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A note on some moduli spaces of Ulrich bundles

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Abstract

We prove that the modular component $\mathcal{M}(r)$, constructed in the Main Theorem in Fania and Flamini (Adv Math 436:109409, 2024. https://doi.org/10.1016/j.aim.2023.109409), of Ulrich vector bundles of rank r and given Chern classes, on suitable threefold scrolls X_e over Hirzebruch surfaces $\mathbb{F}_{e\geq 0}$, which arise as tautological embeddings of projectivization of very-ample vector bundles on $\mathbb{F}_e$, is generically smooth, irreducible and unirational. A stronger result holds for the suitable associated moduli space $\mathcal{M}_{\mathbb{F}_e}(r)$ of vector bundles of rank r and given Chern classes on $\mathbb{F}_e$, Ulrich w.r.t. the very ample polarization $c_1(\mathcal{E}_e) = \mathcal{O}_{\mathbb{F}_e}(3, b_e)$, which turns out to be generically smooth, irreducible and unirational.

Keywords Ulrich bundles · Threefolds · Ruled surfaces · Moduli · Deformations

Mathematics Subject Classification Primary 14J30 · 14J26 · 14J60 · 14C05; Secondary 14N30

Introduction

Let X be a smooth irreducible projective variety of dimension $n \geq 1$, polarized by a very ample divisor H on X. The existence of vector bundles $\mathcal{U}$ on X which are *Ulrich with respect to* $\mathcal{O}_X(H)$ has interested various authors.

For some specific classes of varieties such problem has being attacked, see for instance [1, 2, 9–11, 13]. Whenever such bundles do exist, since they are always *semistable* (in the sense of Gieseker-Maruyama, cf. also § 1 below) and also *slope-semistable* (cf. [6, Def. 2.7, Thm. 2.9-(a)]), one is interested in knowing if these bundles are also *stable*, equivalently *slope-stable* (cf. [6, Def. 2.7, Thm. 2.9-(c)]). Furthermore, from their semi-stability, such rank-r vector bundles give rise to points in a moduli space, say $M := M^{ss}(r; c_1, c_2, \ldots, c_k)$, where

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$k := \min\{r, n\}$, parametrizing (S-equivalence classes of) semistable sheaves of given rank r and given Chern classes c_i on X, $1 \leq i \leq k$ (cf. [6, p. 1250083-9]). Therefore, one is also interested e.g. in understanding: whether M contains at least an irreducible component, say $\mathcal{M}(r)$, which is generically smooth, i.e. reduced, or even smooth; to which sheaf on X corresponds the general point of such a component $\mathcal{M}(r)$; what can be said about the *birational geometry* of $\mathcal{M}(r)$, namely if it is perhaps rational/unirational; finally, if by chance M turns out to be also irreducible, that is, $M = \mathcal{M}(r)$.

In this paper we are interested in some of the aforementioned properties for the moduli spaces of Ulrich vector bundles on a variety X_e which is a 3-fold scroll over a Hirzebruch surface $\mathbb{F}_e$, with $e \geq 0$. More precisely on 3-fold scrolls X_e arising as embedding, via very-ample tautological line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$, of projective bundles $\mathbb{P}(\mathcal{E}_e)$ over $\mathbb{F}_e$, where $\mathcal{E}_e$ are very-ample rank-2 vector bundles on $\mathbb{F}_e$ with Chern classes $c_1(\mathcal{E}_e)$ numerically equivalent to $3C_e + b_e f$ and $c_2(\mathcal{E}_e) = k_e$, where C_e and f are, as customary, generators of $\mathrm{Num}(\mathbb{F}_e)$ as in [14, V, Prop. 2.3] and where b_e and k_e are integers satisfying some natural numerical conditions. We will set $\xi := \mathcal{O}_{X_e}(1)$ the hyperplane line bundle of the embedded 3-fold scroll, which we will also call *tautological polarization of* X_e, as $(X_e, \xi) \cong (\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1))$.

The existence of Ulrich bundles on such threefolds X_e has been considered in [13], where it was proved that X_e does not support any Ulrich line bundle w.r.t. ξ, unless $e = 0$. As to Ulrich vector bundles of rank $r \geq 2$, it was proved in [13] that the moduli space M, in the above sense, arising from rank-r vector bundles $\mathcal{U}_r$ on $X_{e\geq 0}$ which are Ulrich w.r.t. ξ and with first Chern class

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(3, b_e - 3) + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_e - e - 2)\right), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r}{2}, \frac{r}{2}(b_e - e - 2)\right), & \text{if } r \text{ is even.} \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ has been proved to correspond to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 8b_e - k_e - 12e - 3$ (see Theorem 2.5 below, for more details).

As a consequence of such result and a natural one-to-one correspondence among rank-r vector bundles on X_e, of the form $\xi \otimes \varphi^*(\mathcal{F})$, which are Ulrich w.r.t. ξ on X_e, and rank-r vector bundles on $\mathbb{F}_e$, of the form $\mathcal{F}(c_1(\mathcal{E}_e))$, which are Ulrich w.r.t. $c_1(\mathcal{E}_e) = 3C_e + b_e f$, in [13] we have deduced Ulrichness results for vector bundles on the base surface $\mathbb{F}_e$ with respect to naturally associated very ample polarization $c_1(\mathcal{E}_e)$, see Theorem 2.6 for more details.

By a result of Antonelli, [1, Theorem 1.2], if $\mathcal{H}_r$ is a rank-r vector bundle on $\mathbb{F}_e$ which is Ulrich with respect to a very ample polarization of the form $\mathcal{O}_{\mathbb{F}_e}(a, b)$ and with $c_1(\mathcal{H}_r) = \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$, then $\mathcal{H}_r$ must fit into a short exact sequence of the form

$$0 \to \mathcal{O}_{\mathbb{F}_e}(a-1, b-e-1)^{\oplus\gamma} \xrightarrow{\phi} \mathcal{O}_{\mathbb{F}_e}(a-1, b-e)^{\oplus\delta} \oplus \mathcal{O}_{\mathbb{F}_e}(a, b-1)^{\oplus\tau} \to \mathcal{H}_r \to 0,$$

where γ, δ and τ are suitably defined by $r, \alpha, \beta, a, b, e$ (cfr. (3.1)). This fact will be useful in the present note to give further information about our modular components $\mathcal{M}(r)$ as in [13]. Our main results in this paper are the following

Theorem A (cf. Theorem 3.2, below) *For any integer $e \geq 0$, let $\mathbb{F}_e$ be the Hirzebruch surface and let $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$ denote the line bundle $\alpha C_e + \beta f$ on $\mathbb{F}_e$, where C_e and f are generators of* $\mathrm{Num}(\mathbb{F}_e)$ (cf. [14, V, Prop. 2.3]). *Let (X_e, ξ) be a 3-fold scroll over $\mathbb{F}_e$ as above, where $\varphi : X_e \to \mathbb{F}_e$ denotes the scroll map. Then the moduli space of rank-$r \geq 2$ vector bundles $\mathcal{U}_r$ on X_e which are Ulrich w.r.t. ξ and with first Chern class*

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(3, b_e - 3) + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_e - e - 2)\right), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r}{2}, \frac{r}{2}(b_e - e - 2)\right), & \text{if } r \text{ is even} \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$, which is of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, & \text{if } r \text{ is even,} \end{cases}$$

(see Theorem 2.5) and which is moreover unirational.

For the moduli space of rank-$r \geq 2$ bundles on $\mathbb{F}_e$, the base of the scroll X_e, which are Ulrich w.r.t. the polarization $c_1(\mathcal{E}_e) = \mathcal{O}_{\mathbb{F}_e}(3, b_e)$, a stronger result holds; precisely

Theorem B (cf. Theorem 3.1, below) *Let $\mathcal{M}_{\mathbb{F}_e}(r)$ be the moduli space of rank-r vector bundles $\mathcal{H}_r$ on $\mathbb{F}_e$ which are Ulrich w.r.t. $c_1(\mathcal{E}_e)$ and with first Chern class*

$$c_1(\mathcal{H}_r) = \begin{cases} \mathcal{O}_{\mathbb{F}_e}(3(r+1), (r+1)b_e - 3) \otimes \mathcal{O}_{\mathbb{F}_e}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_e - e - 2)\right), & \text{if } r \text{ is odd,} \\ \mathcal{O}_{\mathbb{F}_e}(3r, rb_e) \otimes \mathcal{O}_{\mathbb{F}_e}\left(\frac{r}{2}, \frac{r}{2}(b_e - e - 2)\right), & \text{if } r \text{ is even.} \end{cases}$$

Then $\mathcal{M}_{\mathbb{F}_e}(r)$ is generically smooth, of dimension

$$\dim(\mathcal{M}_{\mathbb{F}_e}(r)) = \begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, & \text{if } r \text{ is even,} \end{cases}$$

(see Theorem 2.6) and moreover it is irreducible *and* unirational.

The above theorems extend unirationality results in [1] and [9].

The paper is structured as follows. In Sect. 1 we fix notation and terminology. In Sect. 2 we recall some of the known results that we will use throughout the paper. In Sect. 3 we state and prove our new main results.

1 Notation and terminology

In this paper we work over $\mathbb{C}$. All schemes will be endowed with the Zariski topology. We will interchangeably use the terms rank-r vector bundle on a smooth, projective variety X and rank-r locally free sheaf. In particular, sometimes, to ease some formulas, with a small abuse of notation we identify divisor classes with the corresponding line bundles, interchangeably using additive and tensor-product notation. The dual bundle of a rank-r vector bundle $\mathcal{F}$ on X will be denoted by $\mathcal{F}^\vee$; thus, if L is of rank-1, i.e. it is a line bundle, we interchageably use $L^\vee$ or $-L$. If M is a *moduli space*, parametrizing objects modulo a given equivalence relation, and if Y is a representative of an equivalence class in M, we will denote by $[Y] \in M$ the point corresponding to Y. For non-reminded general terminology, we refer the reader to [14]).

Because our object will be Ulrich bundles, we recall their definition and basic properties.

Definition 1.1 Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, projective variety of dimension n and let H be a hyperplane section of X. A vector bundle $\mathcal{U}$ on X is said to be *Ulrich* with respect to $\mathcal{O}_X(H)$ if

$$H^i(X, \mathcal{U}(-jH)) = 0 \quad \text{for} \quad i = 0, \cdots, n \quad \text{and} \quad 1 \leq j \leq n.$$

Definition 1.2 Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, projective variety of dimension n polarized by $\mathcal{O}_X(H)$, where H is a hyperplane section of X, and let $\mathcal{U}$ be a rank-2 vector bundle on X which is *Ulrich* with respect to $\mathcal{O}_X(H)$. Then $\mathcal{U}$ is said to be *special* if $c_1(\mathcal{U}) = K_X + (n+1)H$.

For the reader's convenience, we briefly remind facts concerning (semi)stability and slope-(semi)stability properties of Ulrich bundles as in [6, Def. 2.7]. Let X be a smooth, irreducible, projective variety and let $\mathcal{F}$ be a vector bundle on X; recall that $\mathcal{F}$ is said to be *semistable* (in the sense of Gieseker-Maruyama) if for every non-zero coherent subsheaf $\mathcal{G} \subset \mathcal{F}$, with $0 < \mathrm{rk}(\mathcal{G}) :=$ rank of $\mathcal{G} < \mathrm{rk}(\mathcal{F})$, the inequality $\frac{P_{\mathcal{G}}}{\mathrm{rk}(\mathcal{G})} \leq \frac{P_{\mathcal{F}}}{\mathrm{rk}(\mathcal{F})}$ holds true, where $P_{\mathcal{G}}$ and $P_{\mathcal{F}}$ are the *Hilbert polynomials* of the sheaves. Furthermore, $\mathcal{F}$ is *stable* if strict inequality above holds. Similarly, recall that the *slope* of a vector bundle $\mathcal{F}$ (w.r.t. a given polarization $\mathcal{O}_X(H)$ on X) is defined to be $\mu(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\mathrm{rk}(\mathcal{F})}$; the bundle $\mathcal{F}$ is said to be *μ-semistable*, or even *slope-semistable*, if for every non-zero coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$, one has $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$. The bundle $\mathcal{F}$ is *μ-stable*, or *slope-stable*, if strict inequality holds.

The two definitions of (semi)stability are in general related as follows (cf. e.g. [6, § 2]):

$$\text{slope-stability} \Rightarrow \text{stability} \Rightarrow \text{semistability} \Rightarrow \text{slope-semistability}.$$

If $\mathcal{U}$ is in particular a rank-r vector bundle which is Urlich w.r.t. $\mathcal{O}_X(H)$, then $\mathcal{U}$ is always semistable, so also slope-semistable (cf. [6, Thm. 2.9-(a)]); moreover, for $\mathcal{U}$ the notions of stability and slope-stability coincide (cf. [6, Thm. 2.9-(c)]).

As for the projective variety which will be the support of Ulrich bundles we are interested in, throughout this work we will denote it by X_e and it will be a 3-dimensional scroll over the Hirzebruch surface $\mathbb{F}_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, with $e \geq 0$ an integer.

More precisely, let $\pi_e : \mathbb{F}_e \to \mathbb{P}^1$ be the natural projection onto the base. Then, as in [14, V, Prop. 2.3], $\mathrm{Num}(\mathbb{F}_e) = \mathbb{Z}[C_e] \oplus \mathbb{Z}[f]$, where:

- $f := \pi_e^*(p)$, for any $p \in \mathbb{P}^1$, whereas
- C_e denotes either the unique section corresponding to the morphism of vector bundles on $\mathbb{P}^1$
  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-e)$, when $e > 0$, or the fiber of the other ruling different from that induced by f, when otherwise $e = 0$.

In particular

$$C_e^2 = -e, \;\; f^2 = 0, \;\; C_e f = 1.$$

Let $\mathcal{E}_e$ be a rank-2 vector bundle over $\mathbb{F}_e$ and let $c_i(\mathcal{E}_e)$ be its i^{th}-Chern class. Then $c_1(\mathcal{E}_e) \equiv aC_e + bf$, for some $a, b \in \mathbb{Z}$, and $c_2(\mathcal{E}_e) \in \mathbb{Z}$. For the line bundle $\mathcal{L} \equiv \alpha C_e + \beta f$ we will also use the notation $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$.

From now on, we will consider the following:

Assumption 1.3 Let $e \geq 0$, b_e, k_e be integers such that

$$b_e - e < k_e < 2b_e - 4e, \tag{1.1}$$

and let $\mathcal{E}_e$ be a rank-2 vector bundle over $\mathbb{F}_e$, with

$$c_1(\mathcal{E}_e) \equiv 3C_e + b_e f \quad \text{and} \quad c_2(\mathcal{E}_e) = k_e,$$

which fits in the exact sequence

$$0 \to A_e \to \mathcal{E}_e \to B_e \to 0, \tag{1.2}$$

where A_e and B_e are line bundles on $\mathbb{F}_e$ such that

$$A_e \equiv 2C_e + (2b_e - k_e - 2e) f \quad \text{and} \quad B_e \equiv C_e + (k_e - b_e + 2e) f \tag{1.3}$$

From (1.2), in particular, one has $c_1(\mathcal{E}_e) = A_e + B_e$ and $c_2(\mathcal{E}_e) = A_e B_e$.

By results in [13], $\mathcal{E}_e$ as above, turns out to be very ample on $\mathbb{F}_e$. Thus we take X_e to be the 3-fold scroll arising as embedding, via very-ample tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$, of the projective bundle $\mathbb{P}(\mathcal{E}_e)$.

2 Preliminaries

In this section, for the reader convenience, we state some of the known results that we will be using in the sequel.

The following Theorem 2.1, (cf. [12, Theorem 2.4]) states under which conditions an Ulrich bundle on the base of the scroll gives rise to a bundle on the scroll itself which is Ulrich w.r.t. the *tautological polarization* ξ.

Theorem 2.1 ([12, Theorem 2.4]) *Let (S, H) be a polarized surface, with H a very ample line bundle, and let $\mathcal{E}$ be a rank-2 vector bundle on S such that $\mathcal{E}$ is (very) ample and spanned. Let $\mathcal{F}$ be a rank-$r \geq 1$ vector bundle on S. Let $(X, \xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ be a 3-fold scroll over S, where ξ is the tautological polarization, and let $X \xrightarrow{\varphi} S$ denote the scroll map. Then the vector bundle $\mathcal{U} := \xi \otimes \varphi^*(\mathcal{F})$ is Ulrich with respect to ξ if and only if the bundle $\mathcal{F}$ is such that*

$$H^i(S, \mathcal{F}) = 0 \text{ and } H^i(S, \mathcal{F}(-c_1(\mathcal{E}))) = 0, \ 0 \leq i \leq 2. \tag{2.1}$$

In particular, if $c_1(\mathcal{E})$ is very ample on S, then the rank-r vector bundle on X, $\mathcal{U} = \xi \otimes \varphi^*(\mathcal{F})$, is Ulrich with respect to ξ if and only if the rank-r vector bundle on S, $\mathcal{F}(c_1(\mathcal{E}))$, is Ulrich with respect to $c_1(\mathcal{E})$.

Viceversa, starting with a rank-r vector bundle on the 3-fold scroll (X, ξ) which is Ulrich w.r.t. ξ, satisfying suitable properties, we recall how to obtain an Ulrich vector bundle of the same rank on the base S of the scroll.

Let $\varphi : X \to S$ be a 3-fold scroll over a surface S. Let us recall, see [5, Theorem 11.1.2.], that a general hyperplane section $\widetilde{S}$ of X has the structure of a blow-up of the base surface S at $c_2(\mathcal{E})$ points and one can consider the following diagram:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{i} & X \\ & \searrow^{\varphi'} & \downarrow^{\varphi} \\ & & S, \end{array} \tag{2.2}$$

where i is the inclusion and φ' is the blow-up map, where we denote by E_i the exceptional divisors of the latter map. More precisely, if $\widetilde{S} \in |\xi|$ is a general hyperplane section of X, then it corresponds to the vanishing locus of a general global section $\widetilde{\sigma} \in H^0(X,\xi)$; since one has $H^0(X,\xi) \cong H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong H^0(S,\mathcal{E})$, then $\widetilde{\sigma}$ bijectively corresponds to a global section σ of $\mathcal{E}$ whose vanishing locus $Z := V(\sigma)$ is a zero-dimensional subscheme on S which is an element of $c_2(\mathcal{E})$. From [5, Theorem 11.1.2.], $\widetilde{S}$ turns out to be isomorphic to the blow-up of $\varphi' : \widetilde{S} \to S$ at such points Z and, for any $z \in Z$, the φ-fiber $\varphi^{-1}(z) := F_z$ of X is contained in $\widetilde{S}$ as the φ'-exceptional divisor E_z over the point z of such a blow-up φ'.

With this set-up, in [12, Thm. 6.1, Prop. 6.2], the authors gave conditions to get bijective correspondences among rank-r bundles on X which are Ulrich w.r.t. the tautological polarization ξ and rank-r bundles on the base surface S which are Ulrich w.r.t. the naturally related polarization as in Theorem 2.1.

Theorem 2.2 ([12, Theorem 6.1]) *Let $\varphi : X \to S$ be a 3-fold scroll over a surface S and let $\mathcal{G}$ be a rank-r vector bundle on X which is Ulrich with respect to the tautological polarization ξ, i.e. $(X,\xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Let us suppose that $c_1(\mathcal{E})$ is very ample on S. Assume that on the general fiber $F = \varphi^{-1}(s)$, $s \in S$, the vector bundle $\mathcal{G}$ splits as follows: $\mathcal{G}_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$. Then $\varphi_*(\mathcal{G} \otimes i_*(\mathcal{O}_{\widetilde{S}}(\sum_{i=1}^k E_i))$, with $k = |c_2(\mathcal{E})|$, is a rank-r vector bundle on S which is Ulrich w.r.t. $c_1(\mathcal{E})$.*

In the following remark we comment on the hypotheses of Theorem 2.2, in order to better explain the aforementioned Ulrich-bundle bijective correspondence arising from Theorems 2.1 and 2.2 (cf. Proposition 2.4 below).

Remark 2.3 We like to point out that the assumption on the splitting-type of the vector bundle $\mathcal{G}$ on the general fiber F of φ as $\mathcal{G}_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ as in Theorem 2.2 implies that such a splitting-type holds true for all φ-fibers $\varphi^{-1}(u) := F_u$, for u varying in a suitable open dense subset $U \subseteq S$. Thus, from the previous description on the birational structure of a general hyperplane section $\widetilde{S} = V(\widetilde{\sigma})$ of X as in (2.2), the main points to let the Ulrich-bundle bijective correspondence arise are first of all that the zero-dimensional scheme $Z = V(\sigma)$, corresponding to $\widetilde{S} \in |\xi|$ general, is entirely contained in the open set $U \subseteq S$ (so that, for any $z \in Z$, the restriction of $\mathcal{G}$ to $F_z := \varphi^{-1}(z)$ is $\mathcal{G}_{|F_z} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ namely, from (2.2), $\mathcal{G}_{|E_i} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$, for any $1 \le i \le |c_2(\mathcal{E})|$, where $\sum_i E_i$ denotes the total exceptional divisor of the blow-up φ' of S along Z) and then the use of [8, Thm. 4.2].

Arguments described in Remark 2.3 are the principles used in [12] to get the following Proposition.

Proposition 2.4 ([12, Prop. 6.2]) *Let $\varphi : X \to S$ be a 3-fold scroll over a surface S, where $(X,\xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ for some very ample rank-2 vector bundle $\mathcal{E}$ on S. Assume that $c_1(\mathcal{E})$ is very ample on S. Then there exists a bijection:*

$$\left\{ \begin{array}{c} \text{Bundles } \mathcal{F} \text{ of rank } r \text{ on } S \\ \text{which are Ulrich w.r.t. } c_1(\mathcal{E}) \end{array} \right\} \Big/ \cong_{iso} \Leftrightarrow \left\{ \begin{array}{c} \text{Bundles } \mathcal{G} \text{ of rank } r \text{ on } X \\ \text{which are Ulrich w.r.t. } \xi \text{ and such that} \\ \mathcal{G}_{|\varphi^{-1}(s)} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}, \text{ for general } s \in S \end{array} \right\} \Big/ \cong_{iso}$$

the bijection given by the maps

$$\phi : \mathcal{F} \mapsto \mathcal{G} := \xi \otimes \varphi^*(\mathcal{F}(-c_1(\mathcal{E})));$$

and

$$\psi : \mathcal{G} \quad \mapsto \quad \mathcal{F} := \varphi_* \left(\mathcal{G} \otimes i_*(\mathcal{O}_{\widetilde{S}}(\sum_{i=1}^{k} E_i) \right).$$

Because we are interested on moduli spaces of Ulrich bundles on threefolds scrolls X_e over $\mathbb{F}_e$, as well as on moduli spaces of Ulrich bundles on $\mathbb{F}_e$, we recall what was already proved in [13].

Theorem 2.5 ([13, Main Theorem]) *For any integer $e \geq 0$, consider the Hirzebruch surface $\mathbb{F}_e$ and let $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$ denote the line bundle $\alpha C_e + \beta f$ on $\mathbb{F}_e$, where C_e and f are generators of* $\mathrm{Num}(\mathbb{F}_e)$.

Let (X_e, ξ) be a 3-fold scroll over $\mathbb{F}_e$ as in Assumption 1.3, where $\varphi : X_e \to \mathbb{F}_e$ denotes the scroll map. Then:

(a) *X_e does not support any Ulrich line bundle w.r.t. ξ unless $e = 0$. In this latter case, the unique Ulrich line bundles on X_0 are the following:*

(i) *$L_1 := \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, -1)$ and $L_2 := \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, b_0 - 1)$;*

(ii) *for any integer $t \geq 1$, $M_1 := 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, -t - 1)$ and $M_2 := \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t - 1)$, which only occur for $b_0 = 2t$, $k_0 = 3t$.*

(b) *Set $e = 0$ and let $r \geq 2$ be any integer. Then the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X_0 which are Ulrich w.r.t. ξ and with first Chern class*

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, b_0 - 3) + \varphi^*\mathcal{O}_{\mathbb{F}_0}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_0 - 2)\right), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r}{2}, \frac{r}{2}(b_0 - 2)), & \text{if } r \text{ is even.} \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(6b_0 - 4), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_0 - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 8b_0 - k_0 - 3$. If moreover $r = 2$, then $\mathcal{U}_2$ is also special *(cf. Def. 1.2 above).*

(c) *When $e > 0$, let $r \geq 2$ be any integer. Then the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X_e which are Ulrich w.r.t. ξ and with first Chern class*

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(3, b_e - 3) + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_e - e - 2)\right), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}\left(\frac{r}{2}, \frac{r}{2}(b_e - e - 2)\right), & \text{if } r \text{ is even.} \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 8b_e - k_e - 12e - 3$. If moreover $r = 2$, then $\mathcal{U}_2$ is also special.

We want to stress that in [13, Proof of Thm. 5.1] it has been proved that bundles L_1, L_2 and $\mathcal{U}_r$, for any $r \geq 2$, as in Theorem 2.5 split on any φ-fiber of X_e as requested in Theorem 2.2 and in Proposition 2.4, namely for any φ-fiber F, one has $(L_1)_{|F} = (L_2)_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ whereas $(\mathcal{U}_r)_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$ (this is due to the iterative contructions in [13] of such bundles as deformations of iterative extensions). As a direct consequence of Theorem 2.5, Theorem 2.1 and the one–to–one correspondence in Proposition 2.4, in [13] we could prove the following result concerning moduli spaces of rank-r vector bundles on Hirzebruch surfaces $\mathbb{F}_e$, for any $r \geq 1$ and any $e \geq 0$, which are Ulrich w.r.t. the very ample line bundle $c_1(\mathcal{E}_e) = 3C_e + b_e f$, with $b_e \geq 3e + 2$ as it follows from Assumption 1.3 (the case $r = 1, 2, 3$ already known by [1, 2, 7]).

Theorem 2.6 ([13, Theorem 5.1]) *For any integer $e \geq 0$, consider the Hirzebruch surface $\mathbb{F}_e$ and let $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$ denote the line bundle $\alpha C_e + \beta f$ on $\mathbb{F}_e$, where C_e and f are generators of* $\mathrm{Num}(\mathbb{F}_e)$.

Consider the very ample polarization $c_1(\mathcal{E}_e) = \mathcal{O}_{\mathbb{F}_e}(3, b_e)$ on $\mathbb{F}_e$, where $b_e \geq 3e + 2$. Then:

(a) *$\mathbb{F}_e$ does not support any Ulrich line bundle w.r.t. $c_1(\mathcal{E}_e)$ unless $e = 0$. In this latter case, the unique line bundles on $\mathbb{F}_0$ which are Ulrich w.r.t. $c_1(\mathcal{E}_e)$ are*

$$\mathcal{L}_1 := \mathcal{O}_{\mathbb{F}_0}(5, b_0 - 1) \text{ and } \mathcal{L}_2 := \mathcal{O}_{\mathbb{F}_0}(2, 2b_0 - 1).$$

(b) *Set $e = 0$ and let $r \geq 2$ be any integer. Then the moduli space $\mathcal{M}_{\mathbb{F}_0}(r)$ of rank-r vector bundles $\mathcal{H}_r$ on $\mathbb{F}_0$ which are Ulrich w.r.t. $c_1(\mathcal{E}_0)$ and with first Chern class*

$$c_1(\mathcal{H}_r) = \begin{cases} \mathcal{O}_{\mathbb{F}_0}(3(r+1), (r+1)b_0 - 3) \otimes \mathcal{O}_{\mathbb{F}_0}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_0 - 2)\right), & \text{if } r \text{ is odd,} \\ \mathcal{O}_{\mathbb{F}_0}(3r, rb_0) \otimes \mathcal{O}_{\mathbb{F}_0}\left(\frac{r}{2}, \frac{r}{2}(b_0 - 2)\right), & \text{if } r \text{ is even.} \end{cases}$$

is not empty and it contains a generically smooth component of dimension

$$\begin{cases} \frac{(r^2-1)}{4}(6b_0 - 4), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_0 - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

The general point $[\mathcal{H}_r]$ of such a component corresponds to a slope-stable vector bundle.

(c) *When $e > 0$, let $r \geq 2$ be any integer. Then the moduli space $\mathcal{M}_{\mathbb{F}_e}(r)$ of rank-r vector bundles $\mathcal{H}_r$ on $\mathbb{F}_e$ which are Ulrich w.r.t. $c_1(\mathcal{E}_e)$ and with first Chern class*

$$c_1(\mathcal{H}_r) = \begin{cases} \mathcal{O}_{\mathbb{F}_e}(3(r+1), (r+1)b_e - 3) \otimes \mathcal{O}_{\mathbb{F}_e}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_e - e - 2)\right), & \text{if } r \text{ is odd,} \\ \mathcal{O}_{\mathbb{F}_e}(3r, rb_e) \otimes \mathcal{O}_{\mathbb{F}_e}\left(\frac{r}{2}, \frac{r}{2}(b_e - e - 2)\right), & \text{if } r \text{ is even.} \end{cases}$$

is not empty and it contains a generically smooth component of dimension

$$\begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

The general point $[\mathcal{H}_r]$ of such a component corresponds to a slope-stable vector bundle.

3 Moduli spaces

Our aim in this section is to prove that the moduli space $\mathcal{M}_{\mathbb{F}_e}(r)$ of Ulrich bundles on $\mathbb{F}_e$, $e \geq 0$, as in Theorem 2.6 is irreducible, generically smooth and unirational, whereas that the generically smooth modular component $\mathcal{M}(r)$ of Ulrich bundles on X_e, $e \geq 0$, as in Theorem 2.5 is unirational.

Theorem 3.1 *Let $\mathcal{M}_{\mathbb{F}_e}(r)$ be the moduli space of rank-r vector bundles $\mathcal{H}_r$ on $\mathbb{F}_e$ which are Ulrich w.r.t. $c_1(\mathcal{E}_e) = \mathcal{O}_{\mathbb{F}_e}(3, b_e)$ and with first Chern class*

$$c_1(\mathcal{H}_r) = \begin{cases} \mathcal{O}_{\mathbb{F}_e}(3(r+1), (r+1)b_e - 3) \otimes \mathcal{O}_{\mathbb{F}_e}\left(\frac{r-3}{2}, \frac{(r-3)}{2}(b_e - e - 2)\right), & \text{if } r \text{ is odd,} \\ \mathcal{O}_{\mathbb{F}_e}(3r, rb_e) \otimes \mathcal{O}_{\mathbb{F}_e}\left(\frac{r}{2}, \frac{r}{2}(b_e - e - 2)\right), & \text{if } r \text{ is even,} \end{cases}$$

(see Theorem 2.6). Then $\mathcal{M}_{\mathbb{F}_e}(r)$ is generically smooth, irreducible, unirational and of dimension

$$\dim(\mathcal{M}_{\mathbb{F}_e}(r)) = \begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

Proof From Theorem 2.6 we know that the moduli space $\mathcal{M}_{\mathbb{F}_e}(r)$ is not empty.

Let $\mathcal{M} \subseteq \mathcal{M}_{\mathbb{F}_e}(r)$ be any irreducible component and let $[\mathcal{H}_r] \in \mathcal{M}$ be its general point. So $\mathcal{H}_r$ is of rank r and as in the statement of Theorem 3.1.

For simplicity let $c_1(\mathcal{H}_r) = \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$. By [1, Theorem 1.1] $\mathcal{H}_r$ necessarily fits into the following short exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_e}(2, b_e - e - 1)^{\oplus\gamma} \xrightarrow{\phi} \mathcal{O}_{\mathbb{F}_e}(2, b_e - e)^{\oplus\delta} \oplus \mathcal{O}_{\mathbb{F}_e}(3, b_e - 1)^{\oplus\tau} \to \mathcal{H}_r \to 0. \quad (3.1)$$

where $\gamma = \alpha + \beta - r(2 + b_e) - e(\alpha - 3r)$, $\delta = \beta - r(b_e - 1) - e(\alpha - 3r)$, $\tau = \alpha - 2r$ which, after plugging in the value of α and β, become

$\gamma = \frac{(b_e - 2e + 1)r - b_e + 3}{2}$, $\delta = \frac{(r-1)b_e}{2} - er$, $\tau = \frac{3(r+1)}{2}$, if r is odd, and

$\gamma = \frac{(b_e - 2e + 1)r}{2}$, $\delta = \frac{(b_e - 2e)r}{2}$, $\tau = \frac{3r}{2}$, if r is even.

Thus $\mathcal{H}_r$ is expressed as the cokernel of an injective map $\phi \in \mathrm{Hom}_{\mathbb{F}_e}(\mathscr{A}, \mathscr{B})$, where $\mathscr{A} := \mathcal{O}_{\mathbb{F}_e}(2, b_e - e - 1)^{\oplus\gamma}$ and $\mathscr{B} := \mathcal{O}_{\mathbb{F}_e}(2, b_e - e)^{\oplus\delta} \oplus \mathcal{O}_{\mathbb{F}_e}(3, b_e - 1)^{\oplus\tau}$, with γ, δ, τ as above.

On the other hand, by [1, Theorem 1.3], if we take a general map $\phi_{gen} \in \mathrm{Hom}_{\mathbb{F}_e}(\mathscr{A}, \mathscr{B})$ then $\mathrm{coker}(\phi_{gen})$ is a rank-r vector bundle on $\mathbb{F}_e$, in particular locally free, which is Ulrich w.r.t. $c_1(\mathcal{E}_e)$, and with Chern classes $c_1(\mathrm{coker}(\phi_{gen}))$ and $c_2(\mathrm{coker}(\phi_{gen}))$ as those of $\mathcal{H}_r$. Since $\mathscr{A}, \mathscr{B}$ are uniquely determined by r, e, $(3, b_e)$ and $c_1(\mathcal{H}_r)$ and since $\mathrm{Hom}_{\mathbb{F}_e}(\mathscr{A}, \mathscr{B})$ is irreducible, it follows that $\mathcal{M} = \mathcal{M}_{\mathbb{F}_e}(r)$, i.e. $\mathcal{M}_{\mathbb{F}_e}(r)$ is therefore irreducible and moreover it is unirational, being dominated by $\mathrm{Hom}_{\mathbb{F}_e}(\mathscr{A}, \mathscr{B})$.

The generic smoothness of $\mathcal{M}_{\mathbb{F}_e}(r)$ and the formula for its dimension follow as they have already been proved in Theorem 2.6-(b), (c). □

Theorem 3.2 *For any integer $e \geq 0$, let $\mathbb{F}_e$ be the Hirzebruch surface and let $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$ denote the line bundle $\alpha C_e + \beta f$ on $\mathbb{F}_e$, where C_e and f are generators of* $\mathrm{Num}(\mathbb{F}_e)$.

Let (X_e, ξ) be a 3-fold scroll over $\mathbb{F}_e$ as in Assumption 1.3, where $\varphi : X_e \to \mathbb{F}_e$ denotes the scroll map. Then the moduli space of rank-$r \geq 2$ vector bundles $\mathcal{U}_r$ on X_e which are Ulrich w.r.t. ξ and with first Chern class as in Theorem 2.5 is not empty and it contains a generically smooth component $\mathcal{M}(r)$ which is unirational and of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \left(\frac{(r-3)^2}{4} + 2\right)(6b_e - 9e - 4) + \frac{9}{2}(r-3)(2b_e - 3e), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_e - 9e - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

Proof As we have seen in the proof of Theorem 3.1, a general $[\mathcal{H}_r] \in \mathcal{M}_{\mathbb{F}_e}(r)$, turns out to be $\mathcal{H}_r = \mathrm{coker}(\phi)$, with ϕ a general vector bundle morphism as in (3.1).

Now take $\mathscr{A} = \mathcal{O}_{\mathbb{F}_e}(2, b_e - e - 1)^{\oplus \gamma}$, $\mathscr{B} = \mathcal{O}_{\mathbb{F}_e}(2, b_e - e)^{\oplus \delta} \oplus \mathcal{O}_{\mathbb{F}_e}(3, b_e - 1)^{\oplus \tau}$, γ, δ and τ as in the proof of Theorem 3.1; then for $\phi \in \mathrm{Hom}_{\mathbb{F}_e}(\mathscr{A}, \mathscr{B})$ general, one has therefore

$$0 \to \mathscr{A} \xrightarrow{\phi} \mathscr{B} \to \mathcal{H}_r \to 0.$$

We first tensor this exact sequence by $-c_1(\mathcal{E}_e)$, then we pull it back via φ^*, where $\varphi : X_e \to \mathbb{F}_e$ is the scroll map, and the sequence remains exact on the left since $\mathcal{H}_r(-c_1(\mathcal{E}_e))$ is locally free; subsequently we tensor the resulting short exact sequence with ξ, the tautological polarization on X_e, and thus we get the exact sequence

$$0 \to \varphi^*\big(\mathscr{A}(-c_1(\mathcal{E}_e))\big) \otimes \xi \xrightarrow{\overline{\phi}} \varphi^*\big(\mathscr{B}(-c_1(\mathcal{E}_e))\big) \otimes \xi \to \varphi^*\big(\mathcal{H}_r(-c_1)\big) \otimes \xi \to 0, \quad (3.2)$$

defining $\overline{\phi}$. Set $\overline{\mathscr{A}} := \varphi^*\big(\mathscr{A}(-c_1(\mathcal{E}_e)\big) \otimes \xi$ and $\overline{\mathscr{B}} := \varphi^*\big(\mathscr{B}(-c_1(\mathcal{E}_e)\big) \otimes \xi$. Recall that the modular component $\mathcal{M}(r)$ as in Theorem 2.5 has an open dense subset parametrizing isomorphism classes of slope-stable, rank-r vector bundles $\mathcal{U}_r$, which are Ulrich w.r.t. the tautological polarization ξ of X_e and with Chern classes determined by the iterative constructions as in [13] (in particular, the first Chern class c_1 is as reminded in Theorem 2.5); for $[\mathcal{U}_r] \in \mathcal{M}(r)$ general it has also been proved in [13, Proof of Thm. 5.1] that the bundle $\mathcal{U}_r$ has in particular the splitting type requested by Proposition 2.4, namely $(\mathcal{U}_r)_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$, on any φ-fiber F. As a consequence of the bijective correspondence induced by Proposition 2.4, in [13] we deduced therefore that $\mathcal{U}_r = \xi \otimes \varphi^*\big(\mathcal{H}_r(-c_1(\mathcal{E}_e))\big)$, with $\mathcal{H}_r$ Ulrich w.r.t. $c_1(\mathcal{E}_e)$ on $\mathbb{F}_e$ as above.

Then the sequence (3.2) reads

$$0 \to \overline{\mathscr{A}} \xrightarrow{\overline{\phi}} \overline{\mathscr{B}} \to \mathcal{U}_r \to 0. \quad (3.3)$$

In particular, for those morphisms $\overline{\phi} \in \mathrm{Hom}_{X_e}(\overline{\mathscr{A}}, \overline{\mathscr{B}})$ such that $\mathrm{coker}(\overline{\phi}) = \mathcal{U}_r$, one has that $\mathrm{coker}(\overline{\phi})$ is locally free, of rank r and it is moreover Ulrich on X_e w.r.t. the tautological polarization ξ, with Chern classes $c_i(\mathrm{coker}(\overline{\phi})) = c_i(\mathcal{U}_r)$, $1 \leq i \leq 3$, computed by iterative constructions of the vector bundles $\mathcal{U}_r$ as in [13] (e.g. c_1 is reminded in Theorem 2.5 above).

Let $\overline{\phi}_{gen} \in \mathrm{Hom}_{X_e}(\overline{\mathscr{A}}, \overline{\mathscr{B}})$ be general; since

$$\overline{\mathscr{A}}^{\vee} \otimes \overline{\mathscr{B}} = \varphi^*\big(\mathscr{A}^{\vee} \otimes \mathscr{B}\big) = \varphi^*\big(\mathcal{O}_{\mathbb{F}_e}(0, 1)^{\oplus(\gamma\,\delta)} \oplus \mathcal{O}_{\mathbb{F}_e}(1, e)^{\oplus(\gamma\,\tau)}\big),$$

i.e. $\mathscr{A}^{\vee} \otimes \mathscr{B}$ is globally generated, so $\overline{\mathscr{A}}^{\vee} \otimes \overline{\mathscr{B}}$ is also globally generated. Therefore, by [3, Thm. 4.2], (cf. also [4, Thm. 2]) $\overline{\phi}_{gen}$ is injective and it gives rise to an exact sequence

$$0 \to \overline{\mathscr{A}} \xrightarrow{\overline{\phi}_{gen}} \overline{\mathscr{B}} \to \mathrm{coker}(\overline{\phi}_{gen}) \to 0.$$

Since $\overline{\phi} \in \mathrm{Hom}_{X_e}(\overline{\mathscr{A}}, \overline{\mathscr{B}})$ as in (3.3) is such that $\mathrm{coker}(\overline{\phi}) = \mathcal{U}_r$ is locally free, then also $\mathrm{coker}(\overline{\phi}_{gen})$ is locally free, as locally freeness is an open condition on the (irreducible) vector space $\mathrm{Hom}_{X_e}(\overline{\mathscr{A}}, \overline{\mathscr{B}})$. Moreover, the rank of $\mathrm{coker}(\overline{\phi}_{gen})$ is given by $\delta + \tau - \gamma = r = \mathrm{rank}(\mathcal{U}_r)$, with γ, δ, τ as in the proof of Theorem 3.1. Furthermore, once again from the irreducibility of $\mathrm{Hom}_{X_e}(\overline{\mathscr{A}}, \overline{\mathscr{B}})$ and from the constancy of Chern classes in irreducible flat families of vector bundles of given rank (or even from the fact that $\mathcal{U}_r$ and $\mathrm{coker}(\overline{\phi}_{gen})$ are both locally free cokernels of injective vector bundle morphisms in $\mathrm{Hom}_{X_e}(\overline{\mathscr{A}}, \overline{\mathscr{B}})$) one has that

$$c_i(\mathrm{coker}(\overline{\phi}_{gen})) = c_i(\mathcal{U}_r) \text{ for } 0 \leq i \leq 3. \quad (3.4)$$

Finally since $\mathcal{U}_r$ is Ulrich on X_e w.r.t. ξ we have

$$h^i(\mathcal{U}_r(-j\xi)) = 0 \quad \text{for } 0 \le i \le 3 \text{ and } 1 \le j \le 3,$$

then by semicontinuity

$$h^i(\text{coker}(\overline{\phi}_{gen})(-j\xi)) = 0 \quad \text{for } 0 \le i \le 3 \text{ and } 1 \le j \le 3;$$

hence $\text{coker}(\overline{\phi}_{gen})$ is Ulrich w.r.t. ξ.

The fact that $\text{Hom}_{X_e}(\overline{\mathscr{A}}, \overline{\mathscr{B}})$ is irreducible implies that it must dominate the modular component $\mathcal{M}(r)$ (as in Theorem 2.5) containing $[\mathcal{U}_r]$ as its general point, which therefore implies that $\mathcal{M}(r)$ is unirational. The generic smoothness of $\mathcal{M}(r)$ as well as its dimension formula have already being proved in Theorem 2.5-(b), (c) (more precisely in [13, Main Theorem]). □

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