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FOUNDATIONS OF QUANTUM PROBABILITY

0. Introduction.

The main thesis advocated in the present paper is that some well established laws of classical probability theory (typically the theorem of composite probabilities) are model-dependent assertions like some theorems of euclidean geometry. In §§ (1) and (2) we briefly describe how this thesis is framed into the wider context of quantum probability and how it can throw some light on the open debate concerning the interpretation of quantum theory as well as the necessity of the use of a non kolmogorovian model in quantum theory. In §§ (3), (4), (5) we show that the general ideas formulated in §§ (1), (2), when given a precise mathematical form, allow to account for all the non standard features of the quantum probability model ($\mathbb{C}$-numbers, Hilbert spaces, ...) as well as to exhibit statistical models which are neither Kolmogorovian nor quantum.

Only the final results will be reported here. The reader interested in the procedure to reach them is referred to [4] for the results of § (3) and, for those of §§ (4), (5), to the paper [2] of which the present one can be considered as a preliminary version. Our main result is Theorem (5.5) in § (5), which gives a necessary and sufficient condition for a family $\{P(x;y): x,y \in T\}$ (T-an arbitrary set) of transition probability matrices to admit a non-Kolmogorovian model of special type (cf. § (4)). It turns out that this compatibility condition is a generalization of the quantum principle of composite amplitudes.

§ 1. On the necessity of non Kolmogorovian models in probability theory.

Quantum theory was born as a new mechanics, but it was soon realized that it provided a new (atthough "strange" and "antiintuitive") probabilistic formalism.

Probabilities in quantum theory are estimated by means of relative frequencies like in many other branches of physics.

However, the theoretical computations of these probabilities are not carried out with the help of the classical probabilistic models, in which events are represented by subsets of a given set, and probabilities by additive set functions: one has to introduce quantities without a direct physical meaning – e.g. wave functions, which are defined by complex numbers whose square moduli give the probability densities – the only observable things.

The appearance of this new probability model arose formidable problems both of conceptual and technical nature.

The first of these problems was just to describe the mathematical structure of the new statistical formalism. D. Hilbert was one of the first mathematicians to realise the importance of the problem and, as early as in 1926-27 he held a seminar in Göttingen on this topic, whose notes, collected by Nordheim and elaborated with the contribution of von Neumann, were published in a joint paper [13].

In the second stage of this development – the study of the connections between the new statistical model and the traditional one, as well as the analysis and the generalization of the mathematical structure of the new model – the figure of J. von Neumann is absolutely outstanding. He introduced the mathematical structures which allow to deal with the two statistical models within a unified mathematical framework, and his attempts to generalize these structures led ultimately, through the developements due to Segal, Gelfand, Dixmier and many others, to an entirely new branch of the mathematical research (the theory of algebras of operators).

The work of Feynman marks the third main stage in the inner developement of the quantum statistical model. The mathematical difficulties connected with the precise definition of the "Feynman integral" are widely compensated by the tremendons intuitive insight it provides into the quantum mechanical processes, and by the fruitfullness of the mathematical developements which it motivated, starting with the work of M. Kac [18], who recasted the ideas of Feynman within the rigorous framework of the functional integral theory previously developed by N. Wiener, Cameron, Martin, ...

The fourth stage of this developement began with the pioneering works of I.E. Segal - motivated by Quantum Field Theory, and consisted in the attempt to include in the general scheme built by von Neumann, not only classical probability theory, but also the theory of stochastic processes. This attempt led, in the late seventies, to the introduction of some new concepts (conditional density matrix, quantum conditional expectations, ...) and tecniques, the quantum Feynman-Kac formula and, more generally the theory of quantum Markov processes which, in particular, yielded explicit examples of some types of functional integrals which are new even in the classical theory (cf. [1] for a survey of some of these topics and references for the other ones).

The above outline of the main stages of the inner developement of the generalized probabilistic models arising in quantum theory, even if very quick and schematic, gives a clear feeling of the fruitfulness of this line of thought both for mathematics and physics.

However, just because this line of thought consists in the inner developement of a given mathematical model, it yields absolutely no information about the question why this model arises, and leaves open the problem – clearly formulated already in the above mentioned paper by Hilbert, Nordheim and von Neumann [13]: "... Therefore the goal is to formulate so completely the physical requirements, that the analytical apparatus becomes uniquely established...".

The present paper is concerned precisely with this problem. Due to the central role of quantum mechanics in contemporary science this problem has been tackled by many authors, both physicists and mathematicians, among which, just to mention a few names beyond von Neumann, we may recall J. Schwinger [23], I.E. Segal [24], A. Landè [17], G.W. Mackey [20], J.M. Jauch [15], ...

The point of view advocated in the present paper, is rooted in some well established traditions of the quantum mechanical thought: it is based on Bohr's idea that the basic objects in quantum theory are the *transition* (or *conditional*) *probabilities* – and this, among other things, allows to avoid the artificial duality observables – states which obscures the fundamental property of the quantum mechanical transition probabilities of being symmetric with respect to the conditioning and the conditioned event (the role of conditional probabilities in Bohr's thought is discussed, for example, in [14]). Also the Hilbert, Nordheim, von Neumann approach [13] is based on conditional probability (densities), while in all his subsequent papers von Neumann abandons this approach in favour of the state-observable one which, since then, has been almost universally adopted.

It shares, with Jordan [16] and Feynman [9], [10] the conviction that the crucial difference between the classical and the quantum probabilistic formalism lies in the non general validity of the "theorem of composite probabilities" which must be substituted by the "principle of composite amplitudes".

It shares with Dirac [7] the conviction that "... Quantum mechanics is based on two fundamental principles, one of negative character, the indeterminacy principle, and one of positive character, the superposition principle. Beside these ones there is the meaning of the square modulus of the wave function as probability density...".

Finally it uses Schwinger's ideas on the "algebra of measurements" [23] to show the deep mathematical connection among the three statements mentioned by Dirac.

Namely, and this is the main result of the present paper, we show that the positive principle of quantum theory (the superposition principle or equivalently the principle of composite amplitudes), the Hilbert space, the choice of complex numbers, the meaning of the square modulus of the wave function ... – in a word, the whole mathematical apparatus of quantum theory – can be deduced from a detailed mathematical analysis of the transition probabilities (i.e. the empirical data) and from a weak formulation of Heisenberg indeterminacy principle.

However, if our results are based on entirely orthodox presuppositions, there is a point in which our interpretation deviates from the orthodox one: it concerns the validity of the theorem of composite probabilities which has always been considered a law of nature, a mere tautology about relative frequencies. For example Feynman explicitely says (cf. [10], pg. 369) that "... Looking at probability from a frequency point of view (4) (*) simply results from the statement that in each experiment giving a and c, B had some value. The only way (4) could be wrong is the statement, "B had some value", must sometimes be meaningless. Noting that (5) replaces (4) only under the circumstance that we make no attempt to measure B, we are led to say that the statement, "B had some value", may be meaningless whenever we make no attempt to measure B. ... "Thus the only way to explain the violation of the principle of composite probabilities should be to forbid the statement: "an observable has a value even if no-body looks at it".

The point of view clearly expressed in these words of Feynman is by no means an isolated one: it can be traced back to Bohr and to the early Heisenberg and it is shared by many physicists.

Other authors prefer to explain the breakdown of the theorem of composite probabilities in the microscopic world, by appealing to the non vali-

(*) In Feynman's notations A, B, C denote observables; a, b, c - their values; P_{ab}, P_{bc}, ... - the transition probabilities and $\varphi_{ab}, \varphi_{bc}$, ... - the transition amplitudes. With these notations (4) and (5) are respectively:

$P_{ac} = \Sigma_b P_{ab} P_{bc}$ (theorem of composite probabilities for Markov chains)

$\varphi_{ac} = \Sigma_b \varphi_{ab} \varphi_{bc}$ (principle of composite amplitudes).

dity, in that domain, of the classical Aristotelean logic. It is well known that such "explanations" have left many people (among which Einstein, Schrödinger, ...) completely unsatisfied. The unpleasant features of these "explanations" have been analyzed by the author in some detail in the monograph [3] and we will not dwell on them here.

The point of view advocated in the present paper is that, the empirical fact of the breakdown of the theorem of composite probabilities in quantum theory, should be considered as an indication that this theorem is not a law of nature but a model-dependent result, just as the theorem that the sum of the inner angles of a triangle is π. This thesis is much less in contrast with the everyday practice of quantum theorists than with deep rooted beleifs which induced people to repudiate the very concept of physical reality or even the usual propositional calculus, rather than take into account the possibility that the theorem of composite probabilities might be a model dependent statement. Of course, to prove the coherence of such a thesis, it must be shown that it is compatible with the whole established body of classical probability theory, in the same sense as the existence of non euclidean geometries is compatible with classical geometry. The above mentioned monograph [3] is essentially devoted to discuss the various conceptual problems which arise in the attempt of achieving this goal.

In the present paper we will limit ourselves to show that the programme of investigating from the purely mathematical point of view the possibility of such "non-Kolmogorovian models in probability theory "leads to the discovery of some interesting connections between algebra, geometry and probability, as well as to a rather simple theoretical explanation of the occurrence of some mathematical structures such as complex numbers, Hilbert spaces, ... into these new statistical models. We also produce examples of statistical models which are neither of Kolmogorovian nor of quantum type and satisfy a weak form of the Heisenberg principle. For the moment however we don't know whether these new types of non-Kolmogorovian models might be of some use in the description of natural phenomena.

§ 2. Statistical invariants.

Throughout this paper $A, B, C, \ldots$ will denote some observables quantities and the real numbers $(a_\alpha), (b_\beta), (c_\gamma), \ldots$ - their values. We assume that different indices correspond to different values of the observables (non-degeneracy) and that the number of values assumed by each observable is finite (say

n) and independent on the observable.

The basic objects we will be dealing with are the transition (or conditional) probabilities: $P(A=a_\alpha \mid B=b_\beta)$; $P(B=b_\beta \mid C=c_\gamma)$; ... and for simplicity we assume that they are strictly positive, i.e.:

$$P(A=a_\alpha \mid B=b_\beta)>0\ ;\ \dots \tag{1}$$

Heuristically one should think to each of the observables $A, B, C, \dots$ as to a complete set of compatible observables in the quantum mechanical sense, so that each of the events $[A=a_\alpha]$, $[B=b_\beta]$, ... uniquely defines a quantum mechanical (pure) state. Since we are ultimately interested in the comparison with quantum theory, we also assume that the simmetry condition of the quantum mechanical transition probabilities:

$$P(A=a_\alpha \mid B=b_\beta)=P(B=b_\beta \mid A=a_\alpha) \tag{2}$$

is satisfied for any A, B, α, β.

Definition (1.1). A *Kolmogorovian model* for a given set $\{P(A=a_\alpha \mid B=b_\beta)\}$ of conditional probabilities is defined by:

(i) A probability space (Ω, ϑ, μ).

(ii) For each of the observables $A, B, C, \dots$ - a measurable partition (A_α), (B_β), (C_γ), ... of (Ω, ϑ) $(\alpha, \beta, \gamma, \dots = 1, \dots, n)$ such that for any couple of observables A, B and of values a_α, b_β one has:

$$P(A=a_\alpha \mid B=b_\beta)=\frac{\mu(A_\alpha \cap B_\beta)}{\mu(B_\beta)} \tag{3}$$

The applications of classical probability theory are pervaded by the following implicit postulate:

(P.) Any family of transition probabilities arising from the experimental evaluation of relative frequencies concerning observables of a given system in some fixed situation can be described by a Kolmogorovian model.

The pre-quantum level of the above postulate can be emphasized by the remark that it can be equivalently (and in a mathematically more precise form) stated in the form:

(P'.) Given any two events $\mathscr{A}$, $\mathscr{B}$ -the conditional probability $P(\mathscr{A} \mid \mathscr{B})$ can be expressed in terms of the joint probability $P(\mathscr{A} \wedge \mathscr{B})$ by means of the elementary formula:

$$P(\mathscr{A} \mid \mathscr{B})=\frac{P(\mathscr{A} \wedge \mathscr{B})}{P(\mathscr{B})} \tag{4}$$

In classical probability theory the equality (4) is usually considered as a definition of conditional probability (cf. [19], § 3) in the sense that the right hand side defines the left hand side; less frequently it is considered as an empirical fact, meaning by this that the experimental procedures to evaluate the two sides of (4) might be different, but the numerical results coincide. However, as an experimental fact, equation (4) has to be checked against experience, while as a plausible definition of something with an empirical content it requires that the defining entity – i.e. the joint probability – has a physical meaning.

But, as it is well known, quantum theory provides many examples of couples of events $\mathscr{A}$, $\mathscr{B}$ such that the left hand side of (4) has physical meaning while the right hand side has not. For the transition probabilities concerning such events, a Kolmogorovian model can be built only at the expense of introducing same in principle unobservable joint probabilities. There is nothing wrong in the introduction, at an intermediate stage of the computations, of unobservable quantities. However the point is that if the only things available from the experiments are the conditional probabilities, then the *existence* of joint probabilities (even if as purely mathematical and in principle unobservable objects) is not a priori guaranteed. This is the reason why assertion (P'.), in a quantum-theoretical context, should be considered as a postulate, hence subjected to experimental refutation.

More specifically: the fact that a family of transition probabilities comes from an agreeing family of joint probabilities, imposes some *mathematical constraints* among them and therefore the beleif that any set of empirical data admits a Kolmogorovian model – Postulate (P.) – is equivalent to the beleif that the conditional probabilities, which arise from measurements on a physical system, automatically satisfy these mathematical constaints. Like any other beleif on the outer world, this one too has to be subjected to the verdict of experience and, if disproved – given up.

It turns out, as shown in § 2, that the transition probabilities arising in the simplest quantum mechanical systems, and verified by innumerable experiments, do not satisfy these mathematical constraints.

The only conclusion we can draw from these experimental results is then that if we want:

(i) to preserve the current interpretation of quantities like $|\langle\varphi, \psi\rangle|^2$ - i.e. to consider them as transition probabilities between simple events (and not as, say, correlation functions between random variables).

(ii) to preserve the current use of describing all the quantum mechanical transition probabilities, entering the description of a given system, within a single mathematical model (and not to use a different mathematical model for, say, any couple of observables of the given system).

Then we are forced to introduce non-Kolmogorovian models in probability theory.

The implicit prejudice that the laws of classical probability are laws of nature and not model-dependent statements has led many scientists to apply these laws to the transition probabilities which arise in a quantum mechanical context and do not fulfil the mathematical constraints which guarantee the existence of a Kolmogorovian model, hence the validity of these laws. In particular one can show (cf. [3]) that the unjustified application of the theorem of composite probabilities to the transition probabilities which describe quantum phenomena lies at the root of most of the so called "paradoxes" of quantum theory.

Of course not all the laws of classical probability fail at a quantum level. One can ascertain the following facts:

Fact I. Whenever dealing with sets of compatible observables quantum probability reduces to classical probability.

Fact II. The only laws of classical probability which might be empirically invalidated in a quantum theoretical framework are those deduced by an unrestricted use of joint probabilities (e.g. all the laws deduced by the expression (4) of the conditional probability – in particular the theorem of composite probabilities).

Due to the symmetry condition (2), satisfied by the quantum transition probabilities, it is not difficult to show (cf. [1], Proposition (1)) that Fact (I) above can be strengthened in the following way:

Fact III. The quantum transition probabilities concerning the values of any *couple* A, B of (finite valued) quantum observables can always be described by means of a Kolmogorovian model.

Remark 1. Facts (I), (II) do not depend on the range of values of the observables involved. To formulate Fact (III) for observables which can assume infinitely many values, one has to introduce the Renyi axioms of probability theory (cf. [22]), rather then Kolmogorov ones.

Remark 2. An interesting conceptual implication of the fact that the transition probabilities concerning *couples* of quantum observables always admit a Kolmogorovian model, is that the indeterminacy principles of Heisenberg type – which always concern couples of observables or, equivalently by a result of von Neumann [26], of complete sets of compatible observables – have a physical root which cannot be traced back only to the statistics of measurements of values of one of these magnitudes conditioned by values of the other one. This fact will be reflected in our analysis of § 4 where we introduce the additional physical information, contained in the Heisenberg principle, through a weak formulation of the latter.

The analogue of Fact (III) for *triples* of observables is in general false as one can easily see by means of mathematical counterexamples (cf. [1], § 1). The results of § 3 imply, in particular, that there are counterexamples concerning triples of physically meaningful observables (say, spin in three non parallel directions).

Historically the first examples of non Kolmogorovian models for a given set of transition probabilities were provided by the Hilbert space models of quantum theory.

Definition 1.2. A complex (resp. real) Hilbert space model for a given family of transition probabilities $P(A=a_\alpha \mid B=b_\beta)$, $P(B=b_\beta \mid C=c_\gamma)$, ... is defined by:

(i) a complex (resp. real) Hilbert space $\mathcal{H}$.

(ii) for each of the observables $A, B, C, \ldots$ - an orthonormal basis (φ^A_α), (φ^B_β), (φ^C_γ), ... of $\mathcal{H}$ $(\alpha, \beta, \gamma, \ldots = 1, \ldots, n)$ such that for any couple of observables A, B and of values a_α, b_β one has

$$P(A=a_\alpha \mid B=b_\beta) = |\langle \varphi^A_\alpha, \varphi^B_\beta \rangle|^2 \tag{5}$$

With this notations, the above discussion can be synthetized in the following statement: there are in nature sets of statistical data which admit a Hilbert space but not a Kolmogorovian model. At this point the following question naturally arises:

Problem I. Given a set of transition probabilities considered as empirical data, how can one decide wheter this set admits a Kolmogorovian, a $\mathbb{C}$-Hilbert or a $\mathbb{R}$-Hilbert model?

A possible way to answer this question is to mimic a procedure, well established in geometry, which consists in the assignment of mathematical invariants which characterize classes of isomorphic models (we did not give a formal definition of the concepts of isomorphism naturally associated to Definitions (1.1) and (1.2), but they and the affiliated concept of "minimal model", can be easily guessed by inspection of the corresponding definitions).

The search for a complete set of *statistical invariants* relative to a given model and associated to a given family of transition probabilities concerning an arbitrary number of observables is, to a large extent, an open problem at the moment. In § 3 we give a complete solution of this problem in the simple case of *three two-valued* observables. We shall see that the tecnique of statistical invariants, even if applied to such a simple case, allows to through some light on some old aged questions in the foundations of quantum theory.

Having accepted the idea that the description of natural phenomena might require more than one probabilistic model, another obvious question is:

Problem II. Beyond the known generalizations, due essentially to von Neumann, of the Hilbert space models, do there exist statistical models which are neither of Kolmogorovian nor of quantum type?

In § 6 we give a complete classification of a particular class of non-Kolmogorovian models for a given family of transition probabilities relative to an arbitrary set of observables which satisfy the weak form of Heisenberg princple enunciated in § 4. This yields in particular a positive answer to Problem (II).

Finally let us remark that in classical probability there are well known examples (cf. the paper of F. Spitzer [25]) of transition probabilities (even markovian) which do not admit a Kolmogorovian model in the sense of Definition (1.1). However, for the construction of these examples, the technical requirement of *countable additivity* of the involved probability measure is essential. In order to emphasize that the non existence of Kolmogorovian models we refer to in this paper has deeper roots and does not depend on any technical assumption, we have limited our discussion to the case of a finite number of observables with a finite number of values.

§ 3. Examples of statistical invariants.

Let A, B, C be three two-valued observables. In this § we shall use the following notations for the transition probability matrices:

$$\begin{aligned}
P &= P(A \mid B) = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix} = \begin{pmatrix} \cos^2\alpha/2 & \sin^2\alpha/2 \\ \sin^2\alpha/2 & \cos^2\alpha/2 \end{pmatrix} \\
Q &= P(B \mid C) = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} = \begin{pmatrix} \cos^2\beta/2 & \sin^2\beta/2 \\ \sin^2\beta/2 & \cos^2\beta/2 \end{pmatrix} \\
R &= P(C \mid A) = \begin{pmatrix} r & 1-r \\ 1-r & r \end{pmatrix} = \begin{pmatrix} \cos^2\gamma/2 & \sin^2\gamma/2 \\ \sin^2\gamma/2 & \cos^2\gamma/2 \end{pmatrix}
\end{aligned} \tag{1}$$

Condition (1) of § 1 is then equivalent to:

$$0 < p, q, r < 1 \quad ; \quad \text{or:} \ 0 < \alpha, \beta, \gamma < \pi \tag{2}$$

The following result, due to Gutkowski and Masotto [11] (necessary condition), and Corleo, Gutkowski, Masotto, Valdes [12] (sufficient condition), and independently rediscovered in [4], provides the simplest example of a statistical invariant:

Theorem 3.1. The triple of transition probability matrices P, Q, R defined by (1) admits a Kolmogorovian model if and only if:

$$|p + q - 1| \leqslant r \leqslant 1 - |p - q| \tag{3}$$

Remark. For three (or more) observables the Kolmogorovian model, even if existing, needs not to be unique in the sense of the usual stochastic equivalence along stochastic processes (see [8] for example) since the transition probabilities provide an (indirect) information only on the joint probabilities of couples of observables.

Concerning Hilbert space models we can state the following

Theorem 3.2. In the notations above, the transition matrices P, Q, R admit a $\mathbb{C}$-Hilbert space model if and only if

$$-1 \leqslant \frac{p + q + r - 1}{2\sqrt{pqr}} \leqslant 1 \tag{4}$$

Theorem 3.3. In the notations above the transition matrices P, Q, R admit an $\mathbb{R}$-Hilbert space model if and only if:

$$\frac{p + q + r - 1}{2\sqrt{pqr}} = -1 \quad \text{or} \quad \frac{p + q + r - 1}{2\sqrt{pqr}} = 1 \tag{5}$$

Theorem 3.4. In the notations above, the transition matrices P, Q, R admit a quaternion Hilbert space model if and only if they admit a $\mathbb{C}$-Hilbert space model.

An interesting geometrical interpretation of the statistical invariants (4) and (5) can be given in terms of the angles α, β, γ-defined by (1) and (2). In fact one can show (cf. [4]) that condition (4) is equivalent to the inequalities:

$$|\alpha - \beta| \leqslant \gamma \leqslant \alpha + \beta \quad ; \quad \alpha + \beta + \gamma \leqslant 2\pi \tag{6}$$

which are necessary and sufficient conditions for α, β, γ to be adjacent angles of a thriedron in $\mathbb{R}^3$. While condition (5) is equivalent to

$$\alpha + \beta + \gamma = 2\pi \tag{7}$$

which corresponds to a degenerate thriedron, entirely lying on a plane.

Another geometrical interpretation can be based on the concept of "spin model", but for this we refer to [4].

The fact that the statistical invariants in the model considered above are eventually expressed in terms of inequalities on the transition probabilities explains the mathematical meaning of the various inequalities of Bell's type which are known in the literature: they are necessary conditions for a given set of statistical data to admit a Kolmogorovian model. Originally Bell considered, as statistical data, the correlation functions of a composite system [5]; the reformulation of Bell's result in terms of transition probabilities is due to Wigner [27]; and the remark that composite systems, hence locality arguments play no role at all is due to Bub [6]. The above mentioned results, which concern a single system, fully confirm this statement.

The three theorems above show that, while the choice of real and complex numbers in 2-dimensional Hilbert space models have a statistical meaning, the use of quaternion Hilbert space cannot be decided on purely statistical grounds.

The existence of a $\mathbb{C}$-Hilbert space model for a given triple of transition probability matrices $P, Q, R,$ always implies the existence of an $\mathbb{R}$-Hilbert space model; the 2-dimensional case is peculiar in that it is also implied by the existence of a Kolmogorovian model (cf. [4]).

The inclusion $\{$Kolmogorovian models$\} \subseteq \{\mathbb{C}$-Hilbert space models$\}$ is however very specific of the 2-dimensional case; in fact it is not difficult to show that if two three-valued observables are described by the transition probability matrix

$$\begin{pmatrix} \frac{1}{2}\left(\frac{1}{2}+\delta\right) & \frac{1}{2^4} & \frac{11}{2^4}-\frac{1}{2}\delta \\ \frac{1}{2}\left(\frac{1}{2}+\delta\right) & \frac{1}{2}-\frac{1}{2^4}+\delta & 1-2\delta-\left(\frac{11}{2^4}-\frac{1}{2}\delta\right) \\ \frac{1}{2}-\delta & \frac{1}{2}-\delta & 2\delta \end{pmatrix} \tag{6}$$

with $0 < \delta < 1/2^6$, then there exists exactly one (up to stochastic isomorphism) Kolmogorovian model (with symmetric transition probabilities) for (6), but no $\mathbb{C}$-Hilbert space models. This counterexample shows, among other things, that the derivation of the Hilbert space structure due to A. Landè [17] should be supported by some further mathematical assumption.

§ 4. Schwinger's "algebra of measurements".

Let A be an observable with values $a_\alpha : \alpha = 1, \dots, n$, and t denote an instant of time. To the triple (A, a_α, t) we associate an idealized measurement apparatus – denoted $A_\alpha(t)$ – which from an ensamble of independent similar systems selects those for which the value of the observable A at time t is a_α. Such an apparatus will be called an *elementary filter.* Elementary filters are idealizations of first-kind measurement, meaning by this that, if $\epsilon > 0$ is "very small", the measurement $A_\alpha(t)$ and $A_\alpha(t+\epsilon)$ give the same results (cf. equality (3) below). Elementary filters corresponding to compatible observables can be applied at the same time.

The operation corresponding to the simultaneous measurement of the values $a_\alpha, b_\beta, \dots$ of the compatible observables $A, B, \dots$ at time t, will be denoted

$$A_\alpha(t) + B_\beta(t) + \dots \tag{1}$$

Elementary filters corresponding to any two observables can be applied at different times. The resulting operation will be denoted

$$A_\alpha(t) \cdot B_\beta(s) \cdot \dots \quad ; \qquad s < t < \dots \tag{2}$$

The property of first kind measurements can now be symbolically expressed by

$$\text{"}\lim_{\epsilon \to 0}\text{"}\ A_\alpha(t+\epsilon) A_{\alpha'}(t) = \delta_{\alpha\alpha'} A_\alpha(t) \tag{3}$$

Standard arguments (cf. [23] or [21]) show that the operations (1) and

(2) satisfy, when defined, the usual arithmetic rules and that one can naturally introduce the elements 0 (zero) and 1 (identity). The resulting structure is not an algebra, since the algebraic operations are not everywhere defined; however one can embed this structure in a real associative algebra over the reals with identity which, following Schwinger [23], will be called *the algebra of measurements* of the given system of observables.

In the following we shall assume that all the measurement operations we are interested in are performed in a very short interval of time, so that time effects don't play any role. Therefore we shall omit the time parameters from our notations.

Definition 4.1. Two observables A, B are said to satisfy the weak form of Heisenberg principle if for every couple of their values a_α, b_β one has:

$$A_\alpha B_\beta A_\alpha = p_{\alpha\beta} A_\alpha \tag{4}$$

where

$$p_{\alpha\beta} = P(A = a_\alpha \mid B = b_\beta) \tag{5}$$

is the conditional probability that $A = a_\alpha$ given that $B = b_\beta$, and the symbolic notation in the right hand side of (4) means that only the fraction $p_{\alpha\beta}$ of the systems which have possed the first elementary filter of the apparatus $A_\alpha B_\beta A_\alpha$ will pass the third one.

In the classical case all filters are compatible, the multiplication is commutative, and one has:

$$A_\alpha B_\beta A_\alpha = A_\alpha B_\beta \tag{6}$$

This means that, in the classical case, the presence of the filter B_β entails a precise information on the single systems which pass the apparatus $A_\alpha B_\beta A_\alpha$, while, when the weak Heisenberg principle is satisfied, the presence of B_β yields only an information of statistical character, hence on ensambles and not on single particles.

Motivated by these informal, heuristic considerations, we introduce the following:

Definition 4.2. Let T be a set; $\{A(x) : x \in T\}$ - a family of observables each assuming n distinct values (n-independent on x); and let $\forall x,y \in T$, $P(x,y)$ denote the transition probability matrix of $A(x)$ given $A(y)$ whose coefficients are

$$p_{\alpha\beta}(x,y) = P(A(x) = a_\alpha(x) \mid A(y) = a_\beta(y)) \tag{7}$$

An *Heisenberg model for the family* $\{P(x,y)\}$ of transition matrices is defined by the assignement of:

(i) an $\mathbb{R}$-algebra $\mathscr{A}$ (associative with an identity 1)

(ii) $\forall x \in T$, and for each value $a_\alpha(x)$ of A-an element $A_\alpha(x)$ of $\mathscr{A}$ such such that the following conditions are satisfied:

$$A_\alpha(x) \cdot A_{\alpha'}(x) = \delta_{\alpha\alpha'} A_\alpha(x) \ ; \qquad \forall x \in T \tag{8}$$

$$\sum_\alpha A_\alpha(x) = 1 \ ; \qquad \forall x \in T \tag{9}$$

$$A_\alpha(x) \cdot A_\beta(y) \cdot A_\alpha(x) = p_{\alpha\beta}(x,y) A_\alpha(x) \ ; \qquad \forall x,y \in T \tag{10}$$

Remark. An Heisenberg model for the family $\{P(x,y)\}$ is thus a model for the algebra of measurements of the observables $\{A(x) : x \in T\}$ in which every couple of observables $A(x)$, $A(y)$ satisfies the weak form of the Heisenberg principle as formulated in Definition (4.1).

Just as in the case of Kolmogorovian models, there will be mathematical contstraints on the transition probability matrices $P(x,y)$ deriving from the fact that they admit an Heisenberg model; the first problem is then to find these constraints:

Problem I. Given a family of transition probability matrices $\{P(x,y) : x,y \in T\}$, under which conditions does it admit an Heisenberg model?

The second natural question is:

Problem II. Describe the $\mathbb{R}$-algebras which can arise as measurements algebras of some Heisenberg model relative to some family of transition probability matrices.

If on the multiplication among elementary filters one imposes no conditions but (8) and (10), then there is no hope to achieve a complete solution of Problem (II). Let us therefore isolate a simple class of models through the following:

Definition 4.3. In the above notations, an Heisenberg model for the family of transition probabilities $\{P(x,y)\}$ will be called *standard* if for any couple of indices $x, y \in T$, the algebra $\mathscr{A}$ is generated, as a module over its center, by the products

$$A_\alpha(x) \cdot A_\beta(y) \ ; \qquad \alpha, \beta = 1, \dots, n$$

In the following, the center of $\mathscr{A}$ will be denoted κ, and we shall solve

Problems (I) and (II) in the case of standard models. The possible non triviality of the center of $\mathscr{A}$ means that we don't exclude a priori the possibility of superselection rules; however, as shown at the end of § 5, the "minimal size" of the center has a purely statistical character.

§ 5. Structure of the standard Heisenberg models compatible with a given set of transition matrices.

Let us begin to discuss Problems (I) and (II) of § 4 in the case when there are only two observables A, B (i.e. when the set T in Definition (4.2) has only two elements), and therefore only two transition probability matrices: $P(A \mid B) = (p_{\alpha\beta}(A \mid B))$, $P(B \mid A) = (p_{\beta\alpha}(B \mid A))$. In this case, according to Definition (4.3), a standard Heisenberg model is defined by an associative $\mathbb{R}$-algebra $\mathscr{A}$ with identity, and two sets $(A_\alpha), (B_\beta)$ $(\alpha, \beta = 1, \dots, n)$ of elements of $\mathscr{A}$ satisfying:

$$A_\alpha A_{\alpha'} = \delta_{\alpha\alpha'} A_\alpha \quad ; \quad B_\beta B_{\beta'} = \delta_{\beta\beta'} B_\beta \tag{1}$$

$$\sum_\alpha A_\alpha = 1 \quad ; \quad \sum_\beta B_\beta = 1 \tag{2}$$

$$A_\alpha B_\beta A_\alpha = p_{\alpha\beta}(A \mid B) A_\alpha \quad ; \quad B_\beta A_\alpha B_\beta = p_{\beta\alpha}(B \mid A) B_\beta \tag{3}$$

and such that, if κ denotes the center of $\mathscr{A}$ then any element of $\mathscr{A}$ can be expressed as a linear combination of the products $\{A_\alpha B_\beta : \alpha, \beta = 1, \dots, n\}$ with coefficients in κ.

Because of (1) this last condition is equivalent the existence of n^4 elements of κ $(\gamma^{\alpha'\beta'}_{\alpha\beta}$; $\alpha, \beta, \alpha', \beta' = 1, \dots, n)$ such that

$$B_\beta A_\alpha = \sum_{\alpha'\beta'} \gamma^{\alpha'\beta'}_{\alpha\beta} A_{\alpha'} B_{\beta'} \tag{4}$$

The $(\gamma^{\alpha'\beta'}_{\alpha\beta})$'s will be called the *structure constants of $\mathscr{A}$ in the $A_\alpha B_\beta$-basis.* This terminology is justified by the following general fact concerning associative algebras with identity satisfying conditions (1), (2), (4) but not necessarily the weak Heisenberg principle (3):

Proposition 5.1. Let $\mathscr{A}$ be an associative $\mathbb{R}$-algebra with identity and with center κ. Let $(A_\alpha), (B_\beta)$ be elements of $\mathscr{A}$ satisfying (1), (2) and such that $\{A_\alpha B_\beta : \alpha, \beta = 1, \dots, n\}$ is a κ-basis of $\mathscr{A}$.

Then, if $(\gamma^{\alpha'\beta'}_{\alpha\beta})$ are the structure constants of $\mathscr{A}$ in the $A_\alpha B_\beta$-basis, one has:

$$\sum_\alpha \gamma^{\alpha'\beta'}_{\alpha\beta} = \delta_{\beta\beta'} \quad ; \qquad \forall \alpha' \tag{5}$$

$$\sum_\beta \gamma^{\alpha'\beta'}_{\alpha\beta} = \delta_{\alpha\alpha'} \quad ; \qquad \forall \beta' \tag{6}$$

$$\gamma^{\alpha\beta'}_{\alpha'\beta} \cdot \gamma^{\alpha\beta''}_{\alpha''\beta'} = \gamma^{\alpha\beta''}_{\alpha'\beta} \cdot \gamma^{\alpha'\beta''}_{\alpha''\beta'} \tag{7}$$

Conversely, if $\gamma^{\alpha'\beta'}_{\alpha\beta} \in \kappa$ (commutative, associative, $\mathbb{R}$-algebra with identity) satisfy (5), (6), (7), then there exists an associative κ-algebra with identity, $\mathscr{A}$ and elements $(A_\alpha), (B_\beta)$ $(\alpha, \beta = 1, \dots, n)$ of $\mathscr{A}$ satisfying (1), (2), (4) and such that the products $\{A_\alpha B_\beta : \alpha, \beta = 1, \dots, n\}$ are a κ-basis of $\mathscr{A}$.

Example. The structure constants of the usual, Kolmogorovian models, are $\gamma^{\alpha'\beta'}_{\alpha\beta} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}$.

One can also show that:

Proposition 5.2. In the above notations, if $(A_\alpha), (B_\beta)$ satisfy also condition (3) then necessarily

$$p_{\alpha\beta}(A \mid B) = p_{\beta\alpha}(B \mid A)$$

In other terms, the quantum mechanical symmetry condition

$$P(A = a_\alpha \mid B = b_\beta) = P(B = b_\beta \mid A = a_\alpha)$$

is a consequence of our weak formulation of the Heisenberg principle therefore in the following the transition probabilities relative to any couple of observables will be supposed to satisfy this symmetry condition. In fact this result is a consequence of a much deeper connection between the transition probability matrix $P = P(A \mid B) = {}^tP(B \mid A)$ and the structure constants $\gamma^{\alpha'\beta'}_{\alpha\beta}$, which is clarified by the following theorem:

Theorem 5.3. In the notations above, let $\mathscr{A}$, $(A_\alpha), (B_\beta)$, κ define a standard Heisenberg model for the transition probability matrix $P = P(A \mid B) = {}^tP(B \mid A)$, and let $(\gamma^{\alpha'\beta'}_{\alpha\beta})$ be the structure constants of $\mathscr{A}$ in the $A_\alpha B_\beta$-basis. Then there exists a κ-valued matrix $U = U(A \mid B) \equiv (u_{\alpha\beta}(A \mid B))$ such that

$$\gamma^{\alpha'\beta'}_{\alpha\beta} = \frac{u_{\alpha'\beta}(A \mid B) \cdot u_{\alpha\beta'}(A \mid B)}{u_{\alpha\beta}(A \mid B) \cdot u_{\alpha'\beta'}(A \mid B)} \cdot p_{\alpha\beta}(A \mid B) \tag{8}$$

$$\sum_\alpha \frac{p_{\alpha\beta}(A \mid B)}{u_{\alpha\beta}(A \mid B)} \cdot u_{\alpha\beta'}(A \mid B) = \delta_{\beta\beta'} \tag{9}$$

$$\sum_\beta u_{\alpha'\beta}(A \mid B) \cdot \frac{p_{\alpha\beta}(A \mid B)}{u_{\alpha\beta}(A \mid B)} = \delta_{\alpha'\alpha} \tag{10}$$

Conversely, any κ-valued matrix $(u_{\alpha\beta}(A|B))$ satisfying (9), (10), defines through (8) the structure constants of a standard Heisenberg model relative to the transition probability matrix $P = P(A|B) = {}^tP(B|A)$.

Remark. In the theorem above, and in the following, we shall limit ourselves to the generic case in which the $\gamma_{\alpha\beta}^{\alpha'\beta'}$, $u_{\alpha\beta}(A|B)$, $p_{\alpha\beta}(A|B)$ are all invertible.

In analogy with the quantum mechanical terminology let us introduce the following:

Definition 5.4. Let $P \equiv (p_{\alpha\beta})$ be a $(n \times n)$ transition probability matrix, and let κ be an associative and commutative $\mathbb{R}$-algebra with unit.

A matrix $U = (u_{\alpha\beta})$ with coefficients in κ satisfying

$$\sum_\alpha (u_{\alpha\beta})^* u_{\alpha\beta'} = \delta_{\beta\beta'} \tag{11}$$

$$\sum_\beta u_{\alpha\beta} (u_{\alpha'\beta})^* = \delta_{\alpha\alpha'} \tag{12}$$

with

$$(u_{\alpha\beta})^* = p_{\alpha\beta}/u_{\alpha\beta} \tag{13}$$

will be called a *κ-valued transition amplitude matrix for P.*

With these notations our main result can be stated as follows:

Theorem 5.5. Let: T be an arbitrary set; $\{P(x,y): x,y \in T\}$-be a family of $(n \times n)$ transition probability matrices; κ-an associative commutative $\mathbb{R}$-algebra with identity.

A standard Heisenberg model for the family of transition probability matrices $\{P(x,y)\}$ with center κ exists if and only if for every $x,y \in T$ there exists a κ-valued $(n \times n)$ transition amplitude matrix $U(x,y)$ for $P(x,y)$ such that

$$U(x,y) \cdot U(y,z) = U(x,z) \quad ; \qquad \forall x,y,z \in T \tag{14}$$

with the convention that

$$U(x,x) = 1 \quad ; \qquad \forall x \in T \tag{15}$$

Remark 1. Theorem (5.5) shows in particular the statistical meaning of the algebra κ: in fact a given set of transition probability matrices might admit κ-valued but not κ_0-valued transition amplitude matrices satisfying (14), if κ_0 is a sub-algebra of κ. The examples discussed in § 3 illustrate this situation in the case $\kappa = \mathbb{C}$, $\kappa_0 = \mathbb{R}$.

Remark 2. Equation (14) shows that the "reversibility of the quantum mechanical evolution law" has its origins in a much more general and purely statistical phenomenon.

Remark 3. In classical probability theory one knows essentially two types of compatibility conditions among statistical data relative to a family of random variables $\{A(x): x \in T\}$: one, concerning the joint probabilities – which gives rise to theorems of Daniell-Kolmogorov type; the other, concerning the conditional probabilities, which gives rise to the problematic of Dobruscin theory. A particular example of the latter type is the Chapman-Kolmogorov equation:

$$P(r,s) \cdot P(s,t) = P(r,t) \quad ; \qquad r < s < t \ ; \ r,s,t \in \mathbb{R}^+$$

which is a necessary (and sufficient) condition for the family $\{P(s,t): s<t;\ s,t \in \mathbb{R}^+\}$ of transition probability matrices to be the transition matrices of a Markov process. Theorem (5.5) shows that in some non-Kolmogorovian models there may arise compatibility conditions not directly expressed in terms of the transition probabilities, but of what we have called "transition amplitudes". In particular, expressing (14) in coefficients, one finds:

$$\sum_\beta u_{\alpha\beta}(x,y)\, u_{\beta\gamma}(y,z) = u_{\alpha\gamma}(x,z) \tag{16}$$

which is a generalized form of the well known quantum mechanical "principle of composite amplitudes".

Finally, let ut consider the problem of characterizing among all the standard Heisenberg models, those which give rise to the usual Hilbert space structure of quantum mechanics. To this aim first remark that:

Proposition 5.6. The family $\{P(x,y): x,y \in T\}$ of transition probability matrices has a standard Heisenberg model with center κ if and only if there exist:

i) a κ-module $\mathcal{H}$
ii) $\forall x \in T$-a basis $\{|a_\alpha(x)\rangle: \alpha = 1, \dots, n\}$ of $\mathcal{H}$ over κ
iii) for each $x \in T$ and $\alpha = 1, \dots, n$-a κ-linear operator $A_\alpha(x)$ such that for every $x,y \in T$, $\alpha, \beta = 1, \dots, n$

$$A_\alpha(x)|a_{\alpha'}(x)\rangle = \delta_{\alpha\alpha'}|a_\alpha(x)\rangle \tag{17}$$

$$A_\alpha(x) A_\beta(y) A_\alpha(x) = p_{\alpha\beta}(x,y) A_\alpha(x) \tag{18}$$

Let now $\{P(x,y)\}$, $\mathcal{H}$, $(A_\alpha(x))$, $|a_\alpha(x)\rangle$, be as in Proposition (5.6).

Denote $\mathcal{H}^*$ the set of κ-linear functions $\mathcal{H} \to \kappa$ and $\langle \cdot , \cdot \rangle$ the κ-valued duality $\langle \mathcal{H}^*, \mathcal{H} \rangle$. For each of the bases $\{ |a_\alpha(x)\rangle : \alpha = 1, \dots, n \}$ $(x \in T)$, the dual basis $\{ \langle a_\alpha(x)| : \alpha = 1, \dots, n \}$ in $\mathcal{H}^*$ is uniquely defined by the property:

$$\langle a_\alpha(x) \, , \, a_{\alpha'}(x) \rangle = \delta_{\alpha\alpha'}$$

and for any $x \in T$ the map

$$J_x : |a_\alpha(x)\rangle \in \mathcal{H} \longrightarrow \langle a_\alpha(x)| \in \mathcal{H}^*$$

uniquely extends to a κ-linear isomorphism $\mathcal{H} \to \mathcal{H}^*$.

Definition 5.7. In the notations above, the standard Heisenberg model for $\{P(x,y)\}$ defined by κ, the κ-module $\mathcal{H}$, $A_\alpha(x)$, $|a_\alpha(x)\rangle$ $(\alpha = 1, \dots, n;$ $x \in T)$ is called a κ-Hilbert space model if there exists and additive map $J : \mathcal{H} \to \mathcal{H}^*$ and an involution $j : \kappa \to \kappa$ (both independent on $x \in T$) such that $\forall \alpha = 1, \dots, n;$ $\forall x \in T$:

$$J(\lambda |a_\alpha(x)\rangle) = j(\lambda) | \langle a_\alpha(x)| \quad ; \qquad \forall \lambda \in \kappa \tag{19}$$

Proposition 5.8. A standard Heisenberg model for the family $\{P(x,y)\}$ of transition probability matrices, defined by the family $\{U(x,y)\}$ of κ-valued transition amplitude matrices, is a κ-Hilbert space model if and only if there exists a linear involution $j : \kappa \to \kappa$ such that for any $x,y \in T$ and $\alpha, \beta =$ $= 1, \dots, n$ one has

$$j(u_{\alpha\beta}(x,y)) = \frac{p_{\alpha\beta}(x,y)}{u_{\alpha\beta}(x,y)} \tag{20}$$

In other words, among all the standard Heisenberg models, the usual quantum models are characterized by the fact that the involutions

$$u_{\alpha\beta}(x,y) \longmapsto p_{\alpha\beta}(x,y)/u_{\alpha\beta}(x,y)$$

on the coefficients of the transition amplitude matrices, do not depend on the statistics, i.e. on the $p_{\alpha\beta}(x,y)$'s.

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