

Wachs permutations, Bruhat order and weak order ¹

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1. Introduction

Wachs permutations are a class of permutations first introduced (in the even case) in [22] to study the signed Eulerian numbers and the signed major index enumerator of the symmetric groups. This class was extended in [4] to the odd case and to signed permutations in order to study the enumerators of the odd and even major indices of classical Weyl groups twisted by their one-dimensional characters. In this work we study Wachs (and signed Wachs) permutations in their own right. More precisely, we study the partial orders induced on them by the Bruhat and weak orders of the symmetric and hyperoctahedral groups. These orders are fundamental objects in algebraic combinatorics and have important connections to algebra and geometry (see, e.g., [2, Chaps.2 and 3], [1, Chap.2], [16, Chap.1], [11, Chap.5], [7, Chap.10], [17, Chap.2], and the references cited there). Many subsets of the symmetric and hyperoctahedral groups have been studied as posets under the Bruhat order such as, for example, quotients, descent classes, and generalized quotients [3], complements of quotients [19], involutions [9, 12, 13, 15, 18], conjugation-invariant sets of involutions [5, 8], and twisted identities [10]. In this paper we show that Wachs permutations possess many nice properties when partially ordered by Bruhat and weak orders. More precisely, we show that they form a graded poset, determine their rank function, characterize the ordering and covering relations, and compute the characteristic polynomial, when partially ordered by Bruhat order, and determine their structure explicitly when partially ordered by right weak order.

The organization of the paper is as follows. In the next section we collect some notation and results that are used in the sequel. In §3 we study the poset obtained by partially ordering

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Wachs permutations with respect to Bruhat order. We show that these posets are always graded (Theorem 3.9), characterize their order and covering relations (Theorem 3.13 and Proposition 3.12), and compute their characteristic polynomials (Corollary 3.16). In §4, using some of the results in §3, we obtain analogous results for the poset induced on signed Wachs permutations by Bruhat order. More precisely, we show that these posets are graded (Theorem 4.10), characterize their order and covering relations (Theorem 4.13, Proposition 4.2, Corollary 4.12, and Corollary 4.6), and compute the characteristic polynomials (Corollary 4.16). In §5 we study the posets obtained by partially ordering Wachs and signed Wachs permutations under right weak order. We show that they are always isomorphic to the direct product of a Boolean algebra with the weak order on the whole group in rank $\lceil \frac{n}{2} \rceil$ (Theorems 5.1 and 5.3). Finally, in §6, we discuss some conjectures and open problems arising from the present work and the evidence that we have about them.

2. Preliminaries

In this section we recall some notation, definitions, and results that are used in the sequel. As \mathbb{N} we denote the set of non-negative integers and as \mathbb{P} the set of positive integers. If $n \in \mathbb{N}$, $[n] := \{1, 2, \dots, n\}$; in particular $[0] = \emptyset$. For $n \in \mathbb{P}$, in the polynomial ring $\mathbb{Z}[q]$ the q -analogue of n is defined by $[n]_q := \sum_{i=0}^{n-1} q^i$ and the q -factorial by $[n]_q! := \prod_{i=1}^n [i]_q$. For a set A and $f : A \rightarrow \mathbb{N}$ we define

$$A(x, f) := \sum_{w \in A} x^{f(w)}$$

and $f(u, v) := f(v) - f(u)$, for all $u, v \in A$. The cardinality of a set X will be denoted by $|X|$ and the power set of X by $\mathcal{P}(X)$. Given a cartesian product $X \times Y$ of two sets X and Y , we indicate with π_1 and π_2 the projections on X and Y respectively.

Let $n \in \mathbb{N}$, $i \in \mathbb{Z}$, $q \in \mathbb{Q}$ and $J \subseteq [n]$; then we define $J_e := \{j \in J : j \equiv 0 \pmod{2}\}$, $J_o := \{j \in J : j \equiv 1 \pmod{2}\}$, $J + i := \{i + j : j \in J\} \cap [n]$, and $qJ := \bigcup_{j \in J} \{qj\} \subseteq \mathbb{Q}$.

We follow Chapter 3 of [20] for notation and terminology concerning posets. We just recall some definitions. Given two posets P and Q , their *ordinal product* $P \otimes Q$ is defined by ordering the set $P \times Q$ in the following manner: $(x, y) \leq (x', y')$ if and only if $x = x'$ and $y \leq y'$ or $x < x'$, for all $(x, y), (x', y') \in P \times Q$. If μ is the Möbius function of a graded poset P with minimum $\hat{0}$, maximum $\hat{1}$ and rank function ρ , then the characteristic polynomial of P in the indeterminate x is defined by

$$C_P(x) := \sum_{z \in P} \mu(\hat{0}, z) x^{\rho(z, \hat{1})}.$$

Next recall some basic results in the theory of Coxeter groups which will be useful in the sequel. The reader can consult [2] or [11] for further details. Let (W, S) be a Coxeter system. The length of an element $z \in W$ with respect of the given presentation is denoted as $\ell(z)$. If $J \subseteq S$, we let

$$\begin{aligned} W^J &:= \{w \in W : \ell(ws) > \ell(w) \forall s \in J\}, \\ {}^J W &:= \{w \in W : \ell(sw) > \ell(w) \forall s \in J\}, \\ D(w) &:= \{s \in S : \ell(ws) < \ell(w)\}, \end{aligned}$$

and, more generally, for any $A \subseteq W$ we let $A^J := A \cap W^J$. The subgroup $W_J \subseteq W$ is the group generated by J . In particular $W_S = W$ and $W_\emptyset = \{e\}$, being e the identity of W . We consider on W the Bruhat order \leq (see, e.g., [2, Chapter 2] or [11, Chapter 5]) and on any subset we consider the induced order. When the group W is finite, there exists a unique maximal element w_0 of maximal length. We recall this characterizing property of the Bruhat order, known as the *lifting property* (see [2, Proposition 2.2.7]):

Proposition 2.1. *Let $v, w \in W$ be such that $v < w$ and $s \in D(w) \setminus D(v)$. Then $v \leq ws$ and $vs \leq w$.*

For $J \subseteq S$, each element $w \in W$ has a unique expression $w = w_J w^J$, where $w^J \in W^J$ and $w_J \in W_J$ (see [2, Proposition 2.4.4]). Often we consider the projection $P^J : W \rightarrow W^J$ defined by $P^J(w) = w^J$. This map is order preserving ([2, Proposition 2.5.1]). We let $T := \{ws w^{-1} : w \in W, s \in S\}$. The following lemma will be useful in the next section; for a proof see [2, Lemma 2.2.10].

Lemma 2.2. *Suppose that $x < xt$ and $y < ty$, for $x, y \in W$, $t \in T$. Then, $xy < xty$.*

For $w \in W$ we set $T_L(w) := \{t \in T : \ell(tw) < \ell(w)\}$; the *right weak order* \leq_R on W is the partial order whose cover relations are defined by letting $u \triangleleft_R v$ if and only if $v^{-1}u \in S$ and $\ell(v) = \ell(u) + 1$, for all $u, v \in W$ (see [2, Chapter 3]). Then the following characterization holds (see [2, Proposition 3.1.3]):

Proposition 2.3. *Let (W, S) be a Coxeter system. Then $u \leq_R v$ if and only if $T_L(u) \subseteq T_L(v)$, for all $u, v \in W$.*

Now, for any $n \in \mathbb{P}$, let S_n be the group of all bijections of the set $[n]$. It is well known that it is a Coxeter group with set of generators $\{s_1, s_2, \dots, s_{n-1}\}$, being s_i the simple inversion given, in one line notation, by $12\dots(i+1)i\dots n$. Given a permutation $\sigma = \sigma(1)\sigma(2)\dots\sigma(n) \in S_n$, the action of s_i on the right is given by $\sigma s_i = \sigma(1)\sigma(2)\dots\sigma(i+1)\sigma(i)\dots\sigma(n)$, for all $i \in [n-1]$. For $i, j \in [n]$, the action on the right of a transposition (i, j) , $i \neq j$, then is given by $\sigma(i, j) = \sigma(1)\sigma(2)\dots\sigma(i-1)\sigma(j)\dots\sigma(j-1)\sigma(i)\dots\sigma(n)$, for all $\sigma \in S_n$. As a Coxeter group, identifying $\{s_1, s_2, \dots, s_{n-1}\}$ with $[n-1]$, we have that (see e.g. [2, Propositions 1.5.3 and 1.5.2]) $D(\sigma) = \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$ and $\ell(\sigma) = \ell_A(\sigma)$, where $\ell_A(\sigma) := \text{inv}(\sigma) = |\{(i, j) \in [n]^2 : i < j, \sigma(i) > \sigma(j)\}|$, for all $\sigma \in S_n$. Then, given $J \subseteq [n-1]$, $S_n^J = \{\sigma \in S_n : \sigma(i) < \sigma(i+1) \forall i \in J\}$.

For $i \in [n]$ and $A \subseteq S_n$ define $A(i) := \{\sigma \in A : \text{pos}(\sigma) = i\}$, being $\text{pos} : S_n \rightarrow [n]$ the function defined by $\text{pos}(\sigma) := \sigma^{-1}(n)$, for all $\sigma \in S_n$. We find it convenient to define the following involution on $[n]$:

$$i^* := \begin{cases} i-1, & \text{if } i \equiv 0 \pmod{2}; \\ i+1, & \text{if } i \equiv 1 \pmod{2} \text{ and } i+1 \in [n]; \\ n, & \text{otherwise,} \end{cases}$$

and the simple inversion $s_i^* := (i, i^*)$, for all $i \in [n]$. Given a permutation $\sigma \in S_n$, $k \in [n]$ and $i \in [k]$, define $\sigma_{i,k}$ as the i -th element in the increasing rearrangement of $\{\sigma(1), \sigma(2), \dots, \sigma(k)\}_{<}$ and $\sigma_{i,k}^{-1}$ as the position of $\sigma(i)$ in $\{\sigma(1), \sigma(2), \dots, \sigma(k)\}_{<}$. So $\sigma_{j,k} = \sigma(i)$ if $j = \sigma_{i,k}^{-1}$. The following

theorem, known as the “tableau criterion”, characterizes the Bruhat order on S_n (see [2, Theorem 2.6.3]).

Theorem 2.4. *Let $n \in \mathbb{P}$ and $\sigma, \tau \in S_n$. The following are equivalent:*

1. $\sigma \leq \tau$;
2. $\sigma_{i,k} \leq \tau_{i,k}$ for all $k \in D(\sigma)$ and $i \in [k]$;
3. $\sigma_{i,k} \leq \tau_{i,k}$ for all $k \in [n-1] \setminus D(\tau)$ and $i \in [k]$.

We indicate by w_n the maximum of the poset (S_n, \leq) , i.e., in one line notation, $w_n = n\dots 321$.

The *descent number* and the *major index* are the functions $\text{des} : S_n \rightarrow \mathbb{N}$ and $\text{maj} : S_n \rightarrow \mathbb{N}$ defined respectively by $\text{des}(\sigma) := |D(\sigma)|$ and $\text{maj}(\sigma) := \sum_{i \in D(\sigma)} i$. A famous result of McMahon asserts that ℓ_A and maj are equidistributed over S_n (see, e.g. [20, Proposition 1.4.6]), i.e. $S_n(x, \ell_A) = S_n(x, \text{maj})$, and this rank-generating function is known to be (see, e.g [20, Corollary 1.3.10])

$$S_n(x, \ell_A) = [n]_x! \tag{1}$$

Following [4] we define functions from S_n to \mathbb{N} by letting

$$\text{odes}(\sigma) := |D(\sigma)_o|, \quad \text{emaj}(\sigma) := \sum_{i \in D(\sigma)_e} \frac{i}{2},$$

for all $\sigma \in S_n$, and call these functions *odd descent number* and *even major index* respectively.

Following [4], for $n \in \mathbb{P}$, we let

$$\mathcal{W}(S_n) := \{\sigma \in S_n : |\sigma^{-1}(i) - \sigma^{-1}(i^*)| \leq 1 \text{ if } i \in [n-1]\}$$

and call the elements of $\mathcal{W}(S_n)$ *Wachs permutations*. It is not hard to see that, if n is even, $\mathcal{W}(S_n) = \{\sigma \in S_n : |\sigma(i) - \sigma(i^*)| \leq 1 \text{ if } i \in [n-1]\}$.

Let $[\pm n] := \{-n, \dots, -1, 1, \dots, n\}$. We denote by B_n the group of bijective functions $\sigma : [\pm n] \rightarrow [\pm n]$ satisfying $-\sigma(i) = \sigma(-i)$, for all $i \in [n]$. We use the window notation. So, for example, the element $[-2, 1] \in B_2$ represents the function $\sigma : [\pm 2] \rightarrow [\pm 2]$ such that $\sigma(1) = -2 = -\sigma(-1)$ and $\sigma(2) = 1 = -\sigma(-2)$. We let $\text{Neg}(\sigma) := \{i \in [n] : \sigma(i) < 0\}$, $\text{neg}(\sigma) = |\text{Neg}(\sigma)|$, $\text{nsp}(\sigma) := |\{(i, j) \in [n]^2 : i < j, \sigma(i) + \sigma(j) < 0\}|$, $s_j^B := (j, j+1)(-j, -j-1)$ for $j = 1, \dots, n-1$, $s_0 := (1, -1)$, and $S_B := \{s_0, s_1^B, \dots, s_{n-1}^B\}$. It is well known that (B_n, S_B) is a Coxeter system and that, identifying S_B with $[0, n-1]$, the following holds (see, e.g., [2, §8.1]).

Proposition 2.5. *Let $\sigma \in B_n$. Then $\ell_B(\sigma) = \ell_A(\sigma) + \text{neg}(\sigma) + \text{nsp}(\sigma)$, and $D(\sigma) = \{i \in [0, n-1] : \sigma(i) > \sigma(i+1)\}$, where $\sigma(0) := 0$.*

We denote by \leq the Bruhat order of B_n and by $\sigma \mapsto \tilde{\sigma}$ the embedding $B_n \hookrightarrow S_{\pm n}$ (where $S_{\pm n}$ is the set of all bijections of $[\pm n]$). The following result is [2, Corollary 8.1.9].

Proposition 2.6. *We have that $\sigma \leq \tau$ in B_n if and only if $\tilde{\sigma} \leq \tilde{\tau}$ in $S_{\pm n}$.*

The next result, which appears in [14, Theorem 5.5], characterizes the cover relations in the Bruhat order of B_n . For $\sigma \in B_n$ and $i, j \in [\pm n]$, $i < j$, we say that (i, j) is a *rise* for σ if $\sigma(i) < \sigma(j)$ (i.e., if (i, j) is not an inversion of σ). Given a rise (i, j) for σ we then say, following [14], that (i, j) is *central* if $(0, 0) \in [i, j] \times [\sigma(i), \sigma(j)]$, that (i, j) is *free* if there is no $i < k < j$ such that $\sigma(i) < \sigma(k) < \sigma(j)$, and that it is *symmetric* if $i = -j$.

Theorem 2.7. *Let $n \in \mathbb{P}$ and $\sigma, \tau \in B_n$. Then $\sigma \triangleleft \tau$ in Bruhat order if and only if either*

1. $\tau = \sigma(i, j)(-i, -j)$ where (i, j) is a non-central free rise of σ or
2. $\tau = \sigma(i, j)$ where (i, j) is a central symmetric free rise of σ .

Following [4], for $n \in \mathbb{P}$, we let

$$\mathcal{W}(B_n) := \{\sigma \in B_n : |\sigma^{-1}(i) - \sigma^{-1}(i^*)| \leq 1 \text{ if } i \in [n-1]\}$$

and call the elements of $\mathcal{W}(B_n)$ *signed Wachs permutations*.

Our aim in this work is to study the sets of Wachs and signed Wachs permutations under the Bruhat order and the weak order.

3. Wachs permutations and Bruhat order

Let (W, S) be a Coxeter system. For any independent set $I \subseteq S$ (i.e. $st = ts$ for all $s, t \in I$) we define a subgroup of W by

$$G_I := \{w \in W : I^w = I\},$$

where $I^w := \{ws w^{-1} : s \in I\}$. Note that G_I is a subgroup of W , and that, since I is independent, $W_I \subseteq G_I$. Furthermore, W_I is normal in G_I .

Proposition 3.1. *$P^I(G_I)$ is a subgroup of G_I , isomorphic to the quotient G_I/W_I . In particular we have the group isomorphisms*

$$G_I \simeq P^I(G_I) \rtimes W_I \simeq S_2 \wr_I P^I(G_I).$$

Proof. Let $w \in G_I$ and write $w = w^I w_I$ where $w^I \in W^I$ and $w_I \in W_I$. Let $s \in I$. Then, by our hypothesis, $ws w^{-1} = w^I w_I s (w_I)^{-1} (w^I)^{-1} = w^I s (w^I)^{-1}$ so $w^I \in G_I$. So $P^I(G_I) \subseteq G_I$. Let $u, v \in G_I$ and write $u = u^I u_I$ and $v = v^I v_I$ where $u^I, v^I \in W^I$ and $u_I, v_I \in W_I$, so $u^I, v^I \in P^I(G_I)$. Let $s \in I$. Then there is $t \in I$ such that $v^I s = t v^I$. Furthermore, since $u^I, v^I \in W^I$, $\ell(t v^I) > \ell(v^I)$, and $\ell(u^I t) > \ell(u^I)$. Therefore, by Lemma 2.2 we obtain that $u^I v^I s = u^I t v^I > u^I v^I$. Hence $u^I v^I \in W^I$. But, since $v^I \in G_I$, there exists $\tilde{u}_I \in W_I$ such that $uv = u^I v^I \tilde{u}_I v_I$ so $(uv)^I = u^I v^I$. Hence $u^I v^I \in P^I(G_I)$. In particular, $(u^I)^{-1} = (u^{-1})^I$, so $P^I(G_I)$ is a subgroup of G_I , and $P^I : G_I \rightarrow P^I(G_I)$ is a surjective homomorphism whose kernel is W_I . The last statements follow by the definitions. \square

Let $m \in \mathbb{P}$, $W := S_{2m}$. For any set X , we consider $\mathcal{P}(X)$ as an abelian group, the operation being the symmetric difference $+$, i.e. $A + B := (A \setminus B) \cup (B \setminus A)$, for all $A, B \in X$. Then it is straightforward to see that if we take $I = \{s_i : i \equiv 1 \pmod{2}\} \subseteq S$ then $\mathcal{W}(S_{2m}) = G_I$. The group isomorphisms $P^I(\mathcal{W}(S_{2m})) \simeq S_m$ and $W_I \simeq \mathcal{P}([m])$ then imply

$$\mathcal{W}(S_{2m}) \simeq S_m \rtimes \mathcal{P}([m]) = S_2 \wr S_m.$$

Therefore $\mathcal{W}(S_{2m})$ is isomorphic to the hyperoctahedral group.

We consider S_n with the Bruhat order and the subset $\mathcal{W}(S_n) \subseteq S_n$ with the induced order. It is not difficult to see, using Theorem 2.4 that $P^I(\mathcal{W}(S_{2m})) \simeq S_m$ as posets for all $m > 0$, where the

set S_m is ordered by the Bruhat order. Let $m > 0$. For $u \in S_m$ and $T \subseteq [m]$ let $\phi_{2m}^{-1}(u, T) := v$, where $v \in \mathcal{W}(S_{2m})$ is defined by

$$v(2i-1) = \begin{cases} 2u(i) - 1, & \text{if } i \notin T, \\ 2u(i), & \text{if } i \in T, \end{cases}$$

and

$$v(2i) = \begin{cases} 2u(i), & \text{if } i \notin T, \\ 2u(i) - 1, & \text{if } i \in T, \end{cases}$$

for all $i \in [m]$. The following result follows easily from our definitions and Theorem 2.4, and its proof is omitted.

Proposition 3.2. *Let $m \in \mathbb{P}$. Then*

1. ϕ_{2m} is a bijection;
2. $\phi_{2m} : \mathcal{W}(S_{2m}) \rightarrow S_m \otimes \mathcal{P}([m])$ is order preserving;
3. $\phi_{2m}^{-1} : S_m \times \mathcal{P}([m]) \rightarrow \mathcal{W}(S_{2m})$ is order preserving.

Moreover $\ell(v) = 4\ell(\tau) + |T|$ if $\phi(v) = (\tau, T)$.

Figure 1 shows the Hasse diagram of $(\mathcal{W}(S_5), \leq)$. By the following example we see that ϕ_{2m} and ϕ_{2m}^{-1} are not poset isomorphisms in general.

Example 3.3. Let $m = 3$. Then $(123, \{1, 2, 3\}) \leq (132, \emptyset)$ in $S_m \otimes \mathcal{P}([m])$ but $\phi_{2m}^{-1}(123, \{1, 2, 3\}) = 214365 \not\leq 125634 = \phi_{2m}^{-1}(132, \emptyset)$ in $\mathcal{W}(S_{2m})$.

Let $m = 2$, $u = 2143$ and $v = 3412$. Then $u, v \in \mathcal{W}(S_4)$, $u < v$ and $\phi_{2m}(u) = (12, \{1, 2\}) \not\leq (21, \emptyset) = \phi_{2m}(v)$ in $S_m \times \mathcal{P}([m])$.

We consider now $\mathcal{W}(S_{2m+1})$. For any $m > 0$ it is not difficult to see that the set $\mathcal{W}(S_{2m+1})$ is not a group. The previous general construction in Coxeter systems gives, for $W = S_{2m+1}$, the group $G_I \simeq \mathcal{W}(S_{2m})$ which, as a set, can be included in $\mathcal{W}(S_{2m+1})$. For $n > 1$, define a function $\chi_n : \mathcal{W}(S_n) \rightarrow \mathcal{W}(S_{n-1})$ by

$$\chi_n(v)(i) = \begin{cases} v(i), & \text{if } i < \text{pos}(v); \\ v(i+1), & \text{if } i \geq \text{pos}(v), \end{cases}$$

for all $i \in [n-1]$, $v \in \mathcal{W}(S_n)$. Note that the function χ_n is not necessarily order preserving. For example, if $m = 2$ and $u = 21345$, $v = 51234$ then $u \leq v$ but $\chi_n(u) = 2134 \not\leq 1234 = \chi_n(v)$.

Let $\phi_{2m+1} : \mathcal{W}(S_{2m+1}) \rightarrow [m+1] \times S_m \times \mathcal{P}([m])$ be the function defined by

$$\phi_{2m+1}(v) := ((\text{pos}(v) + 1)/2, \tau, T),$$

if $\chi_{2m+1}(v) = (\tau, T)$, for all $v \in \mathcal{W}(S_{2m+1})$. For example, $\phi_{2m+1}(4312756) = (3, 213, \{1\})$. As in the even case, noting that $u \leq v$ implies $\text{pos}(v) \leq \text{pos}(u)$, we have the following result whose proof follows easily from our definitions and Proposition 3.2.

Proposition 3.4. *Let $m > 0$. Then*

1. ϕ_{2m+1} is a bijection;

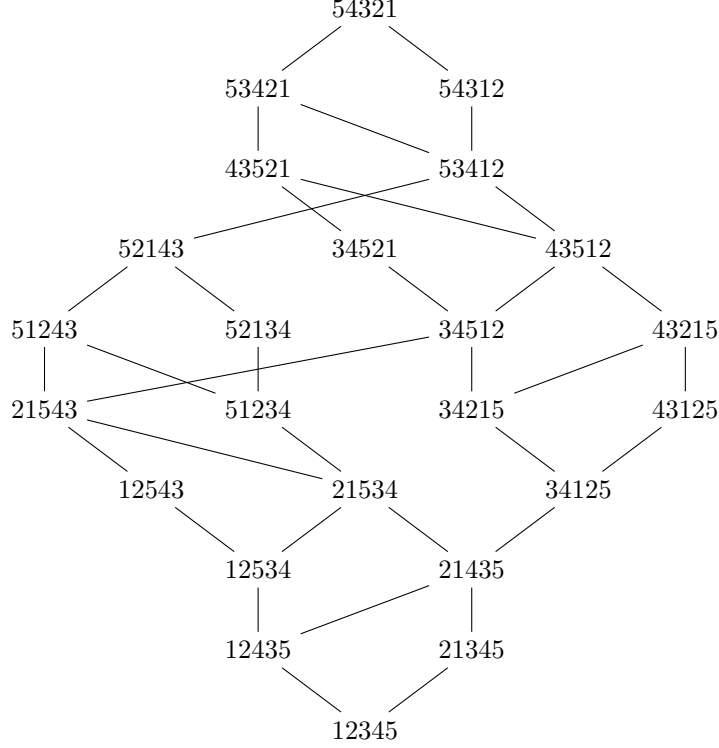


Figure 1: Hasse diagram of $(\mathcal{W}(S_5), \leq)$.

2. $\phi_{2m+1} : \mathcal{W}(S_{2m+1}) \rightarrow [m+1]^* \otimes S_m \otimes \mathcal{P}([m])$ is order preserving;
3. $\phi_{2m+1}^{-1} : [m+1]^* \times S_m \times \mathcal{P}([m]) \rightarrow \mathcal{W}(S_{2m+1})$ is order preserving,

where $i \leq j$ in $[m+1]^*$ if and only if $j \leq i$, for all $i, j \in [m+1]$. Moreover $\ell(v) = 4\ell(\tau) + |T| + 2(m-i+1)$ if $\phi_{2m+1}(v) = (i, \tau, T)$.

Let $n > 0$ and $f_n : \mathcal{W}(S_n) \rightarrow S_{\lfloor \frac{n}{2} \rfloor}$ be the function defined by the assignment $(i, \tau, T) \mapsto \tau$ if n is odd, and $(\tau, T) \mapsto \tau$ if n is even. By Proposition 3.2, if n is even then f_n is order preserving. Corollary 3.8 states that the function f_n is order preserving also in the odd case. We also define a function $\ell_{\mathcal{W}} : \mathcal{W}(S_n) \rightarrow \mathbb{N}$ by

$$\ell_{\mathcal{W}}(v) := \ell(v) - \ell(f_n(v)),$$

for all $v \in \mathcal{W}(S_n)$. For example, $\ell_{\mathcal{W}}(342156) = 5 - 1 = 4$ and $\ell_{\mathcal{W}}(3472156) = 9 - 1 = 8$. The minimum and the maximum of the poset $(\mathcal{W}(S_n), \leq)$ are respectively the identity of S_n , which corresponds to (e, \emptyset) in the even case and to $(m+1, e, \emptyset)$ in the odd one, and the element of maximal length w_n of S_n , which corresponds to $(w_m, [m])$ in the even case and to $(1, w_m, [m])$ in the odd one, where $m := \lfloor \frac{n}{2} \rfloor$. Moreover $\ell_{\mathcal{W}}(e) = 0$ and

$$\ell_{\mathcal{W}}(w_n) = \binom{n}{2} - \binom{\lfloor \frac{n}{2} \rfloor}{2}. \quad (2)$$

Note that if n is even then $\ell_{\mathcal{W}}(w_n) = \frac{(3n-2)n}{8} = \frac{(3m-1)m}{2}$. So $\{\ell_{\mathcal{W}}(w_{2m})\}_{m \in \mathbb{P}}$ is the sequence

of pentagonal numbers (see A000326 in OEIS). If n is odd then $\ell_{\mathcal{W}}(w_n) = \frac{3(n^2-1)}{8} = \frac{3m(m+1)}{2}$; so $\{\ell_{\mathcal{W}}(w_{2m+1})\}_{m \in \mathbb{P}}$ is the sequence of triangular matchstick numbers (see A045943 in OEIS).

Let $n > 0$ and $m := \lfloor \frac{n}{2} \rfloor$; for any $i, j \in [m]$, $i < j$, we define an involution $w_{i,j}^A : \mathcal{W}(S_n) \rightarrow \mathcal{W}(S_n)$ by

$$w_{i,j}^A(v) = \begin{cases} (\tau(i, j), T + \{i, j\}), & \text{if } n \text{ is even and } v = (\tau, T); \\ (k, \tau(i, j), T + \{i, j\}), & \text{if } n \text{ is odd and } v = (k, \tau, T), \end{cases}$$

for all $v \in \mathcal{W}(S_n)$. For example, if $v = 4312765 \in \mathcal{W}(S_7)$, then $w_{2,3}^A(v) = 4356721$.

Lemma 3.5. *Let $m > 0$, $v = (\tau, T) \in \mathcal{W}(S_{2m})$, $i, j \notin T$ and $\tau(i, j) \triangleleft \tau$. Then $w_{i,j}^A(v) < v$ and $\ell_{\mathcal{W}}(v) - \ell_{\mathcal{W}}(w_{i,j}^A(v)) = 1$. If $u = (\sigma, S) \in \mathcal{W}(S_{2m})$, $u < v$ and $\sigma \leq \tau(i, j)$ then $u \leq w_{i,j}^A(v)$.*

Proof. We have that $\hat{v} := w_{i,j}^A(v) = (\tau(i, j), T \cup \{i, j\})$. By the tableau criterion it is easy to deduce that $\hat{v} < v$ and the equality $\ell_{\mathcal{W}}(\hat{v}, v) = 1$ follows easily from Proposition 3.2 and the definition of $\ell_{\mathcal{W}}$. We now show that $u \leq \hat{v}$. We use Theorem 2.4. Note first that, since u and \hat{v} are Wachs permutations and $\sigma \leq \tau(i, j)$, $u_{l,h} \leq \hat{v}_{l,h}$ for all $1 \leq l \leq h$ if h is even. Furthermore, since $u < v$, $u_{l,h} \leq \hat{v}_{l,h}$ for all $l \in [h]$ if $h \leq 2i - 2$ or $h \geq 2j$. So let $k \in [m]$ be such that $k \notin T$ and $i < k < j$. We wish to show that $u_{l,2k-1} \leq \hat{v}_{l,2k-1}$, for all $1 \leq l \leq 2k - 1$.

Let $a := v(2i - 1)$ and $b := v(2j - 1)$. So $a + 1 = v(2i)$ and $a > b + 1 = v(2j)$ as well as $\hat{v}(2i - 1) = b + 1$, $\hat{v}(2i) = b$, $\hat{v}(2j - 1) = a + 1$ and $\hat{v}(2j) = a$. Let $c := v(2k - 1)$ and $d := u(2k - 1)$. Note that, since $\tau(i, j) \triangleleft \tau$, $c \notin [b, a + 1]$ and that $c \equiv 1 \pmod{2}$ since $k \notin T$. Let $r, s \in [k]$ be such that $v_{2r-1,2k-1} = c$ and $u_{2s-1,2k-1} = d$. We have two cases to consider according as to whether $c < b$ or $c > a + 1$.

Say $c > a + 1$.

Let $p, q \in [k]$ be such that $v_{2q-1,2k-1} = a$ and $\hat{v}_{2p-1,2k-1} = b$, so $p \leq q$, $v_{2q,2k-1} = a + 1$ and $\hat{v}_{2p,2k-1} = b + 1$. Then $2r - 1 > 2q$ since $c > a + 1$. Note that

$$v_{l,2k-1} = \hat{v}_{l,2k-1} \tag{3}$$

if $1 \leq l \leq 2p - 2$ or $2q + 1 \leq l \leq 2k - 1$ and that

$$v_{l,2k-1} = \hat{v}_{l+2,2k-1} \tag{4}$$

if $2p - 1 \leq l \leq 2q - 2$. Moreover, since $u \leq v$ by our hypothesis,

$$u_{l,2k-1} \leq v_{l,2k-1}, \tag{5}$$

for all $1 \leq l \leq 2k - 1$.

Note that, by (3) and (5), we have

$$u_{l,2k-1} \leq v_{l,2k-1} = \hat{v}_{l,2k-1}$$

if $1 \leq l \leq 2p - 2$ or $2q + 1 \leq l \leq 2k - 1$. So assume $2p - 1 \leq l \leq 2q$.

If $x, y \in \mathbb{P}$, $x, y \equiv 1 \pmod{2}$, then we find it convenient to write $\{x, x + 1\} \leq \{y, y + 1\}$ if $x \leq y$ and similarly for $<$.

We have two cases to consider.

Let $s > q$. Then, since $\sigma \leq \tau(i, j)$, we have that

$$\{u_{2h-1, 2k-1}, u_{2h, 2k-1}\} \leq \{\hat{v}_{2h-1, 2k-1}, \hat{v}_{2h, 2k-1}\}$$

for all $p \leq h \leq q$, so $u_{l, 2k-1} \leq \hat{v}_{l, 2k-1}$ for all $2p-1 \leq l \leq 2q$.

Let $s \leq q$. If $p \leq h \leq s-1$ then, since $\sigma \leq \tau(i, j)$, we have that

$$\{u_{2h-1, 2k-1}, u_{2h, 2k-1}\} \leq \{\hat{v}_{2h-1, 2k-1}, \hat{v}_{2h, 2k-1}\}$$

so $u_{l, 2k-1} \leq \hat{v}_{l, 2k-1}$ for all $2p-1 \leq l \leq 2s-2$.

We may therefore assume that $\max\{2p-1, 2s-1\} \leq l \leq 2q$. Since $\sigma \leq \tau(i, j)$, we have that

$$u_{l, 2k-2} \leq \hat{v}_{l, 2k-2}$$

for $1 \leq l \leq 2k-2$. But

$$u_{l, 2k-1} = u_{l-1, 2k-2}$$

if $2s \leq l \leq 2k-1$ since $\{u(1), \dots, u(2k-2)\} = \{u(1), \dots, u(2k-1)\} \setminus \{d\}$, and similarly

$$\hat{v}_{l, 2k-1} = \hat{v}_{l, 2k-2}$$

if $1 \leq l \leq 2r-2$. Therefore

$$u_{l, 2k-1} < u_{l+1, 2k-1} = u_{l, 2k-2} \leq \hat{v}_{l, 2k-2} = \hat{v}_{l, 2k-1},$$

if $2s-1 \leq l \leq 2r-2$, so $u_{l, 2k-1} \leq \hat{v}_{l, 2k-1}$ if $2s-1 \leq l \leq 2q$, because $2q \leq 2r-2$.

The case $c < b$ is similar (and slightly simpler) except that one uses $2k$ rather than $2k-2$ and obtains that

$$u_{l, 2k-1} = u_{l, 2k} \leq \hat{v}_{l, 2k} = \hat{v}_{l-1, 2k-1} < \hat{v}_{l, 2k-1}$$

if $2p \leq l \leq 2s-1$. We leave the details to the interested reader. \square

The following lemma can be proved using Lemma 3.5 and the definitions.

Lemma 3.6. *Let $m > 0$, $v = (k, \sigma, S) \in \mathcal{W}(S_{2m+1})$, $i, j, k \notin S$ and $\sigma(i, j) \triangleleft \sigma$. Then $w_{i,j}^A(v) < v$, $v(k, k+2) < v$ and $\ell_{\mathcal{W}}(w_{i,j}^A(v)) = \ell_{\mathcal{W}}(v(k, k+2)) = \ell_{\mathcal{W}}(v) - 1$.*

The next result is the crucial technical tool for the proof of our main theorem, and states that elements of $\mathcal{W}(S_{2m+1})$ are “join-irreducible” in a certain sense.

Proposition 3.7. *Let $m > 0$, $u, v \in \mathcal{W}(S_{2m+1})$, $u < v$ and $i := \text{pos}(v)$. If $u < v$ and $\text{pos}(u, v) > 0$ then $u \leq z$ where $z \in \mathcal{W}(S_{2m+1})$ is defined by*

$$z = \begin{cases} v(i, i+2), & \text{if } v < v(i+1, i+2); \\ v(i+1, i+2), & \text{otherwise.} \end{cases}$$

Proof. Let $h := (i + 1)/2$ and consider first the case $i + 1 \notin D(v)$. Let $a := v(i + 1)$; then $a + 1 = v(i + 2) = z(i)$, $a = z(i + 1)$ and $z(i + 2) = 2m + 1$. Define $c := u(i)$, $r \in [m + 1]$ be such that $u(2r - 1) = 2m + 1$, $p \in [h]$ be such that $z_{2p-1,i} = a + 1$ and $q \in [h]$ be such that $u_{2q-1,i} = c$. Notice that $u(i + 1) \in \{u(i) + 1, u(i) - 1\}$ since $\text{pos}(u, v) > 0$. Assume that $q \leq p$. Note that

$$u_{l,i} \leq v_{l,i} = z_{l,i}$$

if $1 \leq l \leq 2p - 2$, while

$$u_{l,i} = u_{l-1,i-1} \leq v_{l-1,i-1} = z_{l-1,i-1} = z_{l,i}$$

if $2p \leq l$, and

$$u_{2p-1,i} = \begin{cases} u_{2p-2,i-1} \leq v_{2p-2,i-1} = z_{2p-2,i-1} = z_{2p-2,i} < z_{2p-1,i}, & \text{if } q < p, \\ u_{2p-1,i+1} \leq v_{2p-1,i+1} = a < a + 1 = z_{2p-1,i}, & \text{if } q = p \text{ and } u(i) < u(i + 1), \\ u_{2p-1,i+1} + 1 \leq v_{2p-1,i+1} + 1 = a + 1 = z_{2p-1,i}, & \text{if } q = p \text{ and } u(i) > u(i + 1), \end{cases}$$

since $u \leq v$. Moreover $u_{l,i+1} = u_{l-1,i} \leq z_{l-1,i} = z_{l,i+1}$ if $l \geq 2p + 1$ while $u_{l,i+1} \leq v_{l,i+1} = z_{l,i+1}$, if $l \leq 2p - 2$. Let $l \in \{2p - 1, 2p\}$. There are some cases to be considered.

1. $q < p$ and $u(i) > u(i + 1)$: in this case we have that $u_{2p,i+1} = u_{2p-1,i} \leq z_{2p-1,i} = z_{2p,i+1}$ and so $u_{2p-1,i+1} \leq u_{2p,i+1} - 1 \leq z_{2p,i+1} - 1 = z_{2p-1,i+1}$.
2. $q < p$ and $u(i) < u(i + 1)$: in this case we have that $u_{2p-1,i+1} = u_{2p-1,i} - 1 \leq z_{2p-1,i} - 1 = z_{2p-1,i+1}$ and so $u_{2p,i+1} = u_{2p-1,i+1} + 1 \leq z_{2p-1,i+1} + 1 = z_{2p,i+1}$.
3. $q = p$ and $u(i) > u(i + 1)$: in this case $u_{2p,i+1} = u_{2p-1,i} \leq z_{2p-1,i} = z_{2p,i+1}$ and $u_{2p-1,i+1} = u_{2p-1,i} - 1 \leq z_{2p-1,i} - 1 = z_{2p-1,i+1}$.
4. $q = p$ and $u(i) < u(i + 1)$: in this case $u_{2p-1,i+1} \leq v_{2p-1,i+1} = a = z_{2p-1,i+1}$ and $u_{2p,i+1} = u_{2p-1,i+1} + 1 \leq z_{2p-1,i+1} + 1 = z_{2p,i+1}$.

So we have proved that $u \leq z$ whenever $q \leq p$. Consider the case $q > p$. If $l \leq 2p - 2$ or $l \geq 2q$ the result follows as above. Let $2p \leq l \leq 2q - 2$. Then $u_{l,i} = u_{l,i+1} \leq v_{l,i+1} = z_{l,i}$. Moreover

$$u_{2p-1,i} = u_{2p-1,i+1} \leq v_{2p-1,i+1} = a < a + 1 = z_{2p-1,i}.$$

Let $u(i) < u(i + 1)$. Then

$$u_{2q-1,i} = u_{2q-1,i+1} \leq v_{2q-1,i+1} = z_{2q-1,i}.$$

If $u(i) > u(i + 1)$ we have that $u_{2q-1,i} = u_{2q-1,i+1} + 1 \leq v_{2q-1,i+1} + 1 = z_{2q-1,i}$ and then we have proved that $u \leq z$ also in case $q > p$.

Let's consider the case $s_{i+1} \in D(v)$. Define $a := v(i + 2)$ and $v_{2p-1,i+1} := a + 1$. If $l \neq 2p - 1$ we have that $u_{l,i+1} \leq v_{l,i+1} = z_{l,i+1}$. Let $l = 2p - 1$. Since $u_{2p-1,i+1} \equiv 1 \pmod{2}$, $a \equiv 1 \pmod{2}$ and $u_{2p-1,i+1} \leq a + 1$, we conclude that $u_{2p-1,i+1} \leq a = z_{2p-1,i+1}$. □

Corollary 3.8. *Let $n \in \mathbb{P}$. Then $f_n : \mathcal{W}(S_n) \rightarrow S_{\lfloor \frac{n}{2} \rfloor}$ is order preserving.*

Proof. If n is even the result was already observed. If n is odd let $u, v \in \mathcal{W}(S_n)$ and $u \leq v$. We prove the result by induction on $\text{pos}(u, v)$. If $\text{pos}(u, v) = 0$ by the tableau criterion we conclude that $\chi_n(u) \leq \chi_n(v)$ and then the result follows by the even case. Let $\text{pos}(u, v) > 0$ and $i := \text{pos}(v)$. In this case, by Proposition 3.7, $u \leq v(i, i+2) < v$ or $u \leq v(i+1, i+2)(i, i+2) < v(i+1, i+2) < v$. Hence, by the inductive hypothesis, $f_n(u) \leq f_n(v(i, i+2)) = f_n(v)$ or $f_n(u) \leq f_n(v(i+1, i+2)(i, i+2)) = f_n(v(i+1, i+2)) = f_n(v)$. \square

We can now prove the main result of this section.

Theorem 3.9. *Let $n > 0$. Then $(\mathcal{W}(S_n), \leq)$ is graded, of rank $\binom{n}{2} - \binom{\lfloor \frac{n}{2} \rfloor}{2}$, and its rank function is $\ell_{\mathcal{W}}$.*

Proof. Assume first $n = 2m$, $m > 1$. Let $u, v \in \mathcal{W}(S_{2m})$ with $u < v$, $\phi(u) = (\sigma, S)$ and $\phi(v) = (\tau, T)$. We prove that, if $\ell_{\mathcal{W}}(u, v) \neq 1$, then there exists $z \in \mathcal{W}(S_{2m})$ such that $u < z < v$ and $\ell_{\mathcal{W}}(z, v) = 1$. Note that, by Proposition 3.2, $\sigma \leq \tau$. We have two cases to consider.

1. $\ell(\sigma, \tau) = 0$: then $\sigma = \tau$ so, by Proposition 3.2, $S \subseteq T$. Therefore there exists $i \in T \setminus S$. So $s_{2i-1} \in D(v) \setminus D(u)$ hence, by the Lifting Property, $u < vs_{2i-1} < v$, $vs_{2i-1} \in \mathcal{W}(S_{2m})$ and $\ell_{\mathcal{W}}(vs_i, v) = 1$.
2. $\ell(\sigma, \tau) > 0$: in this case, since $\sigma < \tau$, there exist $1 \leq x < y \leq m$ such that $\sigma \leq \tau(x, y) \triangleleft \tau$. Let $i := 2x - 1$ and $j := 2y - 1$. There are three cases to consider.

- (a) $s_i \in D(v)$: let $z := vs_i$ and $w := (\tau(x, y), T)$. We want to prove that $u < z$. Let $a + 1 := v(i) = v_{2p-1, i}$, $b := w(i) = v(j) = w_{2q-1, i}$ and $c := u(i) = u_{2s-1, i}$. In particular $q \leq p$. We only have to prove that $u_{l, i} \leq (vs_i)_{l, i}$ for all $l \leq i$. Since $(vs_i)_{l, i} = v_{l, i}$ if $l \neq 2p - 1$, it is enough to prove that $u_{2p-1, i} \leq (vs_i)_{2p-1, i}$. Note that $u_{l, i+1} \leq v_{l, i+1} = (vs_i)_{l, i+1}$ for all $1 \leq l \leq i + 1$.

If $s \leq p$ then

$$(vs_i)_{2p-1, i} > (vs_i)_{2p-2, i} = w_{2p-1, i} = w_{2p, i+1} \geq u_{2p, i+1} \geq u_{2p-1, i}$$

because $b < a$, and $\sigma \leq \tau(x, y)$.

If $s > p$ then $u_{2p-1, i} = u_{2p-1, i+1} \leq (vs_i)_{2p-1, i+1} = (vs_i)_{2p-1, i}$. Since $\ell_{\mathcal{W}}(vs_i, v) = 1$ the result follows.

- (b) $s_i \notin D(v)$ and $s_j \in D(v)$: we then claim that $u < vs_j$. Let $b + 1 = v(j) = v_{2p-1, j}$; then $b = (vs_j)(j) = (vs_j)_{2p-1, j}$. As in the case above it is sufficient to prove that $u_{2p-1, j} \leq (vs_j)_{2p-1, j}$. If $u(j) > u_{2p-1, j}$ then $u_{2p-1, j} = u_{2p-1, j+1} = u_{2p, j+1} - 1 \leq v_{2p, j+1} - 1 = b$. Let $u(j) \leq u_{2p-1, j}$; since $\sigma \leq \tau(x, y)$ we have $\{u_{2p-1, j-1}, u_{2p, j-1}\} \leq \{w_{2p-1, j-1}, w_{2p, j-1}\}$. Then, $u_{2p-1, j} < u_{2p, j} = u_{2p-1, j-1} \leq w_{2p-1, j-1} = b$, since $a > b + 1$.
- (c) $s_i, s_j \notin D(v)$: in this case, by Lemma 3.5, $u < w_{x, y}^A(v) < v$ and $w_{x, y}^A(v) \in \mathcal{W}(S_{2m})$, $\ell_{\mathcal{W}}(w_{x, y}^A(v), v) = 1$.

Now assume $n = 2m + 1$, $m > 0$. Let $u \leq v$, $i := \text{pos}(v)$, and $\ell_{\mathcal{W}}(u, v) > 1$. We prove that there exists $z \in \mathcal{W}(S_{2m+1})$ such that $u < z < v$ and $\ell_{\mathcal{W}}(z, v) = 1$. If $\text{pos}(u, v) = 0$ then the result follows by the previous point. If $\text{pos}(u, v) > 0$ we have, by Proposition 3.7, $u < v(i, i+2) < v$ if $s_{i+1} \notin D(v)$ and $u < vs_{i+1} < v$ otherwise. \square

Remark 3.10. In general $\mathcal{W}(S_n) \cap S_n^J$ is not graded; one can see this by considering $J = \{s_1\}$ and the interval [124365, 561234] in $\mathcal{W}(S_6) \cap S_6^J$.

We can now compute the rank-generating function of $(\mathcal{W}(S_n), \leq)$.

Corollary 3.11. *Let $m > 0$. Then*

$$\mathcal{W}(S_{2m})(x, \ell_{\mathcal{W}}) = (1+x)^m [m]_{x^3}!,$$

$$\mathcal{W}(S_{2m+1})(x, \ell_{\mathcal{W}}) = (1+x)^m [m+1]_{x^2} [m]_{x^3}!.$$

Moreover

$$\mathcal{W}(S_{2m})(x, \ell_{\mathcal{W}}) = \mathcal{W}(S_{2m})(x, 3\text{emaj} + \text{odes})$$

and

$$\mathcal{W}(S_{2m+1})(x, \ell_{\mathcal{W}}) = \mathcal{W}(S_{2m+1})(x, (3\text{emaj} + \text{odes}) \circ \chi_{2m+1} + \text{pos}).$$

Proof. The result follows by (1), Theorem 3.2 and the definition of $\ell_{\mathcal{W}}$. In fact, $\mathcal{W}(S_{2m+1})(x, \ell_{\mathcal{W}}) = [m+1]_{x^2} \mathcal{W}(S_{2m})(x, \ell_{\mathcal{W}})$ and $\ell(v) = \text{odes}(v) + 4\ell(f_{2m}(v))$, for all $v \in \mathcal{W}(S_{2m})$. \square

From Proposition 3.11 we find that the polynomials $\mathcal{W}(S_n)(x, \ell_{\mathcal{W}})$ are reciprocal, i.e. $x^{\ell_{\mathcal{W}}(w_n)} \mathcal{W}(S_n)(x^{-1}, \ell_{\mathcal{W}}) = \mathcal{W}(S_n)(x, \ell_{\mathcal{W}})$. In fact the poset $(\mathcal{W}(S_n), \leq)$ is self-dual, for all $n \in \mathbb{P}$, since the map $v \mapsto vw_n$ is an antiautomorphism of $(\mathcal{W}(S_n), \leq)$ such that $\ell_{\mathcal{W}}(vw_n) = \ell_{\mathcal{W}}(w_n) - \ell_{\mathcal{W}}(v)$ (see [2, Propositions 2.3.2 and 2.3.4]).

From the combinatorial description of the rank function of $\mathcal{W}(S_{2m+1})$ we can deduce a description of its cover relations.

Proposition 3.12. *Let $m \in \mathbb{P}$, $u, v \in \mathcal{W}(S_{2m+1})$, $u = (i, \sigma, S)$ and $v = (j, \tau, T)$. Then $u \triangleleft v$ if and only if either*

1. $i = j$, $\sigma = \tau$ and $S \triangleleft T$, or
2. $j = i - 1$, $\sigma = \tau$, $i - 1 \in S$ and $T = S \setminus \{i - 1\}$, or
3. $i = j$, $\sigma \triangleleft \tau$, $T \cap \{a, b\} = \emptyset$ and $S = T \cup \{a, b\}$,

where $(a, b) := \tau^{-1}\sigma$. In particular, if $u, v \in \mathcal{W}(S_{2m})$, $u = (\sigma, S)$ and $v = (\tau, T)$, then $u \triangleleft v$ if and only if either $\sigma = \tau$ and $S \triangleleft T$, or $\sigma \triangleleft \tau$, $T \cap \{a, b\} = \emptyset$ and $S = T \cup \{a, b\}$.

Proof. If point 1 or 2 hold then $\ell_{\mathcal{W}}(v) - \ell_{\mathcal{W}}(u) = 1$ and by Theorem 3.4 there follows that $u \leq v$. If point 3 holds then the result follows by Lemma 3.6.

Conversely let $u \triangleleft v$. Then $\sigma \leq \tau$ by Corollary 3.8 and $\ell_{\mathcal{W}}(u, v) = 1$. If $i - j \geq 2$ then, by Proposition 3.7 there is $z \in \mathcal{W}(S_{2m+1})$ such that $u \leq z < v$ and $\ell_{\mathcal{W}}(z, v) = 1$; so $z = u$, which is a contradiction since $k - j \leq 1$, being $z = (k, \rho, R)$. Hence assume $i - j \leq 1$.

If $\sigma < \tau$ then there exists a reflection (r, s) such that $\sigma \leq \tau(r, s) \triangleleft \tau$. If $i - j = 1$ then by Proposition 3.7 $u = v(2j - 1, 2j + 1)$ and $\sigma = \tau$, which is a contradiction. Therefore $i = j$. As in the proof of Theorem 3.9, if $r \in T$ then $(\sigma, S) \leq (\tau, T \setminus \{r\})$; if $r \notin T$ and $s \in T$ then $(\sigma, S) \leq (\tau, T \setminus \{s\})$. Moreover $\ell_{\mathcal{W}}((\tau, T \setminus \{r\}), (\tau, T)) = 1$ and $\ell_{\mathcal{W}}((\tau, T \setminus \{s\}), (\tau, T)) = 1$ in these cases. Then $(\sigma, S) = (\tau, T \setminus \{r\})$ or $(\sigma, S) = (\tau, T \setminus \{s\})$, a contradiction. Therefore $r, s \notin T$ and, as in the

proof of Theorem 3.9, $(\sigma, S) \leq (\tau(r, s), T \cup \{r, s\}) \triangleleft (\tau, T)$, and $\ell_{\mathcal{W}}((\tau(r, s), T \cup \{r, s\}), (\tau, T)) = 1$. So we conclude that $S = T \cup \{r, s\}$ and $\sigma = \tau(r, s) \triangleleft \tau$.

Assume now $\sigma = \tau$. If $i = j$ then $S \triangleleft T$ (else $\ell_{\mathcal{W}}(u, v) = \ell(u, v) \geq 2$). If $i - j = 1$ then by Proposition 3.7 we have that $u = v(2j - 1, 2j + 1)$ and $v < vs_{2j}$ and the result follows.

The second statement follows from the first one by observing that $\mathcal{W}(S_{2m})$ is isomorphic to the interval $[e, 2m \dots 321(2m + 1)]$ in $\mathcal{W}(S_{2m+1})$. \square

For example the permutation $782156934 \in \mathcal{W}(S_9)$ covers the Wachs permutations 781256934 , 782156439 and 652187934 , which correspond respectively to cases 1, 2 and 3 in Proposition 3.12.

From Proposition 3.12 we can now prove the second main result of this section, namely a characterization of the Bruhat order relation on Wachs permutations.

Theorem 3.13. *Let $m > 0$ and $u, v \in \mathcal{W}(S_{2m+1})$, $u = (i, \sigma, S)$, $v = (j, \tau, T)$. Then $u \leq v$ if and only if*

$$\sigma \leq \tau, \quad S(u, v) \subseteq T(u, v), \quad \text{and} \quad j \leq i,$$

where, for $X \subseteq [m + 1]$, $X(u, v) := X \cap ([j - 1] \cup [i, m]) \cap F(\sigma, \tau)$, being $F(\sigma, \tau) := \{k \in [m] : \sigma(k) = \tau(k)\}$. Moreover $\ell_{\mathcal{W}}(u) = 3\ell(\sigma) + |S| + 2(m - i + 1)$. In particular, if $u, v \in \mathcal{W}(S_{2m})$, $u = (\sigma, S)$, $v = (\tau, T)$, then $u \leq v$ if and only if $\sigma \leq \tau$ and $S \cap F(\sigma, \tau) \subseteq T$, and $\ell_{\mathcal{W}}(u) = 3\ell(\sigma) + |S|$.

Proof. Let $u \leq v$. We may assume $u \triangleleft v$. There are three cases to consider.

1. $i = j$, $\sigma = \tau$ and $S \triangleleft T$: in this case $F(\sigma, \tau) = [m]$ so the result follows.
2. $j = i - 1$, $\sigma = \tau$ and $S = T \cup \{i - 1\}$, with $i - 1 \notin T$: in this case $F(\sigma, \tau) = [m]$ and $S \cap ([i - 2] \cup [i, m]) \subseteq T$.
3. $i = j$, $\sigma \triangleleft \tau$, $T \cap \{a, b\} = \emptyset$ and $S = T \cup \{a, b\}$, where $(a, b) = \tau\sigma^{-1}$: we have $F(\sigma, \tau) = [m] \setminus \{a, b\}$ and then $S \cap F(\sigma, \tau) \subseteq T$.

Now let $\sigma \leq \tau$, $j < i$ and $S \cap ([j - 1] \cup [i, m]) \cap F(\sigma, \tau) \subseteq T$. In S_m there exists a saturated chain $\sigma = \sigma_0 \triangleleft \sigma_1 \triangleleft \dots \triangleleft \sigma_n = \tau$ with $n = \ell(\sigma, \tau)$. Define $(a_i, b_i) := \sigma_i^{-1}\sigma_{i-1}$ for all $i \in [n]$. We have the following chain in $(\mathcal{W}(S_{2m+1}), \leq)$:

$$\begin{aligned} (i, \sigma, S) &\leq (i, \sigma, S \cup [j, i - 1]) \\ &\triangleleft (i - 1, \sigma, (S \cup [j, i - 2]) \setminus \{i - 1\}) \\ &\triangleleft (i - 2, \sigma, (S \cup [j, i - 3]) \setminus \{i - 2, i - 1\}) \\ &\triangleleft \dots \triangleleft (j + 1, \sigma, (S \cup \{j\}) \setminus [j + 1, i - 1]) \\ &\triangleleft (j, \sigma, S \setminus [j, i - 1]) \\ &\leq (j, \sigma, (S \setminus [j, i - 1]) \cup \{a_1, b_1\}) \\ &\triangleleft (j, \sigma_1, (S \setminus [j, i - 1]) \setminus \{a_1, b_1\}) \\ &\leq (j, \sigma_1, (S \setminus ([j, i - 1] \cup \{a_1, b_1\})) \cup \{a_2, b_2\}) \\ &\triangleleft (j, \sigma_2, (S \setminus [j, i - 1]) \setminus \{a_1, b_1, a_2, b_2\}) \\ &\leq \dots \leq (j, \tau, (S \setminus [j, i - 1]) \setminus \{a_1, b_1, \dots, a_n, b_n\}) \leq (j, \tau, T), \end{aligned}$$

since $\{a_1, b_1, \dots, a_n, b_n\} = [m] \setminus F(\sigma, \tau)$. The length formula follows by Proposition 3.4.

The last statement follows immediately noting that the map $(\sigma, S) \mapsto (m+1, \sigma, S)$ is a poset isomorphism between $\mathcal{W}(S_{2m})$ and $\{(i, \sigma, S) \in \mathcal{W}(S_{2m+1}) : i = m+1\}$. \square

We illustrate the preceding theorem with an example. Let $u = (4, 2431, \{1, 2, 3\}) \in \mathcal{W}(S_9)$ and $v = (3, 3421, \{2\}) \in \mathcal{W}(S_9)$. Then we have that $2431 < 3421$ and $S \cap ([j-1] \cup [i, m]) \cap F(\sigma, \tau) = \{1, 2, 3\} \cap (\{1, 2\} \cup \{4\}) \cap \{2, 4\} = \{2\}$; hence, by Theorem 3.13, $u < v$.

The following lemma can be easily deduced by Theorem 3.13 so we omit its verification.

Lemma 3.14. *Let $m > 0$ and $u, v \in \mathcal{W}(S_{2m+1})$. If $u \leq (i, \sigma, S_1) \leq v$ and $u \leq (i, \sigma, S_2) \leq v$ in $(\mathcal{W}(S_{2m+1}), \leq)$ then $u \leq (i, \sigma, S_1 \cup S_2) \leq v$.*

The characterization obtained in Theorem 3.13 enables us to give an explicit expression for the Möbius function of lower intervals in the poset of Wachs permutations partially ordered by Bruhat order, and shows, in particular, that it has values in $\{0, 1, -1\}$.

Proposition 3.15. *Let $m > 0$, and $v = (j, \tau, T) \in \mathcal{W}(S_{2m+1})$. Then*

$$\mu(e, v) = \begin{cases} (-1)^{|T|}, & \text{if } \tau = e \text{ and } j = m+1; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $v = (\tau, T) \in \mathcal{W}(S_{2m})$ then

$$\mu(e, v) = \begin{cases} (-1)^{|T|}, & \text{if } \tau = e; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on $\ell_{\mathcal{W}}(v)$. If $\tau = e$ and $j = m+1$ then, by Theorem 3.13, the interval $[e, v]$ is isomorphic to a Boolean algebra, so we conclude that $\mu(e, v) = (-1)^{|T|}$, as desired. So assume that either $\tau \neq e$ or $j < m+1$. Then, by Proposition 3.12, $\ell_{\mathcal{W}}(v) \geq 2$, so, by Lemma 3.14 there exists $R \subseteq [m]$, $R \neq \emptyset$, such that $[e, v] \cap \{(k, \rho, U) \in \mathcal{W}(S_{2m+1}) : \rho = e, k = m+1\} = [e, (m+1, e, R)]$. Hence

$$\begin{aligned} \mu(u, v) &= - \sum_{x \in [e, v]} \mu(e, x) \\ &= - \sum_{x \in [e, (i, \sigma, R)]} \mu(e, x) - \sum_{x \in [e, v] \setminus [e, (i, \sigma, R)]} \mu(e, x) \\ &= - \sum_{x \in [e, v] \setminus [e, (i, \sigma, R)]} \mu(e, x) = 0 \end{aligned}$$

by our induction hypothesis, and the fact that $|[e, (i, \sigma, R)]| \neq 1$, where $[u, v] := \{z \in \mathcal{W}(S_{2m+1}) : u \leq z < v\}$.

The statement about Wachs permutations in the even case follows from the odd one as in the proof of Theorem 3.13. \square

We conclude by computing, using Proposition 3.15, the characteristic polynomial of the poset of Wachs permutations.

Corollary 3.16. *The characteristic polynomial of $(\mathcal{W}(S_n), \leq)$ is*

$$(x-1)^{\lfloor \frac{n}{2} \rfloor} x^{\binom{n}{2} - \binom{\lfloor \frac{n}{2} \rfloor + 1}{2}},$$

for all $n \in \mathbb{P}$.

Proof. The result follows from Proposition 3.15 and Theorem 3.9. \square

The following is a commutative diagram that summarizes the poset morphisms considered in this section. The function π_i stands for the canonical projection on the i -th factor of a Cartesian product. Notice that, if A and B are posets, the projection $\pi_1 : A \otimes B \rightarrow A$ is order preserving, whereas $\pi_2 : A \otimes B \rightarrow B$ is not order preserving.

$$\begin{array}{ccccc}
[m+1]^* \times S_m \times \mathcal{P}([m]) & \xrightarrow{\phi_{2m+1}^{-1}} & \mathcal{W}(S_{2m+1}) & \xrightarrow{\phi_{2m+1}} & ([m+1]^* \times S_m) \otimes \mathcal{P}([m]) \\
\downarrow \pi_2 \times \pi_3 & & & \searrow f_{2m+1} & \\
S_m \times \mathcal{P}([m]) & \xrightarrow{\phi_{2m}^{-1}} & \mathcal{W}(S_{2m}) & \xrightarrow{\phi_{2m}} & S_m \otimes \mathcal{P}([m]) \\
& \searrow \pi_1 & \downarrow f_{2m} & \swarrow \pi_1 & \\
& & S_m & &
\end{array}$$

4. Signed Wachs permutations and Bruhat order

For $n > 0$ recall (see [4]) that the set of *signed Wachs permutation* is

$$\mathcal{W}(B_n) := \{\sigma \in B_n : |\sigma^{-1}(i) - \sigma^{-1}(i^*)| \leq 1 \forall i \in [n-1]\}.$$

So, for example, $[-2, -1, 4, 3] \in \mathcal{W}(B_4)$ while $[3, 4, -2, 1] \notin \mathcal{W}(B_4)$. In the even case, as in type A , we have the following group isomorphism (see Proposition 3.1)

$$\mathcal{W}(B_{2m}) \simeq B_m \times \mathcal{P}([m]) = S_2 \wr B_m.$$

We define a bijection $\phi : \mathcal{W}(B_{2m}) \rightarrow B_m \times \mathcal{P}([m])$ as follows. For $\sigma \in B_m$ and $T \subseteq [m]$ let $\phi^{-1}(\sigma, T) := v$, where $v \in \mathcal{W}(B_{2m})$ is defined by

$$v(2i-1) = \begin{cases} 2\sigma(i) - \chi(\sigma(i) > 0), & \text{if } i \notin T, \\ 2\sigma(i) + \chi(\sigma(i) < 0), & \text{if } i \in T, \end{cases}$$

and

$$v(2i) = \begin{cases} 2\sigma(i) + \chi(\sigma(i) < 0), & \text{if } i \notin T, \\ 2\sigma(i) - \chi(\sigma(i) > 0), & \text{if } i \in T, \end{cases}$$

for all $i \in [m]$. For example, let $v := [-3, -4, 1, 2, 6, 5] \in \mathcal{W}(B_6)$; then $\phi(v) = ([-2, 1, 3], \{1, 3\})$. Because of this bijection from now on we freely identify the sets $\mathcal{W}(B_{2m})$ and $B_m \times \mathcal{P}([m])$, so if $v \in \mathcal{W}(B_{2m})$ and $\phi(v) = (\sigma, T)$, then we simply write $v = (\sigma, T)$ and we define

$$\ell_{\mathcal{W}}(v) := \ell_B(v) - \ell_B(\sigma).$$

Recall that we denote by $v \mapsto \tilde{v}$ the natural embedding $B_n \hookrightarrow S_{\pm n}$. Note that if $v \in \mathcal{W}(B_{2m})$ then $\tilde{v} \in \mathcal{W}(S_{\pm 2m})$. Indeed, if $v = (\sigma, T)$ then $\tilde{v} = (\tilde{\sigma}, -T \cup T)$. In fact, by Proposition 2.6, this is an injective group and poset morphism $\mathcal{W}(B_{2m}) \hookrightarrow \mathcal{W}(S_{\pm 2m})$. For example, for $u = [-2, -1, 6, 5, -3, -4] = ([-1, 3, -2], \{2, 3\}) \in \mathcal{W}(B_6)$ we have

$$\tilde{u} = (4, 3, -5, -6, 1, 2, -2, -1, 6, 5, -3, -4) = ((2, -3, 1, -1, 3, -2), \{-3, -2, 2, 3\}) \in \mathcal{W}(S_{\pm 6}).$$

Notice that if n is odd then the image of a signed Wachs permutation is not a Wachs permutation. For example, if $u = [-2, -1, 6, 5, -3, -4, 7] \in \mathcal{W}(B_7)$ we have

$$\tilde{u} = (-7, 4, 3, -5, -6, 1, 2, -2, -1, 6, 5, -3, -4, 7) \notin \mathcal{W}(S_{\pm 7}).$$

It is known that $2\ell_B(v) = \ell_A(\tilde{v}) + \text{neg}(v)$ (see e.g. [2, Exercise 8.2]) so we have that

$$\ell_{\mathcal{W}}(v) = \frac{\ell_{\mathcal{W}}(\tilde{v}) + \text{neg}(\sigma)}{2}, \quad (6)$$

for all $v = (\sigma, T) \in \mathcal{W}(B_{2m})$, because $2 \text{neg}(\sigma) = \text{neg}(v)$.

Proposition 4.1. *The function $\mathcal{W}(B_{2m}) \rightarrow B_m$ defined by the assignment $(\tau, T) \mapsto \tau$ is order preserving.*

Proof. Let $u = (\sigma, S) \in \mathcal{W}(B_{2m})$ and $v = (\tau, T) \in \mathcal{W}(B_{2m})$. We have that $u \leq v$ implies $\tilde{u} \leq \tilde{v}$ and then, by Corollary 3.8, $\tilde{\sigma} \leq \tilde{\tau}$. By Proposition 2.6, this implies $\sigma \leq \tau$. \square

It is easy to characterize, using Theorem 3.13, the Bruhat order relation between signed Wachs permutations in the even case.

Proposition 4.2. *Let $u, v \in \mathcal{W}(B_{2m})$, $u = (\sigma, S)$, $v = (\tau, T)$. Then $u \leq v$ if and only if $\sigma \leq \tau$ in B_m and $S \cap F(\sigma, \tau) \subseteq T$, where $F(\sigma, \tau) := \{i \in [m] : \sigma(i) = \tau(i)\}$.*

Proof. We have that $\tilde{u} = (\tilde{\sigma}, -S \cup S)$ and similarly $\tilde{v} = (\tilde{\tau}, -T \cup T)$. But, by Theorem 3.13, $\tilde{u} \leq \tilde{v}$ if and only if $\tilde{\sigma} \leq \tilde{\tau}$ and $(-S \cup S) \cap F(\tilde{\sigma}, \tilde{\tau}) \subseteq -T \cup T$, which in turn happens if and only if $\sigma \leq \tau$ and $S \cap F(\sigma, \tau) \subseteq T$. \square

The next result is the analogue of Proposition 3.2.

Proposition 4.3. *Let $m > 0$. Then*

1. $\phi : \mathcal{W}(B_{2m}) \rightarrow B_m \otimes \mathcal{P}([m])$ is order preserving;
2. $\phi^{-1} : B_m \times \mathcal{P}([m]) \rightarrow \mathcal{W}(B_{2m})$ is order preserving.

Moreover $\ell_B(\tau, T) = 4\ell_B(\tau) + |T| - \text{neg}(\tau)$, for all $(\tau, T) \in \mathcal{W}(B_{2m})$.

Proof. Points 1. and 2. are direct consequences of Propositions 2.6 and 3.2. The last equality follows by the formula of Proposition 3.2; in fact

$$\begin{aligned} \ell_B(\tau, T) &= \frac{\ell_A(\tilde{\tau}, -T \cup T) + \text{neg}(\tau, T)}{2} = 2\ell_A(\tilde{\tau}) + |T| + \text{neg}(\tau) \\ &= 4\ell_B(\tau) - 2\text{neg}(\tau) + |T| + \text{neg}(\tau). \end{aligned}$$

\square

For any $(i, j) \in [n] \times [\pm n]$, $i \neq j$, we find it convenient to define

$$(i, j)_B := \begin{cases} (i, j)(-i, -j), & \text{if } i \neq |j|; \\ (i, -i), & \text{otherwise.} \end{cases}$$

So the set of reflections of B_n is (see [2, Proposition 8.1.5])

$$T^{B_n} = \{(i, j)_B : 1 \leq i < |j| \leq n\} \cup \{(i, -i)_B : i \in [n]\}.$$

Let $m > 0$. For any reflection $(i, j)_B \in T^{B_m}$ we define an involution $w_{i,j}^B : \mathcal{W}(B_{2m}) \rightarrow \mathcal{W}(B_{2m})$ by letting

$$w_{i,j}^B(\tau, T) := (\tau(i, j)_B, T + \{i, |j|\}),$$

for all $(\tau, T) \in \mathcal{W}(B_{2m})$, where $X + Y$ stands for the symmetric difference between two sets X and Y . For example, if $v = ([-2, 1, 4, 3], \{1, 4\})$ then $w_{3,-3}^B(v) = ([-2, 1, -4, 3], \{1, 3, 4\})$ and $w_{1,-3}^B(v) = ([-4, 1, 2, 3], \{3, 4\})$.

Remark 4.4. We observe that, under the embedding $\mathcal{W}(B_{2m}) \hookrightarrow \mathcal{W}(S_{\pm 2m})$, we have that $w_{i,j}^B(v) \mapsto w_{-i,-j}^A(w_{i,j}^A(\tilde{v}))$, if $i \neq -j$, and $w_{i,-i}^B(v) \mapsto w_{i,-i}^A(\tilde{v})$.

The next technical result enables us to “lift” some order theoretic properties from B_{2m} to $\mathcal{W}(B_{2m})$. Its proof relies on the corresponding result in type A , namely Lemma 3.5.

Lemma 4.5. *Let $m > 0$, $i \in [m]$ and $j \in [\pm m]$. Assume $v = (\tau, T) \in \mathcal{W}(B_{2m})$, $i, |j| \notin T$ and $\tau(i, j)_B \triangleleft \tau$. Then $w_{i,j}^B(v) < v$ and $\ell_{\mathcal{W}}(w_{i,j}^B(v), v) = 1$. Moreover if $u = (\sigma, S) \in \mathcal{W}(B_{2m})$ and $\sigma \leq \tau(i, j)_B$ then $u \leq w_{i,j}^B(v) < v$.*

Proof. Consider the element $\tilde{v} = (\tilde{\tau}, -T \cup T) \in \mathcal{W}(S_{\pm 2m})$. Then by our hypothesis and Theorem 2.7 we have that $\tilde{\tau}(i, j)(-i, -j) \triangleleft \tilde{\tau}(i, j) \triangleleft \tilde{\tau}$ if $i \neq |j|$, and $\tilde{\tau}(i, -i) \triangleleft \tilde{\tau}$ if $i = |j|$. So, by Lemma 3.5, $\tilde{u} \leq w_{-i,-j}^A(w_{i,j}^A(\tilde{v})) < w_{i,j}^A(\tilde{v}) < \tilde{v}$ (since $i, j \notin T$ implies $-i, -j \notin -T$) if $i \neq |j|$, and $\tilde{u} \leq w_{i,-i}^A(\tilde{v}) < \tilde{v}$ if $i = |j|$. By Proposition 2.6 and Remark 4.4 we conclude that $u \leq w_{i,j}^B(v) < v$. Note also that, by Theorem 2.7, $i \neq |j|$ and $v(i, j)_B \triangleleft v$ in B_n imply $\text{neg}(v(i, j)_B) = \text{neg}(v)$, and that $v(i, -i)_B \triangleleft v$ in B_n implies $\text{neg}(v(i, -i)_B) = \text{neg}(v) - 1$. Therefore, if $i, |j| \notin T$ and $\tau(i, j)_B \triangleleft \tau$, then by Lemma 3.5 and (6) we have that $\ell_{\mathcal{W}}(w_{i,j}^B(v), v) = 1$. \square

The following result characterizes the cover relations of the ordering induced by Bruhat order on the signed Wachs permutations in the even case.

Corollary 4.6. *Let $m > 0$, $u, v \in \mathcal{W}(B_{2m})$, $u = (\sigma, S)$ and $v = (\tau, T)$. Then $u \triangleleft v$ if and only if either one of the following conditions is satisfied:*

1. $\sigma = \tau$ and $S \triangleleft T$;
2. $\sigma \triangleleft \tau$, $T \cap \{a, |b|\} = \emptyset$ and $S = T \cup \{a, |b|\}$, where $(a, b)_B := \tau^{-1}\sigma$.

Proof. Let $u \triangleleft v$ in $\mathcal{W}(B_{2m})$. Then $\sigma \leq \tau$ by Corollary 4.1. Moreover, if $\sigma < \omega < \tau$ for some $\omega \in B_m$ then, by Proposition 4.3, $u \leq (\omega, S) \leq v$. So $u \triangleleft v$ implies $\sigma = \tau$ or $\sigma \triangleleft \tau$.

If $\sigma = \tau$ then, by Corollary 4.2, $S \subseteq T$; since $u \triangleleft v$, we have that $S \triangleleft T$. Assume now $\sigma \triangleleft \tau$ and let $(a, b)_B := \tau^{-1}\sigma$. If $a \in T$ or $|b| \in T$ then, by point 2 of the proof of Theorem 3.9, $\tilde{u} \leq \tilde{v}(a, a+1)(-a-1, -a) < \tilde{v}$ and $\tilde{u} \leq \tilde{v}(|b|, |b|+1)(-|b|-1, -|b|) < \tilde{v}$, respectively. Hence $u \triangleleft v$ and $\sigma \triangleleft \tau$ imply $T \cap \{a, |b|\} = \emptyset$ and $u = w_{a,b}^B(v)$, by Lemma 4.5. This implies $S = T \cup \{a, |b|\}$.

The converse can be proved by noting that if condition 1 or 2 are satisfied then, by Proposition 3.12, $[\tilde{u}, \tilde{v}] = \{\tilde{u}, w_1, w_2, \tilde{v}\}$, with $w_1, w_2 \in \mathcal{W}(S_{\pm 2m}) \setminus \mathcal{W}(B_{2m})$, or $[\tilde{u}, \tilde{v}] = \{\tilde{u}, \tilde{v}\}$, where the intervals are taken in $\mathcal{W}(S_{\pm 2m})$. \square

For example $[-2, -1, 3, 4, 6, 5, -7, -8] \triangleleft [-2, -1, 4, 3, 6, 5, -7, -8] \triangleleft [-2, -1, 5, 6, 3, 4, -7, -8]$. By Corollary 4.6 we have that if $u \triangleleft v$ holds in $\mathcal{W}(B_{2m})$ then either $u = v(2i - 1, 2i)_B$ for some $i \in [m]$, or $u = w_{i,j}^B(v)$ for some $(i, j)_B \in T^{B_m}$.

We can now prove the even part of the main result of this section.

Theorem 4.7. *The poset $\mathcal{W}(B_{2m})$ is graded, with rank function $\ell_{\mathcal{W}}$, and its rank is $3m^2$.*

Proof. Since $e, w_0^{B_{2m}} \in \mathcal{W}(B_{2m})$, these are the minimum and maximum of the poset $\mathcal{W}(B_{2m})$, respectively. Let $u = (\sigma, S) \in \mathcal{W}(B_{2m})$ and $v = (\tau, T) \in \mathcal{W}(B_{2m})$ be such that $u \triangleleft v$. Therefore, by Corollary 4.6, either $u = v(2i - 1, 2i)_B$ for some $i \in [m]$ or $u = w_{i,j}^B(v)$, where $\sigma = \tau(i, j)_B \triangleleft \tau$. In both cases $\ell_{\mathcal{W}}(u, v) = 1$, and this proves the first statement. The rank of the poset $\mathcal{W}(B_{2m})$ is, by (6), $\ell_{\mathcal{W}}(w_0(B_{2m})) = 3m^2$, being $\ell_{\mathcal{W}}(\widetilde{w_0(B_{2m})}) = m(6m - 1)$. \square

We now investigate the Bruhat order on $\mathcal{W}(B_n)$ for n odd. Let $v \in \mathcal{W}(B_{2m+1})$. Note that there is a bijection between $\mathcal{W}(B_{2m+1})$ and $\mathcal{W}(B_{2m}) \times [\pm(m+1)]$ given by $v \mapsto (\bar{v}, (\text{pos}(v) + \text{sgn}(\text{pos}(v)))/2)$ where

$$\bar{v}(i) = \begin{cases} v(i), & \text{if } i < v^{-1}(2m+1), \\ v(i+1), & \text{if } i \geq v^{-1}(2m+1), \end{cases}$$

for $i \in [2m]$. Combining this with the bijection between $\mathcal{W}(B_{2m})$ and $B_m \times \mathcal{P}([m])$ explained at the beginning of this section we obtain a bijection ϕ between $\mathcal{W}(B_{2m+1})$ and $[\pm(m+1)] \times B_m \times \mathcal{P}([m])$. If $v \in \mathcal{W}(B_{2m+1})$ and $(i, \sigma, S) \in [\pm(m+1)] \times B_m \times \mathcal{P}([m])$ correspond under this bijection then we write $v = (i, \sigma, S)$, and we define

$$\ell_{\mathcal{W}}(v) := \ell_B(v) - \ell_B(\sigma), \tag{7}$$

and $\check{v} := (\sigma, S)$ (so $\check{v} \in \mathcal{W}(B_{2m})$). So, for example, if $v = [-1, -2, 5, 6, -7, 3, 4,]$ then $v = (-5, [-1, 3, 2], \{-1\})$ so $\ell_{\mathcal{W}}(v) = (9 + 3 + 7) - 2 = 17$, and $\check{v} = [-1, -2, 5, 6, 3, 4]$. Note that, if $u, v \in \mathcal{W}(B_{2m+1})$ are such that $u^{-1}(2m+1) = v^{-1}(2m+1)$ then

$$\ell_{\mathcal{W}}(u, v) = \ell_{\mathcal{W}}(\check{u}, \check{v}). \tag{8}$$

The next result is the analogue of Proposition 3.4.

Proposition 4.8. *Let $m > 0$. Then*

1. $\phi : \mathcal{W}(B_{2m+1}) \rightarrow [\pm(m+1)]^* \otimes B_m \otimes \mathcal{P}([m])$ is order preserving;
2. $\phi^{-1} : [\pm(m+1)]^* \times B_m \times \mathcal{P}([m]) \rightarrow \mathcal{W}(B_{2m+1})$ is order preserving.

Moreover $\ell_B(v) = 4\ell_B(\tau) + |T| - \text{neg}(\tau) + 2(m-i+1) - 3\chi(i < 0) = \ell_B(\check{v}) + 2(m-i+1) - 3\chi(i < 0)$, for all $v = (i, \tau, T) \in \mathcal{W}(B_{2m+1})$.

Proof. The first two points follow easily from Proposition 4.3 and the fact that if $u \leq v$ then $\text{pos}(u) \geq \text{pos}(v)$. The length equality is easy to check using our definitions and the well known fact (see, e.g., [2, Prop. 8.1.1]) that $\ell_B(v) = \text{inv}(v) + \text{neg}(v) + \text{nsp}(v)$ for all v in the hyperoctahedral group. \square

For $v \in \mathcal{W}(B_{2m+1})$ we define an element $c(v) \in \mathcal{W}(B_{2m+1})$ by

$$c(v) := \begin{cases} v(-1, 1)_B, & \text{if } j = -1; \\ v(j+1, j+2)_B, & \text{if } j \neq -1 \text{ and } v(j+1) > v(j+2); \\ v(j, j+2)_B, & \text{if } j \neq -1 \text{ and } v(j+1) < v(j+2); \end{cases} \quad (9)$$

where $j := v^{-1}(2m+1)$. For example, $c([-9, 4, 3, -6, -5, 2, 1, -8, -7]) = [9, 4, 3, -6, -5, 2, 1, -8, -7]$, $c([4, 3, -6, -5, 9, 2, 1, -8, -7]) = [4, 3, -6, -5, 9, 1, 2, -8, -7]$ and $c([3, 4, -9, 1, 2, 6, 5, -7, -8]) = [-9, 4, 3, 1, 2, 6, 5, -7, -8]$. Note that $\ell_{\mathcal{W}}(c(v), v) = 1$. The next result implies that $c(v)$ is the only coatom u of v such that $v^{-1}(2m+1) \neq u^{-1}(2m+1)$, and is the main technical tool in our proof of the fact that $\mathcal{W}(B_n)$, under Bruhat order, is graded.

Theorem 4.9. *Let $m \in \mathbb{P}$, and $u, v \in D(B_{2m+1})$, $u < v$, be such that $v^{-1}(2m+1) < u^{-1}(2m+1)$. Then $u \leq c(v)$.*

Proof. Let $j := v^{-1}(2m+1)$. We distinguish various cases according as to whether $j < -1$, $j = -1$, or $j > 0$. Let, for brevity, $z := c(v)$. Given a signed permutation $w \in B_n$ and $j \in [\pm n]$ we find it convenient to denote by w_j the increasing rearrangement of $\{w(-n), w(-n+1), \dots, w(j)\}$.

Assume first that $j > 0$ and $j+1 \in D_R(v)$. Let $a := v(j+2)$ so $v(j+1) = a+1$. Then $v_k = z_k$ for all $k \in [\pm(2m+1)] \setminus \{j+1, -j-2\}$. Since u, v, z are all Wachs permutations there are $x_1, \dots, x_{\tilde{m}}, y_1, \dots, y_{\tilde{m}+1} \in [\pm(2m)]$, where $\tilde{m} := \frac{2m-1+j}{2}$, such that

$$v_{j+1} = (-2m-1, x_1, x_1+1, \dots, x_p, x_p+1, a+1, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1)$$

$$z_{j+1} = (-2m-1, x_1, x_1+1, \dots, x_p, x_p+1, a, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1)$$

and

$$u_{j+1} = (-2m-1, y_1, y_1+1, \dots, y_{\tilde{m}+1}, y_{\tilde{m}+1}+1),$$

for some $0 \leq p \leq \tilde{m}$. Since $u_{j+1} \leq v_{j+1}$ we have that $y_{p+1} \leq a+1$. But y_{p+1} and a are of the same parity if they have the same sign so $y_{p+1} \leq a$ and hence $u_{j+1} \leq z_{j+1}$.

Similarly, there are $x_1, \dots, x_{\tilde{m}}, y_1, \dots, y_{\tilde{m}} \in [\pm(2m)]$, where $\tilde{m} := \frac{2m-j-1}{2}$ such that

$$v_{-j-2} = (x_1, x_1+1, \dots, x_p, x_p+1, -a, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1)$$

$$z_{-j-2} = (x_1, x_1+1, \dots, x_p, x_p+1, -a-1, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1)$$

and

$$u_{-j-2} = (-2m-1, y_1, y_1+1, \dots, y_{\tilde{m}}, y_{\tilde{m}}+1),$$

for some $0 \leq p \leq \tilde{m}$. Since $u_{-j-2} \leq v_{-j-2}$ we have that $y_p \leq x_p+1$. But $x_p+1 < -a-1$ so $y_p+1 \leq -a-1$ and hence $u_{-j-2} \leq z_{-j-2}$.

Suppose now that ($j > 0$ and) $j+1 \notin D_R(v)$. Consider first $v_j, z_j, u_j, v_{j+1}, z_{j+1}, u_{j+1}$. Since v, u , and z are all Wachs permutations, and by our definition of z , there are $\{x_1, \dots, x_{\tilde{m}}\}_{<}, \{y_1, \dots, y_{\tilde{m}}\}_{<} \subseteq [\pm(2m)]$, where $\tilde{m} := \frac{2m-1+j}{2}$ such that

$$v_j = (-2m-1, x_1, x_1+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1)$$

$$\begin{aligned}
z_j &= (-2m - 1, x_1, x_1 + 1, \dots, x_p, x_p + 1, a + 1, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1) \\
u_j &= (-2m - 1, y_1, y_1 + 1, \dots, y_q, y_q + 1, b^*, y_{q+1}, y_{q+1} + 1, \dots, y_{\tilde{m}}, y_{\tilde{m}} + 1) \\
v_{j+1} &= (-2m - 1, x_1, x_1 + 1, \dots, x_p, x_p + 1, a, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1, 2m + 1) \\
z_{j+1} &= (-2m - 1, x_1, x_1 + 1, \dots, x_p, x_p + 1, a, a + 1, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1) \\
u_{j+1} &= (-2m - 1, y_1, y_1 + 1, \dots, y_q, y_q + 1, b, b + 1, y_{q+1}, y_{q+1} + 1, \dots, y_{\tilde{m}}, y_{\tilde{m}} + 1)
\end{aligned}$$

where $a := v(j+1)$ ($= z(j+1)$), $b^* := u(j)$, and $p, q \in [\tilde{m}]$. By our hypothesis $u \leq v$ so $u_j \leq v_j$ and $u_{j+1} \leq v_{j+1}$ (componentwise). This easily implies that $u_j \leq z_j$ and $u_{j+1} \leq z_{j+1}$ (componentwise) keeping in mind the fact that x_r and y_s have the same parity if they have the same sign for all $r, s \in [\tilde{m}]$ (so $y_s \leq x_r + 1$ implies $y_s \leq x_r$ for all $r, s \in [\tilde{m}]$). For example, if $q < p$, then $y_p \leq a$ (since $u_{j+1} \leq v_{j+1}$), while $y_k \leq x_k + 1$ (since $u_j \leq v_j$) so $y_k \leq x_k$ if $p + 1 \leq k \leq \tilde{m}$. Similarly, if $q > p$, then $y_{p+1} \leq a$ (since $u_{j+1} \leq v_{j+1}$) while $b \leq x_{q+1}$ so $b \leq x_q$, while $y_k \leq x_k + 1$ (since $u_j \leq v_j$) so $y_k \leq x_k$ if $q + 1 \leq k \leq \tilde{m}$, and $y_k \leq x_{k-1} + 1$ (since $u_{j+1} \leq v_{j+1}$) so $y_k \leq x_{k-1}$ if $p + 2 \leq k \leq q$. The case $p = q$ is even simpler, and is therefore omitted.

Consider now $u_{-j-2}, z_{-j-2}, v_{-j-2}$ and $u_{-j-1}, z_{-j-1}, v_{-j-1}$. Then reasoning as above we have that there are $\{x_1, \dots, x_{\tilde{m}}\}_{<} \subseteq \{y_1, \dots, y_{\tilde{m}}\}_{<} \subseteq [\pm(2m)]$, where $\tilde{m} := m - \frac{j+1}{2}$, such that

$$\begin{aligned}
v_{-j-2} &= (x_1, x_1 + 1, \dots, x_p, x_p + 1, -a - 1, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1) \\
z_{-j-2} &= (-2m - 1, x_1, x_1 + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1) \\
u_{-j-2} &= (-2m - 1, y_1, y_1 + 1, \dots, y_{\tilde{m}}, y_{\tilde{m}} + 1) \\
v_{-j-1} &= (x_1, x_1 + 1, \dots, x_p, x_p + 1, -a - 1, -a, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1) \\
z_{-j-1} &= (-2m - 1, x_1, x_1 + 1, \dots, x_p, x_p + 1, -a, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1) \\
u_{-j-1} &= (-2m - 1, y_1, y_1 + 1, \dots, y_q, y_q + 1, -b, y_{q+1}, y_{q+1} + 1, \dots, y_{\tilde{m}}, y_{\tilde{m}} + 1)
\end{aligned}$$

where $p, q \in [\tilde{m}]$. As above, the fact that $u_{-j-2} \leq v_{-j-2}$ and $u_{-j-1} \leq v_{-j-1}$ easily implies that $u_{-j-2} \leq z_{-j-2}$ and $u_{-j-1} \leq z_{-j-1}$. For example, if $q < p$, then $y_k \leq x_k + 1$ (since $u_{-j-2} \leq v_{-j-2}$) so $y_k \leq x_k$ for $1 \leq k \leq p$ and hence $-b < y_{q+1} \leq x_{q+1}$. Similarly, if $q \geq p$.

Consider now the case $j < -1$ (so $j \leq -3$). Assume first that $j + 1 \notin D_R(v)$. Consider $u_j, z_j, v_j, u_{j+1}, z_{j+1}, v_{j+1}$. If $u^{-1}(-2m - 1) > j$ then we conclude exactly as in the case $j > 0$. So assume that $u^{-1}(-2m - 1) < j$. Then reasoning as above we conclude that there are $\{x_1, \dots, x_{\tilde{m}}\}_{<}, \{y_1, \dots, y_{\tilde{m}}\}_{<} \subseteq [\pm(2m)]$, where $\tilde{m} := m + \frac{j+1}{2}$, such that

$$\begin{aligned}
v_j &= (x_1, x_1 + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1, 2m + 1) \\
z_j &= (x_1, x_1 + 1, \dots, x_p, x_p + 1, a + 1, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1) \\
u_j &= (-2m - 1, y_1, y_1 + 1, \dots, y_{\tilde{m}}, y_{\tilde{m}} + 1) \\
v_{j+1} &= (x_1, x_1 + 1, \dots, x_p, x_p + 1, a, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1, 2m + 1) \\
z_{j+1} &= (x_1, x_1 + 1, \dots, x_p, x_p + 1, a, a + 1, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1)
\end{aligned}$$

$$\begin{aligned}
u_{j+1} &= (-2m-1, y_1, y_1+1, \dots, y_q, y_q+1, c, y_{q+1}, y_{q+1}+1, \dots, y_{\tilde{m}}, y_{\tilde{m}}+1) \\
v_{j+2} &= (x_1, x_1+1, \dots, x_p, x_p+1, a, a+1, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \\
u_{j+2} &= (-2m-1, y_1, y_1+1, \dots, y_q, y_q+1, c, c+1, y_{q+1}, y_{q+1}+1, \dots, y_{\tilde{m}}, y_{\tilde{m}}+1)
\end{aligned}$$

where $a := v(j+1)$ ($= z(j+1)$), $c := u(j+1)$, and $p, q \in [\tilde{m}]$. By our hypothesis $u \leq v$ so $u_j \leq v_j$, $u_{j+1} \leq v_{j+1}$, and $u_{j+2} \leq v_{j+2}$ (componentwise). As above, this implies that $u_j \leq z_j$ and $u_{j+1} \leq z_{j+1}$. For example, if $p < q$, then $y_{p+1} \leq a+1$ (so $y_{p+1}+1 \leq x_{p+1}$), $y_p+1 \leq a$, $c \leq x_q+1$, and $y_i \leq x_{i-1}+1$ (so $y_i \leq x_{i-1}$) for $p+2 \leq i \leq q$ (since $u_{j+2} \leq v_{j+2}$), while $y_i \leq x_i+1$ (so $y_i \leq x_i$) for $p+1 \leq i \leq \tilde{m}$ (since $u_j \leq v_j$). Similarly, if $q < p$, then $y_p \leq a$ and $y_i \leq x_i+1$ (so $y_i \leq x_i$) for $p+1 \leq i \leq \tilde{m}$ (since $u_{j+1} \leq v_{j+1}$). Finally, if $q = p$, then $c \leq a+1$, $y_{p+1} \leq a$ and $y_i \leq x_i+1$ (so $y_i \leq x_i$) for $p+1 \leq i \leq \tilde{m}$ (since $u_{j+2} \leq v_{j+2}$).

Consider now $u_{-j-1}, v_{-j-1}, z_{-j-1}, u_{-j-2}, v_{-j-2}, z_{-j-2}$. If $u^{-1}(-2m-1) > j$ then we conclude exactly as in the case $j > 0$. So assume that $u^{-1}(-2m-1) < j$. Then as above, since u, v , and z are all Wachs permutations we conclude that there are $\{x_1, \dots, x_{\tilde{m}}\}_{<} \subseteq [\pm(2m)]$ and $\{y_1, \dots, y_{\tilde{m}}\}_{<} \subseteq [\pm(2m)]$ such that

$$\begin{aligned}
v_{-j-2} &= (x_1, x_1+1, \dots, x_p, x_p+1, -a-1, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \\
z_{-j-2} &= (-2m-1, x_1, x_1+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \\
u_{-j-2} &= (-2m-1, y_1, y_1+1, \dots, y_q, y_q+1, c^*, y_{q+1}, y_{q+1}+1, \dots, y_{\tilde{m}}, y_{\tilde{m}}+1) \\
v_{-j-1} &= (x_1, x_1+1, \dots, x_p, x_p+1, -a-1, -a, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \\
z_{-j-1} &= (-2m-1, x_1, x_1+1, \dots, x_p, x_p+1, -a, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \\
u_{-j-1} &= (-2m-1, y_1, y_1+1, \dots, y_q, y_q+1, c, c+1, y_{q+1}, y_{q+1}+1, \dots, y_{\tilde{m}}, y_{\tilde{m}}+1) \\
v_{-j-3} &= (x_1, x_1+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \\
u_{-j-3} &= (-2m-1, y_1, y_1+1, \dots, y_{\tilde{m}}, y_{\tilde{m}}+1)
\end{aligned}$$

where $\tilde{m} := m - \frac{j+3}{2}$, $-a := v(-j-1)$ ($= z(-j-1)$), and $c^* := u(-j-2)$, for some $p, q \in [\tilde{m}]$. It is then not hard to conclude that $u_{-j-2} \leq z_{-j-2}$ and $u_{-j-1} \leq z_{-j-1}$. For example, if $p \leq q$, then $y_i \leq x_i+1$ (since $u_{-j-2} \leq v_{-j-2}$) so $y_i \leq x_i$ for $1 \leq i \leq p$. If $p > q$ then $y_i \leq x_i+1$ (since $u_{-j-3} \leq v_{-j-3}$) so $y_i \leq x_i$ for $1 \leq i \leq \tilde{m}$ and hence $c+1 < y_{q+1} \leq x_{q+1}$.

Assume now that $(j < -1$ and $j+1 \in D_R(v))$. Then $v_k = z_k$ if $k \in [\pm(2m+1)] \setminus \{-j-2, j+1\}$. Assume first that $u^{-1}(2m+1) \geq -j$. Let $a := v(j+2)$ so $v(j+1) = a+1$. Then there are $x_1, \dots, x_{\tilde{m}}, y_1, \dots, y_{\tilde{m}} \in [\pm(2m)]$, where $\tilde{m} := \frac{2m-3-j}{2}$ such that

$$v_{-j-2} = (x_1, x_1+1, \dots, x_p, x_p+1, -a, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \quad (10)$$

$$z_{-j-2} = (x_1, x_1+1, \dots, x_p, x_p+1, -a-1, x_{p+1}, x_{p+1}+1, \dots, x_{\tilde{m}}, x_{\tilde{m}}+1, 2m+1) \quad (11)$$

and

$$u_{-j-2} = (-2m-1, y_1, y_1+1, \dots, y_q, y_q+1, c^*, y_{q+1}, y_{q+1}+1, \dots, y_{\tilde{m}}, y_{\tilde{m}}+1),$$

for some $0 \leq p, q \leq \tilde{m}$ where $c^* := u(-j-2)$. It is then easy to see that $u_{-j-2} \leq z_{-j-2}$. Indeed, if $q < p$ then, since $u_{-j-2} \leq v_{-j-2}$, $y_p \leq -a$ so, as above, $y_p \leq -a-1$ and hence $u_{-j-2} \leq z_{-j-2}$. If

$q \geq p$ then $y_p \leq x_p + 1$ (since $u_{-j-2} \leq v_{-j-2}$) so $y_p \leq x_p$. But $x_p + 1 < -a$ hence $y_p + 1 \leq -a - 1$ so $u_{-j-2} \leq z_{-j-2}$.

Similarly, there are $\{x_1, \dots, x_{\tilde{m}}\}_<, \{y_1, \dots, y_{\tilde{m}}\}_< \in [\pm(2m)]$, where $\tilde{m} := \frac{2m+j+1}{2}$, such that

$$v_{j+1} = (x_1, x_1 + 1, \dots, x_p, x_p + 1, a + 1, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1, 2m + 1) \quad (12)$$

$$z_{j+1} = (x_1, x_1 + 1, \dots, x_p, x_p + 1, a, x_{p+1}, x_{p+1} + 1, \dots, x_{\tilde{m}}, x_{\tilde{m}} + 1, 2m + 1) \quad (13)$$

and

$$u_{j+1} = (-2m - 1, y_1, y_1 + 1, \dots, y_q, y_q + 1, c^*, y_{q+1}, y_{q+1} + 1, \dots, y_{\tilde{m}}, y_{\tilde{m}} + 1),$$

for some $0 \leq p, q \leq \tilde{m}$, where $c^* := u(j + 1)$, and we conclude exactly as in the last case.

Assume now that $u^{-1}(2m + 1) < -j$. Then there are $x_1, \dots, x_{\tilde{m}}, y_1, \dots, y_{\tilde{m}+1} \in [\pm 2m]$, where $\tilde{m} := \frac{2m+1+j}{2}$ such that (12) and (13) hold and

$$u_{j+1} = (y_1, y_1 + 1, \dots, y_{\tilde{m}+1}, y_{\tilde{m}+1} + 1)$$

and the result follows easily. Similarly, there are $x_1, \dots, x_{\tilde{m}}, y_1, \dots, y_{\tilde{m}} \in [\pm 2m]$, where $\tilde{m} := \frac{2m-3-j}{2}$, such that (10) and (11) hold and

$$u_{-j-2} = (-2m - 1, y_1, y_1 + 1, \dots, y_{\tilde{m}}, y_{\tilde{m}} + 1, 2m + 1)$$

for some $0 \leq p \leq \tilde{m}$. Hence, since $u_{-j-2} \leq v_{-j-2}$, $y_p \leq x_p + 1 < -a - 1$ so $y_p + 1 \leq -a - 1$.

Finally, if $j = -1$ then $z = v(-1, 1)$ so there are $x_1, \dots, x_m, y_1, \dots, y_m \in [\pm 2m]$ such that $v_{-1} = (x_1, x_1 + 1, \dots, x_m, x_m + 1, 2m + 1)$, $z_{-1} = (-2m - 1, x_1, x_1 + 1, \dots, x_m, x_m + 1)$ and $u_{-1} = (-2m - 1, y_1, y_1 + 1, \dots, y_m, y_m + 1)$, so the conclusion follows easily. This concludes the proof. \square

Recall that if $v = (\sigma, S, i) \in \mathcal{W}(B_{2m+1})$ then we let $\check{v} = (\sigma, S) \in \mathcal{W}(B_{2m})$. We can now prove the main result of this section.

Theorem 4.10. $\mathcal{W}(B_n)$ is graded, with rank function $\ell_{\mathcal{W}}$, and its rank is $n^2 - \lfloor \frac{n}{2} \rfloor^2$.

Proof. If n is even then this follows from Theorem 4.7. So assume that $n = 2m + 1$ for some $m > 0$. Since $\mathcal{W}(B_{2m+1})$ has both a maximum and a minimum element it is enough to show that if $u, v \in \mathcal{W}(B_{2m+1})$ and $u \triangleleft v$ then $\ell_{\mathcal{W}}(u, v) = 1$. So let $u, v \in \mathcal{W}(B_{2m+1})$ be such that $u \triangleleft v$. Let $j := u^{-1}(2m + 1)$, and $i := v^{-1}(2m + 1)$. Since $u < v$ we have that $i \leq j$. If $i < j$ then by Theorem 4.9 we have that $u \leq c(v) < v$ so $u = c(v)$ and hence $\ell_{\mathcal{W}}(u, v) = 1$. If $i = j$ then $\check{u} \triangleleft \check{v}$ so, by Theorem 4.7, $\ell_{\mathcal{W}}(\check{u}, \check{v}) = 1$ and hence, by (8), $\ell_{\mathcal{W}}(u, v) = 1$. Finally, it is not difficult to see, by our definition of $\ell_{\mathcal{W}}$, that $\ell_{\mathcal{W}}(w_0(B_{2m+1})) = 3m^2 + 4m + 1$. \square

We remark that the sequence $\{\ell_{\mathcal{W}}(w_0(B_{2m}))\}_{m \in \mathbb{P}}$ gives the number of edges of the complete tripartite graph $K_{m,m,m}$ (see A033428 in OEIS), and $\{\ell_{\mathcal{W}}(w_0(B_{2m+1}))\}_{m \in \mathbb{P}}$ is the sequence of octagonal numbers (see A000567 in OEIS). We can now compute the rank-generating function of $(\mathcal{W}(B_n), \leq)$.

Corollary 4.11. *Let $m > 0$. Then*

$$\mathcal{W}(B_{2m})(x, \ell_{\mathcal{W}}) = (1+x)^m [m]_{x^3}! \prod_{i=1}^m (1+x^{3i-1})$$

and

$$\mathcal{W}(B_{2m+1})(x, \ell_{\mathcal{W}}) = [m+1]_{x^2} (1+x^{2m+1}) (1+x)^m [m]_{x^3}! \prod_{i=1}^m (1+x^{3i-1}).$$

Proof. Note first that by our definition and Proposition 4.3 if $u \in \mathcal{W}(B_{2m})$, $u = (\sigma, S)$, then $\ell_{\mathcal{W}}(u) = 3\ell_B(\sigma) + |S| - \text{neg}(\sigma)$. Therefore

$$\sum_{u \in \mathcal{W}(B_{2m})} x^{\ell_{\mathcal{W}}(u)} = \sum_{\sigma \in B_m} \sum_{S \subseteq [m]} x^{3\ell_B(\sigma) + |S| - \text{neg}(\sigma)} = (1+x)^m \sum_{\sigma \in B_m} x^{3\ell_B(\sigma) - \text{neg}(\sigma)}.$$

Let $J := [m]$. Then by [2, Prop. 2.4.4] every element of $\sigma \in B_m$ may be expressed in a unique way as $\sigma = zw$ where $w \in (B_m)_J$ and $z \in (B_m)^J$. But the elements of $(B_m)_J$ are permutations of S_m and the $z \in (B_m)^J$ are characterized by the fact that $z(1) < z(2) < \dots < z(m)$. Hence these elements z are in bijection with subsets $S \subseteq [m]$, where the subset S is the set of negative values taken by σ . Furthermore, we then have that $\ell_B(\sigma) = \text{inv}(\sigma) + \text{neg}(\sigma) + \text{nsp}(\sigma) = \text{inv}(w) + \text{neg}(z) + \text{nsp}(z) = \text{inv}(w) + |S| + \sum_{s \in S} (s-1) = \text{inv}(w) + \sum_{s \in S} s$. We therefore have that

$$\sum_{\sigma \in B_m} x^{3\ell_B(\sigma) - \text{neg}(\sigma)} = \sum_{w \in S_m} \sum_{S \subseteq [m]} x^{3\text{inv}(w) - |S| + 3\sum_{s \in S} s} = \sum_{w \in S_m} x^{3\text{inv}(w)} \sum_{S \subseteq [m]} x^{\sum_{s \in S} (3s-1)}$$

and the first equation follows. For the second formula we have, using Proposition 4.8 and (7),

$$\begin{aligned} \sum_{u \in \mathcal{W}(B_{2m+1})} x^{\ell_{\mathcal{W}}(u)} &= \left(\sum_{i=-m-1}^{-1} x^{2m-2i-1} + \sum_{i=1}^{m+1} x^{2(m-i+1)} \right) \mathcal{W}(B_{2m})(x, \ell_{\mathcal{W}}) \\ &= [m+1]_{x^2} (1+x^{2m+1}) \mathcal{W}(B_{2m})(x, \ell_{\mathcal{W}}). \end{aligned}$$

□

Theorem 4.10 enables us to explicitly determine the cover relations in $\mathcal{W}(B_n)$ for n odd.

Corollary 4.12. *Let $u, v \in \mathcal{W}(B_{2m+1})$, $u = (i, \sigma, S)$, $v = (j, \tau, T)$ ($\sigma, \tau \in B_m$, $S, T \subseteq [m]$, $i, j \in [\pm(m+1)]$). Then $u \triangleleft v$ if and only if either one of the following conditions is satisfied:*

1. $i = j$, $\sigma = \tau$, and $S \triangleleft T$;
2. $j = i - 1$, $\sigma = \tau$ and either $|j + \chi(j < 0)| \in S$, $T = S \setminus \{|j + \chi(j < 0)|\}$ if $j \neq -1$, or $S = T$ if $j = -1$;
3. $i = j$, $\sigma \triangleleft \tau$, $T \cap \{|a|, |b|\} = \emptyset$ and $S = T \cup \{|a|, |b|\}$, where $(a, b)_B = \tau\sigma^{-1}$.

Proof. Note that, since $\mathcal{W}(B_{2m+1})$ is graded, and its rank function is $\ell_{\mathcal{W}}$, $u \triangleleft v$ if and only if $u \leq v$ and $\ell_{\mathcal{W}}(u, v) = 1$. If either 1. or 2. hold then it is easy to check that $u \leq v$ and $\ell_{\mathcal{W}}(u, v) = 1$.

Assume now that 3. holds. Suppose first that $\tau = \sigma(a, b)(-a, -b)$. Since $\sigma \triangleleft \tau$ we have from Theorem 2.7 that $ab > 0$. We may assume that $0 < a < b$. Again by Theorem 2.7 we have that $\{k \in [a+1, b-1] : \sigma(a) < \sigma(k) < \sigma(b)\} = \emptyset$. This, by 3., implies that $\text{inv}(v) - \text{inv}(u) = 4 - 2$, $\text{neg}(v) = \text{neg}(u)$, and $\text{nsp}(v) = \text{nsp}(u)$. Hence $\ell_{\mathcal{W}}(u, v) = \ell_B(u, v) - \ell_B(\sigma, \tau) = 2 - 1 = 1$. Suppose

now that $\tau = \sigma(c, -c)$. We may clearly assume that $c > 0$. Then by Theorem 2.7 $\{k \in [c-1] : -\sigma(c) < \sigma(k) < \sigma(c)\} = \emptyset$. This, again by 3., implies that $\text{inv}(v) - \text{inv}(u) = -2(\sigma(c) - 1) - 1$, $\text{neg}(v) = \text{neg}(u) + 2$, and $\text{nsp}(v) = \text{nsp}(u) + 2(\sigma(c) - 1) + 1$. Hence $\ell_{\mathcal{W}}(u, v) = \ell_B(u, v) - \ell_B(\sigma, \tau) = 2 - 1 = 1$.

Conversely, assume that $u \triangleleft v$. If $j > i$ then by Theorem 4.9 $u \leq c(v) < v$ so $u = c(v)$. If $j = -1$ then 2. follows easily from (9). If $j \neq -1$ then $v(\text{pos}(v) + 1) < v(\text{pos}(v) + 2)$ (else, by (9), $v^{-1}(2m+1) = c(v)^{-1}(2m+1)$ so $i = j$, a contradiction) and 2. again follows from (9). If $i = j$ and $\sigma = \tau$ then it follows easily from Proposition 2.6 and Theorem 2.4 that $S \subseteq T$ and 1. follows since $u \triangleleft v$. So assume $(i = j \text{ and } \sigma < \tau)$. Then, since $i = j$, $(\sigma, S) \triangleleft (\tau, T)$ in $\mathcal{W}(B_{2m})$ so 3. follows from Corollary 4.6. □

So, for example, in $\mathcal{W}(B_9)$ we have that $[9, 2, 1, -3, -4, 8, 7, -6, -5] \triangleleft [-9, 2, 1, -3, -4, 8, 7, -6, -5] \triangleleft [1, 2, -9, -3, -4, 8, 7, -6, -5] \triangleleft [1, 2, -9, -3, -4, 8, 7, -5, -6] \triangleleft [1, 2, -9, -6, -5, 8, 7, -4, -3]$, where the first two covering relations are of type 2. (with $j = -1$, and $j = -2$, respectively), the third one is of type 1., and the fourth one of type 3., with $(a, b)_B = (2, 4)(-2, -4)$.

The next theorem characterizes the Bruhat order on signed Wachs permutations, in the odd case. Note that it generalizes and puts in perspective the results of Proposition 4.8.

Theorem 4.13. *Let $m > 0$ and $u, v \in \mathcal{W}(B_{2m+1})$, $u = (i, \sigma, S)$, $v = (j, \tau, T)$. Then $u \leq v$ if and only if*

$$\sigma \leq \tau, \quad S(u, v) \subseteq T(u, v), \quad \text{and} \quad j \leq i,$$

where, for $X \subseteq [m]$, $X(u, v) := X \cap (\{\min\{|i|, |j|\} - 1\} \cup \{\max\{|i|, |j|\}, m\}) \cap F(\sigma, \tau)$, being $F(\sigma, \tau) := \{i \in [m] : \sigma(i) = \tau(i)\}$. Moreover $\ell_{\mathcal{W}}(u) = 3\ell_B(\sigma) + |S| - \text{neg}(\sigma) + 2(m - i + 1) - 3\chi(i < 0)$.

Proof. Let $u \leq v$. We may assume $u \triangleleft v$. It is clear from Corollary 4.12 that $\sigma \leq \tau$ and $j \leq i$. Furthermore in case 1 of Corollary 4.12 and in case 2 with $j = -1$ we have that $S \subseteq T$ and then $S(u, v) \subseteq T(u, v)$. In case 2 with $j \neq -1$ we have that i and j have the same sign and $F(\sigma, \tau) = [m]$; hence $T = S \setminus \{j\}$ if $i > 0$, so $T(u, v) = (S \setminus \{j\}) \cap ([j-1] \cup [j+1, m]) = S(u, v)$. If $i < 0$ then $T = S \setminus \{|i|\}$ and $T(u, v) = (S \setminus \{|i|\}) \cap ([|i|-1] \cup [i+1, m]) = S(u, v)$. Finally in case 3 we have that $F(\sigma, \tau) = [m] \setminus \{a, b\}$; since $S = T \cup \{a, b\}$ we obtain $T(u, v) = S(u, v)$.

Now let $\sigma \leq \tau$, $j \leq i$ and $S(u, v) \subseteq T(u, v)$. Assume $j \leq i < 0$. We then have the following chain in $(\mathcal{W}(B_{2m+1}), \leq)$:

$$\begin{aligned} (i, \sigma, S) &\leq (i, \sigma, S \cup [|i|, |j| - 1]) \\ &\triangleleft (i - 1, \sigma, (S \cup [|i| + 1, |j| - 1]) \setminus \{|i|\}) \\ &\triangleleft (i - 2, \sigma, (S \cup [|i| + 2, |j| - 1]) \setminus \{|i|, |i| + 1\}) \\ &\triangleleft \dots \triangleleft (j, \sigma, S \setminus [|i|, |j| - 1]) \leq (j, \tau, T), \end{aligned}$$

where the last inequality follows from Proposition 4.2 and the facts that $(S \setminus [|i|, |j| - 1]) \cap F(\sigma, \tau) = S(u, v) \subseteq T(u, v) \subseteq T$ and $(j, \sigma, S \setminus [|i|, |j| - 1]) \leq (j, \tau, T)$ if and only if $(\sigma, S \setminus [|i|, |j| - 1]) \leq (\tau, T)$.

If $j < 0 < i$ and $|j| < i$ we have the following chain in $(\mathcal{W}(B_{2m+1}), \leq)$:

$$\begin{aligned}
(i, \sigma, S) &\leq (i, \sigma, S \cup [|j|, i-1]) \\
&\triangleleft (i-1, \sigma, (S \cup [|j|, i-2]) \setminus \{i-1\}) \\
&\triangleleft (i-2, \sigma, (S \cup [|j|, i-3]) \setminus \{i-2, i-1\}) \\
&\triangleleft \dots \triangleleft (-j, \sigma, S \setminus [|j|, i-1]) \leq (j, \sigma, S \setminus [|j|, i-1]) \leq (j, \tau, T),
\end{aligned}$$

where the last inequality follows as in the previous case. If $j < 0 < i$ and $|j| > i$ have that $(i, \sigma, S) \leq (-i, \sigma, S) \leq (j, \tau, T)$, where the second inequality follows by the first case above. The case $i > j > 0$ is similar and easier, so we omit it. The length formula follows from Proposition 4.8. \square

We illustrate the previous theorem with an example. Let $m = 4$, $u = [3, 4, -5, -6, 1, 2, 9, -7, -8]$ and $v = [-3, -4, -9, 1, 2, -5, -6, -8, -7]$. Then $u = (4, [2, -3, 1, -4], \{2, 4\})$, and $v = (-2, [-2, 1, -3, -4], \{1, 3\})$, so $\sigma = [2, -3, 1, -4] \leq [-2, 1, -3, -4] = \tau$, $F(\sigma, \tau) = \{4\}$, $j = -2 \leq 4 = i$, and $S(u, v) = \{2, 4\} \cap \{1, 4\} \cap \{4\} = \{4\}$, $T(u, v) = \{1, 3\} \cap \{1, 4\} \cap \{4\} = \emptyset$, so $u \not\leq v$ in $\mathcal{W}(B_9)$.

The following lemma is the analogue of Lemma 3.14 and can be easily deduced from Theorem 4.13 so we omit its verification.

Lemma 4.14. *Let $m > 0$ and $u, v \in \mathcal{W}(B_{2m+1})$. If $u \leq (i, \sigma, S_1) \leq v$ and $u \leq (i, \sigma, S_2) \leq v$ in $(\mathcal{W}(B_{2m+1}), \leq)$ then $u \leq (i, \sigma, S_1 \cup S_2) \leq v$.*

The next result gives an explicit expression for the Möbius function of lower intervals in the poset of signed Wachs permutations partially ordered by Bruhat order, and shows, in particular, that it has values in $\{0, 1, -1\}$. The proof is similar to the one of Proposition 3.15 and we omit it.

Proposition 4.15. *Let $m > 0$, and $v = (j, \tau, T) \in \mathcal{W}(B_{2m+1})$. Then*

$$\mu(e, v) = \begin{cases} (-1)^{|T|}, & \text{if } \tau = e \text{ and } j = m+1; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $v = (\tau, T) \in \mathcal{W}(B_{2m})$ then

$$\mu(e, v) = \begin{cases} (-1)^{|T|}, & \text{if } \tau = e; \\ 0, & \text{otherwise.} \end{cases}$$

By Proposition 4.15 and Theorem 4.10 we deduce the following result.

Corollary 4.16. *The characteristic polynomial of $\mathcal{W}(B_n)$ with the Bruhat order is*

$$(x-1)^{\lfloor \frac{n}{2} \rfloor} x^{n^2 - \lfloor \frac{n}{2} \rfloor^2 - \lfloor \frac{n}{2} \rfloor},$$

for all $n \in \mathbb{P}$.

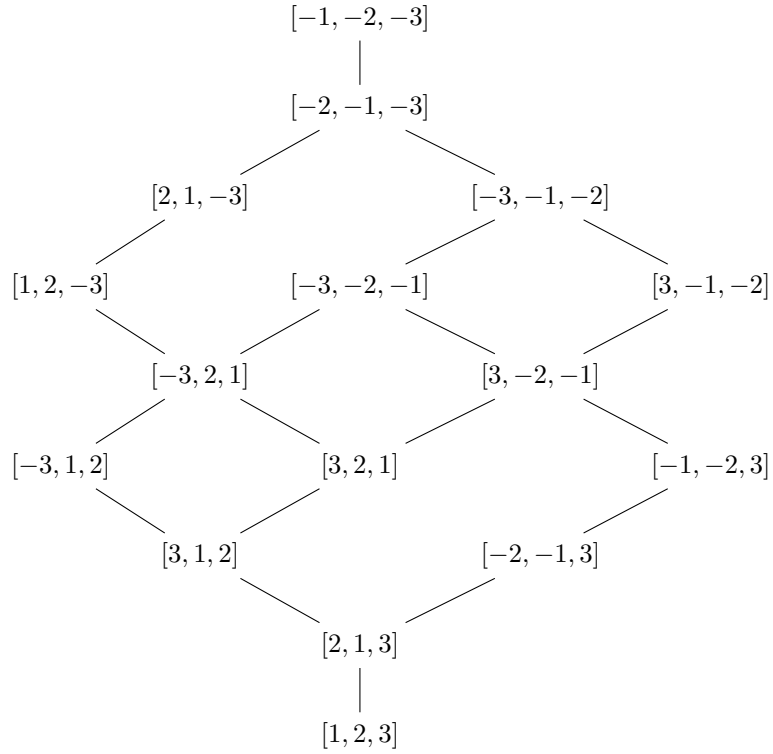


Figure 2: Hasse diagram of $(\mathcal{W}(B_3), \leq)$.

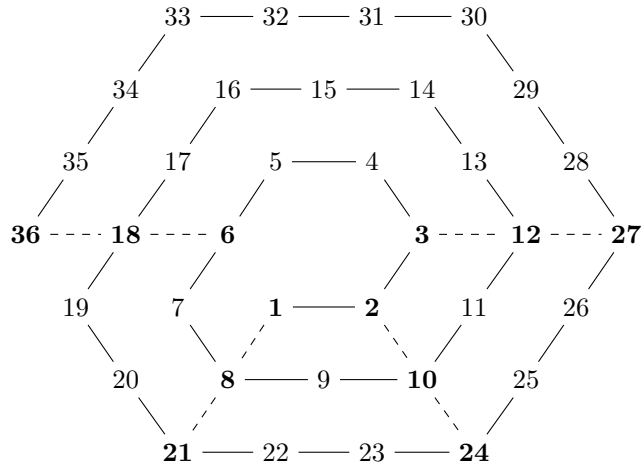


Figure 3: Reading clockwise, the bold numbers on the diagonals of the hexagon are the sequences $\{3, 12, 27, \dots\} = \{\text{rk}(\mathcal{W}(B_{2m}))\}_{m>0}$, $\{2 \text{rk}(\mathcal{W}(S_{2m}))\}_{m>0}$, $\{\text{rk}(\mathcal{W}(B_{2m+1}))\}_{m \geq 0}$ and $\{2 \text{rk}(\mathcal{W}(S_{2m+1}))\}_{m>0}$.

5. Weak orders on Wachs permutations and signed Wachs permutations

In the previous sections we have proved several results concerning the Bruhat order on Wachs permutations and signed Wachs permutations. The Bruhat order of a Coxeter group is a refinement of two fundamental orders, the *left weak order* \leq_L and the right one \leq_R (see, e.g. [2, Chapter 3]), whose Hasse diagrams are isomorphic to the Cayley graph of the group, relative to the considered Coxeter presentation. Then it is natural to ask when our results hold, and in what terms, for the left and right weak orders on Wachs permutations and signed Wachs permutations.

Differently from the Bruhat order, for the right weak order the answer is easily described as a Cartesian product (compare with Propositions 4.3 and 4.8). For reasons that will become clear in the next two pages we begin with signed Wachs permutations. Recall that the set of reflections of B_n is $\{(a, b)(-a, -b) : 1 \leq a < |b| \leq n\} \cup \{(a, -a) : 1 \leq a \leq n\}$. Therefore, if $v \in B_n$, $(a, b)(-a, -b) \in T_L(v)$ if and only if $b > 0$ and b is to the left of a in the complete notation of v , or $b < 0$ and b is to the right of a , while $(a, -a) \in T_L(v)$ if and only if a is to the left of $-a$.

Theorem 5.1. *Let $n > 0$; then $(\mathcal{W}(B_n), \leq_R) \simeq (B_{\lceil \frac{n}{2} \rceil}, \leq_R) \times \mathcal{P}(\lfloor \lfloor \frac{n}{2} \rfloor \rfloor)$. In particular, $(\mathcal{W}(B_n), \leq_R)$ is a complemented lattice.*

Proof. Let n be even; for $v = (\tau, T) \in \mathcal{W}(B_n)$ we have that

$$\begin{aligned} T_L(\tau, T) = & \biguplus_{(a, -a)_B \in T_L(\tau)} \{(2a - 1, -2a + 1)_B, (2a, -2a)_B, (2a - 1, -2a)_B\} \\ & \biguplus_{(a, b)_B \in T_L(\tau): b > 0} \{(2a - 1, 2b - 1)_B, (2a - 1, 2b)_B, (2a, 2b - 1)_B, (2a, 2b)_B\} \\ & \biguplus_{(a, b)_B \in T_L(\tau): b < -a} \{(2a - 1, 2b + 1)_B, (2a - 1, 2b)_B, (2a, 2b + 1)_B, (2a, 2b)_B\} \\ & \uplus \{(v(2a), v(2a - 1))_B : a \in T, \tau(a) > 0\} \\ & \uplus \{(-v(2a - 1), -v(2a))_B : a \in T, \tau(a) < 0\}. \end{aligned}$$

Note that in the first group of reflections the non-symmetric ones (i.e., not of the form $(k, -k)$ for some $k \in [n]$) are never simple and have odd first element, and even and negative second one. The only simple reflections in the second line have even first element and odd positive second one. The ones in the third line are never simple, have negative second element, and if the first one is odd and the second one even then the difference between the absolute values of the two is ≥ 3 (and hence are disjoint from the non-symmetric ones in the first line). The reflections in the last two lines are always simple, and have odd first element and even and positive second one, since $(v(2a), v(2a - 1))_B = (2\tau(a) - 1, 2\tau(a))_B$ if $\tau(a) > 0$ while $(-v(2a - 1), -v(2a))_B = (-2\tau(a) - 1, -2\tau(a))_B$ if $\tau(a) < 0$.

Therefore, if $u = (\sigma, S) \in \mathcal{W}(B_n)$ and $v = (\tau, T) \in \mathcal{W}(B_n)$, we have that $T_L(u) \subseteq T_L(v)$ if and only if $T_L(\sigma) \subseteq T_L(\tau)$ and $\sigma_S \subseteq \tau_T$, where, for $X \subseteq [n/2]$ and $w \in B_{n/2}$, $w_X := \{w(i) : i \in X\}$. Indeed, if $T_L(u) \subseteq T_L(v)$ and $(a, b)_B \in T_L(\sigma)$ is such that (say) $b < -a$ then, by the equation written at the beginning of this proof, $(2a, 2b)_B \in T_L(u)$ so $(2a, 2b)_B \in T_L(v)$ which implies, by the remarks above, that there is $(c, d)_B \in T_L(\tau)$ such that $c < -d$ and $(2a, 2b)_B = (2c, 2d)_B$,

so $(a, b)_B = (c, d)_B \in T_L(\tau)$. Similarly all the other cases. The converse is clear. Hence, by Proposition 2.3, the map $(\tau, T) \mapsto (\tau, \tau_T)$ gives the desired poset isomorphism.

Let n be odd and $v = (j, \tau, T) \in \mathcal{W}(B_n)$. Then similarly

$$\begin{aligned}
T_L(j, \tau, T) = & \biguplus_{(a, -a)_B \in T_L(\bar{\tau}) : a \neq \lceil n/2 \rceil} \{(2a - 1, -2a + 1)_B, (2a, -2a)_B, (2a - 1, -2a)_B\} \\
& \biguplus_{(a, b)_B \in T_L(\bar{\tau}) : 0 < b < \lceil n/2 \rceil} \{(2a - 1, 2b - 1)_B, (2a - 1, 2b)_B, (2a, 2b - 1)_B, (2a, 2b)_B\} \\
& \biguplus_{(a, b)_B \in T_L(\bar{\tau}) : -\lceil n/2 \rceil < b < -a} \{(2a - 1, 2b + 1)_B, (2a - 1, 2b)_B, (2a, 2b + 1)_B, (2a, 2b)_B\} \\
& \biguplus_{(a, b)_B \in T_L(\bar{\tau}) : b = \lceil n/2 \rceil} \{(2a - 1, n)_B, (2a, n)_B\} \\
& \biguplus_{(a, b)_B \in T_L(\bar{\tau}) : b = -\lceil n/2 \rceil, a \neq -b} \{(2a - 1, -n)_B, (2a, -n)_B\} \\
& \uplus \{(n, -n)_B : (\lceil n/2 \rceil, -\lceil n/2 \rceil)_B \in T_L(\bar{\tau})\} \\
& \uplus \{(2\tau(a) - 1, 2\tau(a))_B : a \in T, \tau(a) > 0\} \\
& \uplus \{(2\tau(a) - 1, 2\tau(a))_B : -a \in T, \tau(a) > 0\}
\end{aligned}$$

where

$$\bar{\tau}(k) := \begin{cases} \tau(k), & \text{if } k < |j|; \\ \text{sgn}(j) \cdot (n + 1)/2, & \text{if } k = |j|; \\ \tau(k - 1), & \text{if } k > |j|, \end{cases}$$

for all $k \in \lceil n/2 \rceil$. The considerations made in the even case apply line by line also to this case, with the added remark that the reflections in lines 4, 5, and 6 are the only ones with n or $-n$ in the second position. Therefore, if $u = (i, \sigma, S) \in \mathcal{W}(B_n)$ and $v = (j, \tau, T) \in \mathcal{W}(B_n)$, we have that $T_L(u) \subseteq T_L(v)$ if and only if $T_L(\bar{\sigma}) \subseteq T_L(\bar{\tau})$ and $\sigma_S \subseteq \tau_S$, and we conclude as in the previous case. The last statement follows from the fact that $(B_{\lceil \frac{n}{2} \rceil}, \leq_R)$ is a complemented lattice (see, e.g., [2, Cor. 3.2.2]). \square

Regarding left weak order we have the following observations.

Proposition 5.2. *Let $m > 0$. The posets $(\mathcal{W}(B_{2m}), \leq_R)$ and $(\mathcal{W}(B_{2m}), \leq_L)$ are isomorphic.*

Proof. The assignment $v \mapsto v^{-1}$ defines a bijective function $\mathcal{W}(B_{2m}) \rightarrow \mathcal{W}(B_{2m})$ and $u \leq_R v$ if and only if $u^{-1} \leq_L v^{-1}$, for all $u, v \in B_{2m}$. \square

On the other hand, the poset $(\mathcal{W}(B_3), \leq_L)$ is not graded.

For Wachs permutations the situation is analogous but simpler so we leave to the interested reader the proof of the following result.

Theorem 5.3. *Let $n > 0$; then $(\mathcal{W}(S_n), \leq_R) \simeq (S_{\lceil \frac{n}{2} \rceil}, \leq_R) \times \mathcal{P}(\lceil \lceil \frac{n}{2} \rceil \rceil)$. In particular, $(\mathcal{W}(S_n), \leq_R)$ is a complemented lattice.*

As in the case of signed Wachs permutations, the posets $(\mathcal{W}(S_{2m}), \leq_R)$ and $(\mathcal{W}(S_{2m}), \leq_L)$ are isomorphic, for all $m > 0$. On the other hand, the poset $(\mathcal{W}(S_5), \leq_L)$ is not graded.

6. Open problems

In this section we collect some open problems and conjectures which arise in this work, and the evidence that we have in their favor.

We have proved in Propositions 3.15 and 4.15 that the Möbius function of lower intervals in the posets of Wachs permutations and signed Wachs permutations partially ordered by Bruhat order always has values in $\{0, 1, -1\}$. We feel that this is true in general.

Conjecture 6.1. *Let $n \in \mathbb{P}$. Then*

$$\mu(u, v) \in \{0, 1, -1\}$$

for all $u, v \in \mathcal{W}(S_n)$.

We have verified Conjecture 6.1 for $n \leq 8$.

Conjecture 6.2. *Let $n \in \mathbb{P}$. Then*

$$\mu(u, v) \in \{0, 1, -1\}$$

for all $u, v \in \mathcal{W}(B_n)$.

We have verified Conjecture 6.2 for $n \leq 6$. Note that since $\mathcal{W}(S_n)$ is isomorphic, as a poset, to the interval $[e, [n, \dots, 3, 2, 1]]$ in $\mathcal{W}(B_n)$, Conjecture 6.2 implies Conjecture 6.1.

Recently Davis and Sagan [6] studied the convex hull of various sets of pattern avoiding permutations. Following this idea, it is natural to look at the convex hulls $c(\mathcal{W}(S_n))$ and $c(\mathcal{W}(B_n))$ of Wachs and signed Wachs permutations in \mathbb{R}^n . In this respect, we feel that the following is true.

Conjecture 6.3. *Let $m \in \mathbb{P}$. Then $c(\mathcal{W}(S_{2m}))$ is a simple polytope.*

We have verified Conjecture 6.3 for $m \leq 5$. According to SageMath [21] $c(\mathcal{W}(S_9))$ is not simple.

Conjecture 6.4. *Let $m \in \mathbb{P}$. Then $c(\mathcal{W}(B_{2m}))$ is a simple polytope.*

We have verified Conjecture 6.4 for $m \leq 3$. According to SageMath [21] $c(\mathcal{W}(B_3))$ is not simple.

Regarding left weak order, we feel that the following might be true.

Problem 1. *Is $(\mathcal{W}(S_{2m+1}), \leq_L)$ a lattice for all $m \in \mathbb{P}$?*

We have verified that the answer to Problem 1 is yes if $m \leq 4$.

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