





Parallel computation of interval bases for persistence module decomposition

Alessandro De Gregorio ¹, Marco Guerra ², Sara Scaramuccia ³, and Francesco Vaccarino ^{1,*}

¹Department of Mathematical Sciences, Politecnico di Torino, Italy

²Institut Fourier, Université Grenoble-Alpes CNRS

³Department of Mathematics, Università degli Studi di Roma Tor Vergata, Italy

*Corresponding author

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Abstract

A persistence module M , with coefficients in a field \mathbb{F} , is a finite-dimensional linear representation of an equioriented quiver of type A_n or, equivalently, a graded module over the ring of polynomials $\mathbb{F}[x]$. It is well-known that M can be written as the direct sum of indecomposable representations or as the direct sum of cyclic submodules generated by homogeneous elements. An interval basis for M is a set of homogeneous elements of M such that the sum of the cyclic submodules of M generated by them is direct and equal to M . We introduce a novel algorithm to compute an interval basis for M . Based on a flag of kernels of the structure maps, our algorithm is suitable for parallel or distributed computation and does not rely on a presentation of M . This algorithm outperforms the approach via the presentation matrix and Smith Normal Form. We specialize our parallel approach to persistent homology modules, and we close by applying the proposed algorithm to tracking harmonics via Hodge decomposition.

Keywords— Graded module decomposition; Persistent homology; Hodge decomposition; Hodge Laplacian; Graded Smith Normal Form
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1 Introduction

Persistence Module is a modern name for finite-dimensional representations of an equioriented quiver of type A_n that has become popular within the setting of Topological Data Analysis (TDA) and, more specifically, in connection to *Persistent Homology*, one of the most successful tools in TDA (see [10]).

First, a quiver Q , of type A_n , is the Hasse diagram of the linearly ordered set $[n] := (\{1, \dots, n\}, \leq)$. This is an oriented simple graph whose vertices are indexed by $[n]$ and whose set of arrows is $\{(i, i + 1) : i = 1, \dots, n - 1\}$. A linear representation $\mathcal{M} = \{(M_i, \varphi_i)\}_{i \in \mathbb{N}}$ of Q with coefficients in a field \mathbb{F} is given by the following datum:

- a finite-dimensional \mathbb{F} -vector space M_i , called the i^{th} -step, for each vertex i in $[n]$;
- a linear map $\varphi_i : M_i \rightarrow M_{i+1}$, called i^{th} -structure map, for each arrow $(i, i + 1)$ in $[n]$.

It is a well-known result that persistence modules can be decomposed, uniquely up to isomorphism, into the direct sum of indecomposable modules (see [27])

$$\mathcal{M} \cong \bigoplus_{m=1}^N I_{[b_m, d_m]} \tag{1}$$

where, for all $1 \leq b_m \leq d_m \leq n$, $I_{[b_m, d_m]}$ is the persistence module with steps $(I_{[b_m, d_m]})_i = \mathbb{F}$ for all integers $i \in [b_m, d_m]$ and zero elsewhere; and structure maps the identity for $i \in [b_m, d_m - 1]$ and zero elsewhere. The modules are $I_{[b_m, d_m]}$ and are often called *interval modules* and are the indecomposable representations of A_n . The decomposition of Equation (1) into interval modules has been well known in the quiver representations community since the 70s and somehow neglected and

alessandro.degregorio@polito.it; marco.guerra@univ-grenoble-alpes.fr;
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rediscovered in persistence several years later [43]. The multiset made of the intervals $[b_m, d_m]$ is a complete discrete invariant for the isomorphism classes of finite-dimensional linear representations of type A_n , see the works of Abeasis, Del Fra, and Kraft [2–4]. In particular, they presented in the early 80s the first example of *barcode* that we know, calling it *diagram of boxes* (see, e.g., sec.2 in [4]).

A persistence module \mathcal{M} can be associated with a graded $\mathbb{F}[x]$ -module $\alpha(\mathcal{M})$ under a well-known equivalence of categories [13, 16], in the following way: given \mathcal{M} as above, $\alpha(\mathcal{M})$ is defined as $\bigoplus_{i \in \mathbb{N}} \alpha(M_i) := M_1 \oplus M_2 \cdots \oplus M_n \oplus M_n \oplus M_n \cdots$. The grading structure is obtained by setting $xv = \varphi_i(v)$, for each $i \in [n]$ and $v \in \alpha(M_i) = M_i$ and $xv = v$ for $v \in \alpha(M_j) = M_n$ for $j > n$. Also, in this setting, there is a well-known decomposition where the place of the indecomposable for quiver is taken by cyclic submodules generated by homogeneous elements. Consider indeed the cyclic submodule $I(v)$ of $\alpha(\mathcal{M})$, generated by a homogeneous element $v \in M_b$, for some b . Then there are two possibilities for v : it has torsion, that is, there is $e \in \mathbb{N}$ such that $x^e v = 0_{\mathcal{M}}$, so that $I(v) \cong \mathbb{F}[x]/(x^e)$ or v is torsion-free, so that $I(v) \cong \mathbb{F}[x]$. We denote by $I(b, e)$ the submodule $I(v)$ in the torsion case and by $I(b, \infty)$ in the torsion-free (also named by “free”) case. We call *Interval Modules* the modules of the type $I(*, *)$ as their germane in the quiver representations setting.

The theorem of decomposition of a graded module over a graded principal ideal (see Theorem 1 in [54]) domain can now be restated as

$$\alpha(\mathcal{M}) \cong \bigoplus_{m=1}^N I(b_m, e_m). \quad (2)$$

Now, e_m can be an integer or the ∞ symbol. Exactly as for the quiver representation case, the multisets of intervals (b_m, e_m) occurring in the decomposition is a complete discrete invariant for the isomorphism classes of persistence modules, usually called the *barcode* in TDA.

Strictly related to the above decompositions into interval submodules is the concept of *interval basis*: a finite set $\{v_1, \dots, v_N : v_m \in M_{b_m}, \forall m\}$ of homogenous elements of \mathcal{M} such that $\bigoplus_{m=1}^N I(v_m) = \mathcal{M}$.

By applying the construction in the proof Lemma 6 in [16], here reported in Definition 5 in Appendix B.2, one can always turn a persistence module into a graded module presentation. Once a persistence module is assigned a presentation matrix, the graded Smith Normal Form reduction proposed in [49] (reported in Algorithm 8 in Appendix B.3) returns an interval basis. Details will be treated in Appendix B.

As a guiding example, consider the persistence module $\mathcal{M} = \{(M_i, \varphi_i)\}_{i=1}^3$ with coefficients in a field \mathbb{F} and structure maps

$$\mathcal{M} : 0 \xrightarrow[\begin{smallmatrix} \varphi_0 \\ (0) \end{smallmatrix}]{} \mathbb{F} \xrightarrow[\begin{smallmatrix} \varphi_1 \\ (1) \\ (0) \end{smallmatrix}]{} \mathbb{F}^2 \xrightarrow[\begin{smallmatrix} \varphi_2 \\ (1 \ 1) \end{smallmatrix}]{} \mathbb{F} \xrightarrow[\begin{smallmatrix} \varphi_3 \\ (0) \end{smallmatrix}]{} 0.$$

so $M_1 \cong M_3 \cong \mathbb{F}$ and $M_2 \cong \mathbb{F}^2$. The decomposition $\alpha(\mathcal{M}) \cong \bigoplus_{m=1}^N I(b_m, e_m)$ is then (up to isomorphism) given by:

$$(0 \xrightarrow{0} \mathbb{F} \xrightarrow{1} \mathbb{F} \xrightarrow{1} \mathbb{F} \xrightarrow{0} 0) \oplus (0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{F} \xrightarrow{0} 0 \xrightarrow{0} 0)$$

Consider now, $v_1 = (1) \in M_1$ and $v_2 = (0, 1)^\top \in M_2$, one has:

$$v_1 = (1) \xrightarrow{\varphi_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\varphi_2} (1) \xrightarrow{\varphi_3} (0);$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\varphi_2} (1) \xrightarrow{\varphi_3} (0).$$

A minimal presentation of the associated graded $\mathbb{F}[x]$ -module $\alpha(\mathcal{M})$ is thus obtained as the cokernel of the presentation matrix:

$$S = \begin{pmatrix} x^2 & x^3 \\ -x & 0 \end{pmatrix}, \quad (3)$$

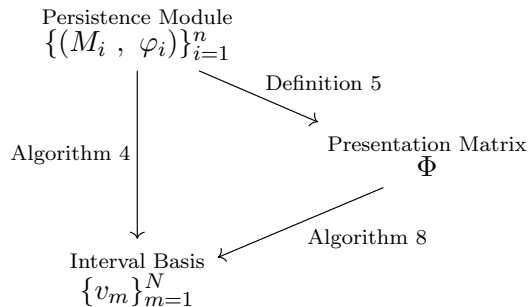
whose columns corresponds to the (homogenous) independent relations satisfied by the homogeneous generators v_1, v_2 of $\alpha(\mathcal{M})$: that is $x^2v_1 = xv_2$ (deg = 3) and $x^3v_1 = 0$ (deg = 4). The elements v_1 and v_2 form a minimal system of generators for $\alpha(\mathcal{M})$, nevertheless they do not form an interval basis for $\alpha(M)$ because $I(v_2) = 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{F} \xrightarrow{1} \mathbb{F} \xrightarrow{0} 0$.

On the contrary, we obtain an interval basis by using $v'_1 = v_1 = (1) \in M_1$ and $v'_2 = (-1, 1)^\top \in M_2$. Considering that $x^k v'_1 = 0$ iff $k \geq 3$ and $xv'_2 = 0$, the corresponding presentation matrix is the following

$$\begin{pmatrix} 0 & x^3 \\ x & 0 \end{pmatrix}. \quad (4)$$

and v'_1 and v'_2 form an interval basis. This basic example shows that not all the minimal systems of homogeneous generator of a graded module over $\mathbb{F}[x]$ are interval bases, while, a fortiori, the opposite is true. Indeed, the presentation associated with an interval basis has special relations, namely, each relation involves a single generator up to multiplication by a homogeneous element in $\mathbb{F}[x]$ as exemplified just above. We can say that interval bases are minimal systems of generators with a flavour.

The main result of this paper is to present Algorithm 4 to find an interval basis of \mathcal{M} without computing a presentation of $\alpha(\mathcal{M})$. Our algorithm is distributed over persistence module steps (Algorithm 3) and avoids explicitly constructing a matrix presentation. A specialization to the case of real coefficient is included in Algorithm 7 in Appendix A.



Persistent Homology Modules

A finite sequence of chain complexes $(C_{\bullet}^i, \partial_{\bullet}^i)$, for $i \in [n]$ connected by chain maps $f^i : C_{\bullet}^i \rightarrow C_{\bullet}^{i+1}$ determines a persistence module $C_k = \{(C_k^i, f_k^i)\}_{i=1}^n$ for each k . Here, we call k^{th} -persistent homology module the persistence module $H_k = \{(H_k^i, \tilde{f}_k^i)\}_{i=1}^n$ obtained by applying to C_k the homology functor in degree k .

Hence, by the term persistent homology module, we do not assume the maps f^i to be necessarily simplicial or injective, which is the typical assumption in persistent simplicial homology.

Persistent homology often focuses on the special but relevant case of persistence modules C_k directly determined by filtered data coming in the form of *filtered simplicial complexes*. In particular, the chain maps f^i are assumed to be injective. This is usually called persistent homology but we call it persistent simplicial homology to avoid confusion. The interested reader is referred to [10, 28], for classical surveys on TDA, and to [14, 44, 51], for more recent ones.

Our parallel decomposition algorithm applies to the special case of persistent simplicial homology.

In Fig. 1, we see an example of a simplicial complex obtained as a triangulation of a portion of a torus filtered by the height function into three steps.

We also specialize this perspective to tracking *harmonic* homology representatives. This furthers the recent trend of exploring the interplay between topological data analysis and the properties of the Hodge Laplacian ([20, 52, 53]). Harmonic

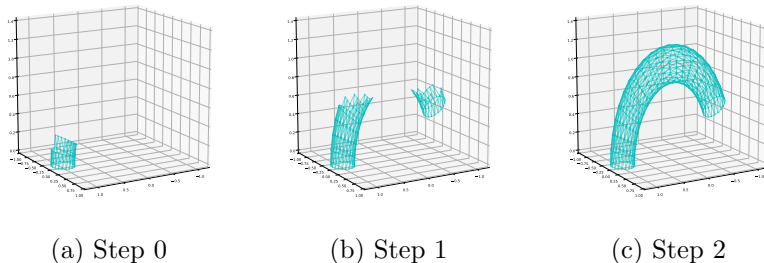


Figure 1: A 3-step filtration by sublevel sets, for the z coordinate, of a tilted and triangulated half torus.

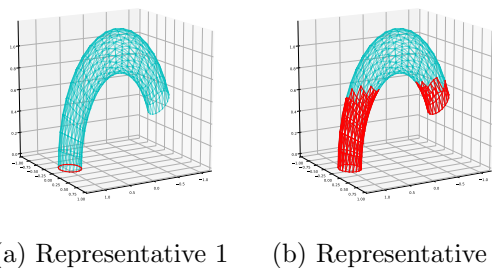
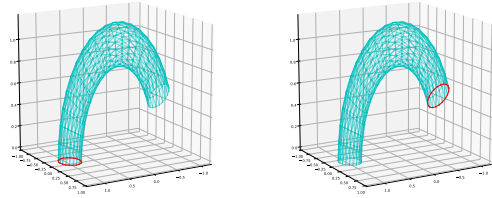


Figure 2: Harmonic representatives via the interval basis algorithms

homology representatives corresponding to interval bases and computed by our methods for the filtered complex in Fig. 1 are depicted in Fig. 2.

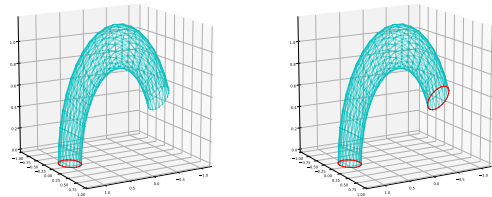
In the same example of filtered complex, one can check that the persistence module isomorphism class in the examples of (3) and (4) is the class of the persistent homology module obtained by applying the 1st-homology functor to the filtered complex in Fig. 1. Further, generators v_1 and v_2 are those associated with the homology classes of the representatives shown in red in Fig. 3, while v'_1 and v'_2 are associated with the homology classes of the representatives in Figure 4.

The tracking of homology representatives along a monotone (equioriented) sequence of simplicial maps that are not necessarily injective is treated in [18]. Each simplicial map is interpreted as a sequence of inclusions and vertex collapses, and a consistent homology basis can be maintained efficiently. Later in [36], a variation of the coning approach of [18] is proposed, which takes a so-called *simplicial tower* and converts it into a filtration while preserving its barcode, with asymptotically small overhead. Here, we can tackle the same problem from the unifying



(a) Representative 1 (b) Representative 2

Figure 3: In red, two representative 1-cycles. Their homology classes form a minimal system of generators of the persistent homology module of the filtration in Fig. 1. These representatives do not in general induce an interval basis.



(a) Representative 1 (b) Representative 2

Figure 4: In red, a different choice of representative 1-cycles. Their homology classes are a different choice of generators for the same persistence module as in Fig. 3. However, these generators do induce an interval basis.

perspective of persistence modules.

With respect to the simplicial filtered complex case, our aim is not that of outperforming computations in persistent homology but to look at the tracking of homology representatives under the lens of the algebraic notion of an interval basis.

The problem of tracking homology representatives along filtered complexes has mainly been studied from a minimality perspective ([17, 31, 35]) in order to geometrically locate the persistent homology features. Through the *standard algorithm* [13] for computing persistent homology, homology representatives can be tracked by storing the operations performed during a matrix reduction. Computing intervals and tracking homology representatives have been optimized in many ways ([8, 15, 19, 26, 42, 48]), including parallel and distributed approaches ([6, 7, 39, 40]). We remark this list is far from being exhaustive. However, not all the mentioned approaches provide an interval basis, and for this purpose, we include the discussion on two relevant cases in Remark 5 and Remark 6.

Contents. In Section 2 and Appendix B, we formalize interval basis as a particular minimal system of generators translated into persistence module terms. We also express the classical interval decomposition result into interval basis terms. In Section 3, we review the literature in the decompositions of graded modules and persistence modules as quiver representations of type A_n . In Section 4, we propose an algorithm computing an interval basis out of a persistence module, by acting in a distributed way over each step in the input persistence module and by avoiding a presentation of the associated graded module. The same algorithm is specialized to the real coefficient case in Appendix A. The latter case is particularly relevant for the harmonic case, later discussed in Section 6.1. In Section 5, we compare the computational cost of our parallel method to the classical Smith Normal Form reduction (pseudocode in Appendix B.3) when specialized to persistence module matrix presentations. In particular, we find that our parallel approach admit an output-dependent estimate and it is more efficient. In Section 6, we describe how to construct in parallel a persistence module from the homology of an monotone sequence of chain maps with homology representatives. Complementary pseudocodes are included in Appendix C. In particular in Section 6.1, we construct in parallel a persistence module from the homology of an monotone sequence of chain maps with harmonic homology representatives. The case of filtered simplicial complexes is treated in Section 6.2.

2 Persistence Modules

In this section, we fix the notation for persistence modules and define interval bases. In Proposition 1, we include the well-known decomposition theorem for equi-oriented quiver representations of type A_n conveniently concerning the interval basis definition.

For the sake of completeness, in Appendix B, we provide further material connecting the interval basis definition to the Smith Normal Form of a module presentation in the isomorphic category of finitely generated graded $\mathbb{F}[x]$ -modules, where $\mathbb{F}[x]$ is the graded ring of polynomials with coefficients in \mathbb{F} and a single indeterminate x .

Following the definition of discrete algebraic persistence module in [16], define a *persistence module* \mathcal{M} as a pair $\{(M_i, \varphi_i)\}_{i=1}^n$ consisting of:

- a finite-dimensional \mathbb{F} -vector space M_i , called the i^{th} -step, for each vertex i in $[n]$;
- a linear map $\varphi_i : M_i \rightarrow M_{i+1}$, called i^{th} -structure map, for each arrow $(i, i+1)$ in $[n]$.

We define $\varphi_{i,j} : M_i \rightarrow M_j$ with $i < j$, as the composition $\varphi_{j-1} \circ \dots \circ \varphi_i$.

A persistence module \mathcal{M} can be associated with a finitely generated graded $\mathbb{F}[x]$ -module $\alpha(\mathcal{M})$ under a well-known equivalence of categories [13, 16]. We will explicitly define $\alpha(\mathcal{M})$ in Appendix B.1 where graded modules are treated. Under the equivalence α , we transpose to persistence modules several notions applying to graded modules, such as isomorphisms, homogeneous elements, direct sums, generators, and submodules.

Let $I(v)$, with $v \in M_b$, be the the persistence module $\{(I_i(v), \psi_i(v))\}_{i=1}^n$ defined by

$$I_i(v) = \begin{cases} \langle \varphi_{b,i}(v) \rangle & \text{if } i \geq b, \\ 0 & \text{otherwise,} \end{cases} \quad \psi_i(v) = \begin{cases} \varphi_i|_{\langle \varphi_{b,i}(v) \rangle} & \text{if } i \geq b, \\ 0 & \text{otherwise,} \end{cases}$$

where the brackets $\langle \cdot \rangle$ denotes the \mathbb{F} -linear space spanned by their argument.

Now, define an (*integer*) *interval* $[b, d]$ with $b \leq d$ to be the finite set of integers i with $b \leq i \leq d$. The *interval module* $I_{[b,d]}$ relative to the interval $[b, d]$ is the persistence module $\{(I_i, \psi_i)\}_{i=1}^n$ such that

$$I_i = \begin{cases} \mathbb{F} & \text{if } b \leq i \leq d, \\ 0 & \text{otherwise,} \end{cases} \quad \psi_i = \begin{cases} \text{id}_{\mathbb{F}} & \text{if } b \leq i < d, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1. Fix a degree b . For each $v \in M_b$, there exists $d \leq n$ such that

$$I(v) \cong I_{[b,d]}.$$

Indeed by construction each step in $I(v)$ is either isomorphic to the vector space \mathbb{F} or to 0. The structure maps in $I(v)$ are either isomorphisms or the null map. If an integer $r \leq d - 1$ exists, such that $I_{r+1}(v) = 0$, we take d to be the minimum of such r 's. Otherwise, $d = n$.

Definition 1. (Interval basis) Given a persistence module $\mathcal{M} = \{(M_i, \varphi_i)\}_{i=1}^n$, a finite family $\{v_1, \dots, v_N\} \subseteq \bigcup_i M_i$ of homogeneous non-zero elements is an *interval basis* for \mathcal{M} if and only if

$$\bigoplus_{m=1}^N I(v_m) = \mathcal{M}.$$

Proposition 1. *Every persistence module \mathcal{M} admits an interval basis.*

Proof. The existence of an interval basis for each \mathcal{M} follows from the interval decomposition corresponding to the Structure Theorem [13] for finitely generated graded $\mathbb{F}[x]$ -modules. Indeed, the interval decomposition implies that \mathcal{M} decomposes into a direct sum of interval modules of the form

$$\mathcal{M} \cong \bigoplus_{m=1}^N I_{[b_m, d_m]}, \quad (5)$$

where the intervals $[b_m, d_m]$ with $b_m \leq d_m \leq n$ are uniquely determined up to reorderings. Let $\Psi = \{\Psi_i\}_{i=1}^n : \bigoplus_{m=1}^N I_{[b_m, d_m]} \longrightarrow \mathcal{M}$ be the persistence module isomorphism of the interval decomposition in Eq. (5). Then, for each summand $I_{[b_m, d_m]}$, the map Ψ_{b_m} detects a vector $v_m \in M_{b_m}$. By Remark 1, we have that $I(v_m) \cong I_{[b_m, r_m]}$ for some $b_m \leq r_m \leq n$. Observe now that, for all indexes i such that $(i, i + 1)$ is an arrow in $[n]$, the decomposition isomorphism satisfies $\varphi_i \circ \Psi_i = \Psi_{i+1} \circ \psi_i$, where ψ_i is the structure map of $I_{[b_m, d_m]}$. This implies that $r_m = d_m$ for all indexes $i \in \mathbb{N}$. \square

Decomposing a persistence module via an interval basis consists of retrieving, given a persistence module \mathcal{M} , an interval basis v_1, \dots, v_N , where N equals the number of interval modules in the interval decomposition of Definition 1.

3 Related Works

The related works comprise methods for the decomposition of persistence and graded modules.

Persistence Module Decompositions

Persistence module decomposition methods can be seen as special instances of methods decomposing quivers of type A_n , hence holding for the so-called *zig-zag persistence modules*. The incremental algorithm introduced in [12] retrieves the interval decomposition by focusing over each step and constructing a flag with respect to images of interval vanishing at that step. The procedure is a dual counterpart with respect to the kernel flag decomposition we propose in this work in Section 4. Differently from our approach, the zig-zag decomposition does not aim at recovering the generators since for general zig-zag persistence modules generators and intervals are not in one-to-one correspondence. More recently [11], a basis suitable for the zig-zag case called *canonical form* has been introduced in a different sense from that of an interval basis. A canonical form consists of a vector space basis for each step in the persistence module. Those bases are selected so that the structure maps connecting the spaces are expressed through matrices in echelon form. When comparable, that is for the case of equi-oriented quivers, an interval basis is equivalent to the canonical form. Specifically, an interval basis encodes data of a canonical form in a compressed way. Indeed, in an interval basis, we represent a single generator per interval belonging to the interval decomposition and the structure maps are the original ones, implicitly encoded by the action of x .

Canonical forms can be computed like proposed in [11] where the decomposition of a zigzag module is tackled from the matrix factorization viewpoint. The approach admits a divide-and-conquer implementation where the module is subdivided into equally-oriented parts. After the matrix factorization, the interval lengths can be retrieved by connecting the pivots in the factorization. Our parallel algorithm instead, leverages graded module presentations to focus on generators rather than changes of basis. Interval basis elements are found already equipped with their associated interval lengths. Furthermore, our distributed method is not a divide-and-conquer approach and instead performs computations independently across all steps in the persistence module.

More recently in [34], a notion similar to the canonical form is called *barcode basis* and it is introduced in order to study the space of transformations from one barcode basis to another as a tool to express in barcode basis terms the decomposition of commutative ladders from [23] and the possibility of defining partial barcode matchings out of a quiver morphism. In [29] authors introduce *persistence*

bases as an isomorphism realizing the interval decomposition (Eq. (5)) with the purpose of defining barcode matchings induced by persistence module morphisms. Finally, our subdivision into homogeneous spaces generated by an interval basis specializes the notion of quotient through the radical functor introduced in [47] where authors characterize tameness conditions in multiparameter persistence.

Graded Module Decompositions

Given a presentation matrix, many methods exist in the literature to retrieve a minimal system of generators for a graded $\mathbb{F}[x]$ -module.

As noticed in [37], extracting a minimal system of generators can be seen as a specialization to $\mathbb{F}[x]$ coefficients of classical Gröbner basis extraction algorithms [24, 25, 46] for multigraded $\mathbb{F}[x_1, \dots, x_n]$ -modules, widely implemented in software packages [1, 9, 22, 30]. See [45] for Gröbner bases of modules and primary decompositions. As already pointed out in Section 1, a minimal system of generators is not in general an interval basis (see (3)).

An interval basis is computable by reducing the presentation matrix into the Smith Normal Form (see Appendix B). To the best of our knowledge the authors of [49] first introduced, for the graded case, an algorithm for SNF reduction. The procedure complexity in terms of time and space depends on the size of the presentation matrix. This motivates us in specializing in Appendix B.1 the same procedure to the case of matrix presentations obtained through the construction in [16] when starting from a persistence module. We provide a complexity evaluation for that specific case which takes advantage of the sparsity in the block subdivision of a matrix presentation of a persistence module. On the contrary, in our parallel decomposition algorithm in Section 4, we take advantage of the independence properties under the action of x of the interval basis in order to propose a parallel approach with output-dependent complexity.

4 Parallel computation of an interval basis

In this section, we present a parallel algorithm for the computation of an interval basis of a persistence module \mathcal{M} (see Algorithm 7 Appendix A for a specialization to the real coefficient case). The whole, length- n persistence module (spaces and structure maps) is assumed to be input to a pool of n processors. Then each step M_i can be processed independently by processor i . The idea is that a *single step decomposition* routine (Algorithm 3) takes care of the bars being born at step i ; to discriminate these from bars that are merely travelling through step i , we make use of the flag of vector spaces given by the kernels of iterated composites of the structure maps. Simple linear algebra then shows that we can recover basis

vectors that form an interval basis of \mathcal{M} . Using such a flag of kernels takes implicit advantage of the death of bars along the barcode, gradually reducing the size of the maps involved, and achieving better efficiency than a method which does not take this into account, such as the graded SNF. This statement is justified in the complexity analysis in Section 5.

Consider $\mathcal{M} = \{(M_i, \varphi_i)\}_{i=1}^n$ a persistence module. Without loss of generality we assume an additional structure map $\varphi_n : M_n \rightarrow M_{n+1}$ to be the null one. This way the treatment of the final step M_n have no qualitative difference from the others. Denote with m_i the dimension of the space M_i and with r_i the dimension of $\text{Im}(\varphi_{i-1})$. For each i there is a flag of vector sub-spaces of M_i given by the kernel of the maps $\varphi_{i,j}$:

$$0 \subseteq \ker(\varphi_{i,i+1}) \subseteq \ker(\varphi_{i,i+2}) \subseteq \cdots \subseteq \ker(\varphi_{i,n+1}) = M_i, \quad (6)$$

where the last equality holds by the assumption above. Denote for simplicity each space $\ker(\varphi_{i,j})$ as V_j^i .

An *adapted basis* for the flag in M_i is given by a set of linearly independent vectors $\mathcal{V}^i = \{v_1, \dots, v_{m_i}\}$, and an index function $J : \mathcal{V}^i \rightarrow \{1, \dots, n - i + 1\}$, such that

$$V_{i+s}^i = \langle \{v \in \mathcal{V}^i \mid J(v) \leq i + s\} \rangle \quad \forall s, 1 \leq s \leq n - i + 1. \quad (7)$$

In words, an adapted basis is an ordered list of vectors in M_i such that for every j , the first $\dim V_j^i$ vectors are a basis of V_j^i (an empty list is a basis of the trivial space). The index function J gives precisely this ordering. Notice that

Lemma 1. *Without loss of generality, it is possible to choose an adapted basis for M_i in such a way that it contains as a subset a basis of $\text{Im}(\varphi_{i-1})$.*

Proof. Let us consider an adapted basis $\mathcal{V}^i = (t_1, \dots, t_{m_i})$ for the flag of kernels in M_i , with the vectors t_1, \dots, t_{m_i} ordered by index function J . We construct the desired basis explicitly: set $\underline{\mathcal{V}}^i = \{t_1\}$. For every $s = 2, \dots, m_i$, if $t_s \notin \langle \underline{\mathcal{V}}^i \rangle + \text{Im}(\varphi_{i-1})$ add the vector t_s to $\underline{\mathcal{V}}^i$. Otherwise, it must hold that $t_s = \sum_{a < s} \lambda_a t_a + x$ with $x \in \text{Im}(\varphi_{i-1})$. Then we add to $\underline{\mathcal{V}}^i$ the vector $x = t_s - \sum_{a < s} \lambda_a t_a$. In this way $\underline{\mathcal{V}}^i$ is another adapted basis, and the elements added by the second route form a basis of $\text{Im}(\varphi_{i-1})$. \square

From now on, therefore, we shall assume that each basis \mathcal{V}^i is in the form of Lemma 1.

Let us introduce two subspaces of $\langle \mathcal{V}^i \rangle$: it holds that $\langle \mathcal{V}^i \rangle = \langle \mathcal{V}_{\text{Birth}}^i \rangle \oplus \langle \mathcal{V}_{\text{Im}}^i \rangle$, where $\mathcal{V}_{\text{Im}}^i$ is the subset of \mathcal{V}^i made of a basis of $\text{Im} \varphi_{i-1}$, and $\mathcal{V}_{\text{Birth}}^i$ is its complement. Our objective is to construct a basis of the whole persistence module using the adapted bases at each step i .

Definition 2. Let us define $\mathcal{V} := \bigcup_i \mathcal{V}_{\text{Birth}}^i$.

\mathcal{V} is the set of elements of the adapted basis in each degree i that are not elements of $\text{Im}(\varphi_{i-1})$. In the following, we prove that \mathcal{V} is in fact an interval basis for $\{M_i, \varphi_i\}_{i=1}^n$.

Lemma 2. For any $i < j$, define the set $T = \langle \{v \in \mathcal{V}^i \mid J(v) > j\} \rangle$. The restriction $\varphi_{i,j}|_T$ of the map $\varphi_{i,j}$ is an injection.

Proof. By definition of T , it holds $M_i = \ker(\varphi_{i,j}) \oplus T$. If the restriction of $\varphi_{i,j}$ onto T were not injective, then T and $\ker(\varphi_{i,j})$ would have nontrivial intersection. This is a contradiction. \square

Lemma 3. For any $i < j \in \mathbb{N}$, it holds

$$\varphi_{i,j}(\langle \mathcal{V}_{\text{Birth}}^i \rangle) \cap \varphi_{i,j}(\langle \mathcal{V}_{\text{Im}}^i \rangle) = \{0\}. \quad (8)$$

Proof. Suppose that the intersection contains a nonzero vector u :

$$0 \neq u = \varphi_{i,j} \left(\sum_{v_k \in \mathcal{V}_{\text{Birth}}^i} \lambda_k v_k \right) = \varphi_{i,j} \left(\sum_{w_l \in \mathcal{V}_{\text{Im}}^i} \mu_l w_l \right).$$

Denote by u_{B} and u_{I} the vectors

$$u_{\text{B}} = \sum_{\substack{J(v_k) > j \\ v_k \in \mathcal{V}_{\text{Birth}}^i}} \lambda_k v_k, \quad u_{\text{I}} = \sum_{\substack{J(w_l) > j \\ w_l \in \mathcal{V}_{\text{Im}}^i}} \mu_l w_l.$$

It holds $u = \varphi_{i,j}(u_{\text{B}}) = \varphi_{i,j}(u_{\text{I}})$, since all the elements v such that $J(v) \leq j$ belong to $\ker(\varphi_{i,j})$. Then, u is the image through $\varphi_{i,j}$ of an element of $T = \langle \{v \in \mathcal{V}^i \mid J(v) > j\} \rangle$. On the other hand also the difference $u_{\text{B}} - u_{\text{I}}$ belongs to the same space and is mapped to zero by $\varphi_{i,j}$. The restriction of $\varphi_{i,j}$ to T is injective because of Lemma 2, therefore it must be $u_{\text{B}} - u_{\text{I}} = 0$. Since $\langle \mathcal{V}^i \rangle = \langle \mathcal{V}_{\text{Birth}}^i \rangle \oplus \langle \mathcal{V}_{\text{Im}}^i \rangle$, it must be $u_{\text{B}} = u_{\text{I}} = 0$, hence $u = 0$. \square

Theorem 1. The set \mathcal{V} is an interval basis for the persistence module M .

Proof. Say that $\mathcal{V} = \{v_1, \dots, v_N\}$. Each vector v_j in the set \mathcal{V} induces an interval module $I(v_j)$. We want to show that $M = \bigoplus_{j=1}^N I(v_j)$. To do so, let us see that for each $0 \leq i \leq n$, the space M_i is exactly $\bigoplus_{j=1}^N I_i(v_j)$. By construction we know that

$$M_i = \text{Im}(\varphi_{i-1}) \oplus \langle \{v \in \mathcal{V}^i \mid v \notin \text{Im}(\varphi_{i-1})\} \rangle = \text{Im}(\varphi_{i-1}) \oplus \bigoplus_{\substack{v \in \mathcal{V} \\ \deg v = i}} I_i(v). \quad (9)$$

All we have to show is that $\text{Im}(\varphi_{i-1})$ can be written as $\bigoplus_{\substack{v \in \mathcal{V} \\ \deg v < i}} I_i(v)$. At first we will see that a sum decomposition holds. By definition, an element in the sum belongs to the image of φ_{i-1} . We can show the converse by induction over the step index $i \in \mathbb{N}$. For $i = 0$, consider $M_0 = \langle \mathcal{V}^0 \rangle$. None of the elements of \mathcal{V}^0 belongs to $\text{Im}(\varphi_{-1})$. It clearly holds that the image of φ_0 is contained in the sum as desired. Suppose by induction that for any $k-1$ the image of φ_{k-2} is contained in the sum. Then, since $M_{k-1} = \langle \{v \in \mathcal{V}^{k-1} \mid v \notin \text{Im}(\varphi_{k-2})\} \rangle \oplus \text{Im}(\varphi_{k-2})$, it holds that

$$\text{Im}(\varphi_{k-1}) = \sum_{\substack{v \in \mathcal{V}^{k-1} \\ v \notin \text{Im}(\varphi_{k-2})}} I_k(v) + \varphi_{k-1}(\text{Im}(\varphi_{k-2})). \quad (10)$$

Therefore, by the induction hypothesis, we have that

$$\text{Im}(\varphi_{k-1}) \subseteq \sum_{\substack{v \in \mathcal{V} \\ \deg v < k}} I_k(v).$$

Now that we have shown the sum decomposition, it remains to see that this sum is direct. Suppose to have a non trivial combination $w_1 + \dots + w_k = 0$, where each w_q belongs to $I_i(v_{t_q})$. Suppose that the w_1, \dots, w_k are ordered according to the degree of the element v_{t_q} that generates the interval module they belong to. Let us say that these elements have a maximum degree $l < i$. Then, it holds

$$w_1 + \dots + w_k = \varphi_{l,i}(x) + \sum_{\substack{v_{t_r} \\ \deg v_{t_r} = l}} \lambda_r \varphi_{l,i}(v_{t_r}),$$

where $x \in \text{Im}(\varphi_{l-1})$. Because of Lemma 3, it must be

$$\varphi_{l,i}(x) = \sum_{\substack{v_{t_r} \\ \deg v_{t_r} = l}} \lambda_r \varphi_{l,i}(v_{t_r}) = 0.$$

On the other hand we also assumed that the w_q are different from zero, therefore the index $J(v_{t_q})$ of the vectors in the adapted basis has to be greater than $i - l$. Hence, because of Lemma 2, it holds that $\sum_{\{v_{t_r} \mid \deg v_{t_r} = l\}} \lambda_r \varphi_{l,i}(v_{t_r}) = 0$ implies $\sum_{\{v_{t_r} \mid \deg v_{t_r} = l\}} \lambda_r v_{t_r} = 0$. Since the $\{v_{t_r} \mid \deg v_{t_r} = l\}$ are linearly independent it must be $\lambda_r = 0$ for any r , and therefore $w_r = 0$. The same idea can be repeated for all the previous elements w_1, \dots, w_s , coming from interval modules generated by

vectors with degree less than l . Since there are finitely many vectors this process has an end and it shows that all the vectors w_1, \dots, w_k are 0 and the sum is direct. \square

We now provide an explicit construction for the set \mathcal{V} . To do so, we must first obtain sets \mathcal{V}^i .

Remark 2. Notice that the construction of each \mathcal{V}^i is independent from the others. Therefore they can be computed simultaneously.

Construction of $\mathcal{V}_{\text{Birth}}^i$

We first recall that a simple basis extension algorithm is given by the procedure described in the following Algorithm 1. The set \mathcal{W} is ordered, and its elements

Algorithm 1: Basis completion algorithm

Input: linearly independent vectors $\mathcal{U} = \{u_1, \dots, u_r\}$, linearly independent vectors $\mathcal{W} = \{w_1, \dots, w_n\}$;

Result: minimal set of vectors $w_{i_1}, \dots, w_{i_p} \notin \langle \mathcal{U} \rangle$ such that

$$\langle \mathcal{U} \cup \{w_{i_1}, \dots, w_{i_p}\} \rangle = \langle \mathcal{U} \cup \mathcal{W} \rangle$$

$\mathcal{R} = \{\}$;

for $i=1, \dots, n$ **do**

if $\text{rank}(\mathcal{U}) < \text{rank}(\mathcal{U} \cup \{w_i\})$ **then**

$\mathcal{U} = \mathcal{U} \cup \{w_i\}$;

$\mathcal{R} = \mathcal{R} \cup \{w_i\}$;

end

end

return \mathcal{R}

are added to \mathcal{U} in their ascending order in \mathcal{W} , so that \mathcal{U} is extended to a basis of $\langle \mathcal{U} \rangle + \langle \mathcal{W} \rangle$. In the following, we refer to the extension of basis \mathcal{U} by the vectors in set \mathcal{W} through Algorithm 1 as $\text{bca}(\mathcal{U}, \mathcal{W})$.

Secondly, we report Algorithm 2 performing the standard left-to-right column reduction on matrix R . The pseudo-code is explicitly reported in order to highlight the input-output representation needed, matrix C and the index of the zeroed-out columns, so that to reduce the size of the matrices treated in Algorithm 3 by discarding already computed zero-columns.

We now give a general algorithm to construct the set $\mathcal{V}_{\text{Birth}}^i$ for a given M_i of persistence module \mathcal{M} . To find an adapted basis \mathcal{V}^i we need only to iteratively

complete a basis of $\ker(\varphi_{i,j})$ to a basis of $\ker(\varphi_{i,j+1})$, using for example Algorithm 1. In general, the basis obtained through Algorithm 1 will not contain a basis of the space $\text{Im}(\varphi_{i-1})$, i.e. it will not be in the form described in Lemma 1. However, this is not necessary, as our goal is only to compute the basis vectors that are not on the image of previous maps.

The full procedure to construct $\mathcal{V}_{\text{Birth}}^i$ from M_i and the structure maps is described in Algorithm 3.

Algorithm 2: Column Reduction

Input: $a \times b$ matrix R , $b \times b$ matrix C , $I \subseteq \{1, \dots, b\}$ set of indices of the zero columns

Result: column-reduced R , change of basis matrix C , updated indices of the new zero columns I'

for $i \in \{1, \dots, b\}$ **do**

$r_i \leftarrow$ the i^{th} -column in R ;

if $i \notin I$ **then**

$J \leftarrow \{1, \dots, i-1\} \setminus I$;

 Perform left-to-right reduction of r_i using columns r_j , for

$j \in J$;

 Perform the same column operations on C ;

if $r_i = 0$ **then**

$I' \leftarrow I' \cup i$

end

end

end

return R, C, I'

Algorithm 3: *ssd* - Single step decomposition of step M_i

Input: map $\varphi_{i-1} : M_{i-1} \rightarrow M_i$;
maps $\{\varphi_j : M_j \rightarrow M_{j+1}\}, i \leq j \leq N$;
Result: $\mathcal{V}_{\text{Birth}}^i$ and its index function J

Reduce φ_{i-1} and find a basis $\mathcal{U} = \{u_1, \dots, u_k\}$ of $\text{Im}(\varphi_{i-1})$;
 $k := \dim \text{Im}(\varphi_{i-1})$;
 $R \leftarrow \text{Id} : M_i \rightarrow M_i$; // Initialize identity matrix
 $C \leftarrow \text{Id} : M_i \rightarrow M_i$; // Initialize identity matrix
 $r \leftarrow \dim(M_i)$;
 $\mathcal{V}_{\text{Birth}}^i \leftarrow \{\}$; // Initialize empty interval basis
 $\text{inds}, \text{newInds} \leftarrow \{\}$; // Sets of indices of zero columns

for $s = 0, \dots, N - i$ **do** // From i to end
 $R \leftarrow \varphi_{i+s} \cdot R$; // Matrix of the map from i to $i + s$
 $R, C, \text{newInds} \leftarrow \text{ColumnReduction}(R, C, \text{inds})$;
 $r' \leftarrow \text{rank}(R) = r - |\text{newInds}|$;
 if $r' < r$ **then** // If some bar has died
 $B \leftarrow \text{basis of } \ker(R) = \{Ce_i, i \in \text{newInds}\}$;
 $B \supseteq B_{\text{new}} \leftarrow \text{bca}(\mathcal{U}, B)$; // Complete \mathcal{U} to B by B_{new}
 $\mathcal{U} \leftarrow \mathcal{U} \cup B_{\text{new}}$; // Update \mathcal{U}
 $\mathcal{V}_{\text{Birth}}^i \leftarrow \mathcal{V}_{\text{Birth}}^i \cup B_{\text{new}}$; // Update interval basis
 for $v \in B_{\text{new}}$ **do**
 $J(v) \leftarrow s + 1$; // Set appropriate index
 end
 $r \leftarrow r'$; // Update "remaining" rank
 $\text{inds} \leftarrow \text{inds} \cup \text{newInds}$; // Update zero columns
 end
 // If all bars dead or enough generators
 if $r = 0$ **or** $|\mathcal{V}_{\text{Birth}}^i| + k = \dim M_i$ **then**
 break;
 end
end
return $\mathcal{V}_{\text{Birth}}^i, J$

In the following, we refer to the construction of $\mathcal{V}_{\text{Birth}}^i$ through Algorithm 3 as $\text{ssd}(M_i)$.

Construction of \mathcal{V}

Once the decomposition of each space is performed, it is immediate to assemble the interval basis \mathcal{V} . Further, we can read the persistence diagram of module $\{(M_i, \varphi_i)\}_i$ off of the interval basis by storing the indices of appearance and death of its elements, without increasing the computational cost. This is the content of Algorithm 4, which summarizes the procedures introduced so far into a single routine that takes a persistence module and returns its interval basis and persistence diagram.

Algorithm 4: Persistence module decomposition

Input: persistence module $\{M_i, \varphi_i\}_{i=1}^n$;
Result: interval basis $\{v_i^s\}$ and persistence diagram
 $\varphi_0 :=$ empty matrix with $\dim M_0$ rows and 0 columns;
 $\varphi_{n+1} :=$ empty matrix with 0 rows and $\dim M_n$ columns;
 $\mathcal{V} = \{\}$;
 $PD = \{\}$;
parfor $i = 1, \dots, n + 1$
 $\mathcal{V}_{\text{Birth}}^i, J = \text{ssd}(\varphi_{i-1}, \{\varphi_j\}_{j \geq i})$;
 for $v \in \mathcal{V}_{\text{Birth}}^i$ **do**
 $\mathcal{V} \leftarrow \mathcal{V} \cup v$;
 $PD \leftarrow PD \cup (i, i + J(v))$;
 end
end
return \mathcal{V}, PD

We refer to the decomposition of Algorithm 4 as $\text{pmd}(M)$.

Lemma 4. (Correctness) *The output of Algorithm 4 is an interval basis.*

Proof. \mathcal{V} is the union of sets $\mathcal{V}_{\text{Birth}}^i$ from Algorithm 3, and each $\mathcal{V}_{\text{Birth}}^i$ is by definition as in Definition 2. Then correctness follows from Theorem 1. \square

Example 1. Consider the same \mathbb{R} -persistence module as in Example 4.

$$0 \xrightarrow[\begin{pmatrix} 0 \end{pmatrix}]{\varphi_0} \mathbb{R} \xrightarrow[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]{\varphi_1} \mathbb{R}^2 \xrightarrow[\begin{pmatrix} 1 & 1 \end{pmatrix}]{\varphi_2} \mathbb{R} \xrightarrow[\begin{pmatrix} 0 \end{pmatrix}]{\varphi_3} 0.$$

We showcase the procedure of Algorithm 4 and compute its interval basis. Notice that this example matches the persistence module generated by persistent homology in Fig. 3. For $i = 0, 1, 2, 3$ we need to compute $\mathcal{V}_{\text{Birth}}^i$. Clearly φ_0 is the null map, so the flag for the first step is trivial and $\mathcal{V}_{\text{Birth}}^0$ is empty.

For $i = 1$, we have $\text{Im}(\varphi_{i-1}) = 0$ and $\ker(\varphi_{1,2}) = \ker(\varphi_{1,3}) = 0$, so $\mathbb{R} = \ker(\varphi_{1,4})$. By `ssd` we extend a basis of $\text{Im}(\varphi_0)$ (which is empty) to a basis of \mathbb{R} , which yields vector 1. Then $\mathcal{V}_{\text{Birth}}^1 = \{1\}$ with persistence pair $(1, 4)$.

For $i = 2$, we have $\text{Im}(\varphi_{i-1}) = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$. Furthermore $\ker(\varphi_{2,3}) = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$, so we extend the basis of $\text{Im}(\varphi_1)$ against the basis of $\ker(\varphi_{2,3})$ obtaining set $\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \}$, which spans \mathbb{R}^2 , so `ssd` terminates setting $\mathcal{V}_{\text{Birth}}^2 = \{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \}$ with persistence pair $(2, 3)$.

For $i = 3$, we have $\text{Im}(\varphi_{i-1}) = \mathbb{R}$, so $\mathcal{V}_{\text{Birth}}^3$ is empty. Finally, the interval basis is $\mathcal{V} = \{1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$, with persistence diagram $PD = \{(1, 4), (2, 3)\}$. It is (up to the irrelevant sign) the same result as in Example 4, as vector 1 in degree 1 corresponds to the first generator g_1 , and vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in degree 2 corresponds to the difference of the second and third $g_2 - g_3 = xg_1 - g_3$.

5 Computational complexity

In this section, we give an estimate of the computational complexity of the parallel algorithm for the decomposition of a persistence module just presented in Section 4. The evaluation of our parallel Algorithm 4 depends on the output barcode, not just in terms of the number of intervals but also in terms of their length. In the final part, we argue that our parallel algorithm has a worst-case complexity lower than that of the known procedure of decomposing a persistence module by reducing in Smith Normal Form a matrix presenting the persistence module. In order to make the comparison precise, we provide an evaluation of the SNF reduction on the graded SNF algorithm presented in [49] and specialized to persistence module matrix presentations as described in [16].

Let us assume that our persistence module has steps M_i , each having dimension m_i for i varying in $\{0, \dots, n+1\}$, where $m_0 = m_{n+1} = 0$, $m = \sum_i m_i$, and $\bar{m} = \max_i m_i$. We assume a parallel implementation of Algorithm 4. Hence, we focus on the single step decomposition performed by Algorithm 3 on step $i = 1, \dots, n$. First, a column reduction is called only once, before entering the outer for-loop, to extract the image of φ_{i-1} and it reduces a matrix of size $m_i \times m_{i-1}$. We observe that inside the inner for-loop the total number of operations depends on the parameter $k_i = m_i - r_i$, where $r_i = \text{rank}(\varphi_{i-1})$, and on the variable parameter

r_s that counts the number of columns that have not yet been reduced to zero. We claim that we can estimate its time complexity as

$$O(\bar{m}V_i), \tag{11}$$

where we defined the *output-dependent* parameter $V_i = \sum_s r_s$.

Indeed within the inner for-loop, for each $s = 0, \dots, n - i$:

- a matrix multiplication is called for matrices of size $m_{i+1+s} \times m_{i+s}$ and $m_{i+s} \times r_s$;
- a column reduction is called for a matrix of size $m_{i+1+s} \times r_s$;
- a basis completion (**bca**), Algorithm 1, is called for a list of r_i vectors and a list of $|\text{newInds}|$ vectors.

First, we observe that the contribution of the calls of the **bca** subroutine can be expressed independently from s . Indeed, the **bca** subroutine is performed in chunks as s increases, but the sum of the $|\text{newInds}|$ eventually amounts to k_i . As the interval generators remain linearly independent (Lemma 2), the total cost of **bca** for the whole for-loop amounts to that of reducing a list of k_i vectors against a list of r_i vectors, each vector being of size m_i . The total cost is therefore $O(m_i r_i k_i)$, where parameters are related via $k_i = m_i - r_i$. By substitution one obtains $m_i^2 r_i - m_i r_i^2$, hence the **bca** subroutine requires $O(m_i^2 r_i)$ operations.

On the contrary, the above mentioned matrix multiplications and column reductions do depend on the iteration parameter s within the inner for-loop. The cost of matrix multiplication between a $m_{i+1+s} \times m_{i+s}$ and a $m_{i+s} \times r_s$ matrix is $O(m_{i+1+s} m_{i+s} r_s)$. Let us consider the worst case \bar{m} for all m_i 's. We get $O(\bar{m}^2 r_s)$ for each step s . The cost of column reduction of a matrix of size $m_{i+1+s} \times r_s$ can be also bounded by $O(\bar{m}^2 r_s)$. Now, the parameter $V_i = \sum_s r_s$ is, intuitively, the “volume” of all bars born at step M_i until their death, and we can express the total cost of matrix multiplication and column reduction by $O(\bar{m}^2 V_i)$. This makes the contribution of **bca** negligible and hence provide the global cost of Algorithm 3.

In order to compare the time complexity of Algorithm 4 to that of the graded SNF algorithm[49] (here included as Algorithm 8 in Appendix B.3) when applied to persistence module presentations, we first express it in terms of the input parameters m and \bar{m} . In the case of our parallel Algorithm 4, we have $V_i \leq (n - i + 1)\bar{m}$. The equality corresponds to a “rectangular” barcode where all interval modules started at step i are non-trivial till step $i = n$. The single step decomposition reaches the worst case when applied to step $i = 1$. In that case, we can change

the output-dependent estimate $O(\bar{m}^2 V_1)$ into $O(\bar{m}^3 n)$. Observe that the parameter m expressing the sum of all m_i 's is now equal to $\bar{m}n$. Hence, we obtain the *input-dependent* estimate

$$O(\bar{m}^2 m). \tag{12}$$

Now, we focus on the graded SNF algorithm specialized to persistence module presentations, that is, we assume the input matrix S to be an $m \times m$ matrix S subdivided into n blocks of size $m_i \times (m_i + m_{i+1})$. First of all, this implies a significantly higher space complexity in the input representation $O(m^2)$ with respect to our parallel approach $O(\bar{m}^2)$. As for the time complexity, the classical estimate of the SNF reduction of an $m \times k$ matrix is $O(m^2 k)$ and the estimate is known to be at most lowered down via optimized algorithms to $O(km^{\omega-1})$ [50] where $2 \leq \omega < 3$ is the lower bound complexity for matrix multiplications. Here, we specialize the general column parameter k . Our assumption on the subdivision into blocks of the matrix S implies a lower complexity estimate. Indeed, for each of the m processed columns (see Algorithm 8 in Appendix B.3) the procedure performs at most m row reductions. However, each row involved has at most $m_i + m_{i+1}$ non-trivial entries. Hence the only non-trivial column reductions which follows are $O(\bar{m})$. This gives $O(\bar{m}m^2)$ as the global worst case, or $O(\bar{m}m^{\omega-1})$ for optimized implementations.

By comparison to the input-dependent evaluation of (12), we have that the worst case of the parallel procedure has still lower time complexity than the graded Smith Normal Form even when specialized to persistence module presentations.

6 Persistent homology modules

In this section, we provide a parallel method to obtain a persistence module by applying the k^{th} -homology functor to an equi-oriented finite sequence of chain complex maps that are not necessarily injective. First, we fix the notation. Then, we show a construction of the k^{th} -persistent homology module by homology representatives, then by harmonics representatives obtained through the combinatorial Laplacian operator. For the general homology representative case, our approach is a simple adaptation of known algorithms independently acting on each step in the input sequence of chain complexes. Here, we simply state the desired properties to be fulfilled by the chosen method.

Our aim is to underline the level of generality of our approach and to exemplify the possibility of being adaptable to a large variety of special homology representatives. We observe that the parallel approach may lead to unnecessary computation repetitions in the case of injective chain maps, namely in the case of persistent ho-

mology. The case of the harmonics instead admits a more efficient construction of the module via simple matrix multiplications thanks to our proof of Theorem 4.

A *chain complex* with coefficients in \mathbb{F} is a sequence $C = (C_\bullet, \partial_\bullet)$ of \mathbb{F} -vector spaces connected by linear maps with $k \in \mathbb{N}$

$$\dots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

such that $\partial_{k+1}\partial_k = 0$ for all $k \in \mathbb{N}$. Each vector space C_k is called the space of *k-chains*. The subspace $Z_k = \ker(\partial_k)$ is called the space of *k-cycles*. The subspace $B_k = \text{Im}(\partial_{k+1})$ is called the space of *k-boundaries*. The condition $\partial_{k+1}\partial_k = 0$ ensures that $B_k \subseteq Z_k$, for all $k \in \mathbb{N}$. The quotient space $H_k = Z_k/B_k$ is the *k-homology space*. A *chain map* $f : (C_\bullet, \partial_\bullet^C) \rightarrow (D_\bullet, \partial_\bullet^D)$ is a collection of linear maps $f_k : C_k \rightarrow D_k$ such that $f_k\partial_{k+1}^C = \partial_{k+1}^D f_{k+1}$, for all $k \in \mathbb{N}$.

A chain map induces linear maps $\tilde{f}_k : H_k^C \rightarrow H_k^D$, for all k , and it can be shown that this implies that the H_k are indeed functors from the category of chain complexes and chain maps to the category of vector spaces over \mathbb{F} and linear mapping, see [32] for a complete account of these facts.

To compute the matrix associated with the map $\tilde{f}_k : H_k^C \rightarrow H_k^D$, we assume to have a basis $\{h_1^C, \dots, h_{\beta_k^C}^C, b_1^C, \dots, b_q^C\}$ of $Z_k(C)$, where $\{b_1^C, \dots, b_q^C\}$ is a basis of $B_k(C)$, and basis $\{h_1^D, \dots, h_{\beta_k^D}^D, b_1^D, \dots, b_r^D\}$ of $Z_k(D)$, such that $\{b_1^D, \dots, b_r^D\}$ is a basis of $B_k(D)$. Such basis can be found applying Algorithm 9 in Appendix C. Then, for each $s = 1, \dots, \beta_k^C$, the following linear system in the variables $\lambda_1^s, \dots, \lambda_{\beta_k^D}^s, \mu_1^s, \dots, \mu_r^s$ has to be solved:

$$f_k(h_i^C) = \sum_{j=1}^{\beta_k^D} \lambda_j^s h_j^D + \sum_{l=1}^r \mu_l^s b_l, \quad (13)$$

defining the matrix with columns $(\lambda_1^s, \dots, \lambda_{\beta_k^D}^s)^T$, with $s = 1, \dots, \beta_k^C$, as the matrix of \tilde{f}_k with respect to the basis induced by the projection of $\{h_1^C, \dots, h_{\beta_k^C}^C\}$ and $\{h_1^D, \dots, h_{\beta_k^D}^D\}$ to their respective homology space (see Algorithm 10 in Appendix C).

The application of the functor H_k to a given sequence of complexes and chain maps

$$C_\bullet^1 \xrightarrow{f^1} \dots \longrightarrow \dots \xrightarrow{f^{i-1}} C_\bullet^i \xrightarrow{f^i} \dots \xrightarrow{f^{n-1}} C_\bullet^n \quad (14)$$

provide a persistence module $\{(H_k^i, \tilde{f}_k^i)\}_{i=0}^n$ for all $k \geq 0$. This persistence module is called the k^{th} -persistent homology of the sequence of chain complexes (14), see [21].

6.1 Parallel construction of the k^{th} -persistent homology module via harmonics

In this section, we describe a parallel construction of the persistence module $\{(\mathcal{H}_k^i, \hat{f}_i)\}_{i \in \mathbb{N}}$ where \mathcal{H}_k^i is the space of k -harmonics at step i and coefficients are taken in \mathbb{R} . We call the persistence module $\{(\mathcal{H}_k^i, \hat{f}_i)\}_{i \in \mathbb{N}}$ the k^{th} -harmonic persistence module.

After some preliminaries on the Hodge Laplacian operator, by means of the Hodge decomposition (Theorem 2, Theorem 3), for each index $i \in \mathbb{N}$, we show that there exists a structure map \hat{f}_i induced by f_k , such that the k^{th} -persistent homology module $\{(H_k^i, \varphi_i)\}$ and the k^{th} -harmonic persistence module $\{(\mathcal{H}_k^i, \hat{f}_i)\}$ are isomorphic. We then provide Algorithm 5 to compute these maps.

The Hodge Laplacian

In this section we fix $\mathbb{F} = \mathbb{R}$. Given a chain complex $(C_\bullet, \partial_\bullet^C)$, we choose an inner product $\langle \cdot, \cdot \rangle_k$ on each space of C_k so that it is well-defined the adjoint of ∂_k , i.e. the map $\partial_k^* : C_{k-1} \rightarrow C_k$ such that $\langle \partial_k(c), d \rangle_{k-1} = \langle c, \partial_k^*(d) \rangle_k$, for all $c \in C_k, d \in C_{k-1}$.

For $k \in \mathbb{N}$, the *Hodge Laplacian* in degree k (Laplacian, for short) is the linear map on k -chains $L_k : C_k \rightarrow C_k$ given by

$$L_k := \partial_{k+1} \partial_{k+1}^* + \partial_k^* \partial_k. \quad (15)$$

The space of k -harmonics of a chain complex is the subspace of C_k

$$\mathcal{H}_k := \ker(L_k). \quad (16)$$

We refer to [33] for more details.

We recall the following theorems, see Section 5.1 of [38].

Theorem 2. *For a chain complex C and for every natural k ,*

$$C_k = \mathcal{H}_k \oplus \text{Im}(\partial_{k+1}) \oplus \text{Im}(\partial_k^*)$$

Moreover, this decomposition is orthogonal and $Z_k = \mathcal{H}_k \oplus \text{Im}(\partial_{k+1})$.

Theorem 3. *The linear map $\psi_k : \mathcal{H}_k \rightarrow H_k$ defined by $\psi_k(h) = [h]$ is an isomorphism, where $[h]$ is the homology class induced by the cycle h .*

An obstacle to persistence of the harmonic space is the following.

Remark 3. A chain map $f : C \rightarrow D$ does not restrict to a map between the harmonic subspaces \mathcal{H}_k^C and \mathcal{H}_k^D .

Indeed, given an element $h \in \mathcal{H}_k^C$, the k -cycle $f(h)$ is not necessarily in \mathcal{H}_k^D . More precisely, $f(h)$ is necessarily a k -cycle but not necessarily a k -cocycle.

However, given a sequence of chain complexes and chain maps as in (14) we want to construct a persistence module, isomorphic to the persistent homology module of the sequence, given by the harmonic spaces of the chain complexes, with maps induced by the chain maps. The following theorem is sufficient to provide such a persistence module.

Theorem 4. *For any chain map $f : C \rightarrow D$ and any $k \in \mathbb{N}$, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{H}_k^C & \xrightarrow{\hat{f}_k} & \mathcal{H}_k^D \\ \downarrow \psi_k^C & & \downarrow \psi_k^D \\ H_k^C & \xrightarrow{\tilde{f}_k} & H_k^D. \end{array}$$

where $\hat{f}_k = \pi_k^D f_k i_k^C$, i_k^C is the natural inclusion of \mathcal{H}_k^C into C_k and π_k^D is the orthogonal projection of D_k onto \mathcal{H}_k^D .

Proof. For any $h \in \mathcal{H}_k^C$, we can see that $\tilde{f}_k(\psi_k^C(h)) = \psi_k^D(\hat{f}_k(h))$. In fact, by the definition of \tilde{f} and \hat{f} , it holds $\tilde{f}_k(\psi_k^C(h)) = \tilde{f}_k([h]_C) = [f_k(h)]_D$ and $\psi_k^D(\hat{f}_k(h)) = [\pi_k^D(f_k(h))]_D$. Since $f_k(h)$ is a cycle in D_k and because of the decomposition in Theorem 2, there is a boundary b of D_k such that $f_k(h) = \pi_k^D(f_k(h)) + b$, hence $[f_k(h)]_D = [\pi_k^D(f_k(h))]_D$ and the diagram commutes. \square

The matrix of \hat{f}_k can then be easily computed with Algorithm 5.

Algorithm 5: Induced map between Laplacian kernels

Input: Chain map $f_k : C_k(C) \rightarrow C_k(D)$, $\{v_1^C, \dots, v_n^C\}$ orthonormal basis of \mathcal{H}_k^C , $\{w_1^C, \dots, w_m^C\}$ orthonormal basis of \mathcal{H}_k^D ;

Result: Matrix Φ representing $\hat{f}_k : \mathcal{H}_k(C) \rightarrow \mathcal{H}_k(D)$

$V_C :=$ matrix with columns v_1^C, \dots, v_n^C ;

$V_D :=$ matrix with columns w_1^D, \dots, w_m^D ;

$\Phi = V_D^T \cdot f_k \cdot V_C$;

return (Φ_i)

6.2 Simplicial complex chains

An (*abstract*) *simplicial complex* Σ on a finite set V is a subset of the power set of V , with the property of being closed under restriction. An element of Σ is called a *simplex* and if $\sigma \in \Sigma$, $\tau \subseteq \sigma$ then $\tau \in \Sigma$. Elements of V are usually called *vertices*. Simplices of cardinality $k + 1$ are called k -simplices. We also say a k -simplex has *dimension* k . We call the k -skeleton of Σ the set of simplices of Σ of dimension $\leq k$. If $\tau \subseteq \sigma$ we say that τ is a face of σ and σ is a coface of τ . The dimension of a simplicial complex is defined as $\dim \Sigma := \max\{\dim \sigma \mid \sigma \in \Sigma\}$. By numbering the vertices in V , we define a *positively oriented* k -simplex $\sigma = [v_0, \dots, v_k]$ as the class of tuples $(v_{p(0)}, \dots, v_{p(k)})$ with p an even permutation. All remaining permutations give the *negatively oriented* simplex σ .

From Σ a simplicial complex and \mathbb{F} any field, we can specialize the chain complex construction of Section 6. We obtain a chain complex $(C_\bullet, \partial_\bullet)$ by defining, for each k , $C_k = C_k(\Sigma)$, where $C_k(\Sigma)$ is the space of k -simplicial chains consisting of finite \mathbb{F} -linear combinations of the oriented k -simplices of Σ and such that $-\sigma$ coincides with the opposite orientation on σ . We define $\partial_k = \partial_k(\Sigma)$, where $\partial_k(\Sigma) : C_k(\Sigma) \rightarrow C_{k-1}(\Sigma)$ is the *simplicial boundary map* defined on an element of the canonical basis $\sigma = [v_0, \dots, v_k] \in \Sigma_k$ by $\partial_k(\sigma) = \sum_{i=0}^k (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$, where \hat{v}_i means that vertex v_i is omitted. It extends to the whole chain space by linearity. A *simplicial map* $s : \Sigma \rightarrow \Sigma'$ is a map satisfying, for each $\sigma \subseteq \tau$, $s(\sigma) \subseteq s(\tau)$. Denote by $(C_\bullet, \partial_\bullet^C)$ the simplicial chain complex of Σ and by $(D_\bullet, \partial_\bullet^D)$ the simplicial chain complex of Σ' . Then the simplicial map s induces a chain map $f : (C_\bullet, \partial_\bullet^C) \rightarrow (D_\bullet, \partial_\bullet^D)$ by setting $f_k(\sigma) = s(\sigma)$ for all $\sigma \in \Sigma_k$.

It is clear that we can apply the persistent homology module constructions presented in this section to any monotone sequence of simplicial maps and hence our parallel algorithm introduced in Section 4 specializes to the simplicial complex case. In the following, we report some examples and discuss interesting points to remark with respect to the simplicial case.

Example 2 (Vertex collapse). Consider the chain map f induced by the vertex collapse in Fig. 5. We apply the above construction to the retrieval of the associated 2-step persistent homology module in degree 1. Depending on the chosen vertex labeling and optimization procedure, our parallel construction of the homology steps might return several choices of homology representatives. For instance, according to the labeling of vertices in black, the already mentioned left-to-right reduction in [13] would return the red and green 1-chains at step 1 and the red 1-chain at step 2. Furthermore, solving the linear systems in Eq. (13) would give a matrix $\begin{pmatrix} 1 & -1 \end{pmatrix}$ representing the linear map \tilde{f} , meaning that the green and red homology representatives from step 1 are both mapped to the red homology representative of step 2 with opposite sign. This concludes the construction of the

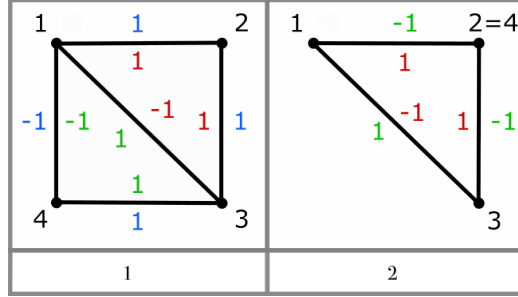


Figure 5: an example of a vertex collapse inducing a chain map $f = (f_k)$ where step 2 is obtained from step 1 by identifying vertexes 2 and 4. Coefficients of 1-chains of possible degree 1 homology representatives are depicted with the same color. The figure shows, at step 1 in red $z_1 = [1, 2] + [2, 3] - [1, 3]$, in green $z_2 = [1, 3] + [3, 4] - [1, 4]$, in blue $z = [1, 2] + [2, 3] + [3, 4] - [1, 4]$; at step 2 in red $\omega_1 = [1, 2] + [2, 3] - [1, 3]$, in green $\omega_2 = [1, 3] - [2, 3] - [1, 2]$. The 1-component f_1 of the chain map f sends z_1 to ω_1 , and z_2 to ω_2 . Hence the image of $z = z_1 + z_2$ is trivial.

desired 2-step persistence module. We observe that the red and green homology representatives do not form an interval basis since, at step 2, they are non-trivial and equivalent one another. The parallel decomposition previously introduced in Section 4 can be applied to the obtained persistence module to get the interval basis formed by the red and blue homology representatives at step 1. This way the blue representative captures the homology class (the sum of the red and green representatives) being born at step 1 and dying at step 2 and the red one captures the class being born at step 1 and still non-trivial at step 2.

Example 3 (Tracking harmonic homology representatives). Consider the chain map f induced by the inclusion in Fig. 6. We apply the construction above to compute harmonic homology representatives in the first degree for the two steps: red and blue on the left (step 1) and red on the right (step 2). In step 1, we notice that the two obtained harmonics coincide with generic homology representatives. This is due to the absence of 2-simplices. However, harmonic representatives are not, in general, preserved by the inclusion of simplicial complexes. Indeed, in step 2, the same homology representatives are no longer harmonic forms. By Algorithm 5, we retrieve that the red 1-chain ω at step 2 is a combination of the inclusions of the red z_1 and blue z_2 1-chains at step 1: $\omega = z_1 + 3z_2$. This concludes constructing the desired 2-step persistent homology module via harmonic representatives. We observe that z_1 and z_2 do not form an interval basis for the obtained persistence

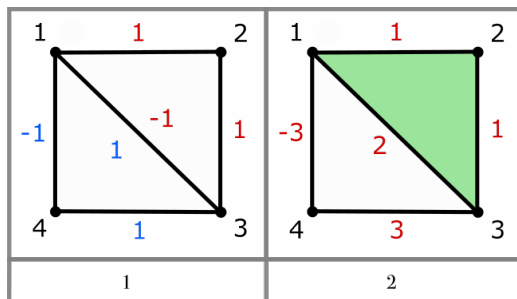


Figure 6: an example of tracking of harmonic homology representatives along the inclusion of simplicial complexes obtained by inserting the 2-simplex [123]. Coefficients of 1-chains forming harmonic homology representatives for the associated 2-step persistence module are depicted with the same color in each step. The figure shows, at step 1 in red $z_1 = [1, 2] + [2, 3] - [1, 3]$, in blue $z_2 = [1, 3] + [3, 4] - [1, 4]$; at step 2 in red $\omega = [1, 2] + 2[1, 3] - 3[1, 4] + [2, 3] + 3[3, 4]$. The 1-component f_1 of the chain map f sends $z_1 + 3z_2$ to ω .

module. Our parallel decomposition allows us to choose $z = z_2 - 3z_1$ and z_2 as harmonic representatives at step 1 so that the inclusion directly maps z to ω and z_2 to 0 and each harmonic representative is kept independent from other representatives along the module steps.

Filtered simplicial complexes

A simplicial complex Σ can be made into a *filtered simplicial complex* (or a *filtration of simplicial complexes*) by taking a finite sequence of subcomplexes $\Sigma^0 \subseteq \Sigma^1 \subseteq \dots \subseteq \Sigma^n = \Sigma$, where a *subcomplex* is a subset and also a simplicial complex. The inclusion maps in the filtration induce a monotone sequence of chain complexes with maps $\{f^i\}$ as in (14) where the induced chain maps are all injective. We fix a dimension k . We get that the filtered chain complex $C_k = \{(C_k^i, f_k^i)\}_{i=1}^n$ is a persistence module.

Remark 4. The $\mathbb{F}[x]$ -graded module $\alpha(C_k)$ associated with the the persistence module of the filtered chain complex C_k is free. Moreover, an interval basis for C_k consists, for each index i , of the k -simplices σ in $\Sigma^i \setminus \Sigma^{i-1}$.

The freeness of $\alpha(C_k)$ implies that also the graded modules associated to the persistence modules of the *filtered k -cycles* Z_k and the persistence modules of the *filtered k -boundaries* B_k are free. To this purpose, we underline the following observations with respect to the left-to-right reduction $\partial = RV$ [13] of the boundary matrix ∂ which is at the heart of most of persistent homology computations,

where R is the reduced matrix and V keeps track of the operations performed on the columns along the reduction.

Remark 5. The collection \mathcal{V} of the homology representatives in the columns of V corresponding to the null columns in R form an interval basis for Z_k . The same collection \mathcal{V} is a minimal system of generators of the persistent homology module H_k which is not in general an interval basis. A presentation of H_k requires further column reductions to find the combinations expressing a system of generators of B_k in terms of \mathcal{V}_Z . An example of \mathcal{V}_Z is in Fig. 3 in Section 1.

Remark 6. The collection \mathcal{V} of the homology representatives in the non-trivial columns of R form an interval basis for B_k . If the graded module associated to the persistent homology module H_k has no free part, the same collection \mathcal{V} is also an interval basis for H_k . Otherwise, the same collection \mathcal{V} can be extended to an interval basis of H_k by adding the elements of the interval basis of Z_k of infinite order in the associated graded module (essential classes). An example is in Fig. 4 in Section 1. The retrieval of this kind of homology representatives is the one implemented, for instance in [5], by the clear optimization procedure [6].

Indeed, one can easily check that with respect to Z_k we get a minimal system of generators \mathcal{V}_Z in the case of Remark 5. By simply noticing that boundaries are cycles, the system \mathcal{V}_B in the case Remark 6 is a minimal system of generators of Z_k . Hence both \mathcal{V}_Z and \mathcal{V}_B are a minimal system of generators for H_k .

By freeness, the two systems form an interval basis of Z_k . Moreover, \mathcal{V}_B contains a minimal system of generators of B_k , and hence an interval basis for B_k . Notice that, the reduction provide a presentation matrix of H_k with respect to generators in \mathcal{V}_B whereas a presentation with respect to \mathcal{V}_Z is less direct to retrieve. For \mathcal{V}_B , the associated presentation matrix has one row per element \mathcal{V}_B and one columns per element of \mathcal{V}_B belonging to the the interval basis of B_k . Rows are graded as elements of Z_k . Columns are graded as elements of B_K .

The subset in \mathcal{V}_B which forms the interval basis for B_k defines the rows containing exactly a non-trivial element in correspondence of the column of the matched boundary. Indeed, if a generator v of degree i in \mathcal{V}_B is matched to w a generator of degree $j \geq i$ of B_k , this implies that $x^{j-i}v = 0$. All other entries are trivial. Hence the presentation matrix with respect to \mathcal{V}_B is in Smith Normal Form according to Definition 4 and hence \mathcal{V}_B is an interval basis of H_k . For \mathcal{V}_Z , if a generator v of degree i in \mathcal{V}_Z is matched to w a generator of degree $j \geq i$ of B_k , this does not imply that $x^{j-i}v = 0$ since there could be another element v' of degree $h \leq i$ such that $x^{j-i}v = x^{j-h}v' \neq 0$.

7 Conclusions and Future Works

In this work, we have formalized interval bases of a persistence module as particular minimal system of generators.

We have introduced the Algorithm 4 as a distributed approach for retrieving an interval basis of a persistence module. Our approach applies to any persistence module as defined in Section 2, hence not necessarily coming from the homology of a filtered chain complex. A specialization for real coefficients based on the SVD matrix decomposition is also available in Appendix A. Through our computational evaluation in Section 5, we have compared our parallel approach to the Smith Normal Form reduction specialized to matrices presenting a persistence module and showed that the Smith Normal Form reduction can sensibly reduce its cost when adapted to persistence modules. However, the advantage of our parallel approach has been quantified through an output dependent computational cost, where the output size is generally lower than the input size. We have discussed the worst output case corresponding to the case with all intervals decomposing the persistence module appearing at the same step, especially at the very first step, and being ever lasting along the entire persistence module. In TDA applications, often dealing with a persistence module obtained through the homology functor in degree 0 via the Vietoris-Rips filtration, we have a single step decomposition, the first one, which would be considerably heavier in cost with respect to other step decompositions. Even though, even in that case, one typically has that most of homology classes disappear soon along the filtration, it is fair to highlight that our parallel approach would have better performances in homology degrees other than 0 due to the higher sparsity of the obtained barcode. Alternatively, we expect our approach to be more effective on other kinds of frameworks, such as the case of harmonics forms treated in Section 6.1.

Afterwards, we have constructively indicated how to obtain a persistent homology module out of a monotone sequence of chain complexes. Above all, we have remarked that each step and structure map in the module can be obtained independently, thus being suitable for parallel and distributed approaches. Such an integration has offered interesting insights to be investigated further. For instance, it has made possible to geometrically locate the interval basis vectors onto a filtered simplicial complex. We have discussed simple examples to make comparisons with two possible kinds of homology representatives obtained through the reduction algorithm [13] and discussed which kind of homology representatives do satisfy the interval basis definition in Remark 5 and Remark 6.

We believe that, for a monotone sequence of chain maps, the descriptive power of the interval basis deserves further study since it encodes implicitly the relations among evolving homology classes. Future directions on the descriptive power of

interval bases include adapting our parallel approach to the retrieval of interval bases of submodules. This might have, in our opinion, applications to the challenge of defining interval matchings induced by persistence module morphisms to be compared with the ones already available in the literature. Other directions may include the study of interval bases of the persistent pipeline applied to multiparameter persistent homology and to non-injective families of simplicial complexes.

As a last point to be addressed in the future, we have seen how working at persistence module level might be favorable for dealing with the persistence of harmonics. In particular, we have shown that, by acting at persistence module level, we managed to overcome the problem expressed in Remark 3 in the harmonic representatives tracking. That case can take advantage of the SVD factorization for the real coefficient case to lower down the computational complexity (Appendix A). From the geometrical point of view, we have shown how the interval basis choice for generators applies to the harmonic case. Hence, our work contributes in combining harmonic generators into the persistent homology framework.

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A Appendix

Complements to Section 4 when $\mathbb{F} = \mathbb{R}$

In case we use the field \mathbb{R} in the persistence module, we can specialise the decomposition of the space described in the previous paragraph. We will use the following notation: given a matrix A with m rows and n columns, $A[:, i]$ denotes the i^{th} column of the matrix, whereas $A[:, : i]$ denotes the submatrix given by the first i columns of A . The same notation is used on the first arguments in the parenthesis to denote operations on rows. We will make use of this simple result in linear algebra.

Lemma 5. *Given three vector spaces V_1, V_2 , and V_3 over \mathbb{R} and two linear maps $\psi_1 : V_1 \rightarrow V_2$ and $\psi_2 : V_2 \rightarrow V_3$ it holds*

$$\ker(\psi_2 \circ \psi_1) = \ker(\psi_1) \oplus \ker\left(\psi_2 \circ \psi_1|_{(\ker(\psi_1))^\perp}\right).$$

Proof. Let x be an element of $\ker(\psi_2 \circ \psi_1)$. It can be written uniquely as $x = v + w$, with $v \in \ker(\psi_1)$ and $w \in (\ker(\psi_1))^\perp$. Since $(\psi_2 \circ \psi_1)(v + w) = 0$ and $v \in \ker(\psi_1)$, it must be $\psi_2(\psi_1(w)) = 0$, therefore $w \in \ker(\psi_2 \circ \psi_1)$. Then, w belongs to $\ker\left(\psi_2 \circ \psi_1|_{(\ker(\psi_1))^\perp}\right)$ and the statement follows. \square

Fix M_0 , and suppose that $\varphi_n = 0$. For each M_i , denote with d_i the number $\dim M_i$. Consider φ_0 and decompose it via the SVD decomposition in $\varphi_0 = U_0 S_0 V_0^T$. If $r_0 = \text{rank } \varphi_0$, then $k_0 = d_0 - r_0$ is the dimension of $\ker \varphi_0$. Notice that S_0 is a matrix $d_1 \times d_0$ with non-zero elements only on the first r_0 positions on the main diagonal. Therefore, if e_i is the i^{th} element of the canonical basis of \mathbb{R}^{d_0} , with $r_0 < i \leq d_0$, then $\varphi_0 V_0 e_i = U_0 S_0 e_i = 0$. Then, a basis of $\ker \varphi_0$ is given by the vectors $\{V_0 e_{r_0+1}, \dots, V_0 e_{d_0}\}$. The index function J attains the value 1 on all of them. All such vectors will be also in the kernel of the maps $\varphi_{0,j}$ for all $j > 0$. In order to avoid repetitions, it will be considered only the restriction of each $\varphi_{0,j}$ on the orthogonal complement of $\ker \varphi_0$. This operation will not change the result because of Lemma 5. To do so, consider the map $\tilde{\varphi}_0 = U_0 \tilde{S}_0$, where $\tilde{S}_0 = S_0[:, : r_0]$, given by the first r_0 columns of S_0 . Repeating the same process, it will be considered $m_1 = \varphi_1 \tilde{\varphi}_0$ instead of $\varphi_{0,2}$. Call $d_1 = d_0 - k_0$. Decompose again $m_1 = U_1 S_1 V_1^T$ and call $r_1 = \text{rank } m_1$ and $k_1 = d_1 - r_1 = \dim \ker m_1$. Again, a basis of $\ker m_1$ is given by the vectors $V_1 e_{r_1+1}, \dots, V_1 e_{d_1}$. Recall that this vectors are expressed in the basis $\{V_0[:, 1], \dots, V_0[:, r_0]\}$ of $\ker \varphi_0^\perp$. To return them in the canonical basis of M_0 it is sufficient to consider the matrix η_0 with d_0 rows and r_0 columns such that $\eta_0[i, j]$ is equal to 1 if $1 \leq i = j \leq r_0$ and 0 otherwise. Then, the vectors in the canonical basis of M_0 are $\{V_0 \eta_0 V_1 e_{r_1+1}, \dots, V_0 \eta_0 V_1 e_{d_1}\}$. In this case

the value of the index function for these vectors will be 2. For the general step j , consider $m_j = \varphi_j \tilde{m}_{j-1} = U_j S_j V_j^T$. The adapted basis of M_0 will be updated with the vectors

$$V_0 \eta_0 \dots V_{j-1} \eta_{j-1} V_j e_x, \quad r_j + 1 \leq x \leq d_j, \quad (17)$$

and it will be $J(V_0 \eta_0 \dots V_{j-1} \eta_{j-1} V_j e_x) = j + 1$ for every $r_j + 1 \leq x \leq d_j$. Once all the vectors are obtained, as in the general case, it is necessary to complete a basis of $\text{Im}(\varphi_{i-1})$ to a basis of M_0 , introducing the vectors in \mathcal{V} in ascending order given by the function J . The resulting vectors will be part of the interval basis.

The procedure is encoded in Algorithm 7, which makes use of the matrix decomposition routine Algorithm 6 and specializes Algorithm 3 to the case of real coefficients. We denote it by $\text{ssdR}(M_i)$. Then, the full decomposition of Algorithm 4 can be specialized to the reals by replacing $\text{ssd}(M_i)$ with $\text{ssdR}(M_i)$.

Algorithm 6: Matrix decomposition

Input: matrix A ;

Result: Restriction of A on the space orthogonal to its kernel with respect to a basis V of the domain, V matrix whose columns are a basis of the domain of A , $\dim(\ker A)^\perp, \dim \ker A$

$U, S, V = \text{SVD}(A)$;

$nz = \text{rank } S, d = \text{number of columns of } A, dk = d - nk$;

$R = US[:, : nz]$;

return R, V, nz, dk

Algorithm 7: single step decomposition on \mathbb{R}

Input: map $\varphi_{i-1} : M_{i-1} \rightarrow M_i$, maps $\{\phi_j : M_j \rightarrow M_{j+1}\}, i \leq j \leq N$

;

Result: Vectors $\mathcal{V}_{\text{Birth}}^i$

$U, S, V = \text{SVD}(\varphi_{i-1})$;

$r := \text{rank}(\varphi_{i-1})$;

$\mathcal{U} = U[:, : r]$ basis of the image of φ_{i-1} ;

$\mathcal{V}_{\text{Birth}}^i = \{\}$; $lk = 0$;

$R = \text{Id} : M_i \rightarrow M_i$;

$d := \dim M_i$;

$V_{\text{tot}} = I_d$;

for $s = 0, \dots, N - i$ **do**

$R = \varphi_{s+1} \cdot R$;

if *number of rows of* $R = 0$ **then**

$k :=$ number of columns of R ;

$V = I_k$;

$nz = 0$;

$dk = k$;

else

$R, V, nz, dk = \text{dec}(R)$;

end

$V_{\text{temp}} = I_d, l = \text{ord}(V), V_{\text{temp}}[:, : l] = V^t$;

$V_{\text{tot}} = V_{\text{tot}} \cdot V_{\text{temp}}$;

if $dk > 0$ **then**

$\mathcal{T} = \text{bca}(\mathcal{U}, V_{\text{tot}}[:, d - lk - dk : d - lk])$;

$\mathcal{U} = \mathcal{U} \cup V_{\text{tot}}[:, d - lk - dk : d - lk]$;

$\mathcal{V}_{\text{Birth}}^i = \mathcal{V}_{\text{Birth}}^i \cup \mathcal{T}$;

$J(t) = s + 1$ for all $t \in \mathcal{T}$;

$lk = lk + dk$;

end

if $nz = 0$ **or** $|\mathcal{V}| + r = d$ **then**

break;

end

end

return $\mathcal{V}_{\text{Birth}}^i, J$

B Appendix

B.1 Graded modules

In this section, we introduce some notation for graded module presentations and decomposition into cyclic modules with the purpose of linking the interval basis notion to the Smith normal form. Since it is not obvious to find in the literature the adaptation of classical decomposition results [41] with notation specialized to the graded module case, we include proofs of classical results.

In the following, we consider the polynomial ring $\mathbb{F}[x]$ endowed with the standard grading structure defined by monomial decomposition of polynomials where the degree $\deg x$ of the indeterminate x is set to 1. This way, $\mathbb{F}[x]$ is seen as a direct sum of the \mathbb{F} -vector spaces $\mathbb{F}[x]_i$ containing monomials of degree i . Moreover, the property $x^j \mathbb{F}[x]_i \subseteq \mathbb{F}[x]_{i+j}$ holds. A graded $\mathbb{F}[x]$ -module is an $\mathbb{F}[x]$ -module admitting a direct sum decomposition into \mathbb{F} -vector spaces, called homogeneous parts of degree i , such that the action of x^j over each homogeneous element of degree i gives a homogeneous element of degree $i + j$. We denote by $\deg v$ the maximum degree of the homogeneous components of $v \in M$. Clearly, $\mathbb{F}[x]$ can be seen as a graded $\mathbb{F}[x]$ -module. This allows us to make explicit the equivalence of categories α already mentioned in Section 1.

A persistence module \mathcal{M} can be associated with a graded $\mathbb{F}[x]$ -module $\alpha(\mathcal{M})$ under a well-known equivalence of categories [13, 16], in the following way: given \mathcal{M} as above, $\alpha(\mathcal{M})$ is defined as $\bigoplus_{i \in \mathbb{N}} \alpha(M_i) := M_1 \oplus M_2 \cdots \oplus M_n \oplus M_n \oplus M_n \cdots$. The structure of graded is obtained by setting $xv = \varphi_i(v)$, for each $i \in [n]$ and $v \in \alpha(M_i) = M_i$ and $xv = v$ for $v \in \alpha(M_j) = M_n$ for $j > n$. The steps M_i is referred to as the homogeneous part of degree i of $\alpha(\mathcal{M})$.

Before proceeding, we introduce the shift notation $\mathbb{F}[x](-d)$ for the graded module $\mathbb{F}[x]$ with standard degrees shifted so that the constant polynomial 1 has degree d . Moreover, we restrict to consider homogeneous homomorphisms with zero degree, i.e., preserving degrees.

Definition 3. Let M be a finitely generated graded $\mathbb{F}[x]$ -module. A *presentation* of M is a choice of

- a finite system of homogeneous *generators* $V = \{v_i\}_{i \in I}$ in M ;
- a finite set of homogeneous equations, called *relations (or syzygies)* $S = \{s_j\}_{j \in J}$ in M ,

such that the following sequence is exact.

$$\bigoplus_{j \in J} \mathbb{F}[x](-\deg s_j) \xrightarrow{\sigma} \bigoplus_{i \in I} \mathbb{F}[x](-\deg v_i) \xrightarrow{\epsilon} M \longrightarrow 0, \quad (18)$$

where the map $\epsilon : \bigoplus_{i \in I} \mathbb{F}[x](-\deg v_i) \rightarrow M$ sends the i^{th} -standard generator e_i to v_i , and $\sigma : \bigoplus_{j \in J} \mathbb{F}[x](-\deg s_j) \rightarrow \bigoplus_{i \in I} \mathbb{F}[x](-\deg v_i)$ expresses the equations s_j with respect to the standard basis $\{e_i\}_{i \in I}$.

In other words, the module M is obtained as the cokernel of σ or $\text{Coker}(S)$ where, with little abuse of notation, the matrix S is set to have as column j the coefficients of s_j . In this case, we say that S is a *presentation matrix* of M , and in the following we will refer to a pair $(\{v_i\}, S)$ as a presentation of M .

Definition 4 (Graded Smith Normal Form). A presentation matrix S for some graded $\mathbb{F}[x]$ -module M is in *graded Smith Normal Form* if and only if each non-zero entry, called pivot, is the unique non-zero entry in its row and column, and the pivot is equal to x^p for some integer $p \geq 0$. We will call $\text{Ones}(S)$ the set of row indices in S with pivots equal to 1.

In order to link a graded Smith Normal Form presentation to an interval decomposition, we set the following notation for the *cyclic module generated* in M by a homogeneous element v

$$\mathbb{F}[x]v \subseteq M, \tag{19}$$

and the order of the cyclic submodule is defined as the maximum exponent p such that $x^{p-1}v \neq 0$, possibly infinite.

Theorem 5. *Let M be a graded $\mathbb{F}[x]$ -module and $(\{v_i\}_{i \in I}, S)$ a presentation for M with the notation of Eq. (18). The matrix S is in graded Smith Normal Form if only if the module M decomposes into cyclic submodules as*

$$M \cong \bigoplus_{m=1}^N \mathbb{F}[x]v_{i_m},$$

with i_1, \dots, i_N the indexing obtained by restricting row indexes to $I \setminus \text{Ones}(S)$. Moreover, if the cyclic submodule $\mathbb{F}[x]v_{i_m}$ is of order p_{i_m} then it is isomorphic to $\mathbb{F}[x](-\deg v_{i_m})/(x^{p_{i_m}})$, otherwise $\mathbb{F}[x]v_{i_m}$ is isomorphic to $\mathbb{F}[x](-\deg v_{i_m})$.

Proof. We reduce to the case $v_i = e_i$, that is M equal to $F / \text{Im}(\sigma)$ where F is freely generated by $\{e_i\}_{i \in I}$ since the standard homomorphism $e_i \mapsto v_i$ realizes the isomorphism to M . Clearly, the elements v_i , for $i \in I$, generate M by definition of presentation. Suppose that the presentation matrix S is in graded Smith Normal Form. Let i_1, \dots, i_N be the indexing obtained by restricting row indexes to $I \setminus \text{Ones}(S)$. The elements v_{i_1}, \dots, v_{i_N} still generate M since a row index $m \in \text{Ones}(S)$ implies that v_{i_m} belongs to the image of S . To prove that the sum is direct, notice

that a null combination of v_{i_1}, \dots, v_{i_N} belongs to the image of S . Indeed, since S is in graded Smith Normal Form, $\text{Im}(\sigma)$ is freely generated by $x^{p_{i_m}}$ for $m = 1, \dots, N$, hence all coefficients are null. The order of the cyclic module of v_{i_m} is either p_{i_m} if defined, or infinite otherwise.

On the contrary, assume that M is a direct sum of cyclic modules. Consider the indexes m such that v_{i_m} is an element of finite period p_{i_m} . For each m , define the column which is zero for all indexes $j = 1, \dots, N$ but in position m where it is $x^{p_{i_m}}$. By construction S is in graded Smith Normal Form with respect generators of M . The direct sum defining M implies that there are no other relations to be added to S to be a presentation of M . \square

Corollary 1. *A Smith Normal Form presentation matrix for $\alpha(\mathcal{M})$ provides an interval basis for \mathcal{M} .*

Proof. The result follows by recalling that an interval basis directly decomposes \mathcal{M} into interval modules which, by definition are, analogues of cyclic graded sub-modules. Then, it suffices to apply Theorem 5 to the associated module $\alpha(\mathcal{M})$. \square

Interval basis via Smith Normal Form

In this section, we propose a method to compute an interval basis, based on a suitable reduction of a presentation matrix. It is based on a combination of two technical ingredients: first, the construction of a presentation matrix S for $\alpha(\mathcal{M})$ out of a persistence module $\mathcal{M} = \{(M_i, \varphi_i)\}_{i=0}^n$. This is done in a way that, to our knowledge, was first explicitly envisaged in a technical passage of [16]. Next, we proceed by the reducing this presentation matrix into graded Smith Normal Form, so that each relation column admits a non-zero entry in correspondence of at most one generator. To do that, we adapt the method from [49].

B.2 From a persistence module to its presentation matrix

Our first task consists in defining a matrix S such that $\text{Coker}(S)$ is isomorphic to the graded module $\alpha(\mathcal{M})$ associated with the persistence module \mathcal{M} .

For each index $i = 0, \dots, n$, fix a basis $\mathcal{B}_i = \{v_1^i, \dots, v_{m_i}^i\}$ of the step M_i and let Φ_i be the $m_{i+1} \times m_i$ matrix expressing φ_i with respect to bases \mathcal{B}_i and \mathcal{B}_{i+1} . Let m be equal to $\sum_{i=0}^n m_i$. Then, we want to define a presentation for $\alpha(\mathcal{M})$ of the kind

$$\bigoplus_{j=1}^m \mathbb{F}[x](-\deg s_j) \xrightarrow{\sigma} \bigoplus_{i=0}^n \bigoplus_{h=1}^{m_i} \mathbb{F}[x](-h) \xrightarrow{\epsilon} \alpha(\mathcal{M}) \longrightarrow 0,$$

where the map ϵ is defined by $e_{i,h} \mapsto v_h^i$, and where we want to determine a square matrix S with columns (s_1, \dots, s_m) representing the map σ .

We follow the construction in Lemma 6 of [16] and specialize it to modules with no free cyclic submodules. Begin by defining S as a matrix of size $m \times m$. We define S by defining some blocks within it. We will use the following notation: given a matrix A , by $A[:, j]$ we indicate the j^{th} column of the matrix, by $A[i, :]$ we indicate the i^{th} row of the matrix, whereas $A[:, : j]$ denotes the submatrix given by the first j columns of A . The same notation is used on the first arguments in the parenthesis to denote operations on rows. By $A[i : i', j : j']$ we indicate the submatrix given by the rows of A from i to i' and columns of A from j to j' .

Let $d_i := \sum_{j < i} m_j + 1$, for each index $i = 0, \dots, n$ (i.e., d_i is the index of the first generator of the i^{th} step). For each step $i = 0, \dots, n$, matrix S contains a $m_i \times m_i$ diagonal block, whose diagonal elements are $-x$:

$$S[d_i : d_{i+1} - 1, d_i : d_{i+1} - 1] = -x \text{Id}_{m_i \times m_i}.$$

Also for each $i = 0, \dots, n$, consider the block S_i below the main diagonal with column indices $d_i, \dots, d_{i+1} - 1$ and row indices $d_{i+1}, \dots, d_{i+2} - 1$. Set

$$S[d_{i+1} : d_{i+2} - 1, d_i : d_{i+1} - 1] = \Phi_i.$$

Notice that S is not a diagonal block matrix. This will impact the computational complexity of the reduction procedure.

Definition 5. Given a persistence module \mathcal{M} , the *persistence module presentation matrix* is the matrix S obtained as above.

Theorem 6. A persistence module \mathcal{M} and its persistence module presentation matrix S satisfy

$$\alpha(\mathcal{M}) = \text{Coker}(S).$$

Proof. The proof follows from the proof of Lemma 6 in [16]. □

Example 4.

$$0 \xrightarrow[\text{(0)}]{\varphi_0} \mathbb{R} \xrightarrow[\text{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}]{\varphi_1} \mathbb{R}^2 \xrightarrow[\text{\begin{pmatrix} 1 & 1 \end{pmatrix}}]{\varphi_2} \mathbb{R} \xrightarrow[\text{(0)}]{\varphi_3} 0$$

Figure 7

Consider the \mathbb{R} -persistence module in Fig. 7, where the matrices below each arrow represent the map above it in the bases \mathcal{B}_i 's. Notice that this corresponds to the homology persistence module of Fig. 1. We ignore the zero steps as they are immaterial to the matrix construction. We say the module has three steps $M_1 = \mathbb{R}$ of degree 1, $M_2 = \mathbb{R}^2$ of degree 2 and $M_3 = \mathbb{R}$ of degree 3. Matrix S is 4×4 , and it holds $d_1 = 1$, $d_2 = 2$, $d_3 = 4$. Then matrix S is as in the following Fig. 8. The ochre blocks are the diagonal blocks, and the cyan and red blocks correspond to matrices φ_1 and φ_2 respectively (see colors in Fig. 7).

$$S = \begin{pmatrix} \text{\begin{matrix} -x \\ 1 \\ 0 \\ 0 \end{matrix}} & \text{\begin{matrix} 0 \\ -x \\ 0 \\ 1 \end{matrix}} & \text{\begin{matrix} 0 \\ 0 \\ -x \\ 1 \end{matrix}} & \text{\begin{matrix} 0 \\ 0 \\ 0 \\ -x \end{matrix}} \end{pmatrix}$$

Figure 8

Notice that the matrix S represents a homogeneous homomorphism with respect to row and column grades. Hence, matrix S can be thought of with entries in \mathbb{F} . The only genuinely relevant information in the matrix is whether an element is zero or not, because other than that its degree is determined by its position.

B.3 From a presentation matrix to its graded Smith Normal Form

In general, the presentation obtained via Definition 5 is far from being minimal, in the sense that several pairs of generator-relation are in excess and can be discarded while maintaining a presentation of the same module. As seen in the previous section, we know that if we can find a suitable presentation, we can obtain the interval generators. This amounts to obtaining a graded version of the structure

theorem via the Smith Normal Form, and to the best of our knowledge has only been explicitly done in [49].

Theorem 7. [49] *Let M be a finitely-generated, graded $\mathbb{F}[x]$ -module, and let $(\{v_i\}, S)$ be a graded presentation of M . There exists an algorithm to obtain another presentation of M , $(\{v'_i\}, S')$ such that S' is in graded Smith Normal Form.*

We apply the algorithm introduced in [49] in reducing the square matrix S of size $m \times m$ with entries in \mathbb{F} . The procedure returns invertible $m \times m$ matrices R, C and $\text{SNF}(S)$ such that the matrix

$$\text{SNF}(S) := RSC$$

is diagonal up to reordering of rows and columns and row and column degrees are preserved.

We sketch the algorithm as follows. By *low* of a column we refer to the index of its last (downward) non-zero entry. Notice no column of S is zero at the beginning. Also, we disregard matrix C , as it is of no interest to us.

Differently from the non-graded case, among the typical elementary operations, swapping rows and columns is not allowed. This explains the possibly non-diagonal final form in the graded counterpart $\text{SNF}(S)$ of the Smith Normal Form of S . For each pivot in $\text{SNF}(S)$ corresponding to row i and column j , we set $J(i) := \deg j - \deg i$. If the row i is null, we set $J(i) = \infty$.

We close this section by linking the matrices $\text{SNF}(S)$ and R to the interval basis of the persistence module \mathcal{M} introduced at the beginning of this section.

Theorem 8. *The columns in R^{-1} of index m such that $J(m) > 0$ form an interval basis for \mathcal{M} .*

Proof. By Theorem 6, The matrix S defines a presentation of $\alpha(\mathcal{M})$ directly encoding \mathcal{M} . As shown in [49], $\text{SNF}(S) = RSC$ still defines a presentation of $\alpha(\mathcal{M})$. Columns in R^{-1} contain the new system of generators with respect to the generators in S . Columns and rows in $\text{SNF}(S)$ contain at most one non-zero entry equal to some power of x , hence the matrix $\text{SNF}(S)$ is in graded Smith Normal Form according to Definition 4. Observe that columns v_m of index $m = 1, \dots, N$ such that $J(m) > 0$ correspond to non-invertible pivots since there are no null rows in $\text{SNF}(S)$. Hence by Corollary 1, the set $\{v_m\}_{m=1}^N$ forms an interval basis for $\alpha(M)$ and each element v_m has associated interval of order $J(m)$. \square

Example 5. (continued from Example 4) From matrix S , let us compute an interval basis. The Smith Normal Form reduction yields

Algorithm 8: Graded Smith Normal Form [49]

Input: Matrix S as per Definition 5 ;
Result: Matrices $\text{SNF}(S)$ and the change of basis matrix R
 $R \leftarrow \text{Id}_{m \times m}$;
for $i = 1, \dots, m$ **do**
 $l \leftarrow$ low of column i of S ;
 $R[l, :] \leftarrow R[l, :] / R[l, i]$;
 $S[l, :] \leftarrow S[l, :] / S[l, i]$;
 for $j = l - 1, \dots, 1$ **do**
 $R[j, :] \leftarrow R[j, :] - S[j, i]R[l, :]$;
 $S[j, :] \leftarrow S[j, :] - S[j, i]S[l, :]$;
 end
 // indexes c can be restricted as discussed in Section
 6
 for $c = i + 1, \dots, m$ **do**
 $S[:, c] \leftarrow S[l, i]S[:, c] - S[l, c]S[:, i]$;
 end
 $\text{SNF}(S) \leftarrow S$;
end
return $\text{SNF}(S), R$

$$\text{SNF}(S) = \begin{pmatrix} 0 & 0 & 0 & x^3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We see that rows 2 and 4 of $\text{SNF}(S)$ correspond to surplus generators, as they contain a unit in $\mathbb{F}[x]$. Row 1 corresponds to a bar born at degree 1, and killed by a relation (column) of degree 4, hence yielding a pair $(1, 4)$. Row 3 corresponds to a bar born at degree 2, and killed by a relation of degree 3, hence yielding a pair $(2, 3)$. The change of basis matrix R is

$$R = \begin{pmatrix} -1 & -x & -x & -x^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whose inverse equals itself

$$R^{-1} = \begin{pmatrix} -1 & -x & -x & -x^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, column 1 and 3 in this matrix, corresponding to non-zero persistence generators, form an interval basis. They are $-v_1^1$ and $-xv_1^1 + v_2^2$. They are indeed the first cycle to be born (up to a minus sign, which is immaterial), and the difference between the first cycle mapped at the second step and the second cycle. We remark that $xv_1^1 = v_1^2$. Notice that when implemented in practice, the terms of positive degree are substituted by their coefficient, as their degree is implicit by their position.

We have implemented this procedure as Python code, as a purely numerical matrix construction and reduction scheme, and plan to render it publicly available soon.

C Appendix

Complementary Results to Section 6

In this section, we include complementary algorithms to implement the construction of a persistence module obtained through the k^{th} -homology functor applied

to a finitely generated chain complex or a chain map between finitely generated chain complexes. In the following, we describe two methods that can possibly admit an implementation which is parallel, and even distributed, over the persistence module steps.

In order to compute the persistent homology $\{(H_k^i, \tilde{f}_i)\}$, for $i = 1, \dots, n$ out a sequence of chain complex maps $\{f_i\}$, we act in parallel over $i = 1, \dots, n$.

Computing the homology steps in parallel

Here, we report a possible algorithm to retrieve the homology of a chain complex over any coefficient field.

Algorithm 9: Computing homology

Input: Boundary matrices $\partial_k, \partial_{k+1}$ of the chain complex C ;
Result: Betti number β_k and basis $\{h_1, \dots, h_{\beta_k}, b_1, \dots, b_r\}$ of Z_k , where $\text{span}\{[h_1], \dots, [h_{\beta_k}]\} = H_k$ and $\text{span}\{b_1, \dots, b_r\} = B_k$.
 Compute the reduction $R_k = \partial_k V_k$;
 Compute the reduction $R_{k+1} = \partial_{k+1} V_{k+1}$;
 $b_1, \dots, b_r :=$ non-zero columns of R_{k+1} ;
 $v_1, \dots, v_s :=$ columns of V_k corresponding to zero columns of R_k ;
 $J :=$ matrix with columns $\{b_1, \dots, b_r, v_1, \dots, v_s\}$;
 $\beta_k = 1$;
for $i = r + 1, \dots, r + s$ **do**
 | **while** $\exists j < i$ s.t. $\text{low}(J[i]) = \text{low}(J[j])$ **do**
 | | $l := \text{low}(J[j])$;
 | | $\gamma := J[l, i] / J[l, j]$;
 | | $J[i] = J[i] - \gamma J[j]$;
 | **end**
 | **if** $J[i]$ is non-zero **then**
 | | $h_{\beta_k} := J[i]$;
 | | $\beta_k = \beta_k + 1$;
 | **end**
end
return β_k , basis $\{h_1, \dots, h_{\beta_k}, b_1, \dots, b_r\}$

Computing the homology structure maps in parallel

Here we report an algorithm to retrieve the map induced between homology spaces given a chain map.

Theorem 9. *The map \tilde{f}_k defined in Algorithm 10 is well-defined and it is the map*

Algorithm 10: Induced map between homology spaces

Input: Chain map $f_k : C_k \rightarrow D_k$, representatives cycles $h_1^C, \dots, h_{\beta_k^C}^C$ of a basis of $H_k(C)$, β_k^D and $\{h_1^D, \dots, h_{\beta_k^D}^D, b_1^D, \dots, b_r^D\}$ output of Algorithm 9 for D ;

Result: map $\tilde{f}_k : H_k(C) \rightarrow H_k(D)$ induced by f_k .

$\tilde{f}_k :=$ zero matrix $\beta_k^D \times \beta_k^C$;

for $i = 1, \dots, \beta_k^C$ **do**

 Solve $f_k(h_i^C) = \sum_{j=1}^{\beta_k^D} \lambda_j h_j^D + \sum_{l=1}^r \mu_l b_l$;
 $\tilde{f}_k[i] = (\lambda_1, \dots, \lambda_{\beta_k^D})^T$

end

return \tilde{f}_k

induced by f_k through the homology functor.

Proof. For all $i = 1, \dots, \beta_k^C$, it holds $\tilde{f}_k([h_i^C]_C) = [f_k(h_i^C)]_D = \sum_{j=1}^{\beta_k^D} \lambda_j [h_j^D]_D$. □