On the Connection Between Costate and the Annihilator of the Hamiltonian Vector Field in Optimal Control Problems

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***Abstract*—Within the framework of finite-horizon optimal control problems involving nonlinear, input-affine dynamics, a connection between the costate variable and generating functions of the annihilating codistribution of the underlying Hamiltonian vector field is established. It is shown that the inverse mapping of any collection of n, such generating functions coincide, for any time and for a certain constant vector, with the costate of the optimal process. In particular, the corresponding constant vector is determined by solving a parameterized boundary value problem in the state space of the original plant alone, rather than in the extended state/costate space of the Hamiltonian dynamics.**

***Index Terms*—Annihilating codistribution, finite-horizon optimal control, Hamiltonian systems, nonlinear systems.**

I. Introduction

A PARTICULARLY desirable objective of any control system consists in ensuring that the state of the plant is steered from a generic initial configuration to a specific value in a safe and *optimal* fashion, according to a certain performance (or cost) criterion, see, e.g., [1]. The above control task can be naturally formulated as an *infinite-horizon optimal control* problem [2], provided that the time interval allocated to complete the transferring of the state between distinct end-points is *sufficiently long* compared with the time scales of the plant. On the contrary, whenever the primary attention is focused on completing the assigned control task within a prescribed horizon, fixed a priori, rather than on reaching a specific configuration at the end of such an interval, the structure of the solution to such a seemingly similar problem becomes significantly different from a mathematical point of view, being for instance intrinsically dependent on the elapsed time, i.e., time-varying [1, Sec. 5.1].

Despite the above difference between finite-horizon and infinite-horizon optimal control, the two problems in fact share common approaches toward the characterization of the underlying solutions. These strategies can be further categorized according to methods that are inspired by the so-called dynamic programming (DP), see, e.g., [3], and those that are based on Pontryagin's minimum (or maximum) principle (PMP) [4]. In the setting of finite-horizon problems, strategies inspired by DP characterize the optimal feedback in terms of the solution to a certain quadratic, time-varying partial differential equation (PDE), i.e., the so-called Hamilton–Jacobi–Bellman (HJB) equation [1]. The latter in particular yields necessary and sufficient conditions for optimality and permits the characterization of the optimal solution, as well as of the optimal cost, for any initial condition in the state space. Methods based on the PMP provide, unless additional assumptions hold, only necessary conditions for optimality. As a consequence, such strategies are typically employed in practice merely to identify *candidate* optimal solutions (*extremals*). Moreover, the computation of the optimal control law is based on the knowledge of the specific initial configuration of the plant, hence essentially leading to *open-loop* strategies. Nonetheless, despite the above drawbacks, the widespread use of such strategies is essentially motivated by the simplicity of the underlying conditions, provided in terms of *ordinary differential equations* (instead of PDEs) that should be satisfied by the optimal process together with an auxiliary variable (*costate*). In fact, the design can be recast in terms of a (two-point) *boundary value problem* (BVP) for a certain nonlinear system. Considering the relevant role played by optimal control formulations in practical applications, it is not surprising that numerous elegant and efficient techniques have been envisioned to address such BVPs. These methods rely either on a *transcription* of the underlying continuous-time optimal control problem into a *nonlinear programming problem* or on the construction of a sequence of *initial value problems* with an iterative refinement of the guess of the initial condition, see, e.g., [5], [6], and [7] for more discussions.

A. Contribution of This Article

The aim of this manuscript consists in discussing how knowledge of *first integrals* can be used to compute optimal control laws. Within the framework of finite-horizon optimal control, recalled in Section II, the main contribution of the manuscript consists in establishing a connection between the *costate* variable and the *first integrals* of the underlying Hamiltonian vector field, which are functions of the state and the costate variables. First, it is shown in Section III that the time evolution of the

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optimal costate coincides, for any time, with the inverse mapping of any collection of n-independent first integrals in which the second argument is replaced by a *constant* vector. The latter depends on the initial condition of the plant and is computed, via a fixed-point characterization, by solving a (parameterized) BVP formulated in the original coordinates of the state alone, rather than in the *extended* state/costate space. The property is established via a preliminary *nonlinear* change of coordinates in such a way that the optimal costate is constant over time in the transformed coordinates. It is further shown in Section V that, in the linear quadratic (LQ) setting, the result can be equivalently achieved via a *linear*, although necessarily *time-varying*, change of coordinates. In a different context (i.e., zero-sum differential games) and limited to LQ problems, a similar intuition has been explored in [8]. Despite the particularly elegant characterization in [8], restricting the attention to *linear, time-varying* change of coordinates prevented the unveiling of much deeper connections between first integrals and the optimal costate.

Furthermore, it is shown that knowledge of a closed-form expression for the mapping between the initial condition of the plant and the corresponding constant vector permits the computation of the underlying *value function* via a *line integral* instead of solving a quadratic, time-varying PDE. Despite the remarkably long history of research in optimal control, it appears that such strong connections between two relevant objects related to Hamiltonian systems, namely, optimal costate and first integrals, have not been observed hitherto.

Finally, it is discussed in Section IV how these abstract properties may pave the way to envision computationally efficient strategies to determine optimal control laws, which rely on premises significantly different from *shooting methods* based on the Hamiltonian dynamics. Indeed, apart from the above dimensional reduction of the required BVP, it is shown that, by hinging on the proposed constructions, the dynamics for which the BVP must be solved naturally inherits the stability properties of the original plant, differently from the case in which the Hamiltonian dynamics are employed.

B. Notation

Given a function $h : \mathbb{R}^n \to \mathbb{R}$, the notation $\nabla_x h(x)$ describes the column vector of partial derivatives of the function h, while $dh(x)$ defines the *gradient*. Moreover, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function, the notation $\nabla_x f(x)$ describes its Jacobian matrix. The subscript in $\nabla_x \cdot$ is neglected whenever the argument is clear from the context. The set $C^\kappa(\mathbb{R}^n)$ contains the functions defined over $\mathbb{R}^n$ with continuous derivatives up to order κ. For a function h and a vector field f, the *Lie derivative* $L_f h$ is defined as $L_f h := dh\, f$. For two vector fields f_1 and f_2, the *Lie bracket* $[f_1, f_2]$ is defined as $[f_1, f_2] := (\nabla f_2) f_1 - (\nabla f_1) f_2$. Given a matrix $M \in \mathbb{R}^{n\times n}$, $\sigma(M)$ denotes the *spectrum* of M. Provided M is symmetric, $M \succ 0$ ($M \succeq 0$) specifies that M is positive (semi)definite. Given an ordinary differential equation $\dot{x} = f(x;\eta)$, $x(t_0) = x_0$, parameterized with respect to $\eta \in \mathbb{R}^q$, the *flow* $\varphi(t, t_0, x_0; \eta)$ denotes the mapping that satisfies $\varphi(t_0, t_0, x_0; \eta) = x_0$ and $\frac{\partial}{\partial t}\varphi(t, t_0, x_0; \eta) = f(\varphi(t, t_0, x_0; \eta); \eta)$, for all $t \geq t_0$.

II. Preliminaries and Problem Statement

The objective of this section is to recall a few standard definitions and results concerning the optimal control problem for nonlinear systems over a finite horizon. To this end, consider nonlinear, input-affine dynamics described by

$$\dot{x} = f(x) + g(x)u, \quad x(t_0) = x_0 \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ denotes the state of the plant and $u : \mathbb{R} \to \mathbb{R}^m$ denotes the input. Suppose that the system possesses an equilibrium at the origin, i.e., $f(0) = 0$. Given a control input $u \in C^0([t_0, t_f])$, defined over an interval $[t_0, t_f] \subset \mathbb{R}$ fixed a priori, the performance of (1) driven by u is evaluated via the cost functional $J : C^0([t_0, t_f]) \to \mathbb{R}$ defined by

$$J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} \left(\ell(x(\tau)) + \|u(\tau)\|_R^2\right) d\tau + m\left(x(t_f)\right) \tag{2}$$

with $R = R^\top \succ 0$. The *running cost* on the state variable is described by the function $\ell : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\ell(0) = 0$, whereas the *terminal cost* is defined by the function $m : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $m(0) = 0$. The value of the functional J is parameterized with respect to the initial condition x_0 in (1). Furthermore, in the following it is implicitly assumed that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, the matrix-valued function $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and the functions ℓ and m are sufficiently smooth. The statement in the following formulates the optimal control problem considered here.

Problem 1: Consider the dynamics (1) and fix $x_0 \in \mathbb{R}^n$. The *finite-horizon optimal control problem* consists in determining a control input $u^\star(t)$, $t \in [t_0, t_f]$ such that $J(u^\star) < J(u)$ for all $u \in C^0([t_0, t_f])$, namely, with the property that the functional (2) is minimized along the trajectories of (1). ∘

A precise characterization of the existence and regularity properties of the solution to Problem 1, which constitutes a challenging task per se and which has attracted significant attention in the literature (see, e.g., [9]), is beyond the scope of this manuscript. Therefore, the following assumption is stated to set the framework for the results in the following. To provide a concise notation, define the matrix-valued function $S : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ according to $S(x) := g(x)R^{-1}g(x)^\top$ for all $x \in \mathbb{R}^n$.

Assumption 1: There exists a unique solution $V^\star : [t_0, t_f] \times \mathbb{R}^n \to \mathbb{R}_{>0}$, $V^\star \in C^2([t_0, t_f] \times \mathbb{R}^n)$, of the HJB PDE

$$\begin{cases} -\nabla_t V = \frac{1}{2}\ell(x) + \nabla_x V^\top f(x) - \frac{1}{2}\nabla_x V^\top S(x)\nabla_x V \\ V(t_f, x) = m(x) \end{cases} \tag{3}$$

for all $t \in [t_0, t_f]$ and all $x \in \mathbb{R}^n$. ∘

The requirements of Assumption 1, which may be relaxed to hold locally in a neighborhood of the origin, ensure the existence of a unique optimal solution to Problem 1, which is obtained in terms of the feedback control law [10, Sec. 4.2]

$$u^\star(t) = -R^{-1} g\left(x^\star(t)\right)^\top \nabla_x V\left(t, x^\star(t)\right). \tag{4}$$

However, the closed-form computation of the *value function* $V^\star$ is typically not viable in practice. To circumvent the latter issue, an alternative *trajectory-based* approach, hence more akin to the formulation of Problem 1, is provided by the theory developed

by Pontryagin to tackle the problem in the spirit of calculus of variations [1]. Toward this end, consider the auxiliary variable $\lambda : \mathbb{R} \to \mathbb{R}^n$ and define the (minimized) Hamiltonian function $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ described by

$$\begin{aligned}\mathcal{H}(x,\lambda) &= \min_u \left\{ \lambda^\top (f(x) + g(x)u) + \frac{1}{2}\ell(x) + \frac{1}{2}\|u\|_R^2 \right\} \\ &= \lambda^\top f(x) + \frac{1}{2}\ell(x) - \frac{1}{2}\lambda^\top S(x)\lambda. \end{aligned} \tag{5}$$

PMP yields necessary conditions for optimality in terms of the solution to the BVP described by (see [4])

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \nabla_\lambda \mathcal{H}(x,\lambda) \\ -\nabla_x \mathcal{H}(x,\lambda) \end{bmatrix} := \mathbf{f}_\mathcal{H}(\chi) \tag{6}$$

with $\chi = (x, \lambda)$, together with the *split* conditions

$$x(0) = x_0, \quad \lambda(t_f) = \nabla_x m\left(x(t_f)\right). \tag{7}$$

Assumption 2: Fix $x_0 \in \mathbb{R}^n$. There exists a unique solution to the BVP (6) and (7) in $[t_0, t_f]$. ○

By the so-called *sensitivity conditions* in optimal control (see, e.g., [9, Ch. 12]), it follows that the unique solution to the BVP (6) and (7) satisfies $\lambda^\star(t) = \nabla_x V^\star(t, x^\star(t))$ for all $t \in [t_0, t_f]$. Furthermore, by combining Assumptions 1 and 2, one has that the unique *extremal* satisfying the necessary conditions (6) and (7) indeed yields the optimal solution in terms of $u^\star(t) = -R^{-1}g(x^\star(t))^\top \lambda^\star(t)$. Finally, since the analysis of the following sections is based on the construction of the *annihilating* codistribution of $\mathbf{f}_\mathcal{H}$, the following definitions are briefly recalled from [11, Sec. 1.3].

Definition 1: Given a collection of vector fields f_1,..., f_d, a *distribution* Δ is a mapping that assigns to each point $x \in \mathbb{R}^n$ a *vector space*, i.e., a subspace of $\mathbb{R}^n$, defined as $\Delta(x) = \operatorname{span}\{f_1(x), \ldots, f_d(x)\}$. The dimension of the distribution corresponds to the dimension of the subspace $\Delta(x)$ at x. Δ is nonsingular if there exist a neighborhood $\mathcal{U}$ and an integer d such that $\dim(\Delta(x)) = d$ for all $x \in \mathcal{U}$. ○

Definition 2: A *codistribution* $\mathcal{C}$ is a mapping that assigns to each point $x \in \mathbb{R}^n$ a *dual vector space*, i.e., a subspace of $(\mathbb{R}^n)^\star$, defined as $\mathcal{C}(x) = \operatorname{span}\{\omega_1(x), \ldots, \omega_d(x)\}$, with $\omega_i(x)$ describing a covector field. Given a nonsingular distribution Δ of dimension d, the *annihilator* is a codistribution $\mathcal{C} := \Delta^\perp = \operatorname{span}\{\omega_1, \ldots, \omega_{n-d}\}$ with the property that $\omega_i(x) f(x) = 0$, for $i = 1, \ldots, n-d$ and for all $f \in \Delta$. ○

Among the set of annihilators of a given distribution Δ, one of particular interest is a codistribution that is spanned by *gradients* only, namely, such that there exist independent functions ϕ_1,... ϕ_{n-d}, with $\phi_i : \mathbb{R}^n \to \mathbb{R}$, such that $\Delta^\perp = \operatorname{span}\{d\phi_1, \ldots, d\phi_{n-d}\}$. Whenever such functions exist, the distribution is said to be (locally) *completely integrable*.

Definition 3: A distribution Δ is said to be *involutive* if $[f, g] \in \Delta$ for any pair of vector fields $f \in \Delta$ and $g \in \Delta$. ○

By Frobenius Theorem (see, e.g., [11, Th. 1.4.1]), a nonsingular distribution is completely integrable if and only if it is involutive. Thus, building on Definitions 1–3 and on Frobenius theorem, the following property[1] holds: any 1-dimensional distribution $\Delta(x) = \operatorname{span}\{f(x)\} \subset \mathbb{R}^n$, i.e., consisting of a single vector field, always admits $n-1$ independent functions ϕ_i with the property that $d\phi_i(x) f(x) = 0$, in a neighborhood of any point such that $f(x)$ is different from zero. The property is derived by recalling that $[f, f] = 0$ for any f, hence $\Delta = \{f\}$ is involutive (see [11, Rmk. 1.3.8]). The assumption in the following requires, therefore, that the optimal process $\chi^\star = (x^\star, \lambda^\star)$ evolves sufficiently *away* from the origin of $\mathbb{R}^n \times \mathbb{R}^n$, in such a way that the underlying vector field remains nonsingular.

Assumption 3: Fix $x_0 \in \mathbb{R}^n$. There exists $\delta > 0$ such that $\min_{t \in [t_0, t_f]} \|\chi^\star(t)\| > \delta$. ○

III. On the Relation Between the Annihilating Codistribution and Optimal Costate

Since $\mathbf{f}_\mathcal{H}$ is a vector field mapping the state into the tangent space subset of $\mathbb{R}^{2n}$, hence it can be interpreted as a 1-dimensional distribution, by Frobenius Theorem the latter vector field always admits, away from the equilibrium point at the origin, an annihilating codistribution of dimension $2n-1$. Therefore, there exist $2n-1$ independent functions $\psi_i(x, \lambda)$ with the property that[2] $d\psi_i \mathbf{f}_\mathcal{H} = 0$. Consider a nonempty set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$, the projection of which on the x-space contains a given initial condition x_0, and let $\Xi = \{\psi_i\}_{i=1,\ldots,2n-1}$ denote a set of independent *generating functions* whose *gradients* span the annihilator of $\mathbf{f}_\mathcal{H}$, namely, $\mathbf{f}_\mathcal{H}^\perp = \operatorname{span}\{d\psi_1, \ldots, d\psi_{2n-1}\}$ for all $(x, \lambda) \in \Omega$. The following statement clarifies the role of the functions in Ξ toward the computation of the optimal solution to Problem 1, by suggesting how first integrals of the Hamiltonian dynamics can be used to construct optimal control laws.

Theorem 1: Consider the nonlinear system (1) together with the cost functional (2) and fix $x_0 \in \mathbb{R}^n$. Suppose that Assumptions 1–3 hold. Consider any selection of n functions in Ξ with the property that $\psi(x, \lambda) := [\psi_{i_1}, \ldots, \psi_{i_n}]^\top$ is such that $\nabla_\lambda \psi$ is nonsingular in $\Omega_0 \subseteq \Omega$. Then, for all the optimal processes $(x^\star, \lambda^\star)$ that remain in Ω_0 for all $t \in [t_0, t_f]$, the optimal solution is equivalently described by[3]

$$u^\star(t) = -R^{-1} g(x^\star(t))^\top \psi^{-1}\left(x^\star(t), \psi\left(\xi, \nabla_x m(\xi)\right)\right) \tag{8}$$

where the constant vector $\xi \in \mathbb{R}^n$ is such that

$$\xi = \varphi_z\left(t_f, t_0, x_0; \xi\right) \tag{9}$$

with $\varphi_z(t, t_0, x_0; \xi)$ denoting the *flow* of the (reduced) system

$$\dot{z} = f(z) - S(z)\psi^{-1}\left(z, \psi\left(\xi, \nabla_x m(\xi)\right)\right), \quad z(0) = x_0 \tag{10}$$

parameterized with respect to ξ. ◇

Remark 1: A few observations about Theorem 1 are in order before the formal proof. First, note that the condition (9) entails

[1] Such a property is also obtained as a straightforward consequence of the so-called *flow-box* or *straightening out* theorem, see, e.g., [12, Th. 1].

[2] It is worth observing that such a property can be put into perspective with respect to the geometric properties of Hamiltonian vector fields by recalling that $d\psi_i \mathbf{f}_\mathcal{H} = \{\psi_i, \mathcal{H}\}$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. The interested reader is referred to [13] where an elegant characterization of such properties is discussed in the setting of infinite-horizon optimal control.

[3] The notation ψ^{-1} describes the (partial) inverse with respect to λ, i.e., a mapping with the property that $\lambda = \psi^{-1}(x, \psi(x, \lambda)) = \psi^{-1}(z, \psi(z, \lambda)) = \psi^{-1}(z, p)$, provided $x = z$ and $p = \psi(x, \lambda)$.

that the optimal solution of Problem 1 can be determined in two steps, by solving first (i) a *linear time-invariant* PDE (to compute the set Ξ of annihilating functions) and subsequently (ii) a *BVP of dimension* n, instead of $2n$ as for (6) and (7), with respect to a vector field parameterized via ξ. Furthermore, the functions in Ξ do not vary with x_0 in Ω_0, hence the solution to step (i) above should not be repeated for different initial conditions. Conversely, to put the complexity of the computations involved in Theorem 1 into perspective, it is worth observing that an implementation via (numerical) integration of (6) and (7) or via the direct solution of (3) would instead hinge upon a *BVP of dimension* $2n$ or on a *quadratic, time-varying* PDE, respectively. In addition, the explicit knowledge of the solution of the BVP of dimension $2n$ for a certain x_0 does not provide any insight on the solution for a different initial condition, and hence, it should be repeated. Finally, note that (9) may be interpreted as the task of computing a *fixed point* of the mapping $\xi \mapsto \varphi_z(t_f, t_0, x_0; \xi)$. The latter intuition is exploited in the following (see Section IV-B) to suggest numerically appealing strategies for its solution. ▲

Proof of Theorem 1: To begin with define the change of coordinates for the Hamiltonian dynamics (6) described by

$$z = x \quad p = \psi(x, \lambda) \tag{11}$$

and note that, by the assumption on non-singularity of the Jacobian matrix $\nabla_\lambda \psi$, the mapping $(x, \lambda) \mapsto (x, \psi(x, \lambda))$ constitutes a local diffeomorphism in the neighborhood of any point in Ω_0 (see [11, Prop. 1.2.3]). Since the components $\psi_i, i = 1, \ldots, n$, of the mapping ψ belong to Ξ, hence the corresponding gradients belong to $\mathbf{f}_{\mathcal{H}}^{\perp}$, one has that, by construction, $d\psi_i \, \mathbf{f}_{\mathcal{H}} = 0$ in Ω for $i = 1, \ldots, n$. Therefore, in the transformed coordinates, the Hamiltonian dynamics (6) becomes

$$\begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \nabla_\chi \psi \, \mathbf{f}_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} f(z) - S(z)\psi^{-1}(z, p) \\ 0 \end{bmatrix} \tag{12}$$

together with the split boundary conditions

$$z(0) = x_0 \tag{13a}$$

$$p(t_f) = \psi\left(z\left(t_f\right), \nabla_z m\left(z(t_f)\right)\right). \tag{13b}$$

Note that (11) does not constitute a *canonical* change of coordinates, hence, as expected, the Hamiltonian structure is not preserved in the transformed coordinates. Nonetheless, by inspecting the second block equation of (12), it follows immediately that the variable $p(t)$ is constant over time in the interval $t \in [t_0, t_f]$, and, hence, by (13b), is equal to

$$p(t) = \psi\left(z(t_f), \nabla_z m\left(z(t_f)\right)\right) \tag{14}$$

for all $t \in [t_0, t_f]$. Replacing (14) into the first block equation of (12), the latter then coincides with (10) where the final state $z(t_f)$ (appearing on the right-hand side of (14)) is described in terms of a generic constant vector $\xi \in \mathbb{R}^n$. The proof is then concluded by observing that, in order to be consistent with (14), and hence with the resulting dynamics in (10), the constant vector $\xi := z(t_f)$ must be precisely characterized by the condition (9). □

Remark 2: The structure of the control law (8) implies that the annihilating codistribution of the underlying Hamiltonian dynamics yields the optimal solution of Problem 1 in terms of a (parameterized) *state feedback*, similarly to techniques inspired by DP [9], [14], whose characterization is, however, formulated in terms of a BVP without the need for solving any PDE, similarly to methods based on PMP [4]. ▲

The following statement establishes a connection between (the inverse mapping of) any collection of n generating functions of the annihilator of the Hamiltonian dynamics (6) and the time history of the optimal costate. More precisely, it is shown that the latter coincides for any time with the former provided the second argument is replaced by a suitably defined constant vector.

Corollary 1: Suppose that the hypotheses of Theorem 1 hold. Then, there exists a constant vector $v \in \mathbb{R}^n$ with the property that

$$\lambda^\star(t) = \psi^{-1}(x^\star(t), v) \tag{15}$$

for all $t \in [t_0, t_f]$, where $\lambda^\star$ denotes the optimal costate. ◇

Proof: The claim follows immediately from the constructions discussed in the proof of Theorem 1 and by uniqueness of the optimal process. In fact, by considering the inverse change of coordinates, it follows that

$$\begin{aligned} \lambda^\star(t) &= \psi^{-1}\left(x^\star(t), p^\star(t)\right) \\ &= \psi^{-1}\left(x^\star(t), \psi\left(\xi, \nabla_x m(\xi)\right)\right) \end{aligned} \tag{16}$$

for all $t \in [t_0, t_f]$, with ξ satisfying the condition (9). The conclusion then follows by letting $v := \psi(\xi, \nabla_x m(\xi))$. □

Remark 3: A combination of the intuitions behind Theorem 1 and Corollary 1 permits the derivation of straightforward algebraic conditions that relate the optimal initialization of the costate variable, i.e., $\lambda^\star(t_0)$, and the vector ξ in (9). In fact, by construction of the change of coordinates, it follows that $p^\star(t_0) = \psi(x_0, \lambda^\star(t_0))$. Therefore, since $p^\star$ is constant in $[t_0, t_f]$, hence $p^\star(t_f) = \psi(\xi, \nabla_x m(\xi)) = p^\star(t_0)$, the vector ξ is obtained, for fixed $(x_0, \lambda^\star(t_0))$, by solving the system of algebraic equations

$$\psi\left(x_0, \lambda^\star(t_0)\right) = \psi\left(\xi, \nabla_x m(\xi)\right) \tag{17}$$

for all $x_0 \in \Omega$ and $t_0 \in \mathbb{R}$. ▲

Remark 4: By further reconciling the conclusions of Corollary 1 with the *sensitivity conditions* arising in optimal control theory, it follows that the composition of the annihilator of the Hamiltonian vector field with its inverse function yields the *sensitivity* of the optimal cost, namely,

$$\nabla_x V^\star\left(t_0, x_0\right) = \psi^{-1}(x_0, \cdot) \circ \psi\left(\xi, \nabla_x m(\xi)\right) \tag{18}$$

provided ξ satisfies (9). It is worth observing that, differently from the left-hand side, the right-hand side of (18) does not depend *explicitly* on the initial time, which is encoded in the value of ξ that solves (9). ▲

The structure of (18) implicitly suggests that the annihilating codistribution of the Hamiltonian dynamics is strictly related also to the value function of Problem 1. However, such a direct computation is viable only if the overall dependence of the right-hand side of (18) on the initial condition x_0 is captured: this includes the fact that the value of ξ depends, in turn, on the specific selection of x_0, as it appears by inspecting (9). This

intuition is summarized in the following statement, in which the concise notation $\bar{\psi}(\xi) := \psi(\xi, \nabla_x m(\xi))$ is employed.

Corollary 2: Suppose that the hypotheses of Theorem 1 hold. In addition, let $\mu : \mathbb{R}^n \to \mathbb{R}^n$ be such that $\mu(x) = \varphi_z(t_f, t_0, x; \mu(x))$ for all $x \in \Omega_0$. Define the function

$$\nu(x) = \int_0^1 \left\langle \psi^{-1}\left(h, \bar{\psi}(\mu(h))|_{h=x_0+s(x-x_0)}\right), x - x_0 \right\rangle ds. \tag{19}$$

Then, the value function satisfies $V^\star(t_0, x) = \nu(x) - \nu(0)$, for all $x \in \Omega_0$. ◇

Proof: Provided the fixed-point condition (9) is satisfied for all $x_0 \in \Omega$, as prescribed by the definition of the function μ, it follows by (18) that the gradient of the optimal value function $V^\star(t_0, x)$, for fixed t_0 and *for all* $x \in \Omega_0$, coincides with $\psi^{-1}(x, \bar{\psi}(\mu(x)))$. Furthermore, note that $\nabla_x \nu(x) = \psi^{-1}(x, \bar{\psi}(\mu(x)))$, by construction, and that $V^\star(t_0, 0) = 0$, by the definition of Problem 1 for all t_0 (by time-invariance of the involved functions). Thus, one has that $\nu(x) - \nu(0)$ yields the optimal value function $V^\star(t_0, x)$ of Problem 1 in Ω_0. □

The statement of Corollary 2 entails that the *value function* may be equivalently computed by solving a linear, time-invariant PDE (to determine first integrals) and a *line integral*, rather than by solving a *quadratic, time-varying PDE*. The claims of Theorem 1 and Corollaries 1 and 2 are illustrated in the following via a numerical example involving a linear system and a quadratic cost functional, for which the constructions can be easily carried out. In more general settings, the computations may, in fact, be obstructed by cumbersome notation and computational issues. These relevant aspects are instead addressed in Section IV-A.

Example 1: Consider an LQ optimal control problem described by the dynamics

$$\dot{x} = u \quad x(0) = x_0 \tag{20}$$

together with the cost functional

$$J(u) = \frac{1}{2} \int_0^1 \left(x(t)^2 + u(t)^2\right) dt \tag{21}$$

hence as in (2) with $\ell(x) = x^2$, $R = 1$ and $m \equiv 0$. In this setting, the optimal solution can be immediately determined by relying on the knowledge, for instance, of the eigenvalues and eigenvectors of the underlying Hamiltonian matrix, which allow to construct the solution to the corresponding differential Riccati equation, see e.g., [1]. More precisely, the optimal solution is $u^\star(t) = -P(t)x(t)$, for $t \in [0, 1]$, with

$$P(t) = \left(1 - e^{-2(1-t)}\right)\left(1 + e^{-2(1-t)}\right)^{-1} \tag{22}$$

solution of $-\dot{P} = 1 - P^2$, $P(1) = 0$, and associated also with the value function according to $V^\star(t, x) = (1/2)x^2 P(t)$. Consider instead the dynamics (6), which becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -\lambda \\ -x \end{bmatrix} \tag{23}$$

together with the boundary conditions $x(0) = x_0, \lambda(1) = 0$. The function $\psi(x, \lambda) = x^2 - \lambda^2$ yields the annihilator of the (linear) vector field in (23). Thus, the associated inverse function is defined as $\psi^{-1}(z, p) = \pm\sqrt{z^2 - p}$ depending on the sign of λ. Therefore, according to Theorem 1, since the variable $p(t)$ is constant in the interval [0,1] and equal to

$$p(t) = \psi(z(1), 0) = z(1)^2 =: \xi^2 \tag{24}$$

the reduced system (10) becomes

$$\dot{z} = \mp\sqrt{z^2 - \xi^2}, \quad z(0) = x_0. \tag{25}$$

By *separation of variables,* the scalar equation (25) yields

$$\left(\frac{z(t) + \sqrt{z(t)^2 - \xi^2}}{x_0 + \sqrt{x_0^2 - \xi^2}}\right)^{\pm 1} = e^t$$

for all $t \in [0, 1]$, which, by solving at $t = 1$ and $z(1) = \xi$ with respect to ξ, satisfies the *fixed-point* condition (9) by selecting

$$\mu(x_0) = \frac{2ex_0}{e^2 + 1}. \tag{26}$$

Therefore, by Theorem 1 and Corollary 1, it follows that the optimal costate variable $\lambda^\star(t)$, $t \in [0, 1]$, is *equivalently* described in terms of $z(t)$ and $\mu(x_0)$ as

$$\lambda^\star(t) = P(t)x^\star(t) = \pm\sqrt{(z^\star(t))^2 - \mu(x_0)^2}$$

where $z^\star(t)$ solves (25) with $\xi = \mu(x_0)$. Furthermore, the optimal value function $V^\star(0, x)$ and the integral (19), namely,

$$\nu(x) - \nu(0) = \frac{x^2}{2}\sqrt{\frac{(e^2+1)^2 + 4e^2}{(e^2+1)^2}}$$

coincide for all $x \in \mathbb{R}$. △

Apart from specially structured classes of optimal control problems, the solution of (6) and (7) can be seldom computed in a closed form. As anticipated in Section I, *shooting methods* aim at providing an accurate estimate of the underlying solution. More precisely, the *split* boundary conditions (7) are satisfied by iterating on (suitably updated) *guesses* of the corresponding initial condition for the costate variable $\lambda(t_0)$, so that the terminal condition eventually holds. Thus, the initial guess for $\lambda(t_0)$ plays a crucial role toward numerical reliability of such methods. In this respect, the property that the origin is an unstable equilibrium point for (6), provided rather standard assumptions hold (see also Section V), renders the majority of shooting methods troublesome, if not impossible, to implement in practice whenever the difference $t_f - t_0$ is large. Within this framework, the following formal statement and numerical simulation illustrate an advantageous feature of the fixed-point condition (9) compared with classic methods.

Proposition 1: Consider the reduced system (10) and suppose that $\sigma(\nabla_z f(z)|_{z=0}) \subset \mathbb{C}^-$. Moreover, let $G_0(\xi) := \nabla_z(S(z)\psi^{-1}(z, \bar{\psi}(\xi)))|_{z=0}$ and suppose that $G_0(0) = 0$. Then, there exists $\varepsilon \in \mathbb{R}_{>0}$ with the property that, for all ξ such that $\|\xi\| < \varepsilon$, $z = 0$ is locally exponentially stable (LES) for system (10). ○

Proof: By isolating the linear terms with respect to z in the vector field of system (10), the latter can be written as

$$\begin{aligned} \dot{z} &= A_0 z + \tilde{f}(z) - G_0(\xi)z - \tilde{G}(\xi, z) \\ &= (A_0 - G_0(\xi))\, z + \tilde{f}(z) - \tilde{G}(\xi, z) \end{aligned} \tag{27}$$

with $A_0 := \nabla_z f(z)|_{z=0}$, where $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ contain higher order terms with respect to z. Since A_0 is Hurwitz by assumption, and hence $\sigma(A_0 - G_0(0)) \subset \mathbb{C}^-$, the claim follows immediately by continuity of the mapping G_0 with respect to ξ and of the eigenvalues of a matrix with respect to variations in its entries. □

Remark 5: The statement of Proposition 1 entails that, for sufficiently small initial conditions x_0 and provided the original system (1) possesses a LES equilibrium point at the origin, it is always possible to initialize an iterative strategy designed to converge to the correct ξ (such as the one proposed in Section IV-B) by means of a selection of ξ_0 that ensures a numerically meaningful terminal condition $z(t_f)$ even for arbitrarily large t_f. Intuitively, the stability properties of the underlying plant are *inherited* by the system employed in the shooting method, and hence, the accuracy of the initial *guess* of the vector ξ is not crucial. This is profoundly different from what happens in general with the (full-order) Hamiltonian dynamics (6) in which, even for arbitrarily small initial conditions for the state, the (random) selection of $\lambda(t_0)$ gives rise with probability one to a trajectory that diverges for large terminal times. In fact, the set of initial guesses for $\lambda(t_0)$ that are associated with bounded trajectories of (6) is of *zero measure*, whereas the selection of an ill-conditioned $\lambda(t_0)$ is *generic*. This aspect is illustrated in Example 2. ▲

Remark 6: The emphasis of Proposition 1 is on the property of rendering the set of well-conditioned initial guesses for the choice of ξ an open set, rather than a zero-measure one. Conversely, the fact that the former set is centered at the origin is not particularly relevant. As a consequence, the assumptions on G_0 may be partially relaxed by requiring instead that there exists $\hat{\xi} \in \mathbb{R}^n$ such that $A_0 - G_0(\hat{\xi})$ is Hurwitz. Then, the conclusions of Proposition 1 would hold for all ξ such that $\|\xi - \hat{\xi}\| < \varepsilon$, for some positive ε. ▲

Example 2: Consider the LQ problem described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -0.5x_2 + u \end{bmatrix} \tag{28}$$

initialized at $x_0 = [1,\ 0.5]^\top$, together with the cost functional

$$J(u) = \int_0^{t_f} u(t)^2\, dt + \frac{1}{2} x_1(t_f)^2. \tag{29}$$

It can be shown that the functions

$$\psi_1(x,\lambda) = \frac{\sqrt{\lambda_1}}{2\lambda_1 + \lambda_2}, \psi_2(x,\lambda) = \sqrt{\lambda_1}\,(3x_2 + 2\lambda_1 + 3\lambda_2) \tag{30}$$

well defined in the positive orthant of the state/costate space are independent functions such that $\mathrm{span}\{d\psi_1, d\psi_2\} \subset \mathbf{f}_{\mathcal{H}}^{\perp}$, where the (linear) vector field $\mathbf{f}_H$ is defined as

$$\mathbf{f}_{\mathcal{H}} := \begin{bmatrix} -x_1 + x_2 \\ -0.5x_2 - \lambda_2 \\ \lambda_1 \\ -\lambda_1 + 0.5\lambda_2 \end{bmatrix}. \tag{31}$$

It is straightforward to notice that the latter vector field possesses an unstable equilibrium point at the origin. The objective of the following numerical simulation consists in assessing the influence of the value of the terminal time t_f toward the implementation of a shooting method either directly for (31) or for (10) with (28) and (30). To this end, suppose that the dynamics described by (31) are forward integrated from the initial condition $x(0) = x_0$ and $\lambda(0) = \lambda^\star_{x_0} + \delta\lambda$, where $\lambda^\star_{x_0}$ denotes the optimal initialization of the costate for the given x_0 and $\delta\lambda$ is a (small) perturbation. To perform a fair comparison, the value of ξ employed to forward propagate the dynamics (10) is instead selected according to (17) with respect to $\lambda(0)$, namely, by solving $\psi(x_0, \lambda^\star_{x_0} + \delta\lambda) = \psi(\xi, \nabla m(\xi))$. Note that the structure of the terminal cost in (29) implies that $\nabla m(\xi) = [\xi_1,\ 0]^\top$. In the following numerical simulations, the perturbation from the nominal value is selected as $\delta\lambda = [0.02,\ -0.01]^\top$. The graphs of Fig. 1 depict the time histories of the Hamiltonian dynamics described by (31) initialized at $\lambda(0) = \lambda^\star_{x_0} + \delta\lambda$ (dashed lines) and of the reduced dynamics (10) with ξ as above (solid lines) for several terminal times, i.e., $t_f = 2$ (top graph), $t_f = 4$ (middle graph), and $t_f = 6$ (bottom graph). The latter graph, in particular, illustrates the fact that forward integration of the Hamiltonian dynamics (6) yields numerically troublesome results for large t_f ($\|\lambda(6) - \nabla_x m(x(6))\| = 15.83$), whereas the forward integration of (10) may still provide meaningful values ($\|z(6) - \xi\| = 0.42$). This feature is employed in Section IV-B to define a hybrid mechanism that converges to the fixed point $\mu(x_0)$. △

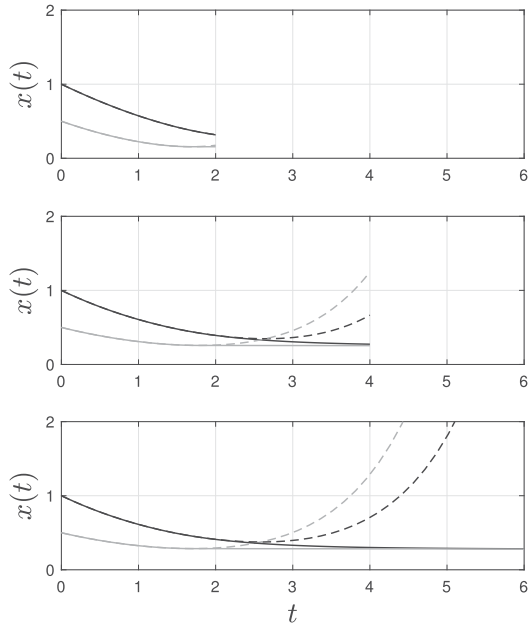

Fig. 1. Time histories of the Hamiltonian dynamics (31) initialized at $\lambda(0) = \lambda^\star_{x_0} + \delta\lambda$ (dashed lines) and of the reduced dynamics (10) with ξ as above (solid lines) for several terminal times, i.e., $t_f = 2$ (top graph), $t_f = 4$ (middle graph), and $t_f = 6$ (bottom graph).

IV. Discussion on Constructive Aspects

While providing an alternative characterization of the optimal solution in terms of a reduced system, the statement of Theorem 1 involves constructive steps that may appear as stumbling blocks in practice. The two daunting requirements

are the computation of the annihilator of the Hamiltonian vector field, which is guaranteed to exist by Frobenius theorem, and of the constant vector ξ that satisfies the condition (9). This latter step in fact must be typically accomplished without the knowledge of a closed-form expression of the flow φ_z. These aspects are addressed in the two following sections, respectively. It is worth mentioning that the purpose of the results in this section is to suggest possible approaches to tackle such constructive challenges. Nonetheless, further refinements or alternative approaches, some of which are hinted to in Section VI, might be envisioned.

A. Approximate Annihilator via Newton's Method

The constructions discussed in this section yield a systematic way of computing a set of functions that approximate, with an *arbitrary degree of accuracy* in a neighborhood of a given state, the generating functions of the annihilator of the Hamiltonian vector field $\mathbf{f}_{\mathcal{H}}$ without the need for solving any differential or algebraic equation. To this end, fix $(x_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}^n$, different from the origin, and suppose, without loss of generality (as explained in the footnote below), that $\mathbf{f}_{\mathcal{H}}^{2n}(x_0, \lambda_0) \neq 0$, where $\mathbf{f}_{\mathcal{H}}^{i}$ denotes the ith component of $\mathbf{f}_{\mathcal{H}}$.

Assumption 4: The components $\mathbf{f}_{\mathcal{H}}^{i}$, $i = 1, \ldots, 2n$, are real analytic functions in a neighborhood of (x_0, λ_0). ○

Assumption 4, although required to prove the following statement, may be relaxed in practice by accepting also sufficiently smooth vector fields. To provide a concise statement of the following result, define the matrix[4] $M \in \mathbb{R}^{2n \times 2n}$ as:

$$M = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{32}$$

Lemma 1: Fix $\chi_0 = (x_0, \lambda_0)$ and suppose that Assumption 4 holds. Define $s \in \mathbb{R}^{2n}$ and

$$h_{r_1}(s_1, \ldots, s_{2n}) = \sum_{k=1}^{r_1} L_{\mathbf{f}_{\mathcal{H}}}^{k} \chi(\chi_0 + Ms) \frac{s_1^k}{k!}. \tag{33}$$

Let $\pi_0(\chi) = \chi_0$, consider the iterations

$$\pi_{j+1} = \pi_j + \nabla h_{r_1}(\pi_j)^{-1} (\chi - h_{r_1}(\pi_j)) \tag{34}$$

$j = 0, 1, \ldots, r_2 - 1$, and define the functions $\tilde{\psi}_i(x, \lambda)$, $i = 1, \ldots, 2n-1$, as

$$\begin{bmatrix} \tilde{\psi}_1 & \ldots & \tilde{\psi}_{2n-1} \end{bmatrix}^{\top} := \begin{bmatrix} 0_{(2n-1)\times 1} & I_{2n-1} \end{bmatrix} \pi_{r_2}(x, \lambda). \tag{35}$$

Then, for any $\varepsilon > 0$ there exist $r_1^\star \in \mathbb{N}$, $r_2^\star \in \mathbb{N}$, and a nonempty neighborhood of χ_0 such that, for all $i = 1, \ldots, 2n-1$

$$\left\| d\tilde{\psi}_i \, \mathbf{f}_{\mathcal{H}}(\chi) \right\| < \varepsilon$$

for all $r_1 > r_1^\star$ and $r_2 > r_2^\star$. ◇

Remark 7: To streamline the proof of Lemma 1, it is worth preliminary recalling the arguments of the (constructive) proof of Frobenius Theorem in [11, Th. 1.4.1], on which the former relies. The latter in particular shows that the annihilator of a distribution $\Delta(x) = \operatorname{span}\{f_1(x), \ldots, f_\varrho(x)\} \subset \mathbb{R}^n$, of dimension ϱ around a certain state $\bar{x} \in \mathbb{R}^n$, can be determined by first completing the distribution with auxiliary vector fields such that $\operatorname{rank}[\Delta(\bar{x}) \; f_{n-\varrho}(\bar{x}) \; \ldots \; f_n(\bar{x})] = n$. Then, one should compute the *flows* $\varphi_{f_i}(s_i, \bar{x})$ of all the vector fields in Δ as well as those of $f_{n-\varrho}$,..., f_n. Then, defining the function $\Psi(s) = \varphi_{f_1}(s_1, \cdot) \circ \ldots \circ \varphi_{f_n}(s_n, \bar{x})$ as the composition of such flows, it is shown in [11, Th. 1.4.1] that the gradients of the last $n - \varrho$ functions in the inverse mapping $x \mapsto \Psi^{-1}(x)$ span the annihilating codistribution $\Delta^{\perp}$. ▲

Proof of Lemma 1: The proof of the claim follows from the premises discussed in Remark 7. To begin with the 1-dimensional distribution defined by the vector field $\mathbf{f}_{\mathcal{H}}$ is complemented by the trivial selection of the vector fields

$$f_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad f_{2n} = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The flows of the above vector fields can be then immediately computed and are such that $\varphi_{f_2}(s_2, \cdot) \circ \ldots \circ \varphi_{f_{2n}}(s_{2n}, \chi_0) = \chi_0 + Ms$. Then, the proof is concluded by relying, first, on the approximation of the flow $\varphi_{\mathbf{f}_{\mathcal{H}}}(s_1, \chi_0 + Ms)$, needed to construct the mapping $\Psi(s)$ introduced in Remark 7 and which is an analytic function of time s_1 by Assumption 4, via the corresponding Taylor expansion of order r_1, see (33) for the definition of h_{r_1}, and subsequently on the approximation of the inverse mapping via Newton's iterations, see (34). □

Remark 8: As a consequence of the constructions discussed in the statement of Lemma 1, it follows that a change of coordinates based on $\tilde{\psi}_i$ in (35) in place of ψ_i (namely, defining $\tilde{p} = \tilde{\psi}(x, \lambda)$) is such that the dynamics of the transformed costate variable can be (locally) uniformly bounded as $\|\dot{\tilde{p}}(t)\| < \tilde{\varepsilon}$, for any $\tilde{\varepsilon} > 0$ and all $t \in [t_0, t_f]$. A similar bound, which may be arbitrarily tuned, is then inherited also by the computation of the fixed-point condition (9), and hence on the optimal solution. The approximation stems from considering that also $\tilde{p}(t)$ is constant in $[t_0, t_f]$ while tackling (9). ▲

Example 3: Consider Hamiltonian dynamics described by

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} x^2 - \lambda \\ -x - 2x\lambda \end{bmatrix} \tag{36}$$

and fix $(x_0, \lambda_0) = (1, 0)$. Fig. 2 depicts $d\tilde{\psi}\, \mathbf{f}_{\mathcal{H}}$ obtained by approximating the flow of the Hamiltonian dynamics via (33) with $r_1 = 4$ and for two different values of r_2, namely, $r_2 = 1$ (light gray) and $r_2 = 3$ (dark gray). The quadratic convergence properties of the algorithm, inherited by Newton's method, can be appreciated from Fig. 2, in which the dark gray surface is almost *flat* in the desired neighborhood. △

[4]Whenever the last component of the vector field $\mathbf{f}_{\mathcal{H}}$ is equal to zero at (x_0, λ_0), the structure of the matrix M should be modified accordingly with the row of zeros corresponding to a nonzero element of $\mathbf{f}_{\mathcal{H}}$.

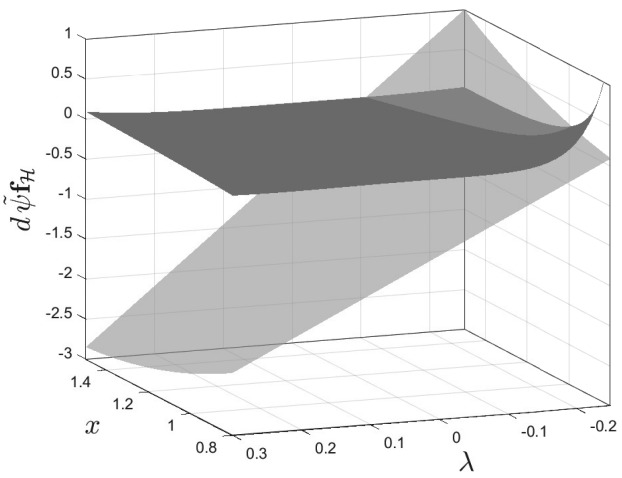

Fig. 2. Graph of $d\tilde{\psi}\,\mathbf{f}_{\mathcal{H}}$ obtained by approximating the flow of the Hamiltonian dynamics via (33) with $r_1 = 4$ and for two different values of r_2, namely, $r_2 = 1$ (light gray) and $r_2 = 3$ (dark gray).

Algorithm 1. Optimal control via first integrals

Parameters: $r_i \in \mathbb{N}$, $i = 1, 2$, $\varepsilon > 0$
Input: $(x_0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}^n$
(1) Let $\pi_0 = (x_0, \lambda_0)$
(2) **For** $j = 0$ **to** r_2
(3) $\quad \pi_{j+1} = \pi_j + \nabla h_{r_1}(\pi_j)^{-1}\,(\chi - h_{r_1}(\pi_j))$
(4) $\quad$ with h_{r_1} defined in (33)
(5) Let $[\tilde{\psi}_1 \dots \tilde{\psi}_{2n-1}]^\top = [0_{(2n-1)\times 1}\ I_{2n-1}]\pi_{r_2}$
(6) **If** $\exists\{i_1, \dots, i_n\} \subset \{1, \dots, 2n-1\}$ such that
(7) $\quad \operatorname{rank}(\nabla_\lambda [\tilde{\psi}_{i_1} \dots \tilde{\psi}_{i_n}]|_{(x_0,\lambda_0)}) = n$
(8) $\quad$ set $\tilde{\psi} = [\tilde{\psi}_{i_1} \dots \tilde{\psi}_{i_n}]^\top$ and **Go to** step (10)
(9) **Else** STOP
(10) Compute $\tilde{\psi}^{-1}(x, p)$
(11) Let (38)–(39) evolve with $\tilde{\psi}^{-1}$ in F until
(12) $\quad \mathcal{T}_{x_0}(\zeta(kt_f, k-1)) =: \mathcal{T}_{x_0}(\zeta_{[k]}) < \varepsilon$
(13) Set $u(t)$ for $t \in [t_0, t_f]$ as
(14) $\quad u(t) = -R^{-1} g(x(t))^\top \tilde{\psi}^{-1}(x(t), \tilde{\psi}(\zeta_{[k]}, \nabla_x m(\zeta_{[k]})))$

B. Fixed-Point Condition via Sensitivity Equations

The objective of this section is to propose a computational approach for determining the value of ξ that satisfies the fixed-point condition (9). More precisely, this is achieved by combining a rather standard approach for computing the partial derivative of the flow of a differential equation with respect to a parameter appearing in the vector field (*sensitivity*) with the framework introduced for the analysis and control of hybrid systems. A comprehensive review about the latter class of systems is beyond the scope of this article (see [15] for detailed discussions). It appears that (9) can be satisfied by minimizing the (static) cost $\zeta \mapsto \mathcal{T}_{x_0}(\zeta)$, $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}$, defined as

$$\mathcal{T}_{x_0}(\zeta) = \|\zeta - \varphi_z\,(t_f, t_0, x_0; \zeta)\,\|^2. \tag{37}$$

The latter task can be, in turn, accomplished by a standard *gradient descent* method, provided one is able of (numerically) evaluating the flow and its derivative with respect to ζ, as discussed in the following result. To provide a concise statement, define $F(z, \zeta) := f(z) - S(z)\psi^{-1}(z, \psi(\zeta, \nabla_x m(\zeta))$ and recall (see [15]) that solutions to hybrid dynamical systems are parameterized with respect to two distinct time variables (t, k), capturing the elapsed continuous time and the number of occurred discrete-time events, respectively.

Proposition 2: Suppose that the hypotheses of Theorem 1 hold. Consider the hybrid system with state $(\tau, \zeta, z, \mathcal{S}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ described by the *flow dynamics*

$$\dot{\tau} = 1 \tag{38a}$$
$$\dot{\zeta} = 0 \tag{38b}$$
$$\dot{z} = F(z, \zeta) \tag{38c}$$
$$\dot{\mathcal{S}} = (\nabla_z F(z, \zeta))\ \mathcal{S} + \nabla_\zeta F(z, \zeta) \tag{38d}$$

jump dynamics

$$\tau^+ = 0 \tag{39a}$$
$$\zeta^+ = \zeta - \gamma(I - \mathcal{S})^\top(\zeta - z) \tag{39b}$$
$$z^+ = x_0 \tag{39c}$$
$$\mathcal{S}^+ = 0 \tag{39d}$$

and with *flow* and *jump sets* described by $C := \{(\tau, \zeta, z, \mathcal{S}) : \tau \leqslant t_f\}$ and $D := \{(\tau, \zeta, z, \mathcal{S}) : \tau \geqslant t_f\}$, respectively. Then, there exist a nonempty set $\mathcal{U}$ and a piecewise constant function $\gamma(t, k)$ such that $\lim_{t+k\to\infty} \|\zeta(t, k) - \mu(x_0)\| = 0$, along all the trajectories of (38) and (39) for all $(\tau(0,0), \zeta(0,0), z(0,0), \mathcal{S}(0,0)) \in \mathcal{U}$. ◇

Proof: The claims of Proposition 2 are obtained as a consequence of a gradient-descent method with variable step size. In fact, note that the dynamics (38d) is such that $\mathcal{S}(t, k) = \nabla_\zeta \varphi_z(t, t_0, x_0; \zeta)$ for any t. Therefore, the right-hand side of the jump dynamics (39b) coincides with a standard gradient-based update $\zeta^+ = \zeta - \gamma(\nabla_\zeta \mathcal{T}_{x_0})^\top$. □

The above strategy represents essentially a *shooting method* with two main differences with respect to classic implementations: first, it requires to "shoot" for the selection of the parameter ξ in (10) to satisfy (9); second, the iterations are formulated within the framework of hybrid systems. The practical implementation of the strategy suggested in Proposition 2 is briefly summarized by the schematic algorithm in the following and subsequently illustrated by the case study in Example 4.

Algorithm 1 provides a systematic strategy to translate the abstract properties discussed in Section III into a design strategy in practice. Nonetheless, it may be possible to replace a few steps therein with alternative approaches. More precisely, the role of steps (1)–(5) could be played by any method that computes first integrals of the Hamiltonian dynamics (see, e.g., the use of algebraic geometry arguments in Example 5), whereas the objective of the steps (11) and (12) could be equivalently achieved by any shooting method on the reduced dynamics (10) with respect to the parameter ξ.

Example 4: Consider a nonlinear system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_2) \\ u \end{bmatrix} \tag{40}$$

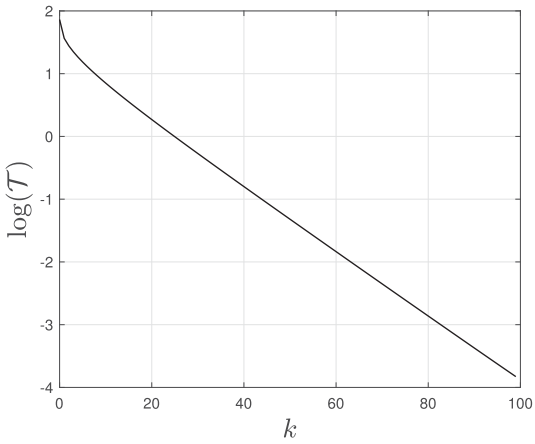

Fig. 3. Time history of the logarithm of the cost function $\mathcal{T}_{x_0}(\zeta)$ along the trajectories of the system (38) and (39) initialized at $\zeta(0,0)=[1,1]^\top$, $z(0,0)=x_0$, and $\mathcal{S}(0,0)=0_{2\times 2}$.

initialized at $x(0)=x_0$, together with the cost functional

$$J(u(\cdot))=\frac{1}{2}\int_0^1 \|u(t)\|^2 dt+\frac{1}{2}\|x(1)\|^2 \tag{41}$$

hence as in (2) with $\ell(x)\equiv 0$, $R=1$, and $m(x)=(1/2)\|x\|^2$. The corresponding Hamiltonian dynamics are described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{\lambda}_1\\ \dot{\lambda}_2\end{bmatrix}=\begin{bmatrix}f_1(x_2)\\ -\lambda_2\\ 0\\ -\frac{df_1}{dx_2}(x_2)\lambda_1\end{bmatrix} \tag{42}$$

with the boundary conditions $x(0)=x_0$ and $\lambda(1)=x(1)$. It can be shown that the functions

$$\psi_1(x,\lambda)=\lambda_1,\quad \psi_2(x,\lambda)=f_1(x_2)\lambda_1-\frac{\lambda_2^2}{2} \tag{43}$$

constitute a pair of independent functions the gradients of which belong to the annihilating codistribution of the vector field in (42). Fig. 3 depicts the time history of the cost function $\mathcal{T}_{x_0}(\zeta)$, in semi-logarithmic scale, along the trajectories of the system (38) and (39) with $f_1(x_2)=x_2^3$ initialized at $\zeta(0,0)=[1,1]^\top$, $z(0,0)=x_0=[5,\ 3]^\top$, and $\mathcal{S}(0,0)=0_{2\times 2}$. The value of ξ that satisfies (9) is, therefore, obtained as $\xi^\star=[6.6894, 0.1918]^\top$ and the optimal costate is

$$\begin{bmatrix}\lambda_1^\star(t)\\ \lambda_2^\star(t)\end{bmatrix}=\begin{bmatrix}\xi_1^\star\\ \sqrt{2((z_2^\star(t))^3\xi_1^\star-(\xi_2^\star)^3\xi_1^\star-(1/2)(\xi_2^\star)^2)}\end{bmatrix}.$$

△

C. Numerical Simulations

The objective of this section is to corroborate the theoretical findings of the previous sections by means of two numerical simulations. In both cases, the claims of Theorem 1 are compared with the optimal solution of the underlying problem, numerically computed via the command *bvp* in MATLAB, which permits the solution to the corresponding nonlinear BVP. In Example 5, the Hamiltonian dynamics admit a closed-form expression of the annihilator, whereas the subsequent example is employed to discuss the use of approximate first integrals whenever these cannot be determined.

Example 5: Consider a nonlinear system described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-x_1-2x_3-u\\ x_1^2+x_2+2x_1x_3+x_3^2\\ x_3+u\end{bmatrix} \tag{44}$$

initialized at $x(0)=x_0$, together with the cost functional

$$J(u(\cdot))=\frac{1}{2}\int_0^1 \|u(t)\|^2 dt+\frac{1}{2}\|x(1)\|^2. \tag{45}$$

The corresponding Hamiltonian dynamics are described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix}=\begin{bmatrix}-x_1-2x_3-\lambda_1+\lambda_3\\ x_1^2+x_2+2x_1x_3+x_3^2\\ x_3+\lambda_1-\lambda_3\\ \lambda_1-2x_1\lambda_2-2x_3\lambda_2\\ -\lambda_2\\ 2\lambda_1-\lambda_3-2x_1\lambda_2-2x_3\lambda_2\end{bmatrix} \tag{46}$$

with the boundary conditions $x(0)=x_0$ and $\lambda(1)=x(1)$. By relying on the notion of *semi-invariant* of a nonlinear system and by borrowing techniques from algebraic geometry (see [16] for more details), it can be shown that (46) admits

$$\psi(x,\lambda)=\begin{bmatrix}(\lambda_1-\lambda_3)\left(x_1^2+2x_1x_3+x_3^2+3x_2\right)\\ \lambda_2\left(x_1^2+2x_1x_3+x_3^2+3x_2\right)\\ \dfrac{3\lambda_1-2\lambda_2\left(x_1+x_3\right)}{x_1^2+2x_1x_3+x_3^2+3x_2}\end{bmatrix} \tag{47}$$

as a collection of $n=3$ generating functions for the annihilating codistribution, yielding the inverse mapping $\psi^{-1}(x,p)$ in (48) (shown at the bottom of this page). Fix $x_0=[1,-1,-2]^\top$. Thus, by replacing the closed-form expressions of ψ and ψ^{-1}, defined in (47) and (48), respectively, into (10), the fixed point $\xi^\star$ is computed via (38) and (39), yielding $\xi^\star=[0.5269,-1.8573,-0.8948]^\top$. Fig. 4 shows the time histories of the reduced system (10) (solid lines) together with the components of $\xi^\star$ (dashed lines). By letting $z^\star$ denote the solution to (10) with $\xi=\xi^\star$, it is verified that the optimal costate variable $\lambda^\star$, numerically computed via *bvp*, indeed, satisfies

$$\psi^{-1}(x,p)=\begin{bmatrix}p_3\left(\frac{x_1^2}{3}+\frac{2}{3}x_1x_3+\frac{x_3^2}{3}+x_2\right)+\dfrac{2p_2\left(x_1+x_3\right)}{3\left(x_1^2+2x_1x_3+x_3^2+3x_2\right)}\\ \dfrac{p_2}{x_1^2+2x_1x_3+x_3^2+3x_2}\\ p_3\left(\frac{x_1^2}{3}+\frac{2}{3}x_1x_3+\frac{x_3^2}{3}+x_2\right)+\dfrac{-3p_1+2p_2\left(x_1+x_3\right)}{3\left(x_1^2+2x_1x_3+x_3^2+3x_2\right)}\end{bmatrix} \tag{48}$$

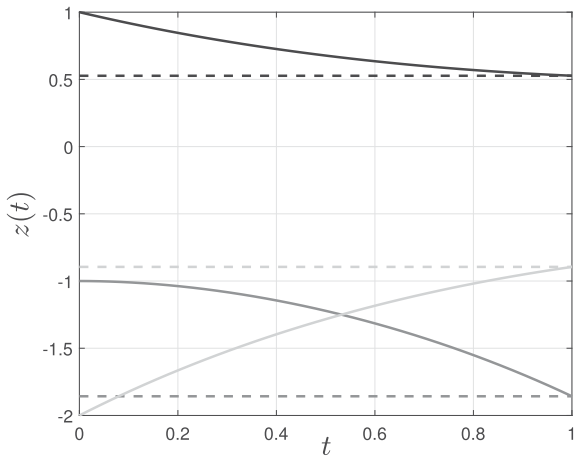

Fig. 4. Time histories of the state of (10) in the case of Example 5, together with the corresponding value of $\xi^\star$ (dashed lines).

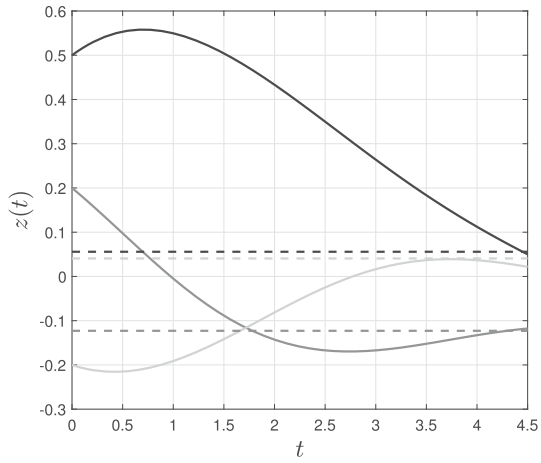

Fig. 5. Time histories of the state of (52) (solid lines), with ξ selected as $x^\star(t_f)$, together with the terminal values of the optimal state $x^\star(t_f)$ (dashed lines).

$\lambda^\star(t) = \psi^{-1}(z^\star(t), p)|_{p=\psi(\xi^\star,\xi^\star)}$ (since $\nabla_x m(x) = x$) for all $t \in [0, 1]$, with ψ in (47) and ψ^{-1} in (48). △

Example 6: Consider a nonlinear system in *strict feedback form* described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 + \varpi_1(x_1) \\ x_2 + \varpi_2(x_1, x_2) \\ u \end{bmatrix} =: f(x) + bu \tag{49}$$

$b = [0,\ 0,\ 1]^\top$, with $\varpi_1(x_1) = -x_1^5$ and $\varpi_2(x_1, x_2) = x_1^2 x_2^2$, together with the cost functional

$$J(u(\cdot)) = \frac{1}{2}\int_0^{t_f} \|u(t)\|^2 dt + \frac{1}{2}\|x(t_f)\|^2, \tag{50}$$

i.e., as in (45) although with $t_f = 4.5$. Letting $x_0 = [0.5,\ 0.2,\ -0.2]^\top$, the (numerically computed) optimal process $(x^\star, \lambda^\star)$ corresponds to the trajectory of the Hamiltonian dynamics, naturally associated with (49) and (50), initialized at $\lambda^\star(0) = [0.0230,\ 0.0866,\ 0.0132]^\top$. Since the generating functions of the annihilator of the Hamiltonian dynamics cannot be easily computed, these are instead approximated by postulating the structure of such functions according to $\psi_i^a(x, \lambda) = a_1^i x_1 + a_2^i x_2 + a_3^i x_3 + a_4^i \lambda_i + a_5^i x_1^2 + a_6^i x_2^2 + a_7^i x_3^2$, for $i = 1, 2, 3$. Given a prescribed point $(x_0, \lambda_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ and by relying on ideas similar to those underlying the expansion (33), the coefficients a_j^i, $i = 1, \ldots, 3$, $j = 1, \ldots, 7$, are determined with the property that $\psi_i^a(x_0, \lambda_0) \neq 0$ while

$$\frac{d^k}{dt^k}\psi_i^a(x(t), \lambda(t))\Big|_{t=0} = 0$$

$k = 1, \ldots, 6$, namely, the first six time derivatives with respect to t of the composite function $\psi_i^a(x(t), \lambda(t))$ are zeroed at (x_0, λ_0). This ensures that the function $\psi_i^a(x, \lambda)$ remains sufficiently small in a neighborhood of (x_0, λ_0) and in the direction of the flow of the underlying Hamiltonian dynamics. The above strategy leads to the mapping ψ^a in (51) shown at the bottom of this page, obtained by selecting (x_0, λ_0) with $\lambda_0 = [0.05,\ 0.09,\ -0.04]^\top$, which approximates $\lambda^\star(0)$. The effect of such an approximation is then assessed according to the following strategy. The mapping ψ^a in (51) and its inverse $\psi^{a,-1}$ with respect to λ, which is immediately computed since ψ^a is linear in λ, are employed to construct (approximate) reduced dynamics (10), i.e.,

$$\dot{z} = f(z) - bb^\top \psi^{a,-1}(z, \psi^a(\xi, \xi)) \tag{52}$$

$z(0) = x_0$, with f and b defined in (49). Then, a measure of the approximation is provided by $e_a := \|z(t_f) - x^\star(t_f)\|^2$, where $z(t_f)$ denotes the solution of (52) with $\xi = x^\star(t_f)$ at $t = t_f = 4.5$, while $x^\star(t_f)$ denotes the terminal value of the (numerically computed) optimal state. In fact, with knowledge of the exact expression of ψ, one obtains $e_a = 0$. Fig. 5 depicts the time histories of the state of (52) (solid lines), with $\xi = x^\star(t_f)$, together with the terminal values of the optimal state $x^\star(t_f)$ (dashed lines). Furthermore, a sensitivity analysis is reported in Fig. 6 , which shows the values of e_a obtained by fixing $\lambda_{0,1} = 0.05$ and by letting $\lambda_{0,2}$ and $\lambda_{0,3}$ vary. △

V. Revisiting the Linear Quadratic Regulator (LQR) Problem

The results discussed in the previous sections are specialized here to the setting of linear dynamics and quadratic cost functionals, which constitute the LQR problem. Toward this end, consider a linear time-invariant (LTI) system described by the

$$\psi^a(x, \lambda) = \begin{bmatrix} -0.9048x_1 - 1.1998x_2 - 1.5559x_3 + 2.5755\lambda_1 + 0.6410x_1^2 + 1.2556x_2^2 + 3.5477x_3^2 \\ 1.3025x_1 + 2.2693x_2 + 3.4895x_3 + 2.3223\lambda_2 - 0.8738x_1^2 - 3.9525x_2^2 - 3.4913x_3^2 \\ 0.2668x_1 - 1.9733x_2 - 1.0869x_3 + 1.9935\lambda_3 + 0.1389x_1^2 + 2.4154x_2^2 + 2.3074x_3^2 \end{bmatrix} \tag{51}$$

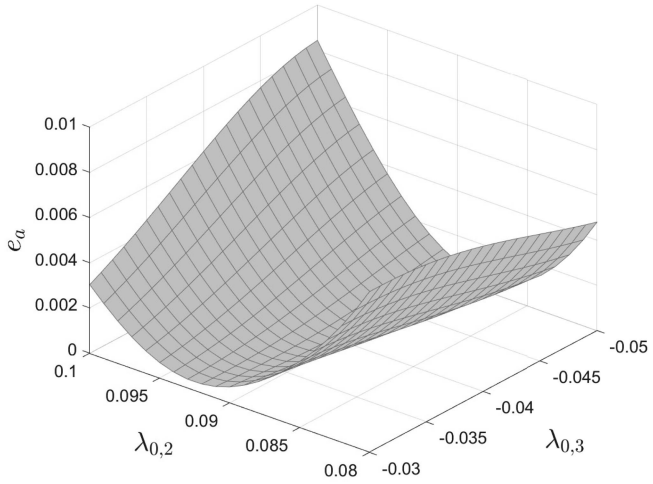

Fig. 6. Graph of e_a in Example 6 obtained by fixing $\lambda_{0,1} = 0.05$ and by letting $\lambda_{0,2}$ and $\lambda_{0,3}$ vary.

equations

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \tag{53}$$

together with the quadratic cost functional

$$J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} \left(\|x(t)\|_Q^2 + \|u(t)\|_R^2\right) dt + \frac{1}{2}\|x(t_f)\|_M^2 \tag{54}$$

with $Q = Q^\top \succeq 0$, $R = R^\top \succ 0$, and $M = M^\top \succeq 0$. The corresponding Hamiltonian dynamics (6) are linear and described by

$$\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix} A & -\bar{S} \\ -Q & -A^\top \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} := H \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{55}$$

with $\bar{S} := BR^{-1}B^\top \succeq 0$. In the rest of this section, the following rather standard structural assumption is considered.

Assumption 5: The pairs (A, B) and (A, C) are reachable and observable, respectively. ○

As a consequence of the requirements of Assumption 5, the Hamiltonian matrix H in (55) possesses a *split spectrum*, namely, having n eigenvalues with positive real part and n eigenvalues with negative real part. Therefore, the underlying linear system (55) is unstable. In the case of LTI systems, such as the Hamiltonian system (55), the *generating functions* for the corresponding annihilator are related to the eigenstructure of the matrix H. For illustrative purposes, suppose that the matrix H is diagonalizable. Then, the system (55) admits $2n-1$ generating function of the annihilator of the form

$$\psi_i(x, \lambda) = \frac{\left(v_i^\top \chi\right)^{a_1}}{\left(v_1^\top \chi\right)^{a_i}} \tag{56}$$

with $\chi = (x, \lambda)$, where the vector v_i and the scalar $a_i \in \mathbb{C}$ denote the ith eigenvector and eigenvalue, respectively, of the matrix $H^\top$, namely, *left eigenvectors* of the matrix H.

Remark 9: The nonlinear structure of (56) (i.e., rational functions of the state) is not surprising. In fact, recall that the *desired* consequence of determining such generating functions and performing the corresponding change of coordinates would be to zero certain components of the vector field. In the linear setting and limiting the search to linear change of coordinates (hence, preserving linearity in the transformed coordinates), the latter structure would correspond to obtaining zero eigenvalues. However, since time-invariant linear changes of coordinates preserve the eigenvalues of the original system, it follows that the above objective could not be achieved by such a transformation. Nonetheless, it is worth observing that instead, by relying on the property of linearity of the underlying dynamical systems, a result identical to Theorem 1 can be equivalently obtained by the use of a *time-varying linear* change of coordinates. The latter in fact may replace the (nonlinear) change of coordinates induced by the annihilating codistribution. ▲

To this end, let $\Pi_x = \begin{bmatrix} I & 0 \end{bmatrix} \in \mathbb{R}^{n\times 2n}$ and $\Pi_\lambda = \begin{bmatrix} 0 & I \end{bmatrix} \in \mathbb{R}^{n\times 2n}$ denote the projection matrices on the state and costate space, respectively.

Proposition 3: Consider the LTI system (53) together with the quadratic cost functional (54) and fix $x_0 \in \mathbb{R}^n$. Define

$$Y(t) = \Pi_\lambda e^{H(t_f - t)} =: \begin{bmatrix} Y_1(t) & Y_2(t) \end{bmatrix} \tag{57}$$

and suppose that $Y_2(t)$ is invertible in $[t_0, t_f]$. Let $X(t) \in \mathbb{R}^{n\times n}$ satisfy

$$\dot{X}(t) = AX(t) - X(t)\tilde{A}(t), \quad X(t_0) = I \tag{58}$$

with $\tilde{A}(t) = A + \bar{S}Y_2(t)^{-1}Y_1(t)$. Then, the *reduced BVP* (9) and (10) becomes $z(t_f) = \xi$ with

$$\dot{z}(t) = Az(t) - X(t)\bar{S}Y_2(t)^{-1}MX(t_f)^{-1}\xi \tag{59}$$

for all $t \in [t_0, t_f]$ and $z(t_0) = x_0$. ◇

Proof: Define the time-varying matrix $T : \mathbb{R} \to \mathbb{R}^{2n\times 2n}$ as

$$T(t) = \begin{bmatrix} X(t) & 0 \\ Y_1(t) & Y_2(t) \end{bmatrix}. \tag{60}$$

Consider then the change of coordinates described by

$$\begin{bmatrix} z \\ p \end{bmatrix} = T(t) \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

which is such that the transformed dynamics become

$$\begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \left(\dot{T}(t) + T(t)H\right) T(t)^{-1} \begin{bmatrix} z \\ p \end{bmatrix}.$$

Similarly to the nonlinear setting, the objective of the change of coordinates consists in transforming the state/costate system (55) into dynamics of the form of

$$\begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & K(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ p \end{bmatrix} =: \Lambda_{\mathrm{d}} \begin{bmatrix} z \\ p \end{bmatrix} \tag{61}$$

where $K : \mathbb{R} \to \mathbb{R}^{n\times n}$ must be determined, together with $T(t)$, such that $\dot{T}(t) + T(t)H = \Lambda_{\mathrm{d}} T(t)$. In fact, the structure of (61) implies that $\dot{p} = 0$. By expanding the latter (matrix) differential equation one obtains the conditions

$$\begin{bmatrix} \dot{X} & 0 \\ \dot{Y}_1 & \dot{Y}_2 \end{bmatrix} + \begin{bmatrix} XA & -X\bar{S} \\ Y_1 A - Y_2 Q & -Y_1\bar{S} - Y_2 A^\top \end{bmatrix}$$

$$= \begin{bmatrix} AX + KY_1 & KY_2 \\ 0 & 0 \end{bmatrix} \tag{62}$$

in which the dependence on time of the time-varying matrices has been removed for brevity. The second block-row equations are satisfied via the selection of $Y_i(t)$, for all $t \in [t_0, t_f]$ and $i = 1, 2$, as in (57). Since $Y(t_f) = \Pi_\lambda$, one has that additionally $p(t_f) = \lambda(t_f) = Mx(t_f) = MX(t_f)^{-1}z(t_f)$. The top-right block is instead dealt with by selecting $K(t) = -X(t)\bar{S}Y_2^{-1}(t)$, which in turn implies that $X(t)$ provided by the initial value problem (58) satisfies the top-left block of (62). Finally, by the structure of $\Lambda_{\rm d}$ achieved with the selection of X, Y_1, and Y_2 as above, it follows that the state p is constant over time in $[t_0, t_f]$ and equal to $p(t) = MX(t_f)^{-1}\xi$, where $\xi \in \mathbb{R}^n$ must verify a fixed-point condition $\xi = z(t_f)$ with $z(t_f)$ denoting the solution of

$$\begin{aligned}\dot{z}(t) &= Az(t) + K(t)p(t)X = Az(t) - X(t)\bar{S}Y_2^{-1}(t)p(t)\\ &= Az(t) - X(t)\bar{S}Y_2^{-1}(t)MX^{-1}(t_f)\xi \end{aligned} \tag{63}$$

at time $t = t_f$, hence concluding the proof. △

Remark 10: Since $Y(t_f) = \Pi_\lambda e^{H(t_f - t)}|_{t=t_f} = [0,\ I]$ and observing that $Y_2(t)$ is a continuous function of time, it follows that the invertibility condition on $Y_2(t)$ holds for all t in the interval $[t_0, t_f]$ provided t_f is sufficiently small. ▲

The equations are significantly simplified whenever the running cost does not impose any penalty on the state, which is captured instead only by the terminal cost, i.e., when $Q = 0$ in (54). The above intuition is discussed in the following result.

Proposition 4: Consider the LTI system (53) together with the quadratic cost functional (54), with $Q = 0$, and fix $x_0 \in \mathbb{R}^n$. Let $\mathcal{G} : \mathbb{R} \to \mathbb{R}^{n\times n}$ denote the controllability Gramian matrix of $(A, R^{-1/2}B)$. Then, the fixed-point condition described by (59) and $z(t_f) = \xi$ is satisfied by

$$\xi = \mu(x_0) := (I + \mathcal{G}(t_f)M)^{-1} e^{At_f} x_0 \tag{64}$$

for all $x_0 \in \mathbb{R}^n$. ◇

Proof: To begin with, in this case, by inspecting the dynamics governing the evolution of $Y_1(t)$ and $Y_2(t)$, it can be noted that they are satisfied by $Y_1 \equiv 0$ and $Y_2(t) = e^{-A^\top(t_f - t)}$. Since Y_1 is identically equal to zero, the equation for $X(t)$ becomes time invariant and described by $\dot{X} = AX - XA$. The latter is then solved by $X(t) = e^{At}X_0e^{-At} = e^{At}e^{-At} \equiv I$, where the boundary condition $X(t_0) = I$ has been used. Finally, the matrix-valued function K reduces instead to $K(t) = -\bar{S}e^{A^\top(t_f - t)}$. Therefore, the transformed optimal costate is equal to $p(t) = Mz(t_f) = Mx(t_f)$ and the dynamics (59) becomes $\dot{z} = Az - \bar{S}e^{A^\top(t_f - t)}M\xi$, $z(0) = x_0$. The latter requires that

$$\begin{aligned}\xi &= e^{At_f}x_0 - \int_{t_0}^{t_f} e^{A(t_f - \tau)}\bar{S}e^{A^\top(t_f - \tau)}d\tau M\xi\\ &= e^{At_f}x_0 - \mathcal{G}(t_f)M\xi \end{aligned} \tag{65}$$

where $\mathcal{G}(t_f)$ denotes the Gramian matrix evaluated at $t = t_f$, which is positive definite by Assumption 5. Therefore, the fixed-point condition can be immediately satisfied in closed form by letting $\xi = \mu(x_0)$ defined in (64). □

Remark 11: The result of Proposition 4 in essence recovers the intuition behind the elegant constructions of [8] in the setting of a class of LQ differential games (*pursuit/evasion*). Therein, a time-varying change of coordinates is proposed to *remove* the dependence of the underlying Hamiltonian function on the state (which instead appears in the boundary condition) with the *byproduct* of inducing a *constant* optimal costate. The results of Section III may be then also interpreted as the extension of ideas similar to those proposed in [8] to the nonlinear setting via the notion of annihilating codistribution of the Hamiltonian vector field. ▲

The section is then concluded by motivating the *need* for the use of the annihilating codistribution, rather than a time-varying change of coordinates, in the setting of nonlinear systems. Toward this end, the following statement first provides an *equivalent* interpretation of the results of Proposition 4 and of [8], although in the setting of optimal control rather than differential game theory.

Proposition 5: Consider the LTI system (53) together with the quadratic cost functional

$$J_r(u(\cdot)) = \frac{1}{2}\int_{t_0}^{t_f} \|u(t)\|_R^2 dt + \frac{1}{2}\|x(t_f)\|_M^2. \tag{66}$$

Then, the optimal control problem defined by (53) and (66) is equivalently described by the *time-varying, state-independent*, Hamiltonian function

$$\tilde{H}(\lambda, u, t) = \lambda^\top \tilde{B}(t)u + \frac{1}{2}u^\top Ru \tag{67}$$

$\tilde{B}(t) := \tilde{M}e^{A(t_f - t)}B$, $\tilde{M}^\top\tilde{M} = M$, the optimal costate of which satisfies $\dot{\lambda} = -\nabla_z\tilde{H} = 0$ for $t \in [t_0, t_f]$. ◇

Proof: Define the transformed variable $z_L(t) = \tilde{M}\Phi_A(t_f, t)x(t) = \tilde{M}e^{A(t_f - t)}x(t)$, where $\tilde{M}$ is such that $\tilde{M}^\top\tilde{M} = M$. Clearly, the cost (66) becomes $J_r(u(\cdot)) = (1/2)\int_{t_0}^{t_f} \|u(t)\|_R^2 + (1/2)\|z_L(t_f)\|^2$. The dynamics is instead described by

$$\begin{aligned}\dot{z}_L(t) &= \tilde{M}\nabla_t\Phi_A(t_f, t)x(t) + \tilde{M}\Phi_A(t_f, t)\dot{x}(t)\\ &= -\tilde{M}\Phi_A(t_f, t)Ax(t) + \tilde{M}\Phi_A(t_f, t)(Ax(t) + Bu(t))\\ &= \tilde{M}\Phi_A(t_f, t)u(t) = \tilde{B}(t)u(t)\end{aligned}$$

from which the structure of $\tilde{H}$ in (67) follows immediately. □

The objective achieved by the auxiliary dynamics $\dot{z}_L$ illustrates the obstruction that prevents the straightforward extension of the ideas in [8] to the nonlinear setting of (1), without resorting to the notion of annihilating codistribution. To this end, consider the latter system and suppose that a mapping $\Lambda : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is sought for to achieve a structure similar to (67). By letting $z_N(t) = \Lambda(t, x(t))$, it follows that

$$\begin{aligned}\dot{z}_N(t) &= \nabla_t\Lambda(t, x(t)) + \nabla_x\Lambda(t, x(t))\dot{x}(t)\\ &= \nabla_t\Lambda(t, x(t)) + \nabla_x\Lambda(t, x(t))(f(x(t)) + g(x(t))u(t))\\ &= \nabla_x\Lambda(t, x(t))g(x(t))u(t) =: \tilde{g}(t, x(t))u(t)\end{aligned}$$

provided $\nabla_t\Lambda(t, x) + \nabla_x\Lambda(t, x)f(x) = 0$ for all $(t, x) \in [t_0, t_f] \times \mathbb{R}^n$, which is indeed described by time-varying dynamics although still function of the original state x. The latter

dependence is inherited by the resulting transformed Hamiltonian, thus preventing the straightforward extension of [8] to the nonlinear setting.

VI. Conclusion

Within the framework of finite-horizon optimal control problems, it has been shown that the evolution of the costate variable coincides, for any time, with the inverse mapping of any collection of n-independent first integrals of the underlying Hamiltonian vector field. This is achieved by interpreting the inverse mapping as a mapping of the original state and constant vector, whose computation is formulated in terms of a BVP, in place of the costate variable. Interestingly, the BVP is defined in the original (state) coordinates, rather than in the (extended) state/costate space as for the classic Hamiltonian dynamics. Such abstract property is subsequently employed to propose a systematic strategy to compute optimal control laws, which is based upon premises significantly different from existing methods. Nonetheless, since the constructions envisioned in Section IV-A may become troublesome for higher dimensional systems, further extensions and refinements are needed. It may be possible, for instance, to entirely circumvent the closed-form constructions of Section IV-A by relying on functional approximators (such as, e.g., neural networks) parameterized in such a way that the required (partial) inversion remains feasible.

Since the resulting conditions appear to be particularly appealing from the computational point of view (considering, for instance, the reduced dimension and the stability properties of the BVP), it would be of interest to further extend similar ideas to the context of differential games, in which, indeed, the notion of open-loop Nash equilibrium is intimately related to the evolution of certain state/costate dynamics for each player. Moreover, a deeper understanding of the connections between first integrals and optimal costate for large terminal times t_f (ideally in the limit for t_f that tends to infinity) would widen the range of applicability of the proposed characterization. For similar purposes, it would be interesting to relax the regularity properties of the involved functions and to generalize the structure of the considered optimal control task, encompassing for instance the case of constraints on the input.

References

[1] D. Liberzon, *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton, NJ, USA: Princeton Univ. Press, 2011.

[2] D. A. Carlson, A. B. Haurie, and A. Leizarowitz, *Infinite Horizon Optimal Control: Deterministic and Stochastic Systems*. Berlin, Germany: Springer, 2012.

[3] R. Bellman, *Dynamic Programming*. Princeton, NJ, USA: Princeton Univ. Press, 1957.

[4] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*. Hoboken, NJ, USA: Wiley, 1962.

[5] J. T. Betts, "Survey of numerical methods for trajectory optimization," *J. Guid., Control, Dyn.*, vol. 21, no. 2, pp. 193–207, 1998.

[6] G. Elnagar, M. A. Kazemi, and M. Razzaghi, "The pseudospectral Legendre method for discretizing optimal control problems," *IEEE Trans. Autom. Control*, vol. 40, no. 10, pp. 1793–1796, Oct. 1995.

[7] D. Garg, W. W. Hager, and A. V. Rao, "Pseudospectral methods for solving infinite-horizon optimal control problems," *Automatica*, vol. 47, no. 4, pp. 829–837, 2011.

[8] Y. Ho, A. Bryson, and S. Baron, "Differential games and optimal pursuit-evasion strategies," *IEEE Trans. Autom. Control*, vol. AC-10, no. 4, pp. 385–389, Oct. 1965.

[9] R. Vinter, *Optimal Control*. Berlin, Germany: Springer, 2010.

[10] A. E. Bryson and Y. C. Ho, *Applied Optimal Control: Optimization, Estimation and Control*. Oxfordshire, U.K.: Taylor and Francis, 1975.

[11] A. Isidori, *Nonlinear Control Systems*. 3rd ed. Berlin, Germany: Springer-Verlag, 1995.

[12] G. J. Pappas and S. Sastry, "Straightening out rectangular differential inclusions," *Syst. Control Lett.*, vol. 35, no. 2, pp. 79–85, 1998.

[13] N. Sakamoto and A. J. van der Schaft, "Analytical approximation methods for the stabilizing solution of the Hamilton–Jacobi equation," *IEEE Trans. Autom. Control*, vol. 53, no. 10, pp. 2335–2350, Nov. 2008.

[14] D. P. Bertsekas, *Dynamic Programming and Optimal Control*. Belmont, MA, USA: Athena Sci., 2005.

[15] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems*. Princeton, NJ, USA: Princeton Univ. Press, 2012.

[16] L. Menini and A. Tornambè, *Symmetries and Semi-Invariants in the Analysis of Nonlinear Systems*. Berlin, Germany: Springer, 2011.

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