Relational structures associated to topological spaces and preserved by image functions

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Abstract. We study relational and algebraic first-order structures on $\mathcal{P}(X)$, for X a topological space, with the further requirement that such structures are preserved by image functions associated to continuous functions. Many of the above structures have arisen independently in disparate and very distant fields.

In particular, we deal with a ternary relation $x \sqsubseteq^z y$ whose intended interpretation is $x \subseteq z \cup Ky$, where K is closure in some topological space. The study provides a smoother, simpler and more general theory, with respect to the formerly studied "basic" binary relation given by $x \subseteq Ky$.

We provide an axiomatization for semilattices with such an "extended" ternary relation, characterizing those structures which can be embedded into a topological model with the above interpretation. More generally, we construct "free extensions" of extended specialization semilattices into closure semilattices. We also take into account the possibility of adding contact and n-ary hypercontact relations. In this way we generalize and uniformize many previous results.

1. Introduction

In [12] we proposed the project of studying those relational and algebraic first-order structures associated to topologies which are preserved "covariantly" by continuous functions. Admittedly, the project has a somewhat narrow range of application: in a strict sense, closure itself is not preserved. Indeed, if $\varphi : X \to Y$ is a continuous function between two topological spaces and $\varphi^{\to}$ is the associate image function, then $\varphi(Kx) \subseteq K(\varphi(x))$, for $x \subseteq X$, where K denotes closure. On the other hand, the reverse inclusion holds only if φ is a closed function.

However, many properties are preserved also in the mentioned strict sense, for example, the notion of adherence between a point and a subset of a topological space. More generally, image functions associated to continuous maps preserve the *contact* or *proximity* relation δ defined by $x \ \delta \ y$ if $Kx \cap Ky \neq \emptyset$. As well-known, such preservation properties provide an equivalent definition for the notion of continuity: a function is continuous if and only if it preserves adherence, equivalently, contact (in the above sense). Such definitions seem

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much closer to intuition than the nowadays more common definition of continuity in terms of preimages, and in fact they played an important role in the hystorical development of topology.

The example we mainly considered in [12] is the binary relation $a \sqsubseteq b$ defined for subsets a, b of some topological space X and whose interpretation is given by $a \subseteq Kb$, where K is the closure of X. In [12] we provided an axiomatization for those semilattices with a further preorder $\sqsubseteq$ which can be represented in the above way, where the join operation corresponds to set theoretical union. We also discovered that such *specialization semilattices* appeared independently in many other fields. See [8, 9, 12, 15] for details. As is the case for the contact mentioned above, the specialization relation $\sqsubseteq$ is sufficient to detect continuity; indeed, a function φ between topological spaces is continuous if and only if the image function $\varphi^{\rightarrow}$ is a $\sqsubseteq$-homomorphism [12, Proposition 2.4]. This is an immediate generalization of the classical fact that a function φ is continuous if and only if φ preserves the adherence relation between points and subsets of topological spaces.

Further representations of specialization semilattices appear in [8, 9, 15].

Here we extend the notion of a specialization semilattice by considering a ternary relation whose intended interpretation is given by $a \subseteq h \cup Kb$. Quite surprisingly, though the notion might appear more complex, proofs in this case turn out to be much simpler. We present axioms for such *extended specialization semilattices* and provide various representation theorems, possibly when specialization is combined with a contact or hypercontact relation. See [6, 8, 9, 12, 15] for further examples and motivations, and [4, 5, 10, 11, 13, 17] for the relevance of contact and hypercontact relations.

2. Preliminaries

Unexplained notions and notation can be found in [2, 3, 6, 8, 9, 12]. Semilattices will always be considered as join semilattice and the join operation will be denoted by $+$. For the sake of simplicity, we will sometimes assume that partially ordered sets (*posets*, for short) have a minimum element, here always denoted by 0. This is generally not strictly necessary, but will simplify notation and some arguments. See Remark 3.7 below. See also [8, Remarks 2.3 and 3.4], [11, Remark 6] and [13, Remark 6.10] for parallel observations.

Homomorphisms and *embeddings* will always be considered in the classical model-theoretical sense [7]. However, we will reproduce the relevant definitions in each case of interest. In case of structures with a 0, we will always assume that homomorphisms send 0 to 0. Formally, 0 is interpreted as a constant in the language. Indeed, when we say that a poset $\mathbf{P}$ (or some other structure) *has a* 0, we will always mean that not only 0 is a minimum element of $\mathbf{P}$, but also that 0 is (the interpretation of) a constant of $\mathbf{P}$, for example, $\mathbf{P} = (P, \leq, 0)$.

We will sometimes deal with functions preserving only part of the structure. For example, if δ is interpreted as a binary relation in a class of posets with further structure, a $\{\leq, \delta\}$-homomorphism is an order-preserving function φ such that, in addition, $a \ \delta \ b$ implies $\varphi(a) \ \delta \ \varphi(b)$, but φ is not required to preserve further structure.

2.1. Structures associated to topologies. We are now going to recall the definitions of certain structures associated to topological spaces. Together with a further kind of structure we will introduce in the next section, these will be frequently combined, and their possible merging is the main subject of the present paper.

In each case, those structures which are directly associated to topological spaces will be called *topological*. Essentially, our main aim is to characterize the larger class of those structures which can be embedded into topological structures in the above sense.

2.2. Closure. Recall that a *pre-closure* operation on some poset $\mathbf{P}$ is a unary operation K which is extensive and isotone, namely, K satisfies $Kx \geq x$, for all $x \in P$ and, moreover, $x \leq y$ implies $Kx \leq Ky$. If K is also idempotent, that is, $KKx = Kx$, then K is called a *closure operation*. If $\mathbf{P}$ has a minimum element 0, K is said to be *normal* if $K0 = 0$. In presence of a 0 we will always assume normality, even without specific mention. A (pre-)closure operation on some semilattice is *additive* if $K(x + y) = Kx + Ky$, for all x and y.

If c is an element of a poset with a pre-closure operation, c is said to be *closed* if $Kc = c$. If both c and d are closed and the meet e of c and d exists, then e is closed, as well. Indeed, $Ke \leq Kc = c$ and $Ke \leq Kc = c$, by isotony, hence $e \leq Ke \leq e$, by extensiveness and since e is the meet of c and d. If K is also assumed to be idempotent, that is, K is a closure operation, then an element c is closed if and only if c has the form $c = Kd$, for some element d.

A *(pre-)closure poset* is a poset together with a (pre-)closure operation. A (pre-)closure poset $\mathbf{S}$ is *normal* if $\mathbf{S}$ has a minimum element 0 and K is normal. *(Pre-)closure semilattices* and *normal (pre-)closure semilattices* are defined in an analogous way. A (pre-)closure semilattice is *additive* if K is additive. Recall that a *closure algebra* is a Boolean algebra together with an additive closure operation, namely, an *additive closure Boolean algebra*, but the concise terminology has become standard in the literature.

We will use the following well-known property of closure semilattices:

$$K(x + y) = K(x + Ky) \tag{2.1}$$

See, e. g., [8, Remark 2.1(b)] for a proof.

A *homomorphism* between two closure semilattices (posets) is a semilattice homomorphism (an order preserving map) η such that $\eta(Ka) = K\eta(a)$, for all elements a in the domain. An *embedding* is an injective homomorphism (in the case of posets, the following implication is also required: $\eta(a) \leq \eta(b)$ implies $a \leq b$). The definition is the same for pre-closure posets and semilattices.

Definition 2.2.1. If X is a topological space with closure K, then $\mathbf{S}(X) = (\mathcal{P}(X), \cup, \emptyset, K)$ is an additive closure semilattice with normal closure, which will be called the *closure semilattice associated to* X. As mentioned at the beginning of this section, closure semilattices which have the form $\mathbf{S}(X)$ as above will be called *topological*.

Note that, when dealing also with morphisms, the correspondence is not exactly functorial in the covariant sense: if X and Y are topological spaces, φ is a function from X to Y and the image function $\varphi^{\rightarrow} : \mathbf{S}(X) \to \mathbf{S}(Y)$ is a homomorphism between the associated closure semilattices, then φ is a continuous function. However, the converse holds only if φ is also closed. See [12].

2.3. Contact relations. A *weak contact poset* is a poset $\mathbf{P}$ with minimum element 0 together with a binary *weak contact relation* δ on P, that is, a binary relation such that

$$a \mathrel{\delta} b \Leftrightarrow b \mathrel{\delta} a \tag{Sym}$$

$$a \mathrel{\delta} b \Rightarrow a > 0 \;\&\; b > 0, \tag{Emp}$$

$$a \mathrel{\delta} b \;\&\; a \leq a_1 \;\&\; b \leq b_1 \Rightarrow a_1 \mathrel{\delta} b_1, \tag{Ext}$$

$$a \neq 0 \Rightarrow a \mathrel{\delta} a, \tag{Ref}$$

for all $a, b, a_1, b_1 \in P$. We write $a \mathrel{\not\delta} b$ to mean that $a \mathrel{\delta} b$ does not hold. A *weak contact semilattice* is defined similarly. The reason for the adjective "weak" is that the following property is frequently required in the definition of a contact relation. An *additive contact relation* on some semilattice is a weak contact relation satisfying the following condition:

$$a \mathrel{\delta} b{+}c \;\Rightarrow\; a \mathrel{\delta} b \;\text{ or }\; a \mathrel{\delta} c. \tag{Add}$$

Throughout, "contact" will always mean "additive contact", while "weak contact" will always mean "not necessarily additive weak contact".

Definition 2.3.1. When symmetry (Sym) of δ is not assumed, we will speak of a *weak pre-contact* (the *pre* here has a meaning unrelated with the *pre* in the definition of a pre-closure operation, however, the terminology is standard.) In the nonsymmetrical case (Add) will be called *right-additivity* and an (additive) *pre-contact* relation is also assumed to satisfy the parallel left-additivity. In any case, we will always assume (Ref) (some authors do not assume (Ref) in the definition of a weak pre-contact).

A *homomorphism* of weak contact or pre-contact posets (semilattices) is an order preserving map (a semilattice homomorphism) which preserves 0 and such that $a \mathrel{\delta} b$ implies $\eta(a) \mathrel{\delta} \eta(b)$. An *embedding* is also required to be injective and to satisfy the converse implication.

Definition 2.3.2. *Various kinds of overlap relations.*

(a) If $\mathbf{S}$ is a poset with 0, then, setting

$$a \;\sigma\; b \text{ if there is } p \in S,\ p > 0 \text{ such that } p \leq a \text{ and } p \leq b, \tag{2.2}$$

we get a weak contact relation called the *overlap relation.* In particular, this applies to $(\mathcal{P}(X), \subseteq, \emptyset)$, for every set X.

(b) If X is a topological space with closure K and we set

$$a \;\delta\; b \quad \text{if} \quad Ka \cap Kb \neq \emptyset, \tag{2.3}$$

then $(\mathcal{P}(X), \cup, \emptyset, \delta)$ is a contact semilattice (with additive contact), which will be called the *contact semilattice associated to* X.

If, instead, we set

$$a \;\alpha\; b \quad \text{if} \quad a \cap Kb \neq \emptyset, \tag{2.4}$$

then $(\mathcal{P}(X), \cup, \emptyset, \alpha)$ is an additive pre-contact semilattice, again, called the *pre-contact semilattice associated to* X. As above, all such structures here in (b) will be called *topological*.

(c) More generally, if in a normal pre-closure poset $\mathbf{P}$ with 0 we set

$$a \;\delta\; b \text{ if there is } p \in P,\ p > 0 \text{ such that } p \leq Ka \text{ and } p \leq Kb, \tag{2.5}$$

then we get a weak contact δ on $\mathbf{P}$. We will call it the *closure-overlap (or K-overlap, for short) weak contact associated to* $\mathbf{P}$. We get a weak pre-contact, called the *K-pre-overlap*, if we set

$$a \;\alpha\; b \text{ if there is } p \in P,\ p > 0 \text{ such that } p \leq a \text{ and } p \leq Kb. \tag{2.6}$$

The assumption that K is extensive can be weakened; it is enough to assume that $p > 0$ implies $Kp > 0$.

Lemma 2.3.3. *If* $\mathbf{S}$ *is a distributive lattice, then the overlap contact defined by* (2.2) *is additive.*

If $\mathbf{S}$ *is a normal additive pre-closure distributive lattice, then the weak contact defined by* (2.5) *and the weak pre-contact defined by* (2.6) *are additive.*

Proof. The first statement appears in [5, Lemma 2, item 1], or see [10, Lemma 2.4]. The proof is somewhat simpler than the following proof of the second statement.

In a lattice, condition (2.5) reads $a \;\delta\; b$ if $Ka \cdot Kb > 0$. Thus $a + a_1 \;\delta\; b$ if and only if $K(a + a_1) \cdot Kb > 0$, if and only if $(Ka + Ka_1) \cdot Kb > 0$, by additivity of K, if and only if $(Ka \cdot Kb) + (Ka_1 \cdot Kb) > 0$, since the lattice is assumed to be distributive. The last inequality means exactly that either $Ka \cdot Kb > 0$ or $Ka_1 \cdot Kb > 0$, that is, either $a \;\delta\; b$ or $a_1 \;\delta\; b$.

The proof dealing with (2.6) is similar; actually, additivity of K is not used in the proof of left additivity of α. $\square$

In comparison to the case of closure, in the case of contact, functoriality is preserved for topological spaces. Indeed, a function between two topological spaces is continuous if and only if the corresponding image function is a homomorphism between the associated contact semilattices.

2.4. Hypercontact relations. See [13] for motivations for the study of hypercontact (non binary) relations.

Definition 2.4.1. In [13] we defined a *hypercontact poset* to be a quadruple $(P, \leq, 0, \Delta)$, where $(P, \leq, 0)$ is a poset with minimal element 0 and Δ is a family of finite subsets of P, satisfying the following conditions, for all $m \in \mathbb{N}^+$, $a_1, a_2, \dots, a_m, b \in P$ and F, G finite subsets of P.

$$\{a_1, a_2, \dots, a_m\} \in \Delta \text{ implies } a_1 > 0, a_2 > 0, \dots, a_m > 0, \tag{Emp$_\Delta$}$$

$$F \in \Delta \text{ and } G \subseteq F \text{ imply } G \in \Delta, \tag{Sub$_\Delta$}$$

$$\text{if } \{a_1, a_2, \dots, a_m\} \in \Delta \text{ and } a_1 \leq b, \text{ then } \{a_1, a_2, \dots, a_m, b\} \in \Delta. \tag{Mon$_\Delta$}$$

$$b \neq 0 \text{ implies } \{b\} \in \Delta. \tag{Ref$_\Delta$}$$

As usual, a *hypercontact semilattice (or lattice, Boolean algebra)* is a join semilattice with 0 (or lattice with 0, or a Boolean algebra) together with a family Δ satisfying the above properties.

A hypercontact semilattice is *additive* if the following holds.

$$\begin{aligned}&\text{If } \{p+q, p_2, \dots, p_m\} \in \Delta, \text{ then}\\ &\text{either } \{p, p_2, \dots, p_m\} \in \Delta, \text{ or } \{q, p_2, \dots, p_m\} \in \Delta.\end{aligned} \tag{Add$_\Delta$}$$

A *homomorphism* of hypercontact posets (semilattices) is an order preserving map (a semilattice homomorphism) which preserves 0 and such that if $\{a_1, \dots, a_m\} \in \Delta$, then $\{\eta(a_1), \dots, \eta(a_m)\} \in \Delta$. An *embedding* is also required to be injective and to satisfy the converse implication.

Parallel to Definition 2.3.2, we can define overlap and K-overlap relations associated to a (pre-closure) poset. In detail, if $\mathbf{S}$ is a poset with 0, the *hypercontact overlap* is obtained by setting $\{a_1, \dots, a_m\} \in \Delta$ if there is $p \in S$, $p > 0$ such that $p \leq a_i$, for all $i \leq m$ (if $m = 0$, we should interpret the quantification over the empty set as furnishing a true value, so that $\emptyset \in \Delta$; also, the definition should be modified in the exceptional case $P = \{0\}$).

In a normal pre-closure poset $\mathbf{S}$ the *closure-hypercontact (or K-hypercontact)* is given by $\{a_1, \dots, a_m\} \in \Delta$ if there is $p \in S$, $p > 0$ such that $p \leq Ka_i$, for all $i \leq m$, again, letting $\emptyset \in \Delta$ in any case.

The hypercontact analogue of Lemma 2.3.3 holds, see [13, Lemma 3.4].

2.5. Specialization semilattices. A *specialization poset* [12] (sometimes called a *basic specialization poset*, when we want to make clear the distinction with the more encompassing notions we are going to define later) is a partially ordered set $(S, \leq)$ together with a further preorder $\sqsubseteq$, called a *specialization*, satisfying the following conditions.

$$a \leq b \Rightarrow a \sqsubseteq b, \tag{S1}$$

$$a \sqsubseteq b \ \&\ b \sqsubseteq c \Rightarrow a \sqsubseteq c, \tag{S2}$$

for all elements $a, b, c \in S$.

A *specialization semilattice* (or a *basic specialization semilattice*) is a triple $(S, +, , \sqsubseteq)$ such that $(S, +)$ is a semilattice, (S1) and (S2) hold with respect to the order induced by $+$ and moreover

$$a \sqsubseteq b \;\&\; a_1 \sqsubseteq b \Rightarrow a + a_1 \sqsubseteq b, \tag{S3}$$

for all elements $a, b, a_1 \in S$.

A *specialization poset (semilattice) with* 0 is further required to have a minimum (with respect to $\leq$) element 0 such that

$$a \sqsubseteq 0 \Rightarrow a = 0, \tag{S0}$$

An element 0 can be generally added (or removed) without modifying the remaining structure. See [8, Remark 2.3]. Compare also Remark 3.7 below in an extended situation and [11, Remark 6], [13, Remark 6.10] for the case of contact relations. Hence we will assume the existence of a 0, when convenient.

It can be shown [12, Remark 3.4(a)] that every specialization semilattice satisfies

$$a \sqsubseteq b \;\&\; a_1 \sqsubseteq b_1 \Rightarrow a + a_1 \sqsubseteq b + b_1. \tag{S7}$$

(the reason in the numbering is that we want to maintain the tags from [12]).

A *homomorphism* of specialization semilattices (posets) is a semilattice homomorphism (an order preserving map) η such that $a \sqsubseteq b$ implies $\eta(a) \sqsubseteq \eta(b)$. An *embedding* is an injective homomorphism satisfying the additional condition that $\eta(a) \sqsubseteq \eta(b)$ implies $a \sqsubseteq b$. When we are considering only the above conditions (disregarding the further structure), we will speak of a $\sqsubseteq$-homomorphism, or of a $\sqsubseteq$-embedding. In the presence of a 0, by convention, homomorphisms are always required to satisfy $\eta(0) = 0$. When some risk of ambiguity might occur, we shall explicitly mention that the homomorphism is 0-*preserving*. If X is a topological space, then $(\mathcal{P}(X), \cup, \emptyset \sqsubseteq)$ is a "topological" specialization semilattice with 0, where $a \sqsubseteq b$ if $a \subseteq Kb$. It can be checked that topological continuity corresponds to the notion of homomorphism between the associated specialization semilattices [12, Proposition 2.4].

Further details about the above notions can be found in [8, 9, 12, 15].

3. Extended specialization semilattices

Definition 3.1. An *extended specialization semilattice*, or *e-specialization semilattice*, for short, is a triple $(S, +, \sqsubseteq^*)$, where $\sqsubseteq^*$ is a ternary relation on S and $*$ stands for the second place. The required conditions are that $(S, +)$ is a join semilattice and

$$a \leq h + b \;\Rightarrow\; a \sqsubseteq^h b, \tag{E1}$$

$$a \sqsubseteq^h b \;\&\; h \sqsubseteq^k c \;\&\; b \sqsubseteq^c c \;\Rightarrow\; a \sqsubseteq^k c, \tag{E2}$$

$$a \sqsubseteq^h b \;\&\; a_1 \sqsubseteq^h b \;\Rightarrow\; a + a_1 \sqsubseteq^h b, \tag{E3}$$

for all elements $a, b, c, h, k, a_1 \in S$, where $\leq$ is the partial order induced by $+$, namely, $a \leq b$ if $a + b = b$.

An *extended specialization semilattice with* 0 has a minimum element 0 such that

$$a \sqsubseteq^h 0 \;\Rightarrow\; a \leq h. \tag{E0}$$

Minimum is always intended with respect to $\leq$.

We can always add a new 0 to an extended specialization semilattice: see Remark 3.7.

Remark 3.2. Assuming (E1), the condition (E2) is equivalent to the conjunction of

$$a \sqsubseteq^h c \;\&\; h \sqsubseteq^k c \;\Rightarrow\; a \sqsubseteq^k c, \text{ and} \tag{E2a}$$

$$a \sqsubseteq^h b \;\&\; b \sqsubseteq^c c \;\Rightarrow\; a \sqsubseteq^h c. \tag{E2b}$$

Indeed, by (E1), $c \sqsubseteq^c c$, hence condition (E2) implies (E2a) taking $b = c$. Again by (E1), $h \sqsubseteq^h c$, hence, taking $k = h$, condition (E2) implies (E2b). Conversely, if (E2b) holds, then from the premises of (E2) we get $a \sqsubseteq^h c$. Applying (E2a) we get $a \sqsubseteq^k c$, the conclusion in (E2).

Definition 3.3. (a) If $\mathbf{S}$ is an extended specialization semilattice, we define a binary relation $\sqsubseteq$ by $a \sqsubseteq b$ if $a \sqsubseteq^b b$. In this way we get a basic specialization relation, as defined in Section 2.5. We will check this in Lemma 3.6(i).

(b) A *homomorphism* of extended specialization semilattices is a semilattice homomorphism η such that $a \sqsubseteq^h b$ implies $\eta(a) \sqsubseteq^{\eta(h)} \eta(b)$. An *embedding* is an injective homomorphism such that also the reverse implication holds. Recall that, by convention, in the presence of a 0, homomorphisms and embeddings are assumed to preserve also 0.

Note that if, as in Definition 3.3(a), we define $a \sqsubseteq b$ by $a \sqsubseteq^b b$, then any $\sqsubseteq^*$-homomorphism (embedding) is a $\sqsubseteq$-homomorphism (embedding).

As we mentioned in the introduction, the intended interpretation of $a \sqsubseteq^h b$ is $a \subseteq h \cup Kb$ in some topological space, or even $a \leq h + Kb$ in some closure semilattice. In the next proposition we show that the conditions (E1) - (E3) are satisfied in the intended models.

Proposition 3.4. *Suppose that* $\mathbf{S}' = (S, +, K)$ *is a closure semilattice and define* $\sqsubseteq^*$ *on* S *by*

$$a \sqsubseteq^h b \quad \text{if} \quad a \leq h + Kb. \tag{3.1}$$

Then

(a) $\mathbf{S} = (S, +, \sqsubseteq^*)$ *is an extended specialization semilattice.*
(b) If 0 *is a minimum for* $(S, +)$ *and* K *is normal, then* $(S, +, 0, \sqsubseteq^*)$ *is an extended specialization semilattice with* 0*.*

(c) *In particular, if X is a topological space with closure K, then $(\mathcal{P}(X), \cup, \emptyset, \sqsubseteq^*)$ is an extended specialization semilattice, where $a \sqsubseteq^h b$ if $a \subseteq h \cup Kb$, for $a, h, b \subseteq X$.*

(d) *Suppose that $\mathbf{S}'$ and $\mathbf{T}'$ are closure semilattices.*

If $\varphi : \mathbf{S}' \to \mathbf{T}'$ is a homomorphism, then the function φ is also a homomorphism between the corresponding extended specialization semilattices $\mathbf{S}$ and $\mathbf{T}$.

More generally, if $\varphi : \mathbf{S}' \to \mathbf{T}'$ is a semilattice homomorphism, then $\varphi : \mathbf{S} \to \mathbf{T}$ is a homomorphism if and only if φ satisfies $\varphi(Kb) \subseteq K\varphi(b)$.

(e) *In particular, if $\varphi : X \to Y$ is a function between topological spaces, then φ is continuous if and only if the image function $\varphi^{\to}$ is a homomorphism between the corresponding extended specialization semilattices, as given by (c).*

Proof. (a) Since K is extensive, $a \leq h + b$ implies $a \leq h + Kb$, thus we get (E1). As far as (E2) is concerned, from $b \leq c + Kc$, the interpretation of $b \sqsubseteq^c c$ through (3.1), we get $b \leq Kc$, since K is extensive, hence $Kb \leq KKc = Kc$, by monotonicity and idempotency of K. Thus from $a \leq h + Kb$ and $h \leq k + Kc$, given, respectively, by $a \sqsubseteq^h b$ and $h \sqsubseteq^k c$, we get $a \leq h + Kb \leq k + Kc + Kc = k + Kc$, that is, $a \sqsubseteq^k c$, by (3.1). (E3) is elementary from the definition of join.

Item (b) is elementary, as well. Item (c) follows from (a) and (b).

(d) If $a \sqsubseteq^h b$ in $\mathbf{S}$, then $a \leq h + Kb$ in $\mathbf{S}'$, by construction. Then $\varphi(a) \leq \varphi(h) + \varphi(Kb) = \varphi(h) + K\varphi(b)$ since φ is a homomorphism of closure semilattices. Thus $\varphi(a) \sqsubseteq^{\varphi(h)} \varphi(b)$ in $\mathbf{T}$, again by construction.

In order to prove the last statement, first notice that we have used only $\varphi(Kb) \subseteq K\varphi(b)$ in the above proof, not the full hypothesis that φ is a K-homomorphism. In the other direction, if $\varphi : \mathbf{S} \to \mathbf{T}$ is a homomorphism, then, since $Kb \sqsubseteq^b b$, we get $\varphi(Kb) \sqsubseteq^{\varphi(b)} \varphi(b)$ from the assumption that φ is a homomorphism of extended specialization semilattices. But this means $\varphi(Kb) \subseteq \varphi(b) + K\varphi(b) = K\varphi(b)$, according to (3.1).

(e) is immediate from the last statement in (d), since, as well-known, a function φ between topological spaces is continuous if and only if the image function $\varphi^{\to}$ satisfies $\varphi(Kx) \subseteq K\varphi(x)$. $\square$

Definition 3.5. (a) Under the assumptions in Proposition 3.4, the extended specialization semilattice $(S, +, \sqsubseteq^*)$ constructed in Proposition 3.4(a) will be called the *e-specialization reduct of* the closure semilattice $\mathbf{S}'$ (this is a slight abuse of terminology; formally, $\mathbf{S}$ is the reduct of a definitional expansion of $\mathbf{S}'$).

(b) As usual by now, an extended specialization semilattice having the form indicated in Proposition 3.4(c) will be called *topological*.

In the basic case, the analogues of Proposition 3.4 are presented in [12, Remark 3.3 and Proposition 2.4].

We now state some elementary properties of extended specialization semilattices.

Lemma 3.6. *Suppose that* $\mathbf{S}$ *is an extended specialization semilattice with ternary relation* $\sqsubseteq^*$ *and* $a, b, c, h, k, \dots \in S$.

(i) *If we define* $a \sqsubseteq b$ *by* $a \sqsubseteq^b b$, *then* $(S, +, \sqsubseteq)$ *is a basic specialization semilattice. If* $\mathbf{S}$ *has a* 0, *this is a* 0 *also for* $\sqsubseteq$, *namely,* (S0) *holds.*

(ii) *If* $a \sqsubseteq^h b$ *and* $b \leq c$, *then* $a \sqsubseteq^h c$.

(iii) *If* $a \sqsubseteq^h c$ *and* $h \leq k$, *then* $a \sqsubseteq^k c$.

(iv) *If both* $a \sqsubseteq^h b$ *and* $a_1 \sqsubseteq^h b_1$, *then* $a + a_1 \sqsubseteq^h b + b_1$.

(v) $a \sqsubseteq^h c$ *if and only if* $a \sqsubseteq^{h+c} c$; *in particular, if* $\mathbf{S}$ *has a* 0, *then* $a \sqsubseteq^0 c$ *if and only if* $a \sqsubseteq c$. *More generally, if* $k \leq c$, *then* $a \sqsubseteq^h c$ *if and only if* $a \sqsubseteq^{h+k} c$.

(vi) *If* $h \sqsubseteq^k c$ *and* $a \leq h$, *then* $a \sqsubseteq^k c$.

(vii) $a \sqsubseteq^b b$ *if and only if the relation* $a \sqsubseteq^h b$ *holds for every* $h \in S$.

Proof. (i) (S1) follows immediately from (E1). (S2) follows from (E2) by taking $h = b$ and $k = c$. (S3) is immediate from (E3) by taking $h = b$. The statement about 0 is also immediate.

(ii) By (E1), $b \sqsubseteq^c c$, hence $a \sqsubseteq^h c$ by (E2a).

(iii) If $h \leq k$, then $h \sqsubseteq^k c$ by (E1). The conclusion follows from (E2a).

(iv) By the assumptions and (ii), $a \sqsubseteq^h b + b_1$ and $a_1 \sqsubseteq^h b + b_1$. The conclusion follows from (E3) with $b + b_1$ in place of b.

(v) The only if part follows from (iii). In the other direction, we have $h+c \sqsubseteq^h c$ by (E1), hence, if $a \sqsubseteq^{h+c} c$, then $a \sqsubseteq^h c$, by (E2a) with $h + c$ in place of h and $k = h$. To prove the nontrivial part in the last statement, if $k \leq c$ and $a \sqsubseteq^{h+k} c$, then $a \sqsubseteq^{h+c} c$ by (iii), hence $a \sqsubseteq^h c$ by the first statement in the present item.

(vi) If $a \leq h$, then $a \sqsubseteq^h c$ by (E1), hence the conclusion follows from (E2a).

(vii) To prove the nontrivial implication, if $a \sqsubseteq^b b$, then $a \sqsubseteq^{h+b} b$, for every h, by taking b in place of h and $h + b$ in place of k in (iii). Then $a \sqsubseteq^h b$ by the first statement in (v). □

Remark 3.7. We can always add a new minimal element 0 to an extended specialization semilattice $\mathbf{S}$ "without 0" by setting in $S \cup \{0\}$, for $a, h, b \neq 0$:

(i) $0 \sqsubseteq^h b$ always,

(ii) $a \sqsubseteq^h 0$ if $a \leq h$, and

(iii) $a \sqsubseteq^0 b$ if either $a = 0$, or $a, b \in S$ and $a \sqsubseteq^b b$,

and letting $\sqsubseteq^*$ remain unchanged when $a, h, b \in S$.

Indeed, the above clauses (i) - (iii) agree in any overlapping case, (E0) holds because of (ii) and (E1) - (E3) are easily verified. For example, if $a, b \in S$ and $a \leq 0 + b$, then $a \leq b + b$, since 0 is minimum, hence $a \sqsubseteq^b b$ by (E1) in $\mathbf{S}$, thus $a \sqsubseteq^0 b$ by (iii). The other cases in (E1) are elementary.

Conversely, if we remove the element 0 from an extended specialization semilattice with 0, we get an extended specialization semilattice satisfying (E1) - (E3).

Results about extended specialization semilattices can be used to give a uniform proof for results about semilattices and basic specialization semilattices. See Corollaries 5.5 and 5.6 below. This will be an application of the the next two propositions. The first proposition has a straightforward proof; it can also be obtained from Lemma 3.6(a)(d) by taking K to be the identity function on S.

Proposition 3.8. *If* $\mathbf{S} = (S, +)$ *is a semilattice, then* $\mathbf{S}_e = (S, +, \sqsubseteq^*)$ *is an extended specialization semilattice, where* $\sqsubseteq^*$ *is defined by*

$$a \sqsubseteq^h b \quad \text{if} \quad a \leq h + b. \tag{3.2}$$

If $\mathbf{S}$ *has a* 0*, then it is a* 0 *also for* $\mathbf{S}_e$*. Semilattice homomorphisms and embeddings are exactly homomorphisms and embeddings for the structures expanded in the above way.*

Proposition 3.9. *Suppose that* $\mathbf{S} = (S, +, \sqsubseteq)$ *is a basic specialization semilattice.*

(a) If we set

$$a \sqsubseteq^h b \text{ if there is } c \in S \text{ such that } a \leq h + c \text{ and } c \sqsubseteq b, \tag{3.3}$$

then $\mathbf{S}_e = (S, +, \sqsubseteq^*)$ *is an extended specialization semilattice.*

(b) If we use $\sqsubseteq^*$ *in order to define* $\sqsubseteq$*, as in Definition 3.3(a), we get back the original specialization* $\sqsubseteq$ *of* $\mathbf{S}$*.*

(c) If $\mathbf{S}$ *has a* 0 *satisfying* (S0)*, then* 0 *satisfies also* (E0) *in* $\mathbf{S}_e$*.*

(d) If $\mathbf{S}$ *and* $\mathbf{T}$ *are basic specialization semilattices, then some function* $\varphi : S \to T$ *is a homomorphism of specialization semilattices from* $\mathbf{S}$ *to* $\mathbf{T}$ *if and only if* φ *is a homomorphism of extended specialization semilattices from* $\mathbf{S}_e$ *to* $\mathbf{T}_e$*, where* $\mathbf{S}_e$ *and* $\mathbf{T}_e$ *are constructed as in (a). If* $\varphi : \mathbf{S}_e \to \mathbf{T}_e$ *is an embedding, then* $\varphi : \mathbf{S} \to \mathbf{T}$ *is an embedding.*

Proof. We first prove (b). The condition $a \sqsubseteq^b b$ means that there is c such that in $\mathbf{S}$ $a \leq b + c$ and $c \sqsubseteq b$, and this implies $b + c \sqsubseteq b$, by $b \sqsubseteq b$ and (S3). By (S1), $a \sqsubseteq b + c$, hence $a \sqsubseteq b$ in $\mathbf{S}$. Conversely, if $a \sqsubseteq b$ in $\mathbf{S}$, just take $h = b$ and $c = a$ in (3.3) in order to get $a \sqsubseteq^b b$ in $\mathbf{S}_e$.

(a) Clause (E1) is immediate by taking $c = b$.

The assumptions of (E2) together with (3.3) provide elements c_1 and c_2 such that, in $\mathbf{S}$, $a \leq h + c_1$, $c_1 \sqsubseteq b$, $h \leq k + c_2$, $c_2 \sqsubseteq c$ and, by the already proved item (b), $b \sqsubseteq c$. Then we get $a \leq k + c_1 + c_2$ and $c_1 \sqsubseteq c$, by (S2) applied to $c_1 \sqsubseteq b$ and $b \sqsubseteq c$. Then (S3) gives $c_1 + c_2 \sqsubseteq c$, hence $c_1 + c_2$ witnesses $a \sqsubseteq^k c$ in $\mathbf{S}_e$.

The assumptions of (E3) and (3.3) imply that there are c and c_1 such that $a \leq h + c$, $c \sqsubseteq b$, $a_1 \leq h + c_1$ and $c_1 \sqsubseteq b$, hence $a + a_1 \leq h + c + c_1$ and $c + c_1 \sqsubseteq b$, by (S3). Thus $c + c_1$ witnesses $a + a_1 \sqsubseteq^h b$.

(c) is straightforward.

(d) If φ is a homomorphism from $\mathbf{S}_e$ to $\mathbf{T}_e$, then φ is a homomorphism from $\mathbf{S}$ to $\mathbf{T}$ in view of (b). Conversely, if φ is a homomorphism from $\mathbf{S}$ to $\mathbf{T}$ and $a \sqsubseteq^h b$ in $\mathbf{S}_e$ is witnessed by (3.3) for some $c \in S$, then $\varphi(c)$ witnesses $\varphi(a) \sqsubseteq^{\varphi(h)} \varphi(b)$ in $\mathbf{T}_e$.

To prove the last statement, if $\varphi(a) \sqsubseteq \varphi(b)$ in $\mathbf{T}$, then $\varphi(a) \sqsubseteq^{\varphi(b)} \varphi(b)$ in $\mathbf{T}_e$, by (b). Since $\varphi : \mathbf{S}_e \to \mathbf{T}_e$ is supposed to be an embedding, $a \sqsubseteq^b b$ in $\mathbf{S}_e$, that is, $a \sqsubseteq b$ in $\mathbf{S}$, again by (b). □

4. Merging the various structures

Motivated by the above topologically induced examples, we are led to consider structures in which also a contact relation is added. For example, a *weak contact closure semilattice* is a quintuple $(S, +, 0, K, \delta)$ such that $(S, +, 0, \delta)$ is a weak contact semilattice and K is a normal closure operation on S. *Weak contact extended specialization semilattices*, etc. are defined in an entirely analogous way. Since weak contact posets have a 0, by definition, we will always assume that specialization posets and semilattices, when endowed also with a contact, have a 0 for which (S0) or (E0) are satisfied.

As in Definition 3.5(a), if $\mathbf{S}' = (S, +, 0, K, \delta)$ is a weak contact closure semilattice, the *weak contact e-specialization reduct* of $\mathbf{S}'$ is $\mathbf{S} = (S, +, 0, \sqsubseteq^*, \delta)$, where $\sqsubseteq^*$ is defined by (3.1).

5. Free embeddings into additive closure semilattices

In this section we show that every extended specialization semilattice, possibly, with a contact relation, has a free extension in the class of additive closure semilattices with respect to the definitional expansion given by (3.1).

Definition 5.1. (A) If $\mathbf{S}$ is an extended specialization semilattice, define a binary relation $\precsim$ on the product $S \times S$ by

$$(a, b) \precsim (c, d) \text{ if, in } \mathbf{S},\ a \sqsubseteq^c d \text{ and } b \sqsubseteq^d d. \tag{5.1}$$

In Proposition 5.3(i) below we will prove that $\precsim$ is reflexive and transitive, thus if we define $\sim$ by

$$(a, b) \sim (c, d) \text{ if both } (a, b) \precsim (c, d) \text{ and } (c, d) \precsim (a, b), \tag{5.2}$$

then $\sim$ is an equivalence relation. Moreover, we will show in Proposition 5.3(ii) that $\sim$ is a congruence on the semilattice product $\mathbf{S} \times \mathbf{S}$, hence $(\mathbf{S} \times \mathbf{S})/\!\sim$ is a semilattice.

Let $\widetilde{S}$ be the set of the $\sim$-equivalence classes. Define $K : \widetilde{S} \to \widetilde{S}$ by

$$K[a,b] = [a, a+b], \tag{5.3}$$

where $[x,y]$ is the $\sim$-class of the pair (x,y). In Proposition 5.3(iv) we will prove that the definition is correct.

Define $\sqsubseteq^*$ on $\widetilde{S}$ using (3.1), namely, in the case at hand, $[a,b] \sqsubseteq^{[h,k]} [c,d]$ if $[a,b] \leq [h,k] + K[c,d]$, where $+$ is the semilattice operation of $(\mathbf{S} \times \mathbf{S})/\!\sim$ and $\leq$ is the induced order.

Suppose now that $\mathbf{S}$ has a 0. Let $\widetilde{\mathbf{S}} = (\widetilde{S}, +, [0,0], \sqsubseteq^*)$, $\widetilde{\mathbf{S}}' = (\widetilde{S}, +, [0,0], K)$, thus $\widetilde{\mathbf{S}}$ is the e-specialization reduct of $\widetilde{\mathbf{S}}'$, in the sense of Definition 3.5(a). Finally, define $\upsilon_{\mathbf{S}} : S \to \widetilde{S}$ by

$$\upsilon_{\mathbf{S}}(a) = [a, 0]. \tag{5.4}$$

(B) Suppose further that δ is a binary relation on S and assume the above constructions in (A). Define $\bar{\delta}$ on $S \times S$ by

$$(a_1, b_1)\ \bar{\delta}\ (e, f) \text{ if there are } s, t \in S \text{ such that } s\ \delta\ t,\ s \sqsubseteq^{a_1} b_1 \text{ and } t \sqsubseteq^{e} f\ . \tag{5.5}$$

We will show in Proposition 5.3(vi) that $\bar{\delta}$ induces a relation $\widetilde{\delta}$ on $\widetilde{S}$. In presence of a weak pre-contact δ on $\mathbf{S}$, we let $\widetilde{\mathbf{S}} = (\widetilde{S}, +, [0,0], \sqsubseteq^*, \widetilde{\delta})$, $\widetilde{\mathbf{S}}' = (\widetilde{S}, +, [0,0], K, \widetilde{\delta})$. It will always be clear from the context whether we are dealing with the definitions of $\widetilde{\mathbf{S}}$ and $\widetilde{\mathbf{S}}'$ in (A) above, or we are using an expanded structure as in (B) here or in (C) below.

(C) Again under the assumptions in (A), if Δ is a family of finite subsets of $\mathbf{S}$, let $\bar{\Delta}$ be the family of finite subsets of $S \times S$ such that

$$\begin{aligned}\{(a_1, b_1), \dots, (a_m, b_m)\} \in \bar{\Delta} &\text{ if and only if there are } s_1, \dots, s_m \in S \\ &\text{such that } \{s_1, \dots, s_m\} \in \Delta \text{ and } s_i \sqsubseteq^{a_i} b_i, \text{ for every } i \leq m.\end{aligned} \tag{5.6}$$

In Proposition 5.3(vii) we will show that $\bar{\Delta}$ induces a family $\widetilde{\Delta}$ on $\widetilde{S}$. Again, when appropriate, let $\widetilde{\mathbf{S}} = (\widetilde{S}, +, [0,0], \sqsubseteq^*, \widetilde{\Delta})$, $\widetilde{\mathbf{S}}' = (\widetilde{S}, +, [0,0], K, \widetilde{\Delta})$.

Remark 5.2. As in [8, p. 107] we intuitively think of $[a,b]$ as $a + Kb$, where Kb is the "new" closure we want to introduce; in particular, $[a,0]$ corresponds to a and $[0,b]$ corresponds to a new element Kb.

We now check that Definition 5.1 is correct.

Proposition 5.3. *Under the notation and the definitions in 5.1, the following statements hold.*

(i) The relation $\precsim$ on $S \times S$ is reflexive and transitive, hence $\sim$ is an equivalence relation.

(ii) The relation $\sim$ is a congruence on the semilattice product $\mathbf{S} \times \mathbf{S}$, hence the quotient inherits a semilattice structure.

(iii) $[a,b] \leq [c,d]$ in the quotient $(\mathbf{S} \times \mathbf{S})/\!\sim$ if and only if $(a,b) \precsim (c,d)$ in $\mathbf{S} \times \mathbf{S}$.

(iv) The operation K is well-defined on the $\sim$-equivalence classes.

(v) If $\mathbf{S}$ *has a* 0*, then* K *satisfies*

$$K[a,b] = [0, a+b] = [a, a+b]. \tag{5.7}$$

(vi) Under the assumptions in Definition 5.1(B), the relation $\bar{\delta}$ *passes to the quotient under* $\sim$*, thus* $\bar{\delta}$ *induces a relation* $\widetilde{\delta}$ *on* $\widetilde{S}$ *defined by*

$$[a,b]\ \widetilde{\delta}\ [e,f] \text{ if there are } s,t \in S \text{ such that } s\ \delta\ t,\ s \sqsubseteq^{a} b \text{ and } t \sqsubseteq^{e} f. \tag{5.8}$$

Moreover,

$$\text{if } (a_1,b_1) \precsim (c_1,d_1) \text{ and } (a_1,b_1)\ \bar{\delta}\ (e,f), \text{ then } (c_1,d_1)\ \bar{\delta}\ (e,f), \tag{5.9}$$

$$\text{if } (a_1,b_1) \precsim (c_1,d_1) \text{ and } (e,f)\ \bar{\delta}\ (a_1,b_1), \text{ then } (e,f)\ \bar{\delta}\ (c_1,d_1). \tag{5.10}$$

(vii) Under the assumptions in Definition 5.1(C), the following statements hold.

$$\begin{gathered}\text{If } j \leq m,\ (a_j,b_j) \precsim (a_j^*,b_j^*) \text{ and } \{(a_1,b_1),\dots,(a_j,b_j),\dots,(a_m,b_m)\} \in \bar{\Delta},\\ \text{then } \{(a_1,b_1),\dots,(a_{j-1},b_{j-1}),(a_j^*,b_j^*),(a_{j+1},b_{j+1}),\dots,(a_m,b_m)\} \in \bar{\Delta}.\end{gathered} \tag{5.11}$$

Thus $\bar{\Delta}$ *induces a family* $\widetilde{\Delta}$ *on the quotient* $\widetilde{S}$ *letting*

$$\{[a_1,b_1],\dots,[a_m,b_m]\} \in \widetilde{\Delta} \quad \text{if} \quad \{(a_1,b_1),\dots,(a_m,b_m)\} \in \bar{\Delta}. \tag{5.12}$$

Proof. (i) The relation $\precsim$ is reflexive; indeed, both $a \sqsubseteq^{a} b$ and $b \sqsubseteq^{b} b$ are immediate from (E1).

In order to check transitivity, assume that $(a,e) \precsim (h,b)$ and $(h,b) \precsim (k,c)$, that is, $a \sqsubseteq^{h} b$, $e \sqsubseteq^{b} b$, $h \sqsubseteq^{k} c$ and $b \sqsubseteq^{c} c$. By (E2) we get $a \sqsubseteq^{k} c$. Recalling Lemma 3.6(i), we have $e \sqsubseteq b$ and $b \sqsubseteq c$, hence $e \sqsubseteq c$, by (S2), that is, $e \sqsubseteq^{c} c$. Together with $a \sqsubseteq^{k} c$, the last inequality implies $(a,e) \precsim (k,c)$.

Since the definition of $\sim$ is symmetric, the relation $\sim$ is symmetric. Moreover, $\sim$ inherits reflexivity and transitivity from $\precsim$.

(ii) It is enough to show that if $(a,b) \precsim (c,d)$, then $(a,b)+(e,f) \precsim (c,d)+(e,f)$, that is, $(a+e, b+f) \precsim (c+e, d+f)$. Indeed, together with the symmetrical statement, this implies that if $(a,b) \sim (c,d)$, then $(a,b)+(e,f) \sim (c,d)+(e,f)$,

By assumption, $a \sqsubseteq^{c} d$. By Lemma 3.6(ii)(iii), $a \sqsubseteq^{c+e} d+f$. By (E1), $e \sqsubseteq^{c+e} d+f$. Thus by (E3) we get $a+e \sqsubseteq^{c+e} d+f$.

By assumption, $b \sqsubseteq^{d} d$. By Lemma 3.6(ii)(iii), $b \sqsubseteq^{d+f} d+f$. By (E1), $f \sqsubseteq^{d+f} d+f$. Thus by (E3) we get $b+f \sqsubseteq^{d+f} d+f$.

We have proved both $a+e \sqsubseteq^{c+e} d+f$ and $b+f \sqsubseteq^{d+f} d+f$, which means $(a+e, b+f) \precsim (c+e, d+f)$.

(iii) $[a,b] \leq [c,d]$ means $[a,b]+[c,d] = [c,d]$, that is $[a+c, b+d] = [c,d]$, that is, $(a+c, b+d) \sim (c,d)$. This last condition is equivalent to $(a+c, b+d) \precsim (c,d)$, since $(c,d) \precsim (a+c, b+d)$ always hold, in view of (E1).

If $(a+c, b+d) \precsim (c,d)$, then $a+c \sqsubseteq^{c} d$ and $b+d \sqsubseteq^{d} d$, by definition, hence $a \sqsubseteq^{c} d$ and $b \sqsubseteq^{d} d$, by Lemma 3.6(vi), that is, $(a,b) \precsim (c,d)$. Conversely, if $(a,b) \precsim (c,d)$, then $a \sqsubseteq^{c} d$ and $b \sqsubseteq^{d} d$, thus $a+c \sqsubseteq^{c} d$ and $b+d \sqsubseteq^{d} d$, by the special case of the argument in (ii) with $e = c$ and $f = d$. This means $(a + c, b + d) \precsim (c, d)$.

In the last paragraph we have shown that $(a+c, b+d) \precsim (c, d)$ is equivalent to $(a, b) \precsim (c, d)$. The conclusion follows from the equivalences proved in the first paragraph of the proof here in (iii).

(iv) It is enough to show that if $(a, b) \precsim (c, d)$, then $(a, a + b) \precsim (c, c + d)$.

By assumption, $a \sqsubseteq^{c} d$, hence $a \sqsubseteq^{c} c + d$ by Lemma 3.6(ii). We also get $a \sqsubseteq^{c+d} c + d$ by Lemma 3.6(iii).

By assumption, $b \sqsubseteq^{d} d$, hence $b \sqsubseteq^{c+d} c + d$, again by Lemma 3.6(ii)(iii). By (E3), we get $a + b \sqsubseteq^{c+d} c + d$, which, together with the already proved $a \sqsubseteq^{c} c + d$, shows $(a, a + b) \precsim (c, c + d)$.

(v) In (iv) we have shown that K is well defined on the equivalence classes, hence it is enough to check that $(a, a+b) \sim (0, a+b)$. This is immediate from the definitions and (E1).

(vi) We first prove (5.9). Recall from Definition 5.1 that $(a_1, b_1) \precsim (c_1, d_1)$ means that $a_1 \sqsubseteq^{c_1} d_1$ and $b_1 \sqsubseteq^{d_1} d_1$. Then $s \sqsubseteq^{c_1} d_1$ by (E2), using $s \sqsubseteq^{a_1} b_1$ given by $(a_1, b_1)\ \bar{\delta}\ (e, f)$ in (5.5) and taking $a = s$, $h = a_1$, $b = b_1$, $k = c_1$ and $c = d_1$. Condition (5.10) is proved symmetrically (of course, this is not necessary if δ is symmetrical). Together with $s\ \delta\ t$ and $t \sqsubseteq^{e} f$, this shows $(c_1, d_1)\ \bar{\delta}\ (e, f)$.

Applying (5.9) twice, we get that if $(a, b) \sim (c, d)$, then $(a, b)\ \bar{\delta}\ (e, f)$ if and only if $(c, d)\ \bar{\delta}\ (e, f)$. Together with the symmetrical argument, this shows that the definition of $\widetilde{\delta}$ in (5.8) is not dependent on the representatives.

(vii) The proof of (5.11) is not essentially different. The assumptions give certain elements $s_1, \dots, s_m$ such that the statements in the second line of (5.6) hold; moreover, $(a_j, b_j) \precsim (a^*_j, b^*_j)$ means that $a_j \sqsubseteq^{a^*_j} b^*_j$ and $b_j \sqsubseteq^{b^*_j} b^*_j$. As above, using $s_j \sqsubseteq^{a_j} b_j$ and by (E2), we get $s_j \sqsubseteq^{a^*_j} b^*_j$. Thus $s_1, \dots, s_m$ witness also $\{(a_1, b_1), \dots, (a_{j-1}, b_{j-1}), (a^*_j, b^*_j), (a_{j+1}, b_{j+1}), \dots, (a_m, b_m)\} \in \bar{\Delta}$. By applying (5.11) twice, we get that, for every $j \leq m$, if $(a_j, b_j) \sim (a^*_j, b^*_j)$, then $\{(a_1, b_1), \dots, (a_j, b_j), \dots, (a_m, b_m)\} \in \bar{\Delta}$ if and only if $\{(a_1, b_1), \dots, (a_{j-1}, b_{j-1}), (a^*_j, b^*_j), (a_{j+1}, b_{j+1}), \dots, (a_m, b_m)\} \in \bar{\Delta}$.

Iterating the above statement, we get that the definition (5.12) does not depend on the representatives. □

Note that Definitions 5.1(B)(C) and Proposition 5.3(vi)(vii) apply to an arbitrary binary relation δ or family Δ, we need no special property of δ or of Δ. The definition of $\bar{\delta}$ in 5.1(B)(C) will allow us to prove the next theorem in a very general form. We mention that in the most interesting case a simpler definition of $\bar{\delta}$ works, namely, $(a_1, b_1)\ \bar{\delta}\ (e, f)$ if $a_1 {+} b_1\ \delta\ e {+} f$. See Lemma 7.3(a) below.

We are now able to prove that, for every extended specialization semilattice $\mathbf{S}$, possibly with a contact or a hypercontact, the structure $\widetilde{\mathbf{S}}'$ is free over $\mathbf{S}$ in the class of additive closure semilattices, modulo the definitional expansion

(3.1). An expression separated by commas within square brackets in the statement of the next theorem can be (uniformly) either added or excluded, so that the theorem actually consists of four theorems at the same time.

Theorem 5.4. *Assume that* $\mathbf{S}$ *is an extended specialization semilattice with* 0 *[and a weak contact, a weak pre-contact, a hypercontact relation]. Let* $\widetilde{\mathbf{S}}$, $\widetilde{\mathbf{S}}'$ *and* $\upsilon_{\mathbf{S}}$ *be as in Definition 5.1(A) [respectively, (B), (B), (C)]. Then the following statements hold.*

(1) $\widetilde{\mathbf{S}}'$ *is an additive closure semilattice with* 0 *[and a weak contact, a weak pre-contact, a hypercontact relation].*

(2) $\upsilon_{\mathbf{S}}$ *is an embedding of* $\mathbf{S}$ *into* $\widetilde{\mathbf{S}}$.

(3) *The pair* $(\widetilde{\mathbf{S}}, \upsilon_{\mathbf{S}})$ *has the following universal property.*

For every additive closure semilattice $\mathbf{T}'$ *with* 0 *[and a weak contact, a weak pre-contact, a hypercontact relation] and every homomorphism* $\eta : \mathbf{S} \to \mathbf{T}$, *where* $\mathbf{T}$ *is the e-specialization reduct of* $\mathbf{T}'$, *there is a unique homomorphism* $\widetilde{\eta} : \widetilde{\mathbf{S}}' \to \mathbf{T}'$ *such that* $\eta = \upsilon_{\mathbf{S}} \circ \widetilde{\eta}$.

$$\begin{tikzcd}
\mathbf{S} \arrow[r, "\upsilon_{\mathbf{S}}"] \arrow[dr, "\eta"'] & \widetilde{\mathbf{S}} \arrow[d, dashed, "\widetilde{\eta}"] & \widetilde{\mathbf{S}}' \arrow[d, dashed, "\widetilde{\eta}"] \\
& \mathbf{T} & \mathbf{T}'
\end{tikzcd}$$

(4) *Suppose that* $\mathbf{U}$ *is another extended specialization semilattice [with a weak contact, a weak pre-contact, a hypercontact relation] and* $\psi : \mathbf{S} \to \mathbf{U}$ *is a homomorphism. Then there is a unique homomorphism* $\widetilde{\psi} : \widetilde{\mathbf{S}}' \to \widetilde{\mathbf{U}}'$ *making the following diagram commute:*

$$\begin{tikzcd}
\mathbf{S} \arrow[r, "\upsilon_{\mathbf{S}}"] \arrow[d, "\psi"'] & \widetilde{\mathbf{S}} \arrow[d, dashed, "\widetilde{\psi}"] & \widetilde{\mathbf{S}}' \arrow[d, dashed, "\widetilde{\psi}"] \\
\mathbf{U} \arrow[r, "\upsilon_{\mathbf{U}}"] & \widetilde{\mathbf{U}} & \widetilde{\mathbf{U}}'
\end{tikzcd}$$

Corollary 5.5. *Theorem 5.4 holds when "extended specialization semilattice" and "e-specialization reduct" are replaced everywhere by, respectively, "semilattice" and "semilattice reduct", with the following further variations: the relation* $\precsim$ *is defined by*

$$(a, b) \precsim (c, d) \text{ if } a \leq c + d \text{ and } b \leq d, \tag{5.13}$$

$\sim$, $\widetilde{S}$, K, $\upsilon_{\mathbf{S}}$, *possibly,* $\widetilde{\delta}$ *or* $\widetilde{\Delta}$, *are correspondingly defined as in Definition 5.1 and* $\widetilde{\mathbf{S}} = (\widetilde{S}, +, [0, 0])$, $\widetilde{\mathbf{S}}' = (\widetilde{S}, +, [0, 0], K)$, *possibly both expanded by adding* $\widetilde{\delta}$ *or* $\widetilde{\Delta}$.

Corollary 5.6. *Theorem 5.4 holds when "extended specialization semilattice" and "e-specialization reduct" are replaced everywhere by, respectively, "basic specialization semilattice" and "specialization reduct", with the following further variations: the relation* $\precsim$ *is defined by*

$$(a, b) \precsim (c, d) \text{ if there is } e \in S \text{ such that } a \leq c + e,\ e \sqsubseteq d \text{ and } b \sqsubseteq d \tag{5.14}$$

$\sim$, $\widetilde{S}$, K, $\upsilon_{\mathbf{S}}$, possibly, $\widetilde{\delta}$ or $\widetilde{\Delta}$, are correspondingly defined as in Definition 5.1 and $\widetilde{\mathbf{S}} = (\widetilde{S}, +, [0,0], \sqsubseteq)$, $\widetilde{\mathbf{S}}' = (\widetilde{S}, +, [0,0], K)$, possibly both expanded by adding $\widetilde{\delta}$ or $\widetilde{\Delta}$.

Proof. We first prove Theorem 5.4. (1) is proved as the Claim in the proof of [8, Theorem 3.2]. We report the details for the reader's convenience. In Proposition 5.3(ii) it is shown that $(\widetilde{S}, +)$ is a semilattice; it remains to check that K is an additive closure. Indeed, by the definition of K, and since the projection from $\mathbf{S} \times \mathbf{S}$ $(\mathbf{S} \times \mathbf{S})/{\sim}$ is a semilattice homomorphism,

$$\begin{aligned}
&[a,b] \leq [a, a+b] = K[a,b],\\
&KK[a,b] = K[a,a+b] = [a, a+a+b] = K[a,b], \text{ and}\\
&K([a,b]+[c,d]) = K[a+c, b+d] = [a+c, a+b+c+d]\\
&\qquad\qquad\qquad\quad = [a,a+b]+[c,c+d] = K[a,b]+K[c,d].
\end{aligned}$$

Note that $[0,0]$ is a 0 of $\widetilde{\mathbf{S}}$, since $(0,0)$ is a neutral element for $\mathbf{S} \times \mathbf{S}$, hence $[0,0]$ is neutral for the quotient $\widetilde{\mathbf{S}} = (\mathbf{S} \times \mathbf{S})/{\sim}$. Moreover, $K[0,0] = [0,0]$, by definition, since 0 is a zero in $\mathbf{S}$.

In case $\mathbf{S}$ has a further binary relation δ, $\widetilde{\delta}$ is well-defined by Proposition 5.3(vi).

If δ is a weak contact, the relation $\widetilde{\delta}$ inherits (Sym), (Emp) and (Ref) from δ. Indeed, $s \sqsubseteq^{0} 0$ implies $s = 0$ by (E0). Furthermore, $a_1 + b_1 \sqsubseteq^{a_1} b_1$ by (E1), hence if $a_1 + b_1 > 0$, then $s = t = a_1 + b_1$ witness $[a_1, b_1] \mathrel{\widetilde{\delta}} [a_1, b_1]$. Moreover, (Ext) follows from (5.9), its symmetric version and Proposition 5.3(iii). Thus $\widetilde{\delta}$ is a weak contact. The case of a weak pre-contact is similar, simply do not deal with symmetry. Also the case of a hypercontact is entirely similar.

(2) The proof that $\upsilon_{\mathbf{S}}$ is an injective semilattice homomorphism preserving 0 is similar to the corresponding part in [8, Theorem 4.3(2)]. Indeed, $\upsilon_{\mathbf{S}}(a+b) = [a+b, 0] = [a,0]+[b,0] = \upsilon_{\mathbf{S}}(a) + \upsilon_{\mathbf{S}}(b)$, hence $\upsilon_{\mathbf{S}}$ is a semilattice homomorphism. Moreover, $\upsilon_{\mathbf{S}}$ is injective, since $\upsilon_{\mathbf{S}}(a) = \upsilon_{\mathbf{S}}(c)$ means $(a,0) \sim (c,0)$ and this happens only if $a \sqsubseteq^{c} 0$ and $c \sqsubseteq^{a} 0$, by the definition of $\sim$. Then by the definition of a 0 in an extended specialization semilattice, $a \leq c$ and $c \leq a$, that is, $a = c$.

We now check that $\upsilon_{\mathbf{S}}$ is an embedding with respect to $\sqsubseteq^*$. Indeed, the following is a chain of equivalent conditions:

(a) $a \sqsubseteq^{h} b$ in $\mathbf{S}$,
(b) $(a,0) \precsim (h,b)$, since $0 \leq b$, hence $0 \sqsubseteq^{b} b$ by (E1),
(c) $[a,0] \leq [h,b]$ in $\widetilde{\mathbf{S}}$, by Proposition 5.3(iii),
(d) $[a,0] \leq [h,b] = [h,0]+[0,b] = [h,0]+K[b,0]$, by Proposition 5.3(v),
(e) $\upsilon_{\mathbf{S}}(a) \sqsubseteq^{\upsilon_{\mathbf{S}}(h)} \upsilon_{\mathbf{S}}(b)$, by the definition of $\sqsubseteq^*$.

If a weak (pre-)contact is present, the semilattice embedding $\upsilon_{\mathbf{S}}$ is also a δ embedding since $\upsilon_{\mathbf{S}}(a) = [a,0] \mathrel{\widetilde{\delta}} [c,0] = \upsilon_{\mathbf{S}}(c)$ if and only if there are s, t such that $s \mathrel{\delta} t$, $s \sqsubseteq^{a} 0$ and $t \sqsubseteq^{c} 0$, that is, $s \leq a$ and $t \leq c$, by (E0). This implies

$a \;\delta\; c$, by (Ext) and since $s \;\delta\; t$. Conversely, if $a \;\delta\; c$, just take $s = a$ and $t = c$ in order to get $[a, 0] \;\widetilde{\delta}\; [c, 0]$. The case of a hypercontact is similar.

(3) Under the assumptions, a function $\widetilde{\eta} : \widetilde{S} \to T$ is such that $\eta = \upsilon_{\mathbf{S}} \circ \widetilde{\eta}$ if and only if $\widetilde{\eta}([a, 0]) = \widetilde{\eta}(\upsilon_{\mathbf{S}}(a)) = \eta(a)$, for every $a \in S$. If, moreover, $\widetilde{\eta} : \widetilde{\mathbf{S}}' \to \mathbf{T}'$ is a homomorphism, then $\widetilde{\eta}([0, b]) =^{(5.7)} \widetilde{\eta}(K[b, 0]) = K\widetilde{\eta}([b, 0]) = K\eta(b)$. Since $\widetilde{\eta}$ is also supposed to be a semilattice homomorphism, it follows that $\widetilde{\eta}([a, b]) = \widetilde{\eta}([a, 0]) + \widetilde{\eta}([0, b]) = \eta(a) + K\eta(b)$, hence if $\widetilde{\eta}$ exists it is unique. It is then enough to show that the above condition

$$\widetilde{\eta}([a, b]) = \eta(a) + K\eta(b) \tag{5.15}$$

actually determines a homomorphism $\widetilde{\eta}$ from $\widetilde{\mathbf{S}}'$ to $\mathbf{T}'$.

First, we need to check that if $(a, b) \sim (c, d)$, then $\eta(a) + K\eta(b) = \eta(c) + K\eta(d)$, so that $\widetilde{\eta}$ is well-defined. In fact, suppose that $(a, b) \precsim (c, d)$ is given by (5.1). By $a \sqsubseteq^{c} d$, we get $\eta(a) \sqsubseteq^{\eta(c)} \eta(d)$ in $\mathbf{T}$, since η is a homomorphism. Hence $\eta(a) \leq \eta(c) + K\eta(d)$ in $\mathbf{T}'$, because of (3.1), since, by definition, $\mathbf{T}$ is the e-specialization reduct of $\mathbf{T}'$. Similarly, from $b \sqsubseteq^{d} d$, we get $\eta(b) \leq \eta(d) + K\eta(d) = K\eta(d)$, since K is extensive. It follows that $\eta(a) + K\eta(b) \leq \eta(c) + K\eta(d)$. Symmetrically, $\eta(a) + K\eta(b) \geq \eta(c) + K\eta(d)$, thus we get equality. Hence $\widetilde{\eta}$ is well-defined.

Verifying that $\widetilde{\eta}$ is a semilattice homomorphism is identical to [8, Theorem 3.2].

$$\begin{aligned}
\widetilde{\eta}([a, b]) + \widetilde{\eta}([c, d]) &= \eta(a) + K\eta(b) + \eta(c) + K\eta(d) \\
&= \eta(a) + \eta(c) + K\eta(b) + K\eta(d) \\
&=^{\mathrm{A}} \eta(a + c) + K(\eta(b) + \eta(d)) \\
&= \eta(a + c) + K\eta(b + d) = \widetilde{\eta}([a + c, b + d]),
\end{aligned}$$

where we have used the definition of $\widetilde{\eta}$, the assumption that η is a semilattice homomorphism and in the identity marked with the superscript A we have used the assumption that K is additive in $\mathbf{T}'$.

Again, the argument showing that $\widetilde{\eta}$ is a K-homomorphism, is similar to [8].

$$\begin{aligned}
\widetilde{\eta}(K[a, b]) =^{(5.7)} \widetilde{\eta}([0, a + b]) = K\eta(a + b) = \\
K(\eta(a) + \eta(b)) =^{(2.1)} K(\eta(a) + K\eta(b)) = K\widetilde{\eta}([a, b]),
\end{aligned}$$

where we have used the definitions of K and $\widetilde{\eta}$, the assumption that η is a semilattice homomorphism and equations (2.1) and (5.7).

We now check that, in the presence of a weak contact relation, $\widetilde{\eta}$ is also a δ-homomorphism. For $a, b, e, f \in S$, each condition in the following list implies the condition below:

(a) $[a, b] \;\widetilde{\delta}\; [e, f]$,
(b) there are $s \;\delta\; t$ such that $s \sqsubseteq^{a} b$ and $t \sqsubseteq^{e} f$, by the definition of $\widetilde{\delta}$,

(c) there are $s, t \in S$ such that $\eta(s)\ \delta\ \eta(t)$, $\eta(s) \sqsubseteq^{\eta(a)} \eta(b)$ and $\eta(t) \sqsubseteq^{\eta(e)} \eta(f)$, since $\eta : \mathbf{S} \to \mathbf{T}$ is a homomorphism, in particular, both a δ- and an e-specialization homomorphism,
(d) there are $s, t \in S$ such that $\eta(s)\ \delta\ \eta(t)$, $\eta(s) \leq \eta(a) + K\eta(b)$ and $\eta(t) \leq \eta(e) + K\eta(f)$, since $\mathbf{T}$ is the weak contact e-specialization reduct of $\mathbf{T}'$,
(e) $\eta(a)+K\eta(b)\ \delta\ \eta(e)+K\eta(f)$, by (Ext) in $\mathbf{T}'$,
(f) $\widetilde{\eta}([a, b])\ \delta\ \widetilde{\eta}([e, f])$, by the definition of $\widetilde{\eta}$.

Thus $\widetilde{\eta}$ is also a δ-homomorphism and this completes the proof. The case of a weak pre-contact relation is the same, since we have never used symmetry in the above argument and the case of a hypercontact is similar.

(4) is a standard categorical argument, e. g. [1, Proposition 8.25] or the proof of clause 4 in [8, Theorem 3.2].

Having proved Theorem 5.4, Corollaries 5.5 and 5.6 are almost immediate from Propositions 3.8 and 3.9. As for Corollary 5.5, if $\mathbf{S}$ is a semilattice, expand $\mathbf{S}$ to $\mathbf{S}_e$ by adding $\sqsubseteq^*$ defined by (3.2) and apply Theorem 5.4 to $\mathbf{S}_e$, getting models, say, $\widetilde{\mathbf{S}_e}$ and $\widetilde{\mathbf{S}_e}'$. Taking the reduct of $\widetilde{\mathbf{S}_e}$ to the language of semilattices with 0, we immediately get (1) and (2) by the last sentence of Proposition 3.8 and observing that, in the special case at hand, (5.1) becomes exactly (5.13). In order to get the universal property (3), if $\mathbf{T}'$ is an additive closure semilattice with 0 and $\mathbf{T}$ is the semilattice reduct of $\mathbf{T}'$, expand $\mathbf{T}$ to an extended specialization semilattice $\mathbf{T}_e$ by using (3.2) again. By Theorem 5.4, there is a unique homomorphism of e-specialization semilattices $\widetilde{\eta}$ from $\widetilde{\mathbf{S}_e}$ to $\mathbf{T}_e$, hence a unique semilattice homomorphism between the reducts $\widetilde{\mathbf{S}_e}$ and $\mathbf{T}_e$ by the last statement in Proposition 3.8.

Of course, a direct proof of Corollary 5.5 along the lines of the proof of Theorem 5.4 is possible; however, we have showed that Theorem 5.4 "incorporates" Corollary 5.5.

Corollary 5.6 can be proved in the same way, by using $\sqsubseteq^*$ as defined by (3.3) and applying Proposition 3.9. A direct proof of Corollary 5.6 appears in [8], but only in the simpler case when no contact relation is present. □

Actually, the above proofs show that Theorem 5.4 and Corollaries 5.5, 5.6 hold when we add simultaneously any number of weak contact, weak pre-contact and hypercontact relations to $\mathbf{S}$. Indeed, the semilattice structure of $\widetilde{S}$ and the homomorphism $\widetilde{\eta}$ do not depend on the relations. We also get that coarseness between pairs of relations of a similar kind is preserved.

From Theorem 5.4(i)(ii) and standard results about closure semilattices, we get that our axiomatization of extended specialization semilattices characterizes those structures which can be embedded into the standard topological example. Recall that an extended specialization semilattice is said to be topological if it has the form $(\mathcal{P}(X), \cup, \emptyset, \sqsubseteq^*)$ for some topological space X with closure K, where $a \sqsubseteq^h b$ if $a \subseteq h \cup Kb$, for $a, h, b \subseteq X$.

Theorem 5.7. *Every extended [basic] specialization semilattice can be embedded into a topological extended [basic] specialization semilattice.*

Proof. If $\mathbf{S}$ is an extended specialization semilattice with 0, let $\widetilde{\mathbf{S}}$, $\widetilde{\mathbf{S}}'$ and $\upsilon_{\mathbf{S}}$ be as in Definition 5.1. By Theorem 5.4(2), $\upsilon_{\mathbf{S}}$ is an embedding of extended specialization semilattices.

By a standard argument, see e. g. [12, Proposition 5.6], the additive closure semilattice $\widetilde{\mathbf{S}}'$ (cf. Theorem 5.4(1)) can be embedded into a topological closure semilattice by, say, an embedding φ. By Proposition 3.4(d), φ is also an embedding between the e-specialization reducts, hence the composition of $\upsilon_{\mathbf{S}}$ and φ is an embedding of $\mathbf{S}$ into a topological extended specialization semilattice.

In the above proof we have assumed that $\mathbf{S}$ has a 0. If $\mathbf{S}$ has not a 0, simply add a zero, as in Remark 3.7. We obtain an extended specialization semilattice $\mathbf{S}_0$ with 0. Applying the above proof to $\mathbf{S}_0$, we get the result also for the 0-less $\mathbf{S}$, since $\mathbf{S}$ embeds into (the 0-less reduct of) $\mathbf{S}_0$.

The case of basic specialization semilattices can be proved in the same way, using Corollary 5.6. Another proof has been given in [12, Theorem 5.7]. $\square$

We will need much more efforts in order to generalize Theorem 5.7 to specialization semilattices endowed with a contact or a hypercontact relation. Put in another way, when a contact or a hypercontact is added, an appropriate version of Theorem 5.7 is not an immediate consequence of Theorem 5.4. Actually, further axioms should be added.

6. Embeddability into topological contact closure semilattices

Lemma 6.1. *Suppose that* $\mathbf{P}$ *is a normal closure poset.*

(a) If δ is the associated K-overlap weak contact, as introduced in Definition 2.3.2(c), equation (2.5)*, then δ satisfies*

$$Ka \; \delta \; Kc \;\Leftrightarrow\; a \; \delta \; c \tag{K$^+$}$$

(b) If α is the associated K-overlap weak pre-contact defined by (2.6)*, then α satisfies*

$$a \; \alpha \; Kc \;\Leftrightarrow\; a \; \alpha \; c \tag{6.1}$$

$$a \; \alpha \; c \;\Rightarrow\; Kc \; \alpha \; a, \tag{6.2}$$

in particular, α is symmetric on closed elements.

Proof. The implications from right to left follow from (Ext), since K is extensive. In the other direction, let us prove (6.1). By definition, $a \; \alpha \; Kc$ if and only if there is $p > 0$ such that $p \leq a$ and $p \leq KKc$. But $KKc = Kc$ since K is idempotent, hence the above condition is equivalent to $a \; \alpha \; c$. The proof of (K$^+$) is similar. In order to prove (6.2), the assumptions give some $p > 0$ such that $p \leq a$ and $p \leq Kc$, hence $p \leq Ka$, thus $Kc \; \alpha \; Ka$, hence $Kc \; \alpha \; a$ by (6.1). $\square$

By Lemma 6.1, the statements in (K^+), resp., (6.1), (6.2), hold in every topological contact closure semilattice, resp., topological pre-contact closure semilattice, as well as in every (pre-)contact closure semilattice embeddable into a topological one. However, the conditions are by no means sufficient, as already shown in the case of contact semilattices (without a closure) treated in [10]. The case of posets, instead of semilattices is somewhat simpler [11], but we will not treat it here.

We are now going to show that the conditions devised in [10] in the case without closure are also sufficient in the case in which closure is also present. Of course, by the above comments, we need to assume also (K^+) or (6.1), (6.2). We first recall the relevant conditions from [10], stated with reference to a weak contact semilattice.

$$\begin{aligned}&\text{For every } b, h, c_0, c_1 \in S, \text{ if } b \leq h + c_0,\ b \leq h + c_1 \text{ and } c_0 \not\delta\, c_1,\\ &\text{then } b \leq h.\end{aligned} \tag{D1}$$

$$\begin{aligned}&\text{For every } n \in \mathbb{N} \text{ and } a, b, c_{1,0}, c_{1,1}, \dots, c_{n,0}, c_{n,1} \in S,\\ &\text{if } c_{1,0} \not\delta\, c_{1,1}, \dots, c_{n,0} \not\delta\, c_{n,1} \text{ and, for every } f : \{1, \dots, n\} \to \{0, 1\},\\ &\text{either } a \leq c_{1,f(1)} + \dots + c_{n,f(n)},\ \text{ or } b \leq c_{1,f(1)} + \dots + c_{n,f(n)},\\ &\text{then } a \not\delta\, b.\end{aligned} \tag{D2}$$

Lemma 6.2. *Suppose that* $\mathbf{T}$ *is a weak contact closure semilattice satisfying* (K^+). *Then the following hold in* $\mathbf{T}$.

(a) Condition (D1) *holds if and only if the restricted version of* (D1) *holds in which* c_0 *and* c_1 *are required to be closed.*

(b) Condition (D2) *holds if and only if the restricted version of* (D2) *holds in which the elements* $c_{1,0}, \dots, c_{n,1}$ *are required to be closed.*

(c) If $\mathbf{T}$ *has additive closure, then condition* (D2) *holds if and only if the restricted version of* (D2) *holds in which all the elements involved are required to be closed.*

Proof. (a) An implication is straightforward. In the other direction, assume that the restricted version of (D1) holds. If the premises of (D1) hold, with c_0 and c_1 arbitrary elements, then such premises hold also with Kc_0 and Kc_1 in place of c_0 and c_1, since K is extensive and by (K^+). Thus the restricted version of (D1) can be applied, and we get the conclusion.

(b) As above, by (K^+) and extensivity of K, if the premises of (D2) hold, then they hold also with each $c_{i,j}$ replaced by $Kc_{i,j}$.

(c) If K is additive, then $Kc_{1,f(1)} + \dots + Kc_{n,f(n)} = K(c_{1,f(1)} + \dots + c_{n,f(n)})$, for every function f, hence, say, $a \leq Kc_{1,f(1)} + \dots + Kc_{n,f(n)}$ if and only if $Ka \leq Kc_{1,f(1)} + \dots + Kc_{n,f(n)}$, since $K(Kc_{1,f(1)} + \dots + Kc_{n,f(n)}) = KK(c_{1,f(1)} + \dots + c_{n,f(n)}) = K(c_{1,f(1)} + \dots + c_{n,f(n)})$. The same holds for inequalities involving b. Hence the premises still hold if we replace everywhere

a and b by Ka and Kb. By the restricted version of (D2) and (b), we get $Ka \not\delta Kb$ hence $a \not\delta b$ by (Ext). $\square$

The next lemma will be of some use, as well.

Lemma 6.3. *(A) Suppose that* $\mathbf{P}$ *and* $\mathbf{Q}$ *are pre-closure posets with the* K*-overlap weak contact* (2.5)*, and* $\varphi : \mathbf{P} \to \mathbf{Q}$ *is a* $\{\leq, K\}$*-homomorphism.*

(a) If $\varphi^{-1}(\{0_Q\}) = \{0_P\}$ *(in particular, this holds if* φ *is an order embedding), then* φ *is a* δ*-homomorphism.*

(b) Suppose further that φ *is a* $\{\leq, K\}$*-embedding and, moreover, for every* $a, b \in P$*, if the meet of* Ka *and* Kb *in* $\mathbf{P}$ *exists and is* 0*, then the meet of* $\varphi(Ka)$ *and* $\varphi(Kb)$ *exists in* $\mathbf{Q}$ *and is* 0*. Then* φ *is a* δ*-embedding.*

(B) The same holds for weak pre-contact closure posets, considering the K*-pre-overlap weak contact* (2.6)*, instead.*

Proof. (A)(a) If $a \;\delta\; b$, then there is $p \in P$ such that $0 < p$, $p \leq Ka$ and $p \leq Kb$, by the definition (2.5) of the K-overlap contact. By the assumption, $0 < \varphi(p)$, hence, since φ is a $\{\leq, K\}$-homomorphism, $\varphi(p) \leq \varphi(Ka) = K\varphi(a)$ and $\varphi(p) \leq \varphi(Kb) = K\varphi(b)$. Thus $\varphi(a) \;\delta\; \varphi(b)$.

(b) If $a \not\delta b$, then the meet of Ka and Kb is 0, since $\mathbf{P}$ has the K-overlap contact (or just by (K^+), (Ref) and (Ext)). By the assumptions, the meet of $K\varphi(a) = \varphi(Ka)$ and $K\varphi(b) = \varphi(Kb)$ is 0, hence $\varphi(a) \not\delta \varphi(b)$, because of the definition of the K-overlap contact. The converse implication is from (a).

The proof of (B) is similar; actually, we never used any special property of δ or of K in the above proof, we just needed that δ is defined according to (2.5) or (2.6) and that φ is a $\{\leq, K\}$-embedding. $\square$

Theorem 6.4. *Suppose that* $\mathbf{S}$ *is a weak contact closure semilattice. Then the following conditions are equivalent, where embeddings are always intended as* $\{+, K, \delta\}$*-embeddings.*

(1) $\mathbf{S}$ *can be embedded into a closure algebra with* K*-overlap contact.*

(1′) $\mathbf{S}$ *can be embedded into a closure algebra with additive contact and satisfying* (K^+)*.*

(2) $\mathbf{S}$ *can be embedded into an additive closure distributive lattice with* K*-overlap contact.*

(2′) $\mathbf{S}$ *can be embedded into an additive closure distributive lattice with additive contact and satisfying* (K^+)*.*

(3) $\mathbf{S}$ *has additive closure and satisfies* (K^+)*,* (D1) *and* (D2)*.*

(4) $\mathbf{S}$ *can be embedded into a complete atomic closure algebra with* K*-overlap contact.*

(5) $\mathbf{S}$ *can be embedded into the contact closure semilattice associated to some topological space, in the sense of Definition 2.3.2(b).*

Proof. (1) $\Rightarrow$ (1′) and (2) $\Rightarrow$ (2′) follow from Lemmas 2.3.3 and 6.1(a). (1) $\Rightarrow$ (2) and (1′) $\Rightarrow$ (2′) are straightforward.

$(2') \Rightarrow (3)$ $\mathbf{S}$ satisfies (D1) and (D2) in view of the corresponding implication in [10, Theorem 3.2], forgetting about the closure. The remaining conditions follow immediately from the corresponding conditions in $(2')$, since the conditions are preserved under taking substructures (and isomorphism).

$(3) \Rightarrow (1)$ Suppose that $\mathbf{S} = (S, \leq, 0, K, \delta)$ is a weak contact additive closure semilattice satisfying the assumptions in (3). Following the proof of [10, Theorem 3.2], consider the Boolean algebra $\mathbf{B}^- = (\mathcal{P}(S), \cup, \cap, \emptyset, S, \complement)$ and let $\varphi : P \to \mathcal{P}(S)$ be the semilattice embedding defined by $\varphi(a) = \not\uparrow a = \{\, x \in S \mid a \not\leq x \,\}$. Note that $\varphi(0) = \emptyset$. On $\mathcal{P}(S)$, set $Kx = \bigcap \{\, \not\uparrow Ka \mid a \in S, x \subseteq \not\uparrow Ka \,\}$. By the proof of [16, Lemma 2.3], K is an additive closure operation, since φ is injective and K on $\mathbf{S}$ is additive. Moreover, φ is a K-homomorphism by construction. See the proof of [12, Proposition 5.6] for more details. So let $\mathbf{B} = (\mathcal{P}(S), \cup, \cap, \emptyset, S, \complement, K)$, with K as just introduced.

As in [10], let $\mathcal{I}$ be the ideal of $\mathbf{B}$ generated by the set of all the elements of the form $\varphi(c) \cap \varphi(d)$, with $c, d \in S$ and $c \not\delta d$. By (K^+) and since K is extensive, $\mathcal{I}$ is equivalently generated by the set of all the elements of the form $\varphi(Kc) \cap \varphi(Kd)$, with $c, d \in S$ and $c \not\delta d$, since, by (K^+), this is equivalent to $Kc \not\delta Kd$. Since φ is a K-homomorphism, the generators of $\mathcal{I}$ can be taken to be of the form $K\varphi(c) \cap K\varphi(d)$, for $c \not\delta d$. Note that $K\varphi(c) \cap K\varphi(d)$ is closed, being the meet of two closed elements; see the comment shortly after the definition of a closed element in Section 2.2. Thus $\mathcal{I}$ has a set of closed generators, hence $i \in \mathcal{I}$ implies $Ki \in \mathcal{I}$, since K is additive. Let $\mathbf{A}$ be the quotient $\mathbf{B}/\mathcal{I}$. Again by additivity of K, the closure is well-defined on A, hence $\mathbf{A}$ is a closure algebra[1].

If $\pi : \mathbf{B} \to \mathbf{A}$ is the quotient homomorphism, then $\kappa = \varphi \circ \pi$ is a $\{+, K\}$-homomorphism from $\mathbf{S}$ to $\mathbf{A}$. Endow $\mathbf{A}$ with the K-overlap contact relation δ_A and with the overlap relation σ_A. The proof of the implication $(3) \Rightarrow (1)$ in [10, Theorem 3.2] shows that κ is an embedding from (S, δ) to (A, σ_A). But this is enough, since the following is a chain of equivalent conditions:

(i) $a \; \delta \; b$ in $\mathbf{S}$,
(ii) $Ka \; \delta \; Kb$, by (K^+) in $\mathbf{S}$, holding by assumption,
(iii) $\kappa(Ka) \; \sigma_A \; \kappa(Kb)$, by the mentioned result from [10],
(iv) $\kappa(Ka) \; \delta_A \; \kappa(Kb)$, since, for closed elements, δ and σ coincide, K being idempotent,
(v) $\kappa(a) \; \delta \; \kappa(b)$, by ($\mathrm{K}^+$) in $\mathbf{S}$, holding by Lemma 6.1

The implication $(4) \Rightarrow (1)$ is immediate. In order to prove $(1) \Rightarrow (4)$, notice that every closure algebra can be extended to a closure algebra which is complete and atomic (as a Boolean algebra). This fact follows from [16, Lemma 2.3] and the corresponding theorem for Boolean algebras; see e. g., [18, Section 2]. Embed (in the sense of closure algebras) the algebra given by (1) into a complete atomic closure algebra, and give this larger algebra, too,

[1] Additivity is necessary: see [18, Section 8].

the K-overlap contact relation. Since Boolean embeddings preserve meets, then the embedding is also a δ-embedding, in view of Lemma 6.3.

(4) $\Leftrightarrow$ (5) Since a complete atomic Boolean algebra $\mathbf{B}$ is isomorphic to a field of sets, say, $\mathcal{P}(X)$, a closure operation on $\mathbf{B}$ is uniquely associated to a topology on X, by Kuratowski characterization. Thus (4) and (5) are essentially the same statement. $\square$

7. Embeddability into topological extended specialization semilattice

Recall that if $\mathbf{S}$ is an extended specialization semilattice, we have set $a \sqsubseteq b$ if $a \sqsubseteq^{b} b$. Recall condition (D2) from the previous section. We let (D2$_{\sqsubseteq}$) be the condition similar to (D2) in which $\leq$ is replaced by $\sqsubseteq$. For a weak contact e-specialization semilattice, the following conditions will also be relevant in what follows.

$$s \mathrel{\delta} t \ \ \& \ \ s \sqsubseteq^{a} b \ \ \& \ \ t \sqsubseteq^{e} f \ \Rightarrow \ a+b \mathrel{\delta} e+f \tag{E$^+$}$$

$$b \sqsubseteq^{h} a+c_0 \ \ \& \ \ b \sqsubseteq^{h} a+c_1 \ \ \& \ \ c_0 \mathrel{\not\delta} c_1 \ \Rightarrow \ b \sqsubseteq^{h} a \tag{D1$_{\sqsubseteq^*}$}$$

Recall that weak contact semilattices have a 0 by definition and that if a specialization structure is also present, we assume that 0 satisfies the requested property (S0) or (E0), relative to the specialization.

Lemma 7.1. *Suppose that* $\mathbf{S}$ *is a weak contact extended specialization semilattice.*

(a) If $\mathbf{S}$ *satisfies* (D1$_{\sqsubseteq^*}$)*, then* $\mathbf{S}$ *satisfies* (D1)*.*
(b) If $\mathbf{S}$ *satisfies* (D2$_{\sqsubseteq}$)*, then* $\mathbf{S}$ *satisfies* (D2)*.*

Proof. (a) If $b \leq h + c_0$, then $b \sqsubseteq^{h} 0+c_0$, by (E1). Similarly, if $b \leq h + c_1$, then $b \sqsubseteq^{h} 0+c_1$. If furthermore $c_0 \mathrel{\not\delta} c_1$, then $b \sqsubseteq^{h} 0$, by (D1$_{\sqsubseteq^*}$), hence $b \leq h$, by (E0). This shows that $\mathbf{S}$ satisfies (D1).

(b) This is immediate from (E1). $\square$

Recall from Definition 3.5(a) that if $\mathbf{S}'$ is a closure semilattice, the e-specialization reduct of $\mathbf{S}$ is the extended specialization semilattice in which $\sqsubseteq^*$ is given by (3.1) in Proposition 3.4, namely, $a \sqsubseteq^{h} b$ if $a \leq h + Kb$.

Lemma 7.2. *Suppose that* $\mathbf{S}$ *is a weak contact extended specialization semilattice and* $\mathbf{S}$ *is the e-specialization reduct of some closure semilattice* $\mathbf{S}'$*. Then*

(a) $\mathbf{S}'$ *satisfies* (K$^+$) *if and only if* $\mathbf{S}$ *satisfies* (E$^+$)*.*

Suppose further that $\mathbf{S}'$ *has additive closure and satisfies* (K$^+$)*. Then*

(b) $\mathbf{S}'$ *(equivalently,* $\mathbf{S}$*) satisfies* (D1) *if and only if* $\mathbf{S}$ *satisfies* (D1$_{\sqsubseteq^*}$)*.*
(c) $\mathbf{S}'$ *(equivalently,* $\mathbf{S}$*) satisfies* (D2) *if and only if* $\mathbf{S}$ *satisfies* (D2$_{\sqsubseteq}$)*.*

Proof. (a) We first prove the “only if” condition. By (3.1), the premises of (E$^+$) read $s \leq a+Kb$ and $t \leq e+Kf$, thus $s \leq a+Kb \leq K(a+Kb) = K(a+b)$, because of (2.1). Similarly, $t \leq K(e+f)$. From $s \ \delta \ t$ and (Ext), we get $K(a+b) \ \delta \ K(e+f)$, hence $a{+}b \ \delta \ e{+}f$, since we are assuming (K^+).

For the converse, take $s = Kb$, $a = b$, $t = Kf$ and $e = f$ in (E^+), getting $b \ \delta \ f$ from $Kb \ \delta \ Kf$. If $b \ \delta \ f$, then $Kb \ \delta \ Kf$ by extensiveness of K and (Ext), with no need of further assumptions.

(b) First, note that (D1) does not deal with specialization or closure, hence it holds in $\mathbf{S}$ if and only if it holds in $\mathbf{S}'$. As far as the other equivalence is concerned, an implication is from Lemma 7.1(a) and needs no assumption on K.

For the other direction, suppose that $\mathbf{S}'$ satisfies (D1). By (3.1), the premises of ($\mathrm{D1}_{\sqsubseteq^*}$) give $b \leq h + K(a{+}c_0) = h + Ka + Kc_0$, by additivity of K, and similarly $b \leq h + Ka + Kc_1$. By (K^+), $Kc_0 \not\delta \ Kc_1$, thus $b \leq h + Ka$, by applying (D1) with $h + Ka$ in place of a. Thus $b \sqsubseteq^h a$, by using (3.1) one more time. This shows that $\mathbf{S}$ satisfies ($\mathrm{D1}_{\sqsubseteq^*}$).

(c) As in (b), the first equivalence uses the fact that (D2) speaks only about the semilattice operation and the weak contact relation.

In the other equivalence, an implication is from Lemma 7.1(b), and does not need the assumption that $\mathbf{S}'$ satisfies (K^+). For the converse, assume the premises of ($\mathrm{D2}_{\sqsubseteq}$). If, say, $a \sqsubseteq c_{1,f(1)} + \dots + c_{n,f(n)}$ holds, for some f, then $a \leq K(c_{1,f(1)} + \dots + c_{n,f(n)}) = Kc_{1,f(1)} + \dots + Kc_{n,f(n)}$ by (3.1), extensivity and additivity of K. By $c_{1,0} \not\delta \ c_{1,1}, \dots$ and (K^+), we get $Kc_{1,0} \not\delta \ Kc_{1,1}, \dots$ Thus we can apply (D2) in $\mathbf{S}'$, getting $a \not\delta \ b$. $\square$

Lemma 7.3. *Suppose that* $\mathbf{S}$ *is a weak contact extended specialization semilattice and let the notation in Definition 5.1(A)(B) be in charge.*

(a) If $\mathbf{S}$ *satisfies* (E^+)*, then*

$$[a,b] \ \widetilde{\delta} \ [e,f] \ \text{in } \widetilde{\mathbf{S}} \text{ if and only if } a{+}b \ \delta \ e{+}f \ \text{in } \mathbf{S}. \tag{7.1}$$

(b) $\widetilde{\mathbf{S}}'$ *satisfies* (K^+) *if and only if* $\mathbf{S}$ *satisfies* (E^+)*.*

(c) Suppose that $\mathbf{S}$ *satisfies* (E^+)*. Then*

(c1) $\widetilde{\mathbf{S}}$ *(equivalently,* $\widetilde{\mathbf{S}}'$*) satisfies* (D1) *if and only if* $\mathbf{S}$ *satisfies* ($\mathrm{D1}_{\sqsubseteq^*}$)*.*

(c2) $\widetilde{\mathbf{S}}$ *(equivalently,* $\widetilde{\mathbf{S}}'$*) satisfies* (D2) *if and only if* $\mathbf{S}$ *satisfies* ($\mathrm{D2}_{\sqsubseteq}$)*.*

Proof. (a) By (5.8), $[a,b] \ \widetilde{\delta} \ [e,f]$ means exactly that the premises of (E^+) hold, thus $a + b \ \delta \ c + d$ by (E^+) itself. Conversely, if $a{+}b \ \delta \ e{+}f$, just take $s = a + b$ and $t = e + f$, in order to have $[a,b] \ \widetilde{\delta} \ [e,f]$, as given by (5.8), using (E1).

(b) Recall from Definition 5.1 that $\widetilde{\mathbf{S}}$ is the e-specialization reduct of $\widetilde{\mathbf{S}}'$. If $\widetilde{\mathbf{S}}'$ satisfies (K^+), then $\widetilde{\mathbf{S}}$ satisfies (E^+), by Lemma 7.2(a). Since (E^+) is a universal statement and $\mathbf{S}$ is embedded in $\widetilde{\mathbf{S}}$, then $\widetilde{\mathbf{S}}$ satisfies (E^+).

Conversely, $K[a,b] \ \widetilde{\delta} \ K[c,d]$ means $[a, a+b] \ \widetilde{\delta} \ [c, c+d]$, by the definition (5.3) of K in $\widetilde{\mathbf{S}}'$. By (a), if $\mathbf{S}$ satisfies (E^+), then $[a, a+b] \ \widetilde{\delta} \ [c, c+d]$ if and only if $a + b \ \delta \ c + d$ if and only if $[a,b] \ \widetilde{\delta} \ [c,d]$, again by ($\mathrm{E}^+$).

(c) As in the previous lemma, note that (D1) and (D2) do not deal with specialization or closure, hence we can equivalently work in $\widetilde{\mathbf{S}}$ or in $\widetilde{\mathbf{S}}'$.

(c1) We first write down explicitly the meaning of (D1) in $\widetilde{\mathbf{S}}$. By (b) and Lemma 6.2(a), we may assume that c_0 and c_1 are closed (in the expansion $\widetilde{\mathbf{S}}'$), hence by Proposition 5.3(v) we my assume that c_0, c_1 have the form $[0, c'_0], [0, c'_1]$. Thus we can write the premises of (D1) in $\widetilde{\mathbf{S}}$ as $[b', b''] \leq [h, a] + [0, c'_0]$, $[b', b''] \leq [h, a] + [0, c'_1]$ and not $[0, c'_0] \ \widetilde{\delta} \ [0, c'_1]$. By Proposition 5.3(iii), the inequalities mean $(b', b'') \precsim (h, a + c'_0)$ and $(b', b'') \precsim (h, a + c'_1)$, that is,

(i) $b' \sqsubseteq^h a + c'_0$,
(ii) $b'' \sqsubseteq a + c'_0$,
(iii) $b' \sqsubseteq^h a + c'_1$, and
(iv) $b'' \sqsubseteq a + c'_1$.

By (a), not $[0, c'_0] \ \widetilde{\delta} \ [0, c'_1]$ is equivalent to

(v) $c'_0 \not\delta \ c'_1$.

Thus, assuming (E^+), $\widetilde{\mathbf{S}}$ satisfies (D1) if and only if the above conditions (i) - (v), stated in terms of elements of S, imply $[b', b''] \leq [h, a]$, that is, $b' \sqsubseteq^h a$ and $b'' \sqsubseteq a$.

If $\widetilde{\mathbf{S}}$ satisfies (D1) and we take $b'' = 0$, then (ii) and (iv) are automatically satisfied; dealing with the remaining conditions means exactly that $\mathbf{S}$ satisfies ($\mathrm{D1}_{\sqsubseteq^*}$). This implication follows also from (b) and Lemma 7.2(b), since ($\mathrm{D1}_{\sqsubseteq^*}$) is a universal statement, so if $\widetilde{\mathbf{S}}$ satisfies ($\mathrm{D1}_{\sqsubseteq^*}$), then $\mathbf{S}$ satisfies ($\mathrm{D1}_{\sqsubseteq^*}$), since it is isomorphic to a substructure of $\widetilde{\mathbf{S}}$.

Conversely, if ($\mathrm{D1}_{\sqsubseteq^*}$) holds in $\mathbf{S}$ and we have (i) - (v), we get $b' \sqsubseteq^h a$ from (i) and (iii). Applying ($\mathrm{D1}_{\sqsubseteq^*}$) with $h = 0$, we get $b'' \sqsubseteq a$ from (ii), (iv) and (v), by Lemma 3.6(v). Thus $(b', b'') \precsim (h, a)$, by (5.1). We have proved that $\widetilde{\mathbf{S}}$ satisfies (D1).

(c2) As in the proof of (c1), by the assumption that $\mathbf{S}$ satisfies (E^+), by (b) and by Lemma 6.2(b) it is enough to deal with closed elements of $\widetilde{\mathbf{S}}'$. Notice that $\widetilde{\mathbf{S}}'$ has additive closure by Theorem 5.4(1). Again by Proposition 5.3(v), closed elements have the form $[0, a]$, hence a condition like, say, $[0, a] \leq [0, c] + [0, d]$, that is, $(0, a) \precsim (0, c + d)$, by Proposition 5.3(ii)(iii), translates to $a \sqsubseteq c + d$, by the definition (5.1) of $\precsim$ and Lemma 3.6(v). $\square$

Theorem 7.4. *A weak contact extended specialization semilattice* $\mathbf{S}$ *can be embedded into a topological one if and only if* $\mathbf{S}$ *satisfies* (E^+)*,* ($\mathrm{D1}_{\sqsubseteq^*}$) *and* *(D2$_{\sqsubseteq}$).*

Proof. A topological weak contact extended specialization semilattice $\mathbf{T}$ satisfies (D1) and (D2) by [10, Theorem 3.2 ($1'$) $\Rightarrow$ (3)], forgetting about the specialization. Note that specialization does not appear in (D1) and (D2); for comparison, the parallel case of [10, Theorem 3.2] in which closure is also taken into account is Theorem 6.4 here. Moreover, considering also the topological closure on $\mathbf{T}$, (K^+) holds, hence $\mathbf{T}$ satisfies (E^+), ($\mathrm{D1}_{\sqsubseteq^*}$) and ($\mathrm{D2}_{\sqsubseteq}$), by Lemma 7.2. Since ($E^+$), ($D1_{\sqsubseteq^*}$) and ($D2_{\sqsubseteq}$) are universal sentences, if $\mathbf{S}$ can be embedded into $\mathbf{T}$, then $\mathbf{S}$ satisfies (E^+), ($D1_{\sqsubseteq^*}$) and ($D2_{\sqsubseteq}$).

Conversely, $\mathbf{S}$ can be embedded into $\widetilde{\mathbf{S}}$, a reduct of $\widetilde{\mathbf{S}}'$, by Theorem 5.4(1). If $\mathbf{S}$ satisfies (E^+), ($D1_{\sqsubseteq^*}$) and ($D2_{\sqsubseteq}$), then $\widetilde{\mathbf{S}}'$ satisfies (K^+), (D1) and (D2) by Lemma 7.3(b)(c). Thus $\widetilde{\mathbf{S}}'$ can be embedded into a topological closure semilattice, by Theorem 6.4(3) $\Rightarrow$ (5). Note that $\widetilde{\mathbf{S}}'$ has additive closure, by Theorem 5.4(1). Considering the e-specialization reducts and composing the two embeddings, we get an embedding of $\mathbf{S}$ into a topological weak contact e-specialization semilattice □

8. Further remarks

Example 8.1. In Lemma 3.6(i) we have seen that to an extended specialization semilattice there is associated a basic specialization semilattice by setting $a \sqsubseteq b$ if $a \sqsubseteq^b b$.

In this example we show that the extended structure cannot be retrieved from the basic structure. This shows that the notion of an extended specialization semilattice is actually more general than the notion of a basic specialization semilattice.

Consider the 5-element semilattice with $S = \{0, h, b, h+b, a\}$ with $h+b < a$. Let K be the closure operation defined by $K(h+b) = a$ and $Kx = x$, for $x \neq h+b$. By Proposition 3.4(a)(b), equation (3.1) induces the structure of an extended specialization semilattice $\mathbf{S}$ on S. In $\mathbf{S}$ the relation $a \sqsubseteq^h b$ fails.

Let $\mathbf{S}_1$ be defined as $\mathbf{S}$, except that we let $a \sqsubseteq^h b$ hold in $\mathbf{S}_1$. We claim that $\mathbf{S}_1$ is an extended specialization semilattice. Of course, this can be checked directly, but we can also use the topological representation given by Proposition 3.4(c).

Let $X = \{h', h'', b', b''\}$ be a topological space such that $K(\{h'\}) = \{h', h''\}$ and $K(\{b'\}) = \{b', b''\}$. The remaining topological structure will not be relevant. Equation (3.1) provides an extended specialization semilattice $\mathbf{T}$ on $\mathcal{P}(X)$. Let $0 = \emptyset$, $h = \{h'\}$ $b = \{b'\}$, $h + b = \{h', b'\}$.

If we further set $a = X$, we get the extended specialization semilattice $\mathbf{S}$, as a substructure of $\mathbf{T}$.

If instead we set $a = \{h', b', b''\}$, we get the extended specialization semilattice $\mathbf{S}_1$.

Symmetrically, we can also get another extended specialization semilattice $\mathbf{S}_2$ in which $a \sqsubseteq^b h$, instead.

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