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CONTINUUM LIMIT OF RANDOM MATRIX PRODUCTS IN STATISTICAL MECHANICS OF DISORDERED SYSTEMS

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Abstract. We consider a particular weak disorder limit (*continuum limit*) of matrix products that arise in the analysis of disordered statistical mechanics systems, with a particular focus on random transfer matrices. The limit system is a diffusion model for which the leading Lyapunov exponent can be expressed explicitly in terms of modified Bessel functions, a formula that appears in the physical literature on these disordered systems. We provide an analysis of the diffusion system as well as of the link with the matrix products. We then apply the results to the framework considered by Derrida and Hilhorst in [12], which deals in particular with the strong interaction limit for disordered Ising model in one dimension and that identifies a singular behavior of the Lyapunov exponent (of the transfer matrix), and to the two dimensional Ising model with columnar disorder (McCoy-Wu model). We show that the continuum limit sharply captures the Derrida and Hilhorst singularity. Moreover we revisit the analysis by McCoy and Wu [31] and remark that it can be interpreted in terms of the continuum limit approximation. We provide a mathematical analysis of the continuum approximation of the free energy of the McCoy-Wu model, clarifying the prediction (by McCoy and Wu) that, in this approximation, the free energy of the two dimensional Ising model with columnar disorder is C^∞ but not analytic at the critical temperature.

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1. Introduction

Products of random matrices can often be interpreted, in a statistical mechanics perspective, as models of disordered systems. The leading Lyapunov exponent may then be identified with some physical quantity such as the free energy density or persistence length. We can also take the opposite viewpoint and ask whether a disordered system can be written in terms of, or at least approximated by, a suitable product of random matrices. It turns out that there are several examples in which this can be done. Examples include essentially all statistical mechanics systems in which there is a natural one dimensional structure, but it goes also beyond this: the literature is too wide to be properly cited here and we refer to the reviews [7, 11]. Of particular interest for us are the examples arising from the transfer matrix approach in the statistical mechanics of disordered systems. For one dimensional (let us say, Ising or Potts) models with finite range interaction one can write the partition function in terms of a product of matrices [2]: if the interactions are only one body and nearest neighbor two body the transfer matrix of an Ising model is a two by two matrix, and the size is larger for Potts and/or longer range models. But

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for two or more dimensional systems the size of the transfer matrix tends to infinity in the thermodynamic limit and the transfer matrix should be thought more as a transfer operator: this is true also in one dimension if the spin variable can take an infinite number of values [38, Ch. 5]. Nevertheless, also in these cases finite dimensional matrix models can be helpful (for numerical approximations for example, but also for rigorous bounds, see for example [21, Ch. 9] and references therein). It is however remarkable that also the solution of the two dimensional Ising model with nearest neighbor interactions and no external field can ultimately be expressed in terms of products of two by two matrices: this is the essence of several formulations of the celebrated solution of Lars Onsager [2, 30]. What is even more remarkable from our viewpoint is that this structure still holds when special types of disorder are introduced, giving a product of *random* matrices [30, 31, 41].

The two by two matrices that arise in the problems we have just mentioned have a particular form: it is

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z & Z \end{pmatrix}, \tag{1.1}$$

where ε is a real number – say $|\varepsilon| \leqslant 1/2$ to keep far from the zero determinant case $\varepsilon = \pm 1$ – and Z is a positive random variable with $\mathbb{E} \log_+(Z) < \infty$. Let us call (informally for the moment) $\widehat{\mathcal{L}}_Z(\varepsilon)$ the Lyapunov exponent of a product of IID matrices of the form (1.1), which appear notably in the following two contexts.

- In the one dimensional Ising model with random external field $h = h_j$ – that is, $\{h_j\}_{j=1,2,\dots}$ is a sequence of independent identically distributed (IID) random variables – and nearest neighbor interaction J, the transfer matrix can be cast in the form (1.1), with $Z = \exp(-2h)$ and $\varepsilon = \exp(-2J)$. The free energy density is therefore precisely $\widehat{\mathcal{L}}_Z(\varepsilon)$ and the $\varepsilon \searrow 0$ limit is the limit of strong ferromagnetic interaction.
- In a much less straightforward way (detailed in Appendix A), also the free energy of the two dimensional Ising model with a special type of disordered nearest neighbor interactions (*columnar disorder*), and no external field, is (essentially) just $\int_0^{1/2} \widehat{\mathcal{L}}_Z(\varepsilon)\,\mathrm{d}\varepsilon$, of course with a proper choice of $Z = Z_\beta$ that contains the inverse temperature β of the system. The phase diagram of this model (that is, the presence and nature of phase transitions) is determined by the regularity of this expression as a function of β; the most notable prediction for this model, which goes now under the name of McCoy-Wu model, is that the second order transition of the two dimensional non-disordered Ising model (for which the second derivative of the free energy diverges at criticality like $-\log|\beta - \beta_c|$) becomes of infinite order when the columnar disorder is introduced: that is, the free energy at the critical point is C^∞ but not analytic. The precise nature of the singularity is characterized in [31] by means of a divergent power series for the free energy at β_c, where the value of β_c depends on the disorder: a summary of the expected effect of disorder on the transition for the two dimensional Ising model is in [22, § 5.3]. The McCoy-Wu model has a prominent role in physics because it can be mapped to the one dimensional quantum spin chain with transversal magnetic field [16] and because it has played a central role in the development of the *real space strong/infinite disorder renormalization group* (see e.g. [16] and [22, § 5.3]).

Other contexts in which (1.1) and $\widehat{\mathcal{L}}_Z(\varepsilon)$ arise include one dimensional random walk in random environment and a number of random hopping problems (see [11] and references therein), and the key issue for us is that all this vast literature focuses on the $\varepsilon \to 0$ behavior of $\widehat{\mathcal{L}}_Z(\varepsilon)$, see notably [11, 12, 30, 31, 33]. From a mathematical viewpoint this limit is of interest because, thanks to [37] (see also [13]), we know that $\varepsilon \mapsto \widehat{\mathcal{L}}_Z(\varepsilon)$ is real analytic if $|\varepsilon| \in (0,1)$ under additional mild hypotheses on Z (for example: $\mathbb{P}(Z = c) = 0$ for every c). But the regularity at $\varepsilon = 0$ is not obvious, as well as if there is a singularity at all. And this is precisely the question addressed in [11, 12, 30, 31, 33]. In particular $\widehat{\mathcal{L}}_Z(\varepsilon)$ is expected to have a fractional or logarithmic scaling when $\varepsilon \to 0$ under the *frustration hypothesis* that $\mathbb{P}(Z > 1)$ and $\mathbb{P}(Z < 1)$ are both positive: this is the case for example of the $\varepsilon^{2\alpha}$ *singularity* found in [12], and proven mathematically in [20], and that we will explain in detail in Section 1.4.

Here we do not address the study of the Lyapunov exponent of products of matrices of the form (1.1). Rather we focus on a continuous time model that arises as a diffusive limit of the matrix product (we call it *continuum limit*). Roughly, the limit is achieved by considering matrices close to the identity: ε is replaced by $\varepsilon\Delta$, with $\Delta \searrow 0$ and we consider $Z = Z^{\Delta}$ that is very concentrated around one: both $\mathbb{E}[Z^{\Delta}]-1$ and $\mathrm{var}(Z^{\Delta})$ are of order Δ. The dynamics will therefore happen on a timescale $1/\Delta$ and it will be governed by a two dimensional stochastic differential system. We then study the leading Lyapunov exponent $\mathcal{L}(\varepsilon)$ of this limit system: we will actually show that $\widehat{\mathcal{L}}_Z(\varepsilon) \sim \Delta\mathcal{L}(\varepsilon)$ for $\Delta \searrow 0$ (we use $\sim$ for asymptotic equivalence: the ratio of left-hand and right-hand sides converges to one). This limit has been already considered in several works and even in greater generality (*matrices close to the identity*: see e.g.[19, 31, 44, 9]), but mathematically rigorous results are lacking (with the exception of [39], whose assumptions however exclude the case we treat): the type of results one finds are expansions of the type

$$\widehat{\mathcal{L}}_{Z^{\Delta}}(\varepsilon) \,=\, c_1(\varepsilon)\Delta + c_2(\varepsilon)\Delta^2 + \dots\,, \tag{1.2}$$

where of course $c_1(\varepsilon) = \mathcal{L}(\varepsilon)$ and expressions or at least procedures to compute the $c_j(\varepsilon)$ are given. To be precise, a full expansion like (1.2) is not expected to hold in general and, even in the cases in which it holds, e.g. [39], and assuming smooth dependence in Δ of the coefficients of the matrix, there is to our knowledge no proof that $\Delta \mapsto \widehat{\mathcal{L}}_{Z^{\Delta}}(\varepsilon) \in C^{\infty}$. We point out, however, a very special example in [44] that has been worked out explicitly and for which the Lyapunov exponent is analytic except at zero where it is nonetheless C^{∞} (note also that [37] cannot be applied because for $\Delta = 0$ the matrix is the identity matrix).

We will focus only on $\mathcal{L}(\varepsilon)$: in other words, the continuum limit we consider captures only the leading order term in (1.2). The first remarkable fact is that $\mathcal{L}(\varepsilon)$ has an explicit expression in terms of a ratio of modified Bessel functions: we provide a proof of this fact, which has long been known in the physics literature. To our knowledge, it is found for the first time in [31, (4.31)], and it then reappears in other works and contexts, see for example [9] to which we refer also for a comprehensive review of the literature. It is rather surprising that, while a detailed analysis of the $\varepsilon \to 0$ limit of $\mathcal{L}(\varepsilon)$ is rather straightforward (the case of $\alpha \in [0,2)$ is worked out in [23, first formula on p. 248]), a full analysis appears to be lacking, as well as an emphasis on the rather striking fact that the $\varepsilon \to 0$ behavior of $\mathcal{L}(\varepsilon)$ *captures all known and conjectured features* of the $\varepsilon \to 0$ behavior of $\widehat{\mathcal{L}}_{Z^{\Delta}}(\varepsilon)$, i.e. for matrix products (without assuming the disorder to be small). In particular, the $\varepsilon^{2\alpha}$ singularity found in [12] is fully present in the continuum limit expression: mathematical results on this issue for $\widehat{\mathcal{L}}_{Z^{\Delta}}(\varepsilon)$ have been recently obtained [20, 24], but the control of the singular term is an open problem for $|\alpha| \geqslant 1$.

Turning to the McCoy-Wu model, we come back to the fact that this model appears prominently in the physical literature, in part of course because of its exactly solvable character. And the conventional wisdom in the mathematical community appears to be that the McCoy-Wu claims are exact. And this is correct as far as the free energy formula (in terms of the Lyapunov exponents) is concerned. The subsequent analysis is less sound: β_c is identified via the equation $\mathbb{E}\log Z = 0$ – the random variable Z depends on the inverse temperature β – and this assertion has some grounds at least at a heuristic level, but then one has to show that the free energy $\int_0^{1/2} \widehat{\mathcal{L}}_{Z^\Delta}(\varepsilon)\,\mathrm{d}\varepsilon$ is not analytic at $\beta = \beta_c$. And this is (ultimately) done by replacing $\widehat{\mathcal{L}}_{Z^\Delta}(\varepsilon)$ with $\Delta\mathcal{L}(\varepsilon)$ and this step is very weak on mathematical grounds because the McCoy-Wu claim (which provides the motivation for the whole exercise) is for $\Delta > 0$ (possibly very small, but non zero): making this step rigorous – possibly by controlling the remainder of the series in (1.2) – appears to be very challenging, and we do not address this in the present work. Once this approximation is done, McCoy and Wu are left with studying the regularity in β of $\int_0^{1/2} \mathcal{L}(\varepsilon)\,\mathrm{d}\varepsilon$. In spite of being a relatively explicit expression, this is still challenging. McCoy and Wu do this by developing the ratio of Bessel functions in the expression for $\mathcal{L}(\varepsilon)$ for β close to β_c – the dependence in β is in the index of the Bessel functions – and by identifying the leading-order (in magnitude) terms in an expansion of the Bessel functions as the most singular part. We provide a proof that $\int_0^{1/2} \mathcal{L}(\varepsilon)\,\mathrm{d}\varepsilon$ is C^∞ but not analytic at $\beta = \beta_c$ and that the asymptotic series at β_c is qualitatively the one found by McCoy and Wu (up to a multiplicative factor that they lost when singling out the most singular term; a similar correction was noted by Luck [28] in a related model): technically, this is the most demanding part of our contribution. We stress however that what we prove does not yield results on the transition for the McCoy-Wu model. The challenging gap pointed out just above remains unclosed. But we believe that our contribution helps understanding the true content of the remarkable McCoy-Wu analysis. As a side remark: the computation of McCoy and Wu is done for a very special form of the disorder distribution while they affirm that they expect the result to be true in great generality. We work under very general assumptions on the disorder, thus substantiating this claim.

We begin by presenting the diffusion system and its analysis. This is a stochastic dynamical system that is interesting in its own right and we provide a detailed analysis that goes beyond the strict purpose of what has been explained up to now. In particular we prove a Central Limit Theorem on the fluctuations of the Lyapunov exponent for the diffusive limit system, with an explicit formula for the variance and an analysis of the singular behavior. We then provide a proof that the Markov chain associated to the matrix product described above does scale to the diffusion system and that to leading order (in Δ) the Lyapunov exponent of the Markov chain is asymptotically proportional to the Lyapunov exponent of the diffusion system, that is $\widehat{\mathcal{L}}_Z(\varepsilon) \sim \Delta\mathcal{L}(\varepsilon)$. The rest of our work focuses on the regularity/singularity properties of $\mathcal{L}(\varepsilon)$ and of the expressions related to it that are of physical relevance.

1.1. **The diffusion model and its leading Lyapunov exponent.** We consider the solution to the stochastic (Itô) differential equations

$$\begin{cases} \mathrm{d}X_1(t) \;=\; \varepsilon X_2(t)\,\mathrm{d}t\,, \\ \mathrm{d}X_2(t) \;=\; \left(\varepsilon X_1(t) + \frac{(1-\alpha)\sigma^2}{2} X_2(t)\right)\,\mathrm{d}t + \sigma X_2(t)\,\mathrm{d}B_t\,, \end{cases} \tag{1.3}$$

where $B_\cdot$ is a standard Brownian motion, $\varepsilon \neq 0$, $\alpha \in \mathbb{R}$ and $\sigma > 0$. We consider deterministic initial condition $(X_1(0), X_2(0)) \in \mathbb{R}^2 \backslash \{(0,0)\}$. The case $(X_1(0), X_2(0)) = (0,0)$, as well as $\varepsilon = 0$, are excluded because they are atypical and trivial. The system (1.3) is linear with a multiplicative noise so, given the initial condition, there exists a unique strong solution. Our focus is on the Lyapunov exponent $\mathcal{L}(\varepsilon) = \mathcal{L}_{\sigma,\alpha}(\varepsilon)$ that we introduce via our first statement in which we use the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^2$ just for definiteness. Before stating it we need to recall one of the definitions of the modified Bessel function of 2^{nd} kind of index $\alpha \in \mathbb{C}$ and argument $x > 0$ [32, §10.25]

$$K_\alpha(x) := \int_0^\infty \exp\left(-x\cosh(t)\right)\cosh(\alpha t)\, \mathrm{d}t = \frac{1}{2}\int_0^\infty \frac{1}{y^{1+\alpha}} \exp\left(-\frac{x}{2}\left(y + \frac{1}{y}\right)\right) \mathrm{d}y\,. \quad (1.4)$$

We note from now that $K_\alpha(x) = K_{-\alpha}(x)$.

Theorem 1.1. *For every $\varepsilon \neq 0$ and every $(X_1(0), X_2(0)) \in \mathbb{R}\backslash\{(0,0)\}$ the limit*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\log\|(X_1(t), X_2(t))\| =: \mathcal{L}_{\sigma,\alpha}(\varepsilon)\,, \quad (1.5)$$

exists and does not depend on $(X_1(0), X_2(0))$. Moreover

1. *the limit is unchanged if we replace $\|(X_1(t), X_2(t))\|$ with $|X_j|$, $j = 1, 2$ as well as if we remove the expectation (in this case the convergence is almost sure);*
2. *if $\varepsilon > 0$ [resp. $\varepsilon < 0$], then $\mathrm{sign}(X_1(t)) = \mathrm{sign}(X_2(t))$ [resp. $\mathrm{sign}(X_1(t)) \neq \mathrm{sign}(X_2(t))$] for all $t \geqslant \tau := \inf\{t \geqslant 0 : \mathrm{sign}(X_1(t)) = \mathrm{sign}(X_2(t))\}$ [resp. $\tau := \inf\{t \geqslant 0 : \mathrm{sign}(X_1(t)) \neq \mathrm{sign}(X_2(t))\}$] and $\mathbb{E}[\tau] < \infty$. Moreover $\mathcal{L}_{\sigma,\alpha}(\varepsilon) = \mathcal{L}_{\sigma,\alpha}(-\varepsilon)$.*
3. *For $\varepsilon > 0$ and every $\alpha \in \mathbb{R}$ we have*

$$\mathcal{L}_{\sigma,\alpha}(\varepsilon) = \frac{\sigma^2}{4}\left(\frac{xK_{\alpha-1}(x)}{K_\alpha(x)}\right), \qquad \text{with} \quad x := \frac{4\varepsilon}{\sigma^2}\,. \quad (1.6)$$

We draw the attention of the reader on the identification of x with $4\varepsilon/\sigma^2$. This shortcut notation is kept in all statements and proofs.

Some of the results in Theorem 1.1 can be understood on the basis of a symmetry enjoyed by our system: If $(X_1(\cdot), X_2(\cdot))$ solves (1.3), then $(-X_1(\cdot), X_2(\cdot))$ solves (1.3) with ε replaced by $-\varepsilon$. We can therefore restrict our analysis to the case $\varepsilon > 0$ and we will show that only the (interior of the) quadrants in which both coordinates have the same sign – first and third quadrant – are recurrent for the dynamics: all the rest is transient. By linearity we can then restrict to the first quadrant.

As already mentioned in the introduction, most of the content of Theorem 1.1 is known in the physical literature and (1.6) appears in number of contexts. Besides the pioneering work [31] that we have already mentioned, (1.6) appears for example also in [28, Sec. 3], which deals with disordered quantum Ising chains with transverse magnetic field: this is not surprising, because this quantum model is mapped exactly (by a Suzuki-Trotter path integral) into a suitable limit of the McCoy-Wu model (this is also exploited [16]). It also appears in the analysis of one dimensional random Schrödinger equation and in the analysis of a diffusion of a particle in a random force field, see e.g. [4, 10], and [19] which is possibly the first work addressing precisely what we refer to as the continuum limit. Mathematical works have also been done in this context and (1.6) appears in a study of the quenched large deviations of diffusions in a random environment [43, Prop.

2.1]: By Kotani's formula (unpublished, 1988) the Laplace transform of hitting times for the diffusion can be expressed in terms of a Riccati equation – the $K_\alpha(\cdot)$ Bessel function appears as solution of this equation – that is equivalent to (2.3) below.

Finally, the first item of Theorem 1.1 is a classical result at the random matrix level, and the second item is an elementary observation. In the continuum set-up, the first two items follow by applying standard tools of stochastic analysis (the proofs turn out to be rather concise and we give full details). The third item is a computation: it is not novel, but it is very short and we provide it for completeness.

We also have a rather explicit representation for the fluctuations:

Proposition 1.2. *The family of random variables*

$$\left\{ \frac{1}{\sqrt{t}} \big(\log \|(X_1(t), X_2(t))\| - t\mathcal{L}_{\sigma,\alpha}(\varepsilon) \big) \right\}_{t\in[0,\infty)} \tag{1.7}$$

converges in law for $t \to \infty$ to a centered Gaussian variable with variance $v_{\sigma,\alpha}(\varepsilon) \in (0,\infty)$,

$$v_{\sigma,\alpha}(\varepsilon) = \frac{2}{\sigma^2 K_\alpha(x)} \int_0^\infty \frac{1}{y^{1-\alpha}} e^{\frac{x}{2}\left(y+\frac{1}{y}\right)} \left(\int_0^y \frac{\varepsilon z - \mathcal{L}_{\sigma,\alpha}(\varepsilon)}{z^{1+\alpha}} e^{-\frac{x}{2}\left(z+\frac{1}{z}\right)} \,\mathrm{d}z \right)^2 \mathrm{d}y\,. \tag{1.8}$$

In physics the behavior of fluctuations for matrix products has beed repeatedly addressed, see [35, 40] and references therein; the same is true for the mathematical literature [3]. Proposition 1.2 is about the fluctuations for the continuum limit: even taking into account the results in [35, 40], we think that understanding how random matrix products fluctuations and continuum limit fluctuations are related is an open issue.

1.2. **Small ε asymptotic expansions.** Thanks to Theorem 1.1 item (3), studying the small ε behavior of $\mathcal{L}_{\sigma,\alpha}(\varepsilon)$ is just a book-keeping exercise that exploits the asymptotic behavior of $K_\alpha(\cdot)$. For $\alpha \in [0,2)$ this result can be found in [23, first formula, p. 248], see also [28, (3.45)] for $\alpha = 0$: we provide the general result and we will explain the relevance of this exercise in Section 1.4. Throughout this work Γ denotes the Gamma function, see [32, §5.2] for definitions and properties.

Proposition 1.3. *Recall that $x = 4\varepsilon/\sigma^2$. For $\alpha \in (0,\infty)\backslash\mathbb{Z}$ we have for $\varepsilon \searrow 0$*

$$\frac{4}{\sigma^2}\mathcal{L}_{\sigma,\alpha}(\varepsilon) \;=\; c_1(\alpha)x^2 + \ldots + c_{\lfloor\alpha\rfloor}(\alpha)x^{2\lfloor\alpha\rfloor} + 2\frac{\Gamma(1-\alpha)}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^{2\alpha} + O\left(x^{\min(2\lceil\alpha\rceil,4\alpha)}\right), \tag{1.9}$$

where $c_j(\cdot)$ is a rational function (for explicit expressions, see (4.4)-(4.7)*).*

For $\alpha \in \{1,2,\ldots\}$ we have

$$\frac{4}{\sigma^2}\mathcal{L}_{\sigma,\alpha}(\varepsilon) \;=\; c_1(\alpha)x^2 + \ldots + c_{\alpha-1}(\alpha)x^{2(\alpha-1)} + (-1)^\alpha \frac{2^{2-2\alpha}}{((\alpha-1)!)^2} x^{2\alpha}\log x + O\left(x^{2\alpha}\right), \tag{1.10}$$

where $c_j(\cdot)$ is the same rational function as in the non integer case.

For $\alpha = 0$ we have

$$\mathcal{L}_{\sigma,\alpha}(\varepsilon) \;=\; \frac{\sigma^2}{4\log(1/x)} + O\left((\log 1/x)^{-2}\right), \tag{1.11}$$

and the result for $\alpha < 0$ is directly recovered from (1.9)-(1.10) *by using the identity*

$$\frac{4}{\sigma^2}\mathcal{L}_{\sigma,\alpha}(\varepsilon) \overset{\alpha\le 0}{=} 2|\alpha| + \frac{4}{\sigma^2}\mathcal{L}_{\sigma,|\alpha|}(\varepsilon)\,. \tag{1.12}$$

The identity (1.12) is a simple consequence of the Bessel identity

$$xK_{1+\alpha}(x) \;=\; 2\alpha K_\alpha(x) + xK_{-1+\alpha}(x)\,, \tag{1.13}$$

that follows from (1.4) by integration by parts, together with the identity $K_\alpha(x) = K_{-\alpha}(x)$.

We have chosen to give these expansions up to the leading singular term: one can of course be much more precise. Keeping only the leading term, Proposition 1.3 implies

$$\mathcal{L}_{\sigma,\alpha}(\varepsilon) \overset{\varepsilon\searrow 0}{\sim} \left(\frac{\sigma^2}{4}\right) \begin{cases} c_1(\alpha)x^2 & \text{if } \alpha > 1\,, \\ \frac{2\Gamma(1-\alpha)}{\Gamma(\alpha)}\left(\frac{x}{2}\right)^{2\alpha} & \text{if } \alpha \in (0,1)\,, \\ 1/\log(1/x) & \text{if } \alpha = 0\,, \\ 2|\alpha| & \text{if } \alpha \in (-\infty, 0)\,; \end{cases} \tag{1.14}$$

recall again that $x = 4\varepsilon/\sigma^2$. It is certainly worth observing for the benefit of those readers who are less at home with special function calculations that (1.14) can be derived by elementary asymptotic methods from (1.4). This is of course also the case for the full Proposition 1.3, but the exercise becomes particularly involved and resorting to the special functions literature is certainly wise, or even necessary.

Remark 1.4. (1.14) *directly entails* $\lim_{\varepsilon\searrow 0}\mathcal{L}_{\sigma,\alpha}(\varepsilon) = \frac{\sigma^2}{2}|\alpha|\mathbf{1}_{\alpha<0}$, *and* $\alpha \mapsto \frac{\sigma^2}{2}|\alpha|\mathbf{1}_{\alpha<0}$ *is singular at the origin, while* $\alpha \mapsto \mathcal{L}_{\sigma,\alpha}(\varepsilon)$ *is real analytic for* $\varepsilon \neq 0$ *(and it is meromorphic in the whole* $\mathbb{C}$*: see beginning of Section 5.2). It is possibly worth observing that we have not defined* $\mathcal{L}_{\sigma,\alpha}(\varepsilon)$ *for* $\varepsilon = 0$ *because of the pathological nature of this case, but the limit in* (1.5) *exists also for* $\varepsilon = 0$ *and* $\mathcal{L}_{\sigma,\alpha}(0) = \frac{\sigma^2}{2}|\alpha|\mathbf{1}_{\alpha<0}$ *(in agreement with* $\lim_{\varepsilon\searrow 0}\mathcal{L}_{\sigma,\alpha}(\varepsilon)$*), but only if* $X(0) \neq 0$*; otherwise the Lyapunov exponent is* $-\infty$*. Moreover the (Laplace) asymptotic behavior of the two components for* $\varepsilon = 0$ *in general does not coincide with the Lyapunov exponent.*

Of course one could wonder about the behavior as $\varepsilon \searrow 0$ of the variance $v_{\sigma,\alpha}(\varepsilon)$ in Proposition 1.2.

Proposition 1.5. *Still with the notation* $x = 4\varepsilon/\sigma^2$*, we have that for every* α *there exists* $C(\alpha) > 0$ *(see* (3.35) *for an explicit expression) such that*

$$v_{\sigma,\alpha}(\varepsilon) \overset{\varepsilon\searrow 0}{\sim} C(\alpha)\frac{\sigma^2}{2} \times \begin{cases} 1 & \text{if } \alpha \leqslant 0\,, \\ x^{2\alpha} & \text{if } \alpha \in (0,2)\,, \\ x^4\log(1/x) & \text{if } \alpha = 2\,, \\ x^4 & \text{if } \alpha > 2\,. \end{cases} \tag{1.15}$$

1.3. **From matrix product to the diffusion model.** We are now going provide rigorous results about how (1.3) emerges as limit of matrix products. Consider, for $\Delta > 0$ and given an IID sequence $\{\mathcal{N}_n\}_{n=1,2,\ldots}$ of standard Gaussian variables, the discrete time stochastic process $\{(X_1^\Delta(n), X_2^\Delta(n))\}_{n=0,1,\ldots}$ defined recursively from the deterministic initial condition $(X_1^\Delta(0), X_2^\Delta(0)) = (X_1(0), X_2(0))$ by

$$\begin{cases} X_1^\Delta(n+1) \;=\; X_1^\Delta(n) + \varepsilon X_2^\Delta(n)\Delta\,, \\ X_2^\Delta(n+1) \;=\; e^{\sigma\sqrt{\Delta}\mathcal{N}_{n+1}-\alpha\frac{\sigma^2}{2}\Delta}\left(X_2^\Delta(n) + \varepsilon X_1^\Delta(n)\Delta\right)\,. \end{cases} \tag{1.16}$$

Defining

$$Z^\Delta(n+1) = e^{\sigma\sqrt{\Delta}\mathcal{N}_{n+1}-\alpha\frac{\sigma^2}{2}\Delta}, \tag{1.17}$$

we can write (1.16) as

$$X^{\Delta}(n+1) = X^{\Delta}(n) + A^{\Delta}(n+1)X^{\Delta}(n), \tag{1.18}$$

where

$$X^{\Delta} = \begin{pmatrix} X_1^{\Delta} \\ X_2^{\Delta} \end{pmatrix}, \qquad A^{\Delta}(n) = \begin{pmatrix} 0 & \varepsilon\Delta \\ \varepsilon\Delta Z^{\Delta}(n) & Z^{\Delta}(n)-1 \end{pmatrix}. \tag{1.19}$$

In different terms: $X^{\Delta}(n)$ results form the product of n independent matrices of the form $I + A^{\Delta}$ and for $\Delta = 1$ we have that the matrix $I + A^1$ coincides with (1.1) when $Z = Z^1$, that is when Z is log-normal. The restriction to log-normal is just for ease of exposition: we are going to prove a result (Theorem 6.1) for much more general distributions.

Note that the determinant of $I + A^{\Delta}$ is $Z^{\Delta}(1 - \varepsilon^2\Delta^2)$ and we want to exclude the degenerate case: since we are going to give a result for $\Delta \searrow 0$, we can assume that this requirement is automatically satisfied. The rate of growth of $X^{\Delta}(n)$ is defined by the Lyapunov exponent

$$\widehat{\mathcal{L}}_{Z^{\Delta}}(\varepsilon) \;=\; \lim_{n\to\infty} \frac{1}{n} \log \|X^{\Delta}(n)\| \,, \tag{1.20}$$

which exists a.s. and is deterministic, see e.g. [3, Th. 4.1 in Ch. 1].

Theorem 1.6. *For $\Delta \searrow 0$, the random process*

$$\left\{\left(X_1^{\Delta}\left(\lfloor t/\Delta\rfloor\right), X_2^{\Delta}\left(\lfloor t/\Delta\rfloor\right)\right)\right\}_{t\in[0,\infty)} \,, \tag{1.21}$$

converges in law to the diffusion $(X_1(\cdot), X_2(\cdot))$ on the Skorokhod space $\mathcal{D}([0,\infty),(0,\infty)^2)$. Moreover,

$$\lim_{\Delta\searrow 0} \frac{\widehat{\mathcal{L}}_{Z^{\Delta}}(\varepsilon)}{\Delta} \;=\; \mathcal{L}_{\sigma,\alpha}(\varepsilon) \,. \tag{1.22}$$

1.4. **Continuum limits and the Derrida-Hilhorst singularity.** In [12], B. Derrida and H. J. Hilhorst study the $\varepsilon \searrow 0$ limit of the Lyapunov exponent $\widehat{\mathcal{L}}_Z(\varepsilon)$ of product of IID matrices of the form (1.1), under the hypothesis that $\mathbb{E}[Z] > 1$ and $\mathbb{E}[\log Z] < 0$. Since $\beta \mapsto \mathbb{E}[Z^{\beta}]$ is convex, by the hypotheses on Z there exists a unique $\alpha \neq 0$ such that $\mathbb{E}Z^{\alpha} = 1$, and one readily realizes that $\alpha \in (0,1)$. It is claimed in [12] that

$$\widehat{\mathcal{L}}_Z(\varepsilon) \overset{\varepsilon\searrow 0}{\sim} C\varepsilon^{2\alpha}\,, \tag{1.23}$$

with a semi-explicit expression for $C = C_Z > 0$, that depends on the law of Z. Such a result directly implies a corresponding result for the case $\mathbb{E}[\log Z] > 0$ and $\mathbb{E}[Z^{-1}] > 1$: note that in this case $\mathbb{E}[Z^{\alpha}] = 1$, $\alpha \neq 0$, is again uniquely solved and $\alpha \in (-1,0)$. So, by writing

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z & Z \end{pmatrix} \;=\; Z \begin{pmatrix} Z^{-1} & \varepsilon Z^{-1} \\ \varepsilon & 1 \end{pmatrix}, \tag{1.24}$$

we see that

$$\widehat{\mathcal{L}}_Z(\varepsilon) \;=\; \mathbb{E}\log Z + \widehat{\mathcal{L}}_{1/Z}(\varepsilon) \overset{(1.23)}{=} \mathbb{E}\log Z + C_{1/Z}\varepsilon^{-2\alpha} + o\left(\varepsilon^{-2\alpha}\right)\,, \tag{1.25}$$

and we recall that $\alpha \in (-1,0)$ now.

Moreover one can find in [12, Sec. 3] an argument telling us that for $\alpha > 1$, $\alpha \notin \mathbb{N}$, one expects

$$\widehat{\mathcal{L}}_Z(\varepsilon) \;=\; c_1\varepsilon^2 + \ldots + \ldots + c_{\lfloor\alpha\rfloor}\varepsilon^{2\lfloor\alpha\rfloor} + C\varepsilon^{2\alpha} + o\left(\varepsilon^{2\alpha}\right)\,, \tag{1.26}$$

for real constants c_j and C that are in principle computable. For the case $\alpha = 0$, i.e. for the case $\mathbb{E}\log Z = 0$ in which the only solution to $\mathbb{E}Z^\alpha = 1$ is $\alpha = 0$, one finds the prediction

$$\widehat{\mathcal{L}}_Z(\varepsilon) \overset{\varepsilon \searrow 0}{\sim} \frac{C}{\log(1/\varepsilon)}, \tag{1.27}$$

with $C > 0$, in more than one reference. We mention here [33, (4.34)] in which (1.27) is found for one dimensional Ising model with random field for a very specific choice of the disorder (the interaction J of [33] corresponds to $\log(1/\varepsilon)$). In the localization context (1.27) has been found for example in [8, (3.17)].

From our perspective, the significance of all this is that:

(1) the continuum limit results of Proposition 1.3 fully match with the expected behaviors (to all orders!), (1.23), (1.26) and (1.27), for the random matrix product. We consider this to be rather striking, and it highlights the richness of the continuum limit;
(2) we are going to review the mathematical results available about (1.23), (1.26) and (1.27), but we want to point out that even at the level of physical predictions some results are more sound than others. Notably, it appears to be rather challenging to capture the $\varepsilon^{2\alpha}$ singularity for $|\alpha| > 1$ and the level of sharpness of the $|\alpha| \in (0, 1)$ prediction (1.23), even leaving aside mathematical rigor, does not appear to be easy to achieve. In this sense, the continuum limit goes beyond what has been established so far for the discrete case.

From a mathematical standpoint a proof of (1.23), and (1.25), (i.e., (1.26) with $|\alpha| \in (0, 1)$) has been achieved only recently and under the assumption that Z has a C^1 density and that the support of Z is bounded and bounded away from zero [20]. It is well known, see e.g. [3], that the problem of computing the Lyapunov exponents boils down to identifying the invariant probability of a Markov chain associated to the matrix product. The arguments in [12] aim at constructing a probability that for ε small is expected to be close to the invariant probability. In [20] this construction is put on rigorous grounds and, above all, it is shown that this probability, although not invariant, is sufficiently close to the invariant one to make it possible to control the Lyapunov exponent with the desired precision. A result about (1.26), i.e. for $|\alpha| \geqslant 1$, has been achieved recently [24], but the the expansion is fully controlled only up to (and excluding) the singular term $C\varepsilon^{2\alpha}$: for the moment results about this term remain very weak.

1.5. **On the two-dimensional Ising model with columnar disorder (McCoy-Wu model).** It is possibly somewhat unexpected, but also computing the free energy of the two dimensional Ising model with columnar disorder (McCoy-Wu model [30, 31]) boils down to analyzing the Lyapunov exponent $\widehat{\mathcal{L}}_Z$. The McCoy-Wu prediction is remarkable and folklore says that their model is the only non trivial exactly solvable disordered statistical mechanics model: we dedicate Appendix A to introducing in detail the model, keeping close to the McCoy-Wu notations. But the key point from the result viewpoint is that L. Onsager celebrated solution of the non disordered case establishes that the free energy, as function of the temperature, has a (logarithmic) divergence in the second derivative at the critical temperature. B. M. McCoy and T. T. Wu predict that if a small amount of columnar disorder (i.e. one dimensional: vertical bounds couplings are random and they are repeated – i.e. no new randomness is introduced – on each line) is introduced the transition persists but disorder is relevant (in the sense of the *Harris criterion*, that is the disorder changes the critical behavior, see e.g. [22, § 5.3]) and the transition becomes C^∞. A precise form of the singularity is also given.

As explained in Appendix A, McCoy and Wu extend Onsager's approach to the columnar disorder case and the free energy can be written, up to additive analytic terms, in terms of an integral in the ε variable of the Lyapunov exponent $\widehat{\mathcal{L}}_Z(\varepsilon)$, with Z that has an explicit expression in terms of the parameters of the Ising model. The analysis by McCoy and Wu of this expression is performed in two steps:

(1) They claim that in the limit of very narrow disorder the relevant – i.e. singular – contribution to the free energy can be written as

$$\mathrm{F}: \alpha \mapsto \int_{(0,\eta)} x \frac{K_{\alpha-1}(x)}{K_\alpha(x)} \, \mathrm{d}x \,, \tag{1.28}$$

with $\eta > 0$ arbitrary (the singular part comes from the small x behavior of the integrand). The integrand is just $4\mathcal{L}_{1,\alpha}(x/4)$, and so it is clear from the estimates in Proposition 1.3 that the integral is well defined for all real α.

(2) They argue, by approximating the integrand by another expression for which the exact integration can be performed, that (1.28) is C^∞ but not analytic at $\alpha = 0$.

The approximation in the first step, see Appendix A, turns out to be precisely the diffusive limit we deal with: this was possibly expected by comparing (1.28) and (1.6). What we do with the next result is providing a rigorous analysis of the second step, that is the analysis of (1.28).

Theorem 1.7. *F is real analytic in $(-1,1)\backslash\{0\}$. Moreover it is C^∞ but not analytic in 0. The radius of convergence of its Taylor series at the origin $\sum_{n=0}^\infty c_n \alpha^n$ is zero: in fact $c_1 = 4\eta$, $c_{2n+1} = 0$ for every $n \in \mathbb{N}$ and the even coefficients satisfy*

$$c_{2n} \overset{n\to\infty}{\sim} 4e^{-\gamma}(-1)^{n+1} \frac{(2n-1)!}{\pi^{2n}} \,, \tag{1.29}$$

with γ the Euler-Mascheroni constant.

We are going to prove more. Namely that (1.28) defines an analytic function for every $\alpha \in \mathbb{C}$ with $0 < |\Re\alpha| < 1$. We believe that the restriction to $0 < |\Re\alpha| < 1$ can be removed to get simply to $|\Re\alpha| > 0$. However this involves a certain number of complications connected to the fact that, with our approach, an *ad hoc* analysis has to be developed for $\Re\alpha \in \mathbb{Z}$. Since the focus is on $\alpha = 0$, we have made the choice not to develop this issue.

A (very) substantial gap remains between where our results lead and the proof of the McCoy-Wu claim that the transition is C^∞, even without the precise claim on the nature of the singularity. What we perform, and what McCoy and Wu do, is capturing the behavior the free energy near criticality when the disorder is vanishing – this is reminiscent of *intermediate disorder limits* [1, 5] in which, like for us, one enters the framework of integrable models – while the true issue is the behavior for (possibly) weak, but non vanishing, disorder.

2. On the Lyapunov exponent: the proof of Theorem 1.1

We use the short-cut notation $\delta := \sigma^2(1-\alpha)/2 \in \mathbb{R}$. We recall that we can assume $\varepsilon > 0$ and let us start by showing that the process does not hit $(0,0)$. Recall that

$(X_1(0), X_2(0)) \neq (0,0)$ and set $\tau_{(0,0)} := \inf\{t > 0 : (X_1(t), X_2(t)) = (0,0)\}$. For this let us consider $R(t) := \sqrt{X_1^2(t) + X_2^2(t)}$. By Itô's formula:

$$\begin{aligned} \mathrm{d}R(t) \;&=\; \frac{X_1}{R}\,\mathrm{d}X_1 + \frac{X_2}{R}\,\mathrm{d}X_2 + \frac{1}{2}\frac{X_1^2}{R^3}\,\mathrm{d}\langle X_2, X_2\rangle \\ &=\; \left(2\varepsilon\frac{X_1X_2}{R} + \delta\frac{X_2^2}{R} + \frac{\sigma^2}{2}\frac{X_1^2X_2^2}{R^3}\right)\mathrm{d}t + \sigma\frac{X_2^2}{R}\,\mathrm{d}B_t \\ &=\; R\left(2\varepsilon\frac{Y}{1+Y^2} + \delta\frac{Y^2}{1+Y^2} + \frac{\sigma^2}{2}\frac{Y^2}{(1+Y^2)^2}\right)\mathrm{d}t + R\left(\sigma\frac{Y^2}{1+Y^2}\right)\mathrm{d}B_t \\ &=:\; R\,D\,\mathrm{d}t + R\,Q\,\mathrm{d}B_t\,, \end{aligned} \tag{2.1}$$

where $Y := X_2/X_1 \in [-\infty, \infty]$ and $D = D(t)$ and $Q = Q(t)$ are uniformly bounded continuous stochastic processes ($\|D\|_\infty \leqslant 2\varepsilon + |\delta| + \sigma^2/2$ and $Q \in [0, \sigma]$), defined up to $\tau_{(0,0)}$. Since, again by Itô's formula, we have

$$\mathrm{d}\log R(t) \;=\; \left(D(t) - \frac{1}{2}Q^2(t)\right)\mathrm{d}t + Q(t)\,\mathrm{d}B_t\,, \tag{2.2}$$

we see that $R(t)/R(0)$ is bounded away from zero on every compact time interval. This readily yields a contradiction if $\mathbb{P}(\tau_{(0,0)} < \infty) > 0$. Hence $\mathbb{P}(\tau_{(0,0)} < \infty) = 0$ and we have proven that the process does not hit the origin.

Now we are going to show that if the initial condition is in the (interior of the) second or fourth quadrant (in the counterclockwise sense), it hits the boundary of these quadrants in an a.s. finite time (in fact, this random time has finite expectation) and enters either the first or third quadrant. And we show also that once the process is in the first (or third) quadrant, it stays there forever.

Without loss of generality let us assume that $X_1(0) < 0$ and $X_2(0) > 0$ (second quadrant). For the analysis it is helpful to consider $Y(t)(< 0)$ up to $t \leqslant \tau_{-\infty} := \inf\{t \geqslant 0 : Y(t) = -\infty\}$, which coincides a.s. with $\inf\{t \geqslant 0 : X_1(t) = 0\}$, and up to $t \leqslant \tau_0 := \inf\{t \geqslant 0 : Y(t) = 0\}$. By Itô formula

$$\mathrm{d}Y \;=\; \left(\varepsilon\left(1 - Y^2\right) + \delta\,Y\right)\mathrm{d}t + \sigma Y\,\mathrm{d}B_t\,, \tag{2.3}$$

and with the specific initial initial conditions we are using is somewhat helpful to work with the positive process $\widetilde{Y} = -Y$:

$$\mathrm{d}\widetilde{Y} \;=\; \left(\varepsilon\left(\widetilde{Y}^2 - 1\right) + \delta\,\widetilde{Y}\right)\mathrm{d}t + \sigma\widetilde{Y}\,\mathrm{d}B_t\,, \tag{2.4}$$

which is in $(0, \infty)$ as long as the two dimensional process does not leave the interior of the quadrant. We use the stopping times $\widetilde{\tau}_0$ and $\widetilde{\tau}_\infty$ with the obvious meaning. We are going to apply the Feller test for explosion to show that $\widetilde{\tau} := \min(\widetilde{\tau}_0, \widetilde{\tau}_\infty)$ is in L^1 so

$$\mathbb{P}\left(\widetilde{\tau} < \infty)\right) \;=\; 1\,, \tag{2.5}$$

which means that, almost surely, the process hits the axes. And if $\widetilde{\tau}_\infty < \widetilde{\tau}_0$, that is if $(X_1(\widetilde{\tau}), X_2(\widetilde{\tau})) = (0, x_2)$, $x_2 > 0$, we readily see from (1.3) that $X_1(\widetilde{\tau} + t) > 0$, at least for $t > 0$ small. If instead $\widetilde{\tau}_0 < \widetilde{\tau}_\infty$, then $(X_1(\widetilde{\tau}), X_2(\widetilde{\tau})) = (x_1, 0)$, $x_1 < 0$, and again from (1.3) one sees that $X_2(\widetilde{\tau} + t) < 0$ for $t > 0$ small: since the equation solved by X_2 is stochastic, the argument is slightly more delicate than for the previous case and we give some details. By the Strong Markov property it suffices to consider $(X_1(0), X_2(0)) = (x_1, 0)$, $x_1 < 0$, and $X_2(t) = \int_0^t (\varepsilon X_1(s) + cX_2(s))\,\mathrm{d}s + M(t)$, with the constant c and the centered Martingale M easily read out of (1.3). Note that M is a time changed Brownian motion. By continuity of $(X_1(\cdot), X_2(\cdot))$ we readily see that $X_2(t) - M(t) \leqslant -|x_1|t/2$ for t small. It is therefore clearly impossible that $\inf\{t > 0 : X_2(t) \neq 0\}$ is positive, because this would imply $X_2(t) < -|x_2|/2$ for small t. Therefore $t \mapsto \int_0^t X_2(s)^2 \, \mathrm{d}s$ is increasing at least for t small, which implies that the time change is non degenerate at least for small times. Hence $M(t)$ becomes negative for arbitrarily small values of t. Therefore $X_2(t) < -t|x_1|/2$ – in particular, it is negative – for arbitrarily small values of t. An application of the Feller test, this time applied to Y and not to $\widetilde{Y}$, actually shows that if Y is in $(0, \infty)$, then it will stay so for all times, that is the interior the first and third quadrants are stable sets for the dynamics.

Let us detail the application of the Feller test. Let Z is a one dimensional diffusion with $Z(0) \in (0, \infty)$ and

$$\mathrm{d}Z(t) \;=\; b(Z(t))\,\mathrm{d}t + q(Z(t))\,\mathrm{d}B_t\,, \tag{2.6}$$

$b(\cdot)$ and $q(\cdot)(> 0)$ differentiable functions. We set $\tau := \inf\{t > 0 : \; Z(t) = 0 \text{ or } Z(t) = \infty\}$ and

$$s(z) \;:=\; \int_1^z \exp\left(-2\int_1^y \frac{b(r)}{q^2(r)}\,\mathrm{d}r\right)\mathrm{d}y \quad \text{and} \quad v(z) \;:=\; \int_1^z s'(y)\left(\int_1^y \frac{2}{s'(r)q^2(r)}\,\mathrm{d}r\right)\mathrm{d}y\,. \tag{2.7}$$

By monotonicity the limits of $v(z)$ for $z \searrow 0$ and $z \nearrow \infty$ exist in $[0, \infty]$ and they will be simply denoted by $v(0)$ and $v(\infty)$. If both $v(0) < \infty$ and $v(\infty) < \infty$ then $\mathbb{E}[\tau] < \infty$ [25, Prop. 5.32, Ch. 5]. On the other hand, if $v(0) = v(\infty) = \infty$ then $\mathbb{P}(\tau = \infty) = 1$ [25, Th. 5.29, Ch. 5] .

Let us start with the $\widetilde{Y}$ case (cf. (2.4)): we have

$$s'(z) \;=\; \frac{C}{z^{1-\alpha}} \exp\left(-\frac{2\varepsilon}{\sigma^2}\left(z + \frac{1}{z}\right)\right)\,, \tag{2.8}$$

so for $z \geqslant 1$

$$s'(z) \asymp z^{-1+\alpha} \exp\left(-\frac{2\varepsilon}{\sigma^2} z\right)\,, \tag{2.9}$$

with the notation $f(z) \asymp g(z)$ if $f(z)/g(z) \in [a, 1/a]$ on the prescribed interval for some $a \in (0, 1)$, and

$$v'(z) \asymp \frac{1}{z^2}\,, \tag{2.10}$$

so $v(\infty) < \infty$. In a very similar way, for $z \in (0, 1]$

$$s'(z) \asymp z^{-1+\alpha} \exp\left(-\frac{2\varepsilon}{\sigma^2}\frac{1}{z}\right)\,, \tag{2.11}$$

and the estimate of $v(0)$ is identical to the one for $v(\infty)$ via a (double) change of variable $z \mapsto 1/z$. Hence $v(0) < \infty$ and the diffusion $\widetilde{Y}$ hits 0 or ∞ at a random time which has finite expectation.

For the case of Y we turn to (2.3) and the difference is that the factor $(z + 1/z)$ in the exponent in (2.8) changes sign. Once again, we can replace $(z + 1/z)$ by z for $z \geqslant 1$, and by $1/z$ for $z \leqslant 1$. This implies that the integral with respect to r in the expression for $v(z)$ in (2.7) stays bounded and bounded away from zero both for $y \nearrow \infty$ and for $y \searrow 0$. The integral with respect to y therefore diverges both for $z \nearrow \infty$ and $z \searrow 0$ (once again, the two computations are identical, up to change of variables). Therefore, almost surely, Y hits neither 0 nor ∞.

We are now going to show that the diffusion Y has a unique invariant probability, that we will make explicit, on $(0, \infty)$. This corresponds to the two (extremal) invariant probabilities for the normalized process $(X_1, X_2)/\sqrt{X_1^2 + X_2^2}$, supported on the intersection of the unit circle with the first (or third) quadrant. For this it is practical to observe that the generator of the evolution (2.3) acts on C^2 functions $f : (0, \infty) \to \mathbb{R}$ as

$$L_\varepsilon f(y) \;=\; \left(\varepsilon(1-y^2) + \delta\, y\right) f'(y) + \frac{\sigma^2}{2} y^2 f''(y) \;=\; \frac{\sigma^2}{2p_\varepsilon(y)} \left(y^2 p_\varepsilon(y) f'(y)\right)' , \tag{2.12}$$

where $p_\varepsilon(\cdot)$ is the probability density

$$p_\varepsilon(y) \;=\; \frac{C_\varepsilon}{y^{1+\alpha}} \exp\left(-\frac{2\varepsilon}{\sigma^2}\left(y + \frac{1}{y}\right)\right) \qquad \text{with } C_\varepsilon^{-1} \;=\; 2K_\alpha\left(4\varepsilon/\sigma^2\right) , \tag{2.13}$$

and $K_\alpha(\cdot)$ is defined in (1.4). This already makes evident the reversible nature of the diffusion Y and, in particular, (2.13) is an invariant probability. The transformation $S(t) := \log Y(t)$ makes things even more straightforward: S is a diffusion on $\mathbb{R}$ with constant diffusion coefficient and a strongly confining potential:

$$\mathrm{d}S \;=\; -U'(S)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t \quad \text{ with } \; U(s) \;:=\; \varepsilon\left(\exp(-s) + \exp(s) - \left(\delta - \frac{\sigma^2}{2\varepsilon}\right)s\right) . \tag{2.14}$$

An invariant probability of this diffusion is $\widetilde{p}_\varepsilon(s) \propto \exp(-2U(s)/\sigma^2)$ and the generator has the familiar symmetric form $\widetilde{L}_\varepsilon g = (\sigma^2/2)(\widetilde{p}_\varepsilon g')'/\widetilde{p}_\varepsilon$, for $g \in C^2(\mathbb{R}, \mathbb{R})$, see e.g [17, p.111].

Uniqueness of this invariant measure as well as ergodic properties can be established in a variety of ways: [29, Th. 5.1] gives a Pointwise Ergodic Theorem that one can directly apply to (2.14), and of course it implies uniqueness. Alternatively one can put (2.3) or (2.14) in *natural scale* via a time change and a scale function, see [36, Ch. V], and apply the Ergodic Theorem [36, Ch. V, Th. 53.1]. Ergodic properties of S are also given in [34].

Therefore for every choice of $Y(0) \in (0, \infty)$, almost surely and in L^1 we have that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log X_1(t) \;&=\; \varepsilon \lim_{t\to\infty} \frac{1}{t} \int_0^t Y(s) ds \\ &=\; \varepsilon \int_0^\infty y\, p_\varepsilon(y)\,\mathrm{d}y \;=\; \frac{\varepsilon K_{\alpha-1}\left(4\varepsilon/\sigma^2\right)}{K_\alpha\left(4\varepsilon/\sigma^2\right)} , \end{aligned} \tag{2.15}$$

where in the first step we have used the first identity in

$$\begin{aligned} X_1(t) \;&=\; X_1(0) \exp\left(\varepsilon \int_0^t Y(s)\,\mathrm{d}s\right) , \\ X_2(t) \;&=\; X_2(0) \exp\left(\varepsilon \int_0^t \frac{1}{Y(s)}\,\mathrm{d}s - \alpha\frac{\sigma^2}{2} t + \sigma B_t\right) , \end{aligned} \tag{2.16}$$

which is directly derived from (1.3) and holds for all $t > 0$ if both $X_1(0)$ and $X_2(0)$ are positive (or both are negative: in general, the formula holds up to the hitting time of the boundary of the quadrant in which $(X_1(0), X_2(0))$ lies). The second step in (2.15) is the application of the Pointwise Ergodic Theorem and the last one is an explicit computation.

In the same way, by using the second identity in (2.16) we get to (with $x = 4\varepsilon/\sigma^2$)

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log X_2(t) &= \varepsilon\lim_{t\to\infty}\frac{1}{t}\left(\int_0^t \frac{1}{Y(s)}ds\right) - \alpha\frac{\sigma^2}{2} \\ &= \varepsilon\int_0^\infty \frac{1}{y}\,p_\varepsilon(y)\,\mathrm{d}y - \alpha\frac{\sigma^2}{2} \\ &= \frac{\sigma^2}{4}\left(\frac{xK_{1+\alpha}(x)}{K_\alpha(x)} - 2\alpha\right) \overset{(1.13)}{=} \frac{\sigma^2}{4}\frac{xK_{1-\alpha}(x)}{K_\alpha(x)} = \frac{\varepsilon K_{\alpha-1}\left(4\varepsilon/\sigma^2\right)}{K_\alpha\left(4\varepsilon/\sigma^2\right)}\,,\end{aligned} \tag{2.17}$$

which coincides with what we found in (2.15). This shows that both components have the same exponential growth rate, hence also the norm of $(X_1(t), X_2(t))$, and (1.6) is proven. If instead of starting from the first quadrant, we were starting from the second quadrant, the result is unchanged because the second quadrant is abandoned after a random time that is in L^1. This completes the proof of Theorem 1.1. □

3. Fluctuations of the Lyapunov exponent: proofs

Proof of Proposition 1.2. Recall that the ratio $Y(t) = X_2(t)/X_1(t)$ converges. Hence it is sufficient to prove the convergence of (1.7) with $X_1(t)$ instead of $\|(X_1(t), X_2(t))\|$ in the logarithm, i.e., to prove convergence in law of

$$\left\{\frac{1}{\sqrt{t}}\left(\int_0^t f(Y(s))ds\right)\right\}_{t\in(0,\infty)}, \tag{3.1}$$

where the function $f(y) = \varepsilon y - \mathcal{L}_{\sigma,\alpha}(\varepsilon)$ is centered for the invariant density $p_\varepsilon(\cdot)$. We start by solving the Poisson equation, $L_\varepsilon g = f$. [34, Th. 1] applies for $S = \log Y$, see (2.14), and shows that the Poisson equation has $g(y) = \int_0^\infty \mathbb{E}_y[f(Y(s))]ds$ as unique solution. We need here an explicit form, and we solve the linear equation

$$\frac{\sigma^2}{2}y^2h' + \big(\varepsilon(1-y^2) + \delta\, y\big)\, h\, = f \tag{3.2}$$

for $h = g'$ by the method of variation of constants. The homogeneous equation – when the right-hand side of (3.2) is equal to 0 – admits $h_0(y) = \big(y^2 p_\varepsilon(y)\big)^{-1}$ as a solution. Looking now for solutions of the form $h(y) = k(y)h_0(y)$ for (3.2) itself, we find that

$$k(y) = \frac{2}{\sigma^2}\int_0^y \big(\varepsilon z - \mathcal{L}_{\sigma,\alpha}(\varepsilon)\big)p_\varepsilon(z)dz + C\,, \tag{3.3}$$

and we choose $C = 0$ (this is the only choice that yields the required integrability properties in what follows). Finally,

$$g'(y) \;=\; \big(y^2 p_\varepsilon(y)\big)^{-1}\frac{2}{\sigma^2}\int_0^y \big(\varepsilon z - \mathcal{L}_{\sigma,\alpha}(\varepsilon)\big)p_\varepsilon(z)\,\mathrm{d}z\,, \tag{3.4}$$

and the value of $g(1)$ does not matter for our purpose. Now, we can follow a standard proof of Central Limit Theorem for reversible diffusions, e.g. [6, Sec. 2]. Note in fact that g is smooth and that for $y \to \infty$ (recall the notation used in (2.9))

$$g'(y) \;\asymp\; y^{\alpha-1}e^{\frac{\varepsilon}{2}y}\left(\int_y^\infty z^{-\alpha}e^{-\frac{\varepsilon}{2}z}\,\mathrm{d}z\right) \;\asymp\; \frac{1}{y}\,, \tag{3.5}$$

where in the first step we use that $\int_0^y(\varepsilon z-\mathcal{L}_{\sigma,\alpha}(\varepsilon))p_\varepsilon(z)\,\mathrm{d}z=\int_y^\infty(\mathcal{L}_{\sigma,\alpha}(\varepsilon)-\varepsilon z)p_\varepsilon(z)\,\mathrm{d}z$ and that $\mathcal{L}_{\sigma,\alpha}(\varepsilon)$ is just a constant. For $y\searrow 0$ instead

$$g'(y)\;\asymp\;y^{\alpha-1}e^{\frac{\varepsilon}{2y}}\left(\int_0^y z^{-1-\alpha}e^{-\frac{\varepsilon}{2z}}\,\mathrm{d}z\right)\;=\;y^{\alpha-1}e^{\frac{\varepsilon}{2y}}\left(\int_{1/y}^\infty z^{-1+\alpha}e^{-\frac{\varepsilon z}{2}}\,\mathrm{d}z\right)\;\asymp\;1\,. \tag{3.6}$$

Therefore $\sup_y|g'(y)|<\infty$ and by Itô's formula we obtain that

$$\begin{aligned}M_t\;:=\;\sigma\int_0^t Y_s g'(Y_s)\,\mathrm{d}B_s\;&=\;g(Y_t)-g(Y_0)-\int_0^t L_\varepsilon g(Y_s)\,\mathrm{d}s\\&=\;g(Y_t)-g(Y_0)-\int_0^t f(Y_s)\,\mathrm{d}s\,,\end{aligned} \tag{3.7}$$

is a martingale with bracket

$$\langle M\rangle_t\;=\;\sigma^2\int_0^t Y_s^2 g'(Y_s)^2\,\mathrm{d}s\,. \tag{3.8}$$

By the ergodic theorem, as $t\to\infty$, almost surely

$$\begin{aligned}\frac{1}{t}\langle M\rangle_t&\longrightarrow\sigma^2\int_0^\infty y^2g'(y)^2p_\varepsilon(y)\,\mathrm{d}y\\&=\frac{4}{\sigma^2}\int_0^\infty\frac{1}{y^2p_\varepsilon}\left(\int_0^y(\varepsilon z-\mathcal{L}_{\sigma,\alpha}(\varepsilon))\,p_\varepsilon(z)\,\mathrm{d}z\right)^2\mathrm{d}y\;=\;v_{\sigma,\alpha}(\varepsilon)\,.\end{aligned} \tag{3.9}$$

This deterministic limit is finite in view of the (exponential) decay of $\int_0^y\big(\varepsilon z-\mathcal{L}_{\sigma,\alpha}(\varepsilon)\big)p_\varepsilon(z)\,\mathrm{d}z$ as $y\to 0$ and $y\to\infty$. Then, the central limit theorem for martingales applies, and $t^{-1/2}M_t$ converges in law to a centered Gaussian with variance given by $v_{\sigma,\alpha}(\varepsilon)$. Now, the first two terms in the last line of (3.7) are bounded in probability, so $-t^{-1/2}\int_0^t f(Y_s)ds$ converges to the same limit as $t^{-1/2}M_t$, and (3.1) is proved. Therefore the proof of Proposition 1.2 is complete. □

Proof of Proposition 1.5. In view of (1.8) and of the fact that we know the asymptotic behavior of $K_\alpha(x)\sim x^{-|\alpha|}\Gamma(|\alpha|)/2^{1-|\alpha|}$ for $\alpha\neq 0$ and $K_0(x)\sim\log(1/x)$, what we have to estimate is

$$\begin{aligned}&\int_0^\infty\frac{1}{y^{1-\alpha}}e^{\frac{x}{2}\left(y+\frac{1}{y}\right)}\left(\int_0^y\frac{\varepsilon z-\mathcal{L}_{\sigma,\alpha}}{z^{1+\alpha}}e^{-\frac{x}{2}\left(z+\frac{1}{z}\right)}\,\mathrm{d}z\right)^2\mathrm{d}y\;=\\&\qquad\int_1^\infty\frac{1}{y^{1-\alpha}}e^{\frac{x}{2}\left(y+\frac{1}{y}\right)}\left(\int_y^\infty\frac{\varepsilon z-\mathcal{L}_{\sigma,\alpha}}{z^{1+\alpha}}e^{-\frac{x}{2}\left(z+\frac{1}{z}\right)}\,\mathrm{d}z\right)^2\mathrm{d}y+\\&\qquad\int_0^1\frac{1}{y^{1-\alpha}}e^{\frac{x}{2}\left(y+\frac{1}{y}\right)}\left(\int_0^y\frac{\varepsilon z-\mathcal{L}_{\sigma,\alpha}}{z^{1+\alpha}}e^{-\frac{x}{2}\left(z+\frac{1}{z}\right)}\,\mathrm{d}z\right)^2\mathrm{d}y\;=:\;T_1(x)+T_2(x)\,.\end{aligned} \tag{3.10}$$

Remark 3.1. *In view of Proposition 1.3 we know that $\mathcal{L}_{\sigma,\alpha}=O(\varepsilon^{\min(2\alpha,2)})$ for $\alpha>0$, except for $\alpha=1$ for which there is a logarithmic correction. Therefore $\mathcal{L}_{\sigma,\alpha}=o(\varepsilon)$ if $\alpha>1/2$ and, since $z\geqslant 1$, in dealing with $T_1(x)$ we can safely neglect the term containing $\mathcal{L}_{\sigma,\alpha}$ for $\alpha>1/2$. On the other hand, in dealing with $T_2(x)$ we can safely neglect the term not containing $\mathcal{L}_{\sigma,\alpha}$ for $\alpha<1/2$. In fact $\mathcal{L}_{\sigma,\alpha}$ is much greater than ε, hence of εz ($z\leqslant 1$ for T_2), for $\alpha<1/2$.*

To make the expressions more compact and readable we choose

$$\sigma^2 \,=\, 2\,; \tag{3.11}$$

the general case is easily recovered by a scaling argument.

We start with the analysis of T_1. By a change of variable we have:

$$T_1(x) \,=\, \left(\frac{x}{2}\right)^\alpha \int_{x/2}^\infty y^{\alpha-1} e^{y+(x/2)^2/y} \left(\int_y^\infty \left(z^{-\alpha} - \mathcal{L}_{\sigma,\alpha} z^{-\alpha-1}\right) e^{-z-(x/2)^2/z}\, \mathrm{d}z \right)^2 \mathrm{d}y\,. \tag{3.12}$$

We claim that for $\alpha < 0$ we simply have

$$T_1(x) \overset{x\searrow 0}{\sim} \left(\frac{x}{2}\right)^\alpha \int_0^\infty y^{\alpha-1} e^y \left(\int_y^\infty \left(z^{-\alpha} - |\alpha| z^{-\alpha-1}\right) e^{-z}\, \mathrm{d}z \right)^2 \mathrm{d}y \,=:\, 2^{|\alpha|}\Gamma(|\alpha|) x^\alpha\,. \tag{3.13}$$

For this choose $\delta \in (0,1)$ and split the integral in y in (3.12) as $\int_{x/2}^\infty \ldots = \int_{x/2}^\delta \ldots + \int_\delta^{1/\delta} \ldots + \int_{x/2}^{1/\delta} \ldots := I_1 + I_2 + I_3$. The limit $x \searrow 0$ is easily taken in I_2 and the dependence on x disappears. Moreover we directly check that $\lim_{\delta\searrow 0}\lim_{x\searrow 0} I_2 \in (0,\infty)$ is the integral in the right-hand side of (3.13). We are left with showing that $\lim_{\delta\searrow 0}\sup_{x\in(0,2\delta)} I_j = 0$ for $j = 1$ and 3. For I_1 recall that in (3.12) we can replace $\int_y^\infty \ldots \mathrm{d}z$ with $\int_0^y \ldots \mathrm{d}z$, so that for δ sufficiently small (so x is small too and we can use the asymptotic approximation of $\mathcal{L}_{\sigma,\alpha} \sim |\alpha|$) we have

$$\sup_{x\in(0,2\delta)} |I_1| \,\leqslant\, \int_0^\delta y^{\alpha-1} \left(\int_0^y \left(z^{-\alpha} + 2|\alpha| z^{-\alpha-1}\right) \mathrm{d}z \right)^2 \mathrm{d}y \overset{\delta\searrow 0}{\longrightarrow} 0\,. \tag{3.14}$$

Moreover (again, δ small)

$$\sup_{x\in(0,2\delta)} |I_3| \,\leqslant\, 2\int_{1/\delta}^\infty y^{\alpha-1} e^y \left(\int_y^\infty \left(z^{-\alpha} + 2|\alpha| z^{-\alpha-1}\right) e^{-z}\, \mathrm{d}z \right)^2 \mathrm{d}y \,\leqslant\,$$

$$\int_{1/\delta}^\infty e^{3y/2} \left(\int_y^\infty e^{-z}\, \mathrm{d}z \right)^2 \mathrm{d}y \overset{\delta\searrow 0}{\longrightarrow} 0\,, \tag{3.15}$$

and (3.13) is proven.

For $\alpha = 0$ we have

$$T_1(x) \,=\, \int_{x/2}^\infty y^{-1} e^{y+(x/2)^2/y} \left(\int_0^y \left(1 - \mathcal{L}_{\sigma,0} z^{-1}\right) e^{-z-(x/2)^2/z}\, \mathrm{d}z \right)^2 \mathrm{d}y\,. \tag{3.16}$$

We anticipate (for future use) that the result we are going to obtain would be the same if $1 - \mathcal{L}_{\sigma,0} z^{-1}$ is replaced by $\mathcal{L}_{\sigma,0} z^{-1}$ Again, $\int_0^y \ldots$ can be replaced by $\int_y^\infty \ldots$ and it suffices the splitting $\int_{x/2}^\infty \ldots = \int_{x/2}^\delta \ldots + \int_\delta^\infty \ldots =: I_1 + I_2$. In fact (recall that $\mathcal{L}_{\sigma,0} = o(1)$ as $x \searrow 0$, in particular $\mathcal{L}_{\sigma,0}$ becomes smaller than one)

$$\sup_{x\in(0,2\delta)} |I_2| \,\leqslant\, 2\int_\delta^\infty y^{-1} e^y \left(\int_y^\infty (1 + z^{-1}) e^{-z}\, \mathrm{d}z \right)^2 \mathrm{d}y\,, \tag{3.17}$$

and the right-hand side is just a finite expression that depends on δ. On the other hand I_1 diverges as $x \searrow 0$. In fact observe that

$$e^{-2\delta}\widetilde{I} \,\leqslant\, I_1 \,\leqslant\, e^{2\delta}\widetilde{I}\,,\ \text{with}\ \widetilde{I} := \int_{x/2}^\delta y^{-1} \left(\int_0^y \left(1 - \mathcal{L}_{\sigma,0} z^{-1}\right) e^{-(x/2)^2/z}\, \mathrm{d}z \right)^2 \mathrm{d}y\,, \tag{3.18}$$

so we can focus on $\widetilde{I}$. By using $\int_0^L (1/x)\exp(-1/x)\,\mathrm{d}x \sim \log L$ for $L \to \infty$, and the fact that $\delta/x^2 \leqslant y/x^2 \leqslant 1/(2x)$ we see that

$$\int_0^y z^{-1} e^{-(x/2)^2/z}\,\mathrm{d}z \sim \log(y/x^2)\,, \tag{3.19}$$

uniformly in the range of y we are using, and as $x \searrow 0$. Therefore

$$\widetilde{I} \sim \left(\frac{1}{2\log(1/x)}\right)^2 \int_{x/2}^{\delta} y^{-1}\left(2\log(1/x) - \log(1/y)\right)^2 \mathrm{d}y \;\sim\; \frac{7}{12}\log(1/x)\,. \tag{3.20}$$

This concludes the $\alpha = 0$ case: $T_1(x) \sim \frac{7}{12}\log(1/x)$.

The case $\alpha > 0$ is quicker to treat for $\alpha > 1/2$ because of Remark 3.1. In reality also for $\alpha \in (0, 1/2]$ the term containing $\mathcal{L}_{\sigma,\alpha}$ does not contribute: we will check this fact after estimating what is giving the main contribution:

$$\left(\frac{x}{2}\right)^{\alpha} \int_{x/2}^{\infty} y^{\alpha-1} e^{y+(x/2)^2/y} \left(\int_y^{\infty} z^{-\alpha} e^{-z-(x/2)^2/z}\,\mathrm{d}z\right)^2 \mathrm{d}y\,, \tag{3.21}$$

and the final result is that for $\alpha \in (0, 2)$

$$T_1(x) \;\sim\; \left(\frac{x}{2}\right)^{\alpha} \int_0^{\infty} y^{\alpha-1} e^{y} \left(\int_y^{\infty} z^{-\alpha} e^{-z}\,\mathrm{d}z\right)^2 \mathrm{d}y \;=:\; q_1(\alpha) x^{\alpha}\,. \tag{3.22}$$

This is proven like before by restricting the integral to $y \in (\delta, 1/\delta)$ and estimating the rest before letting $\delta \searrow 0$. The function $q_1(\cdot)$ can be expressed with a Meijer G-function, but this does not make it much more explicit. Let us quickly verify that the term we neglected for $\alpha \in (0, 1/2]$ is of lower order: by focusing on $y \in (x/2, \delta)$ (otherwise the fact is obvious) we see that an upper bound on this contribution is $O(x^{5\alpha}) \int_{x/2}^{\delta} y^{-\alpha-1}\,\mathrm{d}y = O(x^{4\alpha})$.

The $\alpha = 2$ case generates a logarithmic correction: in fact from (3.21) we see that if we restrict the integral over $y \geqslant \delta$, the contribution is bounded by x^2 times a constant that depends only on δ. The integral with $y \in (x/2, \delta)$ instead is controlled above and below, up to a factor that can be chosen arbitrarily close to one uniformly in $x \searrow 0$ by choosing δ small (like in (3.18)), by

$$\left(\frac{x}{2}\right)^2 \int_{x/2}^{\delta} y \left(\int_y^{\infty} z^{-2} e^{-z}\,\mathrm{d}z\right)^2 \mathrm{d}y \overset{x\searrow 0}{\sim} \left(\frac{x}{2}\right)^2 \log(1/x)\,. \tag{3.23}$$

For $\alpha > 2$ we go back to (3.21)

$$\left(\frac{x}{2}\right)^{\alpha} \int_{x/2}^{\infty} y^{\alpha-1} e^{y+(x/2)^2/y} \left(\int_y^{\infty} z^{-\alpha} e^{-z-(x/2)^2/z}\,\mathrm{d}z\right)^2 \mathrm{d}y \;=\; \left(\frac{x}{2}\right)^{\alpha} \int_1^{\infty} \ldots + \left(\frac{x}{2}\right)^{\alpha} \int_{x/2}^{1} \ldots \leqslant$$
$$C x^{\alpha} + 3\left(\frac{x}{2}\right)^{\alpha} \int_{x/2}^{\infty} y^{\alpha-1} \left(\int_y^{\infty} z^{-\alpha} e^{-z}\,\mathrm{d}z\right)^2 \mathrm{d}y \;\leqslant\; C' x^{\alpha} \left(1 + \int_{x/2}^{\infty} y^{-\alpha+1}\,\mathrm{d}y\right) \;=\; O(x^2)\,, \tag{3.24}$$

where C and C' are constants independent of x.

We collect what we have obtained:

$$T_1(x) \sim \begin{cases} 2^{|\alpha|}\Gamma(|\alpha|)x^\alpha & \text{if } \alpha \in (-\infty,0)\,, \\ \frac{7}{12}\log(1/x) & \text{if } \alpha = 0\,, \\ q_1(\alpha)x^\alpha & \text{if } \alpha \in (0,2)\,, \\ \frac{1}{4}x^2\log(1/x) & \text{if } \alpha = 2\,, \\ O(x^2) & \text{if } \alpha > 2\,. \end{cases} \tag{3.25}$$

We now turn to $T_2(x)$ and the basic expression is after a change of variables (still, $\sigma = \sqrt{2}$)

$$T_2(x) = \left(\frac{x}{2}\right)^{4-\alpha} \int_{x/2}^{\infty} u^{-1-\alpha} e^{u+(x/2)^2/u} \left(\int_u^\infty \left(v^{-2+\alpha} - \left(\frac{2\mathcal{L}_{\sigma,\alpha}}{x^2}\right) v^{-1+\alpha}\right) e^{-v-(x/2)^2/v}\, \mathrm{d}v\right)^2 \mathrm{d}u\,. \tag{3.26}$$

Let us start with $\alpha < 0$ and recall that by Remark 3.1 it suffices to consider

$$\left(\frac{x}{2}\right)^{4-\alpha} \left(\frac{2\mathcal{L}_{\sigma,\alpha}}{x^2}\right)^2 \int_{x/2}^{\infty} u^{-1-\alpha} e^{u+(x/2)^2/u} \left(\int_u^\infty v^{-1+\alpha} e^{-v-(x/2)^2/v}\, \mathrm{d}v\right)^2 \mathrm{d}u\,. \tag{3.27}$$

The pre-factor behaves asymptotically as $\alpha^2(x/2)^{-\alpha}$ and the integral can be bounded by two times

$$\int_{x/2}^{\infty} u^{-1-\alpha} e^{u} \left(\int_u^\infty v^{-1+\alpha} e^{-v}\, \mathrm{d}v\right)^2 \mathrm{d}u \sim \frac{1}{|\alpha|^3}(x/2)^\alpha\,. \tag{3.28}$$

So $T_2(x) = O(1)$ for $\alpha < 0$.

For $\alpha = 0$ the expression to evaluate is

$$\mathcal{L}_{\sigma,0}^2 \int_{x/2}^{\infty} u^{-1} e^{u+(x/2)^2/u} \left(\int_u^\infty v^{-1} e^{-v-(x/2)^2/v}\, \mathrm{d}v\right)^2 \mathrm{d}u\,. \tag{3.29}$$

But this term is minimally different from (3.16) (see observation right after(3.16)) and exactly in the same way we arrive at $T_2(x) \sim T_1(x) \sim \frac{7}{12}\log(1/x)$.

For $\alpha \in (0,2)$ we split the integral with respect to u and the contribution when $u \geqslant 1$ is bounded so the contribution to $T_2(x)$ is $O(x^{4-\alpha})$. For $u < 1$ we make an upper on the contribution to $T_2(x)$:

$$\begin{aligned} &3\left(\frac{x}{2}\right)^{4-\alpha} \int_{x/2}^{1} u^{-1-\alpha} \left(\int_0^u \left(v^{-2+\alpha} + \frac{2\mathcal{L}_{\sigma,\alpha}}{x^2} v^{-1+\alpha}\right) e^{-v}\, \mathrm{d}v\right)^2 \mathrm{d}u \\ &\quad \leqslant C x^{4-\alpha} \left(\int_{x/2}^1 u^{-3+\alpha}\,\mathrm{d}u + \left(\frac{2\mathcal{L}_{\sigma,\alpha}}{x^2}\right)^2 \int_0^1 u^{-1+\alpha}\,\mathrm{d}u\right) \leqslant C'\left(x^2 + \mathcal{L}_{\sigma,\alpha}^2 x^{-\alpha}\right), \end{aligned} \tag{3.30}$$

and since $\mathcal{L}_{\sigma,\alpha} = \max(x^{2\alpha}, x^2)$, except for a logarithmic correction for $\alpha = 1$, we conclue that $T_2(x) = O(\max(x^{3\alpha}, x^2))$ for $\alpha \in (0,2)$.

For $\alpha = 2$ we again split the integral with respect to $u \geqslant \delta$ and $u < \delta$. The integral for $y \geqslant \delta$ is bounded by a constant that depends only on δ. Arguing as in (3.18) we see that what it suffices to control

$$\int_{x/2}^{\delta} u^{-3}\left(\int_0^u \left(1-\left(\frac{2\mathcal{L}_{\sigma,2}}{x^2}\right)v\right)e^{-v}\,\mathrm{d}v\right)^2 \mathrm{d}u \;\sim\; \log(1/x)\,, \tag{3.31}$$

where we have used that $\mathcal{L}_{\sigma,2}=O(x^2)$. Therefore $T_2(x)\sim(x/2)^2\log(1/x)$ for $\alpha=2$.

Finally, for $\alpha>2$ we have

$$T_2(x)\sim\left(\frac{x}{2}\right)^{4-\alpha}\int_0^\infty u^{-1-\alpha}e^u\left(\int_u^\infty\left(v^{-2+\alpha}-\frac{v^{-1+\alpha}}{\alpha-1}\right)e^{-v}\,\mathrm{d}v\right)^2\mathrm{d}u=\frac{2^{\alpha-4}\Gamma(\alpha-2)}{(\alpha-1)^2}x^{4-\alpha}. \tag{3.32}$$

The proof of this claim follows the same line as the proof of (3.13), that is, splitting of the y integral in three parts and taking the limit $\delta\searrow 0$.

We have got to:

$$T_2(x)\;\sim\;\begin{cases} O(1) & \text{if } \alpha<0\,,\\ \frac{7}{12}\log(1/x) & \text{if } \alpha=0\,,\\ O(\max(x^{3\alpha},x^2)) & \text{if } \alpha\in(0,2)\,,\\ \frac14 x^2\log(1/x) & \text{if } x=2\,,\\ \frac{2^{\alpha-4}\Gamma(\alpha-2)}{(\alpha-1)^2}x^{4-\alpha} & \text{if } \alpha>2\,.\end{cases} \tag{3.33}$$

Therefore only T_2 contributes to the final result for $\alpha>2$. Otherwise only T_1 contributes, except at $\alpha=0$ and 2 where they both contribute and exactly with the same amount:

$$T_1(x)+T_2(x)\;\sim\;\begin{cases} 2^{|\alpha|}\Gamma(|\alpha|)x^{\alpha} & \text{if } \alpha\in(-\infty,0)\,,\\ \frac{7}{6}\log(1/x) & \text{if } \alpha=0\,,\\ q_1(\alpha)x^{\alpha} & \text{if } \alpha\in(0,2)\,,\\ \frac12 x^2\log(1/x) & \text{if } \alpha=2\,,\\ \frac{2^{\alpha-4}\Gamma(\alpha-2)}{(\alpha-1)^2}x^{4-\alpha} & \text{if } \alpha>2\,.\end{cases} \tag{3.34}$$

The final result, i.e. (1.15), is recovered by dividing by $K_\alpha(x)$ and using $K_\alpha(x)\sim x^{-|\alpha|}\Gamma(|\alpha|)/2^{1-|\alpha|}$ for $\alpha\neq 0$ and $K_0(x)\sim\log(1/x)$. The constant $C(\alpha)$ in (1.15) is

$$C(\alpha)\;:=\;\begin{cases} 2 & \text{if } \alpha<0\,,\\ 7/6 & \text{if } \alpha=0\,,\\ q_1(\alpha)2^{1-|\alpha|}/\Gamma(|\alpha|) & \text{if } \alpha\in(0,2)\,,\\ 1/4 & \text{if } \alpha=2\,,\\ 1/\big(8(\alpha-1)^3(\alpha-2)\big) & \text{if } \alpha>2\,,\end{cases} \tag{3.35}$$

with $q_1(\alpha)$ given in (3.22). The proof of Proposition 1.5 is therefore complete. □

4. Lyapunov exponent and singularities: the proof of Proposition 1.3

In view of (1.12) we just consider $\alpha\geqslant 0$. We treat first the non integer case.

The case $\alpha\in(0,\infty)\backslash\mathbb{N}$. By the *connection* formula with the other modified Bessel function $I_\alpha(x)$, we have [32, 10.27.4] and [32, 10.25.2]

$$K_\alpha(x)\;=\;\frac{\pi}{2\sin(\pi\alpha)}\left(I_{-\alpha}(x)-I_\alpha(x)\right)\,, \tag{4.1}$$

with

$$I_\alpha(x) \,:=\, \left(\frac{x}{2}\right)^\alpha \sum_{k=0}^\infty \frac{\left(x^2/4\right)^k}{k!\,\Gamma(\alpha+k+1)} \,=:\, \left(\frac{x}{2}\right)^\alpha \widetilde{I}_\alpha(x)\,, \tag{4.2}$$

where $\widetilde{I}_\alpha(x)$ is a non standard notation, but it singles out the analytic part of the $I_\alpha(\cdot)$: in fact $\widetilde{I}_\alpha(\cdot)$ is an entire function. By elementary manipulations we obtain

$$\frac{xK_{\alpha-1}(x)}{K_\alpha(x)} \,=\, 2\,\frac{(x/2)^{2\alpha}\left(\widetilde{I}_{\alpha-1}(x)/\widetilde{I}_{-\alpha}(x)\right) - (x/2)^2\left(\widetilde{I}_{-\alpha+1}(x)/\widetilde{I}_{-\alpha}(x)\right)}{1-(x/2)^{2\alpha}\left(\widetilde{I}_\alpha(x)/\widetilde{I}_{-\alpha}(x)\right)}\,. \tag{4.3}$$

Therefore, aiming at expanding this expression for $x \searrow 0$ up to the first singular term, we obtain

$$\begin{aligned} \frac{xK_{\alpha-1}(x)}{K_\alpha(x)} \,&=\, -2\left(\frac{x}{2}\right)^2\left(\frac{\widetilde{I}_{-\alpha+1}(x)}{\widetilde{I}_{-\alpha}(x)}\right) + \left(\frac{x}{2}\right)^{2\alpha}\frac{2\Gamma(1-\alpha)}{\Gamma(\alpha)} + O(x^{4\alpha}) + O\left(x^{2\alpha+2}\right) \\ &=\, p_{\alpha,\lfloor\alpha\rfloor}\left(x^2\right) + \left(\frac{x}{2}\right)^{2\alpha}\frac{2\Gamma(1-\alpha)}{\Gamma(\alpha)} + O\left(x^{2\lfloor\alpha\rfloor+2}\right) + O(x^{4\alpha})\,, \end{aligned} \tag{4.4}$$

where $p_{\alpha,j}(y)$ is the Taylor expansion up to degree j of $p_\alpha(y) := -(y/2)\widetilde{I}_{-\alpha+1}(\sqrt{y})/\widetilde{I}_{-\alpha}(\sqrt{y})$. It is not difficult to realize that the coefficients of this Taylor expansion are just rational function of α. Let us detail this point that is going to be important also for the passage to α integer: if we introduce for $k \in \mathbb{N} \cup \{0\}$ the Pochhammer's symbol

$$(\nu)_k \,:=\, \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \overset{k=1,2,\ldots}{=} (\nu+k-1)(\nu+k-2)\cdots(\nu+1)\nu\,, \tag{4.5}$$

we see that $p_\alpha(y)$ can be written in terms of Pochhammer's symbols:

$$p_\alpha(y) \,=\, -\frac{y}{2}\left(\sum_{k=0}^\infty \frac{y^k}{k!\,(-\alpha+1)_{k+1}2^{2k}}\right)\Bigg/\left(\sum_{k=0}^\infty \frac{y^k}{k!\,(-\alpha+1)_k 2^{2k}}\right). \tag{4.6}$$

From now the explicit determination of $p_\alpha(y)$ is elementary, but cumbersome (to the point of requiring symbolic computations). We give the first four terms

$$\begin{aligned} p_\alpha(y) \,&=\, \sum_{j=1}^4 c_j(\alpha)y^j + \ldots = \frac{1}{2(\alpha-1)}y - \frac{1}{8(\alpha-2)(\alpha-1)^2}y^2 + \\ &\frac{1}{16(\alpha-3)(\alpha-2)(\alpha-1)^3}y^3 - \frac{5\alpha-11}{128(\alpha-4)(\alpha-3)(\alpha-2)(\alpha-1)^4}y^4 + \ldots \end{aligned} \tag{4.7}$$

This completes the proof for the non integer case.

The case $\alpha = 0, 1, 2, \ldots$. We need to treat separately the case $\alpha = 0$ because it involves $K_{-1}(x)$, which however it is just $K_1(x)$, but it requires an ad hoc (much simpler) analysis. So we start off with the case $\alpha = 1, 2, \ldots$. From now α will be replaced by n and when we write α we mean a quantity that is not integer. By [32, 10.31.1] we have that for $n = 0, 1, 2, \ldots$

$$\left(\frac{x}{2}\right)^n K_n(x) \,=\, \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k} + \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^{2n}\log(x) + O\left(x^{2n}\right)\,, \tag{4.8}$$

and the sum as to be interpreted as empty if $n = 0$. For $n = 1, 2, \ldots$ we write the degree $2n-2$ polynomial in the right-hand side as a $n-1$ degree polynomial with argument x^2:

$$q_n\left(x^2\right) := \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{(n-k-1)!}{k!\,2^{2k}}\left(x^2\right)^k . \tag{4.9}$$

With this notation we have for $n = 1, 2, \ldots$

$$\begin{aligned}\frac{xK_{n-1}(x)}{K_n(x)} &= \frac{2(x/2)^2\frac{q_{n-1}(x^2)}{q_n(x^2)} + \frac{2(-1)^n}{(n-1)!}\frac{(x/2)^{2n}}{q_n(x^2)}\log x + O\left(x^{2n}\right)}{1 + O\left(x^{2n}|\log x|\right)} \\ &= \frac{x^2}{2}t_n(x^2) + \frac{2^{2-2n}(-1)^n}{((n-1)!)^2}x^{2n}\log x + O\left(x^{2n}\right) ,\end{aligned} \tag{4.10}$$

where $t_1(y) := 0$ and, for $n = 2, 3, \ldots$, $t_n(y)$ is the polynomial of degree $n-2$ given by the Taylor expansion of the rational function $q_{n-1}(y)/q_n(y)$.

We are therefore left with showing that the coefficients of the polynomial $yt_n(y)/2$ of degree $n-1$ coincide with the corresponding coefficients of the Taylor polynomial of $p_\alpha(y)$, when $\alpha = n$ (we can consider $n = 2, 3, \ldots$ becaue if $n = 1$ the polynomial is identically zero and our claim is trivially verified). In different terms, we have to show that if we set $\alpha = n = 2, 3, \ldots$ in (4.7) up to the degree $n-1$ (it is readily seen that the n-th Taylor coefficients diverges as $\alpha \to n$), then we obtain $yt_n(y)/2$. To prove this it is useful to remark that, in order to obtain the Taylor expansion of $p_\alpha(y)$ up to order $\lfloor\alpha\rfloor$ it is sufficient to expand the rational function

$$\widetilde{p}_\alpha(y) = -\frac{y}{2}\left(\sum_{k=0}^{\lfloor\alpha\rfloor-1}\frac{y^k}{k!\,(-\alpha+1)_{k+1}2^{2k}}\right)\Bigg/\left(\sum_{k=0}^{\lfloor\alpha\rfloor-1}\frac{y^k}{k!\,(-\alpha+1)_k2^{2k}}\right). \tag{4.11}$$

In this expression we can set $\alpha = n$ obtaining thus the rational function

$$\widetilde{p}_n(y) := -\frac{y}{2}\left(\sum_{k=0}^{n-2}\frac{y^k}{k!\,(-n+1)_{k+1}2^{2k}}\right)\Bigg/\left(\sum_{k=0}^{n-2}\frac{y^k}{k!\,(-n+1)_k2^{2k}}\right). \tag{4.12}$$

and Taylor coefficients up to degree $n-1$ of this function are precisely the limit for $\alpha \to n$ of the Taylor coefficients up to degree $n-1$ of $p_\alpha(y)$, cf. (4.11)-(4.7). We are left with showing that the coefficients up to degree $n-1$ of $\widetilde{p}_n(y)$ coincide with the corresponding coefficients of $yt_n(y)/2$. This is equivalent to showing that

$$-\left(\sum_{k=0}^{n-2}\frac{y^k}{k!\,(-n+1)_{k+1}2^{2k}}\right)\Bigg/\left(\sum_{k=0}^{n-2}\frac{y^k}{k!\,(-n+1)_k2^{2k}}\right) = \frac{q_{n-1}(y)}{q_n(y)} + O\left(y^{n-1}\right) \tag{4.13}$$

and this is implied by the stronger (non asymptotic) condition

$$\begin{aligned}-\left(\sum_{k=0}^{n-2}\frac{y^k}{k!\,(-n+1)_{k+1}2^{2k}}\right)\Bigg/\left(\sum_{k=0}^{n-2}\frac{y^k}{k!\,(-n+1)_k2^{2k}}\right) &= \\ \left(\sum_{k=0}^{n-2}(-1)^k\frac{(n-k-2)!}{k!\,2^{2k}}y^k\right)\Bigg/\left(\sum_{k=0}^{n-2}(-1)^k\frac{(n-k-1)!}{k!\,2^{2k}}y^k\right) &,\end{aligned} \tag{4.14}$$

For $n = 2, 3, \ldots$ this identity is verified directly by using that for $n = 1, 2, \ldots$ and $k \in 0, 1, \ldots$

$$(-n)_k = (-1)^k\frac{n!}{(n-k)!} , \tag{4.15}$$

and this completes the proof in the case of $\alpha = 1, 2, \ldots$.

We are left with the case $\alpha = 0$:

$$\frac{xK_{-1}(x)}{K_0(x)} = \frac{xK_1(x)}{K_0(x)} \overset{(4.8)}{\sim} \frac{1}{\log(1/x)}, \tag{4.16}$$

but this is easily improved by going back to [32, 10.31.1] and using

$$K_0(x) = -\log(x/2) - \gamma + O\left(x^2\right), \tag{4.17}$$

where γ is the Euler-Mascheroni constant. This, with (4.8) for $n = 1$, implies that $xK_{-1}(x)/K_0(x)$ is equal to $1/(\log(1/x) + (\log 2 - \gamma)) + O(x^2)$. This completes the proof of Proposition 1.3. □

5. Lyapunov exponent and singularities: the proof of Theorem 1.7

The proof of Theorem 1.7 is somewhat involved, since the ratio of Bessel functions in the integrand becomes quite singular at one end of the domain of integration. For every fixed $x > 0$ the numerator and denominator are entire functions of α, so the ratio is analytic apart from the zeros of the denominator; these are all on the imaginary axis and they are bounded away from the real axis as long as x is bounded away from zero ([18, Appendix A], Table 1). But integrating over x down to zero adds contributions that are less and less regular as the gap between the origin and the poles of the integrand shrinks when x becomes small.

x	n	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7
1	1	2.96						
1/10	3	1.14	2.04	2.85				
1/100	5	0.64	1.23	1.78	2.30	2.81		
1/1000	7	0.44	0.87	1.27	1.66	2.04	2.42	2.78

Table 1. The zeros of $K_\alpha(x)$ are all for $\alpha = i\nu$ with $\nu \in \mathbb{R}$. Since $K_{i\nu}(x) = K_{-i\nu}(x)$, we put in the table the set $\{\nu \in [0,3] : K_{i\nu}(x) = 0\} = \{\nu_1, \ldots, \nu_n\}$, with $\nu_j = \nu_j(x)$ and $n = n(x)$, for four values of x. The numerical values are rounded to the closest decimal.

One way to obtain a proof is to exploit once again the connection formula (4.1)-(4.2) which gives an expansion of both numerator and denominator: but controlling the ratio is of course not straightforward. And in fact McCoy and Wu approach the problem this way, but keeping only the *leading terms* of the series in the connection formula. The validity of this procedure is not obvious, since *a priori* the resulting correction could be less regular than the leading terms, but it is nonetheless helpful to begin by examining this simplified problem that has the nice feature of leading to a solution in terms of special functions, since we shall see that it correctly illustrates the main features of the proof. With this aim in mind we examine a simplified McCoy-Wu formula – this corresponds to studying a function $\widetilde{\mathrm{F}}$ which is defined, like F, as the integral over $x \in (0, \eta)$ of a suitable function $\widetilde{f}_x(\alpha)$ (see (5.4)) – and

(1) we perform the integration explicitly and discuss the regularity of $\alpha \mapsto \widetilde{\mathrm{F}}(\alpha)$;
(2) we then argue how understanding the location of (some of) the poles of $\widetilde{f}_x$ in the complex plane, and the corresponding residues, gives another way to understand the regularity.

All of this is done in Section 5.1. Then in Section 5.2 we give the proof of Theorem 1.7, based on a treatment of the poles of f_x.

5.1. **Heuristic arguments and idea of the proof.**

The simplified McCoy and Wu problem: exact solution. Much like McCoy and Wu did in [31, p. 642], we can consider the leading contribution to the integrand in (1.28) for $x \searrow 0$ and $\alpha \in \mathbb{C}$ tending also to zero. This step is an uncontrolled approximation that is obtained by keeping *the first terms* in (4.1)-(4.2) and by using $\Gamma(\alpha) \sim 1/\alpha$

$$K_{\alpha-1}(x) \simeq \frac{1}{2}\left(\frac{x}{2}\right)^{\alpha-1}, \tag{5.1}$$

and

$$K_{\alpha}(x) \simeq \frac{1}{2\alpha}\left(\left(\frac{x}{2}\right)^{-\alpha} - \left(\frac{x}{2}\right)^{\alpha}\right), \tag{5.2}$$

so

$$\frac{xK_{\alpha-1}(x)}{K_{\alpha}(x)} \simeq \frac{2\alpha}{\left(\frac{2}{x}\right)^{2\alpha} - 1} = \frac{2\alpha}{\exp(2\alpha L(x)) - 1} =: \widetilde{f}_x(\alpha), \tag{5.3}$$

with $L(x) := \log(2/x)$. We then have

$$\widetilde{\mathrm{F}}(\alpha) := \int_0^{\eta} \frac{2\alpha}{\exp(2\alpha L(x)) - 1}\, \mathrm{d}x. \tag{5.4}$$

for $\eta \in (0, 2)$, cf. [31, (4.44)]. From now onward the analysis is rigorous.

$\widetilde{\mathrm{F}}(\alpha)$ can then be made explicit up to an additive contribution that is analytic near the real axis: in fact if we set

$$\check{\mathrm{F}}(\alpha) := \int_0^{2} \left(\frac{2\alpha}{\exp(2\alpha L(x)) - 1} - \frac{1}{L(x)}\right) \mathrm{d}x, \tag{5.5}$$

we have

$$\widetilde{\mathrm{F}}(\alpha) = \check{\mathrm{F}}(\alpha) - \int_{\eta}^{2} \left(\frac{2\alpha}{\exp(2\alpha L(x)) - 1} - \frac{1}{L(x)}\right) \mathrm{d}x + \int_0^{\eta} \frac{1}{L(x)}\, \mathrm{d}x. \tag{5.6}$$

But $\int_0^{\eta}(1/L(x))\, \mathrm{d}x = \Gamma(0, \log(2/\eta))$ is just a constant and the second addend in the right-hand side is (real) analytic in α. In fact the integrand in is meromorphic with poles on the imaginary axis, precisely for α equal to any integer multiple of $\pm\pi/L(x)$. Therefore the second addend in the right-hand side is analytic for α in $\mathbb{C}\backslash\{iy : |y| \geqslant \pi/L(\eta)\}$. We can therefore focus on $\check{\mathrm{F}}(\alpha)$ and we start by observing that for $\alpha \in \mathbb{R}$

$$\check{\mathrm{F}}(-\alpha) = \check{\mathrm{F}}(\alpha) + 4\alpha, \tag{5.7}$$

as a result of a straightforward manipulation using $1/(1 - e^{-2\alpha L}) = 1/(e^{2\alpha L} - 1) + 1$.

Remark 5.1. *We note that the same argument can be applied directly to* $\widetilde{\mathrm{F}}_1(\alpha)$ *obtaining*

$$\widetilde{\mathrm{F}}(-\alpha) = \widetilde{\mathrm{F}}(\alpha) + 2\eta\alpha. \tag{5.8}$$

But in fact we have also

$$\mathrm{F}(-\alpha) = \mathrm{F}(\alpha) + 2\eta\alpha, \tag{5.9}$$

which follows by applying (1.13) *and* $K_\beta(x) = K_{-\beta}(x)$*. In other words all these functions –* $\check{\mathrm{F}}$*,* $\widetilde{\mathrm{F}}$ *and* F *– are even, up to a linear term.*

Thanks to (5.7) we can focus on the case $\Re\alpha > 0$ and compute (change the variable and move the contour of integration in the complex plane)

$$\breve{\mathrm{F}}(\alpha) \;=\; 2\int_0^\infty \left(\frac{1}{e^v-1}-\frac{1}{v}\right)e^{-v/(2\alpha)}\,\mathrm{d}v \;=\; -2\int_0^\infty \left(\frac{1}{v}-\frac{1}{1-e^{-v}}\right)e^{-v/(2\alpha)}\,\mathrm{d}v - 4\alpha$$
$$= -4\alpha - 2\log(2\alpha) - 2\psi(1/(2\alpha)) \overset{\alpha\searrow 0}{\sim} -2\alpha - \sum_{j=1}^\infty \frac{B_{2j}}{j}(2\alpha)^{2j}\,, \quad (5.10)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ and the notion of $\sim$ is extended here and it has to be interpreted in the sense of asymptotic series (i.e., that the difference of left-hand side and of the series in the right-hand side truncated to $j = n$ is $o(\alpha^{2n})$) : in the third step we have applied [32, 5.9.13] and the asymptotic relation [32, 5.11.2] $\psi(z) \overset{z\to+\infty}{\sim} \log z - 1/(2z) + \sum_{j=1}^\infty (B_{2j}/(2j))z^{2j}$: the rational numbers B_{2n} are the Bernoulli numbers [32, §24.2]. By [32, 24.9.8]

$$B_{2n} \overset{n\to\infty}{\sim} 2(-1)^{n+1}\frac{(2n)!}{(2\pi)^{2n}}\,, \quad (5.11)$$

so the series has radius of convergence zero. Note that (5.7) readily implies that $\breve{\mathrm{F}}(\alpha) \sim -2\alpha - \sum_{j=1}^\infty (B_{2j}/j)(2\alpha)^{2j}$ holds also for $\alpha \nearrow 0$ and not only for $\alpha \searrow 0$. Let us reorder what we have done (and more) into a statement:

Proposition 5.2. *$\breve{\mathrm{F}}(\cdot)$ is defined and analytic in the complex plane without the imaginary axis and it can be extended by continuity on the whole real axis by setting $\breve{\mathrm{F}}(0) = 0$. Then $\breve{\mathrm{F}}$ restricted to $\mathbb{R}$ is C^∞ in the origin. On the other hand, $\breve{\mathrm{F}}$ cannot be continued as an analytic function at any point on the imaginary axis.*

Proof. Let us first show that $\breve{\mathrm{F}}$ is C^∞ at the origin. For this it is more practical (and equivalent, since $\widetilde{\mathrm{F}}-\breve{\mathrm{F}}$ is real analytic) to go back to $\widetilde{\mathrm{F}}$ and observe that with $y = 2\alpha \in \mathbb{R}\backslash\{0\}$ and the change of variable $v = \log(2/x)$ we obtain

$$f(y) \;:=\; \widetilde{\mathrm{F}}(y/2) \;=\; 2\int_c^\infty q(yv)\frac{\exp(-v)}{v}\,\mathrm{d}v\,, \quad (5.12)$$

with $c = \log(2/\eta) \in (0,\infty)$ and $q(u) = \frac{u}{\exp(u)-1}$. It is straightforward to verify that $q(u) \in (0,1]$ for $u \geqslant 0$ and that $q(u) \in [1, u+1]$ for $u \leqslant 0$ and dominated convergence implies that f is C^0 (on the whole $\mathbb{R}$, but of course our attention is at the origin, outside we already know that f is real analytic). To show differentiability it suffices to observe that formally

$$f^{(n)}(y) \;=\; \int_c^\infty q^{(n)}(yv)v^{n-1}\exp(-v)\,\mathrm{d}v\,, \quad (5.13)$$

where $f^{(n)}(y) = (\mathrm{d}/\mathrm{d}y)^n f(y)$. But it is straightforward to verify that $q^{(1)}(u)$ is monotonically decreasing from 1 to zero and $q^{(n)}(u)$, $n = 2$ or larger, vanishes at $\pm\infty$. Hence $\sup_{u\in\mathbb{R}} |q^{(n)}(u)| < \infty$ and this implies that (5.13) is not just formal: $f \in C^\infty$ and its derivatives are given by (5.13).

On the other hand, $\breve{\mathrm{F}}$ is not analytic in zero since the radius of convergence of the series is zero: we record that if the Taylor series for $\breve{\mathrm{F}}(\alpha)$ is $\sum_n c_n\alpha^n$ then

$$c_{2n} \overset{n\to\infty}{\sim} 4(-1)^{n+1}\frac{(2n-1)!}{(\pi)^{2n}}\,, \quad (5.14)$$

while $c_{2n+1} = 0$ for $n = 1, 2, \ldots$: this follows directly from (5.10) and (5.11).

To conclude, we argue that $\check{\mathrm{F}}$ cannot be extended as an analytic function through the imaginary axis. Call $\mathrm{F}^+ : \{z \in \mathbb{C} : \Re(z) > 0\} \longrightarrow \mathbb{C}$ the function that coincides with $\check{\mathrm{F}}$ in its domain of definition (the right half-plane): keep in mind that $\check{\mathrm{F}}$ is defined also in the half plave with negative real part and it is analytic there. Suppose now that there exist $a \in \mathbb{R}$ and $\epsilon > 0$ such that we can define $\check{\mathrm{F}}(\alpha)$ for $\alpha \in \{it : |t-a| < \epsilon\}$, in such a way that $\check{\mathrm{F}}$ is analytic in a neighborhood of ia. We can and do assume that $a > 0$ as well as $a - \varepsilon > 0$ by symmetry and because we already know that $\check{\mathrm{F}}$ is not analytic at the origin. This extends analytically F^+ too. But (5.10) yields $\mathrm{F}^+(\alpha) = -4\alpha - \log 4 - 2\log\alpha - 2\psi(1/(2\alpha))$ in the right-half plane and this expression, containing the function ψ (that is meromorphic on the whole $\mathbb{C}$ and has poles on the negative real axis [32, §5.2]) and a logarithm which is defined and analytic on the whole complex plane except for a cut (that can be for example chosen to be $\{z \in \mathbb{C} : \Im z = \Re z \leqslant 0\}$). But F^+ must coincide with $\check{\mathrm{F}}$ in the whole region where F^+ is extended: this region includes the negative semi-axis where F^+ has poles and $\check{\mathrm{F}}$ is analytic. This is not possible, hence $\check{\mathrm{F}}$ cannot be extended analytically at any point on the imaginary axis. □

A different viewpoint on the McCoy and Wu simplified problem. We restart from (5.4), but we take a different approach: we avoid exact integration. As already noted, the integrand in that expression for $\widetilde{\mathrm{F}}(\alpha)$, for fixed $x > 0$, is a meromorphic function of α. The poles are on the imaginary axis and 0 is not a pole because the singularity is removable: the poles are $n\pi i/L(x)$ for $n \in \mathbb{Z}\backslash\{0\}$, where recall that we write $L(x) := \log(2/x)$ for brevity. The integrand in the expression for $\mathrm{F}_1(\alpha)$ is therefore analytic in a ball of radius $\pi/L(x)$ around zero and the most singular part in the residue expression for the integrand comes from the two closest poles, which are $\pm\pi i/L(x)$. If we compute the residues of these two poles for the integrand we find $\pm 2\pi i/(L(x))^2$ and therefore the contribution to the integrand of these two poles is

$$P_\alpha(x) := \frac{2\pi i}{L(x)^2\left(2\alpha - \frac{2\pi i}{L(x)}\right)} - \frac{2\pi i}{L(x)^2\left(2\alpha + \frac{2\pi i}{L(x)}\right)} = -\frac{2}{L(x)}\frac{1}{\left(\frac{\alpha L(x)}{\pi}\right)^2 + 1} = -2\sum_{n=0}^{\infty}(-1)^n\alpha^{2n}L(x)^{2n-1}\pi^{-2n}, \tag{5.15}$$

where the last equality holds only for $|\alpha L(x)/\pi| < 1$, but it is in any case useful to identify all the derivatives of $P_\alpha(x)$ at $\alpha = 0$. It seems reasonable to believe that the singularity at the origin of $\widetilde{\mathrm{F}}$ is induced by the poles of the integrand and that the two poles that are closest to the origin give the leading part of the singularity. If this is the case $\alpha \mapsto \int_0^\eta p_\alpha(x)\,\mathrm{d}x$ should capture the leading behavior of the singularity of $\widetilde{\mathrm{F}}$. This is confirmed or at least highly suggested by observing that

$$\int_0^\eta L(x)^{2n-1}\,\mathrm{d}x = 2\int_{\log(2/\eta)}^\infty y^{2n-1}e^{-y}\,\mathrm{d}y = 2\Gamma(2n, \log(2/\eta)) \overset{n\to\infty}{\sim} 2\Gamma(2n) = 2(2n-1)!, \tag{5.16}$$

so if we proceed at a completely formal level, by integrating term by term the series in the second line of (5.15) and using (5.16), we directly recover (5.14)!

Approaching the true problem. Going back to the true problem, that is F, for which the integrand is the left-had side of (5.3), we have to identify the zeros of $K_\alpha(x)$: this problem has been studied and, as possibly expected, the heuristics coming from studying the poles of the right-hand side of (5.3) is qualitatively correct, so the poles we have to study, or the zeros of $K_\alpha(x)$, are on the imaginary axis. Moreover they accumulate on the origin when $x \searrow 0$. In fact, to leading order (for n fixed and $x \searrow 0$) they are still in $\pm n\pi i/\log(2/x)$. But, more precisely, they are in $\pm n\pi i/(\log(2/x) - \gamma + o(1))$ as $x \searrow 0$ (γ the Euler constant, see below).

Remark 5.3. *The subleading correction $-\gamma$ to $\log(2/x)$ for the location of the zeros can be inserted in the heuristic argument that we presented just by being keeping one more term in the expansion of the Γ function in the first steps* (5.1)-(5.3)*. The change amounts simply to work with $L(x) = \log(2e^{-\gamma}/x)$, but this small offset in $L(x)$ leads to the multiplicative $e^{-\gamma}$ constant in the final asymptotic results, see Theorem 1.7.*

Apart for the quantitative issue of Remark 5.3, the proof of Theorem 1.7 requires taking care of two main issues:

(1) Even admitting that the two closest poles give the main contribution, we have to set-up a rigorous procedure corresponding to the formal argument that we developed using (5.15) and (5.16). This procedure requires controlling not only location of the two poles, but also the residues that this time are given by ratio of series coming from (5.18).
(2) One needs to control the effect of the poles that are farther from the origin: note that these terms will in any case give contributions that generate divergent series, but this time the distance of the poles from the origin is at least (about) twice the distance of the two $n = 1$ poles. Hence they will contain an exponential factor that is at least twice smaller.

5.2. **The proof.** The proof of Theorem 1.7, that starts here, follows a main line separated by a number of lemmas and corollaries. As a preliminary, we set out precise versions of some of the statements made above about the regularity of

$$f_x : \alpha \mapsto x\frac{K_{1-\alpha}(x)}{K_\alpha(x)}, \tag{5.17}$$

for $x > 0$. $K_\alpha(x)$ is entire as a function of α for all $x \neq 0$ [32, section 10.25(ii)], so f_x is a ratio of two entire functions, and thus it is analytic except at the zeros of $\alpha \mapsto K_\alpha(x)$. For $x > 0$, these zeros are all pure imaginary and located in the region $|\alpha| > x$ [18, Appendix A]. Note that this characterization is already sufficient to show that $\int_X^\eta f_x(\alpha)dx$ is analytic on $\mathbb{C}\setminus \pm i[X, \infty)$ for any $0 < X < \eta < \infty$, so it suffices to prove the statements in Theorem 1.7 with η replaced by some sufficiently small X in the integral defining $\textsc{f}$. Furthermore, f_x is infinitely differentiable on the real axis for any $x > 0$; in the next few lemmata we will bound its derivatives in such a way as to show that this is also true of the integral $\textsc{f}$.

To begin with, note that combining (4.1) and (4.2) gives

$$K_\alpha(x) = \frac{\pi}{2\sin\pi\alpha}\sum_{k=0}^{\infty}\frac{\left(x^2/4\right)^k}{k!}\left[\frac{(x/2)^{-\alpha}}{\Gamma(-\alpha+k+1)} - \frac{(x/2)^{\alpha}}{\Gamma(\alpha+k+1)}\right], \tag{5.18}$$

for $\alpha \in \mathbb{C}\setminus\mathbb{Z}$ and $x > 0$.

Lemma 5.4. *There exist $A, C_1, X > 0$ such that for all $x \in [0, X]$, $\alpha \in \mathbb{C}$ such that $|\alpha| \leqslant \frac{1}{2}$ and $|\Im\alpha| \log 1/x \leqslant A$,*

$$\left|\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right| \geqslant C_1 \, |\alpha| \, x^{-|\Re\alpha|}. \tag{5.19}$$

Proof. Letting $g_x(\alpha) := x^{\alpha}/\Gamma(1+\alpha)$,

$$\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} = g_x(\alpha) - g_x(-\alpha) = \alpha \int_{-1}^{1} g_x'(y\alpha) \, \mathrm{d}y \,, \tag{5.20}$$

and so

$$\left|\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right| \geqslant |\alpha| \left|\int_{-1}^{1} \Re g_x'(y\alpha) \, \mathrm{d}y\right| . \tag{5.21}$$

Note that

$$g_x'(\alpha) = [\log x - \psi(1+\alpha)] \frac{x^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.22}$$

where $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the Psi function [32, §5.2]. $\psi(1+\alpha)/\Gamma(1+\alpha)$ is analytic for $|\alpha| < 1$, so choosing X small enough we can obtain $|\psi(1+\alpha)/\Gamma(1+\alpha)| \leqslant (1-c_1)$ for any fixed $c_1 \in (0,1)$, whence

$$\left|\Re g_x'(\alpha) - \Re\frac{x^{\alpha}\log x}{\Gamma(1+\alpha)}\right| \leqslant \left|g_x'(\alpha) - \frac{x^{\alpha}\log x}{\Gamma(1+\alpha)}\right| = \left|\psi(1+\alpha)\frac{x^{\alpha}}{\Gamma(1+\alpha)}\right| \leqslant (1-c_1) x^{\Re\alpha} \log 1/x, \tag{5.23}$$

and noting that $1/\Gamma(1+\alpha)$ is entire and takes positive real values for $\alpha \in [-1/2, 2/2]$, by choosing A small enough we obtain

$$-\Re\frac{x^{\alpha}\log x}{\Gamma(1+\alpha)} = x^{\Re\alpha}\log 1/x \left[\cos\left(\Im\alpha\log x\right)\Re\frac{1}{\Gamma(1+\alpha)} - \sin\left(\Im\alpha\log x\right)\Im\frac{1}{\Gamma(1+\alpha)}\right] \geqslant c_2 x^{\Re\alpha}\log 1/x \geqslant 0 \,, \tag{5.24}$$

for some $c_2 > 0$. Combining the above observations, we see that

$$\left|\int_{-1}^{1} \Re g_x'(y\alpha) \, \mathrm{d}y\right| \geqslant c_1 c_2 \log 1/x \int_{-1}^{1} x^{y\Re\alpha} \, \mathrm{d}y = c_1 c_2 \frac{x^{-\Re\alpha} - x^{\Re\alpha}}{\Re\alpha}. \tag{5.25}$$

The last expression is an even function of $\Re\alpha$, so without loss of generality we can consider $\Re\alpha = \rho \in [0, 1/2]$. Noting that $\rho \mapsto 1 - x^{2\rho} = 1 - \exp\left(-2\rho\log x\right)$ is concave for all $x > 0$ and checking the boundary cases, it is easy to see that $1 - x^{2\rho} \geqslant \rho$ for all $x \in (0, 1/2)$, $\rho \in [0, 1/2]$. Using this together with (5.21) and (5.25), we obtain

$$\left|\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right| \geqslant c_1 c_2 |\alpha| x^{-|\Re\alpha|}. \tag{5.26}$$

This completes the proof of Lemma 5.4 □

Lemma 5.5. *There exist some $X \in (0,1)$, $A > 0$, and $C_2 > 0$ such that*

$$|f_{\alpha}(x)| \leqslant C_2 \,, \tag{5.27}$$

for all $x \in (0, X]$, $|\alpha| \leqslant 1/2$, and $|\Im\alpha| \leqslant A/\log(2/x)$.

Proof. Rearranging (5.18), we have

$$\begin{aligned} -\frac{2\sin\pi\alpha}{\pi}K_{1-\alpha}(x) &= \frac{(x/2)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=1}^{\infty}\frac{(x/2)^{\alpha+2k-1}}{k!\Gamma(\alpha+k)} - \sum_{k=1}^{\infty}\frac{k}{k-\alpha}\frac{(x/2)^{-\alpha+2k-1}}{k!\Gamma(k-\alpha)} \\ &= \frac{(x/2)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=1}^{\infty}\frac{(x/2)^{2k-1}}{k!}\left[\frac{(x/2)^{\alpha}}{\Gamma(k+\alpha)} - \frac{k}{k-\alpha}\frac{(x/2)^{-\alpha}}{\Gamma(k-\alpha)}\right], \end{aligned} \tag{5.28}$$

Noting that the final sum vanishes term by term when $\alpha = 0$ and that

$$\frac{\partial}{\partial\alpha}\left[\frac{(x/2)^{\alpha}}{\Gamma(k+\alpha)}\right] = \frac{(x/2)^{\alpha}}{\Gamma(k+\alpha)}\left[\log(x/2) - \psi(k+\alpha)\right], \tag{5.29}$$

and

$$\frac{\partial}{\partial\alpha}\left[\frac{(x/2)^{-\alpha}}{\Gamma(k-\alpha)}\frac{k}{k-\alpha}\right] = \frac{\partial}{\partial\alpha}\left[\frac{k(x/2)^{-\alpha}}{\Gamma(1+k-\alpha)}\right] = -\frac{k(\frac{x}{2})^{-\alpha}}{\Gamma(1+k-\alpha)}\left[\log\left(\frac{x}{2}\right) + \psi(1+k-\alpha)\right], \tag{5.30}$$

and that $|\psi(k+\alpha)| \vee |\psi(1+k-\alpha)| \leqslant \text{const.}\log(k+1)$ for $k \geqslant 1$ and $|\alpha| \leqslant 1/2$, we then have also

$$\begin{aligned} &\left|\frac{(x/2)^{\alpha}}{\Gamma(k+\alpha)} - \frac{k}{k-\alpha}\frac{(x/2)^{-\alpha}}{\Gamma(k-\alpha)}\right| \leqslant \\ &\qquad \text{const.}\,|\alpha|\left[\left|\frac{(x/2)^{-\alpha}}{\Gamma(k+\alpha)}\right| \vee \left|\frac{k(x/2)^{\alpha}}{\Gamma(k-\alpha)}\right|\right]\left[\log\left(\frac{x}{2}\right) + \log(k+1)\right]. \end{aligned} \tag{5.31}$$

Noting that $\Gamma(1+k\pm\alpha) = \Gamma(1\pm\alpha)(1\pm\alpha)_k$ (see (4.5)),

$$|(1\pm\alpha)_k| = |1\pm\alpha|\cdots|k\pm\alpha| \geqslant \left(\frac{1}{2}\right)\cdots\left(\frac{2k-1}{2}\right) = 2^{-k}\,(2k-1)!!, \tag{5.32}$$

and that $|\Gamma(1\pm\alpha)|$ is bounded for $|\alpha| \leqslant 1/2$, we have

$$\begin{aligned} &\left|\sum_{k=1}^{\infty}\frac{(x/2)^{2k-1}}{k!}\left[\frac{(x/2)^{\alpha}}{\Gamma(k+\alpha)} - \frac{k}{k-\alpha}\frac{(x/2)^{-\alpha}}{\Gamma(k-\alpha)}\right]\right| \\ &\quad \leqslant \text{const.}|\alpha|\sum_{k=1}^{\infty}\frac{(x/2)^{2k-1}}{k!}\left[\log(k+1) + \log(2/x)\right]\left[\left|\frac{(x/2)^{-\alpha}}{\Gamma(k+\alpha)}\right| \vee \left|\frac{k(x/2)^{\alpha}}{\Gamma(k-\alpha)}\right|\right] \\ &\quad \leqslant \text{const.}|\alpha|(x/2)^{1-|\Re\alpha|}\left[\sum_{k=0}^{\infty}\frac{2^{k+1}\log(k+2)}{k!(2k+1)!!}(x/2)^{2k} + \log(2/x)\sum_{k=0}^{\infty}\frac{2^{k+1}(x/2)^{2k}}{k!(2k+1)!!}\right] \\ &\quad \leqslant \text{const.}|\alpha|\left[1 + \log(2/x)\right](x/2)^{1-|\Re\alpha|}, \end{aligned} \tag{5.33}$$

where the last bound follows from the observation that each of the sums in the preceeding expression is α-independent and defines an entire function of x, and is therefore bounded on any compact interval. Combining this with (5.28) and noting that

$$\left|\frac{(x/2)^{\alpha-1}}{\Gamma(\alpha)}\right| \leqslant \text{const.}|\alpha|(x/2)^{\Re\alpha-1}, \tag{5.34}$$

we have

$$\left|\frac{2\sin\pi\alpha}{\pi}K_{1-\alpha}(x)\right| \leqslant \text{const.}|\alpha|(x/2)^{\Re\alpha-1}. \tag{5.35}$$

As for the denominator, using (5.18)

$$\left|\frac{2\sin\pi\alpha}{\pi}K_\alpha(x)\right| \geqslant \left|\frac{(x/2)^\alpha}{\Gamma(1+\alpha)} - \frac{(x/2)^{-\alpha}}{\Gamma(1-\alpha)}\right| - \sum_{k=1}^{\infty}\frac{(x/2)^{2k}}{k!}\left|\frac{(x/2)^\alpha}{\Gamma(k+1+\alpha)} - \frac{(x/2)^{-\alpha}}{\Gamma(k+1-\alpha)}\right| . \tag{5.36}$$

The sum can be estimated in the same way as the one in (5.28): we have

$$\left|\frac{(x/2)^\alpha}{\Gamma(k+1+\alpha)} - \frac{(x/2)^{-\alpha}}{\Gamma(k+1-\alpha)}\right| \leqslant \text{const.}\frac{2^k\left[\log(k+1)+\log(2/x)\right]}{(2k-1)!!}|\alpha|\left(\frac{x}{2}\right)^{-|\Re\alpha|} , \tag{5.37}$$

and so

$$\sum_{k=1}^{\infty}\frac{(x/2)^{2k}}{k!}\left|\frac{(x/2)^\alpha}{\Gamma(k+1+\alpha)} - \frac{(x/2)^{-\alpha}}{\Gamma(k+1-\alpha)}\right| \leqslant \text{const.}|\alpha|x^{2-|\Re\alpha|}\log(1/x) , \tag{5.38}$$

and we see that this is dominated by the first term, which was bounded from below in Lemma 5.4 above. We then have

$$\left|\frac{2\sin\pi\alpha}{\pi}K_\alpha(x)\right| \geqslant \text{const.} \times |\alpha|\,(x/2)^{-|\Re\alpha|} , \tag{5.39}$$

and combining this with (5.35) we obtain the desired bound and the proof of Lemma 5.5 is complete. □

Letting $C_R(w)$ denote the oriented circle of radius R about w, the Cauchy formula implies that

$$\left|f^{(n)}(w)\right| = \frac{n!}{2\pi}\left|\oint_{C_R(w)}\frac{f(z)}{(z-w)^{n+1}}dz\right| \leqslant \frac{n!}{R^n}\max_{|z-w|=R}|f(z)| , \tag{5.40}$$

for any f which is analytic on an open set containing $C_R(w)$ and its interior. From Lemma 5.5 we thus have

Corollary 5.6. *For the same A, X, C_2 as in Lemma 5.5,*

$$\left|\frac{\partial^n}{\partial a^n}\left[x\frac{K_{1-a}(x)}{K_a(x)}\right]\right| \leqslant C_2 n!\left(\frac{\log(2/x)}{A}\right)^n \tag{5.41}$$

for all $n \in \mathbb{N}$, $a \in [-1/4, 1/4]$.

Since the bounds in Corollary 5.6 are uniformly (in a) integrable (in x), if we take $\eta \leqslant X$ this allows us to take derivatives inside the integral defining F which is therefore infinitely differentiable on the real interval $(-1/4, 1/4)$. This result is going to be crucial for us at 0: the fact that F is C^∞ outside of zero is also a byproduct of the fact that we are going to establish (via the next lemma) that F is real analytic in $(-1,1)\backslash\{0\}$.

Lemma 5.7. *For any $A \in (0, 1/2)$ and $I \in (0, \infty)$, there exist $X \in (0, 2)$ and $C \in (0, \infty)$ such that*

$$|f_\alpha(x)| \leqslant C , \tag{5.42}$$

whenever $x \in (0, X]$, $\Re\alpha \in (A, 1-A)$, and $|\Im\alpha| \leqslant I$.

Proof. Using [32, 5.6.7] and noting that $\Gamma(x) > 1/2$ for all $x > 0$,

$$\left|\frac{1}{\Gamma(z)}\right| \leqslant \frac{(\cosh \pi \Im z)^{1/2}}{\Gamma(\Re z)} \leqslant 2\,(\cosh \pi \Im z)^{1/2}\ , \tag{5.43}$$

whenever $\Re z > 0$. Also using [32, 5.6.6] and noting that the Gamma function is concave for positive real arguments,

$$\left|\frac{1}{\Gamma(2-\alpha)}\right| \geqslant \frac{1}{\Gamma(2-\Re\alpha)} \geqslant \frac{1}{\Gamma(1)} \vee \frac{1}{\Gamma(2)} = 1. \tag{5.44}$$

Applying these two bounds to (5.18), we obtain

$$\begin{aligned}\left|\frac{\pi \sin \pi\alpha}{\pi} K_\alpha(x)\right| &= \left|\frac{(x/2)^{-\alpha}}{\Gamma(1-\alpha)} + \left(\frac{x}{2}\right)^{2-\alpha} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k+1)!\Gamma(k+2-\alpha)}\right. \\ &\qquad \left. - \left(\frac{x}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\Gamma(k+1+\alpha)}\right| \\ &\geqslant |\alpha| \frac{(x/2)^{-\Re\alpha}}{|\Gamma(2-\alpha)|} - 2\left[\left(\frac{x}{2}\right)^{2-\Re\alpha} + \left(\frac{x}{2}\right)^{\Re\alpha}\right] (\cosh \pi I)^{1/2} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!} \\ &\geqslant A\left(\frac{X}{2}\right)^{-A} - 4\left(\frac{X}{2}\right)^{A} \exp\left(\frac{x^2}{4}\right) (\cosh \pi I)^{1/2}\ ,\end{aligned} \tag{5.45}$$

for all relevant α and x; choosing X sufficiently small, the last bound can be made positive.

Similarly, noting also that $\Gamma(x)$ is negative (resp. positive) and decreasing for $x \in (-1/2, 0)$ (resp. $(0, 1/2)$),

$$\begin{aligned}\left|\frac{\pi \sin \pi\alpha}{\pi} K_{1-\alpha}(x)\right| &= \left|\frac{(x/2)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(x/2)^{1-\alpha}}{\Gamma(-\alpha)} + \left(\frac{x}{2}\right)^{\alpha+1} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k+1)!\Gamma(k+1+\alpha)}\right. \\ &\qquad \left. - \left(\frac{x}{2}\right)^{3-\alpha} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k+1)!\Gamma(k+1-\alpha)}\right| \\ &\leqslant 2\,(\cosh \pi I)^{1/2} \left\{\frac{(x/2)^{A-1}}{\Gamma(A)} - \frac{(x/2)^{A}}{\Gamma(-A)} + \left[\left(\frac{x}{2}\right)^{A+1} + \left(\frac{x}{2}\right)^{A+2}\right] \exp\left(\frac{x^2}{4}\right)\right\},\end{aligned} \tag{5.46}$$

and combining this with (5.45) we obtain a suitable bound on $|f_\alpha(x)|$. This completes the proof of Lemma 5.7. □

By Lemma 5.7 we have that for any $\alpha \in \mathbb{C}$ with $\Re\alpha \in (0,1)$, we can choose A, I to obtain such a bound on a neighborhood of α; using the Cauchy formula this impies that f'_x is uniformly bounded on some smaller neighborhood of α, which allows us to exchange differentiation and integration to see that F is holomorphic on that neighborhood. We can then conclude that F is analytic on $\{\alpha \in \mathbb{C} | \Re\alpha \in (0,1)\}$, and by the symmetry noted in (5.9) it is also analytic on $\{\alpha \in \mathbb{C} | \Re\alpha \in (-1,0)\}$.

All that remains is to show that the derivatives of F at the origin grow as stated; since this will imply that the associated Taylor series is divergent, this will also prove that F is not analytic there. We begin by providing a more precise characterization of the poles of f_x for small x.

Lemma 5.8. *There exist $X, C > 0$ and a sequence of functions $\nu_n : (0,\infty) \mapsto (0,\infty)$, satisfying*

$$\left|\nu_n(x) - \frac{n\pi}{\log(2/x) - \gamma}\right| \leqslant \frac{Cn^3}{(\log x)^4} \tag{5.47}$$

and $\nu_1(x) < \nu_2(x) < \dots$, such that for all $x \in (0, X]$, $K_\alpha(x) = 0$ iff $\alpha = \pm i\nu_n(x)$.

Proof. For $x > 0$ and $\nu \in \mathbb{R}$, (5.18) can be rewritten using some properties of the Gamma function ([32, 5.2.5] and [32, 5.4.3]) as

$$K_{i\nu}(x) = -\left(\frac{\pi}{\nu \sinh(\pi\nu)}\right)^{1/2} \sum_{k=0}^{\infty} \frac{\left(x^2/4\right)^k}{k!} \frac{\sin(\theta_k(\nu))}{\sqrt{(1^2+\nu^2)\dots(k^2+\nu^2)}}, \tag{5.48}$$

where

$$\theta_k(\nu) := \nu \log(x/2) - \arg\Gamma(1+k+i\nu), \tag{5.49}$$

(cf. [14, (2.7-8)], where this expression is used to study the x-zeroes and their dependence on ν).

Then the solutions of $K_{i\nu}(x) = 0$ are the nonzero solutions of

$$\sin\theta_0(\nu) = S_x(\nu) := -\sum_{k=1}^{\infty} \frac{\left(x^2/4\right)^k}{k!} \frac{\sin(\theta_k(\nu))}{\sqrt{(1^2+\nu^2)\dots(k^2+\nu^2)}}. \tag{5.50}$$

Using the definition of the ψ function and the expansion [32, 5.7.6],

$$\theta_0'(\nu) = \log(x/2) - \Re\psi(1+i\nu) = \log(x/2) + \gamma - \sum_{m=1}^{\infty} \frac{\nu^2}{m(m^2+\nu^2)} \leqslant \log(x/2) + \gamma, \tag{5.51}$$

so for $0 < x < 2e^{-\gamma} = 1.1229\dots$ θ_0 is strictly decreasing. From its definition in (5.50), it is apparent that S_x can be bounded

$$|S_x(\nu)| \leqslant \sum_{k=1}^{\infty} \frac{\left(x^2/4\right)^k}{(k!)^2} \leqslant \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} \leqslant \cosh x - 1, \tag{5.52}$$

uniformly in ν, and so we see that for each $0 < x < \cosh^{-1}(2) = 1.317\dots$ all solutions ν of (5.50) satisfy

$$-\theta_0(\nu) \in [n\pi - \sin^{-1}(\cosh x - 1), n\pi + \sin^{-1}(\cosh x - 1)]. \tag{5.53}$$

for some integer n, and there is at least one solution for each n. More precisely, for such solutions the derivative of the left hand side of (5.50) satisfies

$$(-1)^{n+1}\frac{\partial}{\partial\nu}\sin(\theta_0(\nu)) = (-1)^{n+1}\theta_0'(\nu)\cos(\theta_0(\nu)) \geqslant |\theta_0'(\nu)|\sqrt{1-(\cosh x - 1)^2}, \tag{5.54}$$

and in light of (5.51), for X small enough this can be bounded from below by any positive number uniformly in ν. As for the right-hand side of (5.50), first note that from [32, 5.5.2] we have

$$\psi(1+k+\alpha) = \psi(1+\alpha) + \sum_{m=1}^{k} \frac{1}{m+\alpha}, \tag{5.55}$$

for all $k \in \mathbb{N}, \alpha \in \mathbb{C}$, and so

$$\begin{aligned} \left|\theta_k'(\nu) - \theta_0'(\nu)\right| &= |\Re\psi(k+1+\alpha) - \Re\psi(1+\alpha)| \leqslant |\psi(k+1+\alpha) - \psi(1+\alpha)| \\ &\leqslant \sum_{m=1}^{k} \frac{1}{m} \leqslant \int_1^{k+1} \frac{dm}{m} = \log(k+1). \end{aligned} \tag{5.56}$$

Using this, we have

$$\begin{aligned}\left|\frac{\partial}{\partial\nu}\frac{\sin(\theta_k(\nu))}{\sqrt{(1^2+\nu^2)\dots(k^2+\nu^2)}}\right| &= \left|\frac{\theta_k'(\nu)\cos\theta_k(\nu)-\sum_{m=1}^k\frac{\nu}{m^2+\nu^2}\sin\theta_k(\nu)}{\sqrt{(1^2+\nu^2)\dots(k^2+\nu^2)}}\right| \\ &\leqslant \frac{|\theta_0'(\nu)|+2\log(k+1)}{k!}\leqslant\frac{|\theta_0'(\nu)|+2}{(k-1)!},\end{aligned} \tag{5.57}$$

which implies

$$\left|S_x'(\nu)\right|\leqslant\sum_{k=1}^\infty\frac{(x^2/4)^{2k}}{k!}\frac{|\theta_0'(\nu)|+2}{(k-1)!}\leqslant\left[\left|\theta_0'(\nu)\right|+2\right]\sum_{k=1}^\infty\frac{x^{2k}}{(2k-2)!}\leqslant\left[\left|\theta_0'(\nu)\right|+2\right]x^2\cosh x, \tag{5.58}$$

which, for x small enough, is smaller than the right-hand side of (5.54) for all ν. This suffices to show that there is only one solution of (5.50) in each of the intervals in (5.53). The solution for $n=0$ must be the trivial solution $\nu=0$ which does not correspond to a solution of $K_{i\nu}(x)=0$. This uniqueness also implies that the solutions for negative and positive n are related by the symmetry $K_{i\nu}(x)=K_{-i\nu}(x)$, so we see that it is possible to relate the zeros to a family of functions as desired.

To see that the functions ν_n satisfy the bound (5.47), we first note that (5.53) and (5.51) together imply that

$$\nu_n(x)\leqslant\frac{n\pi+\cosh x-1}{\inf_{\nu\geqslant0}|\theta_0'(\nu)|}\leqslant\frac{n\pi+\cosh x-1}{\log(2/x)-\gamma}\leqslant\text{const.}\frac{n}{\log 1/x}\,, \tag{5.59}$$

for all $x\in(0,X)$; and also that using the same expansion as in (5.51) we have

$$|\gamma+\psi(1+i\nu)|\leqslant\sum_{m=1}^\infty\frac{\nu^2}{m(m^2+\nu^2)}\leqslant\nu^2\zeta(3)\text{ and so }|\gamma\nu+\arg\Gamma(1+i\nu)|\leqslant\text{const.}|\nu|^3\,, \tag{5.60}$$

for all $\nu\in\mathbb{R}$, where $\zeta(s):=\sum_{m=1}^\infty m^{-s}$ is the Riemann zeta function [32, Section 25.2]. Then recalling the definition of θ_0, this imples

$$|[\log(x/2)-\gamma]\,\nu_n(x)-\theta_0(\nu_n(x))|=|\gamma\nu_n(x)+\arg\Gamma(1+i\nu_n(x))|\leqslant\text{const.}\frac{n^3}{(\log 1/x)^3}\,, \tag{5.61}$$

and restating (5.53) as $|\theta_0(\nu_n(x))-n\pi|\leqslant\text{const. }x^2$ this gives the desired bound and the proof of Lemma 5.8 is complete. □

We denote the residue of f_x at $\pm i\nu_n(x)$ by $\pm R_n(x)$. Letting

$$\begin{aligned}\widetilde{K}_\alpha(x):=\frac{\pi}{2\sin\pi\alpha}\sum_{k=0}^\infty\Bigg\{&\frac{(x/2)^\alpha}{\Gamma(k+1+\alpha)}\left[\log(x/2)-\psi(k+1+\alpha)\right]+ \\ &\frac{(x/2)^{-\alpha}}{\Gamma(k+1-\alpha)}\left[\log(x/2)-\psi(k+1-\alpha)\right]\Bigg\},\end{aligned} \tag{5.62}$$

we have $\frac{\partial}{\partial\alpha}K_\alpha(x)=\widetilde{K}_\alpha(x)-\pi\cot(\pi\alpha)K_\alpha(x)$, and so

$$R_n(x)=x\frac{K_{1-i\nu_n(x)}(x)}{\widetilde{K}_{i\nu_n(x)}(x)}. \tag{5.63}$$

Noting that from Lemma 5.8 and [32, 5.7.4] we have

$$\begin{aligned}\left(\frac{x}{2}\right)^{\pm i\nu_1(x)} &= -1 \pm i\pi\frac{\gamma}{\log(2/x)-\gamma} + O\left(\frac{1}{|\log x|^4}\right),\\ \frac{1}{\Gamma(1\pm i\nu_1)} &= 1 \pm i\gamma\nu_1(x) + O\left(\frac{1}{|\log x|^2}\right),\\ \psi(1\pm i\nu_1(x)) &= -\gamma \pm \zeta(2)\nu_1(x) + O\left(\frac{1}{|\log x|^2}\right),\end{aligned} \tag{5.64}$$

paying attention to cancellations, (5.62) gives

$$\widetilde{K}_{i\nu_1(x)}(x) = i\frac{L^2(x)}{\pi} + O\left(1\right), \tag{5.65}$$

where the $k = 1, 2, \ldots$ terms in the sum are bounded in the same way as the similar sum appearing in the proof of Lemma 5.5. Similarly, noting that

$$1/\Gamma(-i\nu_1(x)) \;=\; -i\nu_1(x)[1 - i\gamma\nu_1(x) + O(\nu_1^2(x)]\,, \tag{5.66}$$

(from [32, 5.7.1]) and expanding $K_{1-\alpha}(x)$ as in (5.28), we have

$$K_{1-i\nu_1(x)}(x) = \frac{1}{x} + O\left(\frac{1}{x|\log x|^2}\right), \tag{5.67}$$

taking advantage of a cancellation between the subleading terms in $(x/2)^{1+i\nu_1(x)}$ and $1/\Gamma(1 + i\nu_1(x))$, and so

$$R_1(x) = -i\frac{\pi}{[\log(2/x)-\gamma]^2} + O\left(\frac{1}{|\log x|^4}\right). \tag{5.68}$$

Lemma 5.9. *For any $a \in (1, 2)$, There exist $X_a, C_a > 0$ such that*

$$\left|x\frac{K_{1-\alpha}(x)}{K_\alpha(x)} + 2iR_1(x)\frac{\nu_1(x)}{\alpha^2+\nu_1^2(x)}\right| < \frac{C_a}{|\log 2/x - \gamma|}\,, \tag{5.69}$$

whenever $x \in (0, X_a]$ and $|\alpha| = a(\pi/|\log(2/x) - \gamma|)$.

Proof. From Lemma 5.8 we see that we can choose $X_a < 2e^{-\gamma}$ such that

$$\nu_2(x) \;>\; a(\pi/|\log(2/x)-\gamma|) \;>\; \nu_1(x)\,, \tag{5.70}$$

for all $x \in (0, X_a]$; then the quantity to be bounded is a continuous function of both α and x for all relevant values except $x = 0$, so we need only check that

$$\limsup_{x\searrow 0}\left|xL(x)\frac{K_{1-\widetilde{\alpha}/L(x)}(x)}{K_{\widetilde{\alpha}/L(x)}(x)} + 2iL(x)R_1(x)\frac{\nu_1(x)}{\left(\frac{\widetilde{\alpha}}{L(x)}\right)^2+\nu_1^2(x)}\right|, \tag{5.71}$$

is bounded uniformly for $|\widetilde{\alpha}| = a\pi$, where for brevity $L(x) := \log 2/x - \gamma$; in fact we will show that both terms in the sum are suitably bounded. In fact

$$\lim_{x\searrow 0}\left(\frac{x}{2}\right)^{\widetilde{\alpha}/\log x} \;=\; e^{\widetilde{\alpha}} \quad\text{and}\quad \lim_{x\searrow 0}\Gamma\left(1+\frac{\widetilde{\alpha}}{\log x}\right) \;=\; 1\,, \tag{5.72}$$

and with (5.18) and (5.52) this implies that

$$\lim_{x\searrow 0}\frac{K_{\widetilde{\alpha}/L(x)}(x)}{L(x)} \;=\; \frac{e^{\widetilde{\alpha}}-e^{-\widetilde{\alpha}}}{2\widetilde{\alpha}}\,. \tag{5.73}$$

Noting that $\lim \Gamma(\widetilde{\alpha}/L(x))/L(x) = 1/\widetilde{\alpha}$ we also have

$$xK_{1-\widetilde{\alpha}/L(x)}(x) \overset{x\searrow 0}{\sim} x\frac{L(x)}{2\widetilde{\alpha}}\frac{(x/2)^{\frac{\widetilde{\alpha}}{L(x)}-1}}{\Gamma\left(\frac{\widetilde{\alpha}}{L(x)}\right)} \sim e^{\widetilde{\alpha}}; \tag{5.74}$$

then

$$\lim_{x\searrow 0} xL(x)\frac{K_{1-\widetilde{\alpha}/L(x)}(x)}{K_{\widetilde{\alpha}/L(x)}(x)} = \frac{2\widetilde{\alpha}}{e^{2\widetilde{\alpha}}-1}, \tag{5.75}$$

which, recalling $|\widetilde{\alpha}| = a\pi \in (\pi, 2\pi)$, is indeed uniformly bounded.

Recalling (5.68), we have

$$\lim_{x\searrow 0} L^2(x)R_1(x) = -i\pi\,, \tag{5.76}$$

and thus

$$\lim_{x\searrow 0} 2iL(x)R_1(x)\frac{\nu_1(x)}{\left(\frac{\widetilde{\alpha}}{L(x)}\right)^2 - \nu_1^2(x)} = 2\frac{\pi^2}{\widetilde{\alpha}^2-\pi^2}\,, \tag{5.77}$$

which is also uniformly bounded in a suitable fashion. Hence (5.71) is proven and therefore also the proof of Lemma 5.9 is complete. □

Noting that

$$-2iR_1(x)\frac{\nu_1(x)}{\alpha^2+\nu_1^2(x)} = \frac{R_1(x)}{\alpha - i\nu_1(x)} - \frac{R_1(x)}{\alpha + i\nu_1(x)}, \tag{5.78}$$

the expression examined above is an analytic function of α in the interior of the circles under consideration apart from removable singularities, and so using Lemma 5.9 and (5.40) we have

Corollary 5.10. *For any $a \in (1, 2)$, There exist some $C_a, X_a > 0$ and a sequence of functions $I_n : (0, X_a] \to \mathbb{C}$ such that*

$$x\frac{K_{1-\alpha}(x)}{K_\alpha(x)} = \sum_{n=0}^{\infty} I_n(x)\alpha^n - 2iR_1(x)\frac{\nu_1(x)}{\alpha^2+\nu_1^2(x)}\,, \tag{5.79}$$

whenever $x \in (0, X_a]$ and $|\alpha| \leqslant a(\pi/|\log x|)$, and

$$|I_n(x)| \leqslant C\left(\frac{|\log(2/x)-\gamma|}{a\pi}\right)^{n-1}\,, \tag{5.80}$$

for all n.

We then have

$$\frac{1}{n!}\frac{\partial^n}{\partial\alpha^n}\left[x\frac{K_{1-\alpha}(x)}{K_\alpha(x)}\right]\Bigg|_{\alpha=0} = I_n(x) - \begin{cases}(-1)^{n/2}\,2i\frac{R_1(x)}{(\nu_1(x))^{n+1}}, & n \text{ even}\\ 0, & n \text{ odd.}\end{cases} \tag{5.81}$$

From (5.47) and (5.68) we have that for any $\epsilon < 2e^{-1-\gamma}$ there exist finite C_R, C_ν such that

$$\left|R_1(x) + \frac{i\pi}{L^2(x)}\right| \leqslant \frac{C_R}{L^4(x)} \text{ and } \left|\frac{1}{\nu_1(x)} - \frac{L(x)}{\pi}\right| \leqslant \frac{C_\nu}{\pi L^2(x)}\,, \tag{5.82}$$

for all $x \in (0, \epsilon]$, and so

$$\left|\frac{R_1(x)}{(\nu_1(x))^{n+1}} + i\frac{L^{n-1}(x)}{\pi^n}\right| = \left|R_1(x)\left(\frac{L(x)}{\pi} + \left(\frac{1}{\nu_1(x)} - \frac{L(x)}{\pi}\right)\right)^{n+1} - i\frac{L^{n-1}(x)}{\pi^n}\right|$$
$$\leqslant \left|R_1(x)\left(\frac{L(x)}{\pi}\right)^{n+1} - i\frac{L^{n-1}(x)}{\pi^n}\right| + |R_1(x)|\frac{1}{\pi^{n+1}}\sum_{m=1}^{n+1}\binom{n+1}{m}L(x)^{n+1-3m}C_\nu^m$$
$$\leqslant \frac{C_R}{\pi^{n+1}}L^{n-3}(x) + \left(\pi + \frac{C_R}{L^2(x)}\right)\frac{1}{\pi^{n+1}}\sum_{m=1}^{n+1}\binom{n+1}{m}L(x)^{n-3m-1}C_\nu^m. \quad (5.83)$$

We have

$$\int_0^\epsilon L^n(x)dx = 2e^{-\gamma}\int_{\log(2/\epsilon)-\gamma}^\infty L^n e^{-L}dL = 2e^{-\gamma}\Gamma(n+1,\bar{\epsilon})\,, \quad (5.84)$$

for $n \geqslant 0$, where $\bar{\epsilon} := \log(2/\epsilon) - \gamma$ for brevity (note $\bar{\epsilon} > 0$ since we have assumed $\epsilon < 2e^{-\gamma}$), and where $\Gamma(n,\epsilon) := \int_\epsilon^\infty t^{n-1}e^{-t}dt$ is the upper incomplete Gamma function [32, Chapter 8]. For $n < 0$, since we have assumed $\epsilon \leqslant 2e^{-\gamma-1}$ we have $L(x) \geqslant 1$ for $x \in (0,\epsilon)$, and thus

$$0 \leqslant \int_0^\epsilon L^n(x)dx \leqslant \epsilon\,. \quad (5.85)$$

Note that by combining [32, 8.8.2] and [32, 8.10.1] we obtain

$$\frac{\Gamma(n+1,\bar{\epsilon})}{\Gamma(n,\bar{\epsilon})} = n + \frac{\bar{\epsilon}^n e^{-\bar{\epsilon}}}{\Gamma(n,\bar{\epsilon})} \geqslant n + \bar{\epsilon} \geqslant n\,, \quad (5.86)$$

which can be applied iteratively to obtain

$$\frac{\Gamma(n+1,\bar{\epsilon})}{\Gamma(n+1-m,\bar{\epsilon})} \geqslant \frac{n!}{(n-m)!}\,. \quad (5.87)$$

for $m \leqslant n$. Then

$$\sum_{m=1}^{n}\binom{n}{m}C_\nu^m\int_0^\epsilon L^{n-3m-1}(x)$$
$$\leqslant \sum_{m=1}^{\lfloor(n-1)/3\rfloor}\binom{n}{m}\frac{(n-3m)!}{(n-1)!}C_\nu^m\Gamma(n,\bar{\epsilon}) + \epsilon\sum_{m=\lfloor(n-1)/3\rfloor+1}^{n}\binom{n}{m}C_\nu^m$$
$$= n\sum_{m=1}^{\lfloor(n-1)/3\rfloor}\frac{1}{m!}\frac{(n-3m)!}{(n-m)!}C_\nu^m\Gamma(n,\bar{\epsilon}) + \epsilon\sum_{m=\lfloor(n-1)/3\rfloor+1}^{n}\binom{n}{m}C_\nu^m \quad (5.88)$$
$$\leqslant n\Gamma(n,\bar{\epsilon})\sum_{m=1}^{\infty}\frac{(n-3m)!}{(n-m)!}\frac{C_\nu^m}{m!} + \epsilon\sum_{m=0}^{n}\binom{n}{m}C_\nu^m$$
$$= \frac{27}{2(2n-3)(2n-6)}\left(e^{C_\nu}-1\right)\Gamma(n,\bar{\epsilon}) + (C_\nu+1)^n\,\epsilon\,,$$

using the observation that

$$\frac{(n-m)!}{(n-3m)!} \geqslant \frac{2}{3}n\left(\frac{2n-3}{3}\right)\left(\frac{2n-6}{3}\right)\,, \quad (5.89)$$

for $1 \leqslant m \leqslant n/3$. Using this to bound the second term on the right-hand side of of Inequality (5.83) and bounding the other two terms similarly, we see that the integral in x from 0 to ϵ of the right-hand side of Inequality (5.83) admits a bound of order $\pi^{-n-1}\Gamma(n,\bar\epsilon)/n^2$ for large n. We also have

$$\left|\int_0^\epsilon I_n(x)dx\right| \leqslant \int_0^\epsilon |I_n(x)|dx \leqslant \frac{2C_a e^{-\gamma}}{a^{n-1}}\frac{\Gamma(n,\bar\epsilon)}{\pi^{n-1}}, \tag{5.90}$$

for any $a \in (1,2)$, and so the dominant behavior of the even Taylor coefficients $\mathrm{F}^{(2n)}(0)/(2n)!$ for n large is that of

$$\begin{aligned}(-1)^{n+1}\frac{2}{\pi^{2n}}\int_0^\epsilon\left(\frac{\log(2/x)-\gamma}{\pi}\right)^{n-1}dx \;&=\; 4e^{-\gamma}(-1)^{n+1}\frac{\Gamma(2n,\bar\epsilon)}{\pi^{2n}}\\ &\sim\; 4e^{-\gamma}(-1)^{n+1}\frac{(2n-1)!}{\pi^{2n}},\end{aligned} \tag{5.91}$$

noting $\Gamma(n,\bar\epsilon)\sim\Gamma(n)=(n-1)!$ [32, 8.2.3, 8.11.4], while the symmetry noted in (5.9) imposes that $\mathrm{F}'(0)=4\eta$ and $\mathrm{F}^{(2n+1)}(0)=0$ for $n=1,2,\ldots$.

The proof of Theorem 1.7 is therefore complete. □

6. Scaling limit of matrix product: proof of Theorem 1.6

As announced, we generalize the set-up of (1.16)-(1.19) in the sense that we prove

Theorem 6.1. *Consider a family of positive random variables* $\{Z^\Delta\}_{\Delta\in(0,\Delta_0)}$ *such that* $\mathbb{P}(Z^\Delta=y)=0$ *for every* y *and such that for some* $\sigma>0$ *and* $\alpha\in\mathbb{R}$ *we have*

$$\lim_{\Delta\searrow 0}\frac{\mathbb{E}\left[Z^\Delta-1\right]}{\Delta}\;=\;\frac{1}{2}\sigma^2(1-\alpha)\qquad\textit{and}\qquad \lim_{\Delta\searrow 0}\frac{\mathbb{E}\left[\left(Z^\Delta-1\right)^2\right]}{\Delta}\;=\;\sigma^2\,. \tag{6.1}$$

Assume moreover that for every $c>0$

$$\lim_{\Delta\searrow 0}\frac{1}{\Delta}\mathbb{P}\left(\left|Z^\Delta-1\right|\,>\,c\right)\;=\;0\,, \tag{6.2}$$

and

$$\limsup_{\Delta\searrow 0}\left|\frac{\mathbb{E}[1/Z^\Delta]-1}{\Delta}\right|\;<\;\infty\,. \tag{6.3}$$

Then if we consider the model (1.18)-(1.19) *with the IID sequence* $\{Z^\Delta(n)\}_{n=1,2,\ldots}$ *generalized to an arbitrary IID sequence with common law satisfying* (6.1)-(6.3)*, then* (1.21) *and* (1.22) *hold true.*

Theorem 6.1 directly implies Theorem 1.6: the cases of two more classes of distributions are treated just before the proof. Note that with (6.1) we are in reality just assuming the existence of the two limits and that the second limit is not zero. The second assumption, i.e. (6.2), barely fails to be a consequence of (6.1). The third assumption, i.e. (6.3), is used to control the amount of the mass of Z^Δ that is close to zero: it is not difficult to realize that, given (6.1), replacing (6.3) with the stronger condition

$$\lim_{\Delta\searrow 0}\frac{\mathbb{E}[1/Z^\Delta]-1}{\Delta}\;=\;\frac{1}{2}(\alpha-1)\sigma^2\,, \tag{6.4}$$

leads to very little loss of generality. Moreover, we have assumed that the law of Z^Δ ha no mass just to be sure that we do not fall into a pathological case for the theory of product of random matrices, but all we need is a condition that guarantees the existence of the limit in (1.20) and that the Markov chain associated to matrix product is ergodic: this is true in greater generality [3].

Before giving the proof let us show two classes of examples to which Theorem 6.1 applies:

(1) The distribution chosen in [31, 30] falls into the class

$$\mathtt{N}\lambda_1^{-\mathtt{N}} y^{\mathtt{N}-1} \mathbf{1}_{(0,\lambda_1)}(y)\,, \tag{6.5}$$

with

$$\lambda_1 \;=\; \lambda_1(\alpha, \mathtt{N}) \;=\; 1 + 1/\mathtt{N} + (1-\alpha)/\mathtt{N}^2 + o(1/\mathtt{N}^2)\,. \tag{6.6}$$

$\mathtt{N}(\to\infty)$ is the parameter that tunes the strength of the disorder and Theorem 6.1 can be applied by setting $\Delta = \mathtt{N}^{-2}$: let us verify the hypotheses. We compute for every ν

$$\mathbb{E}\left[(Z_{\mathtt{N}})^\nu\right] \;=\; \frac{\lambda_1(\alpha,\mathtt{N})^\nu}{1+\frac{\nu}{\mathtt{N}}} \;=\; 1 + \frac{\nu(\nu-\alpha)}{2\,\mathtt{N}^2} + o\left(\frac{1}{\mathtt{N}^2}\right)\,, \tag{6.7}$$

and we directly obtain

$$\lim_{\mathtt{N}\to\infty} \mathtt{N}^2 \mathbb{E}\left[Z_{\mathtt{N}}-1\right] \;=\; \frac{1-\alpha}{2} \qquad \text{and} \qquad \lim_{\mathtt{N}\to\infty} \mathtt{N}^2 \mathbb{E}\left[(Z_{\mathtt{N}}-1)^2\right] \;=\; 1\,, \tag{6.8}$$

and

$$\mathbb{E}\left[Z_{\mathtt{N}}^{\pm 2}\right] \;=\; 1 + \frac{(2\mp\alpha)}{\mathtt{N}^2} + o\left(\frac{1}{\mathtt{N}^2}\right) \;=\; \exp\left(\frac{(2\mp\alpha)}{\mathtt{N}^2}\right) + o\left(\frac{1}{\mathtt{N}^2}\right)\,, \tag{6.9}$$

Moreover or every $c \in (0,1)$ the event $\{|Z_{\mathtt{N}}-1| > c\} = \{Z_{\mathtt{N}} - 1 < -c\}$ if $\mathtt{N}$ is sufficiently large, because $Z_{\mathtt{N}} \leqslant \lambda_1(\alpha,\mathtt{N})$, which tends to one for $\mathtt{N}\to\infty$. On the other hand $\mathbb{P}(Z_{\mathtt{N}} - 1 < -c) = ((1-c)/\lambda_1)^{\mathtt{N}}$, which is bounded by $(1-c)^{\mathtt{N}}$ since $\lambda_1 > 1$.

(2) Choose a centered and compactly supported probability density $p(\cdot)$ and set $\sigma^2 := \int t^2 p(t)\,\mathrm{d}t$. Then the random variable Z^Δ with density given by

$$y \mapsto \frac{1}{\sqrt{\Delta}} p\left(\frac{y-m_\Delta}{\sqrt{\Delta}}\right) \qquad \text{with } m_\Delta := 1 + \frac{1}{2}\sigma^2(1-\alpha)\Delta\,, \tag{6.10}$$

with Δ smaller than a suitable $\Delta_0 > 0$, satisfies the hypotheses of Theorem 6.1.

Proof of Theorem 6.1. We start with the proof of (1.21), which is a direct application of the approximation-diffusion principle: we exploit [42, pp. 266–272], notably [42, Assumptions (2.4)-(2.6), Theorem 11.2.3]. Equivalently, one can resort to [15, Corollary 4.2 in Chapter 7]. The procedure demands three steps:

- compute the local drift at $\underline{x} \in \mathbb{R}^2$: uniformly for $\underline{x} = (x_1, x_2)^{\mathtt{t}}$ in compact sets

$$\begin{aligned} b^\Delta(\underline{x}) \;&=\; \Delta^{-1}\mathbb{E}A^\Delta \underline{x} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon\mathbb{E}\left[Z^\Delta\right] & \frac{\mathbb{E}[Z^\Delta - 1]}{\Delta} \end{pmatrix} \underline{x} \\ &\overset{\Delta\searrow 0}{\longrightarrow}\; b(\underline{x}) \;:=\; b\,\underline{x}, \qquad \text{with } b := \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & (1-\alpha)\frac{\sigma^2}{2} \end{pmatrix}, \end{aligned} \tag{6.11}$$

  where we have applied the first assumption in (6.1);
- compute the diffusion matrix at $\underline{x}$: again uniformly we have

$$a^\Delta(\underline{x}) \;=\; \Delta^{-1}\mathbb{E}\left[A^\Delta \underline{x}\,\underline{x}^{\mathtt{t}} (A^\Delta)^{\mathtt{t}}\right] \overset{\Delta\searrow 0}{\longrightarrow} a(\underline{x}) \;:=\; \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 x_2^2 \end{pmatrix}\,, \tag{6.12}$$

  where we have applied both assumptions in (6.1);
- observe that, by (6.2), $\Delta^{-1}\mathbb{P}(|A^\Delta| \geqslant c) \to 0$ for every $c > 0$.

Then, since the stochastic differential system with drift $b(\cdot)$ and diffusion matrix $a(\cdot)$ has unique (strong) solution, the Markov chain X^Δ converges in law to the diffusion process with drift $b(\cdot)$ and diffusion matrix $a(\cdot)$, which is precisely the solution X to the stochastic differential system (1.3). This completes the proof of (1.21).

In order to prove (1.22) we start by observing that $\widehat{\mathcal{L}}_{Z^\Delta}(\varepsilon) = \widehat{\mathcal{L}}_{Z^\Delta}(-\varepsilon)$, in agreement with the analogous result for $\mathcal{L}_{\sigma,\alpha}(\cdot)$ (Theorem 1.1(2)), because $D(I + A^\Delta)D$, with D the diagonal matrix with $+1$ and -1 on the diagonal, is equal to $I + A^\Delta$ with ε replaced by $-\varepsilon$. Hence we can restrict to $\varepsilon > 0$. Moreover if we set $Y^\Delta(n) := X_2^\Delta(n)/X_1^\Delta(n)$, we have that $Y^\Delta(\lfloor\cdot/\Delta\rfloor) \longrightarrow Y(\cdot)$ in law as $\Delta \searrow 0$ just because of (1.21) and because the map $(x_1, x_2) \mapsto x_2/x_1$, from $(0,\infty)^2$ to $(0,\infty)$, is continuous. Denote by T_t^Δ and T_t the corresponding Markov operator semigroups

$$T_t^\Delta f(y) \,=\, E_y^\Delta\left[f\left(Y^\Delta\left(\lfloor t/\Delta\rfloor\right)\right)\right] , \qquad T_t f(y) = E_y\left[f\left(Y(t)\right)\right] , \tag{6.13}$$

acting on bounded continuous $f : (0,\infty) \to \mathbb{R}$. Note that we have also introduced the notation E^Δ and E for the expectation with respect to the two Markov processes we consider. We claim that:

(1) For bounded continuous $f : (0,\infty) \to \mathbb{R}$ and $t \in [0,\infty)$, we have that

$$T_t^\Delta f(y) \overset{\Delta\searrow 0}{\longrightarrow} T_t f(y) \quad \text{uniformly for } y \text{ in compact subsets of } (0,\infty) . \tag{6.14}$$

(2) For all positive Δ there exists a unique law μ^Δ on $(0,\infty)$ which is invariant for the Markov chain Y^Δ, which is ergodic.

(3) Choosing $\Delta_0 \in (0, 1/\varepsilon)$ we have

$$\sup_{\Delta\in(0,\Delta_0]} \int_0^\infty y^2 \mu^\Delta(\mathrm{d}y) < \infty , \qquad \sup_{\Delta\in(0,\Delta_0]} \int_0^\infty y^{-1} \mu^\Delta(\mathrm{d}y) < \infty . \tag{6.15}$$

Claim (1) is a byproduct of the proof of (1.21) [42, Theorem 11.2.3]. Claim (2) comes from the general theory of products of random matrices. Let us prove (3), and start by writing

$$Y^\Delta(n+1) \,=\, Z^\Delta(n+1)\, u(Y^\Delta(n)) , \qquad u(y) = \frac{y + \varepsilon\Delta}{1 + \varepsilon\Delta y} . \tag{6.16}$$

Observing that for $\Delta \in (0, 1/\varepsilon]$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(u(z^{1/2})^2\right) = -\varepsilon\Delta(1 - \varepsilon^2\Delta^2)\frac{3z + 4\varepsilon\Delta z^{1/2} + 1}{2z^{3/2}(\varepsilon\Delta z^{1/2} + 1)^4} \leqslant 0 , \tag{6.17}$$

we obtain by the Markov property and by Jensen's inequality that for a given initial condition $y > 0$

$$\begin{aligned} E_y\left[Y^\Delta(n+1)^2\right] \,&=\, \mathbb{E}\left[\left(Z^\Delta(n+1)\right)^2\right] E_y\left[u(Y^\Delta(n))^2\right] \\ &\leqslant\, q_{\Delta,+}^2 u\left(E_y\left[Y^\Delta(n)^2\right]^{1/2}\right)^2 , \end{aligned} \tag{6.18}$$

where

$$q_{\Delta,+} \,:=\, \sqrt{\mathbb{E}\left[(Z^\Delta)^2\right]} \overset{(6.1)}{=} 1 + \left(1 - \frac{\alpha}{2}\right)\sigma^2\Delta + o\left(\Delta^2\right) . \tag{6.19}$$

Therefore if we set $x_n := E_y\left[Y^\Delta(n)^2\right]^{1/2}$ we have $x_{n+1} \leqslant q_{\Delta,+} u(x_n)$ which directly entails that $x_n < \infty$ for every n and, since $u(\cdot)$ is bounded and concave increasing with $u(0) > 0$, the application $q_{\Delta,+}u(\cdot)$ has only one positive fixed point that attracts every positive number. The fixed point $x^+_{\alpha,\varepsilon}(\Delta)$ is easily computed:

$$\begin{aligned} x^+_{\alpha,\varepsilon}(\Delta) \;&=\; \frac{1}{2}\left(\frac{q_{\Delta,+}-1}{\varepsilon\Delta}+\sqrt{\left(\frac{q_{\Delta,+}-1}{\varepsilon\Delta}\right)^2+4q_{\Delta,+}}\right) \\ &\overset{\Delta\searrow 0}{\sim} \frac{1}{2}\left(\frac{\left(1-\frac{\alpha}{2}\right)}{\varepsilon}\sigma^2+\sqrt{\left(\frac{\left(1-\frac{\alpha}{2}\right)}{\varepsilon}\sigma^2\right)^2+4}\right). \end{aligned} \tag{6.20}$$

Therefore $\limsup_n x_n \leqslant x^+_{\alpha,\varepsilon}(\Delta)$ and $x^+_{\alpha,\varepsilon}(\Delta)$ is bounded for $\Delta\searrow 0$. Since $\{Y^\Delta_n\}_{n=0,1,\dots}$ converges in law to the random variable Y^Δ_∞ that is distributed according to μ^Δ, by standard measure theory argument we infer that $\mathbb{E}[(Y^\Delta_\infty)^2]=\int_0^\infty y^2\mu^\Delta(\mathrm{d}y)\leqslant(x^+_{\alpha,\varepsilon}(\Delta))^2$ which proves the first claim in (3).

For the other claim in (3) it is useful to note that $\widetilde{Y}^\Delta(n)=Y^\Delta(n)^{-1}$ evolves according to the similar dynamics driven by $1/Z^\Delta$,

$$\widetilde{Y}^\Delta(n+1) \;=\; \left(Z^\Delta(n+1)\right)^{-1}\frac{\widetilde{Y}^\Delta(n)+\varepsilon\Delta}{1+\varepsilon\Delta\widetilde{Y}^\Delta(n)}\,. \tag{6.21}$$

We can now proceed in a simpler way than above and exploit directly the concavity of $u(\cdot)$ to get to

$$\widetilde{x}_{n+1} \;:=\; E_y\left[\widetilde{Y}^\Delta(n+1)\right] \;\leqslant\; q_{\Delta,-}u\left(E_y\left[\widetilde{Y}^\Delta(n)\right]\right)\,, \tag{6.22}$$

and $\limsup_n \widetilde{x}_n\leqslant x^-_{\alpha,\varepsilon}(\Delta)$, with $x^-_{\alpha,\varepsilon}(\Delta)$ defined replacing $q_{\Delta,+}$ with $q_{\Delta,-}$ in the definition (6.20) of $x^+_{\alpha,\varepsilon}(\Delta)$. It is therefore clear that (6.3) tells us that $x^-_{\alpha,\varepsilon}(\Delta)$ remains bounded for $\Delta\searrow 0$ and the second claim in (3) is proven.

Remark 6.2. *Of course if we make the stronger, but in practice almost equivalent, condition on the second moment of* $1/Z^\Delta$ *in* (6.3)*, the argument for the first claim in* (3) *applies and directly yields* $\sup_{\Delta\in(0,\Delta_0]}\int_0^\infty y^{-2}\mu^\Delta(\mathrm{d}y)<\infty$.

With (1)–(3) at hands, we complete the proof of (1.22). By (1.18)-(1.19) and iterating we obtain

$$\begin{aligned} \log X^\Delta_1(n) \;&=\; \log X^\Delta_1(n-1)+\log\left(1+\varepsilon\Delta Y^\Delta(n-1)\right) \\ &=\; \log X^\Delta_1(0)+\sum_{i=1}^n\log\left(1+\varepsilon\Delta Y^\Delta(i-1)\right)\,. \end{aligned} \tag{6.23}$$

Following [3, Th. 4.3 in Ch. III], we express the Lyapunov exponent

$$\begin{aligned} \widehat{\mathcal{L}}_{Z^\Delta}(\varepsilon) \;&=\; \lim_{n\to\infty}\frac{1}{n}\log\|X^\Delta(n)\| \;=\; \lim_{n\to\infty}\frac{1}{n}\log X^\Delta_1(n) \\ &\overset{(6.23)}{=} \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\log\left(1+\varepsilon\Delta Y^\Delta(i-1)\right) \;=\; \int_0^\infty\log\left(1+\varepsilon\Delta y\right)\mu^\Delta(\mathrm{d}y)\,. \end{aligned} \tag{6.24}$$

By (3), the family $\{\mu^\Delta\}_{\Delta\in(0,\Delta_0]}$ of probability measures is tight on $(0,\infty)$. By (1) and [15, Th. 9.10 in Ch. 4], every weak limit of $\{\mu^\Delta\}_{\Delta\in(0,\Delta_0]}$ is invariant for Y, whose unique invariant measure has the density $p_\varepsilon(\cdot)$, implies that $\mu^\Delta(\mathrm{d}y)$ converges weakly to $p_\varepsilon(y)\,\mathrm{d}y$ as $\Delta \searrow 0$. Then,

$$\frac{\widehat{\mathcal{L}}_{Z^\Delta}(\varepsilon)}{\Delta} - \mathcal{L}_{\sigma,\alpha}(\varepsilon) \;=\; \int_0^\infty \left(\frac{\log(1+\varepsilon\Delta y)}{\Delta} - \varepsilon y\right)\mu^\Delta(\mathrm{d}y) + \int_0^\infty \varepsilon y\left(\mu^\Delta(\mathrm{d}y) - p_\varepsilon(y)\,\mathrm{d}y\right) \quad (6.25)$$

The last term vanishes as $\Delta \searrow 0$ by weak convergence and uniform integrability from claim (3). But also the first term in the right-hand side vanishes for the same reasons because

$$\left|\frac{\log(1+\varepsilon\Delta y)}{\Delta} - \varepsilon y\right| \;=\; \int_0^y \frac{\Delta\varepsilon^2 z}{1+\Delta\varepsilon z}\,\mathrm{d}z \;\leqslant\; \sqrt{\int_0^y \varepsilon\,\mathrm{d}z \int_0^y \Delta\varepsilon^2 z\,\mathrm{d}z} \;=\; \frac{\varepsilon^{3/2}}{\sqrt{2}}\Delta^{1/2}y^{3/2}\,, \quad (6.26)$$

where we have used that, for $u \geqslant 0$, $u/(1+u)$ is bounded above both by 1 and by u. This completes the proof of (1.22) and, therefore, also the proof of Theorem 6.1. □

Appendix A. The McCoy-Wu model

In [31], McCoy and Wu examined a two-dimensional Ising model with bond disorder of a particular type (subsequently known as the McCoy-Wu model): the couplings between sites in neighboring columns have a constant strength E_1, while the couplings between neighboring sites in the same column take a random value $E_2(n)$ which is fixed within each row but varies independently – keeping the same distribution – between different rows (Figure 1). They showed that in the thermodynamic limit the free energy per site of this model is given (up to the subtraction of an analytic function of β) by

$$\mathrm{F}_{\mathrm{MW}}(\beta) \;:=\; \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathcal{L}_\beta^{\mathrm{MW}}(\theta)\,\mathrm{d}\theta\,, \quad (A.1)$$

where $\mathcal{L}_\beta^{\mathrm{MW}}(\theta)$ is the Lyapunov exponent of the random matrix

$$M_\beta(\theta) := \begin{pmatrix} 1 & \frac{a}{a^2+b^2} \\ \frac{a}{a^2+b^2}\lambda & \frac{\lambda}{a^2+b^2} \end{pmatrix}, \quad (A.2)$$

with

$$a(\theta) \;=\; -2z_1\frac{\sin(\theta)}{|1+z_1\exp(i\theta)|^2} \quad \text{and} \quad b(\theta) \;=\; \frac{1-z_1^2}{|1+z_1\exp(i\theta)|^2}\,, \quad (A.3)$$

where

$$z_1 \;=\; \tanh(\beta E_1)\,, \quad z_2(n) \;=\; \tanh(\beta E_2(n)) \quad \text{and} \quad \lambda = \lambda(n) = z_2^2(n)\,. \quad (A.4)$$

In [41] a different version of the model has been considered: vertical bounds are random in the horizontal direction and randomness is repeated in each line. This model, that allows frustration, is richer, but the features that are novel with respect to the McCoy-Wu model cannot be appreciated in the weak disorder limit: our analysis applies to [41] as well, but we will not develop this issue here.

To avoid trivialities we assume that $E_1 \neq 0$ as well as that E_2 is a non degenerate random variable: it is immediately clear that the sign of E_2 does not matter and just a little thought reveals that the sign of E_1 is irrelevant too. Therefore we assume that $E_1 \in (0,\infty)$ and that E_2 is a random variable taking values in $(0,\infty)$. It is helpful (mostly to simplify the presentation) to assume that E_2 takes values in $[E_2^-, E_2^+]$, with $0 < E_2^- < E_2^+ < \infty$.

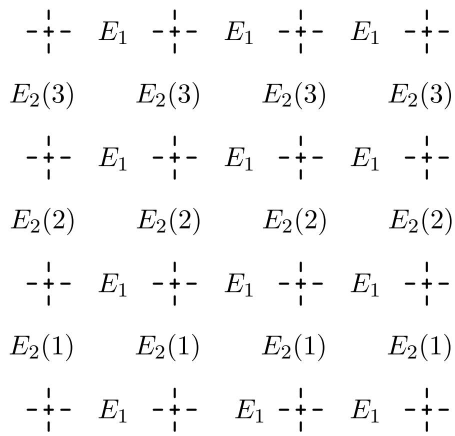

FIGURE 1. The McCoy-Wu disordered version of the two dimensional Ising model: the disordered interactions are in the vertical direction and they are distributed in an IID fashion within one column. This disorder is just copied to all the other columns and the horizontal interactions are non random. The disorder enters the free energy formula via independent copies of the random variable $\lambda = \tanh^2(\beta E_2)$.

Moreover one directly sees that $a(\cdot)$ is odd and $b(\cdot)$ is even, which yields that $\mathcal{L}_\beta^{\mathrm{MW}}(\cdot)$ is even: in fact $D_\pm M_\beta(\theta) D_\pm = M_\beta(-\theta)$, with $D_\pm$ the diagonal matrix with $(+1,-1)$ on the diagonal. Therefore:

$$\mathrm{F}_{\mathrm{MW}}(\beta) := \frac{1}{2\pi}\int_0^\pi \mathcal{L}_\beta^{\mathrm{MW}}(\theta)\,\mathrm{d}\theta\,. \tag{A.5}$$

McCoy and Wu claim that for every $\upsilon \in (0,\pi)$ – our focus is on υ small – the function

$$\beta \mapsto \frac{1}{2\pi}\int_\upsilon^\pi \mathcal{L}_\beta^{\mathrm{MW}}(\theta)\,\mathrm{d}\theta\,. \tag{A.6}$$

is real analytic on $(0,\infty)$. This can be proven by applying the main result in [37] (see also [13]). We sketch the argument here by considering separately the case θ bounded away from 0 and π and the case of θ near π: with $\delta > 0$ small

- For $\theta \in [\delta, \pi-\delta]$ the matrix $M_\beta(\theta)$ (with positive entries) maps the closure of the cone Q – here Q is first quadrant without the axes, that is the set of vectors with positive entries – to $Q \cup \{0\}$. More precisely, by the hypothesis we have made on the suport of Z, for every $\delta \in (0,\pi/2)$ and every $\varrho \in (0,1)$ the closure of Q is mapped into a cone whose closure is a subset of $Q \cup \{0\}$ and this subset is the same for every choice of $\theta \in [\delta, \pi-\delta]$ and every $\beta \in [\varrho, 1/\varrho]$. This uniform cone property implies the real analyticity of $\beta \mapsto \mathcal{L}_\beta^{\mathrm{MW}}(\theta)$ with a convergence radius that is bounded away from zero uniformly in $\theta \in [\delta, \pi-\delta]$ and $\beta \in [\varrho, 1/\varrho]$.
- For $\theta \in [\pi-\delta, \pi]$ we argue by observing first that

$$a(\pi) = 0, \quad b(\pi) = \frac{1-z_1^2}{(1-z_1)^2} = \frac{1+\tanh\beta E_1}{1-\tanh\beta E_1} = e^{2\beta E_1}, \tag{A.7}$$

so

$$M_\beta(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-4\beta E_1}\tanh^2\beta E_2 \end{pmatrix}. \tag{A.8}$$

Since $e^{-4\beta E_1}\tanh^2\beta E_2 < 1$ the action of $M_\beta(\pi)$ contracts uniformly any cone of the form $\{(x,y) : y \geqslant |x|\}$, in the sense there exists $\varrho > 0$ such that $M_\beta(\pi)$

sends $\{(x,y) : y \geqslant |x|\}$ into $\{(x,y) : y \geqslant (1+\varrho)|x|\}$, uniformly in $\beta > 0$ and E_2. Elementary arguments show that this result is only slightly perturbed if we consider $\theta \in [\pi - \delta, \pi]$ with δ sufficiently small. This uniform cone property implies the real analyticity of $\beta \mapsto \mathcal{L}_\beta^{\mathrm{MW}}(\theta)$ with a convergence radius that is bounded away from zero uniformly in $\theta \in [\pi - \delta, \pi]$ and $\beta > 0$.

Therefore the true issue is the regularity (or lack of it) of

$$\beta \mapsto \frac{1}{2\pi} \int_0^{\upsilon} \mathcal{L}_\beta^{\mathrm{MW}}(\theta)\, \mathrm{d}\theta\,, \tag{A.9}$$

for a $\upsilon > 0$ that can be chosen as small as one wishes. At this point McCoy and Wu claim that the only non analytic point of the map in (A.9) can be at β_c defined by

$$2\beta_c E_1 + \mathbb{E}\left[\log \tanh \beta_c E_2\right] \;=\; 0\,. \tag{A.10}$$

To see that this is the only possible candidate, McCoy and Wu point out that

$$a(0) = 0, \quad b(0) = \frac{1 - z_1^2}{(1+z_1)^2} = \frac{1 - \tanh \beta E_1}{1 + \tanh \beta E_2} = e^{-2\beta E_1}, \tag{A.11}$$

so

$$M_\beta(0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{4\beta E_1} \tanh^2 \beta E_2 \end{pmatrix}, \tag{A.12}$$

and so

$$\mathcal{L}_\beta^{MW}(0) = \max\left(0, 4\beta E_1 + 2\mathbb{E}\left[\log \tanh \beta E_2\right]\right) \tag{A.13}$$

for β real. This admits an analytic extension in a neighborhood of any positive β except for β_c.

This is of course far from being close to a proof, since one has to control the integral over $\theta \in (0, \upsilon)$ and not the value in zero. But McCoy and Wu perform also a more subtle analysis that can be understood precisely via the diffusion limit of matrix products that is at the center of our analysis. To explain this let us make a further manipulation to match more sharply our framework.

In fact, as it stands, $M_\beta(\theta)$, cf. (A.2), is not of the form (1.1). But by noting that

$$\frac{1}{a^2(\theta) + b^2(\theta)} = \frac{(1+z_1)^4}{(1-z_1^2)^2} + O(\theta^2) = \left(\frac{1+z_1}{1-z_1}\right)^2 + O(\theta^2) = e^{4\beta E_1} + O(\theta^2)\,, \tag{A.14}$$

and

$$\frac{a}{a^2(\theta) + b^2(\theta)} = -2\left(\frac{1+z_1}{1-z_1}\right)^2 \theta + O(\theta^2) = -2e^{4\beta E_1} + O(\theta^2)\,, \tag{A.15}$$

if we let

$$\widetilde{\varepsilon} := \frac{2z_1}{(1-z_1)^2}\theta\,, \tag{A.16}$$

we see that to leading order as $\theta \searrow 0$

$$\begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ -\widetilde{\varepsilon}\lambda & e^{4\beta E_1}\lambda \end{pmatrix} \tag{A.17}$$

is $M_\beta(\theta)$. The matrix in (A.17) is of the form (1.1) up to a conjugation and a change of variables: in fact

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon Z & Z \end{pmatrix} := \begin{pmatrix} 1 & \widetilde{\varepsilon} e^{-2\beta E_1} \\ \widetilde{\varepsilon} e^{-2\beta E_1}\lambda & e^{4\beta E_1}\lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\beta E_1} \end{pmatrix} \begin{pmatrix} 1 & -\widetilde{\varepsilon} \\ -\widetilde{\varepsilon}\lambda & e^{4\beta E_1}\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -e^{-2\beta E_1} \end{pmatrix}, \tag{A.18}$$

and we observe – recall (A.16) – that $\varepsilon = c_\beta \theta$, with $c_\beta = 2\sinh(2\beta E_1)$.

Remark A.1. *It is important to remark at this stage that the inverse temperature β and our fundamental parameter α – we recall that α is the unique non zero real solution to $\mathbb{E}Z^\alpha = 1$ ($Z = e^{4\beta E_1}\tanh^2(\beta E_2)$ depends on β!) when such a solution exists and otherwise $\alpha = 0$ – should be seen as an analytic change of variable: this is treated in detail in Lemma A.2. In particular $\alpha(\beta_c) = 0$ and therefore $\alpha(\beta) = (\beta-\beta_c)\alpha'(\beta_c) + O((\beta-\beta_c)^2)$, but the* constant *$\alpha'(\beta_c)$ depends of the law of Z (with $\beta = \beta_c$) and this expansion should be done more carefully when the disorder is weak because, as we will see, $\alpha'(\beta_c)$ becomes large in this limit: this is treated in* (A.21)-(A.29).

What McCoy and Wu do at this point is

- making a specific choice of $Z = Z^\Delta = Z^\Delta_\beta$ that satisfies the hypotheses of Theorem 6.1 (say, with $\sigma = 1$ for simplicity); this actually implements two choices:
  (1) the first is evident and it is the fact that disorder can be made weak by making Δ small;
  (2) the second is that $\beta - \beta_c$ is chosen small and, precisely, of the order of Δ. As we will explain, if we set $y = (\beta - \beta_c)/\Delta$ and we keep $y \in \mathbb{R}$ fixed, then $\alpha(\beta) \sim -C_{\beta_c} y$, and the constant $C_{\beta_c} > 0$ will be given explicit in the specific case that we are going to develop, see (A.29).
- they choose also $\upsilon \propto \Delta$: let us fix in an arbitrary fashion $\upsilon = \Delta$.

In physical terms these choices correspond to focusing on the critical window in the limit of weak disorder. Cutting the integral at $\theta = \Delta$ is harmless (as we have discussed before), but of course only as far as Δ is kept fixed.

McCoy and Wu are in the end just dealing (recall (A.18)) with the Lyapunov exponent $\widehat{\mathcal{L}}_{\Delta,\beta_c+y\Delta}(c_{\beta_c} x\Delta)$ of the matrix (we perform the change of variable $\theta = x\Delta$)

$$\begin{pmatrix} 1 & c_{\beta_c} x\Delta \\ c_{\beta_c} x\Delta Z^\Delta_{\beta_c+y\Delta} & Z^\Delta_{\beta_c+y\Delta} \end{pmatrix} . \tag{A.19}$$

But Theorem 6.1 (see also Theorem 1.6) tells us that $\widehat{\mathcal{L}}_{\Delta,\beta_c+y\Delta}(c_{\beta_c} x)$ is asymptotically equivalent for Δ small to $\Delta\mathcal{L}_{1,C_{\beta_c}\alpha}(c_{\beta_c} x)$ so that

$$\int_0^\Delta \mathcal{L}^{\mathrm{MW}}_{\beta_c+y\Delta}(\theta)\,\mathrm{d}\theta \sim \Delta \int_0^1 \widehat{\mathcal{L}}_{\Delta,\beta_c+y\Delta}(c_{\beta_c} x)\,\mathrm{d}x \sim \Delta^2 \int_0^1 \mathcal{L}_{1,C_{\beta_c}\alpha}(c_{\beta_c} x)\,\mathrm{d}x \tag{A.20}$$

and we remind the reader that $\mathcal{L}_{1,C_{\beta_c}\alpha}(c_{\beta_c} x)$ has the explict expression (1.6). Therefore, up to two inessential constants we arrived at (1.28). We did not fully justify the equivalences in (A.20), but this is not really the main problem: the main unresolved mathematical issue is that what we are after is proving that, for a fixed (possibly extremely small) value of Δ, the leftmost term in (A.20) is a C^∞ function of y at 0 and that the same expression is not analytic at zero. McCoy and Wu instead argue (and we prove in Theorem 1.7) that $\alpha \mapsto \int_0^1 \mathcal{L}_{1,C_{\beta_c}\alpha}(c_{\beta_c} x)\,\mathrm{d}x$ has these properties: but this second statement does not imply the first.

We now complement our discussion with the analysis of the specific distribution chosen for the disorder law in [31, 30]. We also discuss more in detail the change of variable $\alpha(\beta)$.

Analysis of the distribution chosen by McCoy and Wu [31, 30]. McCoy and Wu consider the disordered variable $\lambda = \tanh^2(\beta E_2)$ that depends on a parameter that they call $\mathtt{N}$ and it is large: in fact

$$\Delta \,=\, \mathtt{N}^{-2}\,. \tag{A.21}$$

The density of λ is supported on $(0, \lambda_0)$ and equal to $\mathtt{N}\lambda_0^{-\mathtt{N}} y^{\mathtt{N}-1}$ for $y \in (0, \lambda_0)$. Necessarily $\lambda_0 = \lambda_0(\beta) = \tanh^2(\beta E_2^*)$, with E_2^* the maximum value that the random variable E_2 can reach. The density of $Z = Z_{\mathtt{N}}$ (recall that Z is defined in (A.18)) is therefore

$$\mathtt{N}\lambda_1^{-\mathtt{N}} y^{\mathtt{N}-1}\mathbf{1}_{(0,\lambda_1)}(y) \qquad \text{with} \ \ \lambda_1(\beta) \,=\, e^{4\beta E_1}\lambda_0(\beta)\,. \tag{A.22}$$

Note that for every $\nu \in (-\mathtt{N}, \infty)$

$$\mathbb{E}\left[(Z_{\mathtt{N}})^\nu\right] \,=\, \frac{\lambda_1(\beta)^\nu}{1+\frac{\nu}{\mathtt{N}}}\,, \tag{A.23}$$

and we want to solve for $\alpha = \alpha(\beta) \neq 0$ the equation

$$\mathbb{E}\left[(Z_{\mathtt{N}})^\alpha\right] - 1 \,=\, \frac{\lambda_1(\beta)^\alpha - 1 - \frac{\alpha}{\mathtt{N}}}{1+\frac{\alpha}{\mathtt{N}}} \,=\, 0\,. \tag{A.24}$$

On one hand we compute

$$\log(\lambda_1(\beta)) \,=\, \log\tanh^2(\beta_c E_2^*) - \mathbb{E}\left[\log\tanh^2(\beta_c E_2)\right] + 4(\beta-\beta_c) + \log\tanh^2(\beta E_2^*) - \log\tanh^2(\beta_c E_2^*)\,, \tag{A.25}$$

and a straightforward computation yields

$$\mathbb{E}\left[\log\tanh^2(\beta E_2)\right] \,=\, \log\tanh^2(\beta E_2^*) - \frac{1}{\mathtt{N}}\,, \tag{A.26}$$

so for β close β_c we have

$$\log\lambda_1(\beta) \,=\, \frac{1}{\mathtt{N}} + (\beta-\beta_c)\left(4E_1 + \frac{1}{\sinh(2\beta_c E_2^*)}\right) + O\left((\beta-\beta_c)^2\right)\,. \tag{A.27}$$

On the other hand from (A.24) we see that if α is fixed (so we look at β as a function of α) we have

$$\log\lambda_1(\beta) \,=\, \frac{\log\left(1+\frac{\alpha}{\mathtt{N}}\right)}{\alpha} \overset{\mathtt{N}\to\infty}{=} \frac{1}{\mathtt{N}} - \frac{\alpha}{2\mathtt{N}^2} + O\left(\frac{1}{\mathtt{N}^3}\right)\,. \tag{A.28}$$

By comparing (A.27) and (A.28) we see that if $(\beta-\beta_c)\mathtt{N}^2 = O(1)$ then

$$\alpha(\beta) \,=\, -(\beta-\beta_c)\mathtt{N}^2\left(8E_1 + \frac{2}{\sinh(2\beta_c E_2^*)}\right) + O\left(\mathtt{N}^{-1}\right)\,. \tag{A.29}$$

On the relation between β and α. Here are the details of the important map that relates β and α:

Lemma A.2. *Assume that the support of the random variable E_2 is bounded away from zero, so $Z = \exp(4\beta E_2)\tanh^2(\beta E_2)$ is supported on a compact subinterval of $(0,\infty)$. Assume also that E_2 is not constant. Then the equation*

$$\frac{\mathbb{E}\left[Z^\alpha\right] - 1}{\alpha} \,=\, 0\,, \tag{A.30}$$

has a unique real solution α for every $\beta > 0$. This defines a map $\beta \mapsto \alpha(\beta)$ from $(0,\infty)$ to $\mathbb{R}$. This map is decreasing, hence it is a bijection, and it is real analytic.

Proof. Let $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be the function defined by $f(\alpha, \beta) := \frac{\mathbb{E}[Z^\alpha]-1}{\alpha}$, for $\alpha \in \mathbb{R}\backslash\{0\}$ for $\alpha \neq 0$, and $f(0, \beta) := \mathbb{E}[\log Z]$. It is straightforward to see, using the support properties of E_2, that f is real analytic on its entire domain. Then we observe that, for fixed α, Z^α is an increasing function of β and, by the support properties, this implies that $\partial_\beta f(\alpha, \beta) > 0$ for every $\beta > 0$ and $\alpha \in \mathbb{R}$. On the other hand if we set $g_\beta(\alpha) = \mathbb{E}\left[Z^\alpha\right] - 1$ we have that $\partial_\alpha f(\alpha, \beta) = (\alpha g'_\beta(\alpha) - g_\beta(\alpha))/\alpha^2$. But $g_\beta(\cdot)$ is (strictly) convex and $g_\beta(0) = 0$: so $\alpha g'_\beta(\alpha) - g_\beta(\alpha) > 0$ for $\alpha \neq 0$ and therefore $\partial_\alpha f(\alpha, \beta) > 0$ for $\alpha \neq 0$. For $\alpha = 0$ it suffices to perform a Taylor expansion of $g_\beta(\alpha)$ at $\alpha = 0$ to see that $\partial_\alpha f(\alpha, \beta)|_{\alpha=0} = g''_\beta(0)/2 > 0$. The proof is completed by applying the Implicit Function Theorem for real analytic functions [26]. □

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