CENTRAL EXTENSION OF VIRASORO TYPE SUBALGEBRAS OF THE ZAMOLODCHIKOV-w_∞ LIE ALGEBRA

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It is known that the centerless Zamolodchikov–w_∞ $*$–Lie algebra of conformal field theory does not admit nontrivial central extensions, but the Witt $*$–Lie algebra, which is a sub–algebra of w_∞, admits a nontrivial central extension: the Virasoro algebra. Therefore the following question naturally arises: *are there other natural sub–algebras of w_∞ which admit nontrivial central extensions other than the Virasoro one?* We show that for certain infinite dimensional closed subalgebras of w_∞, which are natural generalizations of the Witt algebra the answer is negative.

Keywords: Witt algebra; Virasoro algebra; Central extensions

1. Introduction

The centerless Virasoro (or Witt)-Zamolodchikov-w_∞ $*$–Lie algebra (cf.[3]-[6]) is the infinite dimensional $*$–Lie algebra, with generators

$$\{\hat{B}^n_k \ : \ n \in \mathbb{N}, n \geq 2, k \in \mathbb{Z}\} \tag{1}$$

commutation relations

$$[\hat{B}^n_k, \hat{B}^N_K] = (k\,(N-1) - K\,(n-1))\ \hat{B}^{n+N-2}_{k+K} \tag{2}$$

and involution

$$\left(\hat{B}^n_k\right)^* = \hat{B}^n_{-k} \tag{3}$$

where $\mathbb{N} = \{1, 2, ...\}$.

The central extensions of the w_∞ algebra have been widely studied in the physical literature. In particular, Bakas proved in Ref. 3 that the w_∞ $*$–Lie algebra does not admit non-trivial central extensions. That was done by showing that, after a suitable contraction which yields the w_∞ commutation relations, the central terms appearing in the algebra W_∞, which is defined as a $N \to \infty$ limit of the Zamolodchikov type Lie algebras W_N, vanish. A direct proof of the triviality of all central extensions of w_∞ based on the cocycle definition of a central extension and avoiding the ambiguities that arise from passing to the (non-unique) limit of W_N, was given in Ref. 1.

The $*$–Lie sub–algebra of the w_∞ algebra, generated by the family $\{\hat{B}^2_k : k \in \mathbb{Z}\}$ is the Witt algebra which admits the Virasoro non trivial central extension

$$[\hat{B}^2_m, \hat{B}^2_n] = (m - n)\, \hat{B}^2_{m+n} + \delta_{m+n,0}\, m\,(m^2 - 1)\, E \tag{4}$$

where traditionally $E = c/12$ where $c \in \mathbb{C}$ is the "central charge".

In this paper we examine whether certain infinite dimensional closed sub-algebras of w_∞, which are natural generalizations of the Witt algebra, can also be non-trivially centrally extended.

2. Closed subalgebras of w_∞

In this section we investigate the structure of the Lie sub–algebras of w_∞. More precisely, we investigate which subsets of the generators of w_∞ are such that the Lie algebra (resp. $*$–Lie algebra) generated by them, is a proper sub–algebra of w_∞. To this goal notice that, if $\hat{\mathcal{S}}$ is any subset of the generators of w_∞, then there exists a unique partition $\{\hat{\mathcal{S}}_+ \hat{\mathcal{S}}_0, \hat{\mathcal{S}}_-\}$ of $\hat{\mathcal{S}}$ defined by

$$\begin{aligned}
\hat{\mathcal{S}}_+ &:= \{\hat{B}^n_k \in \hat{\mathcal{S}} \;:\; k > 0\} \\
\hat{\mathcal{S}}_0 &:= \{\hat{B}^n_k \in \hat{\mathcal{S}} \;:\; k = 0\} \\
\hat{\mathcal{S}}_- &:= \{\hat{B}^n_k \in \hat{\mathcal{S}} \;:\; k < 0\}
\end{aligned}$$

From (3) we know that a generator $\hat{B}^n_k$ is self–adjoint if and only if $k = 0$. Therefore $\hat{\mathcal{S}}_0$ is a self–adjoint set. Moreover the set $\hat{\mathcal{S}}$ generates a $*$–sub–algebra if and only if $(\hat{\mathcal{S}}_+)^* = \hat{\mathcal{S}}_-$. Denote $\hat{\mathcal{L}}(\hat{\mathcal{S}})$ the Lie sub–algebra of w_∞ generated by $\hat{\mathcal{S}}$. From (2), we see that the sets $\hat{\mathcal{S}}_+$, $\hat{\mathcal{S}}_-$ generate Lie sub–algebras $\hat{\mathcal{L}}(\hat{\mathcal{S}}_+)$, $\hat{\mathcal{L}}(\hat{\mathcal{S}}_-)$ of $\hat{\mathcal{L}}(\hat{\mathcal{S}})$, while $\hat{\mathcal{S}}_0$ generates a Lie $*$–sub–algebra $\hat{\mathcal{L}}(\hat{\mathcal{S}}_0)$. Denote $\mathbb{N}_{\geq 2} := \{n \in \mathbb{N} : n \geq 2\}$. The map $\hat{B}^n_k \mapsto (n, k) \in \mathbb{N}_{\geq 2} \times \mathbb{Z}$ defines a one–to–one correspondence between the set of generators (1) and the set $\mathbb{N}_{\geq 2} \times \mathbb{Z}$. Therefore the sub–set of generators $\hat{\mathcal{S}}$ will be in one–to–one correspondence with a subset $\mathcal{S} \subseteq \mathbb{N}_{\geq 2} \times \mathbb{Z}$. The images of the subsets $\hat{\mathcal{S}}_\varepsilon$ where $\varepsilon \in \{+, 0, -\}$ under this correspondence will be denoted by $\mathcal{S}_\varepsilon$.

We want to study the following problem: *which subset of* $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ *corresponds to those generators (1) which belong to* $\hat{\mathcal{L}}(\hat{\mathcal{S}})$ *(resp.* $\hat{\mathcal{L}}(\hat{\mathcal{S}}_\varepsilon)$*,* $\varepsilon \in \{+, 0, -\}$)? This sub–set will be denoted by $\mathcal{L}(\mathcal{S})$ (resp. $\mathcal{L}(\mathcal{S}_\varepsilon)$, $\varepsilon \in \{,0, -\}$). The answer to this question, for a generic $\hat{\mathcal{S}}$, is very difficult therefore we begin to analyze a simpler problem, namely: *Can we construct interesting families of subsets* $\hat{\mathcal{S}} \subseteq \mathbb{N}_{\geq 2} \times \mathbb{Z}$ *with the property that the linear span of such a subset is a proper Lie $*$–sub–algebra of* w_∞? Notice that, if $\hat{B}^n_k, \hat{B}^N_K \in \hat{\mathcal{S}}$, then from (2) one sees that, if $k(N-1) - K(n-1) \neq 0$ then the generator $\hat{B}^{n+N-2}_{k+K} \in \hat{\mathcal{L}}(\hat{\mathcal{S}})$.

Moreover, the set $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ is an associative semi–group under the composition law

$$(n, k) \dot{+} (N, K) := (n + N - 2, k + K) \tag{5}$$

In fact, it is the product of the semi–group $\mathbb{N}_{\geq 2}$ with composition law

$$n \dot{+} N := n + N - 2 \tag{6}$$

and the (semi–) group $\mathbb{Z}$ with the usual addition. Thus the set $\mathcal{L}(\mathcal{S})$ will be contained in the sub–semi–group of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ generated by $\mathcal{S}$. Conversely, if $\mathcal{S}_0$ is any sub–semi–group of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$, then the linear span of $\hat{\mathcal{S}}_0 := \{\hat{B}^n_k : (n, k) \in \mathcal{S}_0\}$ is a Lie–sub–algebra of w_∞ and it is a Lie $*$–sub–algebra if and only if $\mathcal{S}_0$ is a self–adjoint subset under the involution

$$(n, k) \in \mathbb{N}_{\geq 2} \times \mathbb{Z} \mapsto (n, -k) \in \mathbb{N}_{\geq 2} \times \mathbb{Z}$$

For this reason, it is interesting to study the sub–semi–groups of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ under the composition law (Ref. 5). An interesting class of these semigroups are those of the form

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \tag{7}$$

where $\mathcal{S}_1$ is a sub–semi–group of $\mathbb{N}_{\geq 2}$ with composition law (6) and $\mathcal{S}_2$ a sub–semi–group of $\mathbb{Z}$. The composition law (6) has an identity, given by the number 2, which is in $\mathbb{N}_{\geq 2}$. Hence $\{2\} \times \mathbb{Z}$ is a self–adjoint sub–semi–group of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$. Therefore the linear span of the set $\{\hat{B}^2_k : k \in \mathbb{Z}\}$ is a Lie $*$–sub–algebra of w_∞ which is precisely the Witt (or centerless Virasoro) algebra.

Notice that $\{2\}$ is the only finite sub–semi–group of $\mathbb{N}_{\geq 2}$. In fact if S is such a semigroup and $n \in S$, then $\forall\, \nu \in \mathbb{N} \cup \{0\}$

$$n \dot{+} \ldots \dot{+} n \qquad (\nu - \text{times}) = \nu n - 2(\nu - 1) = \nu(n-2) + 2 \ \in S \tag{8}$$

and, for varying ν, this is a finite set if and only if $n = 2$. Notice also that the sub–semi–group of $\mathbb{N}_{\geq 2}$ generated by the single element $n \in \mathbb{N}_{\geq 2}$ is the set of elements of the form (8) for $\nu \in \mathbb{N} \cup \{0\}$. Denoting by S_n this semi–group, one has that $S_n \times \mathbb{Z}$ is a self–adjoint sub–semi–group of $\mathbb{N}_{\geq 2} \times \mathbb{Z}$. Therefore $\forall\, n \in \mathbb{N}_{\geq 2}$ the linear span of the set

$$\{\hat{B}^{\nu(n-2)+2}_k \ : \ \nu \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}\}$$

is a closed Lie $*$–sub–algebra of w_∞. Letting $N = n - 2 \geq 0$ and (for fixed N)

$$W^n_k := \hat{B}^{n\,N+2}_k$$

we arrive at the following definition.

Definition 2.1. For any natural integer $N \geq 0$ we denote w_N the $*$–Lie subalgebra of w_∞ defined by

$$w_N := span\,\{W^n_k \ : \ n \in \mathbb{N} \cup \{0\}, k \in \mathbb{Z}\}$$

with Lie brackets (inherited from w_∞)

$$[W^n_k, W^l_m] = ((k\,l - m\,n)\,N + (k - m))\ W^{n+l}_{k+m} \tag{9}$$

For $N = 0$, w_0 is the Witt algebra.

The question of the existence of non–trivial central extensions of w_N is the subject of this paper.

Notice that w_N is a direct generalization of the Witt algebra w_0. Furthermore, notice that the Witt algebra is the vector space generated by the generators of the form $\{\hat{B}_k^{\varphi(k)} \ : \ k \in \mathbb{Z}\}$ where φ is the constant function $\varphi(k) = 2$, $\forall k \in \mathbb{Z}$.

One may wonder if there exist other functions $\varphi : \mathbb{Z} \to \mathbb{N}_{\geq 2}$ with this property. The following Lemma shows that this is not the case.

Lemma 2.1. *Let $\varphi : \mathbb{Z} \to \mathbb{N}_{\geq 2}$ be a function such that the linear span of $\{\hat{B}_k^{\varphi(k)} \ : \ k \in \mathbb{Z}\}$ is a $*$–Lie algebra. Then φ is the constant function $\varphi(k) = 2$, $\forall k \in \mathbb{Z}$.*

Proof. The condition $(\hat{B}_k^{\varphi(k)})^* = \hat{B}_{-k}^{\varphi(k)}$ for all $k \in \mathbb{N}$ implies that $\varphi(k) = \varphi(-k)$. This, together with the condition

$$[\hat{B}_k^{\varphi(k)}, \hat{B}_{-k}^{\varphi(-k)}] = 2k(\varphi(k) - 1)B_0^{\varphi(k)\dot{+}\varphi(-k)} \ ; \quad \forall\, k \in \mathbb{Z}$$

gives that, $\forall\, k \in \mathbb{Z}$

$$\varphi(0) = \varphi(k)\dot{+}\varphi(k) = 2\varphi(k) - 2 \Leftrightarrow 2\varphi(k) = \varphi(0) + 2 \Leftrightarrow \varphi(k) = \frac{1}{2}\,\varphi(0) + 1$$

But then the condition

$$[\hat{B}_0^{\varphi(0)}, \hat{B}_k^{\varphi(k)}] = -k(\varphi(0) - 1)\hat{B}_k^{\varphi(k)\dot{+}\varphi(0)}$$

gives that

$$\varphi(k)\dot{+}\varphi(0) = \varphi(k) \Leftrightarrow \varphi(0) = 2$$

Therefore $\forall\, k \in \mathbb{Z}$, $\varphi(k) = \frac{1}{2}\,\varphi(0) + 1 = 2$. □

A class of examples not of product type, i.e. defined by semi–groups not of the form (7), might be built as follows. Suppose that $[\hat{B}_k^n, \hat{B}_{k'}^{n'}] = 0$, $[\hat{B}_k^n, \hat{B}_{k''}^{n''}] \neq 0$, and $[\hat{B}_{k'}^{n'}, \hat{B}_{k''}^{n''}] \neq 0$. Then the $*$–algebra generated by $\{\hat{B}_k^n, \hat{B}_{k'}^{n'}, \hat{B}_{k''}^{n''}\}$ should not be of product type.

3. Abelian sub–algebras of w_∞

Lemma 3.1. *Any subset of the set*

$$\mathcal{A}_0 := \{\hat{B}_0^n : n \in \mathbb{N}_{\geq 2}\} \tag{10}$$

consists of commuting self–adjoint generators. The set (10) is a maximal set with this property and generates a maximal Abelian $*$–sub–algebra of w_∞.

Proof. The commutativity of the set (10) is clear from (2). The same identity shows that if $X \in W_\infty$, then $\forall\, n \in \mathbb{N}_{\geq 2}$, $[\hat{B}_0^n, X]$ is a linear combination of the (linearly independent) generators of the form $\hat{B}_k^n$ with $k \neq 0$. Therefore either $X \in \mathcal{A}_0$ or X cannot commute with $\mathcal{A}_0$. This proves maximality. That $\mathcal{A}_0$ is a $*$–sub-algebra follows from the fact that the generators are self–adjoint. □

Lemma 3.2. *If a subset $\hat{\mathcal{S}}$ of generators of the form (1) contains an element of the form $\hat{B}_0^n$, then $\hat{\mathcal{S}}$ can be a commutative subset if and only if*

$$\hat{B}_k^m \in \hat{\mathcal{S}} \Rightarrow k = 0 \tag{11}$$

Proof. From Lemma 3.1 we know that (11) is a sufficient condition for commutativity of $\hat{\mathcal{S}}$. Let us prove that, under the conditions of the Lemma, it is also necessary. Suppose that $\hat{B}_k^m \in \hat{\mathcal{S}}$ and that $k \neq 0$. Then (2) implies that $0 = [\hat{B}_0^n, \hat{B}_k^m] = k(m-1)\hat{B}_k^{n+m-2}$. Since by assumption $m, n \geq 2$ and $\hat{B}_k^{n+m-2} \neq 0$, it follows that $k = 0$, against the assumption. □

Lemma 3.3. *Two generators $\hat{B}_k^n$, $\hat{B}_K^N$ with k, $K \neq 0$, commute if and only if sgn $(k) =$ sgn $(K) =: \pm$ and there exist $p, q \in \mathbb{N} \cup \{0\}$ mutually prime, such that, for some k', $K' \geq 1$: $(n, k) = (1+qk', \pm pk')$ and $(n, K) = (1 + qK', \pm pK')$.*

Proof. We have that

$$0 = [\hat{B}_k^n, \hat{B}_K^N] = (k(N-1) - K(n-1))\hat{B}_{k+K}^{n+N-2}$$

Since $\hat{B}_{k+K}^{n+N+2} \neq 0$, this is equivalent to $k(N-1) - K(n-1) = 0$. Since $N, n \geq 2$, this is possible if and only if k and K have the same sign. In this case the condition is equivalent to

$$\frac{k}{n-1} = \frac{K}{N-1} =: \pm\frac{p}{q}$$

where p and q are mutually prime natural integers and the $\pm$ sign is the common sign of k and K. This means that $k = \pm pk'$, $n-1 = qk'$ and $K = \pm pK'$, $N-1 = qK'$ where the sign $\pm$ is the same in both cases and k', $K' \geq 1$. This is equivalent to the statement in the Lemma. $\square$

Definition 3.1. A half–line in $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ is a subset either of the form

$$H_{\varepsilon,p,q} := \{(1+qk, \varepsilon pk) : k \in \mathbb{N} \cup \{0\}\}$$

where $\varepsilon \in \{\pm 1\}$ and q, $p \in \mathbb{N} \cup \{0\}$ are mutually prime, or of the form

$$H_{1,0,q} := \{(1+qk, 0) : k \in \mathbb{N} \cup \{0\}\}$$

Theorem 3.1. *Each of the three sets of indices* $H_{1,0,1} = \{(1+k,0) : k \in \mathbb{N} \cup \{0\}\}$, $H_{+,1,1} = \{(1+k,k) : k \in \mathbb{N} \cup \{0\}\}$ *and* $H_{-,1,1} = \{(1+k,-k) : k \in \mathbb{N} \cup \{0\}\}$ *defines a maximal family of mutually commuting generators.*

Proof. We know from Lemma 3.1 that $H_{1,0,1}$ is a mutually commuting family. The same is true for $H_{+,1,1}$ and $H_{-,1,1}$ because of Lemma 3.3. Now let $\hat{\mathcal{S}}$ be a mutually commuting family of generators (1). If $\hat{\mathcal{S}}$ contains a generator of the form $\hat{B}_0^n$, for some $n \in \mathbb{N}_{\geq 2}$, from Lemma 3.2 we know that $\hat{\mathcal{S}} \subseteq H_{1,0,1}$. If this is not the case, then from Lemma 3.3 we know that $\hat{\mathcal{S}}$ is contained in some half–line $H_{\varepsilon,p,q}$ in $\mathbb{N}_{\geq 2} \times \mathbb{Z}$ with $p \neq 0$. But all half–lines of this type, with $\varepsilon = +1$ (resp. $\varepsilon = -1$), are contained in $H_{+,1,1}$ (resp. $H_{-,1,1}$) and this implies the statement. $\square$

Notice that, of the three families listed in Theorem 3.1, only $H_{1,0,1}$ generates a $*$–sub–algebra.

4. Basic facts on central extensions of Lie algebras

If L and $\widetilde{L}$ are two complex Lie algebras, we say that $\widetilde{L}$ is a one-dimensional *central extension* of L with *central element* E if there is a Lie algebra exact sequence $0 \mapsto \mathbb{C}\,E \mapsto \widetilde{L} \mapsto L \mapsto 0$ where $\mathbb{C}\,E$ is the one-dimensional trivial Lie algebra and the image of $\mathbb{C}\,E$ is contained in the center $Cent(L)$ of $\widetilde{L}$ i.e.,

$$[l_1, E]_{\widetilde{L}} = 0 \qquad , \qquad \forall l_1 \in L$$

where $[\cdot,\cdot]_{\widetilde{L}}$ are the Lie brackets in $\widetilde{L}$. For $*$–Lie algebras we also require that the central element E is self–adjoint, i.e

$$(E)^* = E \tag{12}$$

A 2-*cocycle* on L is a bilinear form $\phi : L \times L \mapsto \mathbb{C}$ on L satisfying, for all $l_1, l_2 \in L$, the skew-symmetry condition

$$\phi(l_1, l_2) = -\phi(l_2, l_1)$$

(in particular $\phi(l, l) = 0$ for all $l \in L$) and the 2-*cocycle* identity:

$$\phi([l_1, l_2]_L, l_3) + \phi([l_2, l_3]_L, l_1) + \phi([l_3, l_1]_L, l_2) = 0 \tag{13}$$

One-dimensional central extensions of L are classified by 2-cocycles in the sense that $\widetilde{L}$ is a central extension of L if and only if, as vector space, it is the direct sum

$$\widetilde{L} = M \oplus \mathbb{C}\, E$$

where M is a Lie algebra isomorphic to L, and there exists a 2-cocycle on L such that, for all $l_1, l_2 \in L$, the Lie brackets in $\widetilde{L}$ are given by

$$[l_1, l_2]_{\widetilde{L}} = [l_1, l_2]_L + \phi(l_1, l_2)\, E \tag{14}$$

where, in the right hand sides of (14), L is identified to $L \oplus \{0\} \subseteq L \oplus \mathbb{C}\, E$, and $\phi : L \times L \mapsto \mathbb{C}$ is a 2-*cocycle* on L,

$$[l_1, l_2]_{\widetilde{L}} = [l_1, l_2]_L + \phi(l_1, l_2)\, E$$

where $[\cdot,\cdot]_L$ are the Lie brackets in L. A central extension is *trivial* if the corresponding 2-cocycle ϕ is uniquely determined by a linear function f : $L \mapsto \mathbb{C}$ through the identity

$$\phi(l_1, l_2) = f([l_1, l_2]_L) \qquad , \qquad \forall l_1, l_2 \in L \tag{15}$$

Such a 2-cocycle is called a 2-*coboundary*, or a *trivial* 2-cocycle. Two extensions are called *equivalent* if each of them is a trivial extension of the other. This is the case if and only if the difference of the corresponding 2-cocycles is a trivial cocycle. A central extension $\widetilde{L}$ of L is called *universal* whenever there exists a homomorphism from $\widetilde{L}$ to any other central extension of L. A Lie algebra L possesses a universal central extension if and only if L is *perfect* (i.e. $L = [L, L]$). In this case, the universal central extension of L is unique up to isomorphism.

Notice that the 2-cocycle identity (13) implies that, if $l_c \in Cent(L)$ is an element of the center of L, then

$$\phi([l_1, l_2]_L, l_c) = 0 \qquad ; \quad \forall l_1, l_2 \in L$$

i.e. l_c is ϕ–orthogonal to the derived set $[L, L]$ of L. Similarly (15) implies that a necessary condition for the 2-cocycle ϕ to be trivial is that the center of L is ϕ–orthogonal to the whole algebra L. Because of (14) this is equivalent to say that the center of L is mapped into the center of $\widetilde{L}$. Therefore a sufficient condition for a 2-cocycle ϕ on L to be non trivial is that there exist $l_c \in Cent(L)$ and $x \in L \setminus [L, L]$ such that

$$\phi(x, l_c) \neq 0$$

This practical rule is useful for Lie algebras L with a *large* derivative $[L, L]$.

5. Central extensions of w_N

Throughout this section we assume that $\widetilde{w_N}$ is a central extension of w_N, where $N > 0$ is fixed. For $N = 0$, the Witt algebra w_0 admits the well-known non-trivial Virasoro central extension

$$[W_k^0, W_m^0] = (k - m)\, W_{k+m}^0 + \delta_{k+m,0}\, m\, (m^2 - 1)\, E$$

We denote by $c(n, k; l, m)$ the value assumed by the corresponding 2–cocycle on the pair of generators (W_k^n, W_m^l), i.e.:

$$c(n, k; l, m) := \phi(W_k^n, W_m^l) \in \mathbb{C} \tag{16}$$

$$[W_k^n, W_m^l] = ((k\, l - m\, n)\, N + (k - m))\; W_{k+m}^{n+l} + c(n, k; l, m)\, E$$

The skew-symmetry of ϕ and the adjointness condition (12) imply respectively that:

$$c(n,k;l,m) = -c(l,m;n,k) \tag{17}$$

$$c(n,k;l,m) = -\overline{c(n,-k;l,-m)} \tag{18}$$

If at least one of n,l is negative we set

$$c(n,k;l,m) = 0 \tag{19}$$

Lemma 5.1. *The derived set of the w_N $*$–Lie algebra is itself.*

Proof. From (9) we see that the derived set of the w_N $*$–Lie algebra is

$$Der(W_N) := \{W_{k+m}^{n+l} \,:\, (k\,l - m\,n)\,N + (k-m) \neq 0, n,l \in \mathbb{N}\cup\{0\}, k,m \in \mathbb{Z}\}$$

Choosing $(n,k) = (0,0)$ we see that $Der(W_N)$ contains the generators of the form W_m^l with $l \in \mathbb{N}\cup\{0\}$ and $m \in \mathbb{Z}\setminus\{0\}$. Choosing $n = 0$ and $(k,m) = (1,-1)$ we see that $Der(W_N)$ also contains the generators of the form W_0^l such that $l\,N + 2 \neq 0$ which is always true for all $l \in \mathbb{N}\cup\{0\}$. $\square$

Combining the remark after equation (15), with Lemma 5.1 one deduces that, in any central extension of W_N, the central element is mapped to the central element of the extension so that, for any $l \in \mathbb{N}\cup\{0\}$ and $m \in \mathbb{Z}$

$$c(0,0;l,m) = 0 \tag{20}$$

Lemma 5.2. *On the w_N generators W_k^n, for the family $\{c(n,k;l,m)\}$ defined by (16), the 2–cocycle identity (13) is equivalent to*

$$\begin{aligned} &((k_1\,n_2 - k_2\,n_1)\,N + (k_1 - k_2))\,c(n_1+n_2, k_1+k_2; n_3, k_3) \\ &+((k_2\,n_3 - k_3\,n_2)\,N + (k_2 - k_3))\,c(n_2+n_3, k_2+k_3; n_1, k_1) \\ &+((k_3\,n_1 - k_1\,n_3)\,N + (k_3 - k_1))\,c(n_3+n_1, k_3+k_1; n_2, k_2) = 0 \end{aligned} \tag{21}$$

Conversely any family $\{c(n,k;l,m)\}$ satisfying (21) defines, through (16), a 2–cocycle on w_N.

Proof. For all n_i, k_i, where $i = 1, 2, 3$, making use of (17) we have

$$0 = \phi([W^{n_1}_{k_1}, W^{n_2}_{k_2}], W^{n_3}_{k_3}) + \phi([W^{n_2}_{k_2}, W^{n_3}_{k_3}], W^{n_1}_{k_1}) + \phi([W^{n_3}_{k_3}, W^{n_1}_{k_1}], W^{n_2}_{k_2})$$

$$= ((k_1\, n_2 - k_2\, n_1)\, N + (k_1 - k_3))\, \phi(W^{n_1+n_2}_{k_1+k_2}, W^{n_3}_{k_3})$$

$$+((k_2\, n_3 - k_3\, n_2)\, N + (k_2 - k_3))\, \phi(W^{n_2+n_3}_{k_2+k_3}, W^{n_1}_{k_1})$$

$$+((k_3\, n_1 - k_1\, n_3)\, N + (k_3 - k_1))\, \phi(W^{n_3+n_1}_{k_3+k_1}, W^{n_2}_{k_2})$$

$$= ((k_1\, n_2 - k_2\, n_1)\, N + (k_1 - k_3))\, c(n_1 + n_2, k_1 + k_2, n_3, k_3)$$

$$+((k_2\, n_3 - k_3\, n_2)\, N + (k_2 - k_3))\, c(n_2 + n_3, k_2 + k_3, n_1, k_1)$$

$$+((k_3\, n_1 - k_1\, n_3)\, N + (k_3 - k_1))\, c(n_3 + n_1, k_3 + k_1, n_2, k_2)$$

The converse is clear due to the linear independence of the generators. □

We notice that the sum of the first and third (resp. second and fourth) arguments in the three 2-cocycle values $c(n_2 + n_3, k_2 + k_3; n_1, k_1)$, $c(n_1 + n_2, k_1+k_2; n_3, k_3)$ and $c(n_3+n_1, k_3+k_1; n_2, k_2)$ appearing in (21) is equal to $n_1+n_2+n_3$ (resp. $k_1+k_2+k_3$). We are thus led to the following definition.

Definition 5.1. Given natural integers $n_1, n_2, n_3 \geq 0$ and $k_1, k_2, k_3 \in \mathbb{Z}$, define $S \in \mathbb{N} \cup \{0\}$ and $M \in \mathbb{Z}$ by:

$$S := n_1 + n_2 + n_3 \qquad ; \qquad M := k_1 + k_2 + k_3$$

and

$$\psi_{S,M}(n_i, k_i) := c(S - n_i, M - k_i; n_i, k_i)\ ;\ i \in \{1, 2, 3\} \tag{22}$$

Corollary 5.1. *The skew-symmetry condition (17) becomes*

$$\psi_{S,M}(n_i, k_i) = -\psi_{S,M}(S - n_i, M - k_i)$$

and (21) is equivalent to

$$\begin{aligned} &((k_1\, n_2 - k_2\, n_1)\, N + (k_1 - k_2))\, c(S - n_3, M - k_3; n_3, k_3) \\ +&((k_2\, n_3 - k_3\, n_2)\, N + (k_2 - k_3))\, c(S - n_1, M - k_1; n_1, k_1) \\ +&((k_3\, n_1 - k_1\, n_3)\, N + (k_3 - k_1))\, c(S - n_2, M - k_2; n_2, k_2) = 0 \end{aligned} \tag{23}$$

or in ψ-form

$$\begin{aligned} &((k_1\, n_2 - k_2\, n_1)\, N + (k_1 - k_2))\, \psi_{S,M}(n_3, k_3) \\ +&((k_2\, n_3 - k_3\, n_2)\, N + (k_2 - k_3))\, \psi_{S,M}(n_1, k_1) \\ +&((k_3\, n_1 - k_1\, n_3)\, N + (k_3 - k_1))\, \psi_{S,M}(n_2, k_2) = 0 \end{aligned} \tag{24}$$

Proof. The proof follows directly from Definition 5.1. □

Proposition 5.1. *For any $\lambda \in \mathbb{R}$ the family $\{c(n, k; l, m)\}$, defined by*

$$c(n, k; l, m) := \delta_{k+m,0}\, \lambda\, k \tag{25}$$

defines, through (16), a 2–cocycle on w_N.

Proof. Condition (17) is verified by inspection and (18) follows from the fact that λ is real. We want to prove that (24) this is satisfied by the family $\{c(n, k; l, m)\}$, defined by (25). Direct substitution shows that, if the family $\{c(n, k; l, m)\}$ is defined by (25), then $\psi_{S,M}$, defined by (22), satisfies (24). Moreover, $\psi_{S,M}(n_i, k_i) = \delta_{M,0}\, \lambda\, k_i$ implies that $c(S - n_i, M - k_i; n_i, k_i) = \delta_{M,0}\, \lambda\, k_i$. For $i = 1$ we get $c(S - n_1, M - k_1; n_1, k_1) = \delta_{M,0}\, \lambda\, k_1$ which for $n_3 = 0$ becomes $c(n_2, k_2 + k_3; n_1, k_1) = \delta_{M,0}\, \lambda\, k_1$. Letting $k_2 + k_3 := K$ we have that

$$c(n_2, K; n_1, k_1) = \delta_{k_1+K,0}\, \lambda\, k_1$$

i.e. $c(n, k; l, m) = \delta_{k+m,0}\, \lambda\, k$. □

Proposition 5.2. *The central extension*

$$[W^n_k, W^l_m] = ((k\,l - m\,n)\,N + (k-m))\;W^{n+l}_{k+m} + \delta_{k+m,0}\,\lambda\,k\,E$$

of w_N *is trivial.*

Proof. We look for a linear complex-valued function f defined on w_N such that

$$f\left([W^n_k, W^l_m]\right) = \delta_{k+m,0}\,k\,\lambda \tag{26}$$

By the w_N commutation relations (9) and the linearity of f, equation (Ref. 26) is equivalent to

$$((k\,l - m\,n)\,N + (k-m))\;\; f\left(W^{n+l}_{k+m}\right) = \delta_{k+m,0}\,k\,\lambda \tag{27}$$

For $k+m \neq 0$ this is equivalent to

$$f\left(W^{n+l}_x\right) = 0 \qquad ; \qquad \forall x \in \mathbb{Z} \setminus \{0\} \tag{28}$$

For $k+m=0 \Leftrightarrow m=-k$ (27) is equivalent to

$$((k\,l + k\,n)\,N + 2k)\;\; f\left(W^{n+l}_0\right) = \; k\,\lambda \Leftrightarrow ((l+n)\,N+2)\;\; f\left(W^{n+l}_0\right) = \; \lambda$$

$$\Leftrightarrow f\left(W^{n+l}_0\right) = \; \frac{\lambda}{(l+n)\,N+2}$$

and this, together with (28) uniquely defines a linear functional f with the required property. Therefore the central extension of w_N is trivial. $\square$

Lemma 5.3. *Let* $z \in \mathbb{C}$. *If* $z = 2\,\bar{z}$ *then* $z = 0$.

Proof. If $z = x + i\,y$, $x, y \in \mathbb{R}$, then $z = 2\,\bar{z}$ implies that $x = 2x$ and $y = -2\,y$. Therefore $x = y = 0$. $\square$

Lemma 5.4. *In the notation of Definition 5.1, let* $S \in \mathbb{N} \cup \{0\}$, $M = 0$ *and* $N > 0$. *Then:*

(i) $\psi_{S,0}(0,1) = c(S,-1;0,1) = 0$

(ii) For all $k \in \mathbb{Z}$, $\psi_{S,0}(0,-k) = c(S,k;0,-k) = 0$

(iii) For all $n \geq 0$ *and* $k \in \mathbb{Z}$, $c(S-n,k;n,-k) = 0$

Notice that (iii) $\Rightarrow$ *(ii)* $\Rightarrow$ *(i).*

Proof. (i) For $n_2 = S - n_1$, $n_3 = 0$, $k_1 = 0$, $k_2 = -1$ and $k_3 = 1$, (21) yields

$$(n_1 N + 1)\, c(S, -1; 0, 1) = \tag{29}$$
$$((S - n_1)\, N + 2)\, c(S - n_1, 0; n_1, 0) + (n_1 N + 1)\, c(S - n_1, -1; n_1, 1)$$

For $n_3 = n_1$, $n_2 = S - 2\, n_1$, $k_1 = 1$, $k_2 = -1$ and $k_3 = 0$, (21) yields

$$((S - n_1)\, N + 2)\, c(S - n_1, 0; n_1, 0) = \tag{30}$$
$$(1 + n_1 N)\, c(S - n_1, -1; n_1, 1) - (n_1 N + 1)\, c(S - 2\, n_1, -1; 2\, n_1, 1)$$

Substituting (30) in (29) we obtain

$$(n_1 N + 1)\, c(S, -1; 0, 1) = (n_1 N + 1)\, c(S - n_1, -1; n_1, 1)$$

$$-(n_1 N + 1)\, c(S - 2\, n_1, -1; 2\, n_1, 1) + (n_1 N + 1)\, c(S - n_1, -1; n_1, 1)$$

which for $n_1 = S$, since by (19) $c(-S, -1; 2\, S, 1) = 0$, after dividing out $(S\, N + 1)$, yields with the use of (17) and (18)

$$c(S, -1; 0, 1) = 2\, c(0, -1; S, 1) = -2\, c(S, 1; 0, -1) = 2\, \overline{c(S, -1; 0, 1)}$$

which, by Lemma 5.3, implies that $c(S, -1; 0, 1) = 0$.

(ii) For $n_1 = S$, $n_2 = 0$, $n_3 = 0$, $k_1 = k$, $k_2 = 1$, $k_3 = -(k + 1)$, letting $a_k := c(S, k; 0, -k)$, since by (i) $a_{-1} = 0$, (21) yields

$$(k - S\, N - 1)\, a_{k+1} = (k + 2)\, a_k$$

which implies that $a_k = 0$ for all k.

(iii) For $k_1 = k \neq 0$, $k_2 = -k$, $k_3 = 0$, $n_1 = S - n$, $n_2 = 0$ and $n_3 = n$, after dividing out $k \neq 0$ and using $c(S, k; 0, -k) = 0$, (21) yields

$$c(S - n, k; n, -k) = -\frac{(S - n)\, N + 2}{n\, N + 1}\, c(S - n, 0; n, 0)$$

for all $k \neq 0$. Similarly, for $k_1 = -k \neq 0$, $k_2 = k$, $k_3 = 0$, $n_1 = 0$, $n_2 = n$ and $n_3 = S - n$, (21) yields

$$c(S-n,k;n,-k) = -\frac{n\,N+2}{(S-n)\,N+1}\,c(S-n,0;n,0)$$

for all $k \neq 0$. Thus

$$\frac{(S-n)\,N+2}{n\,N+1}\,c(S-n,0;n,0) = \frac{n\,N+2}{(S-n)\,N+1}\,c(S-n,0;n,0) \qquad (31)$$

If $S = 2\,n$ then $c(S-n,0;n,0) = c(n,0;n,0) = 0$ by (17). If $S \neq 2\,n$ then $c(S-n,0;n,0) = 0$ by (31). $\square$

Proposition 5.3. *Let $S \in \mathbb{N} \cup \{0\}$ and $M \in \mathbb{Z}$. In the notation of Definition 5.1, all non-trivial 2–cocycles $\psi_{S,M}(n,k)$ on w_N are given by*

$$\psi_{S,M}(n,k) = \delta_{S,0}\,\delta_{M,0}\,k\,(k^2-1)$$

Proof. Case (i): $S = 0$. Then $n_1 + n_2 + n_3 = 0$ and so $n_1 = n_2 = n_3 = 0$ which means that we are reduced to the standard Witt-Virasoro case $W_k^0 = \hat{B}_k^2$. Therefore, the only non-trivial cocycle is

$$\psi_{S,M}(n,k) = \psi_{0,M}(n,k) = \delta_{M,0}\,k\,(k^2-1)$$

Case (ii): $S \neq 0$ and $M \neq 0$. For $n_3 = k_3 = 0$, using $c(n_2,k_2;n_1,k_1) = -c(n_1,k_1;n_2,k_2)$, $n_1 + n_2 = S$ and $k_1 + k_2 = M$, (21) yields

$$(k_1\,(n_2\,N+1) - k_2\,(n_1\,N+1))\,c(S,M;0,0) - (k_2+k_1)\,c(n_1,k_1;n_2,k_2) = 0$$

which, letting $n_2 = n$, $k_2 = k$, $n_1 = S-n$ and $k_1 = M-k$, implies that

$$\psi_{S,M}(n,k) = c(S-n,M-k;n,k)$$

$$= ((M-k)\,(n\,N+1) - k\,((S-n)\,N+1))\,c(S,M;0,0) = 0$$

by (20).

Case (iii): $S \neq 0$ and $M = 0$. For $k_3 = 0$, $n_1 = 0$, $k_1 \neq 0$, using Lemma 5.4 (ii) and (iii), (24) yields

$$\psi_{S,0}(n_2,k_2)=\frac{n_2\,N+1-\frac{k_2}{k_1}}{n_3\,N+1}\,\psi_{S,0}(n_3,0)+\frac{k_2}{k_1}\,\psi_{S,0}(0,k_1)=0 \qquad (32)$$

and the result follows by the arbitrariness of n_2 and k_2. $\square$

The next corollary shows that there are no non-trivial central extensions of w_N other than the Virasoro one.

Corollary 5.2. *The non-trivial central extensions of w_N are given by*

$$[W_k^n,W_m^l]=((k\,l-m\,n)\,N+(k-m))\;W_{k+m}^{n+l}+\delta_{n,0}\,\delta_{l,0}\,\delta_{k+m,0}\,m\,(m^2-1)\,E$$

Thus only the Virasoro sector of w_N can be extended in a non-trivial way.

Proof. By Proposition 5.3, in the notation of Definition 5.1,

$$\psi_{S,M}(n_1,k_1)=c(S-n_1,M-k_1;n_1,k_1)=\delta_{S,0}\,\delta_{M,0}\,k_1\,(k_1^2-1)$$

i.e.,

$$c(n_2+n_3,k_2+k_3;n_1,k_1)=\delta_{n_1+n_2+n_3,0}\,\delta_{k_1+k_2+k_3,0}\,k_1\,(k_1^2-1)$$

which, letting $n_3=k_3=0$, $n_1=n$, $k_1=k$, $n_2=l$ and $k_2=m$ implies that

$$c(n,k;l,m)=\delta_{n+l,0}\,\delta_{k+m,0}\,m\,(m^2-1)=\delta_{n,0}\,\delta_{l,0}\,\delta_{k+m,0}\,m\,(m^2-1) \quad \square$$

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