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Original Paper

Volume Preserving Mean Curvature Flow of Round Surfaces in Asymptotically Flat Spaces

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Abstract. We study the volume preserving mean curvature flow of a surface immersed in an asymptotically flat 3-manifold modeling an isolated gravitating system in General Relativity. We show that, if the ambient manifold has positive ADM-mass and the initial surface is round in a suitable sense, then the flow exists for all times and converges smoothly to a stable CMC-surface. This extends to the asymptotically flat setting a classical result by Huisken-Yau (Invent. Math. 1996) and allows to construct a CMC-foliation of the outer part of the manifold by an alternative approach to the ones by Nerz (Calc. Var. PDE, 2015) or by Eichmair-Koerber (J. Diff. Geometry, 2024).

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1. Introduction

In this paper, we study the evolution by mean curvature of closed surfaces in a smooth Riemannian manifold which is asymptotically flat, according to the following definition.

Definition 1.1. A complete Riemannian 3-manifold $(\mathrm{M}, \overline{\mathrm{g}})$ is called $C^2_{\frac{1}{2}+\delta}$-*asymptotically flat* for some $\delta \in (0, \frac{1}{2}]$ if there exists a compact subset $\emptyset \neq \mathrm{C} \subset \mathrm{M}$, a constant $\overline{c} > 0$ and a diffeomorphism $\vec{x} : \mathrm{M} \setminus \mathrm{C} \to \mathbb{R}^3 \setminus \overline{\mathbb{B}}_1(0)$ such that

$$|\overline{\mathrm{g}}_{\alpha\beta} - \overline{\mathrm{g}}^e_{\alpha\beta}| + |\vec{x}| \left|\partial_\gamma \overline{\mathrm{g}}_{\alpha\beta}\right| + |\vec{x}|^2 \left|\partial_\gamma \partial_\eta \overline{\mathrm{g}}_{\alpha\beta}\right| \leq \overline{c}|\vec{x}|^{-\frac{1}{2}-\delta} \tag{1.1}$$

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and in addition the scalar curvature $\overline{\mathrm{S}}$ satisfies $|\overline{\mathrm{S}}| \leq \overline{c}|\vec{x}|^{-3-\delta}$. Here $\overline{\mathrm{g}}^e$ denotes the Euclidean metric, and the partial derivatives and the norms are the Euclidean ones.

Manifolds of this kind have been extensively studied in General Relativity, since they occur as spacelike time slices of Lorentzian manifolds modeling isolated gravitating systems. A typical example is the well-known Schwarzschild metric of mass m, given by

$$\overline{\mathrm{g}}^S_{\alpha\beta} = \left(1 + \frac{m}{2|\vec{x}|}\right)^4 \overline{\mathrm{g}}^e_{\alpha\beta}, \tag{1.2}$$

which fulfills the above definition with $\delta = \frac{1}{2}$. For a general manifold as in (1.1) one can define the ADM-mass m_{ADM}, see (2.1), which coincides with m in the Schwarzschild case.

In this paper, we study the *volume preserving mean curvature flow* (VPM CF) of closed surfaces immersed in M, namely we consider time-dependent immersions $F : \Sigma \times [0,T) \to \mathrm{M}$, with Σ a closed 2-surface, which evolve according to

$$\frac{\partial F}{\partial t}(p,t) = -[H(p,t) - h(t)]\,\nu(p,t). \tag{1.3}$$

Here H and ν are the mean curvature and the unit normal, while $h(t)$ is the average of the mean curvature on Σ at time t. If we denote by $\Sigma_t = F(\Sigma,t)$ the solution at time t, then the volume of the domain enclosed by Σ_t remains constant, while the area of Σ_t decreases with time, with strict monotonicity unless H is constant. This suggests that, at least for suitable initial data, the surface should converge to a limiting profile given by a stable constant mean curvature (CMC) surface. However, this requires a nontrivial analysis in order to exclude the formation of singularities in finite time due to curvature blowup, and to show smooth convergence as time goes to infinity. Convergence results of the flow to a CMC profile in various kinds of ambient manifolds, either Euclidean or Riemannian, have been obtained by many authors through the years, we recall for instance [1,3,7,10,16,21]. More recently, the long-time convergence has also been studied in the case of weak solutions of the flow, see e.g., [19] and the references therein.

In the context of mathematical relativity, VPMCF was first studied by Huisken and Yau [18] who proved the following result.

Theorem 1.2. *Let* $(\mathrm{M}, \overline{\mathrm{g}})$ *a* C^4_2-asymptotically Schwarzschild 3*-manifold, i.e., such that*

$$|\overline{\mathrm{g}} - \overline{\mathrm{g}}^S| + \sum_{k=1}^{4} |\vec{x}|^k \left|\partial^k \left(\overline{\mathrm{g}}_{\alpha\beta} - \overline{\mathrm{g}}^S_{\alpha\beta}\right)\right| \leq \overline{c}|\vec{x}|^{-2} \tag{1.4}$$

where $\overline{\mathrm{g}}^S$ *is the Schwarzschild metric for some* $m > 0$ *and* ∂^k *denotes derivatives of order* k*. Let* Σ_t *be the solution of* (1.3) *with initial data given by the Euclidean coordinate sphere* $\mathbb{S}_r(0)$*, for a large enough radius* $r > 0$*. Then* Σ_t *exists for all* $t \in [0,\infty)$ *and converges smoothly to a strictly stable CMC-surface* $\Sigma^r_\infty \subset M$ *as* $t \to +\infty$*.*

Huisken and Yau then showed that the union of the surfaces Σ^r_∞ as $r \in (r_0, +\infty)$ forms a foliation by stable CMC-surfaces of the outer part of M, which is uniquely determined. Such a foliation is of interest for the physical model because it defines a canonical system of radial coordinates and allows for a geometric definition of the center of mass. In the following years, various authors have introduced other methods to construct the CMC-foliation and have weakened the hypotheses, as in the papers by Ye [31], Metzger [24] and Huang [12]. An important reference for our purposes is the work of Nerz [26], who strengthened the method of [24] and first proved the result under the general decay assumptions of Definition 1.1, which he then showed to be optimal in [27]. We also mention the recent paper by Eichmair-Koerber [9], who have constructed the CMC-foliation in asymptotically flat spaces of general dimension and have included a survey of the previous results.

In the papers that followed [18], the authors no longer used a curvature flow evolution to construct the CMC-surfaces, and employed instead more classical tools of elliptic theory, such as the implicit function theorem, the continuity method and Lyapunov-Schmidt reduction. This leads to the natural question of whether the leaves of the foliation can also be recovered by the VPMCF under the general assumptions of Definition 1.1. We give an affirmative answer in our main result, Theorem 4.14, where we describe a class of initial data for which the flow has a global solution and converges smoothly to a stable CMC limit as $t \to +\infty$ at an exponential rate. Roughly speaking, the class is characterized by the smallness of suitable integral norms of $H - h$ and of $\overset{\circ}{A}$, where $\overset{\circ}{A}$ is the trace-free part of the second fundamental form, and by a bound on distance of the barycenter from the Euclidean center. Under an additional mild symmetry assumption on the metric, we have the following more explicit statement about the evolution of Euclidean coordinate spheres, which extends Theorem 1.2 to a much more general class of ambient manifolds.

Theorem 1.3. *Let* $(\mathrm{M}, \overline{\mathrm{g}})$ *be a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat 3-manifold with mass* $m_{\mathrm{ADM}} > 0$*, satisfying the* $C^1_{1+\delta}$*-Regge–Teitelboim conditions in Definition 2.1. Then the solution of the flow* (1.3) *starting from a Euclidean coordinate sphere* $\mathbb{S}_r(0)$ *with large enough* $r > 0$ *exists for all times and converges smoothly to a strictly stable CMC-surface as* $t \to +\infty$*.*

The uniqueness result of [26] implies that the limiting CMC-surface of our flow coincides with a leaf of the foliation constructed there. Therefore our analysis provides an alternative approach to the existence of the foliation, which also gives a dynamical stability result for the surfaces under VPMCF. We remark that, as in [18], our construction requires the hypothesis of positive mass, which ensures that the CMC leaves are locally area minimizing. If the mass is negative, there are small perturbations of the leaves with the same volume and smaller area, which cannot converge to the CMC-surface under the flow, since this decreases the area. In this case, one expects that the perturbed surfaces drift away with the flow and do not converge. In contrast, the methods of [9, 26] work for nonzero mass of either sign.

Our strategy of proof has some common ideas with the one by Huisken and Yau [18]. As in that paper, we introduce a suitable class of *round surfaces*, which are close to Euclidean spheres and are well-centered with respect to the Euclidean coordinates, see Definition 2.4. The properties of the class are quantitatively expressed in terms of an approximate radius σ and become more restrictive as $\sigma \to +\infty$ according to the increasingly Euclidean behavior of the ambient metric. The main part of the proof consists of showing that, for suitable initial data, the solution of the flow remains in the roundness class as long at it exists.

On the other hand, the more general assumptions on the ambient manifold pose interesting new challenges in the analysis of the flow (1.3). For instance, in the previous work on mean curvature flow in Riemannian manifolds, e.g., [11,15,22], the convergence results depend on an explicit bound on $\overline{\nabla}\,\overline{\mathrm{Rm}}$, the gradient of the curvature tensor of the ambient manifold. In fact, this term occurs in the evolution equations for the second fundamental form of Σ_t under the flow. For this reason, hypothesis (1.4) in [18] included a control on the decay of the derivatives of the metric up to fourth order. We are assuming instead property (1.1), which gives a decay rate on $\overline{\mathrm{Rm}}$ but not on its derivatives, as it is more natural in the physical model. Therefore we use an approach to the curvature estimates which is inspired by an idea from [24]. Instead of using the maximum principle to control the curvature quantities, we consider suitable integral norms, and use integration by parts to transform the $\overline{\nabla}\,\overline{\mathrm{Rm}}$ terms. The integral estimates allow in turn to obtain a pointwise control on the curvature by using some regularity results from the literature: in particular, the rigidity estimate for nearly umbilical surfaces by De Lellis and Müller [8], together with its L^p version by Perez [28], and the bootstrap for the second fundamental form of nearly CMC-surfaces by Nerz [26]. We observe that integral estimates have been often used to obtain convergence of the mean curvature flow, starting from the pioneering paper [14]. Here, however, we do not need to employ directly a Stampacchia iteration technique to obtain L^∞ estimates, since this step is implicitly contained in Nerz's estimate, which uses a (simpler) integral iteration procedure in the elliptic setting.

Another delicate part of our procedure is the estimate of the possible drift of the barycenter of Σ_t along the flow, which is essential for the convergence and, more importantly, to ensure that all parts of Σ_t remain in a region where the ambient metric is enough flat to ensure that the previous estimates apply. In [18, Proposition 3.4], such a bound is obtained by a barrier argument, based on the properties of the Schwarzschild model and not applicable to our setting. We analyze instead the behavior of $||H-h||_{L^2(\Sigma_t)}$, which controls the speed of the barycenter, in addition to the deviation of Σ_t from being CMC. The time derivative of this norm has a simple expression, see (4.3), which involves the stability operator L associated with Σ_t. In particular, the decay of $H-h$ along the flow is related to the positivity of the operator. The spectral properties of L have been analyzed in detail in [4,26] in the case of round surfaces which are CMC, showing in particular that the smallest eigenvalue on functions with zero mean has the same sign as the ADM-mass of M. Our evolving surfaces Σ_t, on the other hand, are not CMC but only satisfy a smallness assumption on the oscillation of H. This gives rise to some non-negligible additional terms and the positivity of L may fail. However, for our purposes, we only need to apply L to the deformation $H - h$ given by the speed of the flow, and we can show that in this case the remainder terms admit a better estimate thanks to symmetry properties. This leads to an exponential decay in L^2-norm for the speed of the flow, see Proposition 4.9, which gives the desired control on the barycenter if the so-called translational part of H, see Definition 3.5, is enough small at the initial time. By a careful combination of the curvature estimates and the barycenter estimates, we finally obtain Theorem 4.13, where we describe a class of initial data such that the surfaces Σ_t remains round and well-centered along the flow. From this, standard arguments imply the global existence of the flow and the smooth exponential convergence to a CMC limit and lead to our main Theorem 4.14. It is easy to check that, if the ambient space satisfies a weak Regge–Teitelboim condition, the Euclidean coordinate spheres satisfy the requirements of the previous theorems, and this yields Theorem 1.3.

We conclude by describing further related works in the literature. Other convergence results for constrained mean curvature flows in asymptotically Schwarzschild manifolds were obtained in [6] and recently in [11]. An interesting modification of the definition by Huisken-Yau of the center of mass has been given by Cederbaum-Sakovich [4] by considering the so-called space-time mean curvature (STMC): by using a strategy similar to [26], the authors construct a constant STMC foliation which allows to treat certain cases, described in [5], where the center of mass via the CMC-foliation is not well-defined. In this context, the second author [30] has considered the volume preserving STMC-flow and has shown that it drives the CMC leaves constructed in the present paper to a constant STMC limit, providing an alternative construction of the foliation in the positive energy case. Very recently, Kröncke-Wolff [20] have used an area preserving null mean curvature flow to construct foliations of asymptotically Schwarzschildean lightcones by surfaces of constant STMC.

2. Preliminaries

2.1. Definitions and Basic Properties

Throughout the paper, $(\mathrm{M}, \overline{\mathrm{g}})$ will be an asymptotically flat 3-manifold as in Definition 1.1. We denote, respectively, by $\overline{\Gamma}^{\alpha}_{\beta\gamma}$ the Christoffel symbols, by $\overline{\nabla}$ the Riemannian connection, by $\overline{\mathrm{Rm}}, \overline{\mathrm{Ric}}$ the Riemann and Ricci curvature tensors and by $\overline{\mathrm{S}}$ the scalar curvature on $(\mathrm{M}, \overline{\mathrm{g}})$. We call *Euclidean coordinate spheres* the surfaces of the form $\vec{x}^{-1}(\mathbb{S}_R(0))$ for some $R > 0$; by an abuse of notation, we denote them simply by $\mathbb{S}_R(0)$.

The *ADM-mass* of $(\mathrm{M}, \overline{\mathrm{g}})$, first introduced in [2], is defined as

$$\overline{m}_{\mathrm{ADM}} := \lim_{R\to\infty} \frac{1}{16\pi} \int_{\mathbb{S}_R(0)} \left(\partial_\alpha \overline{\mathrm{g}}_{\alpha\beta} - \partial_\beta \overline{\mathrm{g}}_{\alpha\alpha}\right) \nu^\beta \, d\mu, \tag{2.1}$$

where ν is the unit normal vector and $d\mu$ is the measure on $\mathbb{S}_R(0)$ induced by $\overline{\mathrm{g}}$.

We also introduce a weakened form of a symmetry assumption originally stated by Regge and Teitelboim [29]. We remark that this property will only be used in Theorem 1.3 and Lemma 3.14, while it is not needed in the other results of the paper.

Definition 2.1 *($C^1_{1+\delta}$-Regge–Teitelboim conditions).* Let $(\mathrm{M}, \overline{\mathrm{g}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold. We say that this manifold satisfies the $C^1_{1+\delta}$-Regge–Teitelboim conditions if there exists $\overline{c} > 0$ such that

$$\left|\overline{\mathrm{g}}_{\alpha\beta}(\overline{x}) - \overline{\mathrm{g}}_{\alpha\beta}(-\overline{x})\right| + |\overline{x}| \left|\overline{\Gamma}^{\gamma}_{\alpha\beta}(\overline{x}) + \overline{\Gamma}^{\gamma}_{\alpha\beta}(-\overline{x})\right| \le \frac{\overline{c}}{|\overline{x}|^{1+\delta}} \tag{2.2}$$

for every $\overline{x} \in \mathrm{M} \setminus \mathrm{C}$.

In the following, we will consider smooth immersed 2-surfaces $\iota : \Sigma \hookrightarrow \mathrm{M} \setminus \mathrm{C}$. We will always assume that Σ is closed, connected and orientable. On Σ we can consider the physical metric $g := \iota^* \overline{\mathrm{g}}$ and the Euclidean metric $g^e := \iota^* \overline{\mathrm{g}}^e$, where $\overline{\mathrm{g}}^e$ is induced by the diffeomorphism $\vec{x}$. We will use the apex e on each geometric quantity when it is computed with respect to the Euclidean metric, and we write Σ^e as a short notation for (Σ, g^e). We use the latin indices $i, j, k, \dots \in \{1, 2\}$ to denote the coordinates on Σ, while the coordinates in the ambient space are indicated with the greek letters $\alpha, \beta, \gamma, \dots \in \{1, 2, 3\}$.

We denote by ∇ the connection on Σ, by S the scalar curvature and by $d\mu$ the volume form. We denote by ν the unit outer normal, by $A = \{A_{ij}\}$ and by H, respectively, the second fundamental form and the mean curvature, and by κ_i, $i \in \{1, 2\}$ the principal curvatures. In addition, we denote by $\overset{\circ}{A} := A - \frac{H}{2} g$ the traceless part of the second fundamental form. We recall the identities

$$|A|^2 = \kappa_1^2 + \kappa_2^2, \qquad H^2 = (\kappa_1 + \kappa_2)^2, \qquad |\overset{\circ}{A}|^2 = |A|^2 - \frac{1}{2} H^2 = \frac{1}{2}(\kappa_1 - \kappa_2)^2.$$

The property of asymptotic flatness allows to estimate the difference between these quantities and their counterparts computed with respect to the Euclidean metric. We collect the relevant properties in the following lemma, which can be proved by standard computations, see e.g., [23, Lemma 1.5], [4, Lemma 11].

Lemma 2.2. *Let $\iota : \Sigma \hookrightarrow \mathrm{M}$ be a surface immersed in a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat 3-manifold M. Then there exists $C > 0$, only depending on the constant $\overline{c}$ in (1.1), such that*

$$|g - g^e|_g \le C|\vec{x}|^{-\frac{1}{2}-\delta}, \qquad |\Gamma^k_{ij} - (\Gamma^e)^k_{ij}| \le C|\vec{x}|^{-\frac{3}{2}-\delta} \tag{2.3}$$

$$|d\mu - d\mu^e| \le C|\vec{x}|^{-\frac{1}{2}-\delta} d\mu, \tag{2.4}$$

$$|\nu - \nu^e|_g \le C|\vec{x}|^{-\frac{1}{2}-\delta}, \qquad |\nabla\nu - \nabla^e\nu^e|_g \le C|\vec{x}|^{-\frac{3}{2}-\delta}, \tag{2.5}$$

$$|A - A^e|_g \le C\left(|\vec{x}|^{-\frac{3}{2}-\delta} + |\vec{x}|^{-\frac{1}{2}-\delta}|A^e|\right) \tag{2.6}$$

$$|\nabla A - \nabla^e A^e|_g \le C\left(|\vec{x}|^{-\frac{5}{2}-\delta} + |\vec{x}|^{-\frac{1}{2}-\delta}|\nabla^e A^e|\right). \tag{2.7}$$

In addition, if $|A| \leq 10|\vec{x}|^{-1}$, *we have*

$$|H - H^e| \leq C|\vec{x}|^{-\frac{3}{2}-\delta}, \qquad |\mathring{A} - \mathring{A}^e|_g \leq C|\vec{x}|^{-\frac{3}{2}-\delta}. \tag{2.8}$$

We introduce some more notation. We denote the average of the mean curvature by

$$h := \frac{1}{|\Sigma|}\int_\Sigma H \, d\mu,$$

where $|\Sigma| = \int_\Sigma d\mu$. In addition, we define the barycenter of Σ as

$$\vec{z}_\Sigma := \frac{1}{|\Sigma|}\int_\Sigma \iota \, d\mu,$$

that is, the average of the (Euclidean) position vector with respect to the physical metric. This definition differs slightly from the one in [4,26], where the average is taken in the Euclidean metric; however, the two definitions have the same qualitative properties.

Definition 2.3. Let $\iota : \Sigma \hookrightarrow \mathrm{M}$ be a surface immersed in an asymptotically flat 3-manifold M. Then we set

$$r_\Sigma := \min_{x\in\Sigma} |\vec{x}(\iota(x))|, \qquad R_\Sigma := \max_{x\in\Sigma} |\vec{x}(\iota(x))|, \qquad \sigma_\Sigma := \sqrt{\frac{|\Sigma|}{4\pi}}.$$

We call these values *Euclidean radius, Euclidean diameter* and *area radius*, respectively.

As in [4,26] we consider the Sobolev norms on Σ with a radius-dependent weight as follows

$$\|f\|_{W^{0,p}(\Sigma)} := \|f\|_{L^p(\Sigma)}, \qquad \|f\|_{W^{k+1,p}(\Sigma)} := \|f\|_{L^p(\Sigma)} + \sigma_\Sigma \|\nabla f\|_{W^{k,p}(\Sigma)},$$

for $p \in [1,\infty]$ and $k \geq 0$ integer. As usual, we use the notation $H^k = W^{k,2}$.

2.2. Round Surfaces

We now introduce a class of surfaces, which are close to a Euclidean sphere of radius σ in a quantitative way measured by some parameters. The aim is to find a class which is invariant under the volume preserving mean curvature flow for an appropriate choice of the parameters and for large enough radius, or at least such that the possible deterioration of the parameters along the flow can be estimated. Other classes of round surfaces, which are related to the methods used there, have been introduced in [18,24,26]

Definition 2.4 *(Round surfaces).* Let $(\mathrm{M}, \overline{\mathrm{g}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow \mathrm{M}$ be a surface.
(i) For a given approximate radius $\sigma > 1$, and parameters $\eta, B_1, B_2 > 0$, we say that $\iota(\Sigma)$ is a *round surface* in $(\mathrm{M}, \overline{\mathrm{g}})$, and we write $\iota(\Sigma) \in \mathcal{W}^\eta_\sigma(B_1, B_2)$, if the following inequalities are satisfied:

$$|A(t)| < \sqrt{\frac{5}{2}}\sigma^{-1}, \qquad \kappa_i(t) > \frac{1}{2\sigma}, \tag{2.9}$$

$$\frac{7}{2}\pi\sigma^2 < |\Sigma| \leq 5\pi\sigma^2, \qquad \frac{3}{4} < \frac{r_\Sigma}{\sigma} < \frac{R_\Sigma}{\sigma} \leq \frac{5}{4}, \tag{2.10}$$

$$\|\overset{\circ}{A}\|_{L^4(\Sigma)} < B_1 \sigma^{-1-\delta}, \tag{2.11}$$

$$\eta \sigma^{-4} \|H - h\|^4_{L^4(\Sigma)} + \|\nabla H\|^4_{L^4(\Sigma)} < B_2 \sigma^{-8-4\delta}. \tag{2.12}$$

(ii) For given $\sigma > 1$ and $\eta, B_1, B_2, B_{\mathrm{cen}} > 0$, we say that $\iota(\Sigma)$ is a *well-centered round surface*, and we write $\iota(\Sigma) \in \mathcal{B}^\eta_\sigma(B_1, B_2, B_{\mathrm{cen}})$ if it satisfies the above properties and in addition

$$|\vec{z}_\Sigma| < B_{\mathrm{cen}} \sigma^{1-\delta}. \tag{2.13}$$

At first sight it may look redundant to introduce a further scale parameter σ, in addition to the ones in Definition 2.3, since condition (2.10) implies that each of these values controls the others. However, this choice is practical in the study of the curvature flow evolution, where Σ depends on time. Then the radii of Definition 2.3 change with time, and it is convenient to describe the size of the evolving surface in terms of a fixed parameter σ.

Remark 2.5. The decay rates in conditions (2.11)–(2.12) are modeled on the ones of the Euclidean coordinate spheres. In fact, by Lemma 2.2, it is easy to check that if B_1, B_2 are large enough, depending on $\bar{c}$ in (1.1), then $\mathbb{S}_r(0)$ belongs to $\mathcal{B}^\eta_\sigma(B_1, B_2, B_{\mathrm{cen}})$ for r large enough and r/σ enough close to 1. Conversely, we will see in Lemma 2.7(iv) that a round surface is close to a sphere in Euclidean coordinates; in particular, it is embedded and has genus zero. It is also easy to see that a round surface in our sense also belongs to the class of asymptotically concentric surfaces defined in [26, Def. 4.3].

On the Notation for the Constants. Throughout the paper, when deriving estimates on geometric quantities on a surface Σ, we denote by $C, C_1, C_2, \ldots$ constants which only depend on properties of the ambient manifold, such as $\bar{c}, \delta$ in (1.1) or the mass $\overline{m}_{\mathrm{ADM}}$ and by $c, c_1, c_2, \ldots$ constants which in addition depend on the constants $B_1, B_2, B_{\mathrm{cen}}$ in the previous definition. We say that a constant is universal if it is independent on any other parameter of our problem. As usual, the letters c or C will often denote constants which may change from one line to the other, but each time depending on the same parameters.

Let us first observe some easy consequences of the properties of a round surface.

Remark 2.6. (i) Property (2.10) implies that the three radii of Definition 2.3 are comparable among each other and with σ. In particular this property implies, because of the asymptotic flatness of M in (1.1), the bound on the Riemann tensor

$$|\overline{\mathrm{Rm}}| \leq C(\bar{c}) \sigma^{-\frac{5}{2}-\delta} \text{ on } \Sigma. \tag{2.14}$$

(ii) The Michael-Simon inequality in Euclidean space, together with the curvature bound in (2.9) and the error estimates in Lemma 2.2, implies the existence of a universal Sobolev constant $C_S > 0$ such that

$$\|\psi\|_{L^2(\Sigma)} \leq \frac{C_S}{\sigma} \|\psi\|_{W^{1,1}(\Sigma)}, \qquad \forall\ \psi \in W^{1,1}(\Sigma), \tag{2.15}$$

on any round surface Σ with radius $\sigma \geq \sigma_0 = \sigma_0(\overline{c}, \delta) > 0$. From this, the other Sobolev inequalities can be deduced. In particular (see e.g., Lemma 12 in [4] and the references therein) we have, for every $p > 2$,

$$\|\psi\|_{L^\infty(\Sigma)} \leq 2^{\frac{2(p-1)}{p-2}} C_S \sigma^{-\frac{2}{p}} \|\psi\|_{W^{1,p}(\Sigma)}, \qquad \forall \psi \in W^{1,p}(\Sigma), \tag{2.16}$$

and also

$$\|\psi\|_{L^\infty(\Sigma)} \leq 32 C_S^2 \sigma^{-1} \|\psi\|_{H^2(\Sigma)}, \qquad \forall \psi \in H^2(\Sigma). \tag{2.17}$$

In the next Lemma, similar to [26, Proposition 4.4] and [4, Proposition 1], we collect some properties of round surfaces which follow from the integral bounds (2.11) and (2.12) by applying the regularity results by De Lellis and Müller [8] and by Nerz [26].

Lemma 2.7. *Let* $(\mathrm{M}, \overline{\mathrm{g}})$ *be a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat manifold. Fix any* $\eta, B_1, B_2 > 0$*. Then there exists* $\sigma_0 = \sigma_0(B_1, B_2, \eta, \overline{c}, \delta) > 0$ *such that any surface* (Σ, g) *which belongs to* $\mathcal{W}^\eta_\sigma(B_1, B_2)$ *for some* $\sigma > \sigma_0$ *satisfies the following properties.*

(i) *There exists a constant* $c = c(B_2, \eta) > 0$ *such that*

$$\|H - h\|_{L^\infty} \leq c\sigma^{-\frac{3}{2}-\delta}. \tag{2.18}$$

(ii) *There exists* $c = c(B_1, B_2, \eta, \overline{c}) > 0$ *such that*

$$\left| h - \frac{2}{\sigma_\Sigma} \right| \leq c\sigma^{-\frac{3}{2}-\delta}. \tag{2.19}$$

(iii) *There exists a constant* $B_\infty = B_\infty(B_1, B_2, \eta, \overline{c}, \delta)$ *such that* $\|\mathring{A}\|_{L^\infty(\Sigma)} \leq B_\infty \sigma^{-\frac{3}{2}-\delta}$.

(iv) *There exist constants* $c = c(B_1, B_2, \eta, \overline{c}, \delta)$, $c_0 = c_0(B_1, \overline{c}, \delta)$, $\vec{z}_0 \in \mathbb{R}^3$, *and a function* $f : \mathbb{S}_{\sigma_\Sigma}(\vec{z}_0) \to \mathbb{R}$ *such that*

$$\Sigma^e = \mathrm{graph}(f), \qquad \|f\|_{W^{2,\infty}} \leq c\sigma^{\frac{1}{2}-\delta}, \qquad |\vec{z}_0 - \vec{z}_\Sigma| \leq c_0\sigma^{\frac{1}{2}-\delta}. \tag{2.20}$$

(v) *There exists a constant* $\mathrm{c_p} = \mathrm{c_p}(\overline{c}, \delta)$ *such that*

$$\|H - h\|_{L^4(\Sigma)} \leq \mathrm{c_p}\|\mathring{A}\|_{L^4(\Sigma)} + \mathrm{c_p}\sigma^{-1-\delta}.$$

We point out that the interest of part (v) lies in the fact that $\mathrm{c_p}$ is independent on the constant B_2 which appears in (2.12).

Proof. (i) The estimate follows from property (2.12) and the Sobolev immersion (2.16).

(ii) We first observe that (2.10), (2.11) and (2.12) imply,

$$\|\mathring{A}\|_{L^2(\Sigma)} < c_1\sigma^{-\frac{1}{2}-\delta}, \qquad \|H - h\|_{H^1(\Sigma)} < c_2\sigma^{-\frac{1}{2}-\delta}, \tag{2.21}$$

where $c_1 = c_1(B_1)$ and $c_2 = c_2(\eta, B_2)$. Then we recall the estimate by De Lellis and Müller in [8] for surfaces in Euclidean space, which states that

$$\left\| A^e - \frac{1}{\sigma_{\Sigma^e}} g^e \right\|_{L^2(\Sigma^e)} \leq c_{DM} \left\| \mathring{A}^e \right\|_{L^2(\Sigma^e)},$$

where $c_{DM} > 0$ is a universal constant and σ_{Σ^e} is defined by $4\pi\sigma_{\Sigma^e}^2 = |\Sigma^e|$. Passing to the physical metric, using Proposition 2.2 and properties (2.9), (2.10) we deduce

$$\left\| A - \frac{1}{\sigma_\Sigma} g \right\|_{L^2(\Sigma)} \le c_{DM} \left\| \overset{\circ}{A} \right\|_{L^2(\Sigma)} + C(\bar{c})\sigma^{-\frac{1}{2}-\delta} \le c\sigma^{-\frac{1}{2}-\delta}, \tag{2.22}$$

for some $c = c(B_1, \bar{c})$. On the other hand, we have

$$\begin{aligned}\sqrt{2\pi}\sigma_\Sigma \left| h - \frac{2}{\sigma_\Sigma} \right| &= \left\| \frac{h}{2} g - \frac{1}{\sigma_\Sigma} g \right\|_{L^2(\Sigma)} \\ &\le \left\| \frac{h}{2} g - \frac{H}{2} g \right\|_{L^2(\Sigma)} + \left\| \frac{H}{2} g - A \right\|_{L^2(\Sigma)} + \left\| A - \frac{1}{\sigma_\Sigma} g \right\|_{L^2(\Sigma)} \\ &= \frac{1}{\sqrt{2}} \|h - H\|_{L^2(\Sigma)} + \left\| \overset{\circ}{A} \right\|_{L^2(\Sigma)} + \left\| A - \frac{1}{\sigma_\Sigma} g \right\|_{L^2(\Sigma)}.\end{aligned}$$

Taking into account (2.10), (2.21) and (2.22), we obtain (2.19).

(iii) Assumptions (2.11) and (2.12), together with the properties in Remark 2.6, allow to apply the bootstrap regularity for the second fundamental form by Nerz [26, Proposition 4.1], which give the assertion.

(iv) The result of De Lellis-Müller quoted above [8, Thm. 1.1] also gives the existence of a conformal parametrization $\Psi : \mathbb{S}_{\sigma_{\Sigma^e}}(\vec{z}_0) \to \Sigma^e$ for a suitable center $\vec{z}_0 \in \mathbb{R}^3$ such that $\sigma_{\Sigma^e}^{-2} \|\Psi - \mathrm{Id}\|_{H^2(\Sigma^e)} \le C \|\overset{\circ}{A}{}^e\|_{L^2(\Sigma^e)}$ for a universal constant C. Using (2.21), (2.17), and Lemma 2.2, we find

$$\|\Psi - \mathrm{Id}\|_\infty \le C\sigma \|\overset{\circ}{A}{}^e\|_{L^2(\Sigma)} \le c_0 \sigma^{\frac{1}{2}-\delta},$$

for some $c_0 = c_0(\bar{c}, \delta, B_1)$. It follows

$$|\vec{z}_\Sigma - \vec{z}_0| \le \fint_\Sigma |\Psi - \mathrm{Id}| \, d\mu^e + c_0 \sigma^{\frac{1}{2}-\delta} \le c_0 \sigma^{\frac{1}{2}-\delta},$$

for a possibly different c_0 depending on the same values.
In addition, using the L^∞ bound on $|\overset{\circ}{A}|$ from part (iii), we can apply [26, Cor. E.1] which states that Σ can be also written as a graph over the same sphere $\mathbb{S}_{\sigma_{\Sigma^e}}(\vec{z}_0)$, where the graph function f satisfies $\|f\|_{W^{2,\infty}(\mathbb{S}_{\sigma_{\Sigma^e}})} \le c\sigma^{\frac{1}{2}-\delta}$ for a constant c depending on the same parameters as B_∞. Using Lemma 2.2 we see that $|\sigma_\Sigma - \sigma_{\Sigma^e}| \le C\sigma^{\frac{1}{2}-\delta}$ and that f satisfies an analogous $W^{2,\infty}$ estimate when considered as a map on $\mathbb{S}_{\sigma_\Sigma}(\vec{z}_0)$ with the metric g.

(v) The estimate follows from the $p > 2$ generalization of De Lellis-Müller's estimate due to Perez [28, Thm. 1.1], with a remainder term coming from the Riemannian asymptotically flat metric. $\square$

3. Spectral Theory

3.1. Mass and the Stability Operator

In this section, we study the spectral properties of the stability operator associated to a round surface and collect other auxiliary results that will be needed in the analysis of the flow afterward.

Unless explicitly stated, in this section we consider closed surfaces Σ belonging to a roundness class $\mathcal{W}^\eta_\sigma(B_1, B_2)$ for fixed parameters η, B_1, B_2 and a general large σ. It is tacitly meant that the constants c and σ_0 which appear in the statements only depend on η, B_1, B_2 and on the constants $\bar{c}, \delta$ in Definition 1.1.

We begin by recalling the definition of Hawking mass of a surface.

Definition 3.1. Let $(\mathrm{M}, \overline{\mathrm{g}})$ be a 3-dimensional manifold, and $\iota : \Sigma \hookrightarrow \mathrm{M}$ be a closed surface. The *Hawking mass* of Σ is defined as

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, d\mu \right). \tag{3.1}$$

Using the equivalent definition of ADM-mass in terms of the Einstein tensor [25] and the Gauss-Bonnet theorem one can show that the Hawking mass of a round surface Σ is asymptotic to the ADM-mass of M for large radius. More precisely, the results of [26, Appendix A] give the following:

Proposition 3.2. *There exist c and σ_0 such that, for any $\Sigma \in \mathcal{W}^\eta_\sigma(B_1, B_2)$ with $\sigma \geq \sigma_0$, we have*

$$|m_H(\Sigma) - \overline{m}_{\mathrm{ADM}}| \leq c\sigma^{-\delta}.$$

We now introduce the stability operator, which occurs as the second variation of the area functional.

Definition 3.3. Given a surface $\iota : \Sigma \hookrightarrow \mathrm{M}$ and $\mathrm{f} \in H^2(\Sigma)$, the *stability operator* associated to Σ, $L^\Sigma : H^2(\Sigma) \to L^2(\Sigma)$, is defined as

$$L^\Sigma f := -\Delta^\Sigma f - (|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu))f,$$

where Δ^Σ is the Laplace–Beltrami operator on Σ. We simply write Δ, L instead of Δ^Σ, L^Σ whenever the choice of the surface Σ is not ambiguous.

In [26] and [4] the spectral properties of L^Σ are studied under the assumption that Σ is a round surface with constant mean curvature, showing that L^Σ is invertible if $\overline{m}_{\mathrm{ADM}} \neq 0$ and positive definite on functions with zero mean if $\overline{m}_{\mathrm{ADM}} > 0$. Here we generalize this analysis to round surfaces where we only assume that H has a small oscillation as in (2.12). We will see that the positivity property of L^Σ when $\overline{m}_{\mathrm{ADM}} > 0$ is no longer true, but that the error terms admit an estimate that will be enough for our purposes.

We first recall some properties of the Laplace–Beltrami operator on a round sphere $\mathbb{S}_\sigma(0) \subset \mathbb{R}^3$ with the Euclidean metric. On a general closed surface, the eigenvalues of Δ are all positive, except the first one which is zero, with eigenspace given by the constant functions. For the Euclidean sphere, the first nonzero eigenvalue has multiplicity three and is given by

$$\lambda_\alpha^e = \frac{2}{\sigma^2}, \ \alpha = 1, 2, 3.$$

An orthonormal basis for the eigenspace is given by the normalized coordinate functions restricted on $\mathbb{S}_\sigma(0)$

$$f_\alpha^e(x) = \sqrt{\frac{3}{4\pi\sigma^4}} x_\alpha, \ \alpha = 1, 2, 3.$$

The remaining eigenvalues satisfy the bound

$$\lambda_i^e \geq \lambda_4^e = \frac{6}{\sigma^2}, \ \forall i \geq 4.$$

Given a round surface $\Sigma \in \mathcal{W}_\sigma^\eta(B_1, B_2)$, we know from Lemma 2.7(iv) that Σ can be written as a graph over a Euclidean sphere. Similarly to [4], we can use this map to identify functions on Σ with functions on $\mathbb{S}_{\sigma_\Sigma}(0)$. We recall the statement of Lemma 2 of [4], which measures how much the first eigenvalues and the corresponding eigenfunctions of the Laplace–Beltrami operator on a round surface in the physical metric differ from the ones of the approximating sphere in the Euclidean metric.

Lemma 3.4. *There exist $c, \sigma_0 > 0$ such that, for any $\Sigma \in \mathcal{W}_\sigma^\eta(B_1, B_2)$ with $\sigma \geq \sigma_0$, there is a complete orthonormal system in $L^2(\Sigma)$ consisting of eigenfunctions $\{f_\alpha\}_{\alpha=0}^\infty$ such that*

$$-\Delta f_\alpha = \lambda_\alpha f_\alpha, \qquad \text{with } 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

Moreover, after possibly a rotation in the Euclidean coordinates we have, for $\alpha = 1, 2, 3$,

$$\left| \lambda_\alpha - \frac{2}{\sigma_\Sigma^2} \right| \leq c\sigma^{-\frac{5}{2}-\delta}, \quad \|f_\alpha - f_\alpha^e\|_{W^{2,2}(\Sigma)} \leq c\sigma^{-\frac{1}{2}-\delta},$$

where $f_\alpha^e = \sqrt{\frac{3}{4\pi\sigma_\Sigma^4}} x_\alpha$ defined on $\mathbb{R}^3$. In addition, for $\alpha, \beta = 1, 2, 3$ we have

$$\int_\Sigma \left| \langle \nabla f_\alpha, \nabla f_\beta \rangle - \frac{3\delta_{\alpha\beta}}{\sigma_\Sigma^2 |\Sigma|} + \frac{f_\alpha f_\beta}{\sigma_\Sigma^2} \right| \, d\mu \leq c\sigma^{-\frac{5}{2}-\delta}. \tag{3.2}$$

The remaining eigenvalues satisfy

$$\lambda_\alpha > \frac{5}{\sigma_\Sigma^2}, \quad \forall \alpha > 3. \tag{3.3}$$

We observe that in [4] the above statement is given under the additional requirement that Σ has constant space-time mean curvature; however, the proof works in the same way under the assumption that H has a small oscillation as in (2.18).

As in [4,26], we introduce a spectral decomposition for functions on Σ. We write $\langle u, v\rangle_2$ for the $L^2(\Sigma)$ scalar product of functions $u, v : \Sigma \to \mathbb{R}$.

Definition 3.5. Given a surface $\Sigma \in \mathcal{W}^\eta_\sigma(B_1, B_2)$, let $\{f_\alpha\}$, $\alpha = 1, 2, 3$, be as in the previous lemma. For every $\mathrm{w} \in L^2(\Sigma)$ we define

$$\mathrm{w}^t := \sum_{\alpha=1}^{3} \langle \mathrm{w}, f_\alpha \rangle_2 f_\alpha, \qquad \mathrm{w}^d := \mathrm{w} - \mathrm{w}^t. \tag{3.4}$$

We call w^t the *translational part* and w^d the *difference part* of w.

To proceed in the analysis of the stability operator L, the next step is to estimate the contribution of the term containing $\overline{\mathrm{Ric}}(\nu, \nu)$.

Proposition 3.6. *There exist $c, \sigma_0 > 0$ such that, for any $\Sigma \in \mathcal{W}^\eta_\sigma(B_1, B_2)$ with $\sigma \geq \sigma_0$ we have for every $\alpha, \beta \in \{1, 2, 3\}$, with $\alpha \neq \beta$,*

$$\left| \int_\Sigma \left(\overline{\mathrm{Ric}}(\nu, \nu) - \frac{H^2 - h^2}{4} \right) f_\alpha f_\beta \; d\mu \right| \leq c\sigma^{-3-\delta},$$

and for every $\alpha \in \{1, 2, 3\}$

$$\left| \lambda_\alpha - \frac{h^2}{2} - \frac{6m_H(\Sigma)}{\sigma_\Sigma^3} - \int_\Sigma \left(\overline{\mathrm{Ric}}(\nu, \nu) - \frac{H^2 - h^2}{4} \right) f_\alpha^2 \; d\mu \right| \leq c\sigma^{-3-\delta}.$$

Proof. The proof follows the same strategy as in [26, Lemma 4.5] and [4, Lemma 3], so we will only highlight the differences due to the fact that our Σ has not constant mean curvature. We recall that, by Lemma 2.7, we have the bounds

$$H - h = O(\sigma^{-\frac{3}{2}-\delta}), \qquad H^2 - h^2 = O(\sigma^{-\frac{5}{2}-\delta}), \qquad \|\mathring{A}\|^2 = O(\sigma^{-3-2\delta}), \tag{3.5}$$

where the $O(\sigma^\alpha)$ notation means that the quantity is bounded in absolute value by $c\sigma^\alpha$, with c depending on the usual parameters described at the beginning of the section.

By an application of the Bochner-Lichnerowicz formula we obtain, as in formula (41) in [4], the estimate

$$\left| \lambda_\alpha^2 \delta_{\alpha\beta} - \int_\Sigma S \left\langle \nabla f_\alpha, \nabla f_\beta \right\rangle \, d\mu \right| \leq c\sigma^{-5-\delta}.$$

Using the Gauss equations and the bound in (3.5) on $\|\mathring{A}\|^2$, we deduce

$$\left| \lambda_\alpha^2 \delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu, \nu) + \frac{H^2}{2} \right) \left\langle \nabla f_\alpha, \nabla f_\beta \right\rangle \, d\mu \right| \leq c\sigma^{-5-\delta}.$$

By writing $H^2 = h^2 + (H^2 - h^2)$ and using the properties of the eigenfunctions, we find

$$\left| \left(\lambda_\alpha^2 - \lambda_\alpha \frac{h^2}{2} \right) \delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu, \nu) + \frac{H^2 - h^2}{2} \right) \left\langle \nabla f_\alpha, \nabla f_\beta \right\rangle \, d\mu \right| \leq c\sigma^{-5-\delta},$$

which in view of (2.14), (3.2) and (3.5) implies

$$\left| \left(\lambda_\alpha^2 - \lambda_\alpha \frac{h^2}{2} \right) \delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu, \nu) + \frac{H^2 - h^2}{2} \right) \left(\frac{3\delta_{\alpha\beta}}{\sigma_\Sigma^2 |\Sigma|} - \frac{f_\alpha f_\beta}{\sigma_\Sigma^2} \right) d\mu \right| \leq c\sigma^{-5-\delta}. \tag{3.6}$$

Compared to the proof in [4, 26], we have the additional term with $H^2 - h^2$. One part of this contribution can be estimated observing that

$$\int_\Sigma \frac{H^2 - h^2}{2} \frac{3\delta_{\alpha\beta}}{\sigma_\Sigma^2 |\Sigma|} \, d\mu = \frac{3\delta_{\alpha\beta}}{2\sigma_\Sigma^2 |\Sigma|} \int_\Sigma (H-h)^2 d\mu = O(\sigma^{-5-2\delta}), \tag{3.7}$$

since $\int (H^2 - h^2) d\mu = \int (H-h)^2 d\mu$. The term containing $(H^2 - h^2) f_\alpha f_\beta$, on the contrary, cannot be absorbed in the $O(\sigma^{-5-\delta})$ remainder and will be left as it is. The remaining terms in (3.6) can be rewritten in the following way using Gauss-Bonnet theorem and the definition of m_H, as shown in [4, 26],

$$\begin{aligned} &\left(\lambda_\alpha^2 - \lambda_\alpha \frac{h^2}{2}\right) \delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu)\right) \left(\frac{3\delta_{\alpha\beta}}{\sigma_\Sigma^2 |\Sigma|} - \frac{f_\alpha f_\beta}{\sigma_\Sigma^2}\right) d\mu \\ &= \frac{2}{\sigma_\Sigma^2}\left(\lambda_\alpha - \frac{h^2}{2}\right)\delta_{\alpha\beta} - \frac{12 m_H(\Sigma)}{\sigma_\Sigma^5}\delta_{\alpha\beta} - \frac{2}{\sigma_\Sigma^2}\int_\Sigma \overline{\mathrm{Ric}}(\nu,\nu) f_\alpha f_\beta \, d\mu + O(\sigma^{-5-\delta}). \end{aligned} \tag{3.8}$$

Combining formulas (3.6), (3.7), (3.8) and simplifying the factor $2/\sigma_\Sigma^2$ we conclude

$$\left| \left(\lambda_\alpha - \frac{h^2}{2} - \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3}\right)\delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{Ric}}(\nu,\nu) - \frac{H^2 - h^2}{4}\right) f_\alpha f_\beta \, d\mu \right| \leq c\sigma^{-3-\delta},$$

which yields the thesis. □

We can now describe the behavior of the bilinear form associated to the stability operator.

Proposition 3.7. *There exist $c, \sigma_0 > 0$ such that any surface $\Sigma \in \mathcal{W}_\sigma^\eta(B_1, B_2)$ with $\sigma \geq \sigma_0$ satisfies the following: for any $u, \phi, w \in H^2(\Sigma)$ with $u \in \mathrm{span}\{f_1, f_2, f_3\}$, $\varphi \in (\mathrm{span}\{f_0, f_1, f_2, f_3\})^\perp$ and w with zero mean value, we have*

$$\left| \langle Lu, u\rangle_2 - \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3} \|u\|_2^2 + \frac{3h}{2}\int_\Sigma (H-h) u^2 \, d\mu \right| \leq c\sigma^{-3-2\delta} \|u\|_2^2, \tag{3.9}$$

$$\langle L\varphi, \varphi\rangle_2 > \frac{2}{\sigma_\Sigma^2} \|\varphi\|_2^2, \tag{3.10}$$

$$|\langle Lu, w\rangle_2| \leq c\sigma^{-\frac{5}{2}-\delta} \|u\|_2 \|w\|_2. \tag{3.11}$$

Proof. Let $u, v \in \mathrm{span}\{f_1, f_2, f_3\}$. Using the bilinearity of L and Proposition 3.6 we find

$$\begin{aligned} \langle Lu, v\rangle_2 &= \left(\frac{h^2}{2} + \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3}\right)\langle u, v\rangle_2 - \int_\Sigma \left(|A|^2 + \frac{H^2 - h^2}{4}\right) uv \, d\mu \\ &\quad + \mathrm{O}(\sigma^{-3-2\delta}) \|u\|_2 \|v\|_2 \\ &= \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3}\langle u, v\rangle_2 - \int_\Sigma \left(|\mathring{A}|^2 + \frac{3(H^2 - h^2)}{4}\right) uv \, d\mu + \mathrm{O}(\sigma^{-3-2\delta}) \|u\|_2 \|v\|_2. \end{aligned}$$

Since by (3.5) we have $|\mathring{A}|^2 = \mathrm{O}(\sigma^{-3-2\delta})$ and $H^2 - h^2 = 2h(H-h) + \mathrm{O}(\sigma^{-3-2\delta})$, we find

$$\langle Lu, v\rangle_2 = \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3}\langle u, v\rangle_2 - \frac{3h}{2}\int_\Sigma (H-h) uv \, d\mu + \mathrm{O}(\sigma^{-3-2\delta}) \|u\|_2 \|v\|_2. \tag{3.12}$$

By choosing $u = v$, we obtain (3.9).

We now recall that, by Proposition 3.2, the Hawking mass $m_H(\Sigma)$ is bounded uniformly in Σ for σ large. Since $\|H - h\|_{L^\infty(\Sigma)} = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$, we see that (3.12) implies

$$|\langle Lu, v\rangle_2| \leq \mathrm{c}\sigma^{-\frac{5}{2}-\delta}\|u\|_2\|v\|_2. \tag{3.13}$$

Suppose now that $\varphi \in (\mathrm{span}\{f_0, f_1, f_2, f_3\})^\perp$. Then

$$\langle L\varphi, \varphi\rangle_2 \geq \left(\lambda_4 - \sup_\Sigma \left||A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right|\right) \int_\Sigma \varphi^2 \, d\mu,$$

by the characterization of λ_4. We have $\lambda_4 > \frac{5}{\sigma_\Sigma^2}$ by Lemma 3.4 and $\left||A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right| \leq \frac{3}{\sigma_\Sigma^2}$ for σ large by (2.9), (2.10) and (2.14) and so we obtain (3.10).

To prove the last assertion, we observe that, since w has zero mean value, we can write $w = w_1 + w_2$ with $w_1 \in \mathrm{span}\{f_1, f_2, f_3\}$ and $w_2 \in (\mathrm{span}\{f_0, f_1, f_2, f_3\})^\perp$. Then we have $\langle u, w_2\rangle_2 = \langle \Delta u, w_2\rangle_2 = 0$, which implies

$$\langle Lu, w_2\rangle_2 = \int_\Sigma \left(-|A|^2 - \overline{\mathrm{Ric}}(\nu,\nu)\right) uw_2 \, d\mu = \int_\Sigma \left(-|A|^2 - \overline{\mathrm{Ric}}(\nu,\nu) + \frac{h^2}{2}\right) uw_2 d\mu.$$

Since $|A|^2 - \frac{h^2}{2} = |\mathring{A}|^2 + \frac{H^2 - h^2}{2}$, with $|\mathring{A}|^2 = \mathrm{O}(\sigma^{-3-\delta})$ and $H^2 - h^2 = \mathrm{O}(\sigma^{-\frac{5}{2}-\delta})$, we obtain

$$|\langle Lu, w_2\rangle_2| \leq \mathrm{c}\sigma^{-\frac{5}{2}-\delta}\|u\|_2\|w_2\|_2.$$

Combining this with (3.13) with $v = w_1$, we obtain (3.11). □

Remark 3.8. We point out that the last term in the left-hand side in (3.9) is in general of order $O(\sigma^{-\frac{5}{2}-\delta})||u||_2^2$ and cannot be absorbed by the term with the mass which is of order $O(\sigma^{-3})||u||_2^2$. Therefore, even if we are assuming the positivity of the mass, we cannot expect that the stability operator is positive definite on the translational eigenspace for the surfaces of our class $\mathcal{W}_\sigma^\eta(B_1, B_2)$.

We conclude this part with an auxiliary estimate.

Lemma 3.9. *There exist $c, \sigma_0 > 0$ such that on any surface $\Sigma \in \mathcal{W}_\sigma^\eta(B_1, B_2)$ with $\sigma \geq \sigma_0$ we have for $\alpha = 1, 2, 3$*

$$\left|\left\langle L(H-h), \frac{\nu_\alpha}{\sigma}\right\rangle_{L^2(\Sigma)}\right| \leq c\sigma^{-3-2\delta} \tag{3.14}$$

where $\nu_\alpha := \overline{\mathrm{g}}(\nu, \overline{\mathrm{e}}_\alpha)$, with $\{\overline{\mathrm{e}}_\alpha\}_{\alpha=1,2,3}$ the canonical basis in the Euclidean coordinates.

Proof. Let f_α^e, with $\alpha = 1, 2, 3$, be the Euclidean eigenfunctions of $-\Delta$ on the sphere of radius σ_Σ which satisfy, according to Lemma 3.4, $\|f_\alpha - f_\alpha^e\|_{W^{2,2}(\Sigma)} \leq C\sigma^{-\frac{1}{2}-\delta}$. Then we have

$$f_\alpha^e = \sqrt{\frac{3}{4\pi\sigma_\Sigma^4}}x_\alpha = \frac{\sqrt{3}}{\sqrt{|\Sigma|}}\nu_\alpha^e,$$

where we have set $\nu^e_\alpha = \overline{\mathrm{g}}^e(\nu^e, \overline{\mathrm{e}}_\alpha)$. Using Lemma 2.2 we find

$$\|\nu_\alpha - \nu^e_\alpha\|_{H^1(\Sigma)} \le C\sigma\|\nu_\alpha - \nu^e_\alpha\|_{W^{1,\infty}(\Sigma)} \le C\sigma^{\frac{1}{2}-\delta}.$$

Therefore

$$\left\|\sqrt{\frac{|\Sigma|}{3}}f_\alpha - \nu_\alpha\right\|_{H^1(\Sigma)} \le \sqrt{\frac{|\Sigma|}{3}}\|f_\alpha - f^e_\alpha\|_{H^1(\Sigma)} + \|\nu_\alpha - \nu^e_\alpha\|_{H^1(\Sigma)} \le c\sigma^{\frac{1}{2}-\delta}. \tag{3.15}$$

Since $\left| |A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right| \le c\sigma^{-2}$, by definition of L we have, for any $u, v \in H^1(\Sigma)$,

$$|\langle Lu, v\rangle_{L^2}| \le \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + c\sigma^{-2}\|u\|_{L^2}\|v\|_{L^2} \le \sigma^{-2}\|u\|_{H^1}\|v\|_{H^1}. \tag{3.16}$$

Using this, together with (3.15),(3.11) and (2.21), we find

$$\begin{aligned}|\langle L(H-h), \nu_\alpha\rangle_{L^2}| &\le \left|\left\langle L(H-h), \sqrt{\frac{|\Sigma|}{3}}f_\alpha - \nu_\alpha\right\rangle_{L^2}\right| + \left|\left\langle L(H-h), \sqrt{\frac{|\Sigma|}{3}}f_\alpha\right\rangle_{L^2}\right|\\ &\le c\sigma^{-2}\|H-h\|_{H^1}\left\|\sqrt{\frac{|\Sigma|}{3}}f_\alpha - \nu_\alpha\right\|_{H^1}\\ &\quad + c\sigma^{-\frac{5}{2}-\delta}\sqrt{\frac{|\Sigma|}{3}}\,\|H-h\|_{L^2}\\ &\le c\sigma^{-2-\delta},\end{aligned}$$

which implies the assertion. □

3.2. The Translational Part of the Mean Curvature

We analyze now an important property of the translational part of a function (see Definition 3.5). To explain it heuristically, consider first the case of a Euclidean round sphere $\mathbb{S}_\sigma(0) \subset \mathbb{R}^3$ and take any $u \in \mathrm{span}\{f^e_1, f^e_2, f^e_3\}$, i.e., $u = \sum_{i=1}^3 a_\alpha x_\alpha$ for some coefficients a_α. Then, the integral of any odd power of u on $\mathbb{S}_\sigma(0)$ vanishes for symmetry reasons. This allows to obtain a strong bound on the corresponding integral when we consider a round surface in an asymptotically flat space. We focus here on the case of a third power, which is the one that we need in the sequel.

Lemma 3.10. *There exist $c, \sigma_0 > 0$ such that on any surface $\Sigma \in \mathcal{W}^\eta_\sigma(B_1, B_2)$ with $\sigma \ge \sigma_0$ we have, for any $u \in \mathrm{span}\{f_1, f_2, f_3\}$,*

$$\left|\int_\Sigma u^3\ d\mu\right| \le c\sigma^{-\frac{3}{2}-\delta}||u||^3_{L^2(\Sigma)}.$$

Proof. As before, we write Σ as a spherical graph over a Euclidean sphere and we identify correspondingly functions on Σ and on $\mathbb{S}_{\sigma_\Sigma}(0)$. By Lemma 3.4 and the Sobolev immersion (2.17), we have

$$||f_\alpha - f^e_\alpha||_{L^\infty(\Sigma)} \le C\sigma^{-1}||f_\alpha - f^e_\alpha||_{H^2(\Sigma)} \le c\sigma^{-\frac{3}{2}-\delta}. \tag{3.17}$$

If we denote by $d\mu$ and $d\mu^e$, respectively, the Riemannian and Euclidean measure on Σ, and by $d\mu^e_{\mathbb{S}}$ the Euclidean measure on $\mathbb{S}_{\sigma_\Sigma}(0)$ we have that $d\mu - d\mu^e = \mathrm{O}(\sigma^{-\frac{1}{2}-\delta})d\mu$, by Lemma 2.2 and also $d\mu^e - d\mu^e_{\mathbb{S}} = \mathrm{O}(\sigma^{-\frac{1}{2}-\delta})d\mu$, by the $W^{2,\infty}$ bound on the spherical graph function in Lemma 2.7(iv). It follows that $d\mu - d\mu_e^{\mathbb{S}} = \mathrm{O}(\sigma^{-\frac{1}{2}-\delta})d\mu$ as well. We now define the auxiliary function

$$\tilde{u} := \sum_{\alpha=1}^{3} \langle u, f_\alpha \rangle_{L^2(\Sigma)} f_\alpha^e,$$

which is a combination of the Euclidean eigenfunctions f_α^e with the coefficients $\langle u, f_\alpha \rangle_{L^2(\Sigma)}$ occurring in the Riemannian decomposition of u. Since $|\langle u, f_\alpha \rangle_{L^2(\Sigma)}| \leq \|u\|_{L^2(\Sigma)}$ and $\|f_\alpha^e\|_{L^\infty(\Sigma)} \leq c\sigma^{-1}$, we have

$$\|\tilde{u}\|_{L^\infty(\Sigma)} \leq c\sigma^{-1}\|u\|_{L^2(\Sigma)}, \qquad \|\tilde{u}\|_{L^2(\Sigma)} \leq c\|u\|_{L^2(\Sigma)},$$

$$|u - \tilde{u}| = \left| \sum_{\alpha=1}^{3} \langle u, f_\alpha \rangle_{L^2(\Sigma)} (f_\alpha - f_\alpha^e) \right| \leq \|u\|_{L^2(\Sigma)} \sum_{\alpha=1}^{3} |f_\alpha - f_\alpha^e|.$$

By (3.17), we deduce

$$\|u - \tilde{u}\|_{L^\infty(\Sigma)} \leq c\sigma^{-\frac{3}{2}-\delta}\|u\|_{L^2(\Sigma)}.$$

Then we can compute

$$\begin{aligned} \left| \int_\Sigma \left(u^3 - \tilde{u}^3\right) \, d\mu \right| &\leq \frac{3}{2}\|u - \tilde{u}\|_{L^\infty(\Sigma)} \left(\|u\|^2_{L^2(\Sigma)} + \|\tilde{u}\|^2_{L^2(\Sigma)} \right) \\ &\leq c\sigma^{-\frac{3}{2}-\delta}\|u\|^3_{L^2(\Sigma)}. \end{aligned} \tag{3.18}$$

When considered as a function on $\mathbb{S}_{\sigma_\Sigma}(0)$, the function $\tilde{u}$ satisfies for symmetry reasons

$$\int_{\mathbb{S}_{\sigma_\Sigma}} \tilde{u}^3 \, d\mu_e^{\mathbb{S}} = 0.$$

It follows

$$\begin{aligned} \left| \int_\Sigma \tilde{u}^3 \, d\mu \right| = \left| \int_\Sigma \tilde{u}^3 d\mu - \int_{\mathbb{S}_{\sigma_\Sigma}} \tilde{u}^3 \, d\mu_e^{\mathbb{S}} \right| &\leq c\sigma^{-\frac{1}{2}-\delta} \int_\Sigma |\tilde{u}|^3 d\mu \\ \leq c\sigma^{\frac{3}{2}-\delta}\|\tilde{u}\|^3_{L^\infty(\Sigma)} &\leq c\sigma^{-\frac{3}{2}-\delta}\|u\|^3_{L^2(\Sigma)}, \end{aligned}$$

which implies the assertion, thanks to (3.18). □

In particular the above estimate can be applied to the translational part of the mean curvature. Observe that by definition $H^t = (H-h)^t$; we use the longer expression $(H-h)^t$ because it makes more explicit the relation with the speed of the flow studied in the next section and the property of zero mean value.

Corollary 3.11. *There exist $c, \sigma_0 > 0$ such that on any surface $\Sigma \in \mathcal{W}_\sigma^\eta(B_1, B_2)$ with $\sigma \geq \sigma_0$ the translational part of the mean curvature satisfies*

$$\left| \int_\Sigma ((H-h)^t)^3 \, d\mu \right| \leq c\sigma^{-2-2\delta}\|(H-h)^t\|^2_{L^2(\Sigma)}$$

Proof. By (2.21), we have $\|(H-h)^t\|_{L^2} \leq \|H-h\|_{L^2} \leq c\sigma^{-\frac{1}{2}-\delta}$. Then the assertion follows from the previous lemma. □

Remark 3.12. Observe that this result is sharper than the one we would obtain by simply estimating one factor of $(H-h)^t$ with $\|H-h\|_{L^\infty(\Sigma)} = O(\sigma^{-\frac{3}{2}-\delta})$.

The next lemma provides an approximation of $\|(H-h)^t\|_{L^2}$ which will be useful in the sequel.

Lemma 3.13. *There exist $c, \sigma_0 > 0$ such that on any surface $\Sigma \in \mathcal{W}^\eta_\sigma(B_1, B_2)$ with $\sigma \geq \sigma_0$ we have, for any $\varepsilon > 0$,*

$$\left| \frac{4\pi}{3} \|(H-h)^t\|^2_{L^2(\Sigma)} - \sum_{\alpha=1}^{3} \left\langle H-h, \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle^2_{L^2(\Sigma)} \right| \leq c\sigma^{-1-2\delta}(1+\varepsilon^{-1})\|H-h\|^2_{L^2(\Sigma)} + \varepsilon \|(H-h)^t\|^2_{L^2(\Sigma)}.$$

Proof. We have

$$\begin{aligned} &\frac{4\pi}{3} \|(H-h)^t\|^2_{L^2(\Sigma)} - \sum_{\alpha=1}^{3} \left\langle H-h, \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle^2_{L^2(\Sigma)} \\ &= \sum_{\alpha=1}^{3} \left(\frac{4\pi}{3} \langle H-h, f_\alpha \rangle^2_{L^2(\Sigma)} - \left\langle H-h, \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle^2_{L^2(\Sigma)} \right) \\ &= \sum_{\alpha=1}^{3} \left\langle H-h, \sqrt{\frac{4\pi}{3}} f_\alpha - \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle_{L^2(\Sigma)} \left\langle H-h, \sqrt{\frac{4\pi}{3}} f_\alpha + \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle_{L^2(\Sigma)}. \end{aligned}$$

By (3.15), we have

$$\begin{aligned} \left| \left\langle H-h, \sqrt{\frac{4\pi}{3}} f_\alpha - \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle_{L^2(\Sigma)} \right| &\leq \|H-h\|_{L^2(\Sigma)} \left\| \sqrt{\frac{4\pi}{3}} f_\alpha - \frac{\nu_\alpha}{\sigma_\Sigma} \right\|_{L^2(\Sigma)} \\ &\leq c\sigma^{-\frac{1}{2}-\delta} \|(H-h)\|_{L^2(\Sigma)}. \end{aligned}$$

We further observe

$$\begin{aligned} &\left\langle H-h, \sqrt{\frac{4\pi}{3}} f_\alpha + \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle_{L^2(\Sigma)} \\ &= -\left\langle H-h, \sqrt{\frac{4\pi}{3}} f_\alpha - \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle_{L^2(\Sigma)} + 2\sqrt{\frac{4\pi}{3}} \langle H-h, f_\alpha \rangle_{L^2(\Sigma)}, \end{aligned}$$

which implies, by definition of $(H-h)^t$,

$$\begin{aligned} &\left| \left\langle H-h, \sqrt{\frac{4\pi}{3}} f_\alpha + \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle_{L^2(\Sigma)} \right| \\ &\leq c\sigma^{-\frac{1}{2}-\delta} \|(H-h)\|_{L^2(\Sigma)} + 2\sqrt{\frac{4\pi}{3}} \|(H-h)^t\|_{L^2(\Sigma)}. \end{aligned}$$

Putting together the above inequalities we obtain the assertion. □

We conclude the section by observing that the translational part of the mean curvature of a coordinate sphere satisfies an improved estimate if our ambient manifold satisfies the weak Regge–Teitelboim conditions.

Lemma 3.14. *Let* $(\mathrm{M}, \bar{\mathrm{g}})$ *satisfy the* $C^1_{1+\delta}$*-Regge–Teitelboim conditions* (2.2). *Consider the Euclidean coordinate sphere* $\Sigma = \mathbb{S}_r(0)$ *for some* $r > 1$. *There exist* $r_0, C > 0$ *depending only on* $\bar{c}$ *in* (1.1)–(2.2), *such that, if* $r \geq r_0$, *then* $|(H-h)^t|^2_{L^2(\Sigma)} \leq C\sigma_\Sigma^{-2-2\delta}$.

Proof. In this proof, we denote by σ_Σ the area radius of the Euclidean sphere $\Sigma = \mathbb{S}_r(0)$, which is asymptotic to r by Lemma 2.2. A direct computation shows that conditions (2.2) imply, for any $\alpha = 1, 2, 3$,

$$|\nu_\alpha(\bar{x}) - \nu_\alpha(-\bar{x})| \leq Cr^{-1-\delta}, \quad |H(\bar{x}) - H(-\bar{x})| \leq Cr^{-2-\delta},$$

where $\nu_\alpha := \bar{\mathrm{g}}(\nu, \bar{\mathrm{e}}_\alpha)$. From this it also follows $|H(\bar{x})\nu_\alpha(\bar{x}) + H(-\bar{x})\nu_\alpha(-\bar{x})| \leq Cr^{-2-\delta}$. Since by (2.2) $d\mu$ is antipodally symmetric on Σ up to $O(r^{1+\delta})d\mu$, we deduce

$$\left| \int_{S_r(0)} \nu_\alpha \, d\mu \right| \leq Cr^{1-\delta}, \qquad \left| \int_{S_r(0)} H\nu_\alpha \, d\mu \right| \leq Cr^{-\delta},$$

which implies

$$\left| \int_{S_r(0)} (H-h)\nu_\alpha \, d\mu \right| \leq \left| \int_{S_r(0)} H\nu_\alpha \, d\mu \right| + h \left| \int_{S_r(0)} \nu_\alpha \, d\mu \right| \leq Cr^{-\delta}.$$

Using again the asymptotic equivalence of r and σ_Σ, we obtain

$$\left\langle H - h, \frac{\nu_\alpha}{\sigma_\Sigma} \right\rangle_{L^2(\Sigma)} \leq C\sigma_\Sigma^{-1-\delta}.$$

Since by Lemma 2.2 $\|(H-h)\|_{L^2(\Sigma)} \leq C\sigma_\Sigma^{-\frac{1}{2}-\delta}$, we obtain from Lemma 3.13 with $\varepsilon = 1$

$$\frac{4\pi}{3}\|(H-h)^t\|^2_{L^2(\Sigma)} \leq C\sigma_\Sigma^{-2-2\delta} + c\sigma^{-2-4\delta} + \|(H-h)^t\|^2_{L^2(\Sigma)}.$$

From the proof of Lemma 3.13 one sees that in the case of a Euclidean coordinate sphere the constant c in the above inequality only depends on $\bar{c}$ in (1.1), and this implies the assertion. □

4. Volume Preserving Mean Curvature Flow

4.1. Definition of the Flow and Evolution Equations

Definition 4.1. Let $(\mathrm{M}, \bar{\mathrm{g}})$ be a Riemannian manifold, and let $\iota : \Sigma \hookrightarrow \mathrm{M}$ be a closed hypersurface. A time-dependent family of immersions $F_t : \Sigma \hookrightarrow \mathrm{M}$, with $t \in [0, T)$ for some $0 < T \leq +\infty$, which satisfies

$$\begin{cases} \frac{\partial}{\partial t} F_t(\cdot) = -(H(\cdot, t) - h(t))\nu(\cdot, t) \\ F_0 = \iota \end{cases} \tag{4.1}$$

is called a solution to the *volume preserving mean curvature flow*, with initial value ι.

It is well-known that this flow is parabolic and it has a smooth solution at least locally in time. In the following, we always assume that the ambient manifold $(\mathrm{M}, \overline{\mathrm{g}})$ is 3-dimensional and $C^2_{\frac{1}{2}+\delta}$-asymptotically flat. We write $\Sigma_t := F_t(\Sigma)$ to denote the immersed surface at time t, and we call for simplicity Σ_t the "solution of the flow" (4.1) without mentioning explicitly the immersions F_t. We call $g(t)$ the induced metric on Σ at time t, by $d\mu_t$ the corresponding measure and by $A(t)$ the second fundamental form of Σ at time t. Since in what follows we will frequently use complicated integral expressions involving these quantities associated with Σ_t, we will in such cases abbreviate $g(t)$, $d\mu_t$, $A(t)$, etc. simply by g, $d\mu$, A, etc., leaving the dependence on t implicit in the domain of integration Σ_t.

We recall the evolution equations satisfied by the main geometric quantities on Σ_t. We choose at each fixed time a frame $\vec{e}_\alpha$ on the ambient manifold M such that $\vec{e}_1, \vec{e}_2$ are tangent vectors on Σ and $\vec{e}_3 = \nu$. Then the main geometric quantities on Σ_t satisfy the following equations along the flow, see e.g., [17].

Lemma 4.2. *Let $\{F_t\}_{t\in[0,T)}$ be a solution of the flow (4.1). Then we have the equations*

(i) $\frac{\partial g_{ij}}{\partial t} = -2(H-h)A_{ij}$;

(ii) $\frac{\partial \nu}{\partial t} = \nabla H$;

(iii) $\frac{\partial}{\partial t}(d\mu_t) = -(H-h)H\, d\mu_t$;

(iv) $\frac{\partial}{\partial t} A_{ij} = \nabla_i\nabla_j H + (H-h)\left(-A_{ik}A^k_j + \overline{\mathrm{Rm}}_{i3j3}\right)$;

(v) $\frac{\partial H}{\partial t} = \Delta H + (H-h)(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))$.

Observe that the right-hand side of (v) can also be written as $-L(H-h)$, where L is the stability operator associated to Σ_t. As an immediate consequence of the above equations we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Sigma_t| = -\|H-h\|^2_{L^2(\Sigma_t)}, \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|^2_{L^2(\Sigma_t)} = -2\langle L(H-h), H-h\rangle - \int_{\Sigma_t} H(H-h)^3 d\mu. \tag{4.3}$$

We can rewrite the term $\nabla_i\nabla_j H$ in the right-hand side of (iv) by means of the Simons identity, as in [24],

$$\begin{aligned}\Delta A_{ij} &= \nabla_i\nabla_j H + HA^l_iA_{lj} - |A|^2A_{ij} + A^l_i\overline{\mathrm{Rm}}_{kjkl} + A^{lk}\overline{\mathrm{Rm}}_{lijk}\\ &\quad +\nabla_j\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\right) + \nabla_l\left(\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\right).\end{aligned} \tag{4.4}$$

In this form of the equality the derivatives of the Ricci and the Riemann tensor are taken with respect to the connection of Σ, in contrast to the formula used in [18], where they are taken with respect to the ambient space. This allows to deal with these terms inside integral quantities by partial integration on Σ. Using Lemma 4.2 and Simons identity we obtain, by straightforward computations, the following result.

Lemma 4.3. *Along a solution of the volume preserving mean curvature flow we have*

$$\frac{\partial}{\partial t}|\mathring{A}|^2 = \Delta|\mathring{A}|^2 - 2|\nabla \mathring{A}|^2 + \frac{2h}{H}\{|A|^4 - H\mathrm{tr}(A^3)\} + 2|A|^2\left(\frac{H-h}{H}\right)|\mathring{A}|^2$$
$$+2(H-h)\mathring{A}^{ij}\overline{\mathrm{Rm}}_{kilj}\nu^k\nu^l - 2\left(A_i^l\overline{\mathrm{Rm}}^k{}_{jkl} + A^{lk}\overline{\mathrm{Rm}}_{lijk}\right)A^{ij} \quad (4.5)$$
$$-2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\right) + \nabla_l\left(\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\right)\right)\mathring{A}^{ij}.$$

$$\frac{\partial}{\partial t}|\nabla H|^2 = \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2(H-h)A^{ij}\nabla_i H\nabla_j H$$
$$+2(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))|\nabla H|^2 - 2\mathrm{Ric}^\Sigma(\nabla H,\nabla H) \quad (4.6)$$
$$+2(H-h)\langle\nabla|A|^2,\nabla H\rangle + 2(H-h)\langle\nabla\left(\overline{\mathrm{Ric}}(\nu,\nu)\right),\nabla H\rangle,$$

where Ric^Σ *is the Ricci tensor on* Σ *and* $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_g$.

4.2. Evolution of Integral Quantities

In this subsection, we study the evolution of the integral quantities which appear in the definition of round surfaces, with the aim of studying the invariance of the class along the flow. In our statements we will assume that our evolving surfaces satisfy properties (2.9) and (2.10); in some of the results, we further require

$$\|H-h\|_{L^\infty(\Sigma_t)} \le c_\infty\sigma^{-\frac{3}{2}-\delta}, \qquad \left\|\mathring{A}(t)\right\|_{L^\infty(\Sigma)} \le B_\infty\sigma^{-\frac{3}{2}-\delta}, \quad (4.7)$$

for suitable c_∞, B_∞. On the other hand, we do not require a priori properties (2.11) and (2.12). It will be important to keep explicit track of the dependence of the constants which appear in the estimates: in this way we can later fix the parameters of our roundness class in order to have invariance under the flow.

We start by estimating the L^4 norm of $|\mathring{A}|$. For this result, we can replace hypothesis (4.7) by a milder assumption on $|H-h|$.

Proposition 4.4. *Let* Σ_t *be a solution of the flow* (4.1) *for* $t\in[0,T]$ *satisfying properties* (2.9) *and* (2.10) *for some* $\sigma>1$. *Suppose in addition*

$$\|H-h\|_{L^\infty(\Sigma_t)} \le \frac{1}{20\sigma} \quad (4.8)$$

for all $t\in[0,T]$. *Then there exist a constant* $C = C(\bar{c},\delta)>0$ *and a radius* $\sigma_0=\sigma_0(\bar{c},\delta)>0$ *such that if* $\sigma>\sigma_0$ *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}|\mathring{A}|^4\,d\mu \le -2\int_{\Sigma_t}|\mathring{A}|^2|\nabla\mathring{A}|^2\,d\mu - \frac{1}{2\sigma^2}\int_{\Sigma_t}|\mathring{A}|^4\,d\mu + C\sigma^{-6-4\delta}. \quad (4.9)$$

As a consequence, if $\int_{\Sigma_0}|\mathring{A}|^4\,d\mu < B_1\sigma^{-4-4\delta}$ *for some* $B_1>2C$, *then* $\int_{\Sigma_t}|\mathring{A}|^4 d\mu < B_1\sigma^{-4-4\delta}$ *for every* $t\in[0,T]$.

Proof. From Equation (4.5), we deduce, using integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}|\mathring{A}|^4\ d\mu = \int_{\Sigma_t}|\mathring{A}|^4 H(h-H)\ d\mu + 2\int_{\Sigma_t}|\mathring{A}|^2\left(\frac{\partial}{\partial t}|\mathring{A}|^2\right)\ d\mu$$

$$= \int_{\Sigma_t}|\mathring{A}|^4 H(h-H)\ d\mu - 2\int_{\Sigma_t}|\nabla|\mathring{A}|^2|^2\ d\mu - 4\int_{\Sigma_t}|\mathring{A}|^2|\nabla\mathring{A}|^2\ d\mu \tag{4.10}$$

$$+4\int_{\Sigma_t}|\mathring{A}|^2\frac{h}{H}(|A|^4 - H\mathrm{tr}(A^3))\,d\mu + 4\int_{\Sigma_t}|A|^2\left(1-\frac{h}{H}\right)|\mathring{A}|^4\,d\mu \tag{4.11}$$

$$+4\int_{\Sigma_t}(H-h)|\mathring{A}|^2\mathring{A}^{ij}\overline{\mathrm{Rm}}_{kilj}\nu^k\nu^l\ d\mu - 8\int_{\Sigma_t}\overline{\mathrm{Rm}}_{1212}|\mathring{A}|^4\,d\mu \tag{4.12}$$

$$-4\int_{\Sigma_t}\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\right) + \nabla_l\left(\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\right)\right)\mathring{A}^{ij}|\mathring{A}|^2\,d\mu, \tag{4.13}$$

where we have used the identity

$$-2\left(A_i^l\overline{\mathrm{Rm}}_{kjkl} + A^{lk}\overline{\mathrm{Rm}}_{lijk}\right)A^{ij} = -4|\mathring{A}|^2\overline{\mathrm{Rm}}_{1212},$$

which follows from the symmetries of the Riemann tensor.

To estimate the above terms, we first observe that (2.9) and (4.8) imply

$$\frac{1}{\sigma}\le H\le\frac{\sqrt{5}}{\sigma},\qquad\left|1-\frac{h}{H}\right|\le\frac{1}{20},\qquad H(h-H)\le\frac{1}{4\sigma^2}.$$

In addition, we recall the identity

$$|A|^4 - H\mathrm{tr}(A^3) = -2\kappa_1\kappa_2|\mathring{A}|^2.$$

Using again (2.9), we can estimate the positive terms in lines (4.10)-(4.11) as follows

$$\int_{\Sigma_t}|\mathring{A}|^4 H(h-H)\ d\mu + 4\int_{\Sigma_t}|\mathring{A}|^2\frac{h}{H}(|A|^4 - H\mathrm{tr}(A^3))\,d\mu$$
$$+4\int_{\Sigma_t}|A|^2\left(1-\frac{h}{H}\right)|\mathring{A}|^4\,d\mu$$
$$\le\left(\frac{1}{4}-\frac{19}{10}+\frac{1}{2}\right)\frac{1}{\sigma^2}\int_{\Sigma_t}|\mathring{A}|^4\,d\mu\le\frac{1}{\sigma^2}\int_{\Sigma_t}|\mathring{A}|^4\,d\mu. \tag{4.14}$$

We now consider the contribution of (4.12). Using (2.14) we find, for any $a > 0$,

$$4\int_{\Sigma_t}(H-h)|\mathring{A}|^2\mathring{A}^{ij}\overline{\mathrm{Rm}}_{kilj}\nu^k\nu^l\ d\mu - 8\int_{\Sigma_t}\overline{\mathrm{Rm}}_{1212}|\mathring{A}|^4\,d\mu$$
$$\le C\int_{\Sigma_t}\sigma^{-\frac{7}{2}-\delta}|\mathring{A}|^3\ d\mu + C\int_{\Sigma_t}\sigma^{-\frac{5}{2}-\delta}|\mathring{A}|^4\ d\mu$$
$$\le C\int_{\Sigma_t}\left(\left(\frac{a}{\sigma^2}+\sigma^{-\frac{5}{2}-\delta}\right)|\mathring{A}|^4 + \frac{1}{a^3}\sigma^{-8-4\delta}\right)\ d\mu,$$

where we have used Young's inequality and C denotes as usual a constant which can change from line to line, but only depends on $\bar{c},\delta$.

To estimate the term in (4.13), we use integration by parts and find

$$\begin{aligned}
&-4\int_{\Sigma_t}\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\right)+\nabla_l\left(\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\right)\right)\mathring{A}^{ij}|\mathring{A}|^2\,d\mu\\
&=4\int_{\Sigma_t}\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\nabla_j(\mathring{A}^{ij}|\mathring{A}|^2)+\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\nabla_l(\mathring{A}^{ij}|\mathring{A}|^2)\right)d\mu\\
&\le C\int_{\Sigma_t}|\overline{\mathrm{Rm}}||\nabla\mathring{A}||\mathring{A}|^2\,d\mu\le C\int_{\Sigma_t}\sigma^{-\frac{5}{2}-\delta}|\nabla\mathring{A}||\mathring{A}|^2\,d\mu\\
&\le 2\int_{\Sigma_t}|\nabla\mathring{A}|^2|\mathring{A}|^2\,d\mu+C\sigma^{-5-2\delta}\int_{\Sigma_t}|\mathring{A}|^2\,d\mu\\
&\le 2\int_{\Sigma_t}|\nabla\mathring{A}|^2|\mathring{A}|^2\,d\mu+C\int_{\Sigma_t}\left(a\sigma^{-2}|\mathring{A}|^4+\frac{1}{a}\sigma^{-8-4\delta}\right)d\mu.
\end{aligned}$$

The $|\nabla\mathring{A}|^2|\mathring{A}|^2$ term can be absorbed by the corresponding negative term in (4.10). Therefore, by choosing a suitably small and σ large (both depending only on $\bar{c},\delta$) we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathring{A}\|^4_{L^4(\Sigma_t)}\le -2\int_{\Sigma_t}|\mathring{A}|^2|\nabla\mathring{A}|^2\,d\mu-\frac{1}{2\sigma^2}\|\mathring{A}\|^4_{L^4(\Sigma_t)}+C\sigma^{-6-4\delta}\tag{4.15}$$

The last claim in our statement follows by a standard ODE comparison argument. □

We next estimate the rate of change of the volume preserving term $h(t)$ and of the L^4 norm of $H-h$.

Lemma 4.5. *Let Σ_t be a solution of the flow (4.1) for $t\in[0,T]$, satisfying properties (2.9), (2.10) and (4.7). Then there exist a constant $c=c(c_\infty,\bar{c})>0$ and a universal constant $C_1>0$ such that*

$$|\dot{h}(t)|\le c\sigma^{-4-2\delta},\tag{4.16}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t}(H-h)^4\,d\mu&\le -12\int_{\Sigma_t}(H-h)^2|\nabla H|^2\,d\mu\\
&+C_1\sigma^{-2}\int_{\Sigma_t}(H-h)^4\,d\mu+c\sigma^{-\frac{13}{2}-5\delta},
\end{aligned}\tag{4.17}$$

provided $\sigma\ge\sigma_0$, for a suitable $\sigma_0=\sigma_0(\bar{c},c_\infty,B_\infty)$.

Proof. Similar to [21, Lemma 14] we compute,

$$\begin{aligned}
|\Sigma_t|\dot{h}(t)&=\int_{\Sigma_t}\frac{\partial H}{\partial t}\,d\mu+\int_{\Sigma_t}H^2(h-H)\,d\mu+h\int_{\Sigma_t}(H-h)^2\,d\mu\\
&=\int_{\Sigma_t}(H-h)\left(|\mathring{A}|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right)d\mu-\int_{\Sigma_t}(H-h)\left(\frac{H^2}{2}-Hh+h^2\right)d\mu\\
&=\int_{\Sigma_t}(H-h)\left(|\mathring{A}|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right)d\mu-\frac{1}{2}\int_{\Sigma_t}(H-h)^3\,d\mu,
\end{aligned}$$

using the property that $\int_{\Sigma_t} (H-h)\, d\mu = 0$. By (2.14), we have $\left|\overline{\mathrm{Ric}}(\nu,\nu)\right| \leq C\sigma^{-\frac{5}{2}-\delta}$ where $C = C(\bar{c})$. Then we can estimate

$$|\Sigma_t||\dot{h}(t)| \leq |\Sigma_t| \left(c_\infty \sigma^{-\frac{3}{2}-\delta}(B_\infty^2 \sigma^{-3-2\delta} + C\sigma^{-\frac{5}{2}-\delta}) + \frac{1}{2} c_\infty^3 \sigma^{-\frac{9}{2}-3\delta} \right).$$

After simplifying the $|\Sigma_t|$ factor, the lower order term on the right-hand side is $c_\infty C \sigma^{-4-\delta}$, and the other terms can be included in this one if σ is large depending on B_∞, c_∞, C. This proves that (4.16) holds for some c only depending on $c_\infty, \bar{c}$.

We now compute, using Lemma 4.2 and integration by parts,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t} (H-h)^4\, d\mu = &-12\int_{\Sigma_t} (H-h)^2 |\nabla H|^2\, d\mu \\ &+4\int_{\Sigma_t} (H-h)^4 (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\, d\mu \\ &-4\dot{h}\int_{\Sigma_t} (H-h)^3\, d\mu - \int_{\Sigma_t} H(H-h)^5\, d\mu.\end{aligned}$$

By (2.9), and (2.14) we deduce that $\left| |A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right| \leq 3\sigma^{-2}$ if σ is large enough. In addition,(4.7), (2.10) and (4.16) imply

$$\left| \dot{h} \int_{\Sigma_t} (H-h)^3\, d\mu \right| \leq c(c_\infty, \bar{c})\sigma^{-\frac{13}{2}-5\delta}. \tag{4.18}$$

From this we obtain the conclusion, also observing that $|H(H-h)| \leq 5\sigma^{-2}$ in view of (2.9). □

We now analyze the evolution of the L^4 norm of $|\nabla H|$.

Lemma 4.6. *Let Σ_t be a solution to the volume preserving mean curvature flow (4.1) for $t \in [0,T]$, which satisfies properties (2.9) and(2.10). Then there exists a universal constant $C_2 > 0$ and a radius $\sigma_0 = \sigma_0(\bar{c},\delta)$ such that if $\sigma > \sigma_0$ then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t} |\nabla H|^4\, d\mu \leq &-3\int_{\Sigma_t} |\nabla^2 H||\nabla H|^2\, d\mu + C_2\sigma^{-6}\int_{\Sigma_t} (H-h)^4\, d\mu \\ &+C_2\sigma^{-2}\int_{\Sigma_t} |\nabla H|^4\, d\mu.\end{aligned}$$

Proof. From (4.6) we obtain, after integrating by parts,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma_t} |\nabla H|^4\, d\mu = &\int_{\Sigma_t} |\nabla H|^4 H(h-H)\, d\mu - 4\int_{\Sigma_t} \left|\nabla|\nabla H|^2\right|^2\, d\mu \\ &-4\int_{\Sigma_t} |\nabla^2 H|^2 |\nabla H|^2\, d\mu + 4\int_{\Sigma_t} (H-h)A^{ij}\nabla_i H \nabla_j H |\nabla H|^2\, d\mu \\ &-4\int_{\Sigma_t} \mathrm{Ric}^\Sigma(\nabla H, \nabla H)|\nabla H|^2\, d\mu \\ &-4\int_{\Sigma_t} (H-h)\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\nabla\cdot\left(|\nabla H|^2 \nabla H\right)\, d\mu.\end{aligned}$$

By (2.9), we have that $H, |H-h|$ and $|A|$ are all bounded by $C\sigma^{-1}$. Using the asymptotic flatness (2.14) we also obtain that $|\mathrm{Ric}^{\Sigma}| \leq C\sigma^{-2}$ and $\big||A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\big| \leq C\sigma^{-2}$ for σ enough large. Then we have, for $\varepsilon > 0$ arbitrary,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \int_{\Sigma_t} |\nabla H|^4 \, d\mu \leq & -4 \int_{\Sigma_t} |\nabla^2 H|^2 |\nabla H|^2 \, d\mu + C\sigma^{-2} \int_{\Sigma_t} |\nabla H|^4 \, d\mu \\ & + C\sigma^{-2} \int_{\Sigma_t} |H-h||\nabla H|^2 |\nabla^2 H| \, d\mu \\ \leq & \left(\frac{\varepsilon}{2} C - 4\right) \int_{\Sigma_t} |\nabla^2 H|^2 |\nabla H|^2 \, d\mu + C\sigma^{-2} \int_{\Sigma_t} |\nabla H|^4 \, d\mu \\ & + \frac{C}{2\varepsilon} \sigma^{-4} \int_{\Sigma_t} |H-h|^2 |\nabla H|^2 \, d\mu. \end{aligned}$$

The assertion follows choosing $\epsilon = 2/C$ and estimating the last term as follows

$$\sigma^{-4} \int_{\Sigma_t} |H-h|^2 |\nabla H|^2 \, d\mu \leq \sigma^{-6} \int_{\Sigma_t} (H-h)^4 \, d\mu + \sigma^{-2} \int_{\Sigma_t} |\nabla H|^4 \, d\mu.$$

□

We can now estimate the weighted $W^{1,4}$-norm of $H-h$ which appears in condition (2.12). We first prove separately a simple auxiliary inequality.

Lemma 4.7. *Let $\Sigma \subset \mathrm{M}$ be a closed surface. Then we have, for every $\varepsilon > 0$ and $\sigma > 1$,*

$$-\sigma^{-4} \int_{\Sigma} (H-h)^2 |\nabla H|^2 \, d\mu \leq -\frac{\varepsilon}{2\sigma^2} \int_{\Sigma} |\nabla H|^4 \, d\mu + \varepsilon^2 \int_{\Sigma} |\nabla^2 H|^2 |\nabla H|^2 \, d\mu.$$

Proof. Since h is constant, we can write

$$\begin{aligned} \sigma^{-2} \int_{\Sigma} |\nabla H|^4 \, d\mu &= \sigma^{-2} \int_{\Sigma} \langle \nabla (H-h), \nabla H \rangle \, |\nabla H|^2 \, d\mu \\ &= -\sigma^{-2} \int_{\Sigma} (H-h)(\Delta H) |\nabla H|^2 \, d\mu - 2\sigma^{-2} \int_{\Sigma} (H-h) g^{ij} \nabla_j H g^{kl} \nabla_i \nabla_k H \nabla_l H \, d\mu \\ &= \frac{\sqrt{2}+2}{\sigma^2} \int_{\Sigma} |H-h| \, |\nabla^2 H| |\nabla H|^2 \, d\mu \leq 2 \int_{\Sigma} \left(\frac{(H-h)^2}{\varepsilon \sigma^4} + \varepsilon |\nabla^2 H|^2 \right) |\nabla H|^2 d\mu, \end{aligned}$$

which implies the assertion. □

Lemma 4.8. *Let Σ_t be a solution to the volume preserving mean curvature flow for $t \in [0,T]$, which satisfies properties (2.9), (2.10) and (4.7). Suppose in addition that (2.11) holds for $t \in [0,T]$ for some $B_1 > 0$. For $\eta > 0$, let us set*

$$\mathrm{a}_\eta(t) := \eta \sigma^{-4} \|H-h\|^4_{L^4(\Sigma_t)} + \|\nabla H\|^4_{L^4(\Sigma_t)}.$$

Then we can find a universal constant $\eta_{\mathrm{w}} > 0$, a constant $\tilde{c} = \tilde{c}(B_1, \overline{c}, \delta)$ and a radius $\sigma_0 = \sigma_0(B_\infty, B_1, c_\infty, \delta, \overline{c}) > 1$ such that, if $B_2 > \tilde{c}$ and $\sigma > \sigma_0$ we have the implication

$$\mathrm{a}_{\eta_w}(0) < B_2 \sigma^{-8-4\delta} \implies \mathrm{a}_{\eta_w}(t) < B_2 \sigma^{-8-4\delta} \textit{ for every } t \in [0,T].$$

Proof. From the previous Lemmas, we have that

$$\begin{aligned}\dot{\mathrm{a}}_\eta(t) \le& -3\int_{\Sigma_t} |\nabla^2 H||\nabla H|^2 \, d\mu + C_2\sigma^{-6}\int_{\Sigma_t}(H-h)^4 \, d\mu + C_2\sigma^{-2}\int_{\Sigma_t}|\nabla H|^4 \, d\mu + \\ & -12\eta\sigma^{-4}\int_{\Sigma_t}(H-h)^2|\nabla H|^2 \, d\mu + C_1\eta\sigma^{-6}\int_{\Sigma_t}(H-h)^4 \, d\mu + \eta c(c_\infty,\bar{c})\sigma^{-\frac{21}{2}-5\delta} \\ \le& \,(12\eta\varepsilon^2-3)\int_{\Sigma_t}|\nabla^2 H|^2|\nabla H|^2 \, d\mu + (C_2 - 6\eta\varepsilon)\sigma^{-2}\int_{\Sigma_t}|\nabla H|^4 \, d\mu \\ & +(\eta C_1 + C_2)\sigma^{-6}\int_{\Sigma_t}(H-h)^4 \, d\mu + \eta c(c_\infty,\bar{c})\sigma^{-\frac{21}{2}-5\delta},\end{aligned}$$

with C_1, C_2 universal constants. If we now choose

$$\eta = \eta_{\mathrm{w}} := \frac{4}{9}C_2^2, \qquad \varepsilon = \frac{3}{4}C_2 \tag{4.19}$$

the inequality becomes

$$\begin{aligned}\dot{\mathrm{a}}_\eta(t) \; &\le -C_2\sigma^{-2}\int_{\Sigma_t}|\nabla H|^4 \, d\mu + (\eta C_1 + C_2)\sigma^{-6}\int_{\Sigma_t}(H-h)^4 \, d\mu \\ & \quad +\eta c(c_\infty,\bar{c})\sigma^{-\frac{21}{2}-5\delta} \\ &\le -C_2\sigma^{-2}a_\eta(t) + \widetilde{C}\left(\sigma^{-6}\int_{\Sigma_t}(H-h)^4 \, d\mu + \sigma^{-10-4\delta}\right),\end{aligned}$$

for another universal constant $\widetilde{C} > 0$, where we have also used that

$$c(c_\infty,\bar{c})\sigma^{-\frac{21}{2}-5\delta} \le \sigma^{-10-4\delta}$$

if $\sigma \ge \sigma_0$ for a suitable $\sigma_0(c_\infty,\bar{c})$. Now we use point (v) of Lemma 2.7 together with (2.11), to obtain that there exists a constant $\mathrm{c_p} = \mathrm{c_p}(\bar{c},\delta) > 0$ such that

$$\int_{\Sigma_t}(H-h)^4 \, d\mu \le \mathrm{c}_{\mathrm{p}}^4\left(\|\mathring{A}\|^4_{L^4(\Sigma_t)} + \sigma^{-4-4\delta}\right) \le \mathrm{c}_{\mathrm{p}}^4(B_1^4+1)\sigma^{-4-4\delta},$$

for $\sigma \ge \sigma_0$ for a suitable $\sigma_0 = \sigma_0(B_\infty)$. We conclude that

$$\begin{aligned}\dot{\mathrm{a}}_\eta(t) &\le -C_2\sigma^{-2}a_\eta(t) + \widetilde{C}\left(\mathrm{c}_{\mathrm{p}}^4(B_1^4+1)+1\right)\sigma^{-10-4\delta} \\ &= -C_2\sigma^{-2}(a_\eta(t) - \tilde{c}\sigma^{-8-4\delta}),\end{aligned}$$

for $\tilde{c} = \tilde{c}(B_1,\bar{c},\delta)$. The conclusion follows by an ODE comparison argument.

□

From now on, when considering the roundness class $\mathcal{W}^\eta_\sigma(B_1,B_2)$, we fix the parameter η equal to the value η_w given by the previous Lemma, and we will no longer need to specify the dependence on η of the constants in the estimates.

4.3. Evolution of the Barycenter and Convergence

An important assumption in the previous results was the comparability between r_Σ and σ in (2.10), in particular the lower bound on r_Σ which shows that Σ_t stays enough far from the coordinate origin to ensure the desired decay of the ambient curvature. To justify this assumption, we study now the evolution of the barycenter under the flow. We start by proving an important decay estimate on the L^2-norm of $H-h$, which relies on the spectral analysis of Sect. 3.

Proposition 4.9. *Let* $(\mathrm{M},\overline{\mathrm{g}})$ *be a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat manifold with* $\overline{m}_{\mathrm{ADM}}>0$*. Given* $B_1,B_2>0$*, there exists* $\sigma_0=\sigma_0(B_1,B_2,\overline{c},\delta,\overline{m}_{\mathrm{ADM}})$ *such that, if* Σ_t *is a solution to the volume preserving mean curvature flow* (4.1) *which satisfies* $\Sigma_t\in\mathcal{W}^\eta_\sigma(B_1,B_2)$ *for some* $\sigma>\sigma_0$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|^2_{L^2(\Sigma_t)}\le-\frac{4\overline{m}_{\mathrm{ADM}}}{\sigma^3_{\Sigma_t}}\|(H-h)^t\|^2_{L^2(\Sigma_t)}-\frac{2}{\sigma^2_{\Sigma_t}}\|(H-h)^d\|^2_{L^2(\Sigma_t)}.$$

Proof. Let us consider the stability operator associated to $H-h$. To avoid confusion, we point out that in the following formulas the superscript t refers to the spectral decomposition of $H-h$ from Definition 3.5, while the dependence on time will not be written explicitly except in the domain of integration. By writing $H-h=(H-h)^t+(H-h)^d$ and applying Proposition 3.7, we obtain

$$\begin{aligned}&\langle L(H-h),H-h\rangle_2\\ \ge{}&\frac{6m_H(\Sigma_t)}{\sigma^3_\Sigma}\|(H-h)^t\|^2_2-\frac{3h}{2}\int_{\Sigma_t}(H-h)((H-h)^t)^2\,d\mu-c\sigma^{-3-2\delta}\|(H-h)^t\|^2_2\\ &+\frac{2}{\sigma^2_\Sigma}\|(H-h)^d\|^2_2-c\sigma^{-\frac{5}{2}-\delta}\|(H-h)^t\|_2\|(H-h)^d\|_2\\ \ge{}&\frac{5m_H(\Sigma_t)}{\sigma^3_\Sigma}\|(H-h)^t\|^2_2-\frac{3h}{2}\int_{\Sigma_t}(H-h)((H-h)^t)^2\,d\mu+\frac{3}{2\sigma^2_\Sigma}\|(H-h)^d\|^2_2,\end{aligned}$$

for σ large enough. Here we write for simplicity σ_Σ instead of σ_{Σ_t} for the area rdius of Σ_t. Since by Proposition 3.2 $m_H(\Sigma_t)\ge\frac{\overline{m}_{\mathrm{ADM}}}{2}>0$ for σ sufficiently large, we find by (4.3)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|^2_2&=-2\langle L(H-h),H-h\rangle_2-\int_{\Sigma_t}H(H-h)^3\,d\mu\\ &\le-\frac{5\overline{m}_{\mathrm{ADM}}}{\sigma^3_\Sigma}\|(H-h)^t\|^2_2+3h\int_{\Sigma_t}(H-h)((H-h)^t)^2\,d\mu\\ &\quad-\frac{3}{\sigma^2_\Sigma}\|(H-h)^d\|^2_2-h\int_{\Sigma_t}(H-h)^3\,d\mu+c\sigma^{-3-2\delta}\int_{\Sigma_t}(H-h)^2\,d\mu,\end{aligned}$$

where we have used $|H-h|\le c_\infty\sigma^{-\frac{3}{2}-\delta}$ from Lemma 2.7(i). We want to show that the contribution of the $H-h$ integrals can be bounded by the remaining negative terms. By writing again $H-h=(H-h)^t+(H-h)^d$ we obtain

$$\begin{aligned}&3\int_{\Sigma_t}(H-h)((H-h)^t)^2\,d\mu-\int_{\Sigma_t}(H-h)^3\,d\mu\\ ={}&2\int_{\Sigma_t}((H-h)^t)^3\,d\mu-\int_{\Sigma_t}3(H-h)^t((H-h)^d)^2\,d\mu-\int_{\Sigma_t}((H-h)^d)^3\,d\mu.\end{aligned}$$

The first integral was considered in Corollary 3.11. The remaining ones can be estimated as follows

$$\left|\int_{\Sigma_t}3(H-h)^t((H-h)^d)^2\,d\mu+((H-h)^d)^3\,d\mu\right|$$

$$\leq \|3(H-h)^t + (H-h)^d\|_\infty \int_{\Sigma_t} ((H-h)^d)^2 \, d\mu \leq c\sigma^{-\frac{3}{2}-\delta} \int_{\Sigma_t} ((H-h)^d)^2 \, d\mu$$

since also $\|(H-h)^d\|_{L^\infty(\Sigma)} \equiv \|(H-h) - (H-h)^t\|_{L^\infty(\Sigma)} = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$. Combining this with Corollary 3.11, and using $h = O(\sigma^{-1})$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|_2^2 &\leq c\sigma^{-3-2\delta}\|(H-h)^t\|_2^2 + c\sigma^{-\frac{5}{2}-\delta}\|(H-h)^d\|_2^2 \\ &\quad - \frac{5\overline{m}_{\mathrm{ADM}}}{\sigma_\Sigma^3}\|(H-h)^t\|_2^2 - \frac{3}{\sigma_\Sigma^2}\|(H-h)^d\|_2^2 \\ &\leq -\frac{4\overline{m}_{\mathrm{ADM}}}{\sigma_\Sigma^3}\|(H-h)^t\|_2^2 - \frac{2}{\sigma_\Sigma^2}\|(H-h)^d\|_2^2\end{aligned}$$

for σ large enough. □

Remark 4.10. An immediate consequence is that, if σ is sufficiently large, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|_2^2 \leq -\frac{4\overline{m}_{\mathrm{ADM}}}{\sigma_\Sigma^3}\|H-h\|_2^2 \tag{4.20}$$

Let us now define

$$\Pi(t) := \sqrt{\sum_{\alpha=1}^{3} \left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle_{L^2(\Sigma_t)}^2}.$$

Since the L^2-norm of $\frac{\nu_\alpha}{\sigma}$ is uniformly bounded on a round surface for σ large, we obtain that $\Pi(t) \leq C\|H-h\|_{L^2}$ for some universal constant C. We observe that, by Lemma 3.13, $\Pi(t)$ allows to estimate the L^2-norm of the translational part of $H-h$ up to a constant factor. For our purposes it suffices to consider the statement of the Lemma for $\varepsilon = 1$

$$\left|\frac{4\pi}{3}\|(H-h)^t\|_{L^2}^2 - \frac{\sigma^2}{\sigma_\Sigma^2}\Pi(t)^2\right| \leq \|(H-h)^t\|^2 + c\sigma^{-2-4\delta},$$

where we have also used (2.21). Using (2.10) to obtain a rough estimate of σ/σ_Σ we deduce that, for any $c > 0$ we have, for σ large

$$\|(H-h)^t\|_{L^2} \leq c\sigma^{-1-\delta} \implies \Pi(t) \leq 3c\sigma^{-1-\delta} \tag{4.21}$$

and similarly

$$\Pi(t) \leq c\sigma^{-1-\delta} \implies \|(H-h)^t\|_{L^2} \leq c\sigma^{-1-\delta}. \tag{4.22}$$

We can now start our study of the barycenter of the evolving surface.

Lemma 4.11. *Let Σ_t be a solution of the flow (4.1) which belongs to $\mathcal{W}_\sigma^\eta(B_1, B_2)$ for all t. Then we have*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}_{\Sigma_t}\right| \leq C\sigma^{-1}\|H-h\|_{L^2}, \tag{4.23}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Pi(t)\right| \leq c\sigma^{-3-2\delta}, \tag{4.24}$$

for a universal constant C and for a suitable $c = c(B_1, B_2)$, provided σ is large.

Proof. Let us write for simplicity $\vec{z}(t) = \vec{z}_{\Sigma_t}$. A straightforward computation, see for example [6], shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \frac{1}{|\Sigma_t|}\int_{\Sigma_t}(h-H)\left[\nu + H\left(F_t(x)-\vec{z}(t)\right)\right]\,d\mu. \tag{4.25}$$

By Lemma 2.7(iv), we have

$$h = 2\sigma_\Sigma^{-1} + O(\sigma^{-\frac{3}{2}-\delta}), \qquad F_t(x)-\vec{z}(t) = \sigma_{\Sigma_t}\nu + O(\sigma^{\frac{1}{2}-\delta}).$$

which implies $H(F_t(x)-\vec{z}(t)) = 2\nu + O(\sigma^{-\frac{1}{2}-\delta})$. Then we can estimate

$$\begin{aligned}\left|\int_{\Sigma_t}(h-H)\left[\nu + H\left(F_t(x)-\vec{z}(t)\right)\right]\,d\mu\right| &\le \left|3\int_{\Sigma_t}(h-H)\nu\,d\mu\right| \\ &\quad + O(\sigma^{-\frac{1}{2}-\delta})\int_{\Sigma_t}|h-H|d\mu \\ &\le 3\sigma\Pi(t) + c\sigma^{-\frac{1}{2}-\delta}|\Sigma_t|^{\frac{1}{2}}\|H-h\|_{L^2}.\end{aligned}$$

Since $|\Sigma_t| = O(\sigma^2)$, we see that (4.25) implies

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t)\right| \le C\sigma^{-1}\Pi(t) + c\sigma^{-\frac{3}{2}-\delta}\|H-h\|_{L^2},$$

and the right-hand side is smaller than $C\sigma^{-1}\|H-h\|_{L^2}$ for σ large, with C universal constant, which yields (4.23). To derive (4.24) we first compute, by using Lemma 4.2,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle_{L^2(\Sigma_t)} &= \frac{1}{\sigma}\int_{\Sigma_t}(-L(H-h))\nu_\alpha\,d\mu - \frac{1}{\sigma}\int_{\Sigma_t}\dot{h}\nu_\alpha\,d\mu \\ &\quad + \frac{1}{\sigma}\int_{\Sigma_t}(H-h)\langle\nabla H, e_\alpha\rangle_e\,d\mu - \frac{1}{\sigma}\int_{\Sigma_t}(H-h)^2 H\nu_\alpha\,d\mu.\end{aligned}$$

Using (2.21), (3.14) and (4.16) we conclude that the right-hand side is $O(\sigma^{-3-2\delta})$.

□

The next result, which is similar to Proposition 3.4 in [18], gives a bound on the possible change of area of the surface along the flow as long as it remains round.

Lemma 4.12. *Given B_1, B_2, there exist constants $c > 0$ and $\sigma_0 > 1$ such that, if $\sigma > \sigma_0$ and Σ_t is a solution of the flow (4.1) with $\Sigma_t \in \mathcal{W}_\sigma^\eta(B_1, B_2)$ for all $t \in [0, T]$ then*

$$0 \le \sigma_{\Sigma_0} - \sigma_{\Sigma_t} \le c\sigma^{\frac{1}{2}-\delta}$$

for every $t \in [0, T]$.

Proof. By the area-decreasing property (4.2) we have $\sigma_{\Sigma_0} \ge \sigma_{\Sigma_t}$, so we only need to prove the latter inequality. By (2.10), we have $r_{\Sigma_t} < \sigma/2$ and so the Euclidean coordinate sphere $\mathbb{S}_{\frac{\sigma}{2}}(0)$ is enclosed by Σ_t for all t. By definition of our flow, the compact region between Σ_t and $\mathbb{S}_{\frac{\sigma}{2}}(0)$ has constant volume. We call this region Ω_t. We have, using (1.1),

$$|\mathrm{Vol}_e(\Omega_t) - \mathrm{Vol}_g(\Omega_t)| \le C\sigma^{\frac{5}{2}-\delta}.$$

On the other hand, we know from Lemma 2.7(iv) that Σ_t can be written as a graph over a Euclidean sphere of radius σ_{Σ_t} with the radial function of order $O(\sigma^{\frac{1}{2}-\delta})$. It follows

$$\left|\mathrm{Vol}_e(\Omega_t) - \frac{4\pi}{3}(\sigma_{\Sigma_t}^3 - r_0^3)\right| \leq c\sigma^{\frac{5}{2}-\delta}.$$

Since $\mathrm{Vol}_g(\Omega_t)$ is constant, we deduce $|\sigma_{\Sigma_t}^3 - \sigma_{\Sigma_0}^3| \leq c\sigma^{\frac{5}{2}-\delta}$, which implies the assertion. □

We are now ready to prove that, by an appropriate choice of the parameters of the class $\mathcal{B}_\sigma^\eta(B_1, B_2, B_{\mathrm{cen}})$ of well-centered round surfaces, see Definition 2.4, and under suitable conditions on the initial surface, the solution of the flow remains inside the class for arbitrary times. Observe that, in order to control the possible drift of the barycenter, our assumptions on the initial data include an additional smallness requirement on the L^2-norm of the translational part of the mean curvature.

Theorem 4.13. *Let* $(\mathrm{M}, \overline{\mathrm{g}})$ *be a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat manifold with* $\overline{m}_{\mathrm{ADM}} > 0$. *Let* B_1 *be chosen as in* Lemma 4.4 *and* η, B_2 *be chosen as in* Lemma 4.8. *Then, for any* $\mathrm{c_{in}} > 0$ *there exists* $\overline{B} = \overline{B}(\mathrm{c_{in}}, \overline{m}_{\mathrm{ADM}}) > 0$ *and* $\overline{\sigma} = \overline{\sigma}(\overline{c}, \delta, B_1, B_2, \overline{B}, \overline{m}_{\mathrm{ADM}}) > 1$ *such that the following holds. Let* $F_t : \Sigma \hookrightarrow \mathrm{M}$ *be a family of surface immersions which solve the volume preserving mean curvature flow* (4.1) *for* $t \in [0, T]$, *and set* $\Sigma_t = F_t(\Sigma)$. *Suppose that the initial surface* $\Sigma_0 = F_0(\Sigma)$ *satisfies, for some* $\sigma \geq \overline{\sigma}$ *and* $B_{\mathrm{cen}} \geq \overline{B}$:

(i) $\Sigma_0 \in \mathcal{B}_\sigma^\eta(B_1, B_2, B_{\mathrm{cen}})$ *with* $\sigma = \sigma_{\Sigma_0}$,
(ii) $\|(H-h)^t\|_{L^2} \leq \mathrm{c_{in}}\sigma^{-1-\delta}$ *on* Σ_0.

Then $\Sigma_t \in \mathcal{B}_\sigma^\eta(B_1, B_2, 3B_{\mathrm{cen}})$ *for all* $t \in [0, T]$.

Proof. Let η, B_1, B_2 be fixed as in the statement and let $\mathrm{c_{in}} > 0$ be arbitrary. We need to prove that, if Σ_0 satisfies (i) and (ii) for suitably large B_{cen} and σ, whose size will be specified during the proof, then it also satisfies the conclusion.

We argue by contradiction and denote by T' the first time at which the conclusion is violated, that is, $T' \in (0, T]$ is such that

$$\begin{aligned} (\Sigma, g(t)) &\in \mathcal{B}_\sigma^\eta(B_1, B_2, 3B_{\mathrm{cen}}) \text{ for } t \in [0, T'), \\ (\Sigma, g(t)) &\notin \mathcal{B}_\sigma^\eta(B_1, B_2, 3B_{\mathrm{cen}}) \text{ for } t = T'. \end{aligned} \tag{4.26}$$

This means that at least one inequality in the definition of $\mathcal{B}_\sigma^\eta(B_1, B_2, 3B_{\mathrm{cen}})$ becomes an equality at time $t = T'$. The theorem will be proved if we can exclude each of these possibilities.

We first observe that, by Lemma 4.12, we have $|\sigma_{\Sigma_t} - \sigma| \leq c\sigma^{\frac{1}{2}-\delta}$ for $t \in [0, T']$. In addition, using the spherical graph representation in Lemma 2.7 (iv), we have at any point of $\Sigma_{T'}$,

$$\vec{x} = \vec{z}_0 + (\sigma_\Sigma + f)\nu = \vec{z}(t) + \sigma\nu + O(\sigma^{\frac{1}{2}-\delta}),$$

which implies, by (2.13), that

$$||\vec{x}| - \sigma| \leq c\sigma^{1-\delta}$$

for some $c = c(B_1, B_2, B_{\text{cen}}, \bar{c}, \delta)$. From this it follows that, if σ is large enough, the strict inequalities (2.10) hold also at time $t = T'$. Then, by parts (i)-(ii)-(iii) of Lemma 2.7, we deduce that (2.9) holds as well.

Again parts (i) and (iii) of Lemma 2.7 show that assumptions (4.7) and (4.8) are satisfied. Then we can apply first Lemma 4.4 and then Lemma 4.8 to prove that inequalities (2.11) and (2.12) remain strict at time $t = T'$. We conclude that the only property of $\mathcal{B}^{\eta}_{\sigma}(B_1, B_2, 3B_{\text{cen}})$ that can become an equality at time $t = T'$ is the barycenter estimate (2.13), and therefore we have

$$|\vec{z}(T')| = 3B_{\text{cen}}\sigma^{1-\delta}. \tag{4.27}$$

The rest of the proof will be devoted to exclude this last possibility, and this will require a longer work. We first observe that, by (4.21) and assumption (ii), we have

$$\Pi(0) \leq 3\mathrm{c}_{\text{in}}\sigma^{-1-\delta}. \tag{4.28}$$

Let us now denote by t^* the smallest $t \in [0, T']$ such that one of the following properties holds:
(a) $\Pi(t) = (3\mathrm{c}_{\text{in}} + 1)\sigma^{-1-\delta}$,
(b) $\|(H-h)^d\|_2 \leq \|(H-h)^t\|_2$,
(c) $|\vec{z}(t)| = 2B_{\text{cen}}\sigma^{1-\delta}$.
Such a t^* exists, since at least property (c) must hold at some $t < T'$ because of (4.27). We remark that it is possible that $t^* = 0$, because property (b), in contrast with (a) and (c), is compatible with our assumptions on the initial data.

Claim. If B_{cen} and σ are large enough, then at time $t = t^*$ the solution satisfies

$$\vec{z}(t^*) < 2B_{\text{cen}}\sigma^{1-\delta}, \tag{4.29}$$

$$\|H - h\|_{L^2(\Sigma_{t^*})} \leq c\sigma^{-1-\delta}, \tag{4.30}$$

for a suitable $c = c(B_1, B_2, \bar{c}, \mathrm{c}_{\text{in}})$.

If $t^* = 0$ the claim is immediate. In fact, in this case the initial surface satisfies (b) and (4.30) is a consequence of hypothesis (ii) of the theorem, while (4.29) follows from (i). Therefore we assume that $t^* > 0$. By definition of t^*, we have $\|(H-h)^d\|_2 > \|(H-h)^t\|_2$ for $t \in [0, t^*)$. Then Proposition 4.9 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|_2^2 \leq -\frac{2}{\sigma^2_{\Sigma_t}}\|(H-h)^d\|_2^2 \leq -\frac{1}{\sigma^2_{\Sigma_t}}\|(H-h)^t\|_2^2 - \frac{1}{\sigma^2_{\Sigma_t}}\|(H-h)^d\|_2^2,$$

for $t \in [0, t^*]$. By (2.10), we have that $\sigma^2_{\Sigma_t} \geq (4/5)\sigma^2$. Then we can integrate the inequality and obtain

$$\|H-h\|^2_{L^2(\Sigma_t)} \leq \|H-h\|^2_{L^2(\Sigma_0)} e^{-\frac{4}{5\sigma^2}t} \leq c\sigma^{-1-2\delta} e^{-\frac{4}{5\sigma^2}t} \tag{4.31}$$

for every $t \in [0, t^*]$. Therefore, by Lemma 4.11,

$$|\vec{z}(t^*)| - |\vec{z}(0)| \leq \int_0^{t^*} c\sigma^{-1}\|H-h\|_{L^2}\, dt \leq \int_0^{t^*} c\sigma^{-\frac{3}{2}-\delta} e^{-\frac{2}{5\sigma^2}t}\, dt \leq c\sigma^{\frac{1}{2}-\delta}.$$

So if $B_{\text{cen}} > c$ then $|\vec{z}(t^*)| < 2B_{\text{cen}}\sigma^{\frac{1}{2}-\delta}$, proving (4.29). This also shows that case (c) of the definition of t^* cannot occur, and that either (a) or (b) must hold. We prove (4.30) dividing the two cases. Suppose first that (a) holds. Then we have, by Lemma 4.11,

$$\sigma^{-1-\delta} \leq \Pi(t^*) - \Pi(0) \leq \int_0^{t^*} \left| \frac{\mathrm{d}}{\mathrm{d}t}\Pi(t) \right| dt \leq c\sigma^{-3-2\delta} t^*,$$

which implies $t^* \geq c^{-1}\sigma^{2+\delta}$. Substituting in (4.31), we find

$$\|H - h\|^2_{L^2(\Sigma_t)} \leq c\sigma^{-1-2\delta} e^{-\frac{\sigma^\delta}{c}}$$

so that (4.30) is satisfied if σ is large enough. If instead (b) holds at time $t = t^*$, then (4.30) follows directly by using (4.22) to obtain

$$\|H - h\|_{L^2(\Sigma_{t^*})} \leq 2\|(H-h)^t\|_{L^2(\Sigma_{t^*})} \leq 2(3\mathrm{c}_{\mathrm{in}} + 1)\sigma^{-1-\delta}.$$

So we have proved our claim that (4.29)-(4.30) hold at at time $t = t^*$.

To conclude the proof, we study the behavior of the solution for $t \in [t^*, T']$. We can estimate $\sigma^3_\Sigma \leq 2\sigma^3$ using (2.10) and deduce from (4.20), (4.30)

$$\|H - h\|^2_{L^2(\Sigma_t)} \leq \|H - h\|^2_{L^2(\Sigma_{t^*})} e^{-\frac{2\overline{m}}{\sigma^3}(t-t^*)} \leq c\sigma^{-2-2\delta} e^{-\frac{2\overline{m}}{\sigma^3}(t-t^*)}, \qquad t \in [t^*, T'], \tag{4.32}$$

where we have written for simplicity $\overline{m} = \overline{m}_{\mathrm{ADM}}$. Using Lemma 4.11, we then get

$$\begin{aligned} |\vec{z}(T') - \vec{z}(t^*)| &\leq \int_{t^*}^{T'} \left| \frac{\mathrm{d}}{\mathrm{d}t} \vec{z}(t) \right| dt \leq \int_{t^*}^{T'} c\sigma^{-1} \|H - h\|_2 \, dt \\ &\leq \int_{t^*}^{T'} c\sigma^{-2-\delta} e^{-\frac{\overline{m}}{\sigma^3}(t-t^*)} \, dt \\ &\leq c\sigma^{-2-\delta} \left(\frac{\sigma^3}{\overline{m}} \right) \left(e^{-\frac{\overline{m}}{\sigma^3}t^*} - e^{-\frac{\overline{m}}{\sigma^3}T'} \right) \leq c\sigma^{1-\delta}. \end{aligned}$$

If $B_{\mathrm{cen}} > c$ then $|\vec{z}(T')| \leq |\vec{z}(t^*)| + c\sigma^{1-\delta} < (2B_{\mathrm{cen}} + c)\sigma^{1-\delta} < 3B_{\mathrm{cen}}\sigma^{1-\delta}$, in contradiction with (4.27). This shows that (4.27) cannot happen if B_{cen} and σ are large enough, and so it concludes our proof. □

Theorem 4.14. *Let* $(\mathrm{M}, \overline{\mathrm{g}})$ *be a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat manifold with* $\overline{m}_{\mathrm{ADM}} > 0$*. For any given* $\mathrm{c}_{\mathrm{in}} > 0$*, let the parameters* $B_1, B_2, \eta, B_{\mathrm{cen}}, \sigma$ *be chosen as in Theorem 4.13. Then, for any initial data satisfying hypotheses (i) and (ii) of that theorem, the solution to the volume preserving mean curvature flow (4.1) exists for every* $t \in [0, \infty)$*, satisfies* $\Sigma_t \in \mathcal{B}^\eta_\sigma(B_1, B_2, 3B_{\mathrm{cen}})$ *for every* $t \in [0, \infty)$ *and converges exponentially in* C^∞ *to a strictly stable CMC-surface* $\Sigma_\infty \in \mathcal{B}^\eta_\sigma(B_1, B_2, 3B_{\mathrm{cen}})$*.*

Proof. Let us consider an initial surface Σ_0 which satisfies assumptions (i) and (ii) of Theorem 4.13. By the local existence theory, there exists a solution of the flow defined on some maximal time interval $[0, T_{\max})$. Then, Theorem 4.13 shows that Σ_t belongs to $\mathcal{B}^\eta_\sigma(B_1, B_2, 3B_{\mathrm{cen}})$ for all $t \in [0, T_{\max})$. This implies that Σ_t is confined in a compact subset of M, and therefore $\overline{\mathrm{Rm}}$ and each of its derivatives are uniformly bounded on Σ_t. Since the intrinsic curvature of Σ_t

is uniformly bounded by (2.9), well-known estimates based on parabolic regularity, see e.g., [15], show that all derivatives of A are also uniformly bounded. Then a standard continuation argument implies that $T_{\max} = +\infty$.

From estimate (4.32) we see that $\|H - h\|_{L^2(\Sigma_t)}$ decays exponentially as $t \to +\infty$. Since the derivatives of any order of H are uniformly bounded, interpolation estimates imply that they also decay exponentially. Then Sobolev immersion implies that $\|H - h\|_{L^\infty(\Sigma_t)}$ decays exponentially as well. Since $H - h$ is the speed of our flow, this shows that the immersions $F(\cdot, t)$ converge smoothly to a limiting map $F_\infty(\cdot)$. By standard arguments, see Lemma 8.2 in [14], one can show that $F_\infty(\cdot)$ is also a smooth surface immersion and that there is exponential convergence in C^∞ of the curvature. In particular, the limit surface $\Sigma_\infty := F_\infty(\Sigma)$ satisfies $H \equiv h$. Then Proposition 3.7, or also Proposition 4.7 in [26], show that the CMC-surface Σ_∞ is strictly stable, i.e., the stability operator on Σ_∞ is positive definite on functions with zero mean. Finally, the estimates in the proof of Theorem 4.13 for arbitrary $T > 0$ show that the requirements in the definition of $\mathcal{B}^\eta_\sigma(B_1, B_2, 3B_{\text{cen}})$ still hold as strict inequalities on Σ_∞. □

We conclude by considering the explicit example of a Euclidean coordinate sphere $\mathbb{S}_r(0)$ as initial surface for our flow. To ensure that hypothesis (ii) of Theorem 4.13 is satisfied, we have to strengthen the assumptions on our ambient manifold by requiring the $C^1_{1+\delta}$-Regge–Teitelboim conditions in Definition 2.1. Then Theorem 1.3 stated in the introduction is an immediate consequence of the previous results.

Proof of Theorem 1.3. By Lemma 2.2, a Euclidean sphere $\mathbb{S}_r(0)$ satisfies hypothesis (i) of Theorem 4.13 if r is enough large and, if M satisfies the $C^1_{1+\delta}$-Regge–Teitelboim conditions, it also satisfies (ii) by Lemma 3.14. The conclusion then follows from Corollary 4.14. □

Remark 4.15. Once the exponential convergence of the flow starting from spheres of large radius is established, the property that the limiting surfaces form a CMC-foliation can be obtained by a similar strategy as in the previous literature, e.g., [12,18,24]. In fact, a classical application of the implicit function theorem (see for example [13] for a detailed explanation) allows to define a smooth map $F : \mathbb{S}^2 \times (0, h_0) \to \mathrm{M}$ such that $\Sigma^h := F(\mathbb{S}^2, h)$ is the limit of the flow starting from a sphere $\mathbb{S}_r$ for a suitable $r = r(h)$, and is the locally unique CMC-surface with $H \equiv h$. One can then show that these surfaces do not intersect and form a CMC-foliation of the outer part of M, proceeding as in the last part of the proof of [26, Theorem 5.1], see also [4, Lemma 9]. A crucial ingredient in these arguments is the invertibility of the stability operator on the limiting surfaces Σ^h, which follows from the spectral analysis in Sect. 3.

Remark 4.16. We point out that our main convergence result, Theorem 4.14 is independent on the Regge–Teitelboim conditions, which are only needed to prove that large coordinate spheres satisfy assumption (ii) of Theorem 4.13. In this way we provide a flow-based approach to the construction of the foliation alternative to the one of [26], although under slightly stronger assumptions. If we instead take as already known the existence of the foliation from [26], then Theorem 4.14 gives a dynamical stability result for the leaves, with no need to require Regge–Teitelboim conditions: in fact, we show that any small perturbation of the CMC-surfaces, satysfying conditions (i)–(ii) of Theorem 4.13, converges under VPMCF back to a leaf of the foliation.

We observe that Regge–Teitelboim conditions (in a stronger form) have been often used in the previous works on CMC-foliations. In particular, see [5,9,26], the $C^2_{1+\delta}$ version of these conditions is needed to prove the existence of the geometric center of mass, i.e., the limit of the barycenters $\lim_{h\to 0} \vec{z}_{\Sigma^h}$, where Σ^h is the leaf with $H \equiv h$. This has some analogy with our case where, roughly speaking, assumption (ii) of Theorem 4.13 is used instead to study the limit as $t \to \infty$ of $\vec{z}_{\Sigma_t}$, with Σ_t a solution of VPMCF with fixed scale.

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Declarations

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References

[1] Alikakos, N.D., Freire, A.: The normalized mean curvature flow for a small bubble in a Riemannian manifold. J. Differ. Geom. **64**(2), 247–303 (2003)

[2] Arnowitt, R., Deser, S., Misner, C.W.: Coordinate invariance and energy expressions in general relativity. Phys. Rev. **122**(3), 997–1006 (1961)

[3] Cabezas-Rivas, E., Miquel, V.: Volume preserving mean curvature flow in the hyperbolic space. Indiana Univ. Math. J. **56**(5), 2061–2086 (2007)

[4] Cederbaum, C., Sakovich, A.: On center of mass and foliations by constant spacetime mean curvature surfaces for isolated systems in General Relativity. Calc. Var. Partial. Differ. Equ. **60**(6), 214 (2021)

[5] Cederbaum, C., Nerz, C.: Explicit Riemannian manifolds with unexpectedly behaving center of mass. Ann. Henri Poincaré **16**(7), 1609–1631 (2015)

[6] Corvino, J., Wu, H.: On the center of mass of isolated systems. Class. Quantum Gravity **25**(8), 085008 (2008)

[7] de Gennaro, D., Diana, A., Kubin, A., Kubin, A.: Stability of the surface diffusion flow and volume-preserving mean curvature flow in the flat torus. Math. Ann. **390**, 4429–4461 (2024)

[8] De Lellis, C., Müller, S.: Optimal rigidity estimates for nearly umbilical surfaces. J. Differ. Geom. **69**(1), 75–110 (2005)

[9] Eichmair, M., Koerber, T.: Foliations of asymptotically flat 3-manifolds by stable constant mean curvature spheres. J. Differ. Geom. **128**(3), 1037–1083 (2024)

[10] Escher, J., Simonett, G.: The volume preserving mean curvature flow near spheres. Proc. Am. Math. Soc. **126**(9), 2789–2796 (1998)

[11] Gui, Y., Li, Y., Sun, J.: Stability of the area preserving mean curvature flow in asymptotic Schwarzschild space. J. Funct. Anal. **289**(7), 111033 (2025)

[12] Huang, L.-H.: Foliations by stable spheres with constant mean curvature for isolated systems with general asymptotics. Commun. Math. Phys. **300**(2), 331–373 (2010)

[13] Huang, L.-H.: On the Center of Mass and Constant Mean Curvature Surfaces of an Initial Data set, Lecture Notes, 2013 MSRI Summer School in General Relativity

[14] Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differ. Geom. **20**(1), 237–266 (1984)

[15] Huisken, G.: Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature. Invent. Math. **84**(3), 463–480 (1986)

[16] Huisken, G.: The volume preserving mean curvature flow. J. Reine Angew. Math. **382**, 35–48 (1987)

[17] Huisken, G., Polden, A.: Geometric evolution equations for hypersurfaces. In: Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996), pp. 45–84. Springer, Berlin (1999)

[18] Huisken, G., Yau, S.-T.: Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. Invent. Math. **124**(1–3), 281–311 (1996)

[19] Julin, V., Morini, M., Oronzio, F., Spadaro, E.: A sharp quantitative Alexandrov inequality and applications to volume preserving geometric flows in 3D. Arch. Ration. Mech. Anal. **249**, 78 (2025)

[20] Kröncke, K., Wolff, M.: Foliations of asymptotically Schwarzschildean lightcones by surfaces of constant spacetime mean curvature. Preprint arXiv:2412.17563 (2024)

[21] Li, H.: The volume-preserving mean curvature flow in Euclidean space. Pac. J. Math. **243**(2), 331–355 (2009)

[22] Liu, K., Xu, H., Zhao, E.: Mean curvature flow of higher codimension in Riemannian manifolds. Preprint arXiv:1204.0107 (2012)

[23] Metzger, J.: Blätterungen Asymptotisch Flacher Mannigfaltigkeiten Durch Flächen vorgeschriebener Mittleren Krümmung. PhD Thesis, University of Tübingen (2004)

[24] Metzger, J.: Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature. J. Differ. Geom. **77**(2), 201–236 (2007)

[25] Miao, P., Tam, L.-F.: Evaluation of the ADM mass and center of mass via the Ricci tensor. Proc. Am. Math. Soc. **144**(2), 753–761 (2016)

[26] Nerz, C.: Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry. Calc. Var. Partial. Differ. Equ. **54**(2), 1911–1946 (2015)

[27] Nerz, C.: Geometric characterizations of asymptotic flatness and linear momentum in general relativity. J. Funct. Anal. **269**(12), 3812–3867 (2015)

[28] Perez, D. R.: On Nearly Umbilical Hypersurfaces. PhD Thesis, University of Zürich (2011)

[29] Regge, T., Teitelboim, C.: Role of surface integrals in the Hamiltonian formulation of general relativity. Ann. Phys. **88**(1), 286–318 (1974)

[30] Tenan, J.: Volume preserving spacetime mean curvature flow and foliations of initial data sets. J. Funct. Anal. **290**(6), 111313 (2026)

[31] Ye, R.: Foliation by constant mean curvature spheres on asymptotically flat manifolds. arXiv preprint dg-ga/9709020 (1997)

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