IVANOVA CONTACT JOIN-SEMILATTICES ARE NOT FINITELY AXIOMATIZABLE

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Abstract. We show that the class of contact join-semilattices introduced by T. Ivanova (Studia Logica **110**, 1219–1241, 2022) is not finitely axiomatizable. On the other hand, a simple finite axiomatization exists for the class of those join-semilattices with a weak contact relation which can be embedded into the reduct of a weak contact Boolean algebra (equivalently, distributive lattice).

1. Introduction

Structures with a proximity or a contact relation are useful in topology [6], algebraic logic [4, 5], computer science [15][1], image analysis [13], knowledge representation [14], graph theory [1, 12] and are a fundamental tool in region based theory of space [3]. Recall that a contact algebra is a Boolean algebra endowed with a further contact relation. Düntsch, MacCaull, Vakarelov, Winter [7] proposed to study contact relations in algebras with less structure; in particular, Ivanova [9] suggested the naturalness of studying join-semilattices with a contact. See also [11] for further arguments in favor of the use of the semilattice operation only. We refer to the quoted sources and to [10] for further details, motivation, history and references.

Ivanova [9] provided an axiomatization for those contact join-semilattices which admit a representation as a substructure of some field of sets, where two sets are in contact if they have nonempty intersection. This is equivalent to being embeddable into a complete and atomic Boolean algebra with overlap contact. Ivanova also showed that the axiomatization characterizes contact join-semilattices which admit an appropriate topological representation, and asked whether there is a finite axiomatization. In [10] we presented an equivalent axiomatization for Ivanova contact join-semilattices and also axiomatized the class of those weak contact join-semilattices embeddable into a, possibly complete and atomic, weak contact Boolean algebra, equivalently, into a weak contact distributive lattice (here the contact is not necessarily overlap).

Here we use the axiomatizations from [10] in order to show that the former class is not finitely axiomatizable, while the latter is. As a consequence, from [10, Corollary 5.1] we get that in the language of contact join-semilattices the set of logical consequences of the theory of Boolean algebras with an overlap contact relation is not finitely axiomatizable. On the other hand, an easily described finite axiomatization exists if we remove the request that the contact relation is overlap.

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[1]See [12, Remark 6.11] for explanations about the terminology.

We also show that the validity of some of the representation theorems mentioned above is equivalent to the Boolean Prime Ideal Theorem, hence needs a consequence of the axiom of choice. See Proposition 3.6.

As we mentioned, motivations for the study of join-semilattices with a contact relation are presented in [5, 7, 9, 10], among others. Our main technique in the proof of Theorem 4.1 is constructive and is likely to be useful for constructing many more interesting examples of contact join-semilattices. A further possible research direction is the study of join-semilattices with hypercontact n-ary relations, rather than just a binary contact. See [14]. In [12] we show that a large part of the results presented here generalize to the wider context. See Remark 5.3 below for a discussion.

2. Preliminaries

For simplicity, we will always assume that posets and semilattices have a minimum element 0 and that homomorphisms preserve 0. A *weak contact relation* on some poset $\mathbf{S}$ with 0 is a symmetric reflexive binary relation δ on $S \setminus \{0\}$ such that

$$\text{(Ext)} \qquad a \;\delta\; b \;\;\&\;\; a \leq a_1 \;\;\&\;\; b \leq b_1 \;\Rightarrow\; a_1 \;\delta\; b_1,$$

for all $a, b, a_1, b_1 \in S$. We write $a \not\delta\, b$ to mean that $a \;\delta\; b$ does not hold.

A typical example of a weak contact relation is the *overlap* relation, which can be defined on any poset $\mathbf{P}$ with 0. To get the overlap relation, simply set $a \;\delta\; b$ if there is $p \in P$ such that $p > 0$, $p \leq a$ and $p \leq b$.

A *weak contact join-semilattice* is a structure $(S, +, 0, \delta)$, where $(S, +, 0)$ is a join-semilattice with 0 and δ is a weak contact relation, as defined above. Note that in some previous papers we frequently used the word "semilattice" in order to mean "join-semilattice". More generally, a *weak contact lattice* is a lattice together with a weak contact relation. Weak contact Boolean algebras are defined similarly.

We will occasionally also deal with nonsymmetrical relations. A *weak pre-contact relation* on some poset $\mathbf{S}$ with 0 is a reflexive binary relation δ on $S \setminus \{0\}$ satisfying (Ext). Thus in a weak pre-contact we leave out the request that δ is symmetrical. Note that some authors do not assume reflexivity in the definition of a weak pre-contact relation, but here we will always assume reflexivity.

Homomorphisms and embeddings are always intended in the usual sense. For example, a *homomorphism* of weak (pre-)contact join-semilattices is a 0-preserving semilattice homomorphism φ such that (i) $a \;\delta\; b$ implies $\varphi(a) \;\delta\; \varphi(b)$, for all a, b in the domain. An *embedding* of weak contact join-semilattices is an injective homomorphism such that the reverse condition also holds, namely, (ii) $\varphi(a) \;\delta\; \varphi(b)$ implies $a \;\delta\; b$. In certain situations we need to assume that φ preserves only part of the structure, not necessarily all the structure; for example, a *δ-homomorphism*, or a *contact homomorphism* is a function φ such that (i) above holds, but φ is not necessarily required to be, say, order preserving.

The following property is frequently required in the definition of a contact relation on a join-semilattice (this is the reason for the terminology including "weak"). An *additive contact relation* on some join-semilattice is a weak contact relation satisfying the following condition.

$$\text{(Add)} \qquad a \;\delta\; b{+}c \Rightarrow a \;\delta\; b \text{ or } a \;\delta\; c.$$

We will not assume additivity, unless explicitly mentioned.

As mentioned in the introduction, Ivanova [9] axiomatized those contact join-semilattices which admit good set-theoretical and topological representations, calling them “contact join-semilattices”. Here, when we mention a (weak) contact join-semilattice we will only assume the semilattice axioms, together with the properties of (weak) contact listed above. Hence we will refer to the structures considered in [9] as *Ivanova contact join-semilattices*. In [10, Corollary 3.4] we have provided an alternative axiomatization of Ivanova contact join-semilattices, see below for details. In [9] semilattices are assumed to have a maximum 1. As far as the results presented here are concerned, it is irrelevant whether we request or not the existence of 1. See [10, Remark 3.3(a)].

Now we list the relevant conditions, where $\mathbf{S}$ is a weak contact join-semilattice and n varies among positive natural numbers.

(D1+)
For every n and $a, b, c_{1,0}, c_{1,1}, \dots, c_{n,0}, c_{n,1} \in S$,
IF $c_{1,0} \not\delta\, c_{1,1},\ c_{2,0} \not\delta\, c_{2,1},\ \dots,\ c_{n,0} \not\delta\, c_{n,1}$
and $b \leq a + c_{1,f(1)} + \dots + c_{n,f(n)}$, for all $f : \{1, \dots, n\} \to \{0, 1\}$,
THEN $b \leq a$.

($\mathrm{D2}_n$)
For every $a, b, c_{1,0}, c_{1,1}, \dots, c_{n,0}, c_{n,1} \in S$,
IF $c_{1,0} \not\delta\, c_{1,1}, \dots, c_{n,0} \not\delta\, c_{n,1}$ and, for every $f : \{1, \dots, n\} \to \{0, 1\}$,
at least one of the following two inequalities holds
$b \leq c_{1,f(1)} + \dots + c_{n,f(n)}$, or $a \leq c_{1,f(1)} + \dots + c_{n,f(n)}$,
THEN $b \not\delta\, a$.

The special case $n = 1$ of (D1+) has been called (D1) in [10], where we have proved that (D1) implies (D1+). The assumption that ($\mathrm{D2}_n$) holds for every n has been called (D2) in [10, Section 2]. In Section 4 here we will show that, contrary to the case of (D1+), (D2) is not equivalent to a first-order sentence.

Theorem 2.1. [10, Corollary 3.4] *A weak contact join-semilattice* $\mathbf{S}$ *is an Ivanova contact join-semilattice if and only if* $\mathbf{S}$ *satisfies* (D1) *and* ($\mathrm{D2}_n$)*, for every positive integer* n.

As far as the present note is concerned, the reader might take the characterization in Theorem 2.1 as the definition of an Ivanova contact join-semilattice. We refer to [9, 10] for further details about the above notions.

The next useful lemma is implicit in the proof of [10, Theorem 4.1].

Lemma 2.2. *Suppose that* $\mathbf{S} = (S, \leq, 0, \delta_S)$ *is a poset with a weak (pre-)contact relation* δ_S*,* $\mathbf{Q} = (Q, \leq, 0)$ *is a poset with* 0 *and* κ *is a function from* $\mathbf{S}$ *to* $\mathbf{Q}$ *such that* $\kappa(a) = 0$ *implies* $a = 0$*, for every* $a \in S$*. Let* δ_Q *be defined on* $\mathbf{Q}$ *by letting* $b_1\ \delta_Q\ b_2$ *if at least one of the following conditions hold:*

(a) there is $q \in Q$ *such that* $0 < q$*,* $q \leq b_1$ *and* $q \leq b_2$*, or*

(b) there are $a_1, a_2 \in S$ *such that* $a_1\ \delta_S\ a_2$ *and such that* $\kappa(a_1) \leq b_1$ *and* $\kappa(a_2) \leq b_2$*.*

Then

(i) δ_Q *is a weak (pre-)contact on* $\mathbf{Q}$ *and* κ *is a* δ*-homomorphism from* $\mathbf{S}$ *to* $\mathbf{Q}$*. In fact,* δ_Q *is the smallest weak (pre-)contact on* $\mathbf{Q}$ *which makes* κ *a* δ*-homomorphism.*

(ii) Suppose in addition that κ is an order embedding such that, whenever $a_1 \not\delta_S a_2$, the meet of $\kappa(a_1)$ and of $\kappa(a_2)$ in $\mathbf{Q}$ exists and is equal to 0. Then κ is a δ-embedding from $\mathbf{S}$ to $\mathbf{Q}$.

Proof. (i) Symmetry (in the case of a contact relation), reflexivity and (Ext) for δ_Q are immediate. By assumption, if $a \neq 0$, then $\kappa(a) \neq 0$, thus all δ_Q-related elements are nonzero, since all δ_S-related elements are nonzero. Thus δ_Q is a weak (pre-)contact on $\mathbf{Q}$; κ is a δ-homomorphism by construction. Every weak (pre-)contact on $\mathbf{Q}$ must contain all the pairs for which (a) holds, by reflexivity and (Ext). If κ is a δ-homomorphism from (S, δ_S) to (Q, δ'_Q) and $a_1 \ \delta_S \ a_2$, then necessarily $\kappa(a_1) \ \delta'_Q \ \kappa(a_2)$. If (b) holds for b_1 and b_2 with respect to a_1 and a_2 as above, and δ'_Q satisfies (Ext), then $b_1 \ \delta'_Q \ b_2$. Thus δ_Q is the smallest weak (pre-)contact on $\mathbf{Q}$ with the required property.

(ii) Because of (i), we just need to check that if $c_1 \not\delta_S c_2$, then $\kappa(c_1) \not\delta_Q \kappa(c_2)$. By assumption, $\kappa(c_1)\kappa(c_2) = 0$, hence (a) cannot be applied in order to get $\kappa(c_1) \ \delta_Q \ \kappa(c_2)$. If (b) were applicable, there would be $a_1, a_2 \in S$ such that $a_1 \ \delta_S \ a_2$ and $\kappa(a_1) \leq \kappa(c_1)$, $\kappa(a_2) \leq \kappa(c_2)$. Since κ is an order embedding, $a_1 \leq c_1$ and $a_2 \leq c_2$. Thus $c_1 \ \delta_S \ c_2$, by (Ext), a contradiction. □

3. A finite axiomatization in the non overlap case

In this section we present some slight improvements on Theorems 3.2 and 4.1 from [10]. Moreover, we extend [10, Theorem 4.1] to the case of not necessarily symmetric relations. We also show that a consequence of the axiom of choice is necessary in some results from [10].

In what follows we will frequently deal with the situation in which weak (pre-)contact join-semilattices are embedded into models with further structure, e. g., distributive lattices or Boolean algebras. Rather than explicitly saying that a weak contact join-semilattice $\mathbf{S}$ can be embedded into *the contact join-semilattice reduct* of some contact Boolean algebra $\mathbf{B}$, we will simply say that $\mathbf{S}$ can be $\{\delta, +\}$-*embedded* into $\mathbf{B}$. Notice that, on the other hand, we are not assuming that embeddings preserve existing meets, or complements, unless otherwise specified.

Theorem 3.1. *If $\mathbf{S}$ is a weak contact join-semilattice, then the following conditions are equivalent, where embeddings are always intended as $\{\delta, +\}$-embeddings.*

(1) $\mathbf{S}$ can be embedded into a weak contact Boolean algebra.
(2) $\mathbf{S}$ can be embedded into a weak contact distributive lattice.
(3) $\mathbf{S}$ satisfies (D1).
(4) $\mathbf{S}$ can be embedded into a weak contact complete atomic Boolean algebra.

The above equivalences hold if "weak contact" is everywhere replaced by "weak pre-contact".

Proof. For contact relations, the theorem has been proved in [10, Theorem 4.1] with an additional assumption in clause (3). Hence it is enough to prove that (D1) implies the additional assumption, which we reproduce here, relabeling some variables in order to avoid a notational clash in the proof.

(D2-)

For every positive $n \in \mathbb{N}$ and $a^*, b^*, c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, \dots, c_{n,0}, c_{n,1} \in S$,

IF $c_{0,0} \not\delta\, c_{0,1}, c_{1,0} \not\delta\, c_{1,1}, \dots, c_{n,0} \not\delta\, c_{n,1}$ and, for every $f : \{1, \dots, n\} \to \{0, 1\}$,

both $a^* \leq c_{0,0} + c_{1,f(1)} + \dots + c_{n,f(n)}$ and $b^* \leq c_{0,1} + c_{1,f(1)} + \dots + c_{n,f(n)}$,

THEN $a^* \not\delta\, b^*$.

In [10, Lemma 2.3] we have showed that (D1) implies (D1+), hence, assuming the premises of (D2-), we get $b^* \leq c_{0,1}$ by taking $c_{0,1}$ in place of a in (D1+). Similarly, $a^* \leq c_{0,0}$. Since, by the assumptions in (D2-), $c_{0,0} \not\delta\, c_{0,1}$, we get $a^* \not\delta\, b^*$ by (Ext).

We now prove the last statement. The implications (4) ⇒ (1) ⇒ (2) are straightforward. The remaining implications can be obtained by analyzing the proof for the case of a weak contact, observing that we have not actually used symmetry of δ in the above argument and in [10, Theorem 4.1]. Anyway, we are going to show that the result for a pre-contact follows from the already proved result for a contact. We will use the following claim, whose proof is elementary.

Claim 3.2. *If δ is a weak pre-contact relation on some poset $\mathbf{P}$, define δ^s on P by $a \ \delta^s \ b$ if both $a \ \delta \ b$ and $b \ \delta \ a$. Then δ^s is a weak contact relation on $\mathbf{P}$. Moreover, δ satisfies* (D1) *if and only if δ^s satisfies* (D1).

Now we can prove (2) ⇒ (3) in the case of a weak pre-contact. Suppose that the weak pre-contact join-semilattice $\mathbf{S}$ can be embedded into a distributive lattice $\mathbf{L}$ with weak pre-contact δ_L. If we define δ_L^s as in the Claim, then, by the Claim, δ_L^s is a weak contact relation on $\mathbf{L}$, thus, by the theorem in the case of a weak contact, $\mathbf{L}$ with δ_L^s satisfies (D1). By the last statement in the Claim, $\mathbf{L}$ with δ_L satisfies (D1). Property (D1) is clearly preserved under substructures and isomorphism, hence $\mathbf{S}$ satisfies (D1), as well.

To prove (3) ⇒ (4), assume that the join-semilattice $\mathbf{S}$ with weak pre-contact δ satisfies (D1). If we replace δ with δ^s, we get a weak contact join-semilattice $\mathbf{S}^s$ satisfying (D1), by the Claim, hence $\mathbf{S}^s$ can be embedded into a weak contact complete atomic Boolean algebra $\mathbf{Q}$, by the already proved version of the theorem, dealing with weak contact relations. If κ is the above embedding, then κ satisfies the assumptions in (ii) in Lemma 2.2 with respect to δ. Indeed, if $a_1 \not\delta\, a_2$, then also $a_1 \not\delta^s\, a_2$, by definition, hence $\kappa(a_1) \not\delta_Q\, \kappa(a_2)$, where δ_Q is the weak contact on $\mathbf{Q}$. Thus the meet of $\kappa(a_1)$ and $\kappa(a_2)$ in $\mathbf{Q}$ is 0, by (Ext).

By Lemma 2.2(ii) we can thus endow the Boolean reduct of $\mathbf{Q}$ with a weak pre-contact δ^p in such a way that κ is a δ-embedding from $\mathbf{S}$ with δ to $\mathbf{Q}$ with δ^p. The conclusion follows, since the join-semilattice structures on $\mathbf{S}$ and $\mathbf{Q}$ are not affected by the additional contact structure, and κ was assumed to be a semilattice embedding. □

Since (D1), which is the instance $n = 1$ of (D1+), is clearly expressible as a first-order sentence, we immediately get the following corollary.

Corollary 3.3. *The classes of weak contact and weak pre-contact join-semilattices satisfying one of (equivalently, all of) the conditions in Theorem 3.1 are finitely first-order axiomatizable.*

In the next section we will show that no finite axiomatization exists for the class of weak contact join-semilattices embeddable in a contact Boolean algebra with *overlap* contact relation, which by [10, Theorem 3.2 and Corollary 3.4] is precisely the class of Ivanova contact join-semilattices.

We now present a slight extension of [10, Theorem 3.2], giving a few further characterizations of Ivanova contact join-semilattices.

If A is a set, $\mathcal{P}(A)$ denotes the power set of A. If X is a topological space with closure K, the *elementary proximity on* X [6] is the contact relation on $\mathcal{P}(X)$ defined by $a \ \delta_K \ b$ if $Ka \cap Kb \neq \emptyset$, for $a, b \subseteq X$. A contact semilattice is an *elementary topological contact semilattice* if it has the form $(\mathcal{P}(X), \cup, \emptyset, \delta_K)$ with δ_K as above, for some topological space X. Note that the above notion of contact is distinct from another topological notion of contact which is mostly used in region based theory of space, and which is defined on the algebra of regular open, or regular closed subsets of X. See e. g. [9, p. 1221].

Recall that a *pre-closure operation* K on some poset $\mathbf{P}$ is a unary, extensive and isotone operation (the "pre" here bears no connection with the "pre" in pre-contact). If $\mathbf{P}$ has a minimum element 0, we will also require that K is *normal*, that is, $K0 = 0$. If K is also idempotent, it is called a *closure operation*. If $\mathbf{P}$ is a poset with 0 and with a normal pre-closure operation K, the *associated elementary weak contact relation* is defined by $a \ \delta_K \ b$ if there is $p \in P$, $p > 0$ such that both $p \leq Ka$ and $p \leq Kb$. In the above situation, we will say that $(P, \leq, 0, \delta_K)$ is the *elementary contact poset associated to* $\mathbf{P}$. We define similar notions for join-semilattices. In the case of semilattices, a closure operation K is *additive* if $K(x+y) = Kx + Ky$ holds identically. Recall that a δ-embedding between structures endowed with a weak contact relation is an injective function φ such that $a \ \delta \ b$ if and only if $\varphi(a) \ \delta \ \varphi(b)$, for all a, b in the domain.

Theorem 3.4. *If* $\mathbf{S}$ *is a weak contact join-semilattice, then the following conditions are equivalent, where embeddings are always intended as* $\{\delta, +\}$*-embeddings.*

1. *(1)* $\mathbf{S}$ *can be embedded into a Boolean algebra with overlap contact.*
2. *(2)* $\mathbf{S}$ *can be embedded into an additive contact distributive lattice.*
3. *(3)* $\mathbf{S}$ *is an Ivanova contact join-semilattice.*
4. *(4)* $\mathbf{S}$ *can be embedded into a complete atomic Boolean algebra with overlap contact.*
5. *(5)* $\mathbf{S}$ *can be embedded into an elementary topological contact semilattice.*
6. *(6)* $\mathbf{S}$ *can be embedded into the elementary contact join-semilattice associated to some distributive lattice with additive closure.*

Proof. The equivalences of (1) - (4) follow from [9]; see also [10, Theorem 3.2].

(4) $\Rightarrow$ (5) Since a complete atomic Boolean algebra $\mathbf{B}$ is isomorphic to a field of sets, say, $\mathcal{P}(X)$, if we give X the discrete topology, the overlap contact on $\mathbf{B}$ is the same as the elementary proximity on the topological space X.

(5) $\Rightarrow$ (6) $\Rightarrow$ (2) are elementary. Indeed, in a distributive lattice with an additive closure operation K, the associated elementary contact relation is additive. See the next lemma for a slightly more general fact. □

Recall that a lattice with 0 is *meet semidistributive at* 0 if, for all elements p, q, r, $pr = 0$ and $qr = 0$ imply $(p + q)r = 0$. In particular, a distributive lattice with 0 is meet semidistributive at 0 (actually, meet semidistributive, but we will not use this intermediate notion here). More generally, let us say that a join-semilattice with 0 is 2-*semidistributive at* 0 if, whenever the meets of p, r and of q, r both exist and are equal to 0, the meet of $p + q, r$ exists and is equal to 0.

Lemma 3.5. *Suppose that* $\mathbf{S}$ *is a join-semilattice with* 0 *and with a normal additive pre-closure* K*. If* $\mathbf{S}$ *is* 2*-semidistributive at* 0*, then the elementary contact associated to* K *is additive.*

Proof. If $a \not\delta_K b$ and $a \not\delta_K c$ then the meets of Ka, Kb and of Ka, Kc exist and are equal to 0, by the definition of δ_K. By 2-semidistributivity at 0, the meet of Ka and $Kb + Kc$ exists and is equal to 0. Since K is additive, $Kb + Kc = K(b + c)$ and again the definition of δ_K gives $a \not\delta_K (b + c)$. □

Note that a generalization of Theorem 3.4, as it stands, is not possible for weak pre-contact relations, since the overlap relation is necessarily symmetrical. It is an open problem to characterize weak pre-contact join semilattices satisfying Condition 2 in 3.4, where "contact" is replaced by "pre-contact". As far as Clauses (5), (6) are concerned, note that, given a poset $\mathbf{P}$ with a pre-closure operation, we obtain a weak pre-contact relation δ by setting $a \ \delta \ b$ if there is $p \in P$, $p > 0$ such that $p \leq a$ and $p \leq Kb$. Again, it is an open problem to characterize those weak pre-contact join semilattices which are representable in such a way, possibly, when $\mathbf{P}$ is a distributive lattice and K is an additive closure. Note also that the construction in Claim 3.2 does not necessarily preserve additivity (when dealing with pre-contact relations, *additivity* means that both (Add) and its symmetric version hold). Indeed, on a finite Boolean algebra, an additive pre-contact is determined by those atoms which are in pre-contact. Thus, if we consider the 8-element Boolean algebra $\mathbf{B}_8$ with atoms a, b and c, the conditions $a \ \delta \ b$ and $c \ \delta \ a$ uniquely determine an additive pre-contact on B_8. However, if δ^s is defined as in Claim 3.2, then δ^s is not additive. Indeed, $a \ \delta^s \ b+c$, but neither $a \ \delta^s \ b$, nor $a \ \delta^s \ c$.

As we pointed out in [10], we do not need the axiom of choice in order to prove the equivalences of (1) - (3) in Theorems 3.1 and 3.4. However, in both proofs we needed the Stone Representation Theorem, which is equivalent to the Prime Ideal Theorem [8, Form 14], in order to get the equivalence with (4). In the next proposition we point out that in both cases the equivalence of (1) and (4) is indeed equivalent to the Prime Ideal Theorem.

Proposition 3.6. *In ZF, the Zermelo-Fraenkel theory without the axiom of choice, the following statements are equivalent.*

(A) The Prime Ideal Theorem [8, Form 14].
(B) The implication (1) $\Rightarrow$ *(4) in Theorem 3.1 holds.*
(C) The implication (1) $\Rightarrow$ *(4) in Theorem 3.4 holds.*

Proof. We needed only the Prime Ideal Theorem in the proofs of (1) $\Rightarrow$ (4) in [10, Theorems 3.2] and [10, Theorems 4.1], improved here in Theorems 3.4 and 3.1, respectively. Hence (A) implies both (B) and (C).

Suppose that (B) holds and let $\mathbf{C}$ be a Boolean algebra. Endow $\mathbf{C}$ with the overlap contact. If the implication (1) $\Rightarrow$ (4) in Theorem 3.1 holds, then $\mathbf{C}$ can be $\{\delta, +\}$-embedded into some weak contact complete atomic Boolean algebra $\mathbf{D}$. We check that this embedding, call it χ, is also a Boolean embedding. Indeed, if $c \in C$ and c' is the complement of c, then $c \not\delta_C c'$, since δ_C is overlap, hence $\chi(c) \not\delta_D \chi(c')$, since χ is a δ embedding. By reflexivity of δ_D and (Ext), $\chi(c)\chi(c') = 0$; moreover, $\chi(c) + \chi(c') = 1$, since χ is a join-semilattice homomorphism. Hence $\chi(c')$ is the complement of $\chi(c)$ in $\mathbf{D}$, that is, χ preserves complementation. By De Morgan law, meet is expressible in terms of join and complementation, hence χ is a Boolean homomorphism. Taking the Boolean reduct of $\mathbf{D}$, we get an embedding of $\mathbf{C}$ into a complete atomic Boolean algebra. We have proved the Stone Representation Theorem, which is equivalent to the Prime Ideal Theorem [8, Form 14], hence (B) $\Rightarrow$ (A) follows.

Now note that $\mathbf{C}$ in the above argument has indeed overlap contact by construction, hence the argument provides also (C) $\Rightarrow$ (A). ☐

4. Ivanova contact join-semilattices are not finitely axiomatizable

Theorem 4.1. *For every $n \geq 2$, there is a contact join-semilattice satisfying* (D1+) *and* (D2$_m$)*, for every $m < n$, but not satisfying* (D2$_n$)*.*

Proof. Fix $n \geq 2$ and let $\mathbf{C}$ be the Boolean algebra freely generated by n generators $c_{1,0}, c_{2,0}, \dots, c_{n,0}$ and let $c_{1,1} = c'_{1,0}, \ \dots, \ c_{n,1} = c'_{n,0}$. Recall that a prime denotes Boolean complement. By construction, the elements $c_{1,0}, c_{2,0}, \dots, c_{n,0}, c_{1,1}, c_{2,1}, \dots, c_{n,1}$ are pairwise incomparable with respect to the standard Boolean order. Let

$$\bar{a} = \prod \{ \, c_{1,f(1)} + \dots + c_{n,f(n)} \mid f : \{1, \dots, n\} \to \{0,1\}, f(1) + \dots + f(n) \text{ is odd} \, \},$$

$$\bar{b} = \prod \{ \, c_{1,f(1)} + \dots + c_{n,f(n)} \mid f : \{1, \dots, n\} \to \{0,1\}, f(1) + \dots + f(n) \text{ is even} \, \}.$$

Claim. *$\bar{b}$ is the complement of $\bar{a}$ in $\mathbf{C}$, hence, if n is odd*

$$\bar{a} = \sum \{ \, c_{1,f(1)} c_{2,f(2)} \dots c_{n,f(n)} \mid f : \{1, \dots, n\} \to \{0,1\}, f(1) + \dots + f(n) \textit{ is odd} \, \},$$

$$\bar{b} = \sum \{ \, c_{1,f(1)} c_{2,f(2)} \dots c_{n,f(n)} \mid f : \{1, \dots, n\} \to \{0,1\}, f(1) + \dots + f(n) \textit{ is even} \, \},$$

and if n is even the above two displayed identities hold with $\bar{a}$ and $\bar{b}$ swapped.

To prove the first statement in the claim, we have to show that $\bar{a}\bar{b} = 0$ and $\bar{a} + \bar{b} = 1$. Indeed, $\bar{a}\bar{b}$ is the product of all the sums of the form $c_{1,f(1)} + \dots + c_{n,f(n)}$, with $f : \{1, \dots, n\} \to \{0,1\}$, thus, by distributivity

$$\begin{aligned} 0 &= c_{1,0}c_{1,1} + c_{2,0}c_{2,1} + \dots + c_{n,0}c_{n,1} \\ &= (c_{1,0} + c_{2,0}c_{2,1} + \dots + c_{n,0}c_{n,1})(c_{1,1} + c_{2,0}c_{2,1} + \dots + c_{n,0}c_{n,1}) \\ &= (c_{1,0} + c_{2,0} + c_{3,0}c_{3,1} + \dots + c_{n,0}c_{n,1})(c_{1,0} + c_{2,1} + c_{3,0}c_{3,1} + \dots + c_{n,0}c_{n,1}) \\ &\quad\ (c_{1,1} + c_{2,0} + c_{3,0}c_{3,1} + \dots + c_{n,0}c_{n,1})(c_{1,1} + c_{2,1} + c_{3,0}c_{3,1} + \dots + c_{n,0}c_{n,1}) = \dots \\ &= \prod \{ \, c_{1,f(1)} + c_{2,f(2)} + \dots + c_{n,f(n)} \mid f : \{1, \dots, n\} \to \{0,1\} \, \} = \bar{a}\bar{b}. \end{aligned}$$

Moreover, the sum of each factor in the formula defining $\bar{b}$ with each factor in the formula defining $\bar{a}$ gives 1, hence $\bar{a} + \bar{b} = 1$, again by distributivity.

The displayed formulas in the claim then follow by De Morgan's laws. For example, by what we have just proved, $\bar{a} = \bar{b}' = \left(\prod_f c_{1,f(1)} + \dots + c_{n,f(n)} \right)' = \sum_f c'_{1,f(1)} \dots c'_{n,f(n)} = \sum_f c_{1,1-f(1)} \dots c_{n,1-f(n)}$, where f varies among those function $f : \{1, \dots, n\} \to \{0,1\}$ such that $f(1) + \dots + f(n)$ is even. If n is odd, then $f(1) + \dots + f(n)$ is even if and only if $(1 - f(1)) + \dots + (1 - f(n))$ is odd, hence we get the alternative expression for $\bar{a}$. The other cases are treated in a similar way.

Having completed the proof of the claim, we now notice that $\bar{a}$ and $c_{1,0}$ are incomparable, since $c_{1,1} + c_{2,0} + c_{3,0} + \dots + c_{n,0}$ is larger than $\bar{a}$ but not larger than $c_{1,0}$; moreover, $c_{1,1}c_{2,1}c_{3,1} \dots c_{n,1}$ is smaller than $\bar{a}$ but not smaller than $c_{1,0}$.

Similarly, since $n \geq 2$, both $\bar{a}$ and $\bar{b}$ are incomparable with each one of the elements $c_{1,0}, c_{2,0}, \dots, c_{n,0}, c_{1,1}, c_{2,1}, \dots, c_{n,1}$.

Let **S** be the subsemilattice of the (join-semilattice reduct) of **C** generated by the elements 0 and $c_{1,0}, \dots, c_{n,0}, c_{1,1}, \dots, c_{n,1}, \bar{b}, \bar{a}$. In particular, every nonzero element of **S** is $\geq$ than at least one of the elements in the above list. Since we have showed that all the elements in the list are pairwise incomparable, they are distinct atoms of **S**. By construction, the join operation on **S** is the restriction of the join operation on **C**; in particular, the order relation on **S** is the restriction of the order relation on **C**, hence there is no notational issue. In what follows we will frequently use meet and complementation in **C**, however, these are used only in order to get conclusions speaking just of $+$ and $\leq$, hence such conclusions hold in **S**, as well.

In **S** set $c_{1,0} \not\delta\, c_{1,1}$, $c_{1,1} \not\delta\, c_{1,0}$, $c_{2,0} \not\delta\, c_{2,1}$, $\dots$, $c_{n,0} \not\delta\, c_{n,1}$, $c_{n,1} \not\delta\, c_{n,0}$ and let all the other pairs of $\mathbf{S} \setminus \{0\}$ be δ-related; in particular, $\bar{b}\ \delta\ \bar{a}$, since $\bar{b}$ and $\bar{a}$ are both distinct from $c_{1,0}, \dots, c_{n,1}$. Since $c_{1,0}, c_{1,1}, \dots, c_{n,1}$ are distinct atoms of **S**, **S** is a contact join-semilattice. Then the property (D2_n) fails in **S**, by taking $a = \bar{a}$ and $b = \bar{b}$.

Since the inclusion is a join-semilattice embedding from **S** to **C**, we can apply Lemma 2.2 in order to endow **C** with a weak contact relation. Note that the assumptions from Lemma 2.2(ii) are satisfied, since the pairs which are not δ-related in **S** are mutual complements in **C**. Hence the inclusion is also a contact embedding. Thus clause (1) from Theorem 3.1 is satisfied, hence, by the equivalence of (1) and (3) in Theorem 3.1, **S** satisfies (D1), and also (D1+) by [10, Lemma 2.3].

It remains to check that (D2_m) holds in **S**, for every $m < n$. Clearly, in (D2_n) one may assume that each pair $(c_{i,0}, c_{i,1})$ never repeats, since from repeating pairs we only get repeated (or additional) summands in the relevant sums. Moreover, without loss of generality, we can assume that no $c_{i,j}$ is equal to 0. Indeed, assume that, say, $c_{n,0} = 0$. Then the assumptions of (D2_n) are satisfied if and only if the assumptions of (D2_{n-1}) are satisfied, by discarding the pair $c_{n,0}$, $c_{n,1}$.

Since in $\mathbf{S}\setminus\{0\}$ the only δ-unrelated pairs are $(c_{1,0}, c_{1,1})$, $(c_{2,0}, c_{2,1})$, $\dots$, $(c_{n,0}, c_{n,1})$, by symmetry, without loss of generality we can assume that the elements in the premise of (D2_m) are actually the elements $(c_{1,0}, c_{1,1})$, $\dots$, $(c_{m,0}, c_{m,1})$ introduced in the definition of **S**. In other words, we may assume that the notation does not clash, by taking here, of course, m in place of n in (D2_n). So let us assume that, in the above situation, a and b are elements of **S** satisfying the premises of (D2_m).

Using the assumptions in (D2_m), we get $ab \leq \prod\{\, c_{1,f(1)} + \dots + c_{m,f(m)} \mid f : \{1, \dots, m\} \to \{0,1\}\,\}$, hence, arguing as in the proof of $\bar{a}\bar{b} = 0$, we get $ab = 0$ in **C**, hence $b \leq a'$. If $c_{1,0} \leq a$, then $b \leq a' \leq c'_{1,0} = c_{1,1}$. Moreover, if $c_{1,0} < a$, then $a' < c'_{1,0}$, thus $b < c_{1,1}$, but this is impossible if $b \in S$, unless $b = 0$. In conclusion, if $a, b \in S$ and $c_{1,0} \leq a$, then either $b = 0$, or both $a = c_{1,0}$ and $b = c_{1,1}$. Then the conclusion of (D2_m), that is, $b \not\delta\, a$, holds in both cases. The same argument applies if a is $\geq$ than one among the elements $c_{1,1}, c_{2,0}, c_{2,1} \dots, c_{n,1}$.

Since every nonzero element of **S** is $\geq$ than at least one of the elements $c_{1,0}$, $\dots, c_{n,1}, \bar{b}, \bar{a}$, it remains to treat the cases $a \geq \bar{a}$ and $a \geq \bar{b}$. We first claim that $c_{1,f(1)} + \dots + c_{m,f(m)} \geq \bar{a}$, for no $f : \{1, \dots, m\} \to \{0,1\}$. Indeed, for each fixed $f : \{1, \dots, m\} \to \{0,1\}$, let $g : \{1, \dots, n\} \to \{0,1\}$ be any function such that $g(i) \neq f(i)$, for $i = 1, \dots, m$. Then the meet of $c_{1,f(1)} + \dots + c_{m,f(m)}$ and $c_{1,g(1)} c_{2,g(2)} \dots c_{n,g(n)}$ is 0. On the other hand, by the Claim above, and choosing $g(n)$ appropriately (this can be done, since $n > m$), we have $c_{1,g(1)} c_{2,g(2)} \dots c_{n,g(n)} \leq$

$\bar{a}$. Since we are working in the Boolean algebra freely generated by $c_{1,0},\dots,c_{n,0}$ and the $c_{i,1}$'s are their complements, then $c_{1,g(1)}c_{2,g(2)}\dots c_{n,g(n)} > 0$. This shows that $c_{1,f(1)}+\dots+c_{m,f(m)}$ is not $\geq \bar{a}$.

In the assumptions of (D2$_m$), in the nontrivial cases, we have that $a \leq c_{1,f(1)} + \dots + c_{m,f(m)}$, for at least one function $f : \{1,\dots,m\} \to \{0,1\}$. Since we have showed that $\bar{a} \not\leq c_{1,f(1)}+\dots+c_{m,f(m)}$, it is not the case that $\bar{a} \leq a$. Similarly, $\bar{b} \not\leq a$. Hence the only remaining case to be treated is the trivial situation when $b \leq c_{1,f(1)}+\dots+c_{m,f(m)}$, for every function $f : \{1,\dots,m\} \to \{0,1\}$. In this case $b = 0$, hence the conclusion of (D2$_m$) follows. □

The argument in the proof of Theorem 4.1 is very similar in spirit to [10, Example 5.2(d)]. The main simplification here is obtained by constructing **S** directly as a join-subsemilattice of some Boolean algebra, thus Theorem 3.1 can be invoked in order to get (D1), with no need of complicated computations. Notice that, while **S** in the proof is comparatively well-behaved, the weak contact on **C** is not even additive. Indeed, $\bar{a}$ and $\bar{b}$ are in contact, but they are sums of distinct atoms of **C**, by the Claim, and such atoms are not in contact pairwise. Of course, this argument can be carried out in **C** but not in **S**.

Having proved Theorem 4.1, the non-existence of finite axiomatizability of the class of Ivanova contact join semilattices is now an immediate consequence of the compactness theorem

Corollary 4.2. *The class of Ivanova contact join-semilattices, namely, the class of contact join-semilattices satisfying* (D1) *and* (D2)*, is not first-order finitely axiomatizable.*

Proof. Conditions (D1) and (D2$_n$) can be expressed as first-order sentences, say, φ and φ_n, thus the theory $T = \{\varphi\} \cup \{\, \varphi_n \mid n \in \mathbb{N}^+ \,\}$ axiomatizes the class of Ivanova contact join-semilattices. Assume towards a contradiction that the class is finitely axiomatizable. So, there is a sentence ψ axiomatizing it. By the completeness theorem, ψ is a consequence of T; then, by the compactness theorem, ψ is also a consequence of some finite subset of T, say, ψ is a consequence of $\{\varphi\} \cup \{\, \varphi_n \mid n \leq \bar{n} \,\}$, for some $\bar{n}$. But all the sentences of T are consequences of ψ, since ψ axiomatizes the same class, hence all the sentences of T are consequences of $\{\varphi\} \cup \{\, \varphi_n \mid n \leq \bar{n} \,\}$. This contradicts Theorem 4.1. □

An anonymous referee suggested that Corollary 4.2 can be strengthened to non-axiomatizability with finitely many variables. This is indeed the case, and automatically follows from the fact that semilattices, hence also weak contact join-semilattices, are locally finite. Just observe that, modulo the axioms of a locally finite theory in a finite language, for every n, up to logical equivalence, there are only a finite number of sentences containing at most n variables. Thus, for a locally finite theory in a finite language, finite axiomatizability is the same as being axiomatized using only finitely many variables.

5. Further remarks

Proposition 5.1. *If* **S** *is a join-semilattice with overlap weak contact and* **S** *satisfies (D1), then* **S** *satisfies (D2), in particular,* **S** *is additive, by* [10, Remark 2.2].

Proof. Let n be an arbitrary positive integer and let a^* and b^* be two elements of S satisfying the antecedent of (D2$_n$), namely, for any $f : \{1,\dots,n\} \to \{0,1\}$, either the former or the latter is $\leq c_{1,f(1)} + \cdots + c_{n,f(n)}$. Thus, if some $d \in S$ is below both a^* and b^*, then, for every $f : \{1, \ldots, n\} \to \{0, 1\}$, $d \leq c_{1,f(1)} + \cdots + c_{n,f(n)}$. Since by [10, Lemma 2.3] **S** satisfies (D1+), we see that $d \leq 0$ by putting 0 for a and d for b in (D1+). Thus, by the arbitrariness of d, the element 0 is the only lower bound of a^* and b^*, hence it is the meet of a^* and b^*. Since the contact is overlap by assumption, we get $a^* \not\delta\, b^*$. □

Corollary 5.2. *The class of those Ivanova contact join-semilattices which have overlap contact is finitely axiomatizable.*

Proof. By Theorem 2.1 and Proposition 5.1, a weak contact join-semilattice **S** with overlap contact is Ivanova if and only if **S** satisfies (D1). Then notice that both (D1) and "having overlap contact" are properties expressible by a first-order sentence. □

Remark 5.3. We present a final remark about the classes of contact join-semilattices discussed in this note. We have showed that the class of Ivanova contact Join-semilattices is not finitely axiomatizable, while the larger class of weak contact join-semilattices satisfying the conditions in Theorem 3.1 is indeed finitely axiomatizable. This result might suggest the idea that the latter class is more natural, and this is surely a reasonable point of view.

However, the situation may be seen from a different perspective. As we hinted in the introduction, n-ary "hypercontact" relations are much more general than binary relations [14]. In [12] we study classes of join-semilattices with a hypercontact relation, parallel to the classes considered here. The main difference is that in the hypercontact case finite axiomatizability never occurs. Again, this might be an argument in favor of considering binary relations only. On the other hand, one might draw the conclusion that considering just binary relations is an oversimplification. Three regions might be pairwise in contact without being in contact.

We are not taking position in favor of one alternative or the other; rather, we believe that both kinds of structures are interesting for their own sake and each one has its own specific uses and applications. On the other hand, it is surely a significant fact that, in the parallel evolution of the notion of *event structure* in computer science, a decided shift occurred from binary relations [15, Section 8] to n-ary relations [16, Subsec. 2.1.2]. In an even more general applied setting, the advantage of considering n-ary interactions in place of just binary interactions (e. g., hypergraphs instead of graphs) is analyzed in [2]. See the introduction of [12] for further comments and examples.

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