Nonlinear Model Reduction in the Loewner Framework

Joel D. Simard, *Student Member, IEEE*, and Alessandro Astolfi, *Fellow, IEEE*

***Abstract*—We introduce a novel method of model reduction for nonlinear systems by extending the Loewner framework developed for linear time-invariant systems. This objective is achieved by defining Loewner functions obtained by utilizing a state-space interpretation of the Loewner matrices. A Loewner equivalent model using these functions is derived. This allows constructing reduced order models achieving interpolation in the Loewner sense.**

***Index Terms*—Center manifold, interpolation, Loewner matrices, model reduction, nonlinear systems.**

I. Introduction

THE goal of model order reduction is to determine a simplified model of a dynamical system while preserving some desired properties of the system itself, for example, stability or steady-state behavior for selected signals. A variety of approaches to accomplish model reduction have recently been developed. These include moment matching [1]–[6], balanced truncation [7]–[10], and Hankel-norm methods [11]–[16]. Under mild assumptions, these methods, originally developed for linear systems, have been also developed for nonlinear systems (see, e.g., [3], [6]–[8], [14], [17]–[22]).

The Loewner matrix [23] is an important object that has been used in the development of reduced order models for linear time-invariant (LTI) systems, and in the solution of the so-called generalized realization problem for LTI systems [24]. The Loewner matrix, also known as the divided-difference matrix [24], is related to the Hankel matrix [25], [26]. It was first used to solve rational interpolation problems in [27]. The Loewner matrix has an important structure that allows its factorization into two matrices: the tangential generalized controllability matrix and the tangential generalized observability matrix. Once factored, these matrices together can be used to construct LTI models as in [23]. In [28], a data-driven model reduction approach using the Loewner framework is given for linear systems, where frequency-response data are inferred from trajectories of the input and output signals. Note that as a result of the definitions of moments in the time-domain in [3], it has been shown in [5] that the Loewner framework in [24] can be considered as a special case of a two-sided moment-matching procedure [29]. The Loewner framework has been developed for bilinear, quadratic-bilinear, and linear switched systems using a higher order transfer function (frequency domain) approach in [30]–[32], respectively. A similar approach has been pursued in [33]. In this article, we extend the Loewner framework to general nonlinear input-affine systems using an interconnection-based approach.

In [34], new objects that allow for a state-space interpretation of the Loewner matrices have been introduced. These new objects are the left- and the right-Loewner matrices, and they can be interpreted as the input and output gains of a transformed *experimental setup*. This experimental setup involves encoding the interpolation points into two generators that are interconnected with the plant. This interpretation allows for a more sophisticated usage of the tools associated to the Loewner matrices, for example, the authors have used this new interpretation to develop a model order reduction procedure for linear time-varying systems in [35].

In this article, we utilize the state-space interpretation of the Loewner matrices to generalize the Loewner method for model reduction to nonlinear input-affine systems. To accomplish this, Loewner functions are introduced as generalizations of the Loewner matrices, which are then used to construct models that can produce the exact same left- and right-Loewner functions, thus achieving interpolation in a Loewner sense. Locally, the original model and the interpolating model produce the same steady-state response, provided that it exists, when interconnected with generators corresponding to the Loewner functions. Similarly to the linear setting, the Loewner framework for nonlinear systems resembles the two-sided moment matching procedure in [36].

This article is organized as follows. In Section II, we present preliminary results providing a state-space interpretation of the Loewner matrices for linear systems. In Section III, we generalize the notion of Loewner matrices to define Loewner functions for nonlinear systems interconnected with linear generators,

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Joel D. Simard is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K. (e-mail: j.simard18@imperial.ac.uk).

Alessandro Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K., and also with Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata," Via del Politecnico, 1 00133 Roma, Italy (e-mail: a.astolfi@imperial.ac.uk).

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introduce a special set of coordinates, the *Loewner coordinates*, and provide a reduced order model, achieving moment matching, on the basis of the Loewner functions. In Section IV, the results of the previous section are further generalized to allow for nonlinear generators. Finally, in Section V, we conclude this article.

We conclude this introduction by noting that this article has been written in the same spirit as papers such as [3], [7], [8], [24], [29], and [36]; it is a theoretical article introducing ideas and tools for general nonlinear affine systems, and while comparison to other methods and large-scale numerical validation of this work is important, this is the subject of further work relying on the methods built herein.

II. Preliminaries

We use standard notation. The set of complex numbers is denoted by $\mathbb{C}$. The imaginary axis of the complex plane is denoted by $\mathbb{C}_0$. The set of vectors having n rows with complex entries is denoted by $\mathbb{C}^n$. The set of matrices having n rows and m columns with complex entries is denoted by $\mathbb{C}^{n\times m}$. The spectrum of a square matrix A is denoted by $\sigma(A)$.

While the Loewner matrices have been traditionally defined [24], for systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

with state $x(t) \in \mathbb{C}^n$, $u(t) \in \mathbb{C}^m$, $y(t) \in \mathbb{C}^p$, and matrices E, A, B, C, and D of appropriate dimensions, we consider the special case in which $E = I$ and $D = 0$, i.e.

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$
$$y(t) = Cx(t). \quad (2)$$

Note that for simplicity, we consider complex-valued signals and matrices for ease of presentation. These signals and matrices are obtained via coordinate transformations of real-valued signals and matrices.

The following assumptions hold throughout the article.

Assumption 1: The triple of matrices (A, B, C) is a minimal realization of the system (1)–(2), i.e., the system (1)–(2) is reachable and observable.

To pose an interpolation problem, and to define the Loewner matrices, we require the concept of tangential data. Tangential data are data sampled in particular directions, and consist of right tangential data, described by the set

$$\{(\lambda_i, r_i, w_i) \mid \lambda_i \in \mathbb{C},\ r_i \in \mathbb{C}^m,\ w_i \in \mathbb{C}^p,\ i = 1, \ldots, \rho\} \quad (3)$$

and left tangential data, described by the set

$$\{(\mu_j, \ell_j, v_j) \mid \mu_j \in \mathbb{C},\ \ell_j^\top \in \mathbb{C}^p,\ v_j^\top \in \mathbb{C}^m,\ j = 1, \ldots, v\}. \quad (4)$$

We write the right tangential data in a compact form as

$$\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_\rho] \in \mathbb{C}^{\rho\times\rho}$$

$$R = \begin{bmatrix} r_1 & \ldots & r_\rho \end{bmatrix} \in \mathbb{C}^{m\times\rho}$$

$$W = \begin{bmatrix} w_1 & \ldots & w_\rho \end{bmatrix} \in \mathbb{C}^{p\times\rho}$$

and the left tangential data, again in a compact form, as

$$M = \text{diag}[\mu_1, \ldots, \mu_v] \in \mathbb{C}^{v\times v}$$

$$L = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_v \end{bmatrix} \in \mathbb{C}^{v\times p}, \quad V = \begin{bmatrix} v_1 \\ \vdots \\ v_v \end{bmatrix} \in \mathbb{C}^{v\times m}.$$

The following assumption is required to guarantee uniqueness of solution to a number of Sylvester equations arising in this framework.

Assumption 2: The matrices A, Λ, and M have no common eigenvalues, that is

$$\sigma(A) \cap \sigma(\Lambda) = \emptyset,\ \sigma(A) \cap \sigma(M) = \emptyset,\ \sigma(M) \cap \sigma(\Lambda) = \emptyset.$$

The goal of the realization problem is to determine a state-space representation of the form (1)–(2) such that the corresponding rational transfer matrix $H(s) = C(sI - A)^{-1}B$ obeys the right interpolation conditions

$$H(\lambda_i)r_i = w_i,\ i = 1, \ldots, \rho \quad (5)$$

and the left interpolation conditions

$$\ell_j H(\mu_j) = v_j,\ j = 1, \ldots, v. \quad (6)$$

The Loewner matrix and the shifted Loewner matrix [23], $\mathbb{L}$ and $\sigma\mathbb{L}$, respectively, are defined in terms of the tangential data (3) and (4) as

$$\mathbb{L} = \begin{bmatrix} \frac{v_1 r_1 - \ell_1 w_1}{\mu_1 - \lambda_1} & \ldots & \frac{v_1 r_\rho - \ell_1 w_\rho}{\mu_1 - \lambda_\rho} \\ \vdots & \ddots & \vdots \\ \frac{v_v r_1 - \ell_v w_1}{\mu_v - \lambda_1} & \ldots & \frac{v_v r_\rho - \ell_v w_\rho}{\mu_v - \lambda_\rho} \end{bmatrix}$$

and

$$\sigma\mathbb{L} = \begin{bmatrix} \frac{\mu_1 v_1 r_1 - \lambda_1 \ell_1 w_1}{\mu_1 - \lambda_1} & \ldots & \frac{\mu_1 v_1 r_\rho - \lambda_\rho \ell_1 w_\rho}{\mu_1 - \lambda_\rho} \\ \vdots & \ddots & \vdots \\ \frac{\mu_v v_v r_1 - \lambda_1 \ell_v w_1}{\mu_v - \lambda_1} & \ldots & \frac{\mu_v v_v r_\rho - \lambda_\rho \ell_v w_\rho}{\mu_v - \lambda_\rho} \end{bmatrix}$$

which provide the classical frequency domain interpretation of the Loewner matrices. Furthermore, note that if the transfer matrix $H(s)$ generates the data, then the shifted Loewner matrix is the Loewner matrix corresponding to the transfer matrix $sH(s)$. Note also that, by Assumption 2, the Loewner matrix is the unique solution of the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR$$

and the shifted Loewner matrix is the unique solution of the Sylvester equation

$$\sigma\mathbb{L}\Lambda - M\sigma\mathbb{L} = LW\Lambda - MVR.$$

It is also shown in [24] that $\sigma\mathbb{L} - \mathbb{L}\Lambda = VR$ and that $\sigma\mathbb{L} - M\mathbb{L} = LW$.

The definitions of the Loewner and shifted Loewner matrices given so far are independent of any particular state-space representation, i.e., they are defined solely in terms of tangential data. The following definitions assume that the tangential data are generated by a system of the form (1)–(2) according to the relationships given in (5) and (6). We define the tangential generalized observability matrix Y and the tangential generalized controllability matrix X as

$$Y = \begin{bmatrix} \ell_1 C(\mu_1 I - A)^{-1} \\ \vdots \\ \ell_v C(\mu_v I - A)^{-1} \end{bmatrix}$$

and

$$X = \begin{bmatrix} (\lambda_1 I - A)^{-1} B r_1 & \cdots & (\lambda_\rho I - A)^{-1} B r_\rho \end{bmatrix}$$

respectively. These matrices are the unique solution to the Sylvester equations

$$YA + LC = MY \tag{7}$$

and

$$AX + BR = X\Lambda \tag{8}$$

and, furthermore, the Loewner matrix and the shifted Loewner matrix can be expressed in terms of these matrices as

$$\mathbb{L} = -YX, \qquad \sigma\mathbb{L} = -YAX.$$

Note that having defined Y and X, we can now express W as

$$W = CX$$

and V as

$$V = YB.$$

A. State-Space Interpretation

In order to provide a state-space interpretation of the Loewner matrices for the system (1)–(2), we require the definition of a few additional objects. These objects are constructed solely from tangential data. We first define the left-Loewner matrix $\mathbb{L}^\ell$ as the unique solution, by Assumption 2, to the Sylvester equation

$$M\mathbb{L}^\ell - \mathbb{L}^\ell \Lambda = VR \tag{9}$$

and the right-Loewner matrix $\mathbb{L}^r$ as the unique solution, by Assumption 2, to the Sylvester equation

$$\mathbb{L}^r \Lambda - M\mathbb{L}^r = LW. \tag{10}$$

These definitions yield the identity

$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r.$$

In a similar fashion, the shifted left-Loewner matrix $\sigma\mathbb{L}^\ell$ and the shifted right-Loewner matrix $\sigma\mathbb{L}^r$ are defined as the unique solution, again by Assumption 2, to $M\sigma\mathbb{L}^\ell - \sigma\mathbb{L}^\ell\Lambda = MVR$, and $\sigma\mathbb{L}^r\Lambda - M\sigma\mathbb{L}^r = LW\Lambda$, respectively. Furthermore, exploiting these definitions, it is easy to see that

$$\sigma\mathbb{L} = \sigma\mathbb{L}^\ell + \sigma\mathbb{L}^r.$$

Moreover, noting that $M(M\mathbb{L}^\ell) - (M\mathbb{L}^\ell)\Lambda = MVR$, and $(\mathbb{L}^r\Lambda)\Lambda - M(\mathbb{L}^r\Lambda) = LW\Lambda$, by the uniqueness of solution to (9) and (10), we have that $\sigma\mathbb{L}^\ell = M\mathbb{L}^\ell$, and $\sigma\mathbb{L}^r = \mathbb{L}^r\Lambda$.

Remark 1: The left- and right-Loewner matrices are not explicitly required when constructing an interpolant in the Loewner framework, but rather the existence of these objects enhances understanding of how an interpolant in the framework fulfills its purpose. The interpretation that is obtained via these objects does not require frequency domain notions and can be readily used to define interpolants for nonlinear systems.

We now define two auxiliary systems, using the right- and left- tangential interpolation data, as

$$\dot{\zeta}_r(t) = \Lambda\zeta_r(t) + \Delta(t) \tag{11}$$

$$v(t) = R\zeta_r(t) \tag{12}$$

and

$$\dot{\zeta}_\ell(t) = M\zeta_\ell(t) + L\chi(t) \tag{13}$$

$$\eta(t) = \zeta_\ell(t) \tag{14}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$. Consider the interconnected system defined by the interconnection equations $u = v$ and $\chi = y$. This system has a state-space representation given by

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{x} \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ BR & A & 0 \\ 0 & LC & M \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta \tag{15}$$

$$\eta = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix} \tag{16}$$

and is illustrated in Fig. 1. While the interconnection of the three subsystems is primarily meant to provide an interpretation of the Loewner framework that does not rely on frequency data, it could also be considered to be the result of a desired operating environment, i.e., the result of generated signals at the plant input, and filters applied to the output.

To expose an important property of the Loewner matrices, we recall a result from [34].

Theorem 1 ([34]): Consider the interconnected system (15)–(16). The coordinates transformation

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -X & I & 0 \\ \mathbb{L}^\ell & Y & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix}$$

is such that the system in the new coordinates is described by the following equations:

$$\begin{bmatrix} \dot{z}_r \\ \dot{z}_c \\ \dot{z}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} + \begin{bmatrix} I \\ -X \\ \mathbb{L}^\ell \end{bmatrix} \Delta \tag{17}$$

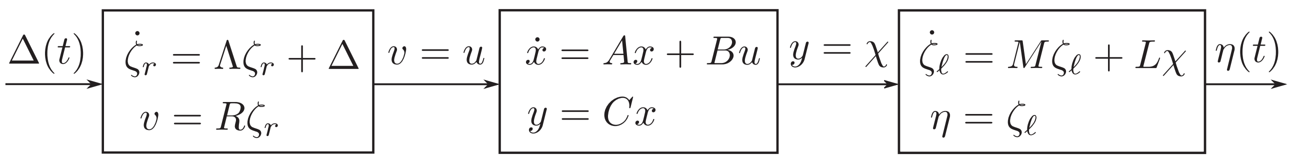

Fig. 1. The interconnected system (15)–(16).

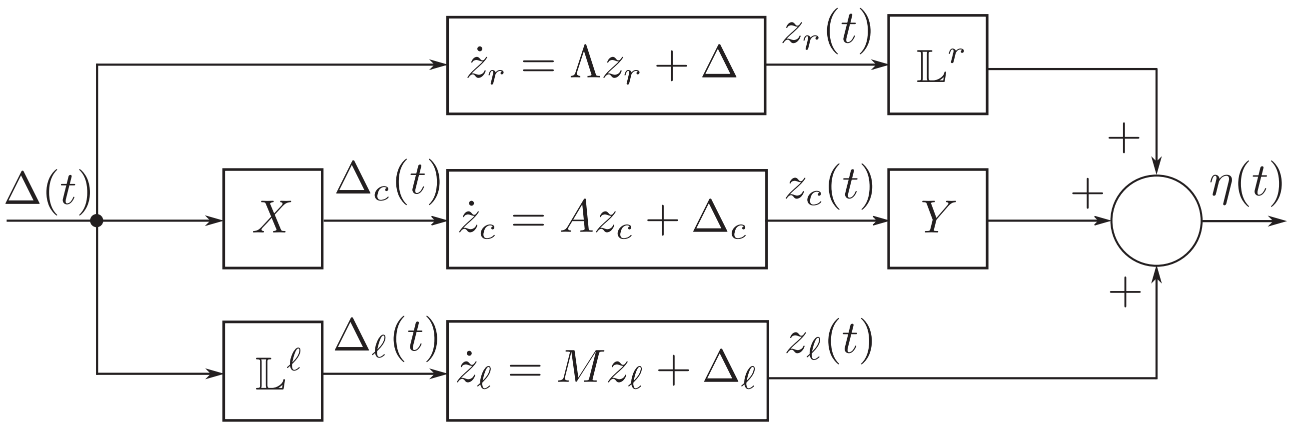

Fig. 2. The transformed, parallel interconnected, system (17)–(18).

$$\eta = \begin{bmatrix} \mathbb{L}^r & -Y & I \end{bmatrix} \begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix}. \tag{18}$$

Theorem 1 lends itself to a state-space interpretation of the Loewner matrices: the Loewner matrices can be viewed as the input and output "gains" of three systems connected in parallel such that the input/output behavior is the same as that of the interconnected system (15)–(16), as illustrated in Figs. 1 and 2.

We now provide a definition which is crucial for the construction of reduced order models in the Loewner sense.

Definition 1 (Loewner Equivalence): Let Σ and $\overline{\Sigma}$ be two systems with left- and right-Loewner matrices $\mathbb{L}^\ell$, $\mathbb{L}^r$, and $\overline{\mathbb{L}}^\ell$, $\overline{\mathbb{L}}^r$, respectively, associated to the generating matrices Λ, R, M, and L. Then, Σ and $\overline{\Sigma}$ are called Loewner equivalent at (Λ, R, M, L) if $\mathbb{L}^\ell = \overline{\mathbb{L}}^\ell$ and $\mathbb{L}^r = \overline{\mathbb{L}}^r$.

The fact that two systems are Loewner equivalent at (Λ, R, M, L) is equivalent to both systems satisfying the conditions (5) and (6).

Considering Theorem 1, and assuming that Δ is bounded and converges to zero, A has only negative eigenvalues, and Λ and M have eigenvalues on the imaginary axis, it is easy to see that the steady-state response, provided it exists, of the system interconnected with the generators is dependent entirely on the generator states and the left- and right-Loewner matrices. Thus, if two exponentially stable systems are Loewner equivalent at (Λ, R, M, L), then there exists an initial condition such that the two systems interconnected with the generators have the same steady-state behavior. It follows that in the state-space interpretation of the Loewner framework η is the signal for which "interpolation" occurs, as any two exponentially stable systems that are Loewner equivalent, or generate the same tangential data, produce the same signal η after some transient period when interconnected with the generators.

We can now formally define what a reduced order model is in the Loewner sense.

Definition 2 (Reduced Order Model): Let Σ and $\overline{\Sigma}$ be two systems of order n and v, respectively. $\overline{\Sigma}$ is called a reduced order model of Σ in the Loewner sense if Σ and $\overline{\Sigma}$ are Loewner equivalent at (Λ, R, M, L) and $v < n$.

Following [24], if $\mathbb{L}$, $\sigma\mathbb{L}$, V, and W are known for the system (1)–(2), with $\rho = v$ and $\mathbb{L}$ is nonsingular, then an interpolating system (i.e., a system that matches the tangential data (3) and (4) exactly) with state $r(t) \in \mathbb{C}^\rho$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$ can be defined as

$$\dot{r} = \mathbb{L}^{-1}\sigma\mathbb{L}r - \mathbb{L}^{-1}V u_r \tag{19}$$

$$y_r = Wr. \tag{20}$$

If the Loewner matrix is nonsingular with rank ρ, the system (19)–(20) is a unique interpolant of degree ρ. Otherwise, there exists a family of interpolants of degree ρ [24].

Remark 2: Consider the interconnected system (15)–(16) with associated Loewner matrices $\mathbb{L}^\ell$, $\mathbb{L}^r$, and $\mathbb{L}$. Let $\overline{X}$ and $\overline{Y}$ be the tangential generalized controllability and observability matrices, and $\overline{\mathbb{L}}^\ell$ and $\overline{\mathbb{L}}^r$ be the left- and right-Loewner matrices, for the system given by the equations (19)–(20) interconnected with the generators (11)–(12) and (13)–(14). Then, the following is true: $\overline{X} = I$, $\overline{Y} = -\mathbb{L}$, $\overline{\mathbb{L}}^{\ell} = \mathbb{L}^{\ell}$, and $\overline{\mathbb{L}}^{r} = \mathbb{L}^{r}$.

Remark 3: The state-space interpretation that is presented in this section has been used to extend the Loewner model reduction framework to linear time-varying systems in [35].

B. Problem Formulation

In the rest of the article, we focus on nonlinear systems described by equations of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{21}$$

$$y(t) = h(x(t)) \tag{22}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, and output $y(t) \in \mathbb{C}^p$, and functions $f : \mathbb{C}^n \to \mathbb{C}^n$, $g : \mathbb{C}^n \to \mathbb{C}^{n\times m}$, and $h : \mathbb{C}^n \to \mathbb{C}^p$ of appropriate dimensions, and such that $f(0) = 0$, $h(0) = 0$, and $f(\cdot)$ is differentiable. Let $A := \frac{\partial f}{\partial x}(0)$. For the ease of presentation, we consider complex valued mappings and signals which are obtained via linear coordinate transformations of real-valued mappings and signals. In addition, with some abuse of terminology, we say, for example, that the zero equilibrium of $\dot{x} = f(x)$, with $x(t) \in \mathbb{C}^n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$, is locally asymptotically stable if the zero equilibrium of the underlying "real" system is locally asymptotically stable. See also Appendix for some additional comments on the use of complex valued signals.

Assumption 3: The unforced system $\dot{x} = f(x)$ is locally exponentially stable at the origin, that is, all eigenvalues of A are in $\mathbb{C}^-$.

The goal of the following sections is to extend the interpolation methods of [24] to nonlinear systems of the form (21)–(22) using the state-space interpretation given by [34] in three scenarios of increasing complexity and generality. To do this, we introduce the notion of Loewner functions which are, in turn, used to introduce the concept of Loewner equivalence at given operating conditions. It is important to note that the following statements regarding the existence of the Loewner functions are all local.

III. Interconnection With Linear Generators

To exploit the state-space interpretation of the Loewner matrices given in [34], we begin by constructing two systems. We start with a simple setup given by two systems of the form

$$\dot{\zeta}_r(t) = \Lambda\zeta_r(t) + \Delta(t) \tag{23}$$

$$v(t) = R\zeta_r(t) \tag{24}$$

and

$$\dot{\zeta}_\ell(t) = M\zeta_\ell(t) + L\chi(t) \tag{25}$$

$$\eta(t) = \zeta_\ell(t) \tag{26}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$, and with matrices $\Lambda \in \mathbb{C}^{\rho\times\rho}$, $R \in \mathbb{C}^{m\times\rho}$, $M \in \mathbb{C}^{v\times v}$, and $L \in \mathbb{C}^{v\times p}$.

Assumption 4: The matrices Λ and M have all eigenvalues on the imaginary axis, and these eigenvalues have geometric multiplicity one.[1]

Consider now the interconnection of the system (21)–(22) with the generators (23)–(24) and (25)–(26), defined via the interconnection equations $u = v$ and $\chi = y$, which yields the state-space representation

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{x} \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda\zeta_r \\ f(x) + g(x)R\zeta_r \\ M\zeta_\ell + Lh(x) \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta \tag{27}$$

$$\eta = \zeta_\ell \tag{28}$$

with state $\begin{bmatrix} \zeta_r^\top & x^\top & \zeta_\ell^\top \end{bmatrix}^\top$, input Δ, and output η.

A. Loewner Functions

Before presenting the main results, we define the nonlinear enhancements of the tangential generalized controllability and observability matrices and of the Loewner matrices. These are defined in terms of the functions and matrices appearing in the interconnected system (27)–(28). The tangential generalized controllability function $X : \mathbb{C}^\rho \to \mathbb{C}^n$ is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial X}{\partial \zeta_r}\Lambda\zeta_r = f(X(\zeta_r)) + g(X(\zeta_r))R\zeta_r, \quad X(0) = 0. \tag{29}$$

The following claim is a direct consequence of Assumptions 3 and 4 and of the center manifold theory [37].

Proposition 1 (Existence of X): Consider the PDE (29) with the boundary condition $X(0) = 0$. Suppose Assumption 3 and Assumption 4 hold. Then, there exists a function $X : \mathbb{C}^\rho \to \mathbb{C}^n$ satisfying the partial differential equation (29) with the given boundary condition.

The tangential generalized observability function $Y : \mathbb{C}^n \to \mathbb{C}^v$ is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial Y}{\partial x}f(x) = MY(x) - Lh(x), \quad Y(0) = 0. \tag{30}$$

To prove the existence of a solution Y, we require the construction of an auxiliary object. To this end, consider the system described by the equations

$$\dot{\zeta}_\ell = M\zeta_\ell + Lh(x) \tag{31}$$

$$\dot{x} = f(x). \tag{32}$$

By the center manifold theory and Assumptions 3 and 4, there exists a map[2] $x = \overline{Y}(-\zeta_\ell)$ satisfying the PDE with boundary condition

$$-\frac{\partial \overline{Y}}{\partial \zeta_\ell}(-\zeta_\ell)\left(M\zeta_\ell + Lh(\overline{Y}(-\zeta_\ell))\right) = f(\overline{Y}(-\zeta_\ell)), \ \overline{Y}(0) = 0. \tag{33}$$

[1]This restriction is imposed because we are interested in bounded signals.

[2]The "−" is key to getting the correct signs in the PDE (30).

Proposition 2 (Existence of Y): Consider the PDE (30) with the boundary condition $Y(0)=0$. Suppose Assumptions 3 and[3] 4 hold. Suppose that the map $\overline{Y}$, solving the PDE (33), has a local differentiable left inverse around the origin. Then, there exists a function $Y:\mathbb{C}^n\to\mathbb{C}^v$ satisfying the partial differential equation (30) with the given boundary condition.

Proof: Recall that $\overline{Y}$ satisfies the PDE

$$f(\overline{Y}(-\zeta_\ell)) = -\frac{\partial \overline{Y}}{\partial \zeta_\ell}(-\zeta_\ell)\left(M\zeta_\ell + Lh(\overline{Y}(-\zeta_\ell))\right)$$

with boundary condition $\overline{Y}(0)=0$. Let Y be the local left inverse of $\overline{Y}$, which exists by assumption, that is

$$Y(\overline{Y}(-\zeta_\ell)) = -\zeta_\ell$$

in a neighborhood of the origin. Note that $Y(0)=0$. Taking the time derivative along the trajectories of the system (31)–(32) yields

$$\frac{\partial Y}{\partial x}\dot{x} = -\frac{\partial Y}{\partial x}(\overline{Y}(-\zeta_\ell))\frac{\partial \overline{Y}}{\partial \zeta_\ell}(-\zeta_\ell)\dot{\zeta}_\ell = -\dot{\zeta}_\ell. \tag{34}$$

Using (34) in $\overline{Y}$ yields

$$\frac{\partial Y}{\partial x}f(x) = -\left(M\zeta_\ell + Lh(x)\right) = MY(x) - Lh(x).$$

Thus, the left inverse of $\overline{Y}$, i.e., Y, solves the PDE (30) in a neighborhood of the origin with the given boundary condition.■

Having defined the tangential generalized observability and controllability functions, the nonlinear enhancements of the tangential data matrices V and W are given by

$$V(\zeta_r) := \frac{\partial Y}{\partial x}(X(\zeta_r))g(X(\zeta_r)),\quad W(\zeta_r) := h(X(\zeta_r)).$$

The nonlinear Loewner function is defined in terms of the tangential generalized controllability and observability functions as

$$\mathbb{L}(\zeta_r) := -Y(X(\zeta_r)).$$

The left-Loewner function $\mathbb{L}^\ell:\mathbb{C}^\rho\to\mathbb{C}^v$ is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Lambda\zeta_r = M\mathbb{L}^\ell(\zeta_r) - V(\zeta_r)R\zeta_r,\quad \mathbb{L}^\ell(0)=0 \tag{35}$$

and the right-Loewner function $\mathbb{L}^r:\mathbb{C}^\rho\to\mathbb{C}^v$ is defined as

$$\mathbb{L}^r(\zeta_r) := \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r).$$

To prove the existence of a solution for the PDE (35), we require a definition from [38].

Definition 3 ([38, Def. 2]): Given an $n\times n$ matrix F, with spectrum $\sigma(F)=\lambda=(\lambda_1,\ldots,\lambda_n)$, and constants $C>0$ and $v>0$, we say that a complex number μ is of type (C,v) with respect to $\sigma(F)$ if for any vector $m=(m_1,m_2,\ldots,m_n)$ of nonnegative integers we have

$$|\mu - m\cdot\lambda| \geq \frac{C}{|m|^v}$$

where $|m|=\sum m_i>0$.

The following claim follows by a direct application of the main theorem of [38].

Proposition 3 (Existence of $\mathbb{L}^\ell$): Consider the PDE (35) with the boundary condition $\mathbb{L}^\ell(0)=0$. Suppose there exist constants $C>0$ and $v>0$ such that all eigenvalues of M are of type (C,v) with respect to $\sigma(\Lambda)$. Then, there exists a function $\mathbb{L}^\ell:\mathbb{C}^\rho\to\mathbb{C}^v$ satisfying the partial differential equation (35) with the given boundary condition.

The definitions introduced thus far show that the Loewner and right-Loewner functions satisfy the PDEs with boundary conditions

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r = M\mathbb{L}(\zeta_r) + LW(\zeta_r) - V(\zeta_r)R\zeta_r,\quad \mathbb{L}(0)=0$$

and

$$\frac{\partial \mathbb{L}^r}{\partial \zeta_r}\Lambda\zeta_r = M\mathbb{L}^r(\zeta_r) + LW(\zeta_r),\quad \mathbb{L}^r(0)=0.$$

The shifted Loewner function $\sigma\mathbb{L}:\mathbb{C}^\rho\to\mathbb{C}^v$ is defined in terms of the left- and right-Loewner functions as

$$\sigma\mathbb{L}(\zeta_r) := M\mathbb{L}^\ell(\zeta_r) + \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\Lambda\zeta_r$$

which implies that

$$\sigma\mathbb{L}(\zeta_r) = M\mathbb{L}(\zeta_r) + LW(\zeta_r) = \frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r + V(\zeta_r)R\zeta_r$$

and

$$\sigma\mathbb{L}(\zeta_r) = -\frac{\partial Y}{\partial x}(X(\zeta_r))f(X(\zeta_r)).$$

Remark 4: If the system (21)–(22) is linear, then the solution to the PDEs (29), (30), and (35) becomes $X(\zeta_r)=X\zeta_r$, $Y(x)=Yx$, and $\mathbb{L}^\ell(\zeta_r)=\mathbb{L}^\ell\zeta_r$, where X, Y, and $\mathbb{L}^\ell$ are the solutions to the Sylvester equations (7), (8), and (9). Thus, the linear Loewner objects are recovered.

B. Loewner Coordinates

To expose the relation between the Loewner functions and the interconnection of systems (27)–(28), we select a specific set of coordinates in a similar fashion as in Theorem 1.

Theorem 2: Consider the system (27)–(28). The coordinates transformation

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} := \begin{bmatrix} \zeta_r \\ x - X(\zeta_r) \\ \zeta_\ell + Y(x) + \mathbb{L}^\ell(\zeta_r) \end{bmatrix}$$

is such that the system in the new coordinates is described by the equations

$$\begin{bmatrix} \dot{z}_r \\ \dot{z}_c \\ \dot{z}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \widetilde{A}(z_c + X(z_r), z_r) & 0 \\ 0 & \widetilde{G}(z_c + X(z_r), z_r) & M \end{bmatrix}\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix}$$

[3]Note that Assumption 4 is not necessary. The proof can also be completed using the approach in the proof of Proposition 3, in which case a type (C,v) condition is required instead (see Definition 3 and Proposition 3).

$$+\begin{bmatrix} I \\ -\frac{\partial X}{\partial \zeta_r}(z_r) \\ \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}(z_r) \end{bmatrix}\Delta$$

$$\eta = \mathbb{L}^r(z_r) - \widetilde{Y}(z_c + X(z_r), z_r)\, z_c + z_\ell$$

where $z_r(t) \in \mathbb{C}^\rho$, $z_c(t) \in \mathbb{C}^n$, $z_\ell(t) \in \mathbb{C}^v$, and where $\widetilde{A}: \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{n\times n}$, $\widetilde{G}: \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$, and $\widetilde{Y}: \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$.

Proof: We proceed by direct differentiation. For z_c, we have

$$\begin{aligned}\dot{z}_c &= \dot{x} - \frac{\partial X}{\partial \zeta_r}\dot{\zeta}_r \\ &= (f(z_c + X(\zeta_r)) - f(X(\zeta_r))) \\ &\quad + (g(z_c + X(\zeta_r)) - g(X(\zeta_r)))\, R\zeta_r - \frac{\partial X}{\partial \zeta_r}\Delta.\end{aligned}$$

For z_ℓ, we have

$$\begin{aligned}\dot{z}_\ell &= \dot{\zeta}_\ell + \frac{\partial Y}{\partial x}\dot{x} + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\dot{\zeta}_r \\ &= M z_\ell + \left(\frac{\partial Y}{\partial x} f(x) - MY(x) + Lh(x)\right) \\ &\quad + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Lambda\zeta_r - M\mathbb{L}^\ell(\zeta_r) + \frac{\partial Y}{\partial x} g(x) R\zeta_r\right) + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Delta.\end{aligned}$$

By the PDEs defining $Y(\cdot)$ and $\mathbb{L}^\ell(\cdot)$, that is, (30) and (35), this becomes

$$\begin{aligned}\dot{z}_\ell &= M z_\ell + \frac{\partial Y}{\partial x}(z_c + X(\zeta_r)) g(z_c + X(\zeta_r)) R\zeta_r \\ &\quad - \frac{\partial Y}{\partial x}(X(\zeta_r)) g(X(\zeta_r)) R\zeta_r + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Delta.\end{aligned}$$

Finally, we have that

$$\begin{aligned}\eta &= z_\ell - Y(z_c + X(\zeta_r)) - \mathbb{L}^\ell(\zeta_r) \\ &= \mathbb{L}^r(\zeta_r) - (Y(z_c + X(\zeta_r)) - Y(X(\zeta_r))) + z_\ell.\end{aligned}$$

The result is then obtained by a direct application of Hadamard's Lemma. ■

Note that, by Assumption 3, for any sufficiently small $x(0)$ and $\zeta_r(0)$, the solutions of the interconnected systems approach the center manifold $x = X(\zeta_r)$ exponentially fast; hence, z_c approaches zero provided Δ is sufficiently small and converges to zero, and the system has a converging input converging state property. On the center manifold, that is, for $x = X(\zeta_r)$, or $z_c = 0$, one has

$$\begin{aligned}\dot{z}_r &= \Lambda z_r + \Delta \\ \dot{z}_\ell &= M z_\ell + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}(z_r)\Delta\end{aligned}$$

and

$$\eta = \mathbb{L}^r(z_r) + z_\ell$$

that is, the system restricted to the center manifold contains only information on the Loewner functions.

C. Loewner Equivalent Model

In this section, the concept of reduced order model in the Loewner sense for nonlinear systems is introduced. In addition, a nonlinear system, reminiscent of the linear systems in [24] and [35], which interpolates the Loewner functions defined by the PDEs (29), (30), and (35), is constructed. Given that the frequency domain interpretations of (5) and (6) hold little meaning in the nonlinear context, we start by describing what we mean by an interpolant when referring to nonlinear systems.

Definition 4 (Loewner Equivalence): Let Σ and $\overline{\Sigma}$ be two systems described by equations of the form (21)–(22) admitting left- and right-Loewner functions $\mathbb{L}^\ell(\cdot), \mathbb{L}^r(\cdot)$, and $\overline{\mathbb{L}}^\ell(\cdot), \overline{\mathbb{L}}^r(\cdot)$, respectively, associated to the matrices Λ, R, M, and L. Then, Σ and $\overline{\Sigma}$ are called Loewner equivalent at (Λ, R, M, L) if $\mathbb{L}^\ell(\zeta_r) = \overline{\mathbb{L}}^\ell(\zeta_r)$ and $\mathbb{L}^r(\zeta_r) = \overline{\mathbb{L}}^r(\zeta_r)$ in a neighborhood of the origin.

Consistently, we say that a nonlinear system interpolates another nonlinear system (in the Loewner sense) at (Λ, R, M, L) if the two systems are Loewner equivalent at (Λ, R, M, L). That is, for the same matrices Λ, R, M, L, the interpolating system possesses the exact same left- and right-Loewner functions.

The property of Loewner equivalence has a strong implication on the steady-state behavior of the system. By Theorem 2, recalling Assumptions 3 and 4, assuming the foregoing stability conditions hold, Δ is sufficiently small, bounded, and converges to zero, and the plant state x has not left the region of attraction of the origin (i.e., x still approaches the center manifold $X(\zeta_r)$), it is easy to see that the steady-state response, provided it exists, of the system interconnected with the generators is dependent entirely on the generator states and the left- and right-Loewner functions. Thus, if two locally exponentially stable systems are Loewner equivalent at (Λ, R, M, L), then there exist initial conditions[4] such that the two systems interconnected with the generators have the same steady-state behavior, provided it exists.

We can now define what a reduced order model is in the Loewner sense.

Definition 5 (Reduced Order Model): Let Σ and $\overline{\Sigma}$ be two systems of order n and v, respectively. $\overline{\Sigma}$ is called a reduced order model of Σ in the Loewner sense if Σ and $\overline{\Sigma}$ are Loewner equivalent at (Λ, R, M, L) and $v < n$.

We now construct a nonlinear system, which is Loewner equivalent at (Λ, R, M, L) to (27)–(28), given that the Loewner functions of (27)–(28) are known.

Theorem 3: Consider the interconnected system (27)–(28) with $\rho = v$. Let $\mathbb{L}^\ell(\cdot), \mathbb{L}^r(\cdot), \mathbb{L}(\cdot), \sigma\mathbb{L}(\cdot), V(\cdot)$, and $W(\cdot)$ be the associated Loewner functions. Assume that $\frac{\partial \mathbb{L}}{\partial \zeta_r}$ is nonsingular. Define the system

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\dot{r} = \sigma\mathbb{L}(r) - V(r)u_r \tag{36}$$

$$y_r = W(r) \tag{37}$$

[4] These initial conditions correspond to points on the manifold $x = X(\zeta_r)$.

with state $r(t) \in \mathbb{C}^\rho$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$. Then, the system (36)–(37) is Loewner equivalent at (Λ, R, M, L) to the system (21)–(22).

Remark 5: The left- and right-Loewner functions are not explicitly used in the construction of the presented interpolant; however, their existence provides straightforward justification of how the interpolant in the nonlinear setting works (namely, via the parallelized representation and the definition of Loewner equivalence). That being said, for LTV plants, the left- and right-Loewner functions are explicitly required when defining the Loewner equivalent interpolant [35].

Proof: Let $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{\mathbb{L}}^\ell(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, and $\overline{\mathbb{L}}^r(\cdot)$ be the set of Loewner functions for the system (36)–(37). We start by rearranging (36) into the form

$$\dot{r} = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1} \sigma\mathbb{L}(r) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1} V(r)u_r.$$

As a result, the functions $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, and $\overline{\mathbb{L}}^\ell(\cdot)$ are solutions to the PDEs, with boundary conditions,

$$\frac{\partial \overline{X}}{\partial \zeta_r}\Lambda\zeta_r = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)^{-1} \sigma\mathbb{L}(\overline{X}(\zeta_r)) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)^{-1} V(\overline{X}(\zeta_r))R\zeta_r, \quad \overline{X}(0) = 0 \tag{38}$$

and

$$\frac{\partial \overline{Y}}{\partial r}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1} \sigma\mathbb{L}(r) = M\overline{Y}(r) - LW(r), \quad \overline{Y}(0) = 0 \tag{39}$$

and

$$\frac{\partial \overline{\mathbb{L}}^\ell}{\partial \zeta_r}\Lambda\zeta_r = \frac{\partial \overline{Y}}{\partial r}(\overline{X}(\zeta_r))\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)^{-1} V(\overline{X}(\zeta_r))R\zeta_r + M\overline{\mathbb{L}}^\ell(\zeta_r), \quad \overline{\mathbb{L}}^\ell(0) = 0 \tag{40}$$

while $\overline{\mathbb{L}}(\cdot)$ and $\overline{\mathbb{L}}^r(\cdot)$ are defined as

$$\overline{\mathbb{L}}(\zeta_r) := -\overline{Y}(\overline{X}(\zeta_r))$$

and

$$\overline{\mathbb{L}}^r(\zeta_r) := \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^\ell(\zeta_r).$$

To prove that (36)–(37) is a Loewner equivalent model, we show that $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ is a solution to the PDEs (38), (39), and (40). Rearranging (38) yields

$$\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)\frac{\partial \overline{X}}{\partial \zeta_r}\Lambda\zeta_r = \sigma\mathbb{L}(\overline{X}(\zeta_r)) - V(\overline{X}(\zeta_r))R\zeta_r$$

while letting $\overline{X}(\zeta_r) = \zeta_r$ yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r = \sigma\mathbb{L}(\zeta_r) - V(\zeta_r)R\zeta_r$$

which holds by definition of $\sigma\mathbb{L}(\cdot)$. Thus, $\overline{X}(\zeta_r) = \zeta_r$ satisfies (38). Letting $\overline{Y}(r) = -\mathbb{L}(r)$ in (39) yields

$$-\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1} \sigma\mathbb{L}(r) = -M\mathbb{L}(r) - LW(r)$$

or

$$\sigma\mathbb{L}(r) = M\mathbb{L}(r) + LW(r)$$

which holds by definition of $\sigma\mathbb{L}(\cdot)$. Thus, $\overline{Y}(r) = -\mathbb{L}(r)$ satisfies (39). Finally, letting $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ in (40) yields

$$\begin{aligned}\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Lambda\zeta_r &= M\mathbb{L}^\ell(\zeta_r) - \frac{\partial \mathbb{L}}{\partial \zeta_r}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\right)^{-1} V(\zeta_r)R\zeta_r \\ &= M\mathbb{L}^\ell(\zeta_r) - \frac{\partial Y}{\partial x}(X(\zeta_r))g(X(\zeta_r))R\zeta_r\end{aligned}$$

which holds by definition of $\mathbb{L}^\ell(\cdot)$. Thus, $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ satisfy (40). Because we have that $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$, we also have that

$$\overline{\mathbb{L}}(\zeta_r) = -\overline{Y}(\overline{X}(\zeta_r)) = \mathbb{L}(\zeta_r)$$

and

$$\overline{\mathbb{L}}^r(\zeta_r) = \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r) = \mathbb{L}^r(\zeta_r)$$

and thus, the system (36)–(37) is Loewner equivalent at (Λ, R, M, L) to the system (21)–(22). ■

Remark 6: Theorem 3 allows constructing reduced order models for the system (21)–(22) in the Loewner sense at (Λ, R, M, L) by simply selecting $\rho = v < n$, and determining the nonlinear Loewner functions of the system (21)–(22).

IV. Interconnection With Nonlinear Generators

In this section, we consider a more general scenario in which the system (21)–(22) is interconnected with two nonlinear systems of the form

$$\dot{\zeta}_r(t) = \lambda(\zeta_r(t)) + \Delta(t) \tag{41}$$

$$v(t) = r(\zeta_r(t)) \tag{42}$$

and

$$\dot{\zeta}_\ell(t) = m(\zeta_\ell(t)) + \ell(\chi(t)) \tag{43}$$

$$\eta(t) = \zeta_\ell(t) \tag{44}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$, and with functions λ, r, m, and ℓ of appropriate dimensions, and such that $\lambda(0) = 0$, $r(0) = 0$, $m(0) = 0$, $\ell(0) = 0$, and $\lambda(\cdot)$ and $m(\cdot)$ are differentiable. Let $\Lambda := \frac{\partial \lambda}{\partial \zeta_r}(0)$ and $M := \frac{\partial m}{\partial \zeta_\ell}(0)$, with Assumption 4 still holding.

To motivate the introduction of nonlinear generators, consider the Van der Pol oscillator (see, e.g., [39]). The limit cycle of the oscillator is stable; however, its linearization at the origin is unstable. If one wanted to determine an interpolant for (21)–(22) when its input is excited by the output of a Van der Pol oscillator,

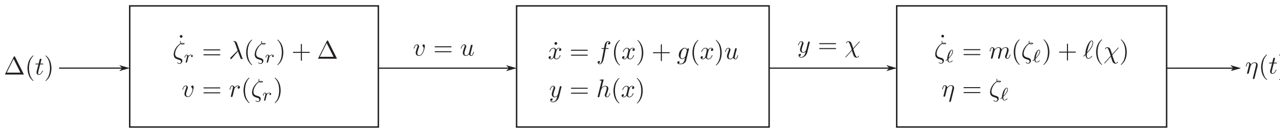

Fig. 3. The interconnected system.

then the linearization at the origin would not be appropriate, as the linearization is unstable. Furthermore, choosing instead a linear generator with poles on the imaginary axis to approximate the oscillator would amount to ignoring nonlinear behaviors that one might be interested in capturing.

Consider now the interconnection of the system (21)–(22) with the generators (41)–(42) and (43)–(44) defined by the interconnection equations $u = v$ and $\chi = y$, which yields the state-space representation

$$\begin{bmatrix}\dot{\zeta}_r\\ \dot{x}\\ \dot{\zeta}_\ell\end{bmatrix} = \begin{bmatrix}\lambda(\zeta_r)\\ f(x)+g(x)r(\zeta_r)\\ m(\zeta_\ell)+\ell(h(x))\end{bmatrix} + \begin{bmatrix}I\\0\\0\end{bmatrix}\Delta \tag{45}$$

$$\eta = \zeta_\ell \tag{46}$$

with state $\begin{bmatrix}\zeta_r^\top & x^\top & \zeta_\ell^\top\end{bmatrix}^\top$, input Δ, and output η. The interconnected system is depicted in Fig. 3.

A. Loewner Functions

We begin by defining the nonlinear enhancements of the tangential generalized controllability and observability matrices, and of the Loewner matrices. These are defined in terms of the functions and matrices making up the interconnected system (45)–(46). The tangential generalized controllability function $X : \mathbb{C}^\rho \to \mathbb{C}^n$ is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial X}{\partial \zeta_r}\lambda(\zeta_r) = f(X(\zeta_r)) + g(X(\zeta_r))r(\zeta_r),\ X(0) = 0. \tag{47}$$

The following claim is a direct consequence of Assumptions 3 and 4 and of the center manifold theory [37].

Proposition 4 (Existence of X): Consider the PDE (47) with the boundary condition $X(0) = 0$. Suppose Assumption 3 and Assumption 4 hold. Then, there exists a function $X : \mathbb{C}^\rho \to \mathbb{C}^n$ satisfying the partial differential equation (47) with the given boundary condition.

The tangential generalized observability function $Y : \mathbb{C}^n \to \mathbb{C}^v$ is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial Y}{\partial x}f(x) = -m(-Y(x)) - \ell(h(x)),\quad Y(0) = 0. \tag{48}$$

To prove the existence of a solution Y, we require the construction of an auxiliary object similar to the object constructed in Section III-A. Consider the system described by the equations

$$\dot{\zeta}_\ell = m(\zeta_\ell) + \ell(h(x)) \tag{49}$$

$$\dot{x} = f(x). \tag{50}$$

Once again, by the center manifold theory and Assumptions 3 and 4, there exists a map $x = \overline{Y}(-\zeta_\ell)$ satisfying the PDE with boundary condition

$$-\frac{\partial \overline{Y}}{\partial \zeta_\ell}(-\zeta_\ell)\left(m(\zeta_\ell) + \ell(h(\overline{Y}(-\zeta_\ell)))\right) = f(\overline{Y}(-\zeta_\ell))$$

$$\overline{Y}(0) = 0. \tag{51}$$

Proposition 5 (Existence of Y): Consider the PDE (48) with the boundary condition $Y(0) = 0$. Suppose Assumptions 3 and 4 hold. Suppose that the map $\overline{Y}$, solving the PDE (51), has a local differentiable left inverse around the origin. Then, there exists a function $Y : \mathbb{C}^n \to \mathbb{C}^v$ satisfying the partial differential equation (48) with the given boundary condition.

Proof: Recall that $\overline{Y}$ satisfies the PDE

$$f(\overline{Y}(-\zeta_\ell)) = -\frac{\partial \overline{Y}}{\partial \zeta_\ell}(-\zeta_\ell)\left(m(\zeta_\ell) + \ell(h(\overline{Y}(-\zeta_\ell)))\right)$$

with boundary condition $\overline{Y}(0) = 0$. Let Y be the local left inverse of $\overline{Y}$, which exists by assumption, that is

$$Y(\overline{Y}(-\zeta_\ell)) = -\zeta_\ell$$

in a neighborhood of the origin. Note that $Y(0) = 0$. Taking the time derivative along the trajectories of the system (49)–(50) yields

$$\frac{\partial Y}{\partial x}\dot{x} = -\frac{\partial Y}{\partial x}(\overline{Y}(-\zeta_\ell))\frac{\partial \overline{Y}}{\partial \zeta_\ell}(-\zeta_\ell)\dot{\zeta}_\ell = -\dot{\zeta}_\ell.$$

Using this equation in $\overline{Y}$ yields

$$\frac{\partial Y}{\partial x}f(x) = -\left(m(\zeta_\ell) + \ell(h(x))\right) = -m(-Y(x)) - \ell(h(x)).$$

Thus, the left inverse of $\overline{Y}$, i.e., Y, solves the PDE (48) in a neighborhood of the origin with the given boundary condition. ∎

Having defined the tangential generalized observability and controllability functions, nonlinear enhancements of V and W are defined as

$$V(\zeta_r) := \frac{\partial Y}{\partial x}(X(\zeta_r))g(X(\zeta_r)),\quad W(\zeta_r) := h(X(\zeta_r)).$$

The nonlinear Loewner function is defined in terms of the tangential generalized controllability and observability functions as

$$\mathbb{L}(\zeta_r) := -Y(X(\zeta_r)).$$

The left-Loewner function $\mathbb{L}^\ell : \mathbb{C}^\rho \to \mathbb{C}^v$ is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) = -m(-\mathbb{L}^\ell(\zeta_r)) - V(\zeta_r)r(\zeta_r)$$

$$\mathbb{L}^\ell(0) = 0 \tag{52}$$

and the right-Loewner function $\mathbb{L}^r : \mathbb{C}^\rho \to \mathbb{C}^v$ is defined as

$$\mathbb{L}^r(\zeta_r) := \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r).$$

A proof that a solution exists for the PDE (52) is now given by extending the proof of the main theorem in [38]. Note that this theorem requires analyticity of $\lambda(\cdot)$, $m(\cdot)$, and $V(\cdot)r(\cdot)$ as the proof makes use of a series expansion.

Proposition 6 (Existence of $\mathbb{L}^\ell$): Consider the PDE (52) with the boundary condition $\mathbb{L}^\ell(0) = 0$ and suppose that λ, m, V, and r are analytic. Suppose there exist constants $C > 0$ and $v > 0$ such that all eigenvalues of M are of type (C, v) with respect to $\sigma(\Lambda)$. Then, there exists a function $\mathbb{L}^\ell : \mathbb{C}^\rho \to \mathbb{C}^v$ satisfying the partial differential equation (52) with the given boundary condition.

In order to prove Proposition 6, we require the following preliminary result.

Lemma 1: Assume that $\kappa : \mathbb{C}^\rho \to \mathbb{C}^\rho$, $h : \mathbb{C}^\rho \to \mathbb{C}^p$, $\beta : \mathbb{C}^p \to \mathbb{C}^v$, and $\epsilon : \mathbb{C}^v \to \mathbb{C}^v$ are analytic vector fields such that $\kappa(0) = 0$, $h(0) = 0$, $\beta(0) = 0$, and $\epsilon(0) = 0$. Let $K = \frac{\partial \kappa}{\partial x}(0)$, $H = \frac{\partial h}{\partial x}(0)$, $B = \frac{\partial \beta}{\partial y}(0)$, and $E = \frac{\partial \epsilon}{\partial z}(0)$. Suppose there exist

1) a $v \times \rho$ matrix T such that $TK = ET - BH$;
2) constants $C > 0$ and $v > 0$ such that all the eigenvalues of E are of type (C, v) with respect to $\sigma(K)$.

Then, locally around $x = 0$, there exists a unique analytic solution θ to the PDE

$$\frac{\partial \theta}{\partial x}\kappa(x) = \epsilon(\theta(x)) - \beta(h(x)).$$

Moreover, $\frac{\partial \theta}{\partial x}(0) = T$.

Proof: The proof extends a similar proof given in [38]; however, the present proof includes the nonlinear term $\epsilon(\theta(\cdot))$. To begin with, by analyticity, expand the functions in the partial differential equation using the Taylor series as

$$\theta(x) = Tx + \sum_{i=2}^{\infty} \theta^{(i)}(x), \quad \kappa(x) = Kx + \sum_{i=2}^{\infty} \kappa^{(i)}(x)$$

$$\beta(h(x)) = BHx + \sum_{i=2}^{\infty} \beta^{(i)}(x)$$

and

$$\epsilon(\theta(x)) = E\theta(x) + \sum_{i=2}^{\infty} \epsilon^{(i)}(\theta(x)).$$

Let $\sigma(K) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of K and let $\sigma(E) = \{\mu_1, \ldots, \mu_\rho\}$ be the spectrum of E. For simplicity, and similarly to [38], we assume that K and E are diagonal; however, this is not necessary. Substituting the expansions into the PDE yields

$$\begin{aligned}&\left(T + \sum_{i=2}^{\infty} \frac{\partial \theta^{(i)}}{\partial x}\right)\left(Kx + \sum_{i=2}^{\infty} \kappa^{(i)}(x)\right)\\ &= \left(E\theta(x) + \sum_{i=2}^{\infty} \epsilon^{(i)}(\theta(x))\right) - \left(BHx + \sum_{i=2}^{\infty} \beta^{(i)}(x)\right)\\ &= (ET - BH)\, x + \sum_{i=2}^{\infty} \left(E\theta^{(i)}(x) - \beta^{(i)}(x)\right)\\ &\quad + \sum_{i=2}^{\infty} \epsilon^{(i)}(\theta(x)). \qquad (53)\end{aligned}$$

Note that $\epsilon^{(m)}(\theta(x))$ contains terms of degree $d \geq m$. Let $\deg(\epsilon^{(m)}(\theta(x)), p)$ denote the terms of degree p from $\epsilon^{(m)}(\theta(x))$. The terms of degree $d = 1$ from (53) are

$$TKx = (ET - BH)x.$$

With some abuse of the summation notation when $d = 2$ (the summation on the LHS is taken to be 0 in this case), the terms of degree $d \geq 2$ are

$$\begin{aligned}T\kappa^{(d)}(x) + \frac{\partial \theta^{(d)}}{\partial x}Kx + \sum_{k=2}^{d-1} \frac{\partial \theta^{(k)}}{\partial x}\kappa^{(d+1-k)}(x)&\\ = E\theta^{(d)}(x) - \beta^{(d)}(x) + \sum_{k=2}^{d} \deg(\epsilon^{(k)}(\theta(x)), d).&\end{aligned}$$

This can be simplified to

$$\frac{\partial \theta^{(d)}}{\partial x}Kx = E\theta^{(d)}(x) - \overline{\beta}^{(d)}(x)$$

where

$$\begin{aligned}\overline{\beta}^{(d)}(x) &:= \beta^{(d)}(x) + T\kappa^{(d)}(x) + \sum_{k=2}^{d-1} \frac{\partial \theta^{(k)}}{\partial x}\kappa^{(d+1-k)}(x)\\ &\quad - \sum_{k=2}^{d} \deg(\epsilon^{(k)}(\theta(x)), d).\end{aligned}$$

It is important to note that $\overline{\beta}^{(d)}$ contains coefficients of θ associated with terms of degree less than d. Therefore, we can expand $\overline{\beta}^d$ and θ^d as

$$\overline{\beta}^{(d)}(x) = \sum_{k=1}^{n} \sum_{|m|=d} \overline{\beta}_{k,m} e_k x^m$$

and

$$\theta^{(d)}(x) = \sum_{k=1}^{n} \sum_{|m|=d} \theta_{k,m} e_k x^m$$

where $x^m = x_1^{m_1} \ldots x_n^{m_n}$. Since K and E are diagonal, $\sigma(K) = \{\lambda_1, \ldots, \lambda_n\}$, and $\sigma(E) = \{\mu_1, \ldots, \mu_n\}$, thus yielding

$$\begin{aligned}-\sum_{k=1}^{n} \sum_{|m|=d} \overline{\beta}_{k,m} e_k x^m &= \sum_{k=1}^{n} \sum_{|m|=d} \theta_{k,m} e_k \frac{\partial x^m}{\partial x}Kx\\ &\quad - \sum_{k=1}^{n} \sum_{|m|=d} \mu_k \theta_{k,m} e_k x^m.\end{aligned}$$

Note that

$$\frac{\partial x^m}{\partial x}Kx = m\lambda x^m$$

hence

$$-\sum_{k=1}^{n} \sum_{|m|=d} \overline{\beta}_{k,m} e_k x^m = \sum_{k=1}^{n} \sum_{|m|=d} \theta_{k,m} e_k m\lambda x^m$$

$$-\sum_{k=1}^{n}\sum_{|m|=d}\mu_k\theta_{k,m}e_k x^m.$$

This leads to the equations

$$\theta_{k,m}m\lambda x^m - \mu_k\theta_{k,m}x^m = -\overline{\beta}_{k,m}x^m$$

hence, $\theta_{k,m}$ can be selected as

$$\theta_{k,m} = (\mu_k - m\lambda)^{-1}\overline{\beta}_{k,m}.$$

Because the eigenvalues of E are of type (C,v) with respect to $\sigma(K)$, $\theta_{k,m}$ is well defined for all k and m. Solving this for each k and for each x^m gives $\theta^{(d)}$ and then determining $\theta^{(d)}$, for $d = 2, 3, \ldots$, yields a solution to the PDE, that is, the function θ. ■

Remark 7: The proof of existence of a solution to the PDE in Lemma 1 is constructive, i.e., existence of solution is proven by building a particular solution in steps. As such, following the steps in the proof yields an explicit solution when the involved functions are analytic and the type-(C,v) condition holds by calculating Taylor series expansions of mappings and constructing each $\overline{\beta}^{(d)}$ term. Scalability for higher dimensional systems is not straightforward; however, this is outside the scope of this article.

Remark 8: The proofs of existence of solution to the PDEs in Propositions 4 and 5 rely on the center manifold theory, and solutions to the involved PDEs can be approximated to any degree of accuracy [37].

Proof (Proof of Proposition 6): It is sufficient to substitute $\mathbb{L}^\ell(\cdot)$, $\lambda(\cdot)$, $m(\cdot)$, and $V(\cdot)r(\cdot)$ into Lemma 1 to complete the proof of Proposition 6. Note that the conditions in Lemma 1 are satisfied because of Assumption 4, and because all eigenvalues of M are of type (C,v) with respect to $\sigma(\Lambda)$. ■

Remark 9: Lemma 1 can also be applied to prove existence of solution for the PDE (48) defining $Y(\cdot)$, when a type-(C,v) condition holds, and to prove existence of solution for the PDE (47) defining $X(\cdot)$, when a type-(C,v) condition holds and $g(x)$ is constant (which is always locally achievable via a coordinates transformation, if the vector field $g(\cdot)$ is involutive). Consequently, if all mappings are analytic and the type-(C,v) conditions hold, then we are able to construct explicit solutions for the considered PDEs via Lemma 1.

Remark 10: The existence conditions based on Lemma 1 have the advantage that the underlying PDEs have solutions even for unstable systems. Namely, unstable nonlinear systems can be analyzed in the nonlinear Loewner framework without stability assumptions. However, when considering unstable systems the relation with the output response is lost.

The definitions introduced thus far show that the Loewner and right-Loewner functions satisfy the PDEs with boundary conditions

$$\frac{\partial\mathbb{L}}{\partial\zeta_r}\lambda(\zeta_r) = m(\mathbb{L}(\zeta_r)) + \ell(W(\zeta_r)) - V(\zeta_r)r(\zeta_r)$$
$$\mathbb{L}(0) = 0$$

and

$$\frac{\partial\mathbb{L}^r}{\partial\zeta_r}\lambda(\zeta_r) = m(\mathbb{L}(\zeta_r)) + m(-\mathbb{L}^\ell(\zeta_r)) + \ell(W(\zeta_r))$$
$$\mathbb{L}^r(0) = 0.$$

The shifted Loewner function $\sigma\mathbb{L} : \mathbb{C}^\rho \to \mathbb{C}^v$ is defined in terms of the left- and right-Loewner functions as

$$\sigma\mathbb{L}(\zeta_r) := -m(-\mathbb{L}^\ell(\zeta_r)) + \frac{\partial\mathbb{L}^r}{\partial\zeta_r}\lambda(\zeta_r)$$

which implies that

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= m(\mathbb{L}(\zeta_r)) + \ell(W(\zeta_r))\\ &= \frac{\partial\mathbb{L}}{\partial\zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r)\end{aligned}$$

and

$$\sigma\mathbb{L}(\zeta_r) = -\frac{\partial Y}{\partial x}(X(\zeta_r))f(X(\zeta_r)).$$

Remark 11: As noted in Remark 4, if the system (21)–(22) is linear, and the auxiliary systems (41)–(42) and (43)–(44) are linear, then the solution to the PDEs (47), (48), and (52) becomes $X(\zeta_r) = X\zeta_r$, $Y(x) = Yx$, and $\mathbb{L}^\ell(\zeta_r) = \mathbb{L}^\ell\zeta_r$, where X, Y, and $\mathbb{L}^\ell$ are the solutions to the Sylvester equations (7), (8), and (9). Thus, the linear Loewner objects are recovered.

B. Loewner Coordinates

To expose the relation between the Loewner functions and the interconnection of systems (45)–(46), we select a specific set of coordinates in a similar fashion as in Theorems 1 and 2.

Theorem 4: Consider the system (45)–(46). The coordinates transformation

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} := \begin{bmatrix} \zeta_r \\ x - X(\zeta_r) \\ \zeta_\ell + Y(x) + \mathbb{L}^\ell(\zeta_r) \end{bmatrix}$$

is such that the system in the new coordinates is described by the equations

$$\begin{bmatrix} \dot z_r \\ \dot z_c \\ \dot z_\ell \end{bmatrix} = \begin{bmatrix} \lambda(z_r) \\ \widetilde{A}\left(z_c + X(z_r), z_r\right) z_c \\ \widetilde{M}\left(z_\ell + \mathbb{L}^r(z_r), z_r\right) z_\ell + \widetilde{G}\left(z_c + X(z_r), z_\ell, z_r\right) z_c \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ m(\mathbb{L}^r(z_r)) - m(\mathbb{L}(z_r)) - m(-\mathbb{L}^\ell(z_r)) \end{bmatrix}$$
$$+ \begin{bmatrix} I \\ -\dfrac{\partial X}{\partial\zeta_r}(z_r) \\ \dfrac{\partial\mathbb{L}^\ell}{\partial\zeta_r}(z_r) \end{bmatrix}\Delta$$

$$\eta = \mathbb{L}^r(z_r) - \widetilde{Y}\left(z_c + X(z_r), z_r\right) z_c + z_\ell$$

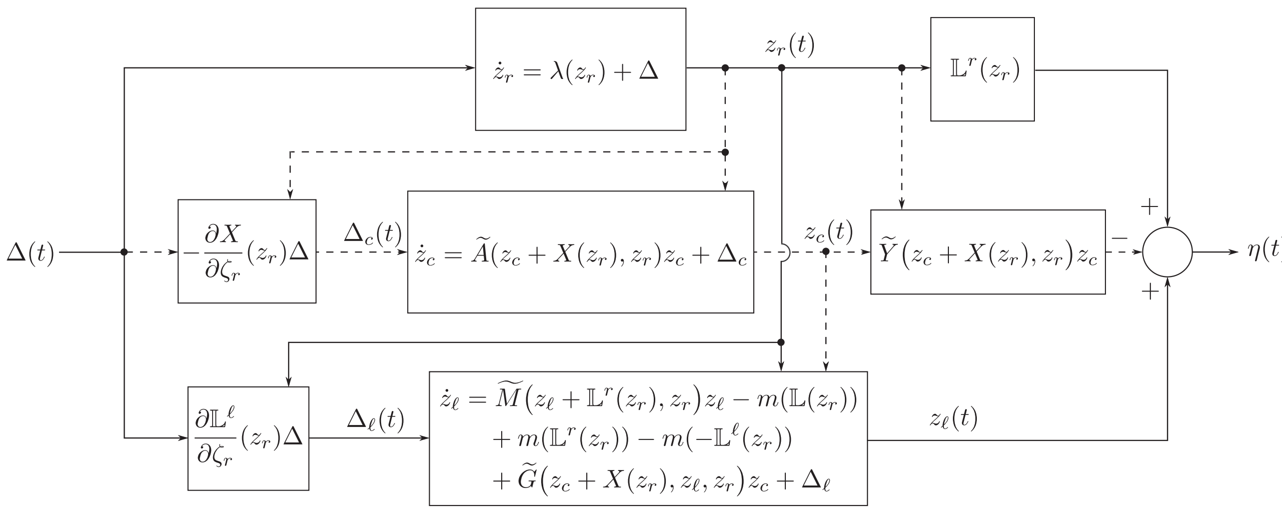

Fig. 4. The transformed, parallel interconnected, system.

where $z_r(t) \in \mathbb{C}^\rho$, $z_c(t) \in \mathbb{C}^n$, $z_\ell(t) \in \mathbb{C}^v$, and $\widetilde{A} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{n\times n}$, $\widetilde{G} : \mathbb{C}^n \times \mathbb{C}^v \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$, $\widetilde{M} : \mathbb{C}^v \times \mathbb{C}^\rho \to \mathbb{C}^{v\times v}$, and $\widetilde{Y} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$.

Proof: We proceed by direct differentiation. For z_c, we have

$$\begin{aligned}\dot z_c &= \dot x - \frac{\partial X}{\partial \zeta_r}\dot\zeta_r \\ &= (f(z_c + X(\zeta_r)) + g(z_c + X(\zeta_r))r(\zeta_r)) \\ &\quad - (f(X(\zeta_r)) + g(X(\zeta_r))r(\zeta_r)) - \frac{\partial X}{\partial \zeta_r}\Delta.\end{aligned}$$

For z_ℓ, we have

$$\begin{aligned}\dot z_\ell &= \dot\zeta_\ell + \frac{\partial Y}{\partial x}\dot x + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\dot\zeta_r \\ &= \left(m(z_\ell - Y(x) - \mathbb{L}^\ell(\zeta_r)) - m(-Y(x)) - m(-\mathbb{L}^\ell(\zeta_r))\right) \\ &\quad + \left(\frac{\partial Y}{\partial x}f(x) + \ell(h(x)) + m(-Y(x))\right) \\ &\quad + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) + \frac{\partial Y}{\partial x}g(x)r(\zeta_r) + m(-\mathbb{L}^\ell(\zeta_r))\right) \\ &\quad + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Delta.\end{aligned}$$

By the PDE-based definitions of $Y(\cdot)$ and $\mathbb{L}^\ell(\cdot)$, (48) and (52), this becomes

$$\begin{aligned}\dot z_\ell &= \left(m(z_\ell - Y(x) - \mathbb{L}^\ell(\zeta_r)) - m(-Y(x)) - m(-\mathbb{L}^\ell(\zeta_r))\right) \\ &\quad + \left(\frac{\partial Y}{\partial x}g(x) - \frac{\partial Y}{\partial x}(X(\zeta_r))g(X(\zeta_r))\right)r(\zeta_r) + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Delta.\end{aligned}$$

Finally, we have that

$$\begin{aligned}\eta &= z_\ell - Y(z_c + X(\zeta_r)) - \mathbb{L}^\ell(\zeta_r) \\ &= \mathbb{L}^r(\zeta_r) - (Y(z_c + X(\zeta_r)) - Y(X(\zeta_r))) + z_\ell.\end{aligned}$$

The result is then obtained by a direct application of Hadamard's Lemma. ■

Note that, by Assumption 3, for any sufficiently small $x(0)$ and $\zeta_r(0)$, the solutions of the interconnected systems approach the center manifold $x = X(\zeta_r)$ exponentially fast; hence, z_c approaches zero provided Δ converges to zero and the system has a converging input converging state property. On the center manifold, that is, for $x = X(\zeta_r)$, or $z_c = 0$, one has

$$\begin{aligned}\dot z_r &= \lambda(z_r) + \Delta \\ \dot z_\ell &= \left(m(z_\ell + \mathbb{L}^r(z_r)) - m(\mathbb{L}(z_r)) - m(-\mathbb{L}^\ell(z_r))\right) \\ &\quad + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}(z_r)\Delta\end{aligned}$$

and

$$\eta = \mathbb{L}^r(z_r) + z_\ell$$

that is, the system restricted to the center manifold contains only information on the Loewner functions. The transformed system is depicted in Fig. 4.

C. Loewner Equivalent Model

In this section, similarly to Section III-C, the concept of a reduced order model in the Loewner sense for nonlinear systems is introduced. In addition, a nonlinear system, reminiscent of the linear systems in [24] and [35], which interpolates the Loewner functions defined by the PDEs (47), (48), and (52), is constructed.

Similarly to Definition 4, we define Loewner equivalence for the more general scenario in which the *objects* defining the generators are functions rather than matrices.

Definition 6 (Loewner Equivalence): Let Σ and $\overline{\Sigma}$ be two systems described by equations of the form (21)–(22) admitting left- and right-Loewner functions $\mathbb{L}^\ell(\cdot), \mathbb{L}^r(\cdot)$, and $\overline{\mathbb{L}}^\ell(\cdot), \overline{\mathbb{L}}^r(\cdot)$, respectively, associated to the functions $\lambda(\cdot), r(\cdot), m(\cdot)$, and $\ell(\cdot)$. Then, Σ and $\overline{\Sigma}$ are called Loewner equivalent at (λ, r, m, ℓ) if $\mathbb{L}^\ell(\zeta_r) = \overline{\mathbb{L}}^\ell(\zeta_r)$ and $\mathbb{L}^r(\zeta_r) = \overline{\mathbb{L}}^r(\zeta_r)$ in a neighborhood of the origin.

Consistently, we say that a nonlinear system interpolates another nonlinear system (in the Loewner sense) at (λ, r, m, ℓ) if the two systems are Loewner equivalent at (λ, r, m, ℓ). That is, for the same functions $\lambda(\cdot), r(\cdot), m(\cdot), \ell(\cdot)$, the interpolating system possesses the exact same left- and right-Loewner functions.

Similarly to Section III-C, the property of Loewner equivalence has a strong implication on the steady-state behavior of the systems. By Theorem 4, recalling Assumptions 3 and 4, assuming the foregoing stability conditions hold, that Δ is sufficiently small, bounded, and converges to zero, it is easy to see that the steady-state response of the system interconnected with the generators is dependent entirely on the generator states and the left- and right-Loewner functions. Thus, if two locally exponentially stable systems are Loewner equivalent at (λ, r, m, ℓ), then there exist initial conditions such that the two systems interconnected with the generators have the same steady-state behavior, provided it exists.

We can now define what a reduced order model is in the Loewner sense. Note that this is a more general version of Definition 5 in which the *objects* defining the generators are functions rather than matrices.

Definition 7 (Reduced Order Model): Let Σ and $\overline{\Sigma}$ be two systems of order n and v, respectively. $\overline{\Sigma}$ is called a reduced order model of Σ in the Loewner sense if Σ and $\overline{\Sigma}$ are Loewner equivalent at (λ, r, m, ℓ) and $v < n$.

We now construct a nonlinear system, which is Loewner equivalent at (λ, r, m, ℓ) to (21)–(22), given that the Loewner functions of (45)–(46) are known. Once again, note that this is a more general version of Theorem 3.

Theorem 5: Consider the interconnected system (45)–(46) with $\rho = v$. Let $\mathbb{L}^\ell(\cdot), \mathbb{L}^r(\cdot), \mathbb{L}(\cdot), \sigma\mathbb{L}(\cdot), V(\cdot)$, and $W(\cdot)$ be the associated Loewner functions. Assume that $\frac{\partial \mathbb{L}}{\partial \zeta_r}$ is nonsingular. Define the system

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\dot{r} = \sigma\mathbb{L}(r) - V(r)u_r \tag{54}$$

$$y_r = W(r) \tag{55}$$

with state $r(t) \in \mathbb{C}^\rho$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$. Then, the system (54)–(55) is Loewner equivalent at (λ, r, m, ℓ) to the system (21)–(22).

Proof: Let $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{\mathbb{L}}^\ell(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, and $\overline{\mathbb{L}}^r(\cdot)$ be the set of Loewner functions for the system (54)–(55). We start by rearranging (54) into the form

$$\dot{r} = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1}\sigma\mathbb{L}(r) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1}V(r)u_r.$$

As a result, the functions $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, and $\overline{\mathbb{L}}^\ell(\cdot)$ are solutions to the PDEs

$$\begin{aligned}\frac{\partial \overline{X}}{\partial \zeta_r}\lambda(\zeta_r) = &\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)^{-1}\sigma\mathbb{L}(\overline{X}(\zeta_r)) \\ &- \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)^{-1}V(\overline{X}(\zeta_r))r(\zeta_r),\ \overline{X}(0) = 0\end{aligned} \tag{56}$$

$$\begin{aligned}\frac{\partial \overline{Y}}{\partial r}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1}\sigma\mathbb{L}(r) = &-m(-\overline{Y}(r)) - \ell(W(r)), \\ &\overline{Y}(0) = 0\end{aligned} \tag{57}$$

and

$$\begin{aligned}\frac{\partial \overline{\mathbb{L}}^\ell}{\partial \zeta_r}\lambda(\zeta_r) = &\frac{\partial \overline{Y}}{\partial r}(\overline{X}(\zeta_r))\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)^{-1}V(\overline{X}(\zeta_r))r(\zeta_r) \\ &- m(-\overline{\mathbb{L}}^\ell(\zeta_r)),\ \overline{\mathbb{L}}^\ell(0) = 0\end{aligned} \tag{58}$$

while $\overline{\mathbb{L}}(\cdot)$ and $\overline{\mathbb{L}}^r(\cdot)$ are defined as

$$\overline{\mathbb{L}}(\zeta_r) := -\overline{Y}(\overline{X}(\zeta_r))$$

and

$$\overline{\mathbb{L}}^r(\zeta_r) := \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^\ell(\zeta_r).$$

To prove that (54)–(55) is a Loewner equivalent model, we show that $\overline{X}(\zeta_r) = \zeta_r, \overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ is a solution to the PDEs (56), (57), and (58). Rearranging (38) yields

$$\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(\overline{X}(\zeta_r))\right)\frac{\partial \overline{X}}{\partial \zeta_r}\lambda(\zeta_r) = \sigma\mathbb{L}(\overline{X}(\zeta_r)) - V(\overline{X}(\zeta_r))r(\zeta_r)$$

while letting $\overline{X}(\zeta_r) = \zeta_r$ yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) = \sigma\mathbb{L}(\zeta_r) - V(\zeta_r)r(\zeta_r)$$

which holds by the definition of $\sigma\mathbb{L}(\cdot)$. Thus, $\overline{X}(\zeta_r) = \zeta_r$ satisfies (56). Letting $\overline{Y}(r) = -\mathbb{L}(r)$ in (57) yields

$$-\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}(r)\right)^{-1}\sigma\mathbb{L}(r) = -m(\mathbb{L}(r)) - \ell(W(r))$$

or

$$\sigma\mathbb{L}(r) = m(\mathbb{L}(r)) + \ell(W(r))$$

which holds by the definition of $\sigma\mathbb{L}(\cdot)$. Thus, $\overline{Y}(r) = -\mathbb{L}(r)$ satisfies (57). Finally, letting $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ in (58) yields

$$\begin{aligned}\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) &= -m(-\mathbb{L}^\ell(\zeta_r)) - \frac{\partial \mathbb{L}}{\partial \zeta_r}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\right)^{-1}V(\zeta_r)r(\zeta_r) \\ &= -m(-\mathbb{L}^\ell(\zeta_r)) - \frac{\partial Y}{\partial x}(X(\zeta_r))g(X(\zeta_r))r(\zeta_r)\end{aligned}$$

which holds by the definition of $\mathbb{L}^\ell(\cdot)$. Thus, $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ satisfy (58). Because we have that $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(r) = -\mathbb{L}(r)$, and $\overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$, we also have that

$$\overline{\mathbb{L}}(\zeta_r) = -\overline{Y}(\overline{X}(\zeta_r)) = \mathbb{L}(\zeta_r)$$

and

$$\overline{\mathbb{L}}^r(\zeta_r) = \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^\ell(\zeta_r) = \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r) = \mathbb{L}^r(\zeta_r)$$

and thus, the system (54)–(55) is Loewner equivalent for (λ, r, m, ℓ) to the system (21)–(22). ■

V. Conclusion

We have presented new methods for the model reduction of nonlinear systems. These methods extend the state-space interpretation of the Loewner matrices, which are classically interpreted in the frequency domain, previously developed by the authors for linear systems in [34] to nonlinear systems. Given that the frequency domain interpretation of the Loewner matrices holds little meaning in the nonlinear setting, we define new objects, the Loewner functions, which are solutions to partial differential equations and are generalizations of the Loewner matrices. Given the Loewner functions for an underlying nonlinear system, we have presented a particular reduced order model that interpolates the underlying system, that is, the reduced order model produces the same Loewner functions as the underlying system. Locally, the two systems produce the same steady-state response, provided it exists, when interconnected with the generators corresponding to the Loewner functions. The relationship between the Loewner framework for nonlinear systems and the two-sided moment matching procedure for nonlinear systems in [36] is currently being investigated.

Appendix
On the Use of Complex-Valued Signals

At first glance, the restriction of Λ and M to diagonal matrices may seem prohibitive. Many important interpolation points, such as those on the imaginary axis of the complex plane, would not be implementable under such a framework. In this section, it is shown that diagonality of the generator matrices is not actually required, and that such interpolation points can, in fact, be achieved.

1) Linear Systems: Let $P \in \mathbb{C}^{\rho\times\rho}$ and $Q \in \mathbb{C}^{v\times v}$ be nonsingular matrices. Consider generators of the form (11)–(12) and (13)–(14) defined by the equations

$$\dot{\zeta}_r = \underbrace{P\Lambda P^{-1}}_{=:\overline{\Lambda}}\zeta_r + \Delta \tag{59}$$

$$v = \underbrace{RP^{-1}}_{=:\overline{R}}\zeta_r \tag{60}$$

and

$$\dot{\zeta}_\ell = \underbrace{QMQ^{-1}}_{=:\overline{M}}\zeta_\ell + \underbrace{QL}_{=:\overline{L}}\,y \tag{61}$$

$$\eta = \zeta_\ell \tag{62}$$

where $\Lambda \in \mathbb{C}^{\rho\times\rho}$ and $M \in \mathbb{C}^{v\times v}$ are diagonal, and $\overline{\Lambda}$, $\overline{R}$, $\overline{M}$, and $\overline{L}$ are real-valued matrices, and therefore, *implementable*. Let $X, Y, W, V, \mathbb{L}$, and $\sigma\mathbb{L}$ denote the *set* of Loewner matrices associated with the generators (11)–(12) and (13)–(14) interconnected with the plant (1)–(2). We now use the real-valued matrices in (59)–(60) and (61)–(62) to construct a new set of Loewner matrices, which we represent in terms of the former set of Loewner matrices. Letting $\overline{X}$, $\overline{Y}$, $\overline{W}$, $\overline{V}$, $\overline{\mathbb{L}}$, and $\overline{\sigma\mathbb{L}}$ denote the new set of Loewner matrices associated with the generators (59)–(60) and (61)–(62) interconnected with the plant (1)–(2), we obtain

$$\overline{X} = XP^{-1},\quad \overline{W} = WP^{-1},\quad \overline{Y} = QY,\quad \overline{V} = QV$$

which yields the new Loewner matrices

$$\overline{\mathbb{L}} = Q\mathbb{L}P^{-1},\quad \overline{\sigma\mathbb{L}} = Q\sigma\mathbb{L}P^{-1}.$$

We then construct the new Loewner equivalent model as

$$\dot{\omega} = \overline{\mathbb{L}}^{-1}\overline{\sigma\mathbb{L}}\omega - \overline{\mathbb{L}}^{-1}\overline{V}u_r$$
$$y_r = \overline{W}\omega$$

which is simplified to

$$\dot{\omega} = P\mathbb{L}^{-1}\sigma\mathbb{L}P^{-1}\omega - P\mathbb{L}^{-1}Vu_r$$
$$y_r = WP^{-1}\omega.$$

This new interpolating model is obtained by a coordinates transformation of the interpolating model (19)–(20); hence, the generators (11)–(12) and (13)–(14), and the generators (59)–(60) and (61)–(62), interconnected with the plant (1)–(2) produce the same interpolant, albeit in different coordinates.

2) Nonlinear Plant With Linear Generators: In a similar fashion, we now consider the set of Loewner functions associated with the generators (59)–(60) and (61)–(62) interconnected with the plant (21)–(22), which we denote $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{W}(\cdot)$, $\overline{V}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^\ell(\cdot)$, $\overline{\mathbb{L}}^r(\cdot)$, and $\overline{\sigma\mathbb{L}}(\cdot)$. Let $X(\cdot)$, $Y(\cdot)$, $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, $\mathbb{L}^\ell(\cdot)$, $\mathbb{L}^r(\cdot)$, and $\sigma\mathbb{L}(\cdot)$ denote the set of Loewner functions associated with the interconnected system (23)–(24), (25)–(26), and (21)–(22). Then, it follows that

$$\overline{X}(\zeta_r) = X(P^{-1}\zeta_r),\quad \overline{W}(\zeta_r) = W(P^{-1}\zeta_r)$$
$$\overline{Y}(x) = QY(x),\quad \overline{V}(\zeta_r) = QV(P^{-1}\zeta_r)$$

which admits the new Loewner functions

$$\overline{\mathbb{L}}(\zeta_r) = Q\mathbb{L}(P^{-1}\zeta_r),\quad \overline{\mathbb{L}}^\ell(\zeta_r) = Q\mathbb{L}^\ell(P^{-1}\zeta_r)$$
$$\overline{\mathbb{L}}^r(\zeta_r) = Q\mathbb{L}^r(P^{-1}\zeta_r),\quad \overline{\sigma\mathbb{L}}(\zeta_r) = Q\sigma\mathbb{L}(P^{-1}\zeta_r).$$

Note also that

$$\frac{\partial\overline{\mathbb{L}}}{\partial\zeta_r} = Q\frac{\partial\mathbb{L}}{\partial\zeta_r}(P^{-1}\zeta_r)P^{-1}.$$

The new Loewner equivalent model is given by

$$\frac{\partial\overline{\mathbb{L}}}{\partial\zeta_r}(\omega)\dot{\omega} = \overline{\sigma\mathbb{L}}(\omega) - \overline{V}(\omega)u_r$$
$$y_r = \overline{W}(\omega).$$

This is simplified to

$$\frac{\partial\mathbb{L}}{\partial\zeta_r}(P^{-1}\omega)P^{-1}\dot{\omega} = \sigma\mathbb{L}(P^{-1}\omega) - V(P^{-1}\omega)u_r$$
$$y_r = W(P^{-1}\omega)$$

which is obtained via a coordinates transformation from the interpolating model (36)–(37).

3) Nonlinear Plant With Nonlinear Generators: Consider now generators of the form (41)–(42) and (43)–(44) defined by the equations

$$\dot{\zeta}_r = \underbrace{P\lambda(P^{-1}\zeta_r)}_{=:\overline{\lambda}(\zeta_r)} + \Delta \tag{63}$$

$$v = \underbrace{r(P^{-1}\zeta_r)}_{=:\overline{r}(\zeta_r)} \tag{64}$$

and

$$\dot{\zeta}_\ell = \underbrace{Qm(Q^{-1}\zeta_\ell)}_{=:\overline{m}(\zeta_\ell)} + \underbrace{Q\ell(y)}_{=:\overline{\ell}(y)} \tag{65}$$

$$\eta = \zeta_\ell \tag{66}$$

where $\lambda : \mathbb{C}^\rho \to \mathbb{C}^\rho$ and $m : \mathbb{C}^v \to \mathbb{C}^v$ are such that $\frac{\partial \lambda}{\partial \zeta_r}(0) = \Lambda$ and $\frac{\partial m}{\partial \zeta_\ell}(0) = M$ are diagonal, and $\overline{\lambda}(\cdot)$, $\overline{r}(\cdot)$, $\overline{m}(\cdot)$, and $\overline{\ell}(\cdot)$ are real-valued maps. We now consider the set of Loewner functions associated with the generators (63)–(64) and (65)–(66) interconnected with the plant (21)–(22), which we denote $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{W}(\cdot)$, $\overline{V}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^\ell(\cdot)$, $\overline{\mathbb{L}}^r(\cdot)$, and $\overline{\sigma\mathbb{L}}(\cdot)$. Let $X(\cdot)$, $Y(\cdot)$, $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, $\mathbb{L}^\ell(\cdot)$, $\mathbb{L}^r(\cdot)$, and $\sigma\mathbb{L}(\cdot)$ denote the set of Loewner functions associated with the interconnected system (41)–(42), (43)–(44), and (21)–(22). Then, it follows that

$$\overline{X}(\zeta_r) = X(P^{-1}\zeta_r), \quad \overline{W}(\zeta_r) = W(P^{-1}\zeta_r)$$
$$\overline{Y}(x) = QY(x), \quad \overline{V}(\zeta_r) = QV(P^{-1}\zeta_r)$$

which admits the new Loewner functions

$$\overline{\mathbb{L}}(\zeta_r) = Q\mathbb{L}(P^{-1}\zeta_r), \quad \overline{\mathbb{L}}^\ell(\zeta_r) = Q\mathbb{L}^\ell(P^{-1}\zeta_r)$$
$$\overline{\mathbb{L}}^r(\zeta_r) = Q\mathbb{L}^r(P^{-1}\zeta_r), \quad \overline{\sigma\mathbb{L}}(\zeta_r) = Q\sigma\mathbb{L}(P^{-1}\zeta_r).$$

Note also that

$$\frac{\partial \overline{\mathbb{L}}}{\partial \zeta_r} = Q\frac{\partial \mathbb{L}}{\partial \zeta_r}(P^{-1}\zeta_r)P^{-1}.$$

The new Loewner equivalent model is given by

$$\frac{\partial \overline{\mathbb{L}}}{\partial \zeta_r}(\omega)\dot{\omega} = \overline{\sigma\mathbb{L}}(\omega) - \overline{V}(\omega)u_r$$
$$y_r = \overline{W}(\omega).$$

This is simplified to

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}(P^{-1}\omega)P^{-1}\dot{\omega} = \sigma\mathbb{L}(P^{-1}\omega) - V(P^{-1}\omega)u_r$$
$$y_r = W(P^{-1}\omega)$$

which is obtained via a coordinates transformation from the interpolating model (54)–(55).

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Joel D. Simard was born in London, ON, Canada, in 1993. He received a B.A.Sc. in electrical engineering from the University of Waterloo, Waterloo, ON, Canada, in 2016. In 2018 he received a M.A.Sc. in electrical engineering studying adaptive control from the University of Waterloo, Waterloo, ON, Canada. In 2018 he joined the Control and Power Research Group at Imperial College London, London, U.K., where he is currently working towards a Ph.D. degree in nonlinear control. His research is mainly focused on model reduction and the analysis of nonlinear systems.

Alessandro Astolfi was born in Rome, Italy, in 1967. He graduated in electrical engineering from the University of Rome in 1991. In 1992 he joined ETH-Zurich where he obtained a M.Sc. in information theory in 1995 and the Ph.D. degree with Medal of Honor in 1995 with a thesis on discontinuous stabilization of nonholonomic systems. In 1996 he was awarded a Ph.D. from the University of Rome La Sapienza for his work on nonlinear robust control. Since 1996 he has been with the Electrical and Electronic Engineering Department of Imperial College London, London (UK), where he is currently Professor of Nonlinear Control Theory and Head of the Control and Power Group. From 1998 to 2003 he was also an Associate Professor at the Department of Electronics and Information of the Politecnico of Milano. Since 2005 he has also been a Professor at Dipartimento di Ingegneria Civile e Ingegneria Informatica, University of Rome Tor Vergata. He has been/is Visiting Lecturer in Nonlinear Control in several universities, including ETHZurich (1995-1996); Terza University of Rome (1996); Rice University, Houston (1999); Kepler University, Linz (2000); SUPELEC, Paris (2001), Northeastern University (2013), University of Cyprus (2018–), Southeast University (2019–).

His research interests are focused on mathematical control theory and control applications, with special emphasis for the problems of discontinuous stabilization, robust and adaptive control, observer design and model reduction. He is the author of more than 165 journal papers, of 30 book chapters and of over 380 papers in refereed conference proceedings. He is the author (with D. Karagiannis and R. Ortega) of the monograph "Nonlinear and Adaptive Control with Applications" (Springer-Verlag).

He is the recipient of the IEEE CSS A. Ruberti Young Researcher Prize (2007), the IEEE RAS Googol Best New Application Paper Award (2009), the IEEE CSS George S. Axelby Outstanding Paper Award (2012), the Automatica Best Paper Award (2017). He is a "Distinguished Member" of the IEEE CSS, IEEE Fellow, IFAC Fellow, IET Fellow, and member of the Academia Europaea. He served as Associate Editor for Automatica, Systems and Control Letters, the IEEE Transactions on Automatic Control, the International Journal of Control, the European Journal of Control and the Journal of the Franklin Institute; as Area Editor for the International J. of Adaptive Control and Signal Processing; as Senior Editor for the IEEE Transactions on Automatic Control; and as Editor-in-Chief for the European Journal of Control. He is currently Editor-in-Chief of the IEEE Transactions on Automatic Control. He served as Chair of the IEEE CSS Conference Editorial Board (2010-2017) and in the IPC of several international conferences. He was/is a Member of the IEEE Fellow Committee (2016/2019-2022).