Frequency-domain stability methods for reset control systems☆

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ABSTRACT

Reset controllers have the potential to enhance the performance of high-precision industrial motion systems. However, similar to other non-linear controllers, the stability analysis for these controllers is complex and often requires parametric model of the system, which may hinder their applicability. In this paper a frequency-domain approach for assessing stability properties of control systems with first and second order reset elements is developed. The proposed approach is also able to determine uniformly bounded-input bounded-state (UBIBS) property for reset control systems in the case of resetting to non-zero values. An illustrative example to demonstrate the effectiveness of the proposed approach in using frequency response measurements to assess stability properties of reset control systems is presented.

1. Introduction

High-tech precision industrial applications have control requirements which are hard to fulfill by means of linear controllers. One way to increase the performance of these systems is to replace linear controllers with non-linear ones, for instance reset controllers. Owing to their simple structure, these controllers have attracted significant attention from academia and industry (Aangenent, Witvoet, Heemels, van de Molengraft, & Steinbuch, 2010; Baños & Barreiro, 2011; Beker, Hollot, Chait, & Han, 2004; Bisoffi et al., 2020; Clegg, 1958; Dastjerdi et al., 2022; Forni, Nešić, & Zaccarian, 2011; Guo, Xie, & Wang, 2015; HosseinNia, Tejado, & Vinagre, 2013; van Loon, Gruntjens, Heertjes, van de Wouw, & Heemels, 2017; Villaverde, Blas, Carrasco, & Torrico, 2011). In particular, reset controllers have been utilized to improve the performance of several mechatronic systems (see, e.g. Beerens et al. (2019), Chen, Saikumar, and HosseinNia (2020), Guo, Wang, and Xie (2009), Hazeleger, Heertjes, and Nijmeijer (2016), Heertjes, Gruntjens, van Loon, Kontaras, and Heemels (2015), Horowitz and Rosenbaum (1975), Saikumar, Sinha, and HosseinNia (2019), Valério, Saikumar, Dastjerdi, Karbasizadeh, and HosseinNia (2019) and Van den Eijnden, Knops, and Heertjes (2018)).

The first reset element was introduced by Clegg (1958) in 1958. The Clegg Integrator (CI) is an integrator which resets its state to zero whenever its input signal is zero. To provide additional design freedom and flexibility, extensions of the CI including First Order Reset Elements (FORE) (Horowitz & Rosenbaum, 1975; Zaccarian, Nesic, & Teel, 2005), Generalized First Order Reset Elements (GFORE) (Saikumar et al., 2019), Second Order Reset Elements (SORE) (Hazeleger et al., 2016), Generalized Second Order Reset Elements (GSORE) (Saikumar et al., 2019), and Second Order Single State Reset Elements (SOSRE) (Karbasizadeh, Ahmadi Dastjerdi, Saikumar, Valerio, & HosseinNia, 2020) have been developed. Moreover, to improve the performances of these controllers several methods such as reset bands (Baños & Davó, 2014; Barreiro, Baños, Dormido, & González-Prieto, 2014), fixed reset instants, partial reset techniques (resetting to a non-zero value or resetting a selection of the controller states) (Zheng, Guo, Fu, Wang, & Xie, 2007), use of shaping filters in the reset instants line (Cai, Dastjerdi, Saikumar, & HosseinNia, 2020), and the PI + CI approach (Zheng et al., 2007) have also been investigated.

Similar to every control system, stability is one of the most essential requirements of reset control systems (Baños & Barreiro, 2011; Baños, Carrasco, & Barreiro, 2010; Beker et al., 2004; Guo et al., 2015; Khalil & Grizzle, 2002; Nešić, Zaccarian, & Teel, 2008; Rifai & Slotine, 2006; van Loon et al., 2017). Stability properties for reset control systems have been studied using quadratic Lyapunov functions (Baños & Barreiro, 2011; Guo et al., 2015; Polenkova, Polderman, & Langerak, 2012; Vettori, Polderman, & Langerak, 2014), reset instants dependent methods (Banos, Carrasco, & Barreiro, 2007; Baños et al., 2010; Paesa, Carrasco, Lucia, & Sagues, 2011), passivity, small gain, and IQC approaches (Carrasco, Baños, & van der Schaft, 2010; Griggs, Anderson, Lanzon, &

Rotkowitz, 2007; Hollot, Zheng, & Chait, 1997; Khalil & Grizzle, 2002). However, most of these methods are complex, require parametric models of the system and the solution of LMI's, and are only applicable to specific types of systems. Thus, since industry often favors the use of frequency-domain methods, these methods are not well matched with the current control design requirements in industry. To overcome this challenge, some frequency-domain approaches for assessing stability properties of reset control systems have been proposed (Beker et al., 2004; Beker, Hollot, Chen, & Chait, 1999; van Loon et al., 2017). A method for determining stability properties of a FORE in closed-loop with a mass–spring–damper system has been developed in Beker et al. (1999). However, this method is only applicable to a specific type of systems. Under the specific reset condition $e(t)u(t) < \dfrac{u^2(t)}{\varepsilon}$, for some $\varepsilon > 0$, in which e and u are the input and the output of the reset element, respectively, the approach in van Loon et al. (2017) is applicable to reset control systems. However, this method is not applicable to traditional reset control systems in which the reset condition is $e(t) = 0$.

The H_β condition is one of the most widely-used methods for assessing stability properties of reset control systems (Baños et al., 2010; Beker et al., 2004; Guo et al., 2015). In spite of its conservatism, as it is based on the Lyapunov function and is applicable to reset control systems with stable base linear systems, it gives the possibility of assessing stability in the frequency-domain. In particular, when the base linear system of the reset element has a first order transfer function, it gives sufficient frequency-domain conditions for uniform bounded-input bounded-state (UBIBS) stability.[1] However, assessing the H_β condition in the frequency-domain is not intuitive, especially for high order transfer function plants. In addition, the effect of a shaping filter in the reset line on the H_β condition has not been studied yet. Furthermore, there is a lack of methods to assess the H_β condition for GSORE using Frequency Response Function (FRF) measurements. Finally, the H_β condition is not applicable to assess UBIBS stability of reset control systems in the case of partial reset techniques. Hence, obtaining a general easy-to-use frequency-domain method for assessing UBIBS stability of reset control systems is an important open question.

In this paper, on the basis of the H_β condition, novel frequency-domain stability conditions for control systems with first and second order reset elements with a shaping filter in the reset line are proposed. This approach allows for assessing UBIBS stability of reset control systems in the frequency-domain. In this approach, the H_β condition does not have to be explicitly tested and stability properties are directly determined on the basis of the FRF measurements of the base linear open-loop system. In addition, the approach can be used in the case of partial reset techniques.

The remainder of the paper is organized as follows. In Section 2 preliminaries about reset elements are presented and the problem is formulated. The frequency-domain approaches for assessing stability properties of control systems with first and second order reset elements are presented in Section 3 and Section 4, respectively. In Section 5 the effectiveness of these approaches is demonstrated via a practical example. Finally, conclusions and suggestions for future studies are given in Section 6.

2. Preliminaries

In this section the description of reset elements and the H_β condition are briefly recalled and some preliminaries are given. The focus of the paper is on the single-input single-output (SISO)

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[1] See Beker et al. (2004) for the definition of UBIBS stability.

control architecture illustrated in Fig. 1. The closed-loop system consists of a linear plant with transfer function $G(s)$ (which we assume strictly proper), linear controllers with proper transfer functions $C_{L_1}(s)$ and $C_{L_2}(s)$, a reset element with base transfer function $C_R(s)$, and a shaping filter with a proper stable transfer function $\mathcal{C}_s(s)$.

The state-space representation of the reset element is

$$
\begin{cases}
\dot{x}_r(t) = A_r x_r(t) + B_r u_1(t), & e_r(t) \neq 0, \\
x_r(t^+) = A_\rho x_r(t), & e_r(t) = 0 \wedge (I - A_\rho) x_r(t) \neq 0, \\
u_r(t) = C_r x_r(t) + D_r u_1(t),
\end{cases}
\tag{1}
$$

in which $x_r(t) \in \mathbb{R}^{n_r}$ is the vector containing the reset state, A_r, B_r, C_r, and D_r are the dynamic matrices of the reset element, A_ρ is the reset matrix, which determines the values of the reset state after the reset action, and $u_1(t) \in \mathbb{R}$ and $u_r(t) \in \mathbb{R}$ are the input and output of the reset element, respectively. The transfer function $C_r(sI - A_r)^{-1}B_r + D_r$ is called the base transfer function of the reset element. The base transfer function in case of GFORE is (in all cases $\omega_r > 0$)

$$
C_R(s) = \frac{1}{\dfrac{s}{\omega_r} + 1},
\tag{2}
$$

for CI and Proportional Clegg Integrator (PCI) one has

$$
C_R(s) = \frac{1}{s},
\tag{3}
$$

$$
C_R(s) = 1 + \frac{\omega_r}{s},
\tag{4}
$$

and for GSORE one has

$$
C_R(s) = \frac{1}{s^2 + 2\xi\omega_r s + \omega_r^2}, \; \xi > 0.
\tag{5}
$$

Thus, for GFORE, $A_r = -C_r = -\omega_r$ (ω_r is the so-called corner frequency), $D_r = 0$, and $B_r = 1$, whereas for the PCI, $A_r = 0$, $C_r = \omega_r$, and $B_r = D_r = 1$. In the case of CI, $A_r = D_r = 0$, $B_r = C_r = 1$, and if we consider the controllable canonical form realization for GSORE, we obtain

$$
A_r = \begin{bmatrix} -2\xi\omega_r & -\omega_r^2 \\ 1 & 0 \end{bmatrix}, \; B_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C_r = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ and } D_r = 0.
\tag{6}
$$

Let $\mathcal{L}$ be the linear time-invariant (LTI) part of the system, see Fig. 1, with input $u_r(t) \in \mathbb{R}$, external disturbance $w(t) = \begin{bmatrix} r(t) & d(t) \end{bmatrix}^T \in \mathbb{R}^2$, and outputs $y(t) \in \mathbb{R}$, $e_r(t) \in \mathbb{R}$, and $u_1(t) \in \mathbb{R}$. The state-space realization of $\mathcal{L}$ is given by equations

$$
\mathcal{L} : \begin{cases}
\dot{\zeta}(t) = A\zeta(t) + B_u u_r(t) + Bw(t), \\
y(t) = C\zeta(t), \\
e_r(t) = C_e \zeta(t) + D_e r(t), \\
u_1(t) = C_u \zeta(t) + D_1 r(t),
\end{cases}
\tag{7}
$$

where $\zeta(t) \in \mathbb{R}^{n_p}$ describes the states of the plant and of the linear controllers (n_p is the number of states of the whole linear part), and A, B, B_u, and C are the corresponding dynamic matrices. The closed-loop state-space representation of the overall system can, therefore, be written as

$$
\begin{cases}
\dot{x}(t) = \bar{A}x(t) + \bar{B}w(t), & e_r(t) \neq 0, \\
x(t^+) = \bar{A}_\rho x(t), & e_r(t) = 0 \wedge (I - \bar{A}_\rho)x(t) \neq 0, \\
y(t) = \bar{C}x(t), \\
e_r(t) = \bar{C}_e x(t) + D_e r(t),
\end{cases}
\tag{8}
$$
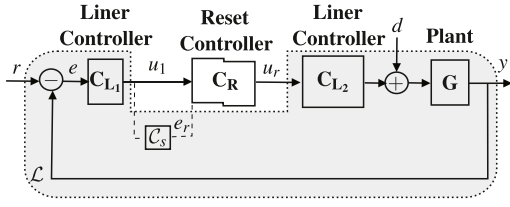
Fig. 1. The closed-loop architecture of a reset control system.

where $x(t) = \begin{bmatrix} x_r(t)^T & \zeta(t)^T \end{bmatrix}^T \in \mathbb{R}^{n_r+n_p}$, $\bar{C} = \begin{bmatrix} 0_{1\times n_r} & C \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0_{n_r\times 2} \\ B \end{bmatrix} + \begin{bmatrix} B_r D_1 & 0_{n_r\times 1} \\ B_u D_r D_1 & 0_{n_p\times 1} \end{bmatrix}$, $\bar{C}_e = \begin{bmatrix} 0_{1\times n_r} & C_e \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A_r & B_r C_u \\ B_u C_r & A + B_u D_r C_u \end{bmatrix}$, and $\bar{A}_\rho = \begin{bmatrix} A_\rho & 0_{n_r\times n_p} \\ 0_{n_p\times n_r} & I_{n_p\times n_p} \end{bmatrix}$.

Definition 1. A time $\bar{T} > 0$ is called a reset instant for the reset control system (8) if $e_R(\bar{T}) = 0 \wedge (I - \bar{A}_\rho)x(T) \neq 0$. For any given initial condition and input w the resulting set of all reset instants defines the reset sequence $\{t_k\}$, with $t_k \leq t_{k+1}$, for all $k \in \mathbb{N}$. The reset instants t_k have the well-posedness property if for any initial condition x_0 and any input w, all the reset instants are distinct, and there exists $\lambda > 0$ such that, for all $k \in \mathbb{N}$, $\lambda \leq t_{k+1}-t_k$ (Baños & Barreiro, 2011; Banos, Mulero, Barreiro, & Davo, 2016).

Lemma 1. *Let $H(s)$ be a proper rational transfer function and assume $det[H(s)+H^T(-s)]$ is not identically zero. Then, $H(s)$ is SPR if and only if (Khalil & Grizzle, 2002):*

- *$H(s)$ is Hurwitz.*
- *$H(j\omega) + H^T(-j\omega)$ is positive definite for all $\omega \in \mathbb{R}^+$,*
- *either $H(\infty) + H^T(\infty)$ is positive definite or*
- *if $H(\infty) + H^T(\infty)$ is positive semi definite, $\lim_{\omega\to\infty} \omega^2 M^T[H(j\omega) + H^T(-j\omega)]M > 0$ for any $p \times (p-q)$ full rank matrix M such that $M^T[H(\infty)+H^T(\infty)]M = 0$, and $q = rank[H(\infty)+H^T(\infty)]$.*

One of the methods for determining stability properties of reset control systems is the H_β condition (Baños & Barreiro, 2011; Baños et al., 2010; Beker et al., 2004; Guo et al., 2015; Hollot, Beker, Chait, & Chen, 2001), which is briefly recalled. Let

$$C_0 = \begin{bmatrix} \varrho & \beta C \end{bmatrix}, \quad B_0 = \begin{bmatrix} I_{n_r\times n_r} \\ 0_{n_p\times n_r} \end{bmatrix}, \quad \varrho = \varrho^T > 0, \quad \varrho \in \mathbb{R}^{n_r\times n_r}, \quad (9)$$

and $\beta \in \mathbb{R}^{n_r \times 1}$. The H_β condition (Baños & Barreiro, 2011; Baños et al., 2010; Beker et al., 2004; Guo et al., 2015; Hollot et al., 2001) is summarized in the following theorem.

Theorem 1. *The zero equilibrium of the reset control system (8) with $C_{L_1} = C_s = 1$ and $w = 0$ is globally uniformly asymptotically stable[2] if there exist $\varrho = \varrho^T > 0$ and β such that the transfer function*

$$H(s) = C_0(sI - \bar{A})^{-1}B_0 \qquad (10)$$

is Strictly Positive Real (SPR), $(\bar{A}, B_0)$ and $(\bar{A}, C_0)$ are controllable and observable, respectively, and

$$A_\rho^T \varrho A_\rho - \varrho \leq 0. \qquad (11)$$

Remark 1. Consider the reset control system (8) and $w(t) = 0$. If Theorem 1 holds, then there exists a positive-definite matrix

$P = P^T > 0$ such that

$$\begin{cases} \bar{A}^T P + P\bar{A} < 0, & e_r(t) \neq 0, \\ \bar{A}_\rho^T P \bar{A}_\rho - P \leq 0, & e_r(t) = 0 \wedge (I - \bar{A}_\rho)x(t) \neq 0. \end{cases} \qquad (12)$$

Evaluating the H_β condition requires finding the parameters ϱ and β, which may be very difficult when the system has a high order transfer function. Furthermore, in the case of GSORE there is no direct frequency-domain method to assess this condition. Besides, the UBIBS property of GSORE and of GFORE have not yet been studied, and the effects of the shaping filter on the H_β condition have not been considered yet. In the current paper, frequency-domain methods to determine stability properties without finding ϱ and β for GFORE and of GSORE with considering the shaping filter are proposed.

Assumption 1. There are infinitely many reset instants and $\lim_{k\to\infty} t_k = \infty$.

Assumption 1 is introduced to rule out a trivial situation. In fact, if there are finitely many reset instants, then there exists a $T_K \in [0, \infty)$ such that for all $t \geq T_K$ the reset control system (8) is a linear stable system provided the H_β condition is satisfied. In addition to Assumption 1, we need the following assumption, which is instrumental to study the UBIBS property of reset control systems.

Assumption 2. In the case of partial reset technique, if A_ρ has the structure

$$A_\rho = \begin{bmatrix} I_{\bar{n}_r} & 0 \\ 0 & A'_{n'_r} \end{bmatrix},$$

then A_r has the structure

$$A_r = \begin{bmatrix} A_{r_1} & A_{r_2} \\ 0_{\bar{n}_r \times n'_r} & A_{r_3} \end{bmatrix}.$$

Remark 2. In the case of GFORE, GSORE, PCI, and CI in which all states of the reset element reset, Assumption 2 holds.

Before stating the main theorem, an important technical lemma, which is instrumental for all proofs, is formulated and proved.

Lemma 2. *Consider the reset control system (8). Suppose that*

- *Assumption 1 holds;*
- *$A_\rho^T \varrho A_\rho - \varrho < 0$;*
- *the H_β condition holds;*
- *at least one of the following conditions holds:*

  1. $C_s = 1$ and Assumption 2 holds;
  2. the reset instants have the well-posedness property.

Then the reset control system (8) has a well-defined unique left-continuous response[3] for any initial condition x_0 and any input w which is a Bohl function.[4] In addition, the reset control system (8) has the UBIBS property and the reset instants have the well-posedness property.

Proof. See Appendix A.

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[2] See Beker et al. (2004) for the definition of global uniform asymptotic stability.

[3] That is for all $a > 0$, $\lim_{t\to a^+} x(t) = x(a)$.

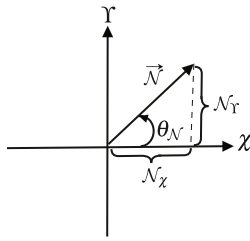[4] See Banos et al. (2016) for the definition of a Bohl function.

Fig. 2. Representation of the NSV in the $\chi - \Upsilon$ plane.

3. Stability analysis of reset control systems with first order reset elements

In this section frequency-domain methods for assessing stability properties of the reset control system (8) with GFORE (2), CI (3), and PCI (4) are proposed on the basis of the H_β condition. To this end, the Nyquist Stability Vector (NSV $= \vec{\mathcal{N}}(\omega) \in \mathbb{R}^2$) in a plane with axis $\chi - \Upsilon$ (see Fig. 2) is defined as follows.

Definition 2. The Nyquist Stability Vector is, for all $\omega \in \mathbb{R}^+$, the vector

$$\vec{\mathcal{N}}(\omega) = [\mathcal{N}_\chi \quad \mathcal{N}_\Upsilon]^T$$
$$= [\Re(L(j\omega)\mathcal{C}_s(j\omega)\kappa(j\omega)) \quad \Re(\kappa(j\omega)C_R(j\omega))]^T,$$

in which $L(s) = C_{L_1}(s)C_R(s)C_{L_2}(s)G(s)$, $L(j\omega) = a(\omega) + b(\omega)j$, and $\kappa(j\omega) = 1 + L^*(j\omega)$ ($L^*(j\omega)$ is the conjugate of $L(j\omega)$).

For simplicity, and without loss of generality, let $\underline{/\vec{\mathcal{N}}(\omega)} = \theta_\mathcal{N} \in [-\frac{\pi}{2}, \frac{3\pi}{2})$ and define the open sets

$$\mathcal{I}_1 = \left\{ \omega \in \mathbb{R}^+ | \ 0 < \underline{/\vec{\mathcal{N}}(\omega)} < \frac{\pi}{2} \right\},$$

$$\mathcal{I}_2 = \left\{ \omega \in \mathbb{R}^+ | \ \frac{\pi}{2} < \underline{/\vec{\mathcal{N}}(\omega)} < \pi \right\},$$

$$\mathcal{I}_3 = \left\{ \omega \in \mathbb{R}^+ | \ \pi < \underline{/\vec{\mathcal{N}}(\omega)} < \frac{3\pi}{2} \right\},$$

$$\mathcal{I}_4 = \left\{ \omega \in \mathbb{R}^+ | \ -\frac{\pi}{2} < \underline{/\vec{\mathcal{N}}(\omega)} < 0 \right\}.$$

Let $L(s)\mathcal{C}_s(s) = \dfrac{K_m s^m + K_{m-1}s^{m-1} + \cdots + K_0}{s^n + K'_{n-1}s^{n-1} + \cdots + K'_0}$ and $\mathcal{C}_s(s) = \dfrac{K_{s_m}s^{m_s} + K_{s_{m-1}}s^{m_s-1} + \cdots + K_{s_0}}{K'_{s_n}s^{n_s} + K'_{s_{n-1}}s^{n_s-1} + \cdots + 1}$. On the basis of the definition of the NSV, systems of Type I and of Type II, which are used to assess stability properties of the reset control system (8), are defined.

Definition 3. The reset control system (8) is of Type I if the following conditions hold.

(1) If $C_{L_1}(s)C_{L_2}(s)G(s)$ has at least one pole at the origin, then $K_{s_0} > 0$.
(2) In the case of CI (3), $K_{s_0} < 0$.
(3) For all $\omega \in \mathcal{M} = \{\omega \in \mathbb{R}^+ | \ \mathcal{N}_\chi(\omega) = 0\}$ one has $\mathcal{N}_\Upsilon(\omega) > 0$.
(4) For all $\omega \in \mathcal{Q} = \{\omega \in \mathbb{R}^+ | \ \mathcal{N}_\Upsilon(\omega) = 0\}$ one has $\mathcal{N}_\chi(\omega) > 0$.
(5) At least one of the following statements is true:

    (a) $\forall \ \omega \in \mathbb{R}^+ : \ \mathcal{N}_\Upsilon(\omega) \geq 0$.
    (b) $\forall \ \omega \in \mathbb{R}^+ : \ \mathcal{N}_\chi(\omega) \geq 0$.
    (c) Let $\delta_1 = \max_{\omega \in \mathcal{I}_4} \left| \dfrac{\mathcal{N}_\Upsilon(\omega)}{\mathcal{N}_\chi(\omega)} \right|$ and $\Psi_1 = \min_{\omega \in \mathcal{I}_2} \left| \dfrac{\mathcal{N}_\Upsilon(\omega)}{\mathcal{N}_\chi(\omega)} \right|$. Then $\delta_1 < \Psi_1$ and $\mathcal{I}_3 = \varnothing$.

Remark 3. Let

$$\theta_1 = \min_{\omega \in \mathbb{R}^+} \underline{/\vec{\mathcal{N}}(\omega)} \text{ and } \theta_2 = \max_{\omega \in \mathbb{R}^+} \underline{/\vec{\mathcal{N}}(\omega)}. \tag{13}$$

Then the conditions identifying Type I systems are equivalent to the following conditions.

(1) If $C_{L_1}(s)C_{L_2}(s)G(s)$ has at least one pole at the origin, then $K_{s_0} > 0$.
(2) In the case of CI (3), $K_{s_0} < 0$.
(3) The condition

$$\left(-\frac{\pi}{2} < \theta_1 < \pi\right) \wedge \left(-\frac{\pi}{2} < \theta_2 < \pi\right) \wedge (\theta_2 - \theta_1 < \pi) \tag{14}$$

holds.

Definition 4. The reset control system (8) is of Type II if the following conditions hold.

(1) If $C_{L_1}(s)C_{L_2}(s)G(s)$ has at least one pole at the origin, then $K_{s_0} < 0$.
(2) In the case of CI (3), $K_{s_0} > 0$.
(3) For all $\omega \in \mathcal{M}$ one has $\mathcal{N}_\Upsilon(\omega) > 0$.
(4) For all $\omega \in \mathcal{Q}$ one has $\mathcal{N}_\chi(\omega) < 0$.
(5) At least, one of the following statements is true:

    (a) $\forall \ \omega \in \mathbb{R}^+ : \ \mathcal{N}_\Upsilon(\omega) \geq 0$;
    (b) $\forall \ \omega \in \mathbb{R}^+ : \ \mathcal{N}_\chi(\omega) \leq 0$;
    (c) Let $\delta_2 = \max_{\omega \in \mathcal{I}_3} \left| \dfrac{\mathcal{N}_\Upsilon(\omega)}{\mathcal{N}_\chi(\omega)} \right|$ and $\Psi_2 = \min_{\omega \in \mathcal{I}_1} \left| \dfrac{\mathcal{N}_\Upsilon(\omega)}{\mathcal{N}_\chi(\omega)} \right|$. Then, $\delta_2 < \Psi_2$ and $\mathcal{I}_4 = \varnothing$.

Remark 4. The conditions identifying Type II systems are equivalent to the following conditions.

(1) If $C_{L_1}(s)C_{L_2}(s)G(s)$ has at least one pole at the origin, then $K_{s_0} < 0$.
(2) In the case of CI (3), $K_{s_0} > 0$.
(3) The condition

$$\left(0 < \theta_1 < \frac{3\pi}{2}\right) \wedge \left(0 < \theta_2 < \frac{3\pi}{2}\right) \wedge (\theta_2 - \theta_1 < \pi) \tag{15}$$

holds.

Theorem 2. *The zero equilibrium of the reset control system (8) with GFORE (2), or CI (3), or PCI (4) is globally uniformly asymptotically stable when $w = 0$, and the system has the UBIBS property for any input w which is a Bohl function if all of the following conditions are satisfied.*

- *The base linear system is stable and the open-loop transfer function does not have any pole-zero cancellation.*
- *In the case of CI (3), $C_{L_1}(s)C_{L_2}(s)G(s)$ does not have any pole at the origin and $n - m = 2$.*
- *The reset control system (8) is either of Type I and/or of Type II.*
- *$A_\rho = \gamma$, $-1 < \gamma < 1$.*
- *$\mathcal{C}_s(s) = 1$ and/or the reset instants have the well-posedness property.*

Proof. See Appendix B.

Corollary 1. *Let $\mathcal{C}_s(s) = 1$, $\theta_L = \underline{/L(j\omega)}$, and $\theta_{C_R} = \underline{/C_R(j\omega)}$. Suppose that the base linear system of the reset control system (8) is stable, $A_\rho = \gamma$, $-1 < \gamma < 1$, $L(s)$ and the open-loop system does not have any pole-zero cancellation. Then the zero equilibrium*
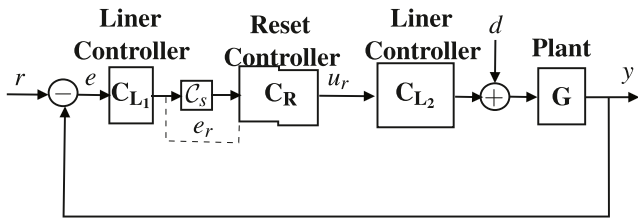
Fig. 3. The closed-loop architecture of a modified reset element.

of the reset control system (8) with GFORE (2), or CI (3), or PCI (4) is globally uniformly asymptotically stable when $w = 0$, and the system has the UBIBS property for any input w which is a Bohl function if at least one of the following conditions hold.

(1) For all $\omega \in \mathbb{R}^+$, $\sin(\theta_L) \geq 0$.
(2) For all $\omega \in \mathbb{R}^+$, $\cos(\theta_L - \theta_{C_R}) \geq 0$ and the reset element is not CI (3).

Proof. When $\mathcal{C}_s(s) = 1$, $\mathcal{N}_\chi(\omega) = a(\omega)^2 + b(\omega)^2 + b(\omega)$. By Hypothesis 1, $b(\omega) \geq 0$, for all $\omega \in \mathbb{R}^+$, which implies that $\mathcal{N}_\chi(\omega) > 0$. Thus, the reset control system (8) is of Type I. In addition, defining $C_R(j\omega) = a_R(\omega) + jb_R(\omega)$, yields $\mathcal{N}_\Upsilon(\omega) = a(\omega)a_R(\omega) + b(\omega)b_R(\omega) + a_R(\omega)$. By Hypothesis 2,

$$\forall\, \omega \in \mathbb{R}^+: \ \cos(\theta_L - \theta_{C_R}) \geq 0 \Rightarrow \frac{a(\omega)a_R(\omega) + b(\omega)b_R(\omega)}{|L(j\omega)C_R(j\omega)|} \geq 0, \tag{16}$$

and since $a_R(\omega) > 0$ in the cases of PCI and GFORE, $\mathcal{N}_\Upsilon(\omega) > 0$, for all $\omega \in \mathbb{R}^+$. Therefore, the reset control system (8) is of Type I and/or Type II, hence the claim.

In Karbasizadeh, Dastjerdi, Saikumar, and HosseinNia (2022) the GFORE, CI and PCI architectures have been modified to improve the performance of reset control systems. Using the same procedure as Theorem 2 a frequency-domain method to assess stability properties of these reset control systems illustrated in Fig. 3 is proposed.

Corollary 2. *Let the NSV vector for the reset control system shown in Fig. 3 be*

$$\vec{\mathcal{N}}_{MF}(\omega) = [\mathcal{N}_{MF_\chi} \quad \mathcal{N}_{MF_\Upsilon}]^T$$
$$= \left[\Re\left(\frac{L'(j\omega)\kappa(j\omega)}{\mathcal{C}_s(j\omega)} \right) \quad \Re(\kappa(j\omega)C_R(j\omega)) \right]^T, \tag{17}$$

in which $L'(s) = C_{L_1}(s)C_R(s)C_{L_2}(s)\mathcal{C}_s(s)G(s)$. Then, the zero equilibrium of the reset control system (8) in the configuration of Fig. 3 with GFORE (2), or CI (3), or PCI (4) is globally uniformly asymptotically stable when $w = 0$, and the system has the UBIBS property for any input w which is a Bohl function if all of the following conditions are satisfied.

- *The base linear system is stable and the open-loop transfer function does not have any pole-zero cancellation.*
- *In the case of CI (3), $C_{L_1}(s)C_{L_2}(s)G(s)$ does not have any pole at the origin and $n - m = 2$.*
- *The reset control system (8) is either of Type I and/or of Type II.*
- $A_\rho = \gamma$, $-1 < \gamma < 1$.
- $\mathcal{C}_s(s) = 1$ *and/or the reset instants have the well-posedness property.*

Proof. See Appendix C.

4. Stability analysis of reset control systems with second order reset elements

4.1. Reset control systems with GSORE

In this section a frequency-domain method for assessing stability properties of the reset control system (8) with GSORE (5), which has the canonical controllable form state-space realization (6), is proposed. In this method the H_β condition is combined with optimization tools to provide sufficient conditions to guarantee stability properties of the reset control system (8). Note that in the case of GSORE, since $\rho \in \mathbb{R}^{2 \times 2}$ and $\beta \in \mathbb{R}^{2 \times 1}$, the relations are more complex than those in the previous section. Before presenting the main result, one preliminary fact, which is useful for assessing stability properties of the reset control system (8) with GSORE (5), is presented.

Proposition 3. *Let $\vec{\mathcal{Q}} \in \mathbb{R}^2$ and $\vec{\mathcal{F}} \in \mathbb{R}^2$ be defined as $\vec{\mathcal{Q}} = [Q_1 \quad Q_2]^T$ and $\vec{\mathcal{F}}(\omega) = [\mathcal{F}_1(\omega) \quad \mathcal{F}_2(\omega)]^T$. Let $\angle\vec{\mathcal{Q}}, \vec{\mathcal{F}}(\omega) = \vartheta(\omega, \frac{Q_2}{Q_1})$, $\omega_p = \{\omega \in \mathbb{R}^+ | \ \mathcal{F}_3(\omega) \geq 0\}$, $\omega_N = \mathbb{R}^+ - \omega_p$,*

$$g_p = \left\{ \frac{Q_2}{Q_1} \in \mathbb{R}| \ \forall \omega \in \omega_p : \ Q_1\mathcal{F}_1(\omega) + Q_2\mathcal{F}_2(\omega) > 0 \right\},$$

and

$$g_N = \left\{ \frac{Q_2}{Q_1} \in \mathbb{R}| \ \forall \omega \in \omega_N : \ Q_1\mathcal{F}_1(\omega) + Q_2\mathcal{F}_2(\omega) > 0 \right\}.$$

Then the condition

$$Q_1\mathcal{F}_1(\omega) + Q_2\mathcal{F}_2(\omega) > \mathcal{F}_3(\omega), \tag{18}$$

holds for all $\omega \in \mathbb{R}$ if and only if

- $\eta_1(\frac{Q_2}{Q_1}) < \sqrt{Q_1^2 + Q_2^2} < \eta_2(\frac{Q_2}{Q_1})$,
- $\frac{Q_2}{Q_1} \in \left\{ \frac{Q_2}{Q_1} \in g_p| \ \eta_1(\frac{Q_2}{Q_1}) < \eta_2(\frac{Q_2}{Q_1}) \right\}$,

where

$$\eta_1\left(\frac{Q_2}{Q_1}\right) = \begin{cases} -\infty & \omega_p = \varnothing, \\ \max_{\omega \in \omega_p} \dfrac{\mathcal{F}_3(\omega)}{\cos(\vartheta)\sqrt{\mathcal{F}_1^2(\omega) + \mathcal{F}_2^2(\omega)}}, & \omega_p \neq \varnothing, \end{cases}$$

$$\eta_2\left(\frac{Q_2}{Q_1}\right) = \begin{cases} +\infty & \frac{Q_2}{Q_1} \in g_N, \vee \omega_N = \varnothing, \\ \min_{\omega \in \omega_N} \dfrac{\mathcal{F}_3(\omega)}{\cos(\vartheta)\sqrt{\mathcal{F}_1^2(\omega) + \mathcal{F}_2^2(\omega)}}, & \frac{Q_2}{Q_1} \notin g_N. \end{cases} \tag{19}$$

Proof. See Appendix D.

Remark 5. The sets g_p and g_N can be easily obtained using the method described in Dastjerdi, Astolfi, and HosseinNia (2020).

Define now $\Gamma(\gamma_1, \gamma_2) = \dfrac{(\gamma_1\gamma_2 - 1)^2}{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}$

$$f_1(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \omega) = \mathcal{X}_1(C_R(j\omega)\kappa(\omega)j\omega)$$
$$+ \mathcal{X}_2(C_R(j\omega)\kappa(\omega))$$
$$+ \mathcal{X}_3(\mathcal{C}_s(j\omega)(a^2 + b^2 + a)),$$

$$f_2(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \omega) = \mathcal{X}_1(C_R(j\omega)\kappa(\omega)(j\omega + 2\xi\omega_r))$$
$$+ \mathcal{X}_3(L(j\omega)\kappa(\omega)\mathcal{C}_s(j\omega)(j\omega + 2\xi\omega_r))$$
$$+ \mathcal{X}_2(C_R(j\omega)\kappa(\omega)(2j\xi\omega_r\omega - \omega^2)$$
$$- (a + 1)^2 - b^2),$$

$$G_1(Q_1, Q_2, Q_3, Q_4) = \sup_{\omega \in (0, \infty)} \left[\frac{f_1\left(Q_2, \frac{Q_2 Q_3}{Q_4}, \frac{Q_2}{Q_4}, -\omega\right) + f_2(Q_2, Q_1, 1, \omega)}{\Re(f_1(Q_1, Q_2, 1, \omega))} \right.$$
$$\left. \times \frac{f_1(Q_4, Q_3, 1, \omega) + f_2\left(Q_4, \frac{Q_1 Q_4}{Q_2}, \frac{Q_4}{Q_2}, -\omega\right)}{\Re(f_2(Q_3, Q_4, 1, \omega))} \right],$$

$$G_2(Q_1, Q_2, Q_3, Q_4) = \sup_{\omega \in [0, \infty)} \left[\frac{f_1\left(Q_2', \frac{Q_2'}{Q_4'}, \frac{Q_2' Q_3'}{Q_4'}, -\omega\right) + f_2(Q_2', 1, Q_1', \omega)}{\Re(f_1(1, Q_2, Q_1, \omega))} \right.$$
$$\left. \times \frac{f_1(Q_4', 1, Q_3', \omega) + f_2\left(Q_4', \frac{Q_4'}{Q_2'}, \frac{Q_1' Q_4'}{Q_2'}, -\omega\right)}{\Re(f_2(1, Q_4, Q_3, \omega))} \right]. \quad (20)$$

We define systems of Type III, of Type IV, and of Type V to assess stability properties of the reset control system (8) with GSORE (6).

Definition 5. The reset control system (8) with GSORE (6) is of Type III if the following conditions hold.

(1) $M < 4$, where $M = \min_{Q_1, Q_2, Q_3, Q_4} G_1(Q_1, Q_2, Q_3, Q_4)$, in which Q_1, Q_2, Q_3, and Q_4 are such that the following constraints hold:

$\mathcal{S}_1 : \forall \omega \in (0, \infty): \ K_{s_0} \Re(f_1(Q_1, Q_2, 1, \omega)) > 0,$

$\mathcal{S}_2 : \forall \omega \in (0, \infty): \ K_{s_0} \Re(f_2(Q_3, Q_4, 1, \omega)) > 0,$

$\mathcal{S}_3 : K_{s_0} \left(\frac{2\xi\omega_r}{Q_1} + \frac{Q_2}{Q_1 Q_4} + \frac{2}{Q_1} \sqrt{\frac{2Q_2 \xi\omega_r}{Q_4} - \frac{Q_2}{K_{s_0}}} \right) > 1,$

$\mathcal{S}_4 : K_{s_0} \left(\frac{2\xi\omega_r}{Q_1} + \frac{Q_2}{Q_1 Q_4} - \frac{2}{Q_1} \sqrt{\frac{2Q_2 \xi\omega_r}{Q_4} - \frac{Q_2}{K_{s_0}}} \right) < 1,$

$\mathcal{S}_5 : \frac{\omega_r^2 Q_1}{Q_2} + 2\omega_r \left(\xi + 2\sqrt{\frac{2Q_1 \xi\omega_r}{Q_2} - 1} \right) > \frac{Q_3}{Q_4},$

$\mathcal{S}_6 : \frac{\omega_r^2 Q_1}{Q_2} + 2\omega_r \left(\xi - 2\sqrt{\frac{2Q_1 \xi\omega_r}{Q_2} - 1} \right) < \frac{Q_3}{Q_4},$

$\mathcal{S}_7 : K_{s_0} Q_i > 0, \ 2\xi\omega_r > \frac{Q_4}{K_{s_0}}, \ 2\xi\omega_r > \frac{Q_2}{Q_1},$

$\mathcal{S}_8 : \frac{Q_1 Q_3}{Q_2 Q_4} > \Gamma(\gamma_1, \gamma_2). \quad (21)$

(2) The pairs $(\bar{A}, C_0)$ and $(\bar{A}, B_0)$ where $B_0 = \begin{bmatrix} 0_{n_p \times 2} \\ I_2 \end{bmatrix}$ and $C_0 = \left[\begin{bmatrix} 1 \\ \frac{Q_2}{Q_4} \end{bmatrix} \bar{C}_e \ \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & \frac{Q_2 Q_3}{Q_4} \end{bmatrix} \right]$ are controllable and observable, respectively.

(3) The open-loop system has at least one pole at the origin and $K_{s_0} \neq 0$.

Definition 6. The reset control system (8) with GSORE (6) is of Type IV if the following conditions hold.

(1) $M < 4$, where $M = \min_{Q_1, Q_2, Q_3, Q_4} G_2(Q_1, Q_2, Q_3, Q_4)$, in which Q_1, Q_2, Q_3, and Q_4 are such that the following

constraints hold:

$\mathcal{S}_1 : \forall \omega \in [0, \infty): \ \Re(f_1(1, Q_2, Q_1, \omega)) > 0,$

$\mathcal{S}_2 : \forall \omega \in [0, \infty): \ \Re(f_2(1, Q_4, Q_3, \omega)) > 0,$

$\mathcal{S}_3 : \omega_r^2 + 2\omega_r \left(\xi Q_2 + 2\sqrt{2Q_2 \xi\omega_r - Q_2^2} \right) > \frac{Q_2}{Q_4},$

$\mathcal{S}_4 : \omega_r^2 + 2\omega_r \left(\xi Q_2 - 2\sqrt{2Q_2 \xi\omega_r - Q_2^2} \right) < \frac{Q_2}{Q_4},$

$\mathcal{S}_5 : Q_4 > 0, \ 0 < Q_2 < 2\xi\omega_r, \ Q_2 Q_4 < \frac{1}{\Gamma(\gamma_1, \gamma_2)}. \quad (22)$

(2) The pairs $(\bar{A}, C_0)$ and $(\bar{A}, B_0)$ where $B_0 = \begin{bmatrix} 0_{n_p \times 2} \\ I_2 \end{bmatrix}$ and $C_0 = \left[\begin{bmatrix} Q_1 \\ \frac{Q_2 Q_3}{Q_4} \end{bmatrix} \bar{C}_e \ \begin{bmatrix} 1 & Q_2 \\ Q_2 & \frac{Q_2}{Q_4} \end{bmatrix} \right]$ are controllable and observable, respectively.

(3) The open-loop system does not have any pole at the origin.

(4) $n - m > 3$.

Definition 7. The reset control system (8) with GSORE (6) is of Type V if the following conditions hold.

(1) $M < 4$, where $M = \min_{Q_1, Q_2, Q_3, Q_4} G_2(Q_1, Q_2, Q_3, Q_4)$, in which Q_1, Q_2, Q_3, and Q_4 are such that the following constraints hold:

$\mathcal{S}_1 : \forall \omega \in [0, \infty): \ \Re(f_1(1, Q_2, Q_1, \omega)) > 0,$

$\mathcal{S}_2 : \forall \omega \in [0, \infty): \ \Re(f_2(1, Q_4, Q_3, \omega)) > 0,$

$\mathcal{S}_3 : \omega_r^2 - K_n Q_1 + 2\xi\omega_r Q_2 + 2\sqrt{2\xi\omega_r^3 Q_2 + \frac{Q_2^2 Q_3 K_n}{Q_4} - \omega_r^2 Q_2^2} > \frac{Q_2}{Q_4},$

$\mathcal{S}_4 : \omega_r^2 - K_n Q_1 + 2\xi\omega_r Q_2 - 2\sqrt{2\xi\omega_r^3 Q_2 + \frac{Q_2^2 Q_3 K_n}{Q_4} - \omega_r^2 Q_2^2} < \frac{Q_2}{Q_4},$

$\mathcal{S}_5 : 2\xi\omega_r^3 Q_2 + \frac{Q_2^2 Q_3 K_n}{Q_4} > \omega_r^2 Q_2^2,$

$\mathcal{S}_6 : Q_2 < 2\xi\omega_r, \ K_n Q_3 < \omega_r^2 Q_4, \ 0 < Q_2 Q_4 < \frac{1}{\Gamma(\gamma_1, \gamma_2)}. \quad (23)$

(2) The pairs $(\bar{A}, C_0)$ and $(\bar{A}, B_0)$ where $B_0 = \begin{bmatrix} 0_{n_p \times 2} \\ I_2 \end{bmatrix}$ and $C_0 = \left[\begin{bmatrix} Q_1 \\ \frac{Q_2 Q_3}{Q_4} \end{bmatrix} \bar{C}_e \ \begin{bmatrix} 1 & Q_2 \\ Q_2 & \frac{Q_2}{Q_4} \end{bmatrix} \right]$ are observable and controllable, respectively.

(3) The open-loop system does not have any pole at the origin.

(4) $n - m = 3$.

Theorem 4. *The zero equilibrium of the reset control system* (8) *with GSORE* (6) *is globally uniformly asymptotically stable when* $w = 0$, *and the system has the UBIBS property for any input* w *which is a Bohl function if all of the following conditions are satisfied.*

- *The base linear system is stable.*
- $A_\rho = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$ *and* $-1 < \gamma_i < 1$, *for* $i = 1, \ 2$.
- *The reset control system is either of Type III, or of Type IV, or of Type V.*
- $\mathcal{C}_s(s) = 1$ *and/or the reset instants have the well-posedness property.*

Proof. See Appendix E.

4.2. Reset control systems with (SOSRE)

In this section stability analysis for the reset control system (8) with the SOSRE (Karbasizadeh et al., 2020) is presented. In Karbasizadeh et al. (2020) GSORE (6) with $A_\rho = \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}$, which is termed SOSRE, is used to improve the performance of the reset control system (8). In the case of SOSRE one state of GSORE is reset and the other state is utilized to reduce the high order harmonics of the reset element.

Corollary 3. *Consider the reset control system (8) with SOSRE. Define the NSV vector as*

$$\vec{\mathcal{N}}_{SOS}(\omega) = [\mathcal{N}_{SOS_\chi} \quad \mathcal{N}_{SOS_\gamma}]^T =$$

$$\left[\Re(L(j\omega)\kappa(j\omega)\mathcal{C}_s(j\omega)) \quad -\Im(\omega\kappa(j\omega)C_R(j\omega)) \right]^T.$$

Suppose that the reset instants have the well-posedness property and $-1 < \gamma < 1$. Then, with this definition of NSV the zero equilibrium of the reset control system (8) with SOSRE is globally uniformly asymptotically stable when $w = 0$, and the system has the UBIBS property for any input w which is a Bohl function if all of the following conditions are satisfied.

- *The base linear system is stable and the open-loop transfer function does not have any pole-zero cancellation.*
- *The reset control system (8) is either of Type I and/or of Type II.*

Proof. Let $\beta' = -\beta$. The transfer function (10) with C_0 as in (B.1) can be rewritten as (see also Fig. E.1, transfer function from r_1 to y_1 with $\varrho_2 = 0$)

$$H(s) = \frac{\beta' L(s)\mathcal{C}_s(s) + \varrho \, sC_R(s)}{1 + L(s)}. \tag{24}$$

Step 1 and Step 4 of the proof of Theorem 2 are repeated with small modifications. When the open-loop system has poles at the origin

$$\lim_{\omega \to 0} \Re(H(j\omega)) = K_{s_0}\beta' > 0. \tag{25}$$

In the case of SOSRE one has $n - m \geq 3$. Consequently,

$$\lim_{\omega \to \infty} \omega^2 \Re(H(j\omega)) = 2 \, \varrho \, \xi \omega_r > 0, \tag{26}$$

and the proof is complete.

Note that it is impossible to satisfy Assumption 2 for this configuration. Thus, the reset instants must have the well-posedness property.

5. Illustrative examples

In this section two examples showing how the proposed methods can be used to study stability properties of reset control systems are presented. In particular, stability properties of a precision positioning system (Saikumar et al., 2019) (knows as a spider stage) controlled by a reset controller are considered. In this system (see Fig. 4), three actuators are angularly spaced to actuate three masses (labeled as B1, B2, and B3) which are constrained by parallel flexures and connected to the central mass D through leaf flexures. Only one of the actuators (A1) is considered and used for controlling the position of the mass B1 attached to the same actuator, which results in a SISO system. For using these stability methods the FRF measurement of the plant (Fig. 5) is needed. In Saikumar et al. (2019) a non-linear phase compensator, which is termed "Constant in gain Lead in phase" (CgLp) (for more details see Chen et al. (2020), Palanikumar,
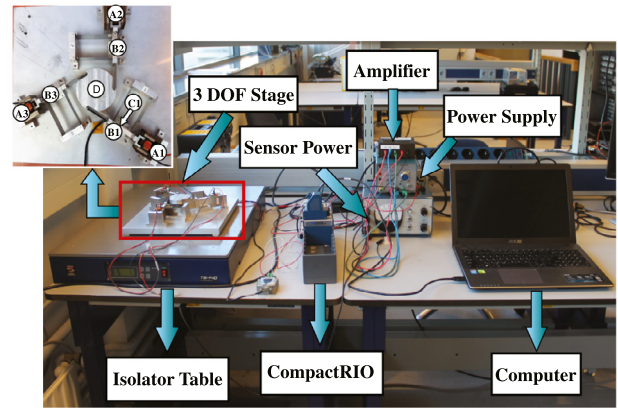

Fig. 4. The whole setup including computer, CompactRio, power supply, sensor power, amplifier, isolator, sensor and, stage.

Saikumar, and HosseinNia (2018) and Saikumar et al. (2019)), has been used to improve the performance of this precision positioning stage. CgLp compensators, consisting of a first/second order lead filter and a GFORE/GSORE, have been utilized along with a PID controller to enhance the precision of the system. In the following, stability properties of two CgLp + PID controllers, one of which has GSORE and the other has SOSRE, are assessed with the proposed methods. The general structure of the controller is

$$C(s) = K_p \overbrace{\left(\underbrace{\frac{1}{s^2 + 2\xi\omega_r s + \omega_r^2}}_{\text{GSORE}} \right)^{A_\rho} \underbrace{\left(\frac{s^2 + 2\xi_d\omega_d s + \omega_d^2}{s^2 + 20\omega_c s + 100\omega_c^2} \right)}_{\text{Lead}}}^{\text{CgLp}}$$

$$\times \underbrace{\overbrace{\left(1 + \frac{\omega_c}{10s} \right)}^{\text{PI}} \overbrace{\left(\frac{\frac{3s}{\omega_c} + 1}{\frac{s}{3\omega_c} + 1} \right)}^{\text{Lead}}}_{\text{PID}}, \tag{27}$$

in which ω_c is the cross-over frequency and K_p, γ, ω_d, ω_r, ξ, and ξ_d are tuning parameters. The PID part is tuned on the basis of Dastjerdi, Saikumar, and HosseinNia (2018), Schmidt, Schitter, and Rankers (2014) and the CgLp part is tuned on the basis of Karbasizadeh et al. (2020), Karbasizadeh, Saikumar, and Hoseinnia (2021), Saikumar et al. (2019), and K_p is set so that $\omega_c = 200\pi$, considering the Describing Function (DF) method (Saikumar et al., 2019). In addition, no shaping filter is used for modifying the performance of the reset controller (i.e. $\mathcal{C}_s(s) = 1$). Note that the tuning of the CgLp compensator is not within the scope of this paper, and we only discuss how to assess stability properties of reset control systems with these compensators.

Remark 6. Suppose that the H_β condition is/is not satisfied for the reset control system (8) with $\mathcal{C}_s(s)$, $C_{L_1}(s)$, $C_{L_2}(s)$, $C_R(s)$, $G(s)$, and A_ρ. Then the H_β condition is/is not satisfied for the reset control system (8) with $\mathcal{C}_s(s)$, $C'_{L_1}(s)$, $C'_{L_2}(s)$, $C_R(s)$, $G'(s)$, and A_ρ if $C'_{L_1}(s)C'_{L_2}(s)G'(s) = C_{L_1}(s)C_{L_2}(s)G(s)$ and $G'(s)$ is strictly proper. In other words, the "position" of the reset element does not change in the H_β condition. However, the "position" of the reset element has effects on the performance of the reset control systems (Cai et al., 2020). In the two following examples, the sequence of control filters is such that the tracking error is the input of the reset element and other linear parts following in series.
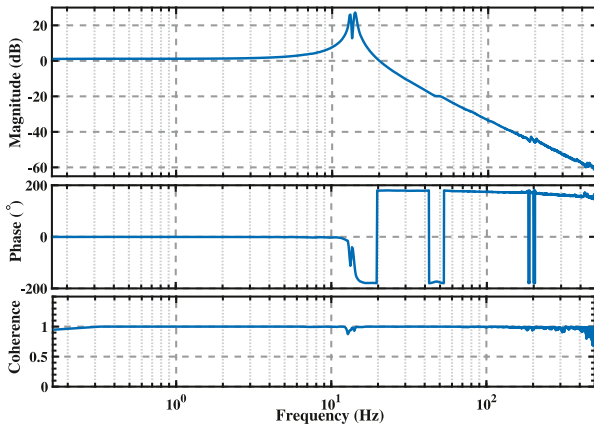
Fig. 5. FRF measurement of considered SISO position of the Spyder stage.

Fig. 6. Step response of the closed-loop system with the designed GSORE for different values of γ_i.

5.1. A reset control system with GSORE

In the case of GSORE, the control parameters are $\gamma_1 = \gamma_2 = 0.5$, $\omega_r = 800\pi$, $\omega_d = 720\pi$, $K_p = 8.5273e^7$, and $\xi = \xi_d = 1$ which leads that the base linear systems have cross-over frequency at 96 Hz with 40° phase margin and 15 dB gain margin. Since the controller has a pole at the origin, we use Definition 5 to assess stability properties of this reset control system. Using Proposition 3 yields $340 < \frac{Q_2}{Q_1} < 5057$ and $1132 < \frac{Q_3}{Q_4}$ for $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Thus, we have to solve the optimization problem $M = \min_{Q_1,Q_2,Q_3,Q_4} G_1(Q_1, Q_2, Q_3, Q_4)$ such that the following constraints hold

$$\mathcal{S}_1 : \forall \omega \in (0, \infty) : \Re(f_1(Q_1, Q_2, 1, \omega)) > 0$$

$$\mathcal{S}_2 : \forall \omega \in (0, \infty) : \Re(f_2(Q_3, Q_4, 1, \omega)) > 0$$

$$\mathcal{S}_3 : \frac{1600\pi}{Q_1} + \frac{Q_2}{Q_1 Q_4} + \frac{2}{Q_1}\sqrt{\frac{1600\pi Q_2}{Q_4} - Q_2} > 1$$

$$\mathcal{S}_4 : \frac{1600\pi}{Q_1} + \frac{Q_2}{Q_1 Q_4} - \frac{2}{Q_1}\sqrt{\frac{1600\pi Q_2}{Q_4} - Q_2} < 1$$

$$\mathcal{S}_5 : \frac{640000\pi^2 Q_1}{Q_2} + 1600\pi\left(1 + 2\sqrt{\frac{1600\pi Q_1}{Q_2} - 1}\right) > \frac{Q_3}{Q_4}$$

$$\mathcal{S}_6 : \frac{640000\pi^2 Q_1}{Q_2} + 1600\pi\left(1 - 2\sqrt{\frac{1600\pi Q_1}{Q_2} - 1}\right) < \frac{Q_3}{Q_4}$$

$$\mathcal{S}_7 : Q_i > 0, \ 1600\pi > Q_4, \ 1600\pi < \frac{Q_2}{Q_1} < 5057, \ 1132 < \frac{Q_3}{Q_4},$$

$$\mathcal{S}_8 : \frac{Q_1 Q_3}{Q_2 Q_4} > 1, \tag{28}$$

This optimization problem is solved using Genetic Algorithm and Proposition 3. The optimal solution is $Q_1 = 13172$, $Q_2 = 12001144$, $Q_3 = 8113151$, and $Q_4 = 1055$, yielding $M = 3.65$ (note that it is not necessary to find the global minimum in these methods). Furthermore, $(\bar{A}, C_0)$ is observable and $(\bar{A}, B_0)$ is controllable. Hence, the reset control system is of Type III and using Theorem 4 this GSORE has the UBIBS property for $A_\rho = \gamma I$, $-1 < \gamma < 1$. Furthermore, since $\frac{Q_1 Q_3}{Q_2 Q_4} > \Gamma(-0.5, 0.5)$ and $\frac{Q_1 Q_3}{Q_2 Q_4} > \Gamma(0.5, -0.5)$, Theorem 4 holds for the considered closed-loop system with $A_\rho = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$ or $A_\rho = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$. In Fig. 6 the step responses of the closed-loop Spider stage (Fig. 4) with
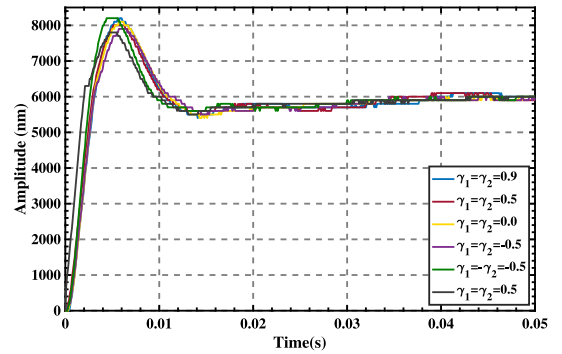
the designed controller for different values of γ_i are displayed. As it can be observed, the values of γ_i have effect on the performance of the system. In the sense of transient response, the reset controller with $\gamma_1 = \gamma_2 = 0.5$ has better performance among other configurations (for more details see Karbasizadeh et al. (2020), Saikumar et al. (2019)). Note that it is possible to solve this problem using a LMI solver if we fit a model to the FRF measurements (Fig. 5). However, this approach has the following disadvantages:

- the fitted model is not as accurate as the FRF measurements;
- solving LMIs may be challenging as the order of the model increases.

5.2. Reset control system with SOSRE

In the case in which the controller is a SOSRE the control parameters are $-1 < \gamma < 1$, $\omega_r = 150\pi$, $\omega_d = 96\pi$, $K_p = 1.135e^6$, and $\xi = \xi_d = 1$ which leads that the base linear systems have cross-over frequency at 70 Hz with 50° phase margin and 14 dB gain margin. Since the controller has a pole at the origin, we use Definition 3 with the NSV defined in Corollary 3 to assess stability properties. The phase of the NSV for this example is shown in Fig. 7. Since the phase of the NSV for this example is between $(-\frac{\pi}{2}, \pi)$ and the difference between its maximum and its minimum is less than π, by Remark 3 the reset control system is of Type I. Moreover, in the implementation, successive reset instants are prevented. In other words, the system resets if the input is zero and reset does not occur at the previous sample which lead to the well-posedness property of this reset control system. Note that this does not affect the results obtained from the H_β condition since the reset instants in this case are a subset of the reset instants considered in the H_β condition. Consequently, by Corollary 3 the designed SOSRE yields a closed-loop system which has the UBIBS property. The step responses of the closed-loop Spider stage (Fig. 4) with the designed controller for different values of γ are shown in Fig. 8. In the sense of transient response, reset control system with $\gamma = 0.5$ has better performance among other controllers. For deeper insights on the performance of closed-loop reset control systems with SOSRE see Karbasizadeh et al. (2020, 2021).

6. Conclusion

In this paper a novel frequency-domain approach based on the H_β condition for assessing stability properties of reset control systems has been proposed. This method can be used to determine stability properties of control systems with first and
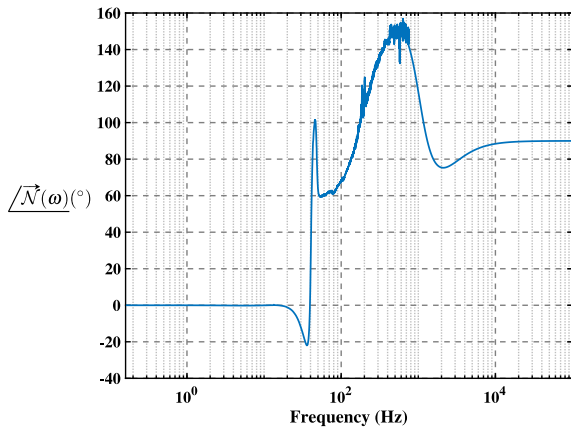
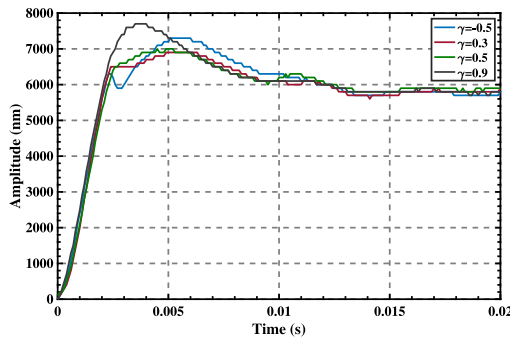**Fig. 7.** $\angle\vec{N}(\omega)$ for the reset control systems with SOSORE.

Fig. 8. Step response of the closed-loop system with the designed SOSRE for different values of γ.

second order reset elements using FRF measurements of their base linear open-loop system. Consequently, the methods do not need an accurate parametric model of the system and the solution of LMIs. In addition, these methods are applicable to the case in which partial reset techniques are used. The effectiveness of the proposed methods have been illustrated with a practical example.

Acknowledgments

Appendix A. Proof of Lemma 2

It has been shown in Beker et al. (2004) that when $A_\rho = 0$, $\mathcal{C}_s(s) = 1$, Assumption 2 holds, and the H_β condition is satisfied, the reset control system has the UBIBS property. In what follows, we provide a slight modification of the proof in Beker et al. (2004) to deal with the case $A_\rho \neq 0$. The base linear dynamic of the reset control system is given by

$$\begin{cases} \dot{x}_l(t) = \bar{A}x_l(t) + \bar{B}w(t), \\ y_l(t) = \bar{C}x_l(t), \end{cases} \tag{A.1}$$

where $x_l(t) = [x_{r_l}(t)^T \; \zeta_l(t)^T]^T \in \mathbb{R}^{n_p + n_r}$. Denoting $z(t) : x(t) - x_l(t) = [z_r(t)^T \; z_p(t)^T]^T$, yields

$$\begin{cases} \dot{z}(t) = \bar{A}z(t), & e(t) \neq 0, \\ z(t^+) = \bar{A}_\rho z(t) + (\bar{A}_\rho - I)x_l(t), & e(t) = 0 \wedge z(t^+) \neq 0. \end{cases} \tag{A.2}$$

According to Beker et al. (2004), it is sufficient to show that $z(t)$ is bounded. Since the H_β condition is satisfied, there exists a matrix $P = P^T > 0$ such that

$$P = \begin{bmatrix} \varrho & \beta\bar{C}_e \\ (\beta\bar{C}_e)^T & P_1 \end{bmatrix}, \; P_1 = P_1^T > 0. \tag{A.3}$$

Consider now the quadratic Lyapunov function $V(t) = z(t)^T P z(t)$. Using the same procedure as in Beker et al. (2004) yields

$$V(t) \leq e^{-\varepsilon(t - t_i)}V(t_i), \quad t \in (t_i, t_{i+1}], \; \varepsilon > 0, \tag{A.4}$$

and

$$V(t_i^+) = V(t_i) + x_r^T(t_i)(A_\rho^T \varrho A_\rho - \varrho)x_r(t_i) \\ + 2x_r^T(t_i)(A_\rho^T - I)\left(\beta\bar{C}_e z_p(t_i) - \varrho \, x_{r_l}(t_i)\right), \tag{A.5}$$

in which t_i are the reset instants. Now, let the maximum eigenvalue of $A_\rho^T \varrho A_\rho - \varrho$ be $\lambda_{\max}$ and note that $\lambda_{\max} < 0$ since $A_\rho^T \varrho A_\rho - \varrho < 0$. As a result

$$V(t_i^+) \leq V(t_i) - |\lambda_{\max}|x_r^T(t_i)x_r(t_i) + 2x_r^T(t_i)(A_\rho^T - I)\left(\beta\bar{C}_e z_p(t_i) \\ - \varrho \, x_{r_l}(t_i)\right) \Rightarrow \\ V(t_i^+) \leq V(t_i) + 2\|x_r(t_i)\| \, \| \, A_\rho^T - I\|(\|\beta\bar{C}_e z_p(t_i)\| \\ + \| \, \varrho \, x_{r_l}(t_i)\|). \tag{A.6}$$

At the reset instants $|\bar{C}_e z_p(t_i)| \leq |D_e r(t)|$ which implies that $|\bar{C}_e z_p(t_i)|$ is bounded. Moreover, since the base linear system is stable, $x_{r_l}(t_i)$ is bounded. Assume that $\lim_{t_i \to \infty} x_r(t_i) = \infty$, by (A.6) we obtain that $V(t^+) < V(t)$. Now, considering (A.4), we conclude that $\lim_{i \to \infty} V(t_i) = 0$. This is a contradiction because $z(t) = 0 \Rightarrow x(t) = x_l(t)$ which implies that the system is a stable linear system with bounded state. Therefore, $x_r(t_i)$ is bounded. Now, we prove that $\dot{x}_r(t_i)$ is bounded. If reset happens when the input of the reset element is zero (i.e. $\mathcal{C}_s(s) = 1$) and Assumption 2 holds, then

$$\frac{dx_r(t)}{dt}\Big|_{t=t_i^-} = A_r\left(e^{A_r(t_i - t_{i-1})}x_r(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{A_r(t_i - \tau)}B_r e(\tau)d\tau\right) \\ = A_r x_r(t_i) \Rightarrow \left|\dot{x}_r(t_i^-)\right| = |A_r x_r(t_i)|. \tag{A.7}$$

Thus, since $|x_r(t_i)|$ is bounded, $\left|\dot{x}_r(t_i^-)\right|$ is bounded. As a result, since $|x_r(t_{i-1}^+)| \leq |A_\rho||x_r(t_{i-1})|$, $|x_r(t_i)|$ and $\left|\dot{x}_r(t_i^-)\right|$ are bounded,

$$\exists \, K_1 > 0, \; \alpha > 0 \text{ such that } |x_r(t_i)| \leq K_1\left(1 - e^{\alpha(t_i - t_{i-1})}\right), \; \forall \, t_i. \tag{A.8}$$

Now we want to proof that there exists $\lambda > 0$ such that, for all $k \in \mathbb{N}$, $\lambda \leq t_{k+1} - t_k$. To prove this, assume that there exist reset instants t_i and t_{i-1} such that for any $\varepsilon > 0$, $t_i - t_{i-1} < \varepsilon$. Thus, by (A.8) and for sufficient small ε, $x_r(t_i) \to 0$. This is a contradiction because $(I - \bar{A}_\rho)x_r(t_i) \to 0$ which means that t_i is not a reset instant. Thus, there exists $\lambda > 0$ such that, for all $k \in \mathbb{N}$, $\lambda \leq t_{k+1} - t_k$. Therefore, the reset instants have the well-posedness property (see Definition 1).

In the case in which $\mathcal{C}_s = 1$ or Assumption 2 does not hold, (A.7) cannot be concluded. However, if the well-posedness property of the reset instants holds, then there exists $\lambda > 0$ such that, for all $k \in \mathbb{N}$, $\lambda \leq t_{k+1} - t_k$. In addition, since $|x_r(t_{i-1}^+)| \leq |A_\rho||x_r(t_{i-1})|$ and $|x_r(t_i)|$ are bounded, we conclude (A.8). Since the system has the well-posedness property, the reset control system (8) has a unique well-defined solution for any initial condition x_0
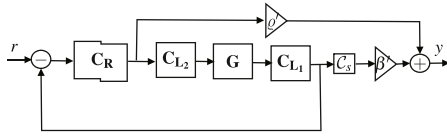
Fig. B.1. The block diagram of the H_β condition for the closed-loop architecture Fig. 1 with GFORE or PCI.

and any input w which is a Bohl function (Banos et al., 2016). The rest of the proof is the same as the proof in Beker et al. (2004).

Appendix B. Proof of Theorem 2

For $w(t) = 0$, for all $t \geq 0$, reset happens when $x(t) \in \ker(\bar{C}_e)$. Looking at the proof of the H_β condition, which is given in Baños and Barreiro (2011), Beker et al. (2004) and Guo et al. (2015), when there is a shaping filter in the reset line, C_0 in the H_β condition is changed to

$$C_0 = [\varrho \quad \beta \bar{C}_e]. \tag{B.1}$$

Theorem 2 is now proved in several steps.

- Step 1: It is shown that there is a β and $\varrho > 0$ such that $\Re(H(j\omega)) > 0$, for all $\omega \in \mathbb{R}^+$.
- Step 2: For systems with poles at the origin it is shown that $\lim_{\omega \to 0} \Re(H(j\omega)) > 0$.
- Step 3: It is shown that either $\lim_{s \to \infty} H(s) > 0$ or $\lim_{\omega \to \infty} \omega^2 \Re(H(j\omega)) > 0$.
- Step 4: It is shown that (A, C_0) and (A, B_0) are observable and controllable, respectively.

Step 1: For simplicity take $\beta' = -\beta$ and $\varrho' = \frac{\varrho}{C_f}$. The transfer function (10) with the modified C_0 as in (B.1) can be rewritten as (see also Fig. B.1)

$$H(s) = \frac{y}{r} = \frac{\beta' L(s) C_s(s) + \varrho' C_R(s)}{1 + L(s)}. \tag{B.2}$$

Thus[5]

$$\Re(H(j\omega)) = \frac{\beta' \mathcal{N}_\chi + \varrho' \mathcal{N}_\Upsilon}{(a+1)^2 + b^2}. \tag{B.3}$$

Define now the vector $\vec{\xi}$ in the $\chi - \Upsilon$ plane as $\vec{\xi} = [\beta' \quad \varrho']^T$. Using Definition 2, Eq. (B.3) can be re-written as

$$\Re(H(j\omega)) = \frac{\vec{\xi} \cdot \vec{\mathcal{N}}}{(a+1)^2 + b^2}. \tag{B.4}$$

Therefore

$$\forall \omega \in \mathbb{R}^+ : \Re(H(j\omega)) > 0 \iff \vec{\xi} \cdot \vec{\mathcal{N}} > 0 \iff \\ -\frac{\pi}{2} < \underline{/(\vec{\xi}, \vec{\mathcal{N}})} < \frac{\pi}{2} \land |\vec{\mathcal{N}}| \neq 0 \land |\vec{\xi}| \neq 0. \tag{B.5}$$

The rest of the proof of this step are the same as the proof of Step 1 provided in Dastjerdi et al. (2020).

Step 2: When the open-loop system has poles at the origin and C_R is a GFORE, Eq. (B.2) becomes

$$\lim_{\omega \to 0} \Re(H(j\omega)) = K_{s_0} \beta' > 0, \tag{B.6}$$

whereas in the case of PCI and CI when $C_{L_1}(s) C_{L_2}(s) G(s)$ does not have any pole at the origin, (B.2) becomes

$$\lim_{\omega \to 0} \Re(H(j\omega)) = K_{s_0} \beta' + \varrho' \frac{\omega_r}{C_{L_1}(0) C_{L_2}(0) G(0)} > 0. \tag{B.7}$$

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[5] Omitting arguments for simplicity.

Setting $\overrightarrow{\mathcal{N}'} = [K_{s_0} \quad \frac{\omega_r}{C_{L_1}(0) C_{L_2}(0) G(0)}]^T$, yields

$$\lim_{\omega \to 0} \Re(H(j\omega)) = \vec{\xi} \cdot \overrightarrow{\mathcal{N}'}. \tag{B.8}$$

In addition

$$\underline{/\mathcal{N}'} = \lim_{\omega \to 0} \underline{/\mathcal{N}} \xrightarrow{(13)} \theta_1 \leq \underline{/\mathcal{N}'} \leq \theta_2. \tag{B.9}$$

As a result, by Step 1, $\lim_{\omega \to 0} \Re(H(j\omega)) = \vec{\xi} \cdot \overrightarrow{\mathcal{N}'} > 0$. For PCI, when $C_{L_1}(s) C_{L_2}(s) G(s)$ has poles at the origin,

$$\lim_{\omega \to 0} \Re(H(j\omega)) = K_{s_0} \beta' > 0. \tag{B.10}$$

Note that for CI in Eqs. (B.7)–(B.9), $\omega_r = 1$. It is therefore concluded that if $C_{L_1}(s) C_{L_2}(s) G(s)$ has poles at the origin, then $K_{s_0} \beta' > 0$. If $C_{L_1}(s) C_{L_2}(s) G(s)$ does not have any pole at the origin, β can be either positive or negative.

Step 3: In the case of GFORE with $n - m = 2$, setting $\overrightarrow{\mathcal{N}''} = [-K_n \quad \omega_r^2]^T$ yields

$$\lim_{\omega \to \infty} \omega^2 \Re(H(j\omega)) = -\beta' K_n + \varrho' \omega_r^2 = \vec{\xi} \cdot \overrightarrow{\mathcal{N}''}. \tag{B.11}$$

In addition,

$$\underline{/\mathcal{N}''} = \lim_{\omega \to \infty} \underline{/\mathcal{N}} \xrightarrow{(13)} \theta_1 \leq \underline{/\mathcal{N}''} \leq \theta_2. \tag{B.12}$$

Thus, by Step 1 $\lim_{\omega \to \infty} \omega^2 \Re(H(j\omega)) = \vec{\xi} \cdot \overrightarrow{\mathcal{N}''} > 0$. For GFORE with $n - m > 2$, $\lim_{\omega \to \infty} \omega^2 \Re(H(j\omega)) = \varrho' \omega_r^2 > 0$. For PCI $\lim_{s \to \infty} H(s) = \varrho' > 0$. Moreover, in the case of CI when $n - m > 2$,

$$\lim_{\omega \to \infty} \omega^2 \Re(H(j\omega)) = 0, \tag{B.13}$$

which implies that $H(s)$ is not SPR in the case of $n - m > 2$. Whereas in the case of CI with $n - m = 2$,

$$\lim_{\omega \to \infty} \omega^2 \Re(H(j\omega)) = -K_{s_0} \beta' > 0, \tag{B.14}$$

which means that in the case of CI, $C_{L_1}(s) C_{L_2}(s) G(s)$ must not have any pole at the origin.

Step 4: In order to show that the pairs (A, C_0) and (A, B_0) are observable and controllable, respectively, it is sufficient to show that the denominator and the numerator of $H(s)$ do not have any common root. Let $a_0 + jb_0$ be a root of the denominator. Then

$$1 + R_L(a_0, b_0) + j I_L(a_0, b_0) = 0 \Rightarrow \begin{cases} R_L(a_0, b_0) = -1, \\ I_L(a_0, b_0) = 0. \end{cases} \tag{B.15}$$

Now, the numerator must not have a root at $a_0 + jb_0$, that is

$$\beta' \left(R_{C_s}(a_0, b_0) + j I_{C_s}(a_0, b_0) \right) \neq \varrho' \left(R_{C_R}(a_0, b_0) + j I_{C_R}(a_0, b_0) \right) \\ \Rightarrow \beta' R_{C_s}(a_0, b_0) \neq \varrho' R_{C_R}(a_0, b_0) \lor \beta' I_{C_s}(a_0, b_0) \neq \varrho' I_{C_R}(a_0, b_0). \tag{B.16}$$

Therefore, using Step 1 and (B.16) it is possible to find a pair (β', ϱ') such that $H(s)$ does not have any pole-zero cancellation. According to Step 1–4, $H(s)$ is SPR (Khalil & Grizzle, 2002), $(\bar{A}, C_0)$ is observable and $(\bar{A}, B_0)$ is controllable, and the base linear system is stable. Moreover, since $-1 < \gamma < 1$, one has that $A_\rho^T \varrho A_\rho - \varrho < 0$. As a result, the H_β condition is satisfied for the reset control system (8) with GFORE (2), or CI (3), or PCI (4). Hence, the zero equilibrium of the reset control system (8) is globally uniformly asymptotically stable when $w = 0$, and according to Lemma 2, it has the UBIBS property for any initial condition x_0 and any input w which is a Bohl function.
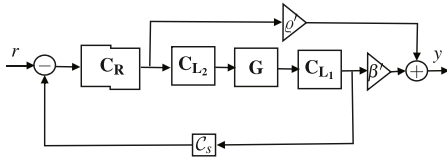
Fig. C.1. The block diagram of H_β condition for the modified architecture Fig. 3 with GFORE or PCI.

Appendix C. Proof of Corollary 2

Let $\beta' = -\beta$ and $\varrho' = \dfrac{\varrho}{C_r}$. By the proof of the H_β condition in Beker et al. (2004) the transfer function (10) for the configuration shown in Fig. 3 can be rewritten as (see also) Fig. C.1

$$H(s) = \frac{\beta' \dfrac{L'(s)}{C_s(s)} + \varrho' \, C_R(s)}{1 + L'(s)}. \tag{C.1}$$

Let $C_{L_1}(s)C_{L_2}(s)C_R(s)G(s) = \dfrac{k_m s^m + k_{m-1}s^{m-1} + \cdots + k_0}{s^n + k'_{n-1}s^{n-1} + \cdots + k'_0}$. Using the NSV defined in (20), one could repeat Steps 1 to 4 of the proof of Theorem 2. Note that $K_{s_0}\beta'$ in (B.6)–(B.10) and (B.14) has to be replaced by $\dfrac{\beta'}{K_{s_0}}$ and K_n has also to be replaced by k_n in (B.11).

Appendix D. Proof of Proposition 3

Consider $Q_1\mathcal{F}_1(\omega) + Q_2\mathcal{F}_2(\omega)$ as the scalar product of the two vectors $\vec{\mathcal{F}}(\omega)$ and $\vec{\mathcal{Q}}$. Thus, for all $\omega \in \mathbb{R}^+$, the condition (18) can be re-written as

$$\sqrt{Q_1^2 + Q_2^2}\sqrt{\mathcal{F}_1^2(\omega) + \mathcal{F}_2^2(\omega)}\cos(\vartheta) > \mathcal{F}_3(\omega). \tag{D.1}$$

As a result, when $\mathcal{F}_3(\omega) \geq 0$, $\cos(\vartheta)$ must be positive and

$$\sqrt{Q_1^2 + Q_2^2} > \max_{\omega \in \omega_p} \frac{\mathcal{F}_3(\omega)}{\cos(\vartheta)\sqrt{\mathcal{F}_1^2(\omega) + \mathcal{F}_2^2(\omega)}} = \eta_1(\frac{Q_2}{Q_1}). \tag{D.2}$$

Positivity of $\cos(\vartheta)$ implies $\dfrac{Q_2}{Q_1} \in g_p$. When $\mathcal{F}_3(\omega) < 0$, there are two solutions for condition (D.1). $\cos(\vartheta) \geq 0$ which requires $\dfrac{Q_2}{Q_1} \in g_N$, or

$$\sqrt{Q_1^2 + Q_2^2} < \min_{\omega \in \omega_N} \frac{\mathcal{F}_3(\omega)}{\cos(\vartheta)\sqrt{\mathcal{F}_1^2(\omega) + \mathcal{F}_2^2(\omega)}} = \eta_2(\frac{Q_2}{Q_1}). \tag{D.3}$$

Therefore, by (D.2) and (D.3) $\eta_2(\dfrac{Q_2}{Q_1}) > \eta_1(\dfrac{Q_2}{Q_1})$ and the proof is complete.

Appendix E. Proof of Theorem 4

Theorem 4 is proved in the following steps.

- Step 1: The transfer function $H(s)$ in (10) for the reset control system (8) with GSORE (6) is calculated. Then, it is shown that $A_\rho^T \varrho A_\rho - \varrho < 0$.
- Step 2: It is shown that $\lim_{\omega \to \infty} \omega^2(H(j\omega) + H(-j\omega)^T) > 0$.
- Step 3: For systems with poles at the origin it is shown that $\lim_{\omega \to 0} H(j\omega) + H(-j\omega)^T > 0$.
- Step 4: It is shown that $H(j\omega) + H(-j\omega)^T > 0$, for all $\omega \in \mathbb{R}^+$.

Step 1: In the case of GSORE let $\beta = -\begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}$ and $\varrho = \begin{bmatrix} \varrho_1 & \varrho_2 \\ \varrho_2 & \varrho_3 \end{bmatrix} > 0$ be such that

$$\beta_i \in \mathbb{R}, \quad \varrho_3 > 0, \quad \varrho_1 > 0, \quad \varrho_1\varrho_3 > \varrho_2^2. \tag{E.1}$$

In addition, since $A_\rho = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$, we have the condition

$$A_\rho^T \varrho A_\rho - \varrho = \begin{bmatrix} (\gamma_1^2 - 1)\varrho_1 & (\gamma_1\gamma_2 - 1)\varrho_2 \\ (\gamma_1\gamma_2 - 1)\varrho_2 & (\gamma_2^2 - 1)\varrho_3 \end{bmatrix} < 0. \tag{E.2}$$

Since $-1 < \gamma_i < 1$, using (E.1) and (E.2), yields

$$\frac{\varrho_1\varrho_3}{\varrho_2^2} > \Gamma(\gamma_1, \gamma_2) = \frac{(\gamma_1\gamma_2 - 1)^2}{(\gamma_1^2 - 1)(\gamma_2^2 - 1)} \geq 1. \tag{E.3}$$

With the considered matrix ϱ and vector β, $H(s)$ in (10) with C_0 as in (B.1) is equal to (see also Fig. E.1)

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}. \tag{E.4}$$

in which $H_{ij}(s)$ with $i, j = 1, 2$ is transfer function from r_j to y_i. Thus, $H(j\omega) + H(-j\omega)^T$ is equal to

$$\begin{bmatrix} 2\Re(H_{11}(j\omega)) & H_{12}(j\omega) + H_{21}(-j\omega) \\ H_{12}(-j\omega) + H_{21}(j\omega) & 2\Re(H_{22}(j\omega)) \end{bmatrix} > 0 \Rightarrow \tag{E.5}$$

$$\frac{1}{|\kappa(\omega)|^2}\begin{bmatrix} 2\Re(f_1(\varrho_1, \varrho_2, \beta_1, \omega)) & f_1(\varrho_2, \varrho_3, \beta_2, \omega) + f_2(\varrho_2, \varrho_1, \beta_1, -\omega) \\ f_1(\varrho_2, \varrho_3, \beta_2, -\omega) + f_2(\varrho_2, \varrho_1, \beta_1, \omega) & 2\Re(f_2(\varrho_3, \varrho_2, \beta_2, \omega)) \end{bmatrix} > 0. \tag{E.6}$$

Step 2: Since the transfer functions $\dfrac{y_i}{r_j}$, with $i, j = 1, 2$, are strictly proper, $\lim_{s \to \infty} H(s) = 0$. Therefore, it is necessary to have $\lim_{\omega \to \infty} \omega^2(H(j\omega) + H(-j\omega)^T) > 0$. Note that in the case of SORE, $n - m \geq 3$. By (E.6), if $n - m > 3$, $\lim_{\omega \to \infty} \omega^2(H(j\omega) + H(-j\omega)^T)$ is equal to

$$\begin{bmatrix} 4 \varrho_1 \xi\omega_r - 2 \varrho_2 & \omega_r^2 \varrho_1 + 2 \varrho_2 \xi\omega_r - \varrho_3 \\ \omega_r^2 \varrho_1 + 2 \varrho_2 \xi\omega_r - \varrho_3 & 2\omega_r^2 \varrho_2 \end{bmatrix}. \tag{E.7}$$

Therefore, the condition $\lim_{\omega \to \infty} \omega^2(H(j\omega) + H(-j\omega)^T) > 0$ is equivalent to

$$2 \varrho_1 \xi\omega_r > \varrho_2, \quad \varrho_2 > 0, \tag{E.8}$$

and

$$4(2 \varrho_1 \xi\omega_r - \varrho_2)(\omega_r^2 \varrho_2) > (\omega_r^2 \varrho_1 + 2 \varrho_2 \xi\omega_r - \varrho_3)^2$$
$$\Downarrow$$
$$\varrho_3^2 - 2 \varrho_3 (\omega_r^2 \varrho_1 + 2\xi\omega_r \varrho_2) + (\omega_r^2 \varrho_1 - 2\xi\omega_r \varrho_2)^2 + 4\omega_r^2 \varrho_2^2 < 0$$
$$\Downarrow$$
$$\left(\varrho_3 > (\omega_r^2 \varrho_1 + 2\xi\omega_r \varrho_2) - 2\omega_r\sqrt{2\xi\omega_r \varrho_1\varrho_2 - \varrho_2^2}\right) \wedge$$
$$\left(\varrho_3 < (\omega_r^2 \varrho_1 + 2\xi\omega_r \varrho_2) + 2\omega_r\sqrt{2\xi\omega_r \varrho_1\varrho_2 - \varrho_2^2}\right). \tag{E.9}$$

When $n - m = 3$, condition (E.7) is re-written as

$$\begin{bmatrix} 4 \varrho_1 \xi\omega_r - 2 \varrho_2 & \omega_r^2 \varrho_1 + 2 \varrho_2 \xi\omega_r - \varrho_3 - K_n\beta_1 \\ \omega_r^2 \varrho_1 + 2 \varrho_2 \xi\omega_r - \varrho_3 - K_n\beta_1 & 2\omega_r^2 \varrho_2 - 2K_n\beta_2 \end{bmatrix} > 0, \tag{E.10}$$

which is equivalent to

$$2 \varrho_1 \xi\omega_r > \varrho_2, \quad \omega_r^2 \varrho_2 > K_n\beta_2, \tag{E.11}$$

and

$$4(2\,\varrho_1\,\xi\omega_r - \varrho_2)(\omega_r^2\,\varrho_2 - K_n\beta_2) > (\omega_r^2\,\varrho_1 + 2\,\varrho_2\,\xi\omega_r - \varrho_3 - K_n\beta_1)^2$$
$$\Downarrow$$
$$\left(\varrho_3 > (\omega_r^2\,\varrho_1 - K_n\beta_1 + 2\xi\omega_r\,\varrho_2) - 2\sqrt{2\xi\omega_r^3\,\varrho_1\varrho_2 + K_n\,\varrho_2\,\beta_2 - \omega_r^2\,\varrho_2^2}\right) \wedge$$
$$\left(\varrho_3 < (\omega_r^2\,\varrho_1 - K_n\beta_1 + 2\xi\omega_r\,\varrho_2) + 2\sqrt{2\xi\omega_r^3\,\varrho_1\varrho_2 + K_n\,\varrho_2\,\beta_2 - \omega_r^2\,\varrho_2^2}\right) \wedge$$
$$\left(2\xi\omega_r^3\,\varrho_1\varrho_2 + K_n\,\varrho_2\,\beta_2 > \omega_r^2\,\varrho_2^2\right). \tag{E.12}$$

Step 3: When $L(s)$ has at least one pole at the origin, by (E.6) $\lim_{\omega\to 0} H(j\omega) + H(-j\omega)^T$ is equal to

$$\begin{bmatrix} 2K_{s_0}\beta_1 & K_{s_0}\beta_2 + 2K_{s_0}\beta_1\xi\omega_r - \varrho_1 \\ K_{s_0}\beta_2 + 2K_{s_0}\beta_1\xi\omega_r - \varrho_1 & 4K_{s_0}\beta_2\xi\omega_r - 2\,\varrho_2 \end{bmatrix} > 0, \tag{E.13}$$

which is equivalent to

$$K_{s_0}\beta_1 > 0, \quad 2K_{s_0}\beta_2\xi\omega_r > \varrho_2, \tag{E.14}$$

and

$$4(K_{s_0}\beta_1)(2K_{s_0}\beta_2\xi\omega_r - \varrho_2) > (K_{s_0}\beta_2 + 2K_{s_0}\beta_1\xi\omega_r - \varrho_1)^2$$
$$\Downarrow$$
$$\left(\varrho_1 > K_{s_0}(2\beta_1\xi\omega_r + \beta_2) - 2\sqrt{2K_{s_0}^2\xi\omega_r\beta_1\beta_2 - K_{s_0}\beta_1\,\varrho_2}\right) \wedge \tag{E.15}$$
$$\left(\varrho_1 < K_{s_0}(2\beta_1\xi\omega_r + \beta_2) + 2\sqrt{2K_{s_0}^2\xi\omega_r\beta_1\beta_2 - K_{s_0}\beta_1\,\varrho_2}\right).$$

Step 4: In the case in which $L(s)$ has poles at the origin, denote $Q_1 = \frac{\varrho_1}{\beta_1}$, $Q_2 = \frac{\varrho_2}{\beta_1}$, $Q_3 = \frac{\varrho_3}{\beta_2}$ and $Q_4 = \frac{\varrho_2}{\beta_2}$. Furthermore, since $K_{s_0}\beta_1$, $K_{s_0}\beta_2$, and $|\kappa(\omega)|^2$ are positive, condition (E.6) is equal to

$$\begin{bmatrix} 2K_{s_0}\Re(f_1(Q_1, Q_2, 1, \omega)) & \\ f_1(Q_4, Q_3, 1, \omega) + f_2(Q_4, \frac{Q_1Q_4}{Q_2}, \frac{Q_4}{Q_2}, -\omega) & \\ f_1(Q_2, \frac{Q_2Q_3}{Q_4}, \frac{Q_2}{Q_4}, -\omega) + f_2(Q_2, Q_1, 1, \omega) & \\ \frac{2}{K_{s_0}}\Re(f_2(Q_3, Q_4, 1, \omega)) & \end{bmatrix} > 0. \tag{E.16}$$

Therefore, for all $\omega \in (0, \infty)$, there exist Q_1, Q_2, Q_3, and Q_4 such that

$$K_{s_0}\Re(f_1(Q_1, Q_2, 1, \omega)) > 0, \quad K_{s_0}\Re(f_2(Q_3, Q_4, 1, \omega)) > 0, \tag{E.17}$$

and since $\Re(f_1(Q_1, Q_2, 1, \omega))\Re(f_2(Q_3, Q_4, 1, \omega)) > 0$,

$$4 > \frac{f_1(Q_2, \frac{Q_2Q_3}{Q_4}, \frac{Q_2}{Q_4}, -\omega) + f_2(Q_2, Q_1, 1, \omega)}{\Re(f_1(Q_1, Q_2, 1, \omega))}$$
$$\times \frac{f_1(Q_4, Q_3, 1, \omega) + f_2(Q_4, \frac{Q_1Q_4}{Q_2}, \frac{Q_4}{Q_2}, -\omega)}{\Re(f_2(Q_3, Q_4, 1, \omega))}. \tag{E.18}$$

Thus, since the condition (E.18) must hold for all $\omega \in (0, \infty)$, $\min_{Q_i} G_1(Q_1, Q_2, Q_3, Q_4) < 4$, with $i = 1, 2, 3, 4$. Moreover, re-writing Eqs. (E.1), (E.3) (E.9), and (E.15) using the variables Q_1, Q_2, Q_3, and Q_4, the constraints $\mathcal{S}_3 - \mathcal{S}_8$ of Definition 5 are obtained.

When $L(s)$ does not have any pole at the origin, let $Q_1' = \frac{\beta_1}{\varrho_1}$, $Q_2' = \frac{\varrho_2}{\varrho_1}$, $Q_3' = \frac{\beta_2}{\varrho_3}$ and $Q_4' = \frac{\varrho_2}{\varrho_3}$. With this change of variables, since ϱ_3, ϱ_1 and $|\kappa(\omega)|^2$ are positive, condition (E.6) is equivalent to

$$\begin{bmatrix} 2\Re(f_1(1, Q_2', Q_1', \omega)) & \\ f_1(Q_4', 1, Q_3', \omega) + f_2(Q_4', \frac{Q_4'}{Q_2'}, \frac{Q_1'Q_4'}{Q_2'}, -\omega) & \end{bmatrix}$$
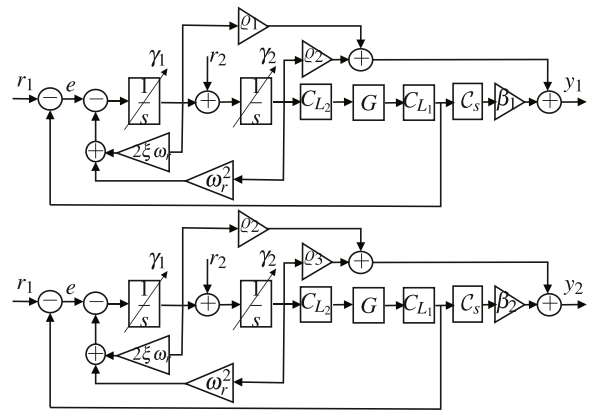

Fig. E.1. The block diagram of the H_β condition for the closed-loop architecture Fig. 1 with GSORE.

$$\begin{bmatrix} f_1(Q_2', \frac{Q_2'}{Q_4'}, \frac{Q_2'Q_3'}{Q_4'}, -\omega) + f_2(Q_2', 1, Q_1', \omega) & \\ 2\Re(f_2(1, Q_4', Q_3', \omega)) & \end{bmatrix} > 0. \tag{E.19}$$

This implies that $\Re(f_1(1, Q_2', Q_1', \omega)) > 0$, $\Re(f_2(1, Q_4', Q_3', \omega)) > 0$, and since $\Re(f_1(1, Q_2', Q_1', \omega))\Re(f_2(1, Q_4', Q_3', \omega)) > 0$,

$$4 > \frac{f_1(Q_2', \frac{Q_2'}{Q_4'}, \frac{Q_2'Q_3'}{Q_4'}, -\omega) + f_2(Q_2', 1, Q_1', \omega)}{\Re(f_1(1, Q_2', Q_1', \omega))}$$
$$\times \frac{f_1(Q_4', 1, Q_3', \omega) + f_2(Q_4', \frac{Q_4'}{Q_2'}, \frac{Q_1'Q_4'}{Q_2'}, -\omega)}{\Re(f_2(1, Q_4', Q_3', \omega))}. \tag{E.20}$$

Therefore, since condition (E.20) must hold for all $\omega \in [0, \infty)$, $\min_{Q_i'} G_2(Q_1', Q_2', Q_3', Q_4') < 4$, with $i = 1, 2, 3, 4$. Re-writing Eqs. (E.3) and (E.9) with the variables Q_1', Q_2', Q_3', and Q_4', the constraints $\mathcal{S}_3 - \mathcal{S}_5$ of Definition 6 are achieved. Similarly, using these variables in Eqs. (E.3) and (E.12), the constraints $\mathcal{S}_3 - \mathcal{S}_6$ of Definition 7 are obtained.

By Steps 1–4, $A_\rho^T\,\varrho\,A_\rho - \varrho < 0$, $H(s)$ is SPR (Khalil & Grizzle, 2002), $(\bar{A}, C_0)$ is observable and $(\bar{A}, B_0)$ is controllable, and the base linear system is stable. Thus, the H_β condition is satisfied for the reset control system (8) with GSORE (6). Hence, the zero equilibrium of the system is globally uniformly asymptotically stable when $w = 0$ and according to Lemma 2, it has the UBIBS property for any initial condition x_0 and any input w which is a Bohl function.

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