

Existence of normalized solutions to mass supercritical Schrödinger equations with negative potential

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Abstract

We study the existence of positive solutions with prescribed L^2 -norm for the Schrödinger equation

$$-\Delta u - V(x)u + \lambda u = |u|^{p-2}u \quad \lambda \in \mathbb{R}, \quad u \in H^1(\mathbb{R}^N),$$

where $V \geq 0$, $N \geq 1$ and $p \in (2 + \frac{4}{N}, 2^*)$, $2^* := \frac{2N}{N-2}$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1, 2$. We treat two cases. Firstly, under an explicit smallness assumption on V and no condition on the mass, we prove the existence of a mountain pass solution at positive energy level, and we exclude the existence of solutions with negative energy. Secondly, requiring that the mass is smaller than some explicit bound, depending on V , and that V is not too small in a suitable sense, we find two solutions: a local minimizer with negative energy, and a mountain pass solution with positive energy. Moreover, a nonexistence result is proved.

Keywords: Nonlinear Schrödinger equations, normalized solutions, positive solutions.

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1 Introduction

We consider the problem

$$\begin{cases} -\Delta u - V(x)u + \lambda u = u^{p-1} \\ \lambda \in \mathbb{R}, \quad u \in S_\rho, \quad u \geq 0, \end{cases} \quad (P_\rho)$$

where $N \geq 1$ and

$$S_\rho = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 dx = \rho^2 \right\}, \quad \rho > 0.$$

Throughout the paper we assume

$$2 + \frac{4}{N} < p < 2^* \quad \text{and} \quad V \geq 0, \quad V \not\equiv 0, \quad (1.1)$$

up to further restrictions on some Lebesgue norm of the measurable potential V (as usual $2^* := \frac{2N}{N-2}$ if $N \geq 3$ and $2^* := +\infty$ if $N = 1, 2$).

Problems of the form (P_ρ) come from the study of standing waves for the nonlinear Schrödinger equation

$$iw_t + \Delta w + V(x)w = f(w), \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad (1.2)$$

that is solutions of the form

$$w(x, t) = e^{i\lambda t} u(x), \quad (x, t) \in \mathbb{R}^N \times (0, \infty) \quad (1.3)$$

where u is a real function. Here we consider the model case of a power nonlinearity $f(w) = |w|^{p-2}w$. A lot of efforts have been done studying problem (1.2), (1.3) for a fixed frequency $\lambda \in \mathbb{R}$; the literature in this direction is huge and we do not even make an attempt to summarize it here (see e.g. the recent papers [14, 23] and references therein).

On the other hand, after the seminal paper by Jeanjean [18], only recently more work has been devoted to the natural question of considering problem (1.2) when the mass of the particle is known, i.e. the real function u in (1.3) is in S_ρ , $\rho > 0$ fixed, and λ is an unknown of the problem. A natural approach to such questions is by variational methods. Indeed, when $V \in L^r(\mathbb{R}^N)$, for some $r \in [N/2, +\infty)$, $r \geq 1$ ($r > 1$ if $N = 2$), solutions of (P_ρ) can be found as critical points of the energy functional

$$F(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 - V(x)u^2) dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx \quad u \in H^1(\mathbb{R}^N),$$

constrained on S_ρ . Here λ comes out as a Lagrange multiplier. The functional F is unbounded from below, as it is readily seen testing the functional on $u_h(x) := h^{\frac{N}{2}} \bar{u}(hx)$, $h > 0$, for a fixed $\bar{u} \in S_\rho$. Hence the problem cannot be solved by (global) minimization arguments.

The unboundedness from below is due to the nonlinearity growth $p > 2 + \frac{4}{N}$: indeed, if the so called mass-subcritical case $p \in (2, 2 + \frac{4}{N})$ is involved, then the problem can be faced by minimization. In this respect, we refer the reader to the classical paper [19], and to the recent paper [16], where the existence of global minimizers is obtained for more general nonlinearities and for negative potentials, that is $V \geq 0$ in our framework. Several related results about the minimization of the nonlinear Schrödinger energy have been obtained on metric graphs, also in the mass-critical case [1, 2, 28].

Turning to the mass super-critical case, as we already mentioned the reference paper is [18], which deals with the autonomous case: the key idea in [18] is to obtain a mountain pass solution on S_ρ , by exploiting a natural constraint related to the Pohozaev identity; since then, more general autonomous equations and systems have been considered, also refining and developing this initial strategy, see [3–5, 7, 8, 15, 17, 30, 31] and references therein. On the other hand, equations on bounded domains and/or with (step well) trapping potentials, i.e. potentials as in (P_ρ) satisfying

$$\lim_{|x| \rightarrow +\infty} -V(x) = +\infty$$

(or even $-V \equiv +\infty$ outside some bounded Ω), have been considered in [10, 24–26, 29]. In this case, the trapping nature of the potential provides enough compactness to cause the existence of solutions which are local minimizers of F on S_ρ also in the mass-supercritical case, at least when ρ is sufficiently small. On the other hand, for non-trapping potentials very few results are available: in particular, weakly repulsive potentials, i.e.

$$-V(x) \geq \liminf_{|x| \rightarrow +\infty} -V(x) > -\infty,$$

were considered in the very recent paper [6]. In this case, the mountain pass structure by Jeanjean is destroyed, but a new variational principle exploiting the Pohozaev identity can be used to obtain existence of solutions with high Morse index. To conclude this discussion about the previous literature,

since the results we have discussed so far are all of variational nature, let us mention that also topological methods have been applied, for instance in [9, 13, 27], also in connection with ergodic Mean Field Games systems.

In this paper we consider a class of mass-supercritical, and Sobolev sub-critical, problems with weakly attractive potential, that is

$$-V(x) \leq \limsup_{|x| \rightarrow +\infty} -V(x) < +\infty$$

(notice that, up to subtracting the constant $\rho^2 \limsup_{|x| \rightarrow +\infty} -V(x)$ in F , this corresponds to assuming (1.1)). Under suitable assumptions, we obtain two families of solution. On the one hand we show that the mountain pass solution of Jeanjean still exists also in this situation; on the other hand, we show that such mountain pass structure also provides a local minimizer, as in [26, 29], even though the potential is not trapping at all. More precisely, we first show that, under an explicit smallness assumption on V and no condition on the mass, a mountain pass solution at positive energy level exists; under the same smallness assumption we also show that no solution at negative energy values can exist. Secondly, requiring that the mass is smaller than some explicit bound, depending on V , and that V is not too small in a suitable sense, we find two solutions: a local minimizer with negative energy, and a mountain pass solution with positive energy. Notably, this second result holds true also in dimensions $N = 1, 2$.

In order to state our results we define the auxiliary function

$$W(x) = V(x)|x| \tag{1.4}$$

Theorem 1.1. *Let $N \geq 3$ and (1.1) hold true. There exists a positive explicit constant $L = L(N, p)$ such that if*

$$\max\{\|V\|_{N/2}, \|W\|_N\} < L \tag{1.5}$$

then (P_ρ) has a mountain pass solution for every $\rho > 0$, at a positive energy level, while no solution with negative energy exists.

Theorem 1.2. *Let $N \geq 1$ and (1.1) hold true, and let $r \in (\max(1, \frac{N}{2}), +\infty]$, $s \in (\max(2, N), +\infty]$.*

1. There exist positive explicit constants $\sigma = \sigma(N, p, r)$ and $K = K(N, p, r)$ such that, if

$$\text{either } r < +\infty, \text{ or } r = +\infty \text{ and } \lim_{|x| \rightarrow +\infty} V(x) = 0, \tag{1.6}$$

$$\|V\|_r \cdot \rho^\sigma < K \text{ and} \tag{1.7}$$

$$\text{there exists } \varphi \in S_\rho : \int_{\mathbb{R}^N} (|\nabla \varphi|^2 - V(x)\varphi^2) dx \leq 0, \tag{1.8}$$

then (P_ρ) has a solution, which corresponds to a local minimizer of F on S_ρ with negative energy.

2. There exist positive explicit constants $\sigma_i = \sigma_i(N, p, r)$, $\bar{\sigma}_i = \bar{\sigma}_i(N, p, s)$, $i = 1, 2$, and $\tilde{L} = \tilde{L}(N, p, r, s)$ such that, if

$$\max\{\|V\|_r \cdot \rho^{\sigma_i}, \|W\|_s \cdot \rho^{\bar{\sigma}_i}\} < \tilde{L}, \quad i = 1, 2, \tag{1.9}$$

then (P_ρ) has a mountain pass solution at a positive energy level.

Remark 1.3. We point out that our results are not perturbative, indeed all the constants in the above theorems can be made explicit with respect to the structural parameters N, p, r, s , see e.g. (3.28)–(3.32), or (4.70)–(4.72).

Remark 1.4. Under the assumption of Theorem 1.2, it is standard to prove that the non-empty set of local minimizers is (conditionally) orbitally stable for the associated evolution equation.

Remark 1.5. It is well known that a sufficient condition for (1.8) to hold true is that

$$\inf_{B_R} V \geq \eta, \quad \text{with} \quad \begin{cases} \eta > 0, R > 0 & N = 1, 2 \\ R^2 \eta \geq N(N-2) & N \geq 3. \end{cases} \quad (1.10)$$

In particular, no condition is required in dimension $N = 1, 2$ as long as $V \geq 0$, $V \not\equiv 0$. See Lemma 4.5 for further details.

Remark 1.6. Notice that, for every V (either satisfying (1.8) or not), it is always possible to choose ρ sufficiently small so that both (1.7) and (1.9) hold true. Moreover, if $N \geq 3$, assumption (1.10) just requires

$$\|V\|_r^r \geq \eta^r |B_R| \geq \eta^{r-\frac{N}{2}} \frac{\omega_N}{N} [N(N-2)]^{\frac{N}{2}}.$$

In particular, since $r > \frac{N}{2}$, one can exhibit potentials with arbitrarily small L^r norm fulfilling the assumptions of Theorem 1.2 (with sufficiently small ρ and large R).

Remark 1.7. The mountain pass geometry in Theorems 1.1 and 1.2 is essentially the same, therefore most probably the mountain pass solutions coincide. As a matter of fact, the explicit dependence on r, s show that $\sigma_1 = \sigma_2 = 0$ if $r = N/2$ and $s = N$, so that in this case (1.9) and (1.5) coincide. On the other hand, if $r = N/2$ then also $\sigma = 0$, and also (1.7) reduces to (1.5). Nonetheless, in this case (1.5) and (1.8) are not compatible, so that the minimizer with negative energy does not exist.

Remark 1.8. In principle, for our results we only need V, W to belong to suitable Lebesgue spaces. On the other hand, if we also have $V \in C_{\text{loc}}^{0,\alpha}(\mathbb{R}^N)$ then all the solutions we find are classical and, by the strong maximum principle, they are strictly positive in \mathbb{R}^N .

Remark 1.9. The main difficulty to prove the existence of the mountain pass solution is the analysis of the behavior of a bounded Palais-Smale sequence related to the mountain pass level. To overcome this difficulty, we prove that the Lagrange multiplier associated our PS-sequence is positive and then we use an almost classical splitting result in the unconstrained sub-critical case (see [11]).

Observe that in [6] an existence result is proved when the potential V in (P_ρ) is negative, while in the present paper we find solutions for nonnegative V . In the following proposition we give another contribution in the study of the problem with a nonexistence result, analogous to [12, Theorem 1.1].

Proposition 1.10. *Let $p \in (2, 2^*)$, $V \in L^\infty(\mathbb{R}^N)$ and assume that there exists $\frac{\partial V}{\partial \nu} \in L^s(\mathbb{R}^N)$ for some $\nu \in \mathbb{R}^N \setminus \{0\}$ and $s \in [N/2, \infty]$. If $\frac{\partial V}{\partial \nu} \geq 0$ and $\frac{\partial V}{\partial \nu} \not\equiv 0$, then problem*

$$-\Delta u - V(x)u = |u|^{p-2}u \quad u \in S_\rho$$

has no smooth solutions.

Remark 1.11. It is worthwhile noticing that our results are compatible with Proposition 1.10: indeed, this is clearly the case for Point 1 in Theorem 1.2 when $r = +\infty$, as V is explicitly required to vanish at infinity; on the other hand, observe that the assumption on W in Point 2 imply some decay on V even in the case $r = s = +\infty$.

Another result that could have some interest, for example in stating constraints to work with, is the following necessary condition for critical points of F on S_ρ .

Proposition 1.12. *Let $p \in (2, 2^*)$ and $V \in L^r(\mathbb{R}^N)$ for some $r \in [N/2, \infty]$. If $u \in S_\rho$ is a critical point for F constrained on S_ρ , then*

$$\int_{\mathbb{R}^N} V(x) u \frac{\partial u}{\partial \nu} dx = 0$$

for every direction $\nu \in \mathbb{R}^N \setminus \{0\}$.

Before concluding this introduction, we want to observe that in exterior domains or in some domains with unbounded boundary the splitting Lemma 2.3 holds true again. Moreover, one can analyze the displacement of the potential and the mass ρ in order to recover the mountain pass geometry. Hence, it would be interesting to investigate whether the mountain pass solution exists, taking into account that in our approach the Pohozaev identity plays a crucial role. We refer the reader to [20–22] for existence results in this framework, in the the unconstrained case.

The paper is organized as follows: in Section 2 we introduce the main notations and some preliminary results, and prove Proposition 1.10, in Section 3 we prove Theorem 1.1 while Section 4 is devoted to the proof of Theorem 1.2.

2 Notation and preliminary results

For $p \in [1, +\infty]$, we denote by $L^p(\mathbb{R}^N)$ the Lebesgue's space with norm $\|\cdot\|_p$ and by $H^1(\mathbb{R}^N)$, $D^{1,2}(\mathbb{R}^N)$ the usual Sobolev spaces with the norm $\|\cdot\|_H$ and $\|\nabla u\|_2$, respectively; S will denote the Sobolev constant, namely:

$$S = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_H^2}{\|u\|_{2^*}^2} = \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}; \quad (2.11)$$

c, C are constants which may vary from line to line (structural constants will depend on N, p, r, s , while the dependence on V, W and ρ will be made explicit whenever useful). We fix the constant

$$\gamma = \frac{N}{2}(p-2) > 2. \quad (2.12)$$

We recall that for every $\rho > 0$ there exists a unique solution Z_ρ , up to translations, for the limit problem

$$\begin{cases} -\Delta Z_\rho + \lambda_\rho Z_\rho = Z_\rho^{p-1} \\ Z_\rho \in S_\rho, Z_\rho > 0, \end{cases} \quad (2.13)$$

with $\lambda_\rho > 0$. The function Z_ρ is radial and it is a mountain pass critical point for

$$F_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx \quad u \in H^1(\mathbb{R}^N)$$

constrained on S_ρ .

By scaling, Z_ρ can be expressed in terms of the unique positive solution $U \in H^1(\mathbb{R}^N)$ of

$$\begin{cases} -\Delta U + U = U^{p-1} \\ U > 0, U(0) = \|U\|_\infty. \end{cases}$$

More precisely, setting $\rho_0 = \|U\|_2$, for $\rho > 0$ we define:

$$\mu_\rho = \left(\frac{\rho}{\rho_0} \right)^{\frac{2(p-2)}{N(p-2)-4}},$$

then in (2.13) we have

$$Z_\rho(x) := \mu_\rho^{-\frac{2}{p-2}} U(x/\mu_\rho), \quad \lambda_\rho = \mu_\rho^{-2} = \left(\frac{\rho_0}{\rho} \right)^{\frac{4(p-2)}{N(p-2)-4}} > 0. \quad (2.14)$$

Setting $m_\rho = F_\infty(Z_\rho)$, so that $m_{\rho_0} = F_\infty(U)$, observe that

$$m_\rho = m_{\rho_0} \left(\frac{\rho_0}{\rho} \right)^{2 \frac{2N-p(N-2)}{N(p-2)-4}} = \frac{N(p-2)-4}{4N-2p(N-2)} \left(\frac{\rho_0}{\rho} \right)^{\frac{4(p-2)}{N(p-2)-4}} \rho^2 = \frac{N(p-2)-4}{4N-2p(N-2)} \lambda_\rho \rho^2. \quad (2.15)$$

In the following lemma we recall the well-known Gagliardo-Nirenberg inequality.

Lemma 2.1. For every $N \geq 3$, $2 \leq q \leq 2^*$ there exists $G_q > 0$, depending on N and q , such that

$$\|u\|_q \leq G_q \|u\|_2^{1 - \frac{N(q-2)}{2q}} \|\nabla u\|_2^{\frac{N(q-2)}{2q}} \quad (2.16)$$

for every $u \in H^1(\mathbb{R}^N)$. The inequality holds true also in $N = 1, 2$, for every $2 \leq q < +\infty$.

In particular, if $q = p$:

$$\|u\|_p \leq G \|u\|_2^{1 - \frac{\gamma}{p}} \|\nabla u\|_2^{\frac{\gamma}{p}} \quad (2.17)$$

where γ is defined in (2.12) and $G = G_p$.

Of course, if $N \geq 3$, the above inequality holds true also for $q = 2^*$, reducing to the first Sobolev inequality in (2.11) (with $S = G_{2^*}^{-2}$). It is well known, see [32], that G is achieved by Z_ρ , for any ρ . Recalling (2.15), standard calculations (see e.g. the appendix in [6]) yield

$$G = \frac{\|U\|_p}{\|U\|_2^{1 - \frac{\gamma}{p}} \|\nabla U\|_2^{\frac{\gamma}{p}}} = \frac{(2p)^{\frac{1}{p}}}{(2N - p(N - 2))^{\frac{p-2}{2p}} (N(p-2))^{\frac{\gamma}{2p}}} \left(\frac{N(p-2) - 4}{2} \right)^{\frac{p-2}{2p}} m_{\rho_0}^{-\frac{p-2}{2p}}. \quad (2.18)$$

Next, we recall some basic estimates involving V and W (defined in (1.4)), that immediately follow from Hölder, Gagliardo-Nirenberg and Sobolev inequalities.

Lemma 2.2. For every $2 \leq q < 2^*$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} V(x) u^2 dx \right| &\leq \|V\|_{\frac{q}{q-2}} \|u\|_q^2 \leq G_q^2 \|V\|_{\frac{q}{q-2}} \|u\|_2^{2 - \frac{N(q-2)}{q}} \|\nabla u\|_2^{\frac{N(q-2)}{q}}, \\ \left| \int_{\mathbb{R}^N} V(x) u(x) \nabla u(x) \cdot x dx \right| &\leq \|W\|_{\frac{2q}{q-2}} \|u\|_q \|\nabla u\|_2 \leq G_q \|W\|_{\frac{2q}{q-2}} \|u\|_2^{1 - \frac{N(q-2)}{2q}} \|\nabla u\|_2^{1 + \frac{N(q-2)}{2q}}. \end{aligned}$$

Furthermore, if $N \geq 3$, then the above inequalities hold also for $q = 2^*$:

$$\left| \int_{\mathbb{R}^N} V(x) u^2 dx \right| \leq \|V\|_{\frac{N}{2}} \|u\|_{2^*}^2 \leq S^{-1} \|V\|_{\frac{N}{2}} \|\nabla u\|_2^2, \quad (2.19)$$

$$\left| \int_{\mathbb{R}^N} V(x) u(x) \nabla u(x) \cdot x dx \right| \leq \|W\|_N \|u\|_{2^*} \|\nabla u\|_2 \leq S^{-1/2} \|W\|_N \|\nabla u\|_2^2. \quad (2.20)$$

Let $\lambda \in \mathbb{R}$, we denote by $I_\lambda, I_{\infty, \lambda} : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ the functionals

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx,$$

$$I_{\infty, \lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx.$$

Proof of Proposition 1.10. By Lemma 2.2 the functional F is well defined and of class C^1 . If u is a critical point of F on S_ρ then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u is a critical point for I_λ on $H^1(\mathbb{R}^N)$. Hence

$$\frac{d}{dt} I_\lambda(u(x + t\nu))|_{t=0} = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{\partial V}{\partial \nu} u^2 dx = 0. \quad (2.21)$$

Since $\frac{\partial V}{\partial \nu}$ has constant sign and $\text{meas}\{x \in \mathbb{R}^N : u(x) = 0\} = 0$, a contradiction arises by (2.21). \square

Proof of Proposition 1.12. Arguing as in the previous proof, for every direction $\nu \in \mathbb{R}^N \setminus \{0\}$ we have

$$\int_{\mathbb{R}^N} V(x) u \frac{\partial u}{\partial \nu} dx = \frac{d}{dt} I_\lambda(u(x + t\nu))|_{t=0} = 0.$$

□

In order to get the compactness result for Palais-Smale sequences, we state a Splitting Lemma for I_λ , in our framework. Its proof is very close to that given in [11, Lemma 3.1] for exterior domains, so we only sketch it.

Lemma 2.3. *Let us assume that*

1. $N \geq 3$: $V \in L^{N/2}(B_1(0))$, $V \in L^{\tilde{r}}(\mathbb{R}^N \setminus B_1(0))$ for $\tilde{r} \in [N/2, +\infty]$,
2. $N = 1, 2$: $V \in L^r(B_1(0))$, $V \in L^{\tilde{r}}(\mathbb{R}^N \setminus B_1(0))$ for $r, \tilde{r} \in (1, +\infty]$,
3. in case $\tilde{r} = +\infty$, V further satisfies $\lim_{|x| \rightarrow +\infty} V(x) = 0$,
4. $\lambda > 0$.

If $(v_n)_n$ is a bounded Palais-Smale sequence for I_λ in $H^1(\mathbb{R}^N)$, then, up to a subsequence, v_n weakly converge to a function $v \in H^1(\mathbb{R}^N)$ and if the convergence is not strong then there exist an integer $k \geq 1$, k nontrivial solutions $w^1, \dots, w^k \in H^1(\mathbb{R}^N)$ to the limit equation

$$-\Delta w + \lambda w = |w|^{p-2} w \quad (2.22)$$

and k sequences $\{y_n^j\}_n \subset \mathbb{R}^N$, $1 \leq j \leq k$, such that $|y_n^j| \rightarrow \infty$ as $n \rightarrow \infty$, $|y_n^{j_1} - y_n^{j_2}| \rightarrow \infty$, for $j_1 \neq j_2$, as $n \rightarrow \infty$, and

$$v_n = v + \sum_{j=1}^k w^j(\cdot - y_n^j) + o(1) \quad \text{strongly in } H^1(\mathbb{R}^N). \quad (2.23)$$

Moreover, we have

$$\|v_n\|_2^2 = \|v\|_2^2 + \sum_{j=1}^k \|w^j\|_2^2 + o(1) \quad (2.24)$$

and

$$I_\lambda(v_n) \rightarrow I_\lambda(v) + \sum_{j=1}^k I_{\infty, \lambda}(w^j) \quad \text{as } n \rightarrow \infty. \quad (2.25)$$

Remark 2.4. We notice that the assumptions on V in this lemma easily follow from those in our main results. In particular, for Point 2 in Theorem 1.2, they are implied by those on W , by Hölder inequality.

Proof of Lemma 2.3. In this proof we argue up to suitable subsequences. Let v be the weak limit of v_n and set $v_{1,n} := v_n - v$. Then, $v_{1,n} \rightarrow 0$ weakly in $H^1(\mathbb{R}^N)$, strongly in $L_{\text{loc}}^2(\mathbb{R}^N)$, $L_{\text{loc}}^p(\mathbb{R}^N)$, and a.e. in \mathbb{R}^N . Moreover

$$\int_{\mathbb{R}^N} V(x) v_{1,n}^2 dx = \int_{B_1(0)} V(x) v_{1,n}^2 dx + \int_{\mathbb{R}^N \setminus B_1(0)} V(x) v_{1,n}^2 dx = I + II. \quad (2.26)$$

Assume first $N \geq 3$ and $\tilde{r} \in [N/2, +\infty)$. We deduce by Egorov's Theorem that $v_{1,n}^2 \rightarrow 0$ weakly in $L^{N/(N-2)}(B_1(0))$ and in $L^{\tilde{r}'}(\mathbb{R}^N \setminus B_1(0))$, because v_n^2 is bounded in $L^{N/(N-2)}(B_1(0))$ and in $L^{\tilde{r}'}(\mathbb{R}^N \setminus B_1(0))$ and goes to 0 a.e.. Hence both I and II converge to zero as $n \rightarrow \infty$. Also in the case $\tilde{r} = +\infty$ the addendum II goes to zero, because $v_{1,n}^2 \rightarrow 0$ in $L_{\text{loc}}^2(\mathbb{R}^N)$, $\|v_{1,n}\|_2$ is bounded and

$\lim_{|x| \rightarrow +\infty} V(x) = 0$. If $N = 1, 2$ the arguments above again prove that I and II converge to zero as $n \rightarrow \infty$. Summing up, in any case we have that

$$\int_{\mathbb{R}^N} V(x)v_{1,n}^2 dx \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (2.27)$$

and the sequence $\tilde{v}_{1,n}$ turns out to be a PS sequence for $I_{\infty,\lambda}$.

If $v_{1,n} \rightarrow 0$ in $H^1(\mathbb{R}^N)$ we are done, otherwise we can assume that $\|v_{1,n}\|_H \geq d_1$ for a suitable constant $d_1 > 0$. As a consequence, we deduce the existence of a constant $\tilde{d}_1 > 0$ such that

$$\|v_{1,n}\|_p \geq \tilde{d}_1 \quad \forall n \in \mathbb{N}.$$

Indeed, suppose by contradiction that $\|v_{1,n}\|_p \rightarrow 0$. Then, since $v_{1,n}$ is a bounded PS sequence for I_λ and taking into account (2.27), we get

$$\|\nabla v_{1,n}\|_2^2 + \lambda \|v_{1,n}\|_2^2 = \|v_{1,n}\|_p^p + o(1) = o(1),$$

contrary to $\|v_{1,n}\|_H \geq d_1$.

Now, let $\{Q_i\}_{i \in \mathbb{N}}$ be a decomposition of \mathbb{R}^N by unitary dyadic cubes, and set

$$l_n = \max_{i \in \mathbb{N}} \|v_{1,n}\|_{L^p(Q_i)}.$$

Then there exists a constant $l > 0$ such that $l_n \geq l$, for all $n \in \mathbb{N}$, because

$$\begin{aligned} 0 < \tilde{d}_1 \leq \|v_{1,n}\|_p^p &= \sum_{i=1}^{\infty} \|v_{1,n}\|_{L^p(Q_i)}^p \\ &\leq l_n^{p-2} \sum_{i=1}^{\infty} \|v_{1,n}\|_{L^p(Q_i)}^2 \leq c_1 l_n^{p-2} \sum_{i=1}^{\infty} \|v_{1,n}\|_{H^1(Q_i)}^2 \\ &\leq c_2 l_n^{p-1}, \end{aligned}$$

for suitable positive constants c_1, c_2 depending only on the Sobolev constant and the upper bound of $\|v_n\|_H^2$. Let y_n^1 be the center of a cube Q_{i_n} such that $d_n = \|v_{1,n}\|_{L^p(Q_{i_n})}^p$ and observe that $|y_n^1| \rightarrow \infty$, by $v_{1,n} \rightarrow 0$ in $L_{\text{loc}}^p(\mathbb{R}^N)$. Setting

$$\tilde{v}_{1,n} = v_{1,n}(\cdot + y_n^1),$$

it turns out that $\tilde{v}_{1,n}$ is a bounded PS sequence for $I_{\infty,\lambda}$. So, $\tilde{v}_{1,n} \rightarrow w^1$ weakly in $H^1(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$, in $L_{\text{loc}}^p(\mathbb{R}^N)$ and a.e. in \mathbb{R}^N , where w^1 is a weak solution of (2.22), non trivial because $\|w^1\|_{L^p(B_{\sqrt{N}}(0))} \geq l > 0$. Moreover, in view of (2.27),

$$\begin{aligned} v_n = v + v_{1,n} &= v + \tilde{v}_{1,n}(\cdot - y_n^1) = v + w^1(\cdot - y_n^1) + [\tilde{v}_{1,n}(\cdot - y_n^1) - w^1(\cdot - y_n^1)], \\ \|v_n\|_H^2 &= \|v\|_H^2 + \|v_{1,n}\|_H^2 + o(1) = \|v\|_H^2 + \|w^1\|_H^2 + \|\tilde{v}_{1,n} - w^1\|_H^2 + o(1), \\ I_\lambda(v_n) &= I_\lambda(v) + I_{\infty,\lambda}(v_{1,n}) + o(1) = I_\lambda(v) + I_{\infty,\lambda}(w^1) + I_{\infty,\lambda}(\tilde{v}_{1,n} - w^1) + o(1). \end{aligned}$$

Iterating the procedure, taking into account that v_n is bounded and that the action of the ground state solution is positive, the proof is completed (see also [22, Lemma 3.2] for more details). \square

Finally, we recall the following well-known fact, see e.g. [6, Appendix A].

Lemma 2.5. *Let $w \in H^1(\mathbb{R}^N)$ be a non-trivial solution of*

$$-\Delta w + \lambda w = |w|^{p-2}w,$$

for some $\lambda > 0$. Then

$$\lambda \geq \lambda_{\|w\|}, \quad F_\infty(w) \geq m_{\|w\|_2} > 0,$$

where λ_ρ is defined in (2.14) and m_ρ in (2.15), for every $\rho > 0$.

3 Proof of Theorem 1.1

In this section we assume that $L > 0$ is such that assumption (1.5) implies the following explicit bounds on V and W , for some fixed $\delta \in (0, 1)$:

$$\|V\|_{N/2} < (1 - \delta)S; \quad (3.28)$$

$$N|4 - p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N < B, \quad (3.29)$$

$$[AMN|4 - p| + (N - 2)D]S^{-1}\|V\|_{\frac{N}{2}} + [4AM + 2D]S^{-1/2}\|W\|_N < ABM, \quad (3.30)$$

where

$$\begin{aligned} A &= [2N - (N - 2)p], & B &= N(p - 2) - 4, & D &= N(p - 2)^2, \\ M &= \left[\frac{\delta}{\gamma} \right]^{\frac{\gamma}{\gamma-2}} \left[\frac{\gamma}{2} - 1 \right] \left(\frac{p}{G^p} \right)^{\frac{2}{\gamma-2}} \frac{1}{m_{\rho_0} \rho_0^s}, \end{aligned} \quad (3.31)$$

with $s = 2 \frac{2N - (N - 2)p}{N(p - 2) - 4}$; moreover:

$$3(p - 4)^+ S^{-1}\|V\|_{N/2} + 4S^{-1/2}\|W\|_N \leq N(p - 2) - 4. \quad (3.32)$$

Notice that $(p - 4)^+ = 0$ if $N \geq 4$.

We prove that F has a mountain pass geometry, which ensures by Proposition 3.3 the existence of a Palais-Smale sequence. Then, in order to recover compactness for this sequence, we use the Splitting Lemma 2.3, and, to apply this, we need to prove that the limit of the sequence of the Lagrange multipliers related to the PS-sequence is positive.

To start with, we focus on the geometric structure of F , observing first the following scaling property. For every $u \in S_\rho$ and $h > 0$ we define the function $u_h \in S_\rho$ by

$$u_h(x) = h^{\frac{N}{2}} u(hx).$$

Since $\nabla_x u_h(x) = h^{\frac{N}{2}+1} \nabla_y u(hx)$, $y = hx$, we get

$$F(u_h) = \frac{h^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{h^{\frac{N}{2}(p-2)}}{p} \int_{\mathbb{R}^N} |u|^p dx - \int_{\mathbb{R}^N} V\left(\frac{x}{h}\right) u^2(x) dx. \quad (3.33)$$

For fixed $u \in S_\rho$ we infer:

$$\int_{\mathbb{R}^N} V(x) u_h^2(x) dx \leq h^2 \|V\|_{N/2} \|u\|_2^2 \longrightarrow 0, \quad \text{as } h \rightarrow 0. \quad (3.34)$$

Therefore, for every $u \in S_\rho$ it follows:

$$\lim_{h \rightarrow 0^+} \|\nabla u_h\|_2 = 0, \quad \lim_{h \rightarrow +\infty} \|\nabla u_h\|_2 = \infty, \quad (3.35)$$

$$\lim_{h \rightarrow 0^+} F(u_h) = 0, \quad \lim_{h \rightarrow +\infty} F(u_h) = -\infty. \quad (3.36)$$

The following lemma gives a lower estimate for F , useful to prove that F has a mountain pass geometry. Assumption (3.28) and inequalities (2.17) and (2.19) imply

Lemma 3.1.

$$F(u) \geq \frac{\delta}{2} \|\nabla u\|_2^2 - c(\rho) \|\nabla u\|_2^\gamma, \quad \forall u \in S_\rho, \quad (3.37)$$

where $c(\rho) = \frac{G^p}{p} \rho^{p-\gamma}$, with G defined in (2.18).

Hence from (3.37) we infer that $\bar{R} > 0$ exists such that

$$\mathcal{M} := \inf\{F(u) : u \in S_\rho, \|\nabla u\|_2 = \bar{R}\} > 0.$$

Now, let us consider the function $Z_\rho \in S_\rho$ introduced in (2.14). By (3.35) and (3.36) there exist $0 < h_0 < 1 < h_1$ such that

$$\begin{aligned} \|\nabla(Z_\rho)_{h_0}\|_2 &< \bar{R}, & F((Z_\rho)_{h_0}) &< \mathcal{M}, \\ \|\nabla(Z_\rho)_{h_1}\|_2 &> \bar{R}, & F((Z_\rho)_{h_1}) &< 0. \end{aligned}$$

Then, we define in a standard way the mountain pass value

$$m_{V,\rho} := \inf_{\xi \in \Gamma} \max_{t \in [0,1]} F(\xi(t)) \quad (3.38)$$

where

$$\Gamma = \{\xi : [0, 1] \rightarrow S_\rho : \xi(0) = (Z_\rho)_{h_0}, \xi(1) = (Z_\rho)_{h_1}\}. \quad (3.39)$$

Remark 3.2. Since

$$m_\rho = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F_\infty(\xi(t))$$

(see [18]), it is immediately seen that

$$m_{V,\rho} < m_\rho \quad (3.40)$$

(it is sufficient to use the test path $\xi(t) = (Z_\rho)_{h_0(1-t)+h_1t}$ and use assumption (1.1)). Moreover, it holds

$$m_{V,\rho} \geq M m_\rho \quad \forall \rho > 0. \quad (3.41)$$

In fact, if we set $f(t) = \frac{\delta}{2}t^2 - c(\rho)t^\gamma$, by (3.37) we infer $F(u) \geq f(|\nabla u|)$ and hence $m_{V,\rho}$ is greater than the maximum of f , which is achieved for $\bar{t}_\rho = \left(\frac{\delta}{\gamma c(\rho)}\right)^{\frac{1}{\gamma-2}}$, getting

$$f(\bar{t}_\rho) = \delta^{\frac{\gamma}{\gamma-2}} \left[\frac{1}{2\gamma^{\frac{2}{\gamma-2}}} - \frac{1}{\gamma^{\frac{\gamma}{\gamma-2}}} \right] \frac{1}{c(\rho)^{\frac{2}{\gamma-2}}}.$$

Recalling that $c(\rho) = \frac{G^p}{p} \rho^{p-\gamma}$, by (2.15) we obtain (3.41).

Proposition 3.3. *There exists a Palais-Smale sequence $(v_n)_n$ for F constrained on S_ρ at the level $m_{V,\rho}$, namely*

$$F(v_n) \rightarrow m_{V,\rho}, \quad \nabla_{S_\rho} F(v_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.42)$$

such that

$$\|\nabla v_n\|_2^2 - \frac{N(p-2)}{2p} \|v_n\|_p^p - \frac{1}{2} \int_{\mathbb{R}^N} V(x)(Nv_n^2 + 2v_n \nabla v_n \cdot x) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.43)$$

$$\lim_{n \rightarrow \infty} \|(v_n)^-\|_2 = 0. \quad (3.44)$$

Moreover, the sequence $(v_n)_n$ is bounded and the related Lagrange multipliers

$$\lambda_n := -\frac{DF(v_n)[v_n]}{\rho^2} \quad (3.45)$$

are bounded and verify, up to a subsequence, $\lambda_n \rightarrow \lambda$, with $\lambda > 0$.

Proof. The existence of a PS-sequence that verifies (3.43) and (3.44) follows as in [6, Proposition 3.11], see also [18].

Step 1. Boundedness of the Palais-Smale sequence

We set

$$a_n := \|\nabla v_n\|_2^2, \quad b_n := \|v_n\|_p^p, \quad c_n := \int_{\mathbb{R}^N} V(x)v_n^2 dx, \quad d_n := \int_{\mathbb{R}^N} V(x)v_n \nabla v_n \cdot x dx. \quad (3.46)$$

By (3.42), (3.43) and (3.45) we get

$$a_n - c_n - \frac{2}{p}b_n = 2m_{V,\rho} + o(1) \quad (3.47)$$

$$a_n - c_n + \lambda_n \rho^2 = b_n + o(1)(a_n^{1/2} + 1) \quad (3.48)$$

$$a_n - \frac{N(p-2)}{2p}b_n - \frac{N}{2}c_n - d_n = o(1). \quad (3.49)$$

The term $(a_n^{1/2} + 1)$ is in (3.48) because we do not know that v_n is bounded in $H^1(\mathbb{R}^N)$ yet. By (3.47) and (3.49) we obtain

$$\frac{N(p-2)-4}{2p}b_n = 2m_{V,\rho} - \frac{N-2}{2}c_n - d_n + o(1)$$

and, recalling the definition of B in (3.31), we infer:

$$\begin{aligned} a_n &= \frac{4}{N(p-2)-4} \left(2m_{V,\rho} - \frac{N-2}{2}c_n - d_n \right) + c_n + 2m_{V,\rho} + o(1) \\ &= \frac{N(p-2)}{B} 2m_{V,\rho} - \frac{N(4-p)}{B}c_n - \frac{4}{B}d_n + o(1). \end{aligned} \quad (3.50)$$

Since $m_{V,\rho} < m_\rho$, $c_n \leq S^{-1}\|V\|_{\frac{N}{2}}a_n$ and $|d_n| \leq S^{-1/2}\|W\|_N a_n$, we have:

$$\begin{aligned} 0 \leq Ba_n &\leq N(p-2)2m_\rho + N|4-p|S^{-1}\|V\|_{\frac{N}{2}}a_n + 4S^{-1/2}\|W\|_N a_n + o(1) \\ &= N(p-2)2m_\rho + \left[N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N \right] a_n + o(1) \end{aligned} \quad (3.51)$$

and hence:

$$\left(B - \left[N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N \right] \right) a_n \leq N(p-2)2m_\rho. \quad (3.52)$$

By assumption (3.29) we have $B - \left[N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N \right] > 0$, so that:

$$a_n \leq \frac{N(p-2)2m_\rho}{B - \left[N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N \right]} + o(1). \quad (3.53)$$

Step 2. Positivity of the Lagrange multiplier

By the previous step, we can assume that the sequences a_n, b_n, c_n, d_n and λ_n converge, up to a subsequence, to suitable a, b, c, d and λ , respectively.

Since $c_n \leq S^{-1}\|V\|_{\frac{N}{2}}a_n$ and $|d_n| \leq S^{-1/2}\|W\|_N a_n$, recalling (3.41), the bound on a_n given by (3.53) and the definitions in (3.31), by (3.47)–(3.49) we get:

$$\begin{aligned}
\lambda\rho^2 &= \frac{p-2}{p}b - 2m_{V,\rho} \\
&= \frac{2(p-2)}{N(p-2)-4} \left(2m_{V,\rho} - \frac{N-2}{2}c - d \right) - 2m_{V,\rho} \\
&= \frac{2N - (N-2)p}{N(p-2)-4} 2m_{V,\rho} - \frac{(N-2)(p-2)}{N(p-2)-4}c - \frac{2(p-2)}{N(p-2)-4}d \\
&\geq \frac{1}{B} \left\{ [2N - (N-2)p]M \right. \\
&\quad \left. - \frac{N(N-2)(p-2)^2 S^{-1}\|V\|_{\frac{N}{2}}}{B - [N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N]} - \frac{2N(p-2)^2 S^{-1/2}\|W\|_N}{B - [N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N]} \right\} 2m_\rho \\
&= \frac{1}{B} \left\{ AM - \frac{[(N-2)S^{-1}\|V\|_{\frac{N}{2}} + 2S^{-1/2}\|W\|_N]D}{B - [N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N]} \right\} 2m_\rho \\
&= \frac{1}{B} \frac{ABM - \{[AMN|4-p| + (N-2)D]S^{-1}\|V\|_{\frac{N}{2}} + [4AM + 2D]S^{-1/2}\|W\|_N\}}{B - [N|4-p|S^{-1}\|V\|_{\frac{N}{2}} + 4S^{-1/2}\|W\|_N]} 2m_\rho
\end{aligned} \tag{3.54}$$

and hence hypothesis (3.30) ensures the positivity of λ . \square

Lemma 3.4. *Let v be a weak solution of (P_ρ) , for some $\rho > 0$. If (3.32) holds then*

$$F(v) \geq 0. \tag{3.55}$$

Proof. The function v satisfies the Pohozaev identity:

$$\frac{1}{p}\|v\|_p^p = \frac{2}{N(p-2)}\|\nabla v\|_2^2 - \frac{1}{p-2} \int_{\mathbb{R}^N} V(x)v^2 dx - \frac{2}{N(p-2)} \int_{\mathbb{R}^N} V(x)v \nabla v \cdot x dx \tag{3.56}$$

and we get:

$$\begin{aligned}
F(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} V(x)v^2 dx - \frac{1}{p}\|v\|_p^p \\
&= \left(\frac{1}{2} - \frac{2}{N(p-2)} \right) \|\nabla v\|_2^2 + \frac{4-p}{2(p-2)} \int_{\mathbb{R}^N} V(x)v^2 dx + \frac{2}{N(p-2)} \int_{\mathbb{R}^N} V(x)v \nabla v \cdot x dx.
\end{aligned} \tag{3.57}$$

If $N \geq 4$ or $N = 3$ and $p \in (\frac{10}{3}, 4]$ then, using (2.20), (3.57) gives

$$\begin{aligned}
F(v) &\geq \left(\frac{1}{2} - \frac{2}{N(p-2)} \right) \|\nabla v\|_2^2 - \frac{2}{N(p-2)} S^{-1/2}\|W\|_N \|\nabla v\|_2^2 \\
&= \left(\frac{1}{2} - \frac{2}{N(p-2)} - \frac{2}{N(p-2)} S^{-1/2}\|W\|_N \right) \|\nabla v\|_2^2,
\end{aligned}$$

so, since by assumption (3.32)

$$\frac{1}{2} - \frac{2}{N(p-2)} - \frac{2}{N(p-2)} S^{-1/2}\|W\|_N \geq 0,$$

inequality (3.55) follows.

If $N = 3$ and $p \in (4, 6)$ then (3.57) gives

$$\begin{aligned} F(v) &\geq \left(\frac{1}{2} - \frac{2}{3(p-2)} \right) \|\nabla v\|_2^2 - \frac{p-4}{2(p-2)} S^{-1} \|V\|_{3/2} \|\nabla v\|_2^2 - \frac{2}{3(p-2)} S^{-1/2} \|W\|_3 \|\nabla v\|_2^2 \\ &= \left[\frac{3(p-2)-4}{6(p-2)} - \left(\frac{p-4}{2} S^{-1} \|V\|_{3/2} + \frac{2}{3} S^{-1/2} \|W\|_3 \right) \frac{1}{p-2} \right] \|\nabla v\|_2^2, \end{aligned}$$

and the claim follows from (3.32). \square

End of the proof of Theorem 1.1. Now, let us consider the bounded Palais-Smale sequence v_n given by Proposition 3.3. Since v_n is bounded in $H^1(\mathbb{R}^N)$, there exists $v \in H^1(\mathbb{R}^N)$ such that, up to a subsequence, v_n converges weakly in $H^1(\mathbb{R}^N)$ and a.e. in \mathbb{R}^N to a function $v \in H^1(\mathbb{R}^N)$, which turns out to be a weak solution of

$$-\Delta v + (\lambda - V)v = |v|^{p-2}v \quad (3.58)$$

with $\|v\|_2 \leq \rho$. To prove the theorem we will show that actually v_n converge to v strongly in H^1 . Then we are done, because in such a case $\|v\|_2 = \rho$ and $v \geq 0$ by (3.44).

Now, since

$$\int_{\mathbb{R}^N} \nabla v_n \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) v_n \varphi \, dx - \int_{\mathbb{R}^N} |v_n|^{p-2} v_n \varphi \, dx = -\lambda_n \int_{\mathbb{R}^N} v_n \varphi \, dx + o(1) \|\varphi\|$$

for every $\varphi \in H^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \nabla v_n \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) v_n \varphi \, dx - \int_{\mathbb{R}^N} |v_n|^{p-2} v_n \varphi \, dx = -\lambda \int_{\mathbb{R}^N} v_n \varphi \, dx + (\lambda - \lambda_n) \int_{\mathbb{R}^N} v_n \varphi \, dx + o(1) \|\varphi\|$$

and hence

$$\int_{\mathbb{R}^N} \nabla v_n \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) v_n \varphi \, dx - \int_{\mathbb{R}^N} |v_n|^{p-2} v_n \varphi \, dx = -\lambda \int_{\mathbb{R}^N} v_n \varphi \, dx + o(1) \|\varphi\|,$$

because v_n is bounded in $H^1(\mathbb{R}^N)$. Therefore, v_n is also a Palais-Smale sequence for I_λ at level $m_{V,\rho} + \frac{\lambda}{2}\rho^2$, so that we can apply the Splitting Lemma 2.3, getting:

$$v_n = v + \sum_{j=1}^k w^j(\cdot - y_n^j) + o(1),$$

with w^j being solutions to

$$-\Delta w^j + \lambda w^j = |w^j|^{p-2} w^j$$

and $|y_n^j| \rightarrow \infty$. Assume by contradiction that $k \geq 1$ or, equivalently, that $\mu := \|v\|_2 < \rho$. Recall that by (2.15) and by Lemma 2.5

$$m_\alpha > m_\beta \quad \text{if } 0 < \alpha < \beta \quad \text{and} \quad F_\infty(w^j) \geq m_{\alpha_j}, \quad (3.59)$$

where $\alpha^j := \|w_j\|_2$, $j \in \{1, \dots, k\}$. The condition $F(v_n) \rightarrow m_{V,\rho}$ and (2.25) implies

$$m_{V,\rho} + \frac{\lambda}{2}\rho^2 = F(v) + \frac{\lambda}{2}\mu^2 + \sum_{j=1}^k F_\infty(w^j) + \frac{\lambda}{2} \sum_{j=1}^k \alpha_j^2. \quad (3.60)$$

By (2.24) we have

$$\rho^2 = \mu^2 + \sum_{j=1}^k \alpha_j^2,$$

and (3.60) becomes

$$m_{V,\rho} = F(v) + \sum_{j=1}^k F_\infty(w^j). \quad (3.61)$$

Using (3.55),(3.59) and the fact that $\alpha_j < \rho$, we infer that the right-hand side of (3.61) is strictly greater than m_ρ . This contradicts the fact that the left-hand side of (3.61) is strictly less than m_ρ , proving our result. \square

4 Proof of Theorem 1.2

4.1 Existence of a local minimizer

In order to prove the first part of Theorem 1.2, we first show that, under (1.7), F restricted on S_ρ admits a mountain pass structure which depends on $\|V\|_r$ but is uniform with respect to ρ . More precisely, we have the following

Proposition 4.1. *Let $N \geq 1$ and $r \in (\max(1, \frac{N}{2}), +\infty]$. There exist positive explicit constants σ , K , Θ and Υ , only depending on N, p, r , such that, if (1.7) holds true then*

$$\inf\{F(u) : u \in S_\rho, R_* - \varepsilon \leq \|\nabla u\|_2 \leq R_*\} > 0.$$

where

$$R_* = \Theta \cdot \|V\|_r^\Upsilon$$

and $\varepsilon > 0$ is sufficiently small, depending only on (a bound from above on) ρ .

To prove the proposition, we use the following elementary lemma.

Lemma 4.2. *Let A, B, s, α, β be positive parameters, with $\alpha \leq 1$, and define*

$$f_z(t) = t - Az^s t^{1-\alpha} - Bzt^{1+\beta}, \quad z, t > 0.$$

Let z_* and t_* be defined as

$$z_* = \left(\frac{\alpha}{B}\right)^{\frac{\alpha}{\alpha+s\beta}} \left(\frac{\beta}{A}\right)^{\frac{\beta}{\alpha+s\beta}} (\alpha + \beta)^{-\frac{\alpha+\beta}{\alpha+s\beta}}, \quad t_* = \left(\frac{\alpha}{B}\right)^{\frac{s}{\alpha+s\beta}} \left(\frac{A}{\beta}\right)^{\frac{1}{\alpha+s\beta}} (\alpha + \beta)^{\frac{1-s}{\alpha+s\beta}}. \quad (4.62)$$

Then

$$0 < z < z_* \implies f_z(t_*) > 0. \quad (4.63)$$

Proof. By direct calculation, it follows that $f_{z_*}(t_*) = 0$. Then (4.63) follows, as $f_z(\cdot)$ is (pointwise) decreasing with respect to z . \square

Proof of Proposition 4.1. Let

$$u \in S_\rho, \quad \|\nabla u\|_2 = R, \quad r = \frac{q}{q-2} \text{ (with } 2 \leq q < 2^* \text{)}.$$

By Lemma 2.2 and (2.17) we know that

$$2F(u) \geq R^2 - G_q^2 \|V\|_{\frac{q}{q-2}} \rho^{2-\frac{N(q-2)}{q}} R^{\frac{N(q-2)}{q}} - \frac{2}{p} G_p^p \rho^{p-\frac{N(p-2)}{2}} R^{\frac{N(p-2)}{2}}. \quad (4.64)$$

Thus we can apply Lemma 4.2, with the corresponding notations, writing

$$t = R^2, \quad z = \rho^{\frac{2N-(N-2)p}{2}}, \quad A = G_q^2 \|V\|_{\frac{q}{q-2}}, \quad B = \frac{2}{p} G_p^p, \quad (4.65)$$

and

$$\alpha = \frac{2N - q(N - 2)}{2q} \leq 1, \quad \beta = \frac{N(p - 2) - 4}{4}, \quad s = \frac{2}{q} \cdot \frac{2N - q(N - 2)}{2N - p(N - 2)}. \quad (4.66)$$

In particular, only A depends on $\|V\|_{\frac{q}{q-2}}$, while B, α, β and s just depend on N, p and r (via q). Then

$$\alpha + s\beta = \frac{p - 2}{q} \cdot \frac{2N - q(N - 2)}{2N - p(N - 2)},$$

and we can write

$$\frac{\rho_*^{\frac{2N-p(N-2)}{2}}}{\rho_*} = z_* = C(N, p, q) \cdot \|V\|_{\frac{q}{q-2}}^{-\frac{N(p-2)-4}{4} \cdot \frac{q}{p-2} \cdot \frac{2N-p(N-2)}{2N-q(N-2)}},$$

in such a way that $z < z_*$ is equivalent to (1.7), with a suitable choice of σ and K (the choice is explicit, by combining (4.62), (4.65) and (4.66)). Moreover

$$R_*^2 = t_* = C(N, p, q) \cdot \|V\|_{\frac{q}{q-2}}^{\frac{q}{p-2} \cdot \frac{2N-p(N-2)}{2N-q(N-2)}},$$

and the proposition follows, with a suitable choice of Θ, Υ (again, the choice is explicit, by combining (4.62), (4.65) and (4.66)). \square

Hereafter we assume that ρ, V satisfy (1.7), that is

$$0 < \rho < \rho_* := H \cdot \|V\|_r^{-\tau},$$

for suitable H, τ . By Proposition 4.1 we know that, for every $\alpha < \rho_*$,

$$\inf\{F(u) : u \in S_\alpha, \|\nabla u\|_2 = R_*\} > 0, \quad (4.67)$$

where $R_* = \Theta \cdot \|V\|_r^\Upsilon$ is independent of α . We define

$$c_{V,\alpha} := \inf\{F(u) : u \in S_\alpha, \|\nabla u\|_2 \leq R_*\}. \quad (4.68)$$

The proof of the first part of Theorem 1.2 is based on the following proposition.

Proposition 4.3. *If $c_{V,\rho} < 0$ then $c_{V,\rho}$ is achieved by a solution of (P_ρ) .*

In turn, the proof of the proposition is based on the following lemma.

Lemma 4.4. *If $c_{V,\rho} < 0$ then*

$$0 < \alpha \leq \rho \quad \implies \quad c_{V,\alpha} \geq c_{V,\rho}.$$

Proof. If $c_{V,\alpha} \geq 0$ then the lemma is trivial. On the contrary, let $0 > c' > c_{V,\alpha}$ and $u \in S_\alpha$ such that $\|\nabla u\|_2 < R_*$ and $F(u) < c'$. For every $t \geq 1$ we have

$$tu \in S_{t\alpha} \quad \text{and} \quad F(tu) = \frac{t^2}{2} \left(\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} V(x)u^2 dx \right) - \frac{t^p}{2} \|u\|_p^p \leq t^2 F(u) < c' < 0.$$

We claim that $\|\nabla \frac{\rho}{\alpha} u\|_2 \leq R_*$. If not, there exists $\bar{t} \in (1, \frac{\rho}{\alpha})$ such that $\|\nabla \bar{t}u\|_2 = R_*$, $\|\bar{t}u\|_2 < \rho$ and $F(\bar{t}u) < 0$, in contradiction with (4.67). Thus, by definition,

$$c_{V,\rho} \leq F\left(\frac{\rho}{\alpha}u\right) < c'.$$

Since $c' > c_{V,\alpha}$ is arbitrary, the lemma follows. \square

Proof of Proposition 4.3. Let $(u_n)_n$ be a minimizing sequence for $c_{V,\rho}$. By Proposition 4.1 we know that $\|\nabla u_n\|_2 \leq R_* - \varepsilon$, for some $\varepsilon > 0$ suitably small. Therefore, by Ekeland's principle, we can assume that $(u_n)_n$ is a Palais-Smale sequence for F constrained on S_ρ , i.e.

$$F(u_n) \rightarrow c_{V,\rho}, \quad \nabla_{S_\rho} F(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.69)$$

Since both the functional and the constraint are even, we can choose each u_n to be non-negative. Furthermore, $(u_n)_n$ is bounded by construction, and therefore also the Lagrange multipliers

$$\lambda_n := -\frac{DF(u_n)[u_n]}{\rho^2}$$

are bounded. Up to subsequences, we obtain that $u_n \rightharpoonup u \geq 0$ weakly in $H^1(\mathbb{R}^N)$, and $\lambda_n \rightarrow \lambda \in \mathbb{R}$. By (4.69),

$$o(1) = \frac{1}{2} \left(\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} V u_n^2 dx - \|u_n\|_p^p \right) + \frac{1}{2} \lambda_n \rho^2 \leq F(u_n) + \frac{1}{2} \lambda_n \rho^2 = c_{V,\rho} + \frac{1}{2} \lambda \rho^2 + o(1),$$

which forces

$$\lambda \geq -\frac{2c_{V,\rho}}{\rho^2} > 0.$$

Arguing as in the proof of Theorem 1.1 we have that (4.69) implies that $(u_n)_n$ is a (free) Palais-Smale sequence for the action functional I_λ , with $\lambda > 0$. Then the Splitting Lemma 2.3 applies, yielding

$$u_n = u + \sum_{j=1}^k w^j(\cdot - y_n^j) + o(1) \quad \text{strongly in } H^1(\mathbb{R}^N),$$

where

$$-\Delta u - V u + \lambda u = u^{p-1}, \quad -\Delta w^j + \lambda w^j = |w^j|^{p-2} w^j, \quad 1 \leq j \leq k,$$

and

$$\|u\|_2^2 + \sum_{j=1}^k \|w^j\|_2^2 = \rho^2, \quad I_\lambda(u) + \sum_{j=1}^k I_{\infty,\lambda}(w^j) = c_{V,\rho} + \frac{1}{2} \lambda \rho^2, \quad F(u) + \sum_{j=1}^k F_\infty(w^j) = c_{V,\rho}.$$

Writing $\|u\|_2 = \alpha \leq \rho$, we have that $\|\nabla u\|_2 \leq \liminf \|\nabla u_n\|_2 < R_*$, thus $F(u) \geq c_{V,\alpha}$. Lemma 4.4 yields

$$c_{V,\rho} = F(u) + \sum_{j=1}^k F_\infty(w^j) \geq c_{V,\alpha} + \sum_{j=1}^k F_\infty(w^j) \geq c_{V,\rho} + \sum_{j=1}^k F_\infty(w^j),$$

forcing

$$\sum_{j=1}^k F_\infty(w^j) \leq 0.$$

By Lemma 2.5 we deduce that $k = 0$, so that $u_n \rightarrow u$ strongly in $H^1(\mathbb{R}^N)$ and the proposition follows. \square

To show that $c_{V,\rho} < 0$ we will use the fact that the bottom of the spectrum of the operator $-\Delta - V$ is non-positive. As we mentioned, a sufficient condition in this direction is contained in Remark 1.5. For the reader's convenience we sketch such result in the following lemma.

Lemma 4.5. *Let $V \geq 0$ satisfy (1.10). Then (1.8) holds true.*

Proof. Let B_R and η be as in assumption (1.10), and without loss of generality let us assume that B_R is centered at 0.

In case $N = 1$ it is sufficient to choose, for a normalizing constant $t^* > 0$,

$$t^* \varphi(x) = \begin{cases} 1 & |x| \leq R \\ \left(\frac{kR-|x|}{(k-1)R}\right)^+ & |x| \geq R \end{cases}$$

with $k > 1$ sufficiently large. In case $N = 2$ it is sufficient to choose

$$t^* \varphi(x) = \begin{cases} 1 & |x| \leq R \\ \left(\frac{\ln(k-1)R - \ln|x|}{\ln(k-1)}\right)^+ & |x| \geq R \end{cases}$$

with $k > 2$ sufficiently large.

Finally, if $N \geq 3$, let

$$t^* \varphi(x) = \begin{cases} 1 & |x| \leq R \\ R^{N-2}|x|^{2-N} & |x| \geq R. \end{cases}$$

Letting $|\partial B_1| = \omega_N$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (|\nabla\varphi|^2 - V(x)\varphi^2) dx &\leq \int_{\mathbb{R}^N \setminus B_R} |\nabla\varphi|^2 dx - \int_{B_R} \gamma\varphi^2 dx \\ &= R^{2(N-2)}(N-2)^2\omega_N \int_R^{+\infty} r^{2(1-N)} \cdot r^{N-1} dr - \frac{\omega_N}{N} R^N \gamma \\ &= \frac{R^{N-2}\omega_N}{N} (N(N-2) - R^2\gamma) \leq 0 \end{aligned}$$

by (1.10). □

Now, let us prove that a local minimum solution exists. By Proposition 4.3, we just need to show that, under the assumptions of the theorem, $c_{V,\rho} < 0$. Let φ be as in (1.8). Notice that, for every $t > 0$,

$$F(t\varphi) \leq -\frac{t^p}{p} \|\varphi\|_p^p < 0.$$

Let $\bar{t} = \frac{R_*}{\|\nabla\varphi\|_2}$. Then $\|\nabla\bar{t}\varphi\|_2 = R_*$ and $F(\bar{t}\varphi) < 0$, hence (4.67) implies that

$$\rho_* \leq \|\bar{t}\varphi\|_2 = \frac{R_*}{\|\nabla\varphi\|_2} \rho, \quad \implies \quad \|\nabla\varphi\|_2 \leq \frac{\rho}{\rho_*} R_* < R_*.$$

Resuming, we have that both $\varphi \in S_\rho$ and $\|\nabla\varphi\|_2 \leq R_*$. By definition we infer

$$c_{V,\rho} \leq F(\varphi) < 0$$

and the theorem follows. □

4.2 Mountain pass solution

The proof of the second part of Theorem 1.2, i.e. the existence of a mountain pass solution, can be obtained arguing as in the proof of Theorem 1.1. Some changes are in order, especially for the Palais-Smale condition, because in this framework Lemma 3.4 cannot work. In this case, instead of working with a mountain pass geometry uniform in ρ , as we did to find the minimizer, it is more convenient to use a mountain pass geometry dependent on ρ ; this will allow a direct comparison between the

mountain pass level and m_ρ , allowing to show that the Lagrange multipliers of the Palais-Smale sequence are eventually positive. Observe that, by Remark 2.4, (1.9) ensures that the assumptions of the Splitting Lemma 2.3 holds also for $r = +\infty$.

In the following, we assume that $\sigma_i, \bar{\sigma}_i, i = 1, 2$, and \tilde{L} in (1.9) are such that

$$\|V\|_r \cdot \rho^{(2-\frac{N}{r})\frac{2(p-2)}{N(p-2)-4}} \leq L_1, \quad \|W\|_s \cdot \rho^{(1-\frac{N}{s})\frac{2(p-2)}{N(p-2)-4}} \leq L_1, \quad (4.70)$$

$$\|V\|_r \cdot \rho^{2-\frac{N}{r}} \leq L_2, \quad \|W\|_s \cdot \rho^{1-\frac{N}{s}} \leq L_2, \quad (4.71)$$

$$\|V\|_r \cdot \rho^{2-\frac{N}{r}} \leq L_3 m_\rho, \quad \|W\|_s \cdot \rho^{1-\frac{N}{s}} \leq L_3 m_\rho, \quad (4.72)$$

for suitable positive constants L_i to be chosen below independently of V, W and ρ (this is possible in view of (2.15)). Moreover we notice that (4.70) is equivalent to

$$\|V\|_r^{\frac{2r}{2r-N}} \leq L'_1 \cdot \frac{m_\rho}{\rho^2}, \quad \|W\|_s^{\frac{2s}{s-N}} \leq L'_1 \cdot \frac{m_\rho}{\rho^2}. \quad (4.73)$$

Mountain pass geometry - revisited

We recall that by (4.64) we have, for every $u \in S_\rho$,

$$F(u) \geq \frac{1}{2} \|\nabla u\|_2^2 - \frac{G^2}{2} \|V\|_{\frac{q}{q-2}}^2 \rho^{2-\frac{N(q-2)}{q}} \|\nabla u\|_2^{\frac{N(q-2)}{q}} - \frac{1}{p} G^p \rho^{p-\gamma} \|\nabla u\|_2^\gamma, \quad (4.74)$$

where, as usual, $q \in [2, 2^*)$ satisfies $\frac{q}{q-2} = r$. Let $\tilde{R} > 0$ be such that

$$\tilde{\mathcal{M}}_0 := \frac{1}{2} \tilde{R}^2 - \frac{1}{p} G^p \rho^{p-\gamma} \tilde{R}^\gamma = \max_{t \geq 0} \frac{1}{2} t^2 - \frac{1}{p} G^p \rho^{p-\gamma} t^\gamma.$$

Then $\tilde{R} = C \rho^{-\frac{p-\gamma}{\gamma-2}}$, where C depends only on N and p . By a direct computation and (2.15) we have that $\tilde{\mathcal{M}}_0 = 2\tilde{M}m_\rho$ for a suitable constant $\tilde{M} > 0$ independent on ρ . Then we can choose $L_1 > 0$, only depending on N, p and q , such that

$$\frac{G^2}{2} \|V\|_{\frac{q}{q-2}}^2 \rho^{2-\frac{N(q-2)}{q}} \tilde{R}^{\frac{N(q-2)}{q}} \leq \tilde{M}m_\rho \quad \iff \quad \|V\|_r \rho^{(2-\frac{N}{r})\frac{2(p-2)}{N(p-2)-4}} \leq L_1. \quad (4.75)$$

By (4.74), (4.75) and (4.70) we obtain

$$\tilde{\mathcal{M}} := \frac{1}{2} \tilde{R}^2 - \frac{G^2}{2} \|V\|_{\frac{q}{q-2}}^2 \rho^{2-\frac{N(q-2)}{q}} \tilde{R}^{\frac{N(q-2)}{q}} - \frac{1}{p} G^p \rho^{p-\gamma} \tilde{R}^\gamma \geq \tilde{M}m_\rho. \quad (4.76)$$

Now, let us fix $u_0 = (Z_\rho)_{h_0}, u_1 = (Z_\rho)_{h_1} \in S_\rho$ such that

$$\|\nabla u_0\|_2 < \tilde{R}, \quad \|\nabla u_1\|_2 > \tilde{R}, \quad F(u_0) < \tilde{\mathcal{M}}, \quad F(u_1) < 0,$$

and define the mountain pass value

$$m_{V,\rho} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F(\xi(t)), \quad \Gamma = \{\xi : [0,1] \rightarrow S_\rho : \xi(0) = u_0, \xi(1) = u_1\}.$$

By (4.76) and (1.1) we infer

$$\tilde{M}m_\rho \leq m_{V,\rho} < m_\rho \quad (4.77)$$

(the strict inequality follows as in Remark 3.2). As in Proposition 3.3, we get a Palais-Smale sequence $(v_n)_n$, at the level $m_{V,\rho}$, that satisfies (3.43), (3.44) and we have to verify that it is bounded, the

related Lagrange multipliers $(\lambda_n)_n$ are bounded and converge, up to a subsequence, to a positive value.

Bounded Palais-Smale sequence

The proof of this step goes on as the proof of Proposition 3.3 until (3.50), with analogous notation. In this framework, by Lemma 2.2 we have

$$\begin{aligned} [N(p-2) - 4]a_n &\leq 2N(p-2)m_{V,\rho} + N|4-p|c_n + 4|d_n| + o(1) \\ &\leq 2N(p-2)m_\rho + N|4-p| \left(G_q^2 \rho^{2-\frac{N}{r}} \|V\|_r \right) a_n^{\frac{N}{2r}} \\ &\quad + 4 \left(G_{q_1} \|W\|_s \rho^{1-\frac{N}{s}} \right) a_n^{\frac{1}{2}(1+\frac{N}{s})} + o(1) \end{aligned} \quad (4.78)$$

where $q_1 \in [2, 2^*)$ satisfies $\frac{2q_1}{q_1-2} = s$. By assumption,

$$\frac{N}{2r} < 1 \quad \text{and} \quad \frac{1}{2} \left(1 + \frac{N}{s} \right) < 1, \quad (4.79)$$

hence when $a_n \geq 1$

$$\left\{ [N(p-2) - 4] - N|4-p|G_q^2 \rho^{2-\frac{N}{r}} \|V\|_r - 4G_{q_1} \rho^{1-\frac{N}{s}} \|W\|_s \right\} a_n \leq 2N(p-2)m_\rho + o(1). \quad (4.80)$$

Then we can choose L_2 in (4.71) small, in such a way that

$$a_n \leq \max \left\{ 1, \frac{3N(p-2)}{N(p-2) - 4} m_\rho \right\}, \quad (4.81)$$

and in particular the sequence $(a_n)_n$ is bounded. We deduce that the sequences b_n, c_n, d_n and λ_n are bounded as well, and they all converge, up to subsequences, to suitable a, b, c, d and λ , respectively. Then we focus on the sign of the Lagrange multiplier λ .

Lower bound for the Lagrange multiplier

As in (3.54), by (4.77) and the estimates of Lemma 2.2 we obtain

$$\begin{aligned} \lambda \rho^2 &= \frac{2N - (N-2)p}{N(p-2) - 4} 2m_{V,\rho} - \frac{(N-2)(p-2)}{N(p-2) - 4} c - \frac{2(p-2)}{N(p-2) - 4} d \\ &\geq C_1 \cdot 2\widetilde{M}m_\rho - C_2 \rho^{2-\frac{N}{r}} \|V\|_r a^{\frac{N}{2r}} - C_3 \rho^{1-\frac{N}{s}} \|W\|_s a^{\frac{1}{2}(1+\frac{N}{s})}, \end{aligned} \quad (4.82)$$

where the nonnegative constants C_i only depend on N, p, q, q_1 . Now, if $a \geq 1$, then (4.81) implies that $Cm_\rho \geq 1$ too. Then (4.82), (4.79) and (4.81) imply

$$\lambda \rho^2 \geq \left[C'_1 - C'_2 \rho^{2-\frac{N}{r}} \|V\|_r - C'_3 \rho^{1-\frac{N}{s}} \|W\|_s \right] m_\rho,$$

and using again (4.71), with a possibly smaller value of L_2 , we infer that $\lambda > 0$. If instead $a \leq 1$, then we can use (4.72) to write (4.82) as

$$\lambda \rho^2 \geq C_1 \cdot 2\widetilde{M}m_\rho - C_2 \rho^{2-\frac{N}{r}} \|V\|_r - C_3 \rho^{1-\frac{N}{s}} \|W\|_s = \left[C_1 \cdot 2\widetilde{M} - (C_2 + C_3)L_3 \right] m_\rho,$$

and also in this case $\lambda > 0$, provided L_3 in (4.72) is chosen sufficiently small.

Palais-Smale condition

Up to now, we have shown that the Palais-Smale sequence $(v_n)_n$ provided by the mountain pass geometry and the corresponding sequence of Lagrange multipliers $(\lambda_n)_n$ satisfy:

$$F(v_n) \rightarrow m_{V,\rho}, \quad F'(v_n)\phi = \lambda_n\phi + o(1)\|\phi\|_{H^1}, \quad \lambda_n \rightarrow \lambda > 0.$$

Then $(v_n)_n$ is a Palais-Smale sequence also for I_λ and by the Splitting Lemma we can write

$$v_n(x) = v(x) + \sum_{i=1}^k w^i(x - y_n^i) + o(1) \quad \text{in } H^1(\mathbb{R}^N)$$

where each $w^i \in H^1(\mathbb{R}^N)$ satisfies $-\Delta w + \lambda w = |w|^{p-2}w$, for $i \in \{1, \dots, k\}$, and the weak limit v satisfies $-\Delta w + (\lambda - V)w = |w|^{p-2}w$. Arguing as in (3.61) we get

$$m_{V,\rho} = F(v) + \sum_{i=1}^k F_\infty(w^i). \quad (4.83)$$

Let us assume by contradiction that $v_n \not\rightarrow v$ strongly in H^1 , that is $k > 0$. Then we denote

$$\mu = \|v\|_2, \quad \alpha_i = \|w^i\|_2, \quad \text{so that } \mu^2 + \sum_{i=1}^k \alpha_i^2 = \rho^2.$$

Now, by Lemma 2.5 and (2.15),

$$\sum_{i=1}^k F_\infty(w^i) \geq \sum_{i=1}^k m_{\|w^i\|_2} \geq m_{\alpha_1} = m_\rho \left(\frac{\alpha_1}{\rho} \right)^{-2\theta}, \quad \theta := \frac{2N - p(N-2)}{N(p-2) - 4} > 0. \quad (4.84)$$

We claim that, taking into account (4.73), we have:

$$F(v) \geq -\theta m_\rho \left(\frac{\mu}{\rho} \right)^2. \quad (4.85)$$

If this is the case, using (4.84) and the fact that $\mu^2 + \alpha_1^2 \leq \rho^2$, we obtain

$$\begin{aligned} F(v) + \sum_{i=1}^k F_\infty(w_i) &\geq -\theta m_\rho \left(\frac{\mu}{\rho} \right)^2 + m_\rho \left(\frac{\alpha_1}{\rho} \right)^{-2\theta} \geq m_\rho \left[-\theta + \theta \left(\frac{\alpha_1}{\rho} \right)^2 + \left(\frac{\alpha_1}{\rho} \right)^{-2\theta} \right] \\ &\geq m_\rho \min_{0 < x \leq 1} [-\theta + \theta x + x^{-2\theta}] = m_\rho, \end{aligned} \quad (4.86)$$

which completes the proof of the compactness, because it is in contradiction with (4.83) and the fact that $m_{V,\rho} < m_\rho$ by (4.77).

To conclude, we are left to show (4.85). To this aim, let us set $a = \|\nabla v\|_2^2$, $b = \|v\|_p^p$, $c = \int_{\mathbb{R}^N} V(x)v^2 dx$, $d = \int_{\mathbb{R}^N} V(x)v\nabla v \cdot x dx$. Taking into account the Pohozaev identity for v :

$$a - \frac{N(p-2)}{2p}b - \frac{N}{2}c - d = 0,$$

we infer

$$\begin{aligned} F(v) &= \frac{1}{2}a - \frac{1}{2}c - \frac{1}{p}b \geq \frac{N(p-2) - 4}{2N(p-2)}a - \frac{p-4}{2(p-2)}c + \frac{2}{N(p-2)}d \\ &\geq C_0 [2a - C_1c - C_2|d|], \end{aligned} \quad (4.87)$$

where the nonnegative constants C_i only depend on N and p . Now, using Lemma 2.2, we have that

$$a - C_1 c \geq a - C'_1 \|V\|_r \mu^{2-\frac{N}{r}} a^{\frac{N}{2r}} \geq -C''_1 \|V\|_r^{\frac{2r}{2r-N}} \mu^2, \quad (4.88)$$

where in the last step we used the elementary inequality

$$a > 0, 0 < \tau < 1, k > 0 \quad \implies \quad a - ka^\tau \geq -(1-\tau)\tau^{\frac{\tau}{1-\tau}} k^{\frac{1}{1-\tau}}.$$

Analogously,

$$a - C_2 |d| \geq a - C'_2 \|W\|_s \mu^{1-\frac{N}{s}} a^{\frac{1}{2}(1+\frac{N}{s})} \geq -C''_2 \|W\|_s^{\frac{2s}{s-N}} \mu^2. \quad (4.89)$$

Substituting (4.88) and (4.89) into (4.87), and exploiting assumption (4.73) we finally obtain

$$F(v) \geq -CL'_1 m_p \frac{\mu^2}{\rho^2},$$

and (4.85) follows by taking L'_1 sufficiently small. □

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