma)$.*

Proof. For part (1), see, e.g., [Caporaso and Viviani 2010, Lemma 2.4.3(1)] and the references therein. Part (2) is a reformulation of [Caporaso and Viviani 2010, Lemma 2.4.3(2)]. The only difference is that part (2) is proved in [loc. cit.] under the additional hypothesis that $E(\Gamma)_{\text{sep}} = \varnothing$. Note, however, that each of the conditions (a), (b), and (d) imply that $E(\Gamma)_{\text{sep}} = \varnothing$; hence, we deduce part (2) as stated above. □

The following well-known lemma can be thought of as a modification of (c) above. We no longer require that the oriented circuits on Γ be oriented compatibly. The statement is essentially that any cycle c in $H_1(\Gamma, \mathbb{Z})$ is a positive linear combination of cycles associated to circuits supported on c.

Lemma 2.4. *Let Γ be a graph, and let $c \in \vec{C}_1(\Gamma, \mathbb{Z})$. Then $c \in H_1(\Gamma, \mathbb{Z})$ if and only if c can be expressed as*

$$c = \sum_{\gamma \in \text{Cir}_{\phi_c}(\Gamma_c)} n_c(\gamma)[\gamma] \tag{2-2}$$

for some natural numbers $n_c(\gamma) \in \mathbb{N}$.

Proof. A direct proof follows from the definitions and is left to the reader. Alternatively, one can use the fact that a covector of an oriented matroid can be written as

a composition of cocircuits conformal to it [Björner et al. 1999, Proposition 3.7.2] together with Remark 2.2. □

The oriented circuits can be used to define a simplicial complex that will be used in Section 5B.

Definition 2.5. Two oriented circuits $\gamma = (\Delta, \phi)$ and $\gamma' = (\Delta', \phi')$ are said to be *concordant*, written $\gamma \asymp \gamma'$, if for any $e \in E(\Delta) \cap E(\Delta')$ we have $\phi(e) = \phi'(e)$. We write $\gamma \not\asymp \gamma'$ if γ and γ' are not concordant.

Definition 2.6. The *simplicial complex of concordant circuits* $\Delta(\overrightarrow{\mathrm{Cir}}(\Gamma))$ is defined to be the (abstract) simplicial complex whose elements are collections $\sigma \subseteq \overrightarrow{\mathrm{Cir}}(\Gamma)$ of oriented circuits on Γ with the property that any two circuits are concordant (i.e., if $\gamma_1, \gamma_2 \in \sigma$, then $\gamma_1 \asymp \gamma_2$).

2C. ***The poset $\mathbb{OP}_\Gamma$ of totally cyclic orientations.*** Totally cyclic orientations naturally form a poset. We recall the definition for the sake of completeness.

Definition 2.7 [Caporaso and Viviani 2010, Definition 5.2.1]. The poset $\mathbb{OP}_\Gamma$ of *totally cyclic orientations* of Γ is the set of pairs (T, ϕ) where $T \subset E(\Gamma)$ and $\phi : E(\Gamma \setminus T) \to \vec{E}(\Gamma \setminus T)$ is a totally cyclic orientation of $\Gamma \setminus T$,[3] endowed with the partial order

$$(T', \phi') \leq (T, \phi) \iff \Gamma \setminus T' \subseteq \Gamma \setminus T \text{ and } \phi' = \phi|_{E(\Gamma \setminus T')}.$$

We call T the *support* of the pair (T, ϕ).

Using Lemma 2.3(2)(d), we get that

$$(T', \phi') \leq (T, \phi) \iff \mathrm{Cir}_{\phi'}(\Gamma \setminus T') \subseteq \mathrm{Cir}_{\phi}(\Gamma \setminus T). \tag{2-3}$$

The set $\mathrm{Cir}_\phi(\Gamma \setminus T)$ is a collection of concordant cycles. Another connection between orientations and totally cyclic orientations is given by the following definition:

Definition 2.8. Let $\sigma \in \Delta(\overrightarrow{\mathrm{Cir}}(\Gamma))$ be a collection of concordant circuits. To σ we associate the pair $(T_\sigma, \phi_\sigma) \in \mathbb{OP}_\Gamma$, which is defined as follows. Set T_σ equal to the set of all edges that are *not* contained in a circuit $\gamma \in \sigma$. The orientation ϕ_σ of $\Gamma \setminus T_\sigma$ is defined by setting

$$\phi_\sigma(e) := \begin{cases} \vec{e} & \text{if } (\vec{e}, [\gamma]) > 0 \text{ for all } \gamma \in \sigma, \\ \overleftarrow{e} & \text{if } (\overleftarrow{e}, [\gamma]) > 0 \text{ for all } \gamma \in \sigma. \end{cases}$$

Observe that the orientation ϕ_σ on $\Gamma \setminus T_\sigma$ is a totally cyclic orientation by Lemma 2.3(2)(d) and that $\sigma \subseteq \mathrm{Cir}_{\phi_\sigma}(\Gamma \setminus T_\sigma)$. The following lemma, whose proof is left to the reader, will be useful in the sequel:

[3] The choice of orientation on the complement of T, rather than on T itself, has to do with the importance of the notion of spanning subgraphs of Γ, all of which are of this form. In graph theory, it is customary to denote spanning subgraphs in this way, so we follow that convention.

Lemma 2.9. *The maximal elements of the poset $\mathcal{OP}_\Gamma$ are given by $(E(\Gamma)_{\text{sep}}, \phi)$ as ϕ varies among the totally cyclic orientations of $\Gamma \setminus E(\Gamma)_{\text{sep}}$.* □

Remark 2.10. The poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations is isomorphic to the poset of covectors of the cographic oriented matroid $M^*(\Gamma)$ of Γ [Björner et al. 1999, §3.7]. Equivalently, the poset obtained from $\mathcal{OP}_\Gamma$ by adding an element $\hat{1}$ and declaring that $\hat{1} \geq (T, \phi)$ for any $(T, \phi) \in \mathcal{OP}_\Gamma$ is isomorphic to the big face lattice $\mathcal{F}_{\text{big}}(M^*(\Gamma))$ of the cographic oriented matroid $M^*(\Gamma)$ [Björner et al. 1999, §4.1].

3. Comparing posets: the cographic arrangement, the Voronoi polytope, and totally cyclic orientations

In this section, we prove that the poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations of Γ is isomorphic to the poset of cones (ordered by inclusion) of the cographic fan $\mathcal{F}_\Gamma^\perp$, which we also show is the normal fan of the Voronoi polytope Vor_Γ of Γ.

3A. ***Cographic arrangement.*** Let us start by describing the cographic arrangement $\mathcal{C}_\Gamma^\perp$ associated to Γ in the language of totally cyclic orientations.

For every edge $e \in E(\Gamma)$, we can consider the linear subspace of $H_1(\Gamma, \mathbb{R})$

$$\{(\,\cdot\,, e) = 0\} := \{v \in H_1(\Gamma, \mathbb{R}) : (v, e) = 0\}.$$

This subspace is a proper subspace (i.e., a hyperplane) precisely when e is not a separating edge, and the collection of all such hyperplanes is defined to be the cographic arrangement. Similarly, for any oriented edge $\vec{e} \in \vec{E}(\Gamma)$, we set

$$\{(\,\cdot\,, \vec{e}) \geq 0\} := \{v \in H_1(\Gamma, \mathbb{R}) : (v, \vec{e}) \geq 0\}.$$

As mentioned, the elements of the cographic arrangement partition $H_1(\Gamma, \mathbb{R})$ into a finite collection of rational polyhedral cones. These cones, together with their faces, form the cographic fan $\mathcal{F}_\Gamma^\perp$. We can enumerate these cones and make their relation to totally cyclic orientations more explicit by introducing some notation.

Given a collection T of edges and an orientation ϕ of $\Gamma \setminus T$ (not necessarily totally cyclic), we define (possibly empty) cones $\sigma(T, \phi)$ and $\sigma^{\text{o}}(T, \phi)$ by

$$\sigma(T, \phi) := \bigcap_{e \notin T} \{(\,\cdot\,, \phi(e)) \geq 0\} \cap \bigcap_{e \in T} \{(\,\cdot\,, e) = 0\}, \tag{3-1}$$

$$\sigma^{\text{o}}(T, \phi) := \bigcap_{e \notin T} \{(\,\cdot\,, \phi(e)) > 0\} \cap \bigcap_{e \in T} \{(\,\cdot\,, e) = 0\}. \tag{3-2}$$

The cone $\sigma^{\text{o}}(T, \phi)$ is a subcone of $\sigma(T, \phi)$, and it is the relative interior of $\sigma(T, \phi)$ provided $\sigma^{\text{o}}(T, \phi)$ is nonempty. The cone $\sigma(T, \phi)$ is an element of the cographic fan, and every cone in the fan can be written in this form. While every element of $\mathcal{F}_\Gamma^\perp$ can be written as $\sigma(T, \phi)$, the pair (T, ϕ) is *not* uniquely determined

by the cone. The pair (T, ϕ) is, however, uniquely determined if we further require that $(T, \phi) \in \mathcal{OP}_\Gamma$. This fact is proven in the following proposition, which is essentially a restatement of some results of Greene and Zaslavsky [1983, §8]:

Proposition 3.1. (i) *Every cone $\sigma \in \mathscr{F}_\Gamma^\perp$ can be written as $\sigma = \sigma(T, \phi)$ for a unique element $(T, \phi) \in \mathcal{OP}_\Gamma$.*

(ii) *For any $(T, \phi) \in \mathcal{OP}_\Gamma$, the linear span of $\sigma(T, \phi)$ is equal to*

$$\langle \sigma(T, \phi) \rangle = \bigcap_{e \in T} \{(\cdot\,, e) = 0\} = H_1(\Gamma \setminus T, \mathbb{R})$$

and has dimension $b_1(\Gamma \setminus T)$.

(iii) *For any $(T, \phi) \in \mathcal{OP}_\Gamma$, the extremal rays of $\sigma(T, \phi)$ are the rays generated by the elements $[\gamma]$ for $\gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)$.*

Proof. Part (i) follows from [Greene and Zaslavsky 1983, Lemma 8.2]. Note that in [ibid.] the authors assume that $E(\Gamma)_{\mathrm{sep}} = \varnothing$. However, it is easily checked that the inclusion map $\Gamma \setminus E(\Gamma)_{\mathrm{sep}} \subseteq \Gamma$ induces natural isomorphisms $\mathscr{F}_{\Gamma \setminus E(\Gamma)_{\mathrm{sep}}}^\perp \cong \mathscr{F}_\Gamma^\perp$ and $\mathcal{OP}_{\Gamma \setminus E(\Gamma)_{\mathrm{sep}}} \cong \mathcal{OP}_\Gamma$. Therefore, the general case follows from the special case treated in [ibid.].

Let us now prove part (ii). The linear subspace $\bigcap_{e \in T} \{(\cdot\,, e) = 0\} \subseteq H_1(\Gamma, \mathbb{R})$ is generated by all the cycles of Γ that do not contain edges $e \in T$ in their support and is therefore equal to $H_1(\Gamma \setminus T, \mathbb{R})$, which has dimension equal to $b_1(\Gamma \setminus T)$. Now, to complete the proof, let us establish that $\langle \sigma(T, \phi) \rangle = \bigcap_{e \in T} \{(\cdot\,, e) = 0\}$. First, if $\sigma(T, \phi)^\circ = \varnothing$, i.e., if $\sigma(T, \phi) = \{0\}$, then $b_1(\Gamma \setminus T) = 0$ by Lemma 2.3(2)(d). But then $\bigcap_{e \in T} \{(\cdot\,, e) = 0\} = H_1(\Gamma \setminus T, \mathbb{R}) = 0$, and we are done. On the other hand, if $\sigma(T, \phi)^\circ \neq \varnothing$, then $\sigma^{\mathrm{o}}(T, \phi)$ is the relative interior of $\sigma(T, \phi)$, and hence, the linear span of $\sigma(T, \phi)$ is equal to $\bigcap_{e \in T} \{(\cdot\,, e) = 0\}$.

Finally, let us prove part (iii). From [Greene and Zaslavsky 1983, Lemma 8.5], it follows that the extremal rays of $\sigma(T, \phi)$ are among the rays generated by the elements $[\gamma]$ for $\gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)$. We conclude by showing that for any $\gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)$, the ray generated by $[\gamma]$ is extremal for $\sigma(T, \phi)$. By contradiction, suppose that we can write

$$[\gamma] = \sum_{\substack{\gamma' \in \mathrm{Cir}_\phi(\Gamma \setminus T) \\ \gamma' \neq \gamma}} m_{\gamma'} [\gamma'] \tag{3-3}$$

for some $m_{\gamma'} \in \mathbb{R}_{\geq 0}$. Consider a cycle $\gamma_0 \in \mathrm{Cir}_\phi(\Gamma \setminus T) \setminus \{\gamma\}$ such that $m_{\gamma_0} > 0$ (which clearly exists since $[\gamma] \neq 0$). Since γ and γ_0 are concordant and distinct, there should exist an edge $e \in E(\gamma_0) \setminus E(\gamma)$. Now returning to the expression (3-3), on the left-hand side, neither the oriented edge $\vec{e}$ nor $\overleftarrow{e}$ can appear. On the other hand, on the right-hand side, the oriented edge $\phi(e)$ appears with positive multiplicity

because it appears with multiplicity $m_{\gamma_0} > 0$ in $m_{\gamma_0}[\gamma_0]$ and all the oriented circuits appearing in the summation are concordant. This is a contradiction. □

Corollary 3.2. *The association*

$$(T, \phi) \mapsto \sigma(T, \phi)$$

defines an isomorphism between the poset of $\mathcal{OP}_\Gamma$ *and the poset of cones of* $\mathcal{F}_\Gamma^\perp$ *ordered by inclusion.*[4] *In particular, the number of connected components of the complement of* $\mathcal{C}_\Gamma^\perp$ *in* $H_1(\Gamma, \mathbb{R})$ *is equal to the number of totally cyclic orientations on* $\Gamma \setminus E(\Gamma)_{\text{sep}}$.

Proof. According to Proposition 3.1(i), the map in the statement is bijective. We have to show that

$$\sigma(T, \phi) \subseteq \sigma(T', \phi') \iff (T, \phi) \leq (T', \phi').$$

The implication $\Longleftarrow$ is clear by the definition (3-1) of $\sigma(T, \phi)$.

Conversely, assume that $\sigma(T, \phi) \subseteq \sigma(T', \phi')$. There is nothing to show if $\sigma(T, \phi) = \{0\}$ is the origin. Otherwise, by Proposition 3.1(ii), the relative interior $\sigma^{\circ}(T, \phi)$ of $\sigma(T, \phi)$ is nonempty, so pick $c \in \sigma^{\circ}(T, \phi)$. By formula (3-2), for every $e \notin T$, we have that $(c, \phi(e)) > 0$. Since $c \in \sigma(T', \phi')$, by definition (3-1), we must have $e \notin T'$ and $\phi'(e) = \phi(e)$. This shows that $T \supseteq T'$ and that $\phi'_{\Gamma \setminus T} = \phi$ or in other words that $(T, \phi) \leq (T', \phi')$.

The last assertion follows from the first one using the fact that the connected components of the complement of $\mathcal{C}_\Gamma^\perp$ in $H_1(\Gamma, \mathbb{R})$ are the maximal cones in $\mathcal{F}_\Gamma^\perp$ and Lemma 2.9. □

Remark 3.3. The last assertion of Corollary 3.2 is due to Green and Zaslavsky [1983, Lemma 8.1]. Moreover, Greene and Zaslavsky [1983, Theorem 8.1] give a formula for the number of totally cyclic orientations of a graph free from separating edges.

The following well-known result plays a crucial role in the proof of Theorem 6.1:

Corollary 3.4. *Let*

$$c = \sum_{e \in E} a_e \vec{e} \quad \text{and} \quad c' = \sum_{e \in E} a'_e \vec{e}$$

be cycles in $H_1(\Gamma, \mathbb{Z})$. *Then there is a cone of* $\mathcal{F}_\Gamma^\perp$ *containing* c *and* c' *if and only if, for all* $e \in E$, $a_e a'_e \geq 0$.

Proof. From Proposition 3.1(i), it follows that c and c' belong to the same cone of $\mathcal{F}_\Gamma^\perp$ if and only if there exists $(T, \phi) \in \mathcal{OP}_\Gamma$ such that $c, c' \in \sigma(T, \phi)$. We conclude by looking at the explicit description (3-1). □

[4] Note that the poset of cones of $\mathcal{F}_\Gamma^\perp$ is anti-isomorphic to the face poset $\mathcal{L}(\mathcal{C}_\Gamma^\perp)$ of the arrangement $\mathcal{C}_\Gamma^\perp$ [Orlik and Terao 1992, Definition 2.18].

Remark 3.5. Corollary 3.2 together with Remark 2.10 imply that the cographic oriented matroid $M^*(\Gamma)$ is represented by the cographic hyperplane arrangement $\mathscr{C}_\Gamma^\perp$ in the sense of [Björner et al. 1999, §1.2(c)]. Using this, Corollary 3.4 is a restatement of the fact that two elements of $H_1(\Gamma, \mathbb{Z})$ belong to the same cone of $\mathscr{F}_\Gamma^\perp$ if and only if their associated covectors are conformal [Björner et al. 1999, §3.7].

3B. ***Voronoi polytope.*** The following description of the faces of Vor_Γ is a restatement, in our notation, of a result of Omid Amini [Amini 2010], which gives a positive answer to a conjecture of Caporaso and Viviani [2010, Conjecture 5.2.8(i)]:

Proposition 3.6 (Amini). (i) *Every face of the Voronoi polytope* Vor_Γ *is of the form*

$$F(T,\phi) := \{v \in \mathrm{Vor}_\Gamma : (v,[\gamma]) = \tfrac{1}{2}([\gamma],[\gamma]) \text{ for any } \gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)\} \quad (3\text{-}4)$$

for some uniquely determined element $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$.

(ii) *For any* $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$, *the dimension of the affine span of* $F(T,\phi)$ *is equal to* $b_1(\Gamma(T)) = b_1(\Gamma) - b_1(\Gamma \setminus T)$.

(iii) *For any* $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$, *the codimension-*1 *faces of* Vor_Γ *containing* $F(T,\phi)$ *are exactly those of the form* $F(S,\psi)$, *where* $(S,\psi) \le (T,\phi)$ *and* $b_1(\Gamma \setminus S) = 1$.

Proof. Part (i) follows by combining [Amini 2010, Theorem 1, Lemma 7]. Part (ii) follows from the remark after [Amini 2010, Lemma 10]. Part (iii) follows from [Amini 2010, Lemma 7]. □

Corollary 3.7 (Amini). *The association*

$$(T,\phi) \mapsto F(T,\phi)$$

defines an isomorphism of posets between the poset $\mathbb{O}\mathscr{P}_\Gamma$ *and the poset of faces of* Vor_Γ *ordered by reverse inclusion. In particular, the number of vertices of* Vor_Γ *is equal to the number of totally cyclic orientations on* $\Gamma \setminus E(\Gamma)_{\mathrm{sep}}$.

Proof. The first statement is a reformulation of [Amini 2010, Theorem 1]. The last assertion follows from the first one together with Lemma 2.9. □

We now show that the cographic fan $\mathscr{F}_\Gamma^\perp$ is the *normal fan* $\mathscr{N}(\mathrm{Vor}_\Gamma)$ of the Voronoi polytope Vor_Γ. The cones of the normal fan, ordered by inclusion, form a poset that is clearly isomorphic to the poset of faces of Vor_Γ, ordered by reverse inclusion.

Proposition 3.8. *The cographic fan* $\mathscr{F}_\Gamma^\perp$ *is equal to* $\mathscr{N}(\mathrm{Vor}_\Gamma)$, *the normal fan of the Voronoi polytope* Vor_Γ.

Proof. By Propositions 3.1 and 3.6, it is enough to show that, for any $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$, the normal cone in $\mathscr{N}(\mathrm{Vor}_\Gamma)$ to the face $F(T,\phi) \subset \mathrm{Vor}_\Gamma$ is equal to $\sigma(T,\phi)$. Fix a face $F(T,\phi)$ of Vor_Γ for some $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$. If (T,ϕ) is equal to the minimal element $\underline{0} = (E(\Gamma)_{\mathrm{sep}}, \varnothing)$ of the poset $\mathbb{O}\mathscr{P}_\Gamma$, then $F(\underline{0}) = \mathrm{Vor}_\Gamma$ and its normal cone is equal to the origin in $H_1(\Gamma, \mathbb{R})$, which is equal to $\sigma(\underline{0})$.

Suppose now that $b_1(\Gamma \setminus T) \geq 1$. Denote by $\{(S_i, \psi_i)\}$ all the elements of $\mathcal{OP}_\Gamma$ such that $(S_i, \psi_i) \leq (T, \phi)$ and $b_1(\Gamma \setminus S_i) = 1$. Let γ_i be the unique oriented circuit of Γ such that $\mathrm{Cir}_{\psi_i}(\Gamma \setminus S_i) = \{\gamma_i\}$. According to Proposition 3.6(iii), the codimension-1 faces of Vor_Γ containing $F(T, \phi)$ are exactly those of the form $F(S_i, \psi_i)$. Therefore, the normal cone of $F(T, \phi)$ is the cone whose extremal rays are the normal cones to the faces $F(S_i, \psi_i)$, which, using (3-4), are equal to $\sigma(S_i, \psi_i) = \mathbb{R}_{\geq 0} \cdot [\gamma_i]$. By Proposition 3.1(iii), the cone whose extremal rays are given by $\mathbb{R}_{\geq 0} \cdot [\gamma_i]$ is equal to $\sigma(T, \phi)$, which completes the proof. □

Combining Corollaries 3.2 and 3.7 and Proposition 3.8, we get the following incarnations of the poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations:

Corollary 3.9. *The following posets are isomorphic*:

(1) *the poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations*,

(2) *the poset of faces of the Voronoi polytope* Vor_Γ, *ordered by reverse inclusion*,

(3) *the poset of cones in the normal fan $\mathcal{N}(\mathrm{Vor}_\Gamma)$, ordered by inclusion, and*

(4) *the poset of cones in the cographic fan $\mathcal{F}_\Gamma^\perp$, ordered by inclusion.*

Remark 3.10. Corollary 3.9 together with Remark 2.10 imply that the cographic oriented matroid $M^*(\Gamma)$ is represented by the Voronoi polytope Vor_Γ (which is a zonotope; see, e.g., [Erdahl 1999]) in the sense of [Björner et al. 1999, §2.2].

4. Geometry of toric face rings

Let $H_\mathbb{Z}$ be a free $\mathbb{Z}$-module of finite rank b, and let $\mathcal{F}$ be a fan of (strongly convex rational polyhedral) cones in $H_\mathbb{R} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$. The aim of this section is to study the toric face ring $R(\mathcal{F}) = R_k(\mathcal{F})$ associated to $\mathcal{F}$ as in Definition 1.2. We will pay special attention to fans $\mathcal{F}$ that are *complete*, i.e., such that every $x \in H_\mathbb{R}$ is contained in some cone $\sigma \in \mathcal{F}$, or *polytopal*, i.e., the normal fans of rational polytopes in $H_\mathbb{R}^*$. Note that a polytopal fan is complete, but the converse is false if $b \geq 3$ (see [Oda 1988, p. 84] for an example). In the subsequent sections, we will apply the results of this section to the cographic fan $\mathcal{F}_\Gamma^\perp$ of a graph Γ, which is polytopal by Proposition 3.8.

Note that the fan $\mathcal{F}$ is naturally a poset: given $\sigma, \sigma' \in \mathcal{F}$, we say that $\sigma \geq \sigma'$ if $\sigma \supseteq \sigma'$. The poset $(\mathcal{F}, \geq)$ has some nice properties, which we now describe. Recall the following standard concepts from poset theory. A (finite) poset $(P, \leq)$ is called a *meet-semilattice* if every two elements $x, y \in P$ have a meet (i.e., an element, denoted by $x \wedge y$, that is uniquely characterized by conditions $x \wedge y \leq x, y$ and, if $z \in P$ is such that $z \leq x, y$, then $z \leq x \wedge y$). In a meet-semilattice, every finite subset of elements $\{x_1, \ldots, x_n\} \subset P$ admits a meet, denoted by $x_1 \wedge \cdots \wedge x_n$. A meet-semilattice is called *bounded* (from below) if it has a minimum element $\underline{0}$. A bounded meet-semilattice is called *graded* if, for every element $x \in P$, all maximal

chains from $\underline{0}$ to x have the same length. If this is the case, we define a function, called the *rank function*, $\rho : P \to \mathbb{N}$ by setting $\rho(x)$ equal to the length of any maximal chain from $\underline{0}$ to x. A graded meet-semilattice is said to be *pure* if all the maximal elements have the same rank, and this maximal rank is called the *rank* of the poset and is denoted by $\operatorname{rk} P$. A graded meet-semilattice is said to be *generated in maximal rank* if every element of P can be obtained as the meet of a subset consisting of maximal elements.

Having made these preliminary remarks, we now collect some of the properties of the poset $(\mathscr{F}, \geq)$ that we will need later.

Lemma 4.1. *The poset $(\mathscr{F}, \geq)$ has the following properties*:

(i) *$(\mathscr{F}, \geq)$ is a meet-semilattice, where the meet of two cones is equal to their intersection.*

(ii) *$(\mathscr{F}, \geq)$ is bounded with minimum element $\underline{0}$ given by the zero cone* $\{0\}$.

(iii) *$(\mathscr{F}, \geq)$ is a graded semilattice with rank function given by $\rho(\sigma) := \dim \sigma$.*

(iv) *If $\mathscr{F}$ is complete, then $(\mathscr{F}, \geq)$ is pure of rank $\operatorname{rk} \mathscr{F} = b$.*

(v) *If $\mathscr{F}$ is complete, then $(\mathscr{F}, \geq)$ is generated in maximal rank.* □

We will denote by $\mathscr{F}_{\max}$ the subset of $\mathscr{F}$ consisting of the *maximal cones* of $\mathscr{F}$.

4A. ***Descriptions of $R(\mathscr{F})$ as an inverse limit and as a quotient.*** In this subsection, we give two descriptions of the toric face ring $R(\mathscr{F})$.

The first description of $R(\mathscr{F})$ is as an inverse limit of affine semigroup rings. For any cone $\sigma \in \mathscr{F}$, consider the semigroup

$$C(\sigma) := \sigma \cap H_{\mathbb{Z}} \subset H_{\mathbb{Z}}, \tag{4-1}$$

which, according to Gordan's lemma (e.g., [Bruns and Herzog 1993, Proposition 6.1.2]), is a positive normal affine semigroup, i.e., a finitely generated semigroup isomorphic to a subsemigroup of $\mathbb{Z}^d$ for some $d \in \mathbb{N}$ such that 0 is the unique invertible element and such that if $m \cdot z \in C(\sigma)$ for some $m \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ then $z \in C(\sigma)$.

Definition 4.2. We define $R_k(\sigma) := k[C(\sigma)]$ to be the affine semigroup ring associated to $C(\sigma)$ (in the sense of [Bruns and Herzog 1993, §6.1]), i.e., the k-algebra whose underlying vector space has basis $\{X^c : c \in C(\sigma)\}$ and whose multiplication is defined by $X^c \cdot X^{c'} := X^{c+c'}$. We will write $R(\sigma)$ if we do not need to specify the base field k. If $\mathscr{F}_\sigma$ is the fan induced by σ (consisting of the cones in $\mathscr{F}$ that are faces of σ), then clearly $R(\sigma) = R(\mathscr{F}_\sigma)$.

The following properties are well-known.

Lemma 4.3. *$R(\sigma)$ is a normal, Cohen–Macaulay domain of dimension equal to* $\dim \sigma$.

Proof. By definition, we have $R(\sigma) \subset k[H_{\mathbb{Z}}] = k[x_1^{\pm 1}, \ldots, x_b^{\pm 1}]$; hence, $R(\sigma)$ is a domain. $R(\sigma)$ is normal by [Bruns and Herzog 1993, Theorem 6.1.4] and Cohen–Macaulay by a theorem of Hochster [loc. cit., Theorem 6.3.5(a)]. Finally, it follows easily from [loc. cit., Proposition 6.1.1] that the (Krull) dimension of $R(\sigma)$ is equal to $\dim \sigma$. □

Given two elements $\sigma, \sigma' \in \mathcal{F}$ such that $\sigma \geq \sigma'$, or equivalently such that $\sigma \supseteq \sigma'$, there exists a natural projection map between the corresponding affine semigroup rings of Definition 4.2

$$r_{\sigma/\sigma'} : R(\sigma) \twoheadrightarrow R(\sigma'), \quad X^c \mapsto \begin{cases} X^c & \text{if } c \in \sigma' \subseteq \sigma, \\ 0 & \text{if } c \in \sigma \setminus \sigma'. \end{cases}$$

With respect to these maps, the set $\{R(\sigma) : \sigma \in \mathcal{F}\}$ forms an inverse system of rings. From [Bruns et al. 2008, Proposition 2.2], we deduce the following description of $R(\mathcal{F})$:

Proposition 4.4. *Let $\mathcal{F}$ be a fan. We have an isomorphism*

$$R(\mathcal{F}) = \varprojlim_{\sigma \in \mathcal{F}} R(\sigma).$$

We denote by $r_\sigma : R(\mathcal{F}) \to R(\sigma)$ the natural projection maps.

The second description of $R(\mathcal{F})$ is as a quotient of a polynomial ring. For any cone $\sigma \in \mathcal{F}$, the semigroup $C(\sigma) = \sigma \cap H_{\mathbb{Z}}$ has a unique minimal generating set, called the *Hilbert basis* of $C(\sigma)$ and denoted by $\mathcal{H}_\sigma$ [Miller and Sturmfels 2005, Proposition 7.15]. Therefore, we have a surjection

$$\pi_\sigma : k[V_\alpha : \alpha \in \mathcal{H}_\sigma] \twoheadrightarrow R(\sigma), \quad V_\alpha \mapsto X^\alpha. \tag{4-2}$$

In the terminology of [Sturmfels 1996, Chapter 4], the kernel of π_σ, which we denote by I_σ, is the *toric ideal* associated to the subset $\mathcal{H}_\sigma$. In the terminology of [Miller and Sturmfels 2005, Chapter II.7], I_σ is the *lattice ideal* associated with the kernel of the group homomorphism

$$p_\sigma : \mathbb{Z}^{\mathcal{H}_\sigma} \to H_{\mathbb{Z}}, \quad \underline{u} = \{u_\alpha\}_{\alpha \in \mathcal{H}_\sigma} \mapsto \sum_{\alpha \in \mathcal{H}_\sigma} u_\alpha \alpha.$$

From [Sturmfels 1996, Lemma 4.1] (see also [Miller and Sturmfels 2005, Theorem 7.3]), we get that I_σ is a binomial ideal with the explicit presentation

$$I_\sigma = \langle V^{\underline{u}} - V^{\underline{v}} : \underline{u}, \underline{v} \in \mathbb{N}^{\mathcal{H}_\sigma} \subset \mathbb{Z}^{\mathcal{H}_\sigma} \text{ with } p_\sigma(\underline{u}) = p_\sigma(\underline{v}) \rangle, \tag{4-3}$$

where, for any $\underline{u} = (u_\alpha)_{\alpha \in \mathcal{H}_\sigma} \in \mathbb{N}^{\mathcal{H}_\sigma}$, we set $V^{\underline{u}} := \prod_{\alpha \in \mathcal{H}_\sigma} V_\alpha^{u_\alpha} \in k[V_\alpha : \alpha \in \mathcal{H}_\sigma]$.

If we set $\mathcal{H}_{\mathcal{F}} := \bigcup_{\sigma \in \mathcal{F}} \mathcal{H}_\sigma$, then, from Definition 1.2, it follows that we have a surjection

$$\pi_{\mathcal{F}} : k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}] \twoheadrightarrow R(\mathcal{F}), \quad V_\alpha \mapsto X^\alpha. \tag{4-4}$$

We denote by $I_{\mathcal{F}}$ the kernel of $\pi_{\mathcal{F}}$. In order to describe the ideal $I_{\mathcal{F}}$, we introduce the abstract simplicial complex $\Delta_{\mathcal{F}}$ on the vertex set $\mathcal{H}_{\mathcal{F}}$ whose faces are the collections

of elements of $\mathscr{H}_{\mathscr{F}}$ that belong to the same cone of $\mathscr{F}$. The minimal nonfaces of $\Delta_{\mathscr{F}}$ are formed by pairs $\{\alpha, \alpha'\}$ of elements of $\mathscr{H}_{\mathscr{F}}$ such that α and α' do not belong to the same cone of $\mathscr{F}$; hence, $\Delta_{\mathscr{F}}$ is a flag complex [Stanley 1996, Chapter III, §4]. Consider the Stanley–Reisner ring (or face ring)

$$k[\Delta_{\mathscr{F}}] := \frac{k[V_\alpha : \alpha \in \mathscr{H}_{\mathscr{F}}]}{(V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathscr{F}})}$$

associated to the flag complex $\Delta_{\mathscr{F}}$ (see [Stanley 1996, Chapter II] for an introduction to Stanley–Reisner rings). Observe that if $\{\alpha, \alpha'\} \notin \Delta_{\mathscr{F}}$, then $X^{[\alpha]} \cdot X^{[\alpha']} = 0$ by Definition 1.2. This implies that the surjection $\pi_{\mathscr{F}}$ factors as

$$\pi_{\mathscr{F}} : k[V_\alpha : \alpha \in \mathscr{H}_{\mathscr{F}}] \twoheadrightarrow \frac{k[V_\alpha : \alpha \in \mathscr{H}_{\mathscr{F}}]}{(V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathscr{F}})} = k[\Delta_{\mathscr{F}}] \twoheadrightarrow R(\mathscr{F})$$

or in other words that $(V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathscr{F}}) \subseteq I_{\mathscr{F}}$.

Moreover, observe also that the surjection $\pi_{\mathscr{F}}$ of (4-4) is compatible with the surjections π_σ of (4-2) for every $\sigma \in \mathscr{F}$ in the sense that we have a commutative diagram

$$\begin{array}{ccc} k[V_\alpha : \alpha \in \mathscr{H}_{\mathscr{F}}] & \xrightarrow{\pi_{\mathscr{F}}} & R(\mathscr{F}) \\ {\scriptstyle\theta}\big\downarrow\big\uparrow{\scriptstyle s} & & \big\downarrow{\scriptstyle r_\sigma} \\ k[V_\alpha : \alpha \in \mathscr{H}_\sigma] & \xrightarrow{\pi_\sigma} & R(\sigma) \end{array} \tag{4-5}$$

where θ is the surjective ring homomorphism given by sending $V_\alpha \mapsto V_\alpha$ if $\alpha \in \mathscr{H}_\sigma \subseteq \mathscr{H}_{\mathscr{F}}$ and $V_\alpha \mapsto 0$ if $\alpha \in \mathscr{H}_{\mathscr{F}} \setminus \mathscr{H}_\sigma$. Both the vertical surjections have natural sections: the left map has a section s obtained by sending $V_\alpha \mapsto V_\alpha$ for any $\alpha \in \mathscr{H}_\sigma \subset \mathscr{H}_{\mathscr{F}}$, and the left map has a section obtained by sending X^c into X^c for any $c \in C(\sigma) = \sigma \cap H_{\mathbb{Z}} \subset H_{\mathbb{Z}}$. Therefore, we can regard I_σ as an ideal of $k[V_\alpha : \alpha \in \mathscr{H}_{\mathscr{F}}]$ by extensions of scalars and, by the above commutative diagram, we have that $I_\sigma \subseteq I_{\mathscr{F}}$.

From [Bruns et al. 2008, Propositions 2.3 and 2.6], we get the following description of the ideal $I_{\mathscr{F}}$:

Proposition 4.5. *Let $\mathscr{F}$ be a fan. The kernel $I_{\mathscr{F}}$ of the map $\pi_{\mathscr{F}}$ of (4-4) is given by*

$$I_{\mathscr{F}} = (V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathscr{F}}) + \sum_{\sigma \in \mathscr{F}} I_\sigma = (V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathscr{F}}) + \sum_{\sigma \in \mathscr{F}_{\max}} I_\sigma,$$

where, as usual, $\mathscr{F}_{\max}$ denotes the subset of $\mathscr{F}$ consisting of the maximal cones.

4B. ***Prime ideals of $R(\mathscr{F})$.*** We now want to describe the prime ideals of the ring $R(\mathscr{F})$. Observe that, from the Definition 1.2, it follows that $R(\mathscr{F})$ has a natural $\mathbb{Z}^b \cong H_{\mathbb{Z}}$-grading.

Recall the following notions for a $\mathbb{Z}^n$-graded ring R (see, e.g., [Uliczka 2009]). A *graded ideal* is an ideal I of R with the property that for any $x \in I$ all homogenous components of x belong to I as well; this is equivalent to I being generated by

homogenous elements. For any ideal I of R, the *graded core* I^* of I is defined as the ideal generated by all homogenous elements of I. It is the largest graded ideal contained in I. If $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p}^*$ is a prime ideal [Uliczka 2009, Lemma 1.1(ii)].

For any $\sigma \in \mathcal{F}$, the kernel of the natural projection map $r_\sigma : R(\mathcal{F}) \twoheadrightarrow R(\sigma)$, which is explicitly equal to

$$\mathfrak{p}_\sigma := (\{X^c : c \notin \sigma\}), \tag{4-6}$$

is graded since it is generated by homogeneous elements and is prime by Lemma 4.3. From [Ichim and Römer 2007, Lemma 2.1], we deduce the following description of the graded ideals of $R(\mathcal{F})$:

Proposition 4.6. *The assignment $\sigma \mapsto \mathfrak{p}_\sigma$ gives an isomorphism between the poset $(\mathcal{F}, \geq)$ and the poset of graded prime ideals of $R(\mathcal{F})$ ordered by reverse inclusion. In particular, $\mathfrak{m} = \mathfrak{p}_{\{0\}}$ is the unique graded maximal ideal of $R(\mathcal{F})$, which is also a maximal ideal in the usual sense.*

From Proposition 4.6, we can deduce a description of the minimal primes of $R(\mathcal{F})$.

Corollary 4.7. *The minimal primes of $R(\mathcal{F})$ are the primes $\mathfrak{p}_\sigma$ as σ varies among all the maximal cones of $\mathcal{F}$. In particular, if $\mathcal{F}$ is complete, then $R(\mathcal{F})$ is of pure dimension b.*

Proof. Observe that if $\mathfrak{p}$ is a minimal ideal of $R(\mathcal{F})$, then $\mathfrak{p}^* = \mathfrak{p}$ by the minimality of $\mathfrak{p}$; hence, $\mathfrak{p}$ is graded. Conversely, if $\mathfrak{p}$ is a graded ideal of $R(\mathcal{F})$ that is minimal among the graded ideals of $R(\mathcal{F})$, then $\mathfrak{p}$ is also a minimal ideal of $R(\mathcal{F})$: indeed, if $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q}^* = \mathfrak{p}$ by the minimality properties of $\mathfrak{p}$; hence, $\mathfrak{q} = \mathfrak{p}$.

It is now clear that the first assertion follows from Proposition 4.6. The last assertion follows from the first one together with Lemmas 4.1(iv) and 4.3. □

Definition 4.8. The poset of *strata* of $R(\mathcal{F})$, denoted by $\mathrm{Str}(R(\mathcal{F}))$, is the set of all the ideals of $R(\mathcal{F})$ that are sums of minimal primes with the order relation given by reverse inclusion.

Geometrically, the poset $\mathrm{Str}(R(\mathcal{F}))$ is the collection of all scheme-theoretic intersections of irreducible components of $\mathrm{Spec}\, R(\mathcal{F})$ ordered by inclusion.

Corollary 4.9. *If $\mathcal{F}$ is complete, then the assignment $\sigma \mapsto \mathfrak{p}_\sigma$ gives an isomorphism between $(\mathcal{F}, \geq)$ and* $\mathrm{Str}(R(\mathcal{F}))$.

Proof. The statement will follow from Proposition 4.6 if we show that the ideals that are sums of minimal primes of $R(\mathcal{F})$ are exactly those of the form $\mathfrak{p}_\sigma$ for some $\sigma \in \mathcal{F}$. Indeed, given minimal primes $\mathfrak{p}_{\sigma_i}$ for $i = 1, \ldots, n$ (see Corollary 4.7), we have that $\bigcap_{i=1}^n \sigma_i = \sigma$ for some $\sigma \in \mathcal{F}$ and, from (4-6), it follows that

$$\sum_{i=1}^n \mathfrak{p}_{\sigma_i} = \left(X^c : c \notin \bigcap_{i=1}^n \sigma_i\right) = \mathfrak{p}_\sigma. \tag{4-7}$$

Conversely, every cone $\sigma \in \mathcal{F}$ is the intersection of the maximal dimensional cones σ_i containing it by Lemma 4.1(v). Therefore, (4-7) shows that $\mathfrak{p}_\sigma \in \mathrm{Str}(R(\mathcal{F}))$. □

4C. ***Gorenstein singularities.*** The aim of this subsection is to prove the following:

Theorem 4.10. *If $\mathcal{F}$ is a polytopal fan, then $R(\mathcal{F})$ is a Gorenstein ring and its canonical module $\omega_{R(\mathcal{F})}$ is isomorphic to $R(\mathcal{F})$ as a graded module.*

Proof. This is a consequence of two results from [Ichim and Römer 2007]. The first is Theorem 1.1, stating that a toric face ring $R(\mathcal{F})$ is Cohen–Macaulay provided that the fan $\mathcal{F}$ is *shellable* (see p. 252 of that paper for the definition). The second is Theorem 1.4, stating that $R(\mathcal{F})$ is Gorenstein and its canonical module $\omega_{R(\mathcal{F})}$ is isomorphic to $R(\mathcal{F})$ as a graded module provided that $R(\mathcal{F})$ is Cohen–Macaulay and $\mathcal{F}$ is *Eulerian* (see Definition 6.4 in the same paper).

Now it is enough to recall that a polytopal fan is Eulerian (see, e.g., [Stanley 1994, p. 302]) and shellable by the Bruggesser–Manni theorem [Bruns and Herzog 1993, Theorem 5.2.14]. □

4D. ***The normalization.*** In this subsection, we prove that the toric face ring of any fan is seminormal and we describe its normalization.

Recall that, given a reduced ring R with total quotient ring $Q(R)$, the *normalization* of R, denoted by $\bar{R}$, is the integral closure of R inside $Q(R)$. R is said to be normal if $R = \bar{R}$ (see [Huneke and Swanson 2006, Definition 1.5.1], for example). Moreover, we need the following:

Definition 4.11. Let R be a Mori ring, i.e., a reduced ring such that $\bar{R}$ is finite over R. The *seminormalization* of R, denoted by ${}^+R$, is the biggest subring of $\bar{R}$ such that the induced pull-back map $\mathrm{Spec}({}^+R) \to \mathrm{Spec}\, R$ is bijective with trivial residue field extension. We say that R is seminormal if ${}^+R = R$.

For the basic properties of seminormal rings, we refer to [Greco and Traverso 1980; Swan 1980]. Observe that $R(\mathcal{F})$ is a Mori ring since it is reduced and finitely generated over a field k (see Remark 1.3).

Theorem 4.12. *Let $\mathcal{F}$ be any fan.*

(i) *The normalization of $R(\mathcal{F})$ is equal to*

$$\overline{R(\mathcal{F})} = \prod_{\sigma \in \mathcal{F}_{\max}} R(\sigma),$$

where $\mathcal{F}_{\max}$ is the subset of $\mathcal{F}$ consisting of all the maximal cones of $\mathcal{F}$.

(ii) *$R(\mathcal{F})$ is a seminormal ring.*

Proof. Let us first prove part (i). By [Huneke and Swanson 2006, Corollary 2.1.13] and Corollary 4.7, we get that the normalization of $R(\mathcal{F})$ is equal to

$$\overline{R(\mathcal{F})} = \prod_{\sigma \in \mathcal{F}_{\max}} \overline{R(\sigma)}.$$

We conclude by Lemma 4.3, which says that each domain $R(\sigma)$ is normal.

Let us now prove part (ii). According to Proposition 4.4 and Lemma 4.3, the ring $R(\mathcal{F})$ is an inverse limit of normal domains. Then the seminormality of $R(\mathcal{F})$ follows from [Swan 1980, Corollary 3.3]. □

4E. ***Semi log canonical singularities.*** In this subsection, we prove that Spec $R(\mathcal{F})$ has semi log canonical singularities provided that $\mathcal{F}$ is a polytopal fan.

We first recall the definitions of log canonical and semi log canonical pairs (see [Kollár and Mori 1998] for log canonical pairs and [Abramovich et al. 1992; Fujino 2000] for semi log canonical pairs). For the relevance of slc singularities in the theory of compactifications of moduli spaces, see [Kollár 2010].

Definition 4.13. Let X be an S_2 variety (i.e., such that the local ring $\mathcal{O}_{X,x}$ of X at any (schematic) point $x \in X$ has depth at least $\min\{2, \dim \mathcal{O}_{X,x}\}$) of pure dimension n over a field k and Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

(i) We say that the pair (X, Δ) is log canonical (or *lc* for short) if

- X is smooth in codimension 1 (or equivalently X is normal) and
- there exists a log resolution $f : Y \to X$ of (X, Δ) such that

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i,$$

where E_i are divisors on Y and $a_i \geq -1$ for every i.

We say that X is lc if the pair $(X, 0)$ is lc, where 0 is the zero divisor.

(ii) We say that the pair (X, Δ) is semi log canonical (or *slc* for short) if

- X is nodal in codimension 1 (or equivalently, X is seminormal and Gorenstein in codimension 1) and
- if $\mu : X^\mu \to X$ is the normalization of X and Θ is the $\mathbb{Q}$-Weil divisor on X given by

$$K_{X^\mu} + \Theta = \mu^*(K_X + \Delta), \tag{4-8}$$

then the pair (X^μ, Θ) is lc.

We say that X is slc if the pair $(X, 0)$ is slc, where 0 is the zero divisor.

Theorem 4.14. *If $\mathcal{F}$ is a polytopal fan, then the variety* Spec $R(\mathcal{F})$ *is slc.*

Proof. Observe that Spec $R(\mathscr{F})$ is Gorenstein by Theorem 4.10 and seminormal by Theorem 4.12(ii); hence, in particular, it is S_2 and nodal in codimension 1 [Greco and Traverso 1980, §8]. Moreover, Spec $R(\mathscr{F})$ is of pure dimension $\operatorname{rk}\mathscr{F}$ by Corollary 4.7. Consider now the normalization morphism (see Theorem 4.12(i))

$$\mu : \operatorname{Spec} \overline{R(\mathscr{F})} = \coprod_{\sigma \in \mathscr{F}_{\max}} \operatorname{Spec} R(\sigma) \to \operatorname{Spec} R(\mathscr{F}).$$

If we apply the formula (4-8) to the above morphism μ and we use the fact that $\Delta = 0$ (by hypothesis) and $K_X = 0$ by Theorem 4.10, then we get that the divisor Θ restricted to each connected component Spec $R(\sigma)$ of the normalization Spec $\overline{R(\mathscr{F})}$ is equal to $-K_{\operatorname{Spec} R(\sigma)}$. Therefore, from Definition 4.13(ii), we get that Spec $R(\mathscr{F})$ is slc if and only if the pair $(\operatorname{Spec} R(\sigma), -K_{\operatorname{Spec} R(\sigma)})$ is lc for every $\sigma \in \mathscr{F}_{\max}$. Therefore, we conclude using the fact that for any toric variety Z the pair $(Z, -K_Z)$ is lc [Fujino and Sato 2004, Proposition 2.10; Cox et al. 2011, Corollary 11.4.25]. □

4F. ***Embedded dimension.*** In this subsection, we compute the embedded dimension of $R(\mathscr{F})$ at its unique graded maximal ideal $\mathfrak{m}$. In doing this, we also compute the embedded dimension of the affine semigroup ring $R(\sigma)$ of Definition 4.2 at the maximal ideal $(X^c : c \in C(\sigma) \setminus \{0\})$, which, by a slight abuse of notation, we also denote by $\mathfrak{m}$.

Recall that given a maximal ideal $\mathfrak{m}$ of a ring R with residue field $k := R/\mathfrak{m}$, the embedded dimension of R at $\mathfrak{m}$ is the dimension of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. Geometrically, the embedded dimension of R at $\mathfrak{m}$ is the dimension of the Zariski tangent space of $\operatorname{Spec}(R)$ at the point $\mathfrak{m} \in \operatorname{Spec}(R)$.

Theorem 4.15. *Let $\mathscr{F}$ be a fan.*

(i) *The embedded dimension of $R(\sigma)$ at $\mathfrak{m}$ is equal to the cardinality of the Hilbert basis $\mathscr{H}_\sigma$* (see Section 4A).

(ii) *The embedded dimension of $R(\mathscr{F})$ at $\mathfrak{m}$ is equal to the cardinality of $\mathscr{H}_{\mathscr{F}}$ $(= \bigcup_{\sigma \in \mathscr{F}} \mathscr{H}_\sigma)$.*

Proof. Consider the presentation (4-2) of the ring $R(\sigma)$. Since the elements of the Hilbert basis $\mathscr{H}_\sigma$ cannot be written in a nontrivial way as $\mathbb{N}$-linear combinations of elements in the semigroup $C(\sigma)$ [Miller and Sturmfels 2005, proof of Proposition 7.15], we get that the ideal $I_\sigma = \ker \pi_\sigma$ satisfies

$$I_\sigma \subset \mathfrak{n}^2, \tag{4-9}$$

where $\mathfrak{n} := (V_\alpha : \alpha \in \mathscr{H}_\sigma) \subset k[V_\alpha : \alpha \in \mathscr{H}_\sigma]$. Part (i) now follows from (4-2) and (4-9).

In order to prove part (ii), consider the presentation (4-4) of the ring $R(\mathcal{F})$. It is enough to prove that the ideal $I_{\mathcal{F}} = \ker \pi_{\mathcal{F}}$ satisfies

$$I_{\mathcal{F}} \subset \mathfrak{o}^2, \tag{4-10}$$

where $\mathfrak{o} := (V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}) \subset k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}]$. Consider the generators of $I_{\mathcal{F}}$ given in Proposition 4.5. Clearly the generators of the form $V_\alpha V_{\alpha'}$ (for $\{\alpha, \alpha'\} \notin \Delta_{\mathcal{F}}$) belong to $\mathfrak{o}^2$. In order to deal with the other generators of $I_{\mathcal{F}}$, consider the diagram (4-5). As in the discussion that precedes Proposition 4.5, we view I_σ as included in $I_{\mathcal{F}}$ via the section s. By applying the section s to the inclusion (4-9) and using the obvious inclusion $s(\mathfrak{n}^2) \subseteq \mathfrak{o}^2$, we get the desired inclusion (4-10). □

4G. ***Multiplicity.*** In this subsection, we study the multiplicity $e_{\mathfrak{m}}(R(\mathcal{F}))$ of $R(\mathcal{F})$ at its unique graded maximal ideal $\mathfrak{m}$.

Recall (see, e.g., [Serre 1965, Chapter IIB, Theorem 3]) that the Hilbert–Samuel function

$$n \mapsto \dim_k R(\mathcal{F})/\mathfrak{m}^n$$

is given, for large values of $n \in \mathbb{N}$, by a polynomial (called the Hilbert–Samuel polynomial) that is denoted by $P_{\mathfrak{m}}(R(\mathcal{F}); n)$. The degree of $P_{\mathfrak{m}}(R(\mathcal{F}); n)$ is equal to $\dim R(\mathcal{F})$ [Serre 1965, Chapter IIIB, Theorem 1]. We can therefore write

$$P_{\mathfrak{m}}(R(\mathcal{F}); n) = e_{\mathfrak{m}}(R(\mathcal{F}))\frac{n^{\dim R(\mathcal{F})}}{\dim R(\mathcal{F})!} + O(n^{\dim R(\mathcal{F})-1}),$$

where $O(n^t)$ denotes a polynomial of degree less than or equal to t and $e_{\mathfrak{m}}(R(\mathcal{F}))$ is, by definition, the multiplicity of $R(\mathcal{F})$ at $\mathfrak{m}$ [Serre 1965, Chapter VA]. The following result is a special case of [Matsumura 1989, Theorem 14.7]:

Theorem 4.16. *If $\mathcal{F}$ is a fan of dimension d (i.e., such that the maximum of the dimension of the cones in $\mathcal{F}$ is d) in $\mathbb{R}^b$, then $R(\mathcal{F})$ has dimension d and its multiplicity is equal to*

$$e_{\mathfrak{m}}(R(\mathcal{F})) = \sum_{\dim \sigma = d} e_{\mathfrak{m}}(R(\sigma)),$$

where $\mathfrak{m}$ is the unique graded maximal ideal of the rings in question.

Proof. The theorem is the special case of [Matsumura 1989, Theorem 14.7], where $A = R(\mathcal{F})$ and $\mathfrak{q} = \mathfrak{m}$. Indeed, the rings $R(\sigma)$ are the localizations of $R(\mathcal{F})$ at minimal primes $\mathfrak{q}$ satisfying $\dim R(\mathcal{F})/\mathfrak{q} = d$ by Corollary 4.7. □

The above result reduces the computation of the multiplicity of $R(\mathcal{F})$ at $\mathfrak{m}$ (for a complete fan $\mathcal{F}$) to that of the affine semigroup rings $R(\sigma)$ at $\mathfrak{m}$ for σ a cone of $\mathcal{F}$ of maximal dimension. These latter multiplicities can be computed geometrically

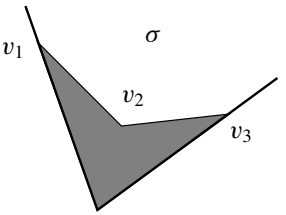

Figure 1. A two-dimensional cone σ whose associated semigroup $C(\sigma)$ has Hilbert basis $\mathscr{H}_\sigma = \{v_1, v_2, v_3\}$. The shaded region is the subdiagram part $K_-(C(\sigma))$ of $C(\sigma)$.

from the affine semigroup $C(\sigma)$ as we now explain following Gel′fand, Kapranov, and Zelevinsky [Gel′fand et al. 1994].

To that aim, we need to recall some definitions. Given a cone $\sigma \in \mathscr{F}$, set $C(\sigma)_\mathbb{Z} := \langle\sigma\rangle \cap H_\mathbb{Z}$ and $C(\sigma)_\mathbb{R} := \langle\sigma\rangle \cap H_\mathbb{R}$. We denote by $\mathrm{vol}_{C(\sigma)}$ the unique translation-invariant measure on $C(\sigma)_\mathbb{R}$ such that the volume of a standard unimodular simplex Δ (i.e., Δ is the convex hull of a basis of $H_\mathbb{Z}$ together with 0) is 1. Following [Gel′fand et al. 1994, p. 184], denote by $K_+(C(\sigma))$ the convex hull of the set $C(\sigma) \setminus \{0\}$ and $K_-(C(\sigma))$ the closure of $\sigma \setminus K_+(C(\sigma))$. The set $K_-(C(\sigma))$ is a bounded (possibly not convex) lattice polyhedron in $C(\sigma)_\mathbb{R}$ that is called the *subdiagram part* of $C(\sigma)$.

Definition 4.17 [Gel′fand et al. 1994, Chapter 5, Definition 3.8]. The *subdiagram volume* of $C(\sigma)$ is the natural number

$$u(C(\sigma)) := \mathrm{vol}_{C(\sigma)_\mathbb{Z}}(K_-(C(\sigma))).$$

The multiplicity of $R(\sigma)$ at $\mathfrak{m}$ can be computed in terms of the subdiagram volume of $C(\sigma)$ as asserted by the following result, whose proof can be found in [Gel′fand et al. 1994, Chapter 5, Theorem 3.14]:

Theorem 4.18. *The multiplicity of $R(\sigma)$ at $\mathfrak{m}$ is equal to*

$$e_\mathfrak{m}(R(\sigma)) = u(C(\sigma)).$$

5. Geometry of cographic rings

The aim of this section is to describe the properties of the cographic ring $R(\Gamma)$ associated to a graph Γ. The main results are Theorem 5.7 and the descriptions of the cographic ring in Section 5B. Recall from Definition 1.4 that $R(\Gamma)$ is the toric face ring associated to the cographic fan $\mathscr{F}_\Gamma^\perp$ in $H_1(\Gamma, \mathbb{R})$, which is a polytopal fan by Proposition 3.8.

According to Proposition 3.1(i), every cone of $\mathscr{F}_\Gamma^\perp$ is of the form

$$\sigma(T,\phi) := \bigcap_{e\notin T}\{(\,\cdot\,,\phi(e))\geq 0\}\cap\bigcap_{e\in T}\{(\,\cdot\,,e)=0\}$$

for some uniquely determined element $(T,\phi)\in\mathbb{OP}_\Gamma$, i.e., a totally cyclic orientation ϕ on $\Gamma\setminus T$. We will denote the positive normal affine semigroup associated to $\sigma(T,\phi)$ as in (4-1) by

$$C(\Gamma\setminus T,\phi) := C(\sigma(T,\phi)) = \sigma(Y,\phi)\cap H_1(\Gamma,\mathbb{Z})$$

and its associated affine semigroup ring (as in Definition 4.2)

$$R(\Gamma\setminus T,\phi) := k[C(\Gamma\setminus T,\phi)].$$

5A. ***Affine semigroup rings $R(\Gamma\setminus T,\phi)$.*** Let us look more closely at the affine semigroup rings $R(\Gamma\setminus T,\phi)$ for a fixed $(T,\phi)\in\mathbb{OP}_\Gamma$.

The ring $R(\Gamma\setminus T,\phi)$ is a normal, Cohen–Macaulay domain of dimension equal to $\dim\sigma(T,\phi) = b_1(\Gamma\setminus T)$ as follows from Lemma 4.3 and Proposition 3.1(ii). However, the ring $R(\Gamma\setminus T,\phi)$ need not be Gorenstein and indeed not even $\mathbb{Q}$-Gorenstein as the following example shows:

Example 5.1. Consider the totally cyclic oriented graph (Γ,ϕ) depicted in Figure 2.

Consider the pointed rational polyhedral cone $\sigma(\varnothing,\phi)\subset H_1(\Gamma,\mathbb{R})$ and its dual cone $\sigma(\varnothing,\phi)^\vee\subset H_1(\Gamma,\mathbb{R})^\vee$ defined by

$$\sigma(\varnothing,\phi)^\vee := \{\ell\in H_1(\Gamma,\mathbb{R})^\vee : \ell(v)\geq 0 \text{ for every } v\in\sigma(\varnothing,\phi)\}.$$

Since for any edge $e\in E(\Gamma)$, the graph $\Gamma\setminus\{e\}$ with the orientation induced by ϕ is totally cyclic, we get that the cone $\sigma(\varnothing,\phi)$ has five codimension-1 faces defined by the equations $\{(\,\cdot\,,\phi(e_i))=0\}$ for $i=1,\dots,5$ (see Corollary 3.2). This implies that the extremal rays of $\sigma(\varnothing,\phi)^\vee$ are the rays generated by $(\,\cdot\,,\phi(e_i))$ for $i=1,\dots,5$.

It follows from [Dais 2002, proof of Theorem 3.12] that $R(\Gamma,\phi)$ is $\mathbb{Q}$-Gorenstein if and only if there exists an element $m\in H_1(\Gamma,\mathbb{Q})$ such that $(m,\phi(e_i))=1$ for every

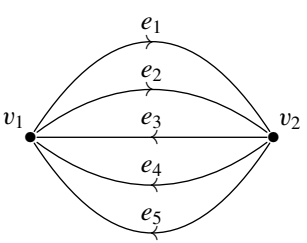

Figure 2. A totally cyclic oriented graph (Γ,ϕ) with $R(\Gamma,\phi)$ not $\mathbb{Q}$-Gorenstein.

$i = 1, \dots, 5$. However, these conditions force m to be equal to $m = \sum_{i=1}^{5} \phi(e_i)$, which is a contradiction since $\partial\big(\sum_{i=1}^{5} \phi(e_i)\big) = v_1 - v_2 \neq 0$.

Denote by $\mathscr{H}_{(\Gamma\setminus T,\phi)}$ the Hilbert basis (i.e., the minimal generating set) of the positive affine normal semigroup $C(\Gamma \setminus T, \phi)$. From Lemma 2.4, we get the following explicit description of $\mathscr{H}_{(\Gamma\setminus T,\phi)}$:

Proposition 5.2. *The Hilbert basis of $C(\Gamma \setminus T, \phi)$ is equal to*

$$\mathscr{H}_{(\Gamma\setminus T,\phi)} := \{[\gamma] : \gamma \in \mathrm{Cir}_{\phi}(\Gamma \setminus T)\} \subset H_1(\Gamma \setminus T, \mathbb{Z}) \subseteq H_1(\Gamma, \mathbb{Z}).$$

The Hilbert basis $\mathscr{H}_{(\Gamma\setminus T,\phi)}$ of $C(\Gamma \setminus T, \phi)$ enjoys the following remarkable properties:

Lemma 5.3. *Let $(T, \phi) \in \mathcal{OP}_{\Gamma}$.*

(i) *The group $\mathbb{Z} \cdot \mathscr{H}_{(\Gamma\setminus T,\phi)} \subseteq H_1(\Gamma \setminus T, \mathbb{Z})$ generated by $\mathscr{H}_{(\Gamma\setminus T,\phi)}$ coincides with $H_1(\Gamma \setminus T, \mathbb{Z})$.*

(ii) *The ray $\mathbb{R}_{\geq 0} \cdot [\gamma]$ is extremal for the cone $\sigma(T, \phi) = \mathbb{R}_{\geq 0} \cdot \mathscr{H}_{(\Gamma\setminus T,\phi)}$ for each $[\gamma] \in \mathscr{H}_{(\Gamma\setminus T)}$.*

Proof. Part (i) follows from Lemma 2.3(2)(c). Part (ii) follows from Proposition 3.1(iii). □

We warn the reader that the Hilbert basis $\mathscr{H}_{(\Gamma\setminus T,\phi)}$ need not be unimodular as we show in Example 5.4 below. Recall that a subset $\mathscr{A} \subset \mathbb{Z}^d$ is said to be *unimodular* if $\mathscr{A}$ spans $\mathbb{R}^d$ and, moreover, if we represent the elements of $\mathscr{A}$ as column vectors of a matrix A with respect to a basis of $\mathbb{Z}^d$, then all the nonzero $d \times d$ minors of A have the same absolute value [Sturmfels 1996, p. 70].

Example 5.4. Consider the totally cyclic oriented graph (Γ, ϕ) depicted in Figure 3. One can check that $b_1(\Gamma) = 4$ and that $\mathscr{H}_{(\Gamma,\phi)}$ consists of the eight elements

$$[\gamma_{ijk}] = \phi(e_i^1) + \phi(e_j^2) + \phi(e_k^3)$$

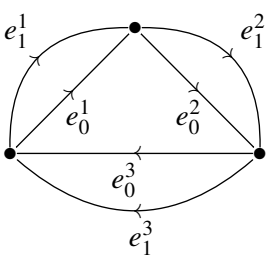

Figure 3. A totally cyclic oriented graph (Γ, ϕ) with $\mathscr{H}_{(\Gamma,\phi)}$ not totally unimodular.

for $i, j, k \in \{0, 1\}$. The elements $\mathcal{B} := \{[\gamma_{000}], [\gamma_{100}], [\gamma_{010}], [\gamma_{001}]\}$ form a basis of $H_1(\Gamma, \mathbb{Z})$. If we order the elements of $\mathcal{H}_{(\Gamma,\phi)}$ as

$$\{[\gamma_{000}], [\gamma_{100}], [\gamma_{010}], [\gamma_{001}], [\gamma_{110}], [\gamma_{101}], [\gamma_{011}], [\gamma_{111}]\},$$

then the elements of $\mathcal{H}_{(\Gamma,\phi)}$, with respect to the basis $\mathcal{B}$, are the column vectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The minor A_{1234} (i.e., the minor corresponding to the first four columns) is equal to 1 while the minor A_{2348} is equal to 2; hence, $\mathcal{H}_{(\Gamma,\phi)}$ is not unimodular.

According to (4-2) and (4-3), the affine semigroup ring $R(\Gamma \setminus T, \phi)$ admits the presentation

$$R(\Gamma \setminus T, \phi) := \frac{k[V_\gamma : \gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)]}{I_{(\Gamma\setminus T,\phi)}}, \tag{5-1}$$

where $I_{(\Gamma\setminus T,\phi)} := I_{\sigma(T,\phi)}$ is a binomial ideal, called the toric ideal associated to $\mathcal{H}_{(\Gamma\setminus T,\phi)}$ in the terminology of [Sturmfels 1996, Chapter 4]. The following problem seems interesting:

Problem 5.5. *Find generators for the binomial toric ideal* $I_{(\Gamma\setminus T,\phi)}$.

We warn the reader that the toric ideal $I_{(\Gamma\setminus T,\phi)}$ need not to be homogeneous as shown by the following example:

Example 5.6. Consider the totally cyclicly oriented graph (Γ, ϕ) depicted in Figure 4.

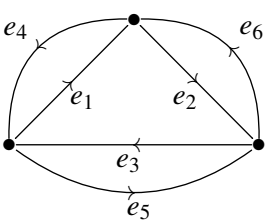

Figure 4. A totally cyclic oriented graph (Γ, ϕ) with $I_{(\Gamma,\phi)}$ not homogeneous.

It is easy to see that $b_1(\Gamma) = 4$ and that $\mathscr{H}_{(\Gamma,\phi)}$ consists of the five elements

$$\begin{aligned}
[\gamma_1] &:= \phi(e_1) + \phi(e_4),\\
[\gamma_2] &:= \phi(e_2) + \phi(e_6),\\
[\gamma_3] &:= \phi(e_3) + \phi(e_5),\\
[\gamma_4] &:= \phi(e_1) + \phi(e_2) + \phi(e_3),\\
[\gamma_5] &:= \phi(e_4) + \phi(e_4) + \phi(e_6).
\end{aligned}$$

The binomial ideal $I_{(\Gamma,\phi)}$ is generated by $V_{\gamma_1}V_{\gamma_2}V_{\gamma_3} - V_{\gamma_4}V_{\gamma_5}$; hence, it is not homogeneous.

5B. ***Descriptions of $R(\Gamma)$ as an inverse limit and as a quotient.*** Using the general results of Section 4A, the ring $R(\Gamma)$ admits two explicit descriptions.

The first description of $R(\Gamma)$ is as an inverse limit of affine semigroup rings (see Proposition 4.4):

$$R(\Gamma) = \varprojlim_{(T,\phi)\in\mathbb{OP}_\Gamma} R(\Gamma \setminus T, \phi). \tag{5-2}$$

The second description is a presentation of $R(\Gamma)$ as a quotient of a polynomial ring. In order to make this explicit for $R(\Gamma)$, observe first that the union of all the Hilbert bases of the cones $\sigma(T,\phi)$, as (T,ϕ) varies in $\mathbb{OP}_\Gamma$, is equal to the set of all oriented circuits of Γ, i.e.,

$$\mathscr{H}_{\mathscr{F}_\Gamma^\perp} = \overrightarrow{\mathrm{Cir}}(\Gamma). \tag{5-3}$$

Moreover, Corollary 3.4 implies that the simplicial complex $\Delta_{\mathscr{F}^\perp}$ introduced in Section 4A coincides with the simplicial complex $\Delta(\overrightarrow{\mathrm{Cir}}(\Gamma))$ of concordant circuits as in Definition 2.5, or in symbols,

$$\Delta_{\mathscr{F}^\perp} = \Delta(\overrightarrow{\mathrm{Cir}}(\Gamma)).$$

From (4-4), Proposition 4.5, and Lemma 2.9, we get the presentation of $R(\Gamma)$

$$R(\Gamma) = \frac{k[V_\gamma : \gamma \in \overrightarrow{\mathrm{Cir}}(\Gamma)]}{I_\Gamma}, \tag{5-4}$$

where $I_\Gamma := I_{\mathscr{F}_\Gamma^\perp}$ is explicitly given by

$$\begin{aligned}
I_\Gamma = (V_\gamma V_{\gamma'} : \gamma \not\asymp \gamma') + \sum_{(T,\phi)\in\mathbb{OP}_\Gamma} I_{(\Gamma\setminus T,\phi)} &= (V_\gamma V_{\gamma'} : \gamma \not\asymp \gamma')\\
&\quad + \sum_{(E(\Gamma)_{\mathrm{sep}},\phi)\in\mathbb{OP}_\Gamma} I_{(\Gamma\setminus E(\Gamma)_{\mathrm{sep}},\phi)}.
\end{aligned} \tag{5-5}$$

From Proposition 4.6, we get that the graded prime ideals of $R(\Gamma)$ are given by

$$\mathfrak{p}_{(T,\phi)} := (\{X^c : c \notin \sigma(T,\phi)\}) \tag{5-6}$$

as (T,ϕ) varies in $\mathbb{OP}_\Gamma$.

5C. ***Singularities of $R(\Gamma)$.*** In this subsection, we analyze the singularities of the ring $R(\Gamma)$.

Theorem 5.7. *Let Γ be a graph and $R(\Gamma)$ its associated cographic ring. Then we have the following*:

(i) *$R(\Gamma)$ is a reduced finitely generated k-algebra of pure dimension equal to $b_1(\Gamma)$. The minimal prime ideals of $R(\Gamma)$ are given by $\mathfrak{p}_{(E(\Gamma)_{\text{sep}},\phi)}$ as ϕ varies among all the totally cyclic orientations of $\Gamma \setminus E(\Gamma)_{\text{sep}}$.*

(ii) *$R(\Gamma)$ is Gorenstein, and its canonical module $\omega_{R(\Gamma)}$ is isomorphic to $R(\Gamma)$ as a graded module.*

(iii) *$R(\Gamma)$ is a seminormal ring.*

(iv) *The normalization of $R(\Gamma)$ is equal to*

$$\overline{R(\Gamma)} = \prod_{\phi} R(\Gamma \setminus E(\Gamma)_{\text{sep}}, \phi),$$

where the product is over all the totally cyclic orientations ϕ of $E(\Gamma) \setminus E(\Gamma)_{\text{sep}}$.

(v) *The variety* Spec $R(\Gamma)$ *is slc.*

(vi) *The embedded dimension of $R(\Gamma)$ at $\mathfrak{m}$ is equal to the cardinality of $\overrightarrow{\text{Cir}}(\Gamma)$, the set of oriented circuits on Γ.*

(vii) *The multiplicity of $R(\Gamma)$ at $\mathfrak{m}$ is equal to*

$$e_{\mathfrak{m}}(R(\Gamma)) = \sum_{\phi} e_{\mathfrak{m}}(R(\Gamma \setminus E(\Gamma)_{\text{sep}}, \phi)) = \sum_{\phi} u(C(\Gamma \setminus E(\Gamma)_{\text{sep}}, \phi)),$$

where the sum is over all the totally cyclic orientations ϕ of $\Gamma \setminus E(\Gamma)_{\text{sep}}$ and $\mathfrak{m}$ is the unique graded maximal ideal of the rings in question.

Proof. Part (i) follows from Remark 1.3, Corollary 4.7, and Lemma 2.9. Part (ii) follows Theorem 4.10 using that $\mathscr{F}_\Gamma^\perp$ is a polytopal fan by Proposition 3.8. Part (iii) follows from Theorem 4.12(ii). Part (iv) follows from Theorem 4.12(i) and Lemma 2.9. Part (v) follows from Theorem 4.14 using that $\mathscr{F}_\Gamma^\perp$ is polytopal. Part (vi) follows from Theorem 4.15(ii) and (5-3). Part (vii) follows from Theorem 4.16, Theorem 4.18, and Lemma 2.9. □

Problem 5.8. *Express the multiplicity of $R(\Gamma)$ at $\mathfrak{m}$ in terms of well-known graph invariants.*

Problem 5.9. *Characterize the graphs Γ that have the property that* Spec$(R(\Gamma))$ *is semi divisorial log terminal.* (*See* [*Fujino 2000, Definition 1.1*] *for the definition of semi divisorial log terminal.*)

Problem 5.9 is motivated by moduli theory. The singularities of $R(\Gamma)$ are the singularities that appear on compactified Jacobians, and compactified Jacobians arise as limits of abelian varieties. Fujino [2011] shows that, in a suitable sense, it is possible to degenerate an abelian variety to a semi divisorial log terminal variety. If $R(\Gamma)$ is semi divisorial log terminal, then compactified Jacobians are examples of Fujino's degenerations. For a general discussion of singularities and their role in moduli theory, we direct the reader to [Kollár 2010].

Following the proof of Theorem 4.14, Problem 5.9 is equivalent to the following one: characterize the totally cyclic orientations ϕ of a graph Γ that have the property that the pair $(\operatorname{Spec} R(\Gamma,\phi), -K_{R(\Gamma,\phi)})$ is divisorial log terminal (in the sense of [Kollár and Mori 1998]). Note that the pair $(\operatorname{Spec} R(\Gamma,\phi), -K_{R(\Gamma,\phi)})$ does not satisfy the stronger condition of being Kawamata log terminal (and so $\operatorname{Spec} R(\Gamma)$ is not semi Kawamata log terminal) because $-K_{R(\Gamma,\phi)}$ is effective and nonzero.

6. The cographic ring $R(\Gamma)$ as a ring of invariants

In [Casalaina-Martin et al. 2011], the completion of the ring $R(\Gamma)$ with respect to the maximal ideal $\mathfrak{m} = \mathfrak{p}_{\underline{0}}$ appears naturally as a ring of invariants. In this section, we explain this connection. Consider the multiplicative group

$$T_\Gamma := \prod_{v\in V(\Gamma)} \mathbb{G}_m.$$

The elements of $T_\Gamma(S)$ for a k-scheme S can be written as $\lambda = (\lambda_v)_{v\in V(\Gamma)}$ with $\lambda_v \in \mathbb{G}_m(S) = \mathcal{O}_S^*$.

Consider the ring

$$A(\Gamma) := \frac{k[U_{\vec{e}}, U_{\vec{e}} : e \in E(\Gamma)]}{(U_{\vec{e}}U_{\vec{e}} : e \in E(\Gamma))}.$$

If we make the group T_Γ act on $A(\Gamma)$ via

$$\lambda \cdot U_{\vec{e}} = \lambda_{s(\vec{e})} U_{\vec{e}} \lambda_{t(\vec{e})}^{-1},$$

then the invariant subring is described by the following theorem:

Theorem 6.1. *The invariant subring $A(\Gamma)^{T_\Gamma}$ is isomorphic to the cographic toric ring $R(\Gamma)$.*

Proof. We prove the theorem by exhibiting a k-basis for the invariant subring that is indexed by $H_1(\Gamma,\mathbb{Z})$ in such a way that multiplication satisfies Equation (1-1). We argue as follows. Grade $A(\Gamma)$ by the $\vec{C}_1(\Gamma,\mathbb{Z})$-grading induced by the obvious grading of $k[U_{\vec{e}}, U_{\vec{e}} : e \in E(\Gamma)]$ (so the weight of $U_{\vec{e}}$ is $\vec{e}$).

This grading is preserved by the action of T_Γ on $A(\Gamma)$, so the invariant subring is generated by invariant homogeneous elements. Furthermore, given a homogeneous

element $M^c = \prod U_{\vec{e}}^{a(\vec{e})}$ of weight $c = \sum a(\vec{e})\vec{e}$, an element $\lambda \in T_\Gamma$ acts as

$$\lambda \cdot M^c = \prod_{\vec{e}} \lambda_{s(\vec{e})} U_{\vec{e}} \lambda_{t(\vec{e})}^{-1} = \left(\prod_v \lambda_v^{b(v)} \right) M^c,$$

where $b(v)$ is defined by $\partial(c) = \sum b(v)v$. In particular, we see that M^c is invariant if and only if $\partial(c) = 0$, or in other words, $c \in H_1(\Gamma, \mathbb{Z})$.

We can conclude that the invariant subring is generated by the homogeneous elements M^c whose weight c lies in $H_1(\Gamma, \mathbb{Z})$. In fact, these elements freely generate the invariant subring because distinct elements have distinct weights.

To complete the proof, observe that multiplication satisfies

$$M^c \cdot M^{c'} = \begin{cases} 0 & \text{if } (c, \vec{e}) > 0 \text{ and } (c', \vec{e}) < 0 \text{ for some } \vec{e}, \\ M^{c+c'} & \text{otherwise.} \end{cases} \tag{6-1}$$

The condition that there exists an oriented edge $\vec{e}$ with $(c, \vec{e}) > 0$ and $(c', \vec{e}) < 0$ is equivalent to the condition that c and c' do not lie in a common cone by Corollary 3.4. We can conclude that the rule $X^c \mapsto M^c$ defines an isomorphism between the cographic ring $R(\Gamma)$ and the invariant subring of $A(\Gamma)$. □

7. A Torelli-type result for $R(\Gamma)$

In this section, we investigate when two graphs give rise to the same cographic toric face ring. Before stating the result, we need to briefly recall some operations in graph theory introduced in [Caporaso and Viviani 2010, §2]. Two graphs Γ and Γ' are said to be *cyclic equivalent* (or 2-isomorphic) if there exists a bijection $\epsilon : E(\Gamma) \to E(\Gamma')$ inducing a bijection on the circuits. The cyclic equivalence class of Γ is denoted by $[\Gamma]_{\text{cyc}}$. Given a graph Γ, a 3-*edge connectivization* of Γ is a graph that is obtained from Γ by contracting all the separating edges of Γ and by contracting, for every separating pair of edges, one of the two edges. While a 3-edge connectivization of Γ is not unique (because of the freedom that we have in performing the second operation), its cyclic equivalence class is well-defined; it is called the 3-*edge connected class* of Γ and denoted by $[\Gamma]^3_{\text{cyc}}$.

Theorem 7.1. *Let Γ and Γ' be two graphs. Then $R(\Gamma) \cong R(\Gamma')$ if and only if $[\Gamma]^3_{\text{Cyc}} = [\Gamma']^3_{\text{Cyc}}$.*

Proof. Assume first that $[\Gamma]^3_{\text{Cyc}} = [\Gamma']^3_{\text{Cyc}}$. From [Caporaso and Viviani 2010, proof of Proposition 3.2.3], it follows that $\mathscr{C}_\Gamma^\perp \cong \mathscr{C}_{\Gamma'}^\perp$, i.e., that there exists an $\mathbb{R}$-linear isomorphism $\phi : H_1(\Gamma, \mathbb{R}) \to H_1(\Gamma', \mathbb{R})$ that sends $H_1(\Gamma, \mathbb{Z})$ isomorphically onto $H_1(\Gamma', \mathbb{Z})$ and such that ϕ sends the hyperplanes of $\mathscr{C}_\Gamma^\perp$ bijectively onto the hyperplanes of $\mathscr{C}_{\Gamma'}^\perp$. Since $\mathscr{F}_\Gamma^\perp$ is the fan induced by the arrangement of hyperplanes $\mathscr{C}_\Gamma^\perp$, the above map ϕ will send the cones of $\mathscr{F}_\Gamma^\perp$ bijectively onto the cones

of $\mathscr{F}_{\Gamma'}^{\perp}$. Therefore, the map

$$R(\Gamma) \to R(\Gamma'), \quad X^c \mapsto X^{\phi(c)}$$

is an isomorphism of rings.

Conversely, if $R(\Gamma) \cong R(\Gamma')$, then clearly $\mathrm{Str}(R(\Gamma)) \cong \mathrm{Str}(R(\Gamma'))$ (see Definition 4.8). By Corollary 4.9, we deduce that $\mathcal{OP}_\Gamma \cong \mathcal{OP}_{\Gamma'}$, which implies that $[\Gamma]^3_{\mathrm{Cyc}} = [\Gamma']^3_{\mathrm{Cyc}}$ by [Caporaso and Viviani 2010, Theorem 5.3.2]. □

Acknowledgements

Casalaina-Martin was partially supported by the NSF grant DMS-1101333. Kass was partially supported by the NSF grant RTG DMS-0502170. Viviani is a member of the Centre for Mathematics of the University of Coimbra and is supported by FCT project Espaços de Moduli em Geometria Algébrica (PTDC/MAT/111332/2009) and by the MIUR project Spazi di moduli e applicazioni (FIRB 2012).

This work began when we were visiting the MSRI, in Berkeley, for the special semester in algebraic geometry in the spring of 2009; we would like to thank the organizers of the program as well as the institute for the excellent working conditions and the stimulating atmosphere. We would like to thank Bernd Sturmfels for his interest in this work, especially for some useful suggestions regarding multiplicity, and for pointing out a mistake in a previous version of this paper. We thank Farbod Shokrieh for some comments on an early draft of this manuscript and for pointing out the connection between some of our results and the theory of oriented matroids. We thank the referees for many useful comments and for pointing out a mistake in a previous version of this paper.

References

[Abramovich et al. 1992] D. Abramovich, L.-Y. Fong, J. Kollár, and J. McKernan, "Semi log canonical surfaces", pp. 139–158 in *Flips and abundance for algebraic threefolds* (Salt Lake City, UT, 1991), Astérisque **211**, Société Mathématique de France, Paris, 1992. Zbl 0799.14017

[Alexeev 2002] V. Alexeev, "Complete moduli in the presence of semiabelian group action", *Ann. of Math.* (2) **155**:3 (2002), 611–708. MR 2003g:14059 Zbl 1052.14017

[Alexeev 2004] V. Alexeev, "Compactified Jacobians and Torelli map", *Publ. Res. Inst. Math. Sci.* **40**:4 (2004), 1241–1265. MR 2006a:14016 Zbl 1079.14019

[Alexeev and Nakamura 1999] V. Alexeev and I. Nakamura, "On Mumford's construction of degenerating abelian varieties", *Tohoku Math. J.* (2) **51**:3 (1999), 399–420. MR 2001g:14013 Zbl 0989.14003

[Altmann and Hille 1999] K. Altmann and L. Hille, "Strong exceptional sequences provided by quivers", *Algebr. Represent. Theory* **2**:1 (1999), 1–17. MR 2000h:16019 Zbl 0951.16006

[Amini 2010] O. Amini, "Lattice of integer flows and poset of strongly connected orientations", preprint, 2010. arXiv 1007.2456

[Bacher et al. 1997] R. Bacher, P. de la Harpe, and T. Nagnibeda, "The lattice of integral flows and the lattice of integral cuts on a finite graph", *Bull. Soc. Math. France* **125**:2 (1997), 167–198. MR 99c:05111 Zbl 0891.05062

[Beck and Zaslavsky 2006] M. Beck and T. Zaslavsky, "The number of nowhere-zero flows on graphs and signed graphs", *J. Combin. Theory Ser. B* **96**:6 (2006), 901–918. MR 2007k:05084 Zbl 1119.05105

[Björner et al. 1999] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler, *Oriented matroids*, 2nd ed., Encyclopedia of Mathematics and its Applications **46**, Cambridge University Press, 1999. MR 2000j:52016 Zbl 0944.52006

[Breuer and Dall 2010] F. Breuer and A. Dall, "Viewing counting polynomials as Hilbert functions via Ehrhart theory", pp. 545–556 in *22nd international conference on formal power series and algebraic combinatorics (FPSAC 2010)* (San Francisco, CA, 2010), Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010. MR 2012m:52023

[Breuer and Sanyal 2012] F. Breuer and R. Sanyal, "Ehrhart theory, modular flow reciprocity, and the Tutte polynomial", *Math. Z.* **270**:1-2 (2012), 1–18. MR 2012m:05176 Zbl 1235.05070

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993. MR 95h:13020 Zbl 0788.13005

[Bruns et al. 2008] W. Bruns, R. Koch, and T. Römer, "Gröbner bases and Betti numbers of monoidal complexes", *Michigan Math. J.* **57** (2008), 71–91. MR 2010a:13045 Zbl 1180.13026

[Caporaso 1994] L. Caporaso, "A compactification of the universal Picard variety over the moduli space of stable curves", *J. Amer. Math. Soc.* **7**:3 (1994), 589–660. MR 95d:14014 Zbl 0827.14014

[Caporaso and Viviani 2010] L. Caporaso and F. Viviani, "Torelli theorem for graphs and tropical curves", *Duke Math. J.* **153**:1 (2010), 129–171. MR 2011j:14013 Zbl 1200.14025

[Casalaina-Martin et al. 2011] S. Casalaina-Martin, J. L. Kass, and F. Viviani, "The local structure of compactified Jacobians", preprint, 2011. arXiv 1107.4166

[Casalaina-Martin et al. 2012] S. Casalaina-Martin, J. L. Kass, and F. Viviani, "On the singularities of the universal compactified Jacobian", in prepration, 2012.

[Chen 2010] B. Chen, "Orientations, lattice polytopes, and group arrangements, I: Chromatic and tension polynomials of graphs", *Ann. Comb.* **13**:4 (2010), 425–452. MR 2011d:05184 Zbl 1229.05120

[Cox et al. 2011] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics **124**, American Mathematical Society, Providence, RI, 2011. MR 2012g:14094 Zbl 1223.14001

[Dais 2002] D. I. Dais, "Resolving 3-dimensional toric singularities", pp. 155–186 in *Geometry of toric varieties*, edited by L. Bonavero and M. Brion, Sémin. Congr. **6**, Société Mathématique de France, Paris, 2002. MR 2005e:14004 Zbl 1047.14038

[Erdahl 1999] R. M. Erdahl, "Zonotopes, dicings, and Voronoi's conjecture on parallelohedra", *European J. Combin.* **20**:6 (1999), 527–549. MR 2000d:52014 Zbl 0938.52016

[Faltings and Chai 1990] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergeb. Math. Grenzgeb. (3) **22**, Springer, Berlin, 1990. MR 92d:14036 Zbl 0744.14031

[Fujino 2000] O. Fujino, "Abundance theorem for semi log canonical threefolds", *Duke Math. J.* **102**:3 (2000), 513–532. MR 2001c:14032 Zbl 0986.14007

[Fujino 2011] O. Fujino, "Semi-stable minimal model program for varieties with trivial canonical divisor", *Proc. Japan Acad. Ser. A Math. Sci.* **87**:3 (2011), 25–30. MR 2012j:14023 Zbl 1230.14016

[Fujino and Sato 2004] O. Fujino and H. Sato, "Introduction to the toric Mori theory", *Michigan Math. J.* **52**:3 (2004), 649–665. MR 2005h:14037 Zbl 1078.14019

[Gel′fand et al. 1994] I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, MA, 1994. MR 95e:14045 Zbl 0827.14036

[Greco and Traverso 1980] S. Greco and C. Traverso, "On seminormal schemes", *Compositio Math.* **40**:3 (1980), 325–365. MR 81j:14030 Zbl 0412.14024

[Greene and Zaslavsky 1983] C. Greene and T. Zaslavsky, "On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs", *Trans. Amer. Math. Soc.* **280**:1 (1983), 97–126. MR 84k:05032 Zbl 0539.05024

[Huneke and Swanson 2006] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series **336**, Cambridge University Press, 2006. MR 2008m:13013 Zbl 1117.13001

[Ichim and Römer 2007] B. Ichim and T. Römer, "On toric face rings", *J. Pure Appl. Algebra* **210**:1 (2007), 249–266. MR 2008a:13032 Zbl 1117.05113

[Kollár 2010] J. Kollár, "Moduli of varieties of general type", preprint, 2010. arXiv 1008.0621

[Kollár and Mori 1998] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134**, Cambridge University Press, 1998. MR 2000b:14018 Zbl 0926.14003

[Matsumura 1989] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1989. MR 90i:13001 Zbl 0666.13002

[Miller and Sturmfels 2005] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics **227**, Springer, New York, 2005. MR 2006d:13001 Zbl 1066.13001

[Mumford 1972] D. Mumford, "An analytic construction of degenerating abelian varieties over complete rings", *Compositio Math.* **24** (1972), 239–272. MR 50 #4593 Zbl 0241.14020

[Namikawa 1980] Y. Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Mathematics **812**, Springer, Berlin, 1980. MR 82a:32034 Zbl 0466.14011

[Novik et al. 2002] I. Novik, A. Postnikov, and B. Sturmfels, "Syzygies of oriented matroids", *Duke Math. J.* **111**:2 (2002), 287–317. MR 2003b:13023 Zbl 1022.13002

[Oda 1988] T. Oda, *Convex bodies and algebraic geometry: an introduction to the theory of toric varieties*, Ergeb. Math. Grenzgeb. (3) **15**, Springer, Berlin, 1988. MR 88m:14038 Zbl 0628.52002

[Oda and Seshadri 1979] T. Oda and C. S. Seshadri, "Compactifications of the generalized Jacobian variety", *Trans. Amer. Math. Soc.* **253** (1979), 1–90. MR 82e:14054 Zbl 0418.14019

[Orlik and Terao 1992] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren Math. Wiss. **300**, Springer, Berlin, 1992. MR 94e:52014 Zbl 0757.55001

[Pandharipande 1996] R. Pandharipande, "A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles", *J. Amer. Math. Soc.* **9**:2 (1996), 425–471. MR 96f:14014 Zbl 0886.14002

[Serre 1965] J.-P. Serre, *Algèbre locale: multiplicités*, 2nd ed., Lecture Notes in Mathematics **11**, Springer, Berlin, 1965. MR 34 #1352 Zbl 0142.28603

[Serre 1980] J.-P. Serre, *Trees*, Springer, Berlin, 1980. MR 82c:20083 Zbl 0548.20018

[Simpson 1994] C. T. Simpson, "Moduli of representations of the fundamental group of a smooth projective variety, I", *Inst. Hautes Études Sci. Publ. Math.* **79** (1994), 47–129. MR 96e:14012 Zbl 0891.14005

[Stanley 1994] R. P. Stanley, "A survey of Eulerian posets", pp. 301–333 in *Polytopes: abstract, convex and computational* (Scarborough, ON, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **440**, Kluwer, Dordrecht, 1994. MR 95m:52024 Zbl 0816.52004

[Stanley 1996] R. P. Stanley, *Combinatorics and commutative algebra*, 2nd ed., Progress in Mathematics **41**, Birkhäuser, Boston, MA, 1996. MR 98h:05001 Zbl 0838.13008

[Sturmfels 1996] B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series **8**, American Mathematical Society, Providence, RI, 1996. MR 97b:13034 Zbl 0856.13020

[Swan 1980] R. G. Swan, "On seminormality", *J. Algebra* **67**:1 (1980), 210–229. MR 82d:13006 Zbl 0473.13001

[Uliczka 2009] J. Uliczka, "A note on the dimension theory of $\mathbb{Z}^n$-graded rings", *Comm. Algebra* **37**:10 (2009), 3401–3409. MR 2010k:13001 Zbl 1181.13010

Communicated by Bernd Sturmfels
Received 2011-12-22 Revised 2012-12-04 Accepted 2012-12-05

casa@math.colorado.edu *Department of Mathematics, University of Colorado Boulder, Campus Box 395, Boulder, CO, 80309-0395, United States*
http://math.colorado.edu/~sbc21/

kass@math.uni-hannover.de *Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany*
http://www2.iag.uni-hannover.de/~kass/

filippo.viviani@gmail.com *Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma, Italy*
http://ricerca.mat.uniroma3.it/users/viviani/

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2013 is US $200/year for the electronic version, and $350/year (+$40, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 7 No. 8 2013

The geometry and combinatorics of cographic toric face rings 1781
Sebastian Casalaina-Martin, Jesse Leo Kass and Filippo Viviani

Essential p-dimension of algebraic groups whose connected component is a torus 1817
Roland Lötscher, Mark MacDonald, Aurel Meyer and Zinovy Reichstein

Differential characterization of Wilson primes for $\mathbb{F}_q[t]$ 1841
Dinesh S. Thakur

Principal W-algebras for GL($m|n$) 1849
Jonathan Brown, Jonathan Brundan and Simon M. Goodwin

Kernels for products of L-functions 1883
Nikolaos Diamantis and Cormac O'Sullivan

Division algebras and quadratic forms over fraction fields of two-dimensional henselian domains 1919
Yong Hu

The operad structure of admissible G-covers 1953
Dan Petersen

The p-adic monodromy theorem in the imperfect residue field case 1977
Shun Ohkubo

On the Manin–Mumford and Mordell–Lang conjectures in positive characteristic 2039
Damian Rössler