Mathematical Notes, Vol. 68, No. 6, 2000

Hilbert Module Realization of the Square of White Noise and Finite Difference Algebras

L. Accardi and M. Skeide

UDC 517

Abstract. We develop an approach to the representations theory of the *algebra of the square of white noise* based on the construction of Hilbert modules. We find the unique Fock representation and show that the representation space is the usual symmetric Fock space. Although we started with *one degree of freedom* we end up with *countably many degrees of freedom*. Surprisingly, our representation turns out to have a close relation to Feinsilver's *finite difference algebra*. In fact, there exists a holomorphic image of the finite difference algebra in the algebra of square of white noise. Our representation restricted to this image is the Boukas representation on the finite difference Fock space. Thus we extend the Boukas representation to a bigger algebra, which is generated by creators, annihilators, and number operators.

Key words: Fock space, creation, annihilation and number processes, white noise, Feinsilver's finite difference algebra, Hilbert module, Boukas representation, Kolmogorov decomposition.

1. Introduction

Following [1], by *white noise* we understand operator-valued distributions b_t^+ and b_t (indexed by the variable $t \in \mathbb{R}$) which fulfill the canonical commutation relations (CCR)

$$[b_t, b_s^+] = \delta(t-s).$$

Formally, the squares of the white noise should be operator-valued distributions $B_t^+ = {b_t^+}^2$ and $B_t = b_t^2$ fulfilling the commutation relations which follow from the CCR.

Unfortunately, it turns out that the objects B_t^+ and B_s are too singular. This manifests itself in the fact that their formal commutator has the factor $\delta^2(t-s)$ which *a priori* does not make sense. To overcome this trouble, it was proposed in [1] to consider a *renormalization* of the singular object δ^2, which replaces δ^2 by $2c\delta$ with $c > 0$. This choice is motivated by a *regularization procedure*, where δ is approximated by functions δ_ε such that $\delta_\varepsilon^2 \to 2c\delta$ in a suitable sense, and the constant c may be even complex.

After the renormalization, the commutator $[B_t, B_s^+] = \delta(t-s)(2c + 4b_t^+ b_t)$ has an operator part in the right-hand side, namely, the number density $N_t = b_t^+ b_t$. Since $[N_t, B_s^+] = \delta(t-s)2B_t^+$, we obtain a closed Lie algebra. Smearing out the densities by setting

$$B_f^+ = \int f(t) B_t^+ \, dt \qquad \text{and} \qquad N_a^+ = \int a(t) N_t^+ \, dt,$$

and computing the formal commutators, we find the following relations:

$$[B_f, B_g^+] = 2c \operatorname{Tr}(\overline{f} g) + 4N_{\overline{f}g}, \qquad f, g \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{1.1a}$$

$$[N_a, B_f^+] = 2B_{af}^+, \qquad a \in L^\infty(\mathbb{R}), \quad f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{1.1b}$$

and $[B_f^+, B_g^+] = [N_a, N_{a'}] = 0$, where we set $\operatorname{Tr} f = \int f(t)\, dt$. Our aim is to find a representation of the $*$-algebra generated by these relations.

In [1], the representation of this algebra was constructed with the help of the Kolmogorov decomposition for a certain positive definite kernel. It was not so difficult to define the correct kernel, but it was difficult

Translated from *Matematicheskie Zametki*, Vol. 68, No. 6, pp. 803–818, December, 2000.
Original article submitted October 9, 1999 .

0001–4346/2000/6856–0683$25.00

to show that it is positive definite. In [2], Sniady found an explicit form of the kernel which is obviously positive. In the present paper, we proceed in a different way, motivated by the following observations. Relation (1.1a) looks like the usual CCR with the exception that the inner product in the right-hand side takes values in the algebra generated by the number operators N_a. It is therefore natural to try a realization on a Hilbert module over the algebra of number operators. Additionally, on a Hilbert module, we have the chance to also realize the relation (1.1b) by representing explicitly a suitable left multiplication by number operators.

Since the number operators are unbounded, we cannot use the theory of Hilbert modules over C^*-algebras, but on the contrary, the theory of pre-Hilbert modules over $*$-algebras is required, as described in [3]. In Sec. 2, we summarize the necessary notions. In Sec. 3, we show how one can define representations of the algebra of number operators just by fixing the values of the representation on N_a. This is essential for the definition of left multiplication and allows to identify the algebra generated by N_a not just as an abstract algebra, but concretely as an algebra of number operators.

The main part is Sec. 4, where we construct a two-sided pre-Hilbert module E, and show that it is possible to construct a symmetric Fock module $\Gamma(E)$ over E. As we will see, the natural creation operators $a^*(f)$ on this Fock module and the natural left multiplication by N_a fulfill the relation (1.1b) and the relation (1.1a) up to an additive term. By the tensor product construction, we obtain a pre-Hilbert space in which the relations (1.1a) and (1.1b) are fulfilled. This representation coincides with the one constructed in [1].

In Sec. 5, we show that our representation space is isomorphic to the usual symmetric Fock space over $L^2(\mathbb{R}, \ell^2)$. In the final Sec. 6, we show that our representation may be regarded as an extension of the Boukas representation of Feinsilver's finite difference algebra [4] on the finite difference Fock space. The calculus based on the square of white noise generalizes the Boukas calculus [5]. In [6], Parthasarathy and Sinha realized the finite difference algebra by operators on a symmetric Fock space. However, they do not consider the question whether this representation is equivalent to the Boukas representation. It is very likely that the algebra of square of white noise allows a similar representation. It would also be interesting to know whether this representation is faithful.

2. Hilbert modules over $*$-algebras

In the sequel, we will need the notion of Hilbert module over $*$-algebras of unbounded operators. This makes the definition of positivity somewhat tricky. For a C^*-algebra, there are many equivalent ways to define positive elements and positive linear functionals. For a general $*$-algebra, different definitions give rise to different notions of positivity. For instance, the algebraic definition, where positive elements are those in the convex cone generated by all elements of the form b^*b, does not contain enough positive elements for our purposes. On the other hand, a weak definition which says "an element b is positive, if $\varphi(b) \geq 0$ for all positive functionals $\varphi(\cdot)$" is uncontrollable, because it does not allow to show that the inner product on a tensor product of Hilbert modules is again positive.

Here we follow the approach of [3], where a generating set of positive elements is introduced axiomatically, and we consider a certain cone generated by these elements. This definition of positivity remains controllable, because it is algebraic. However, it makes it necessary to involve directly the existence of a left multiplication on a two-sided Hilbert module. Therefore, we do not give a definition of a right Hilbert module, but immediately define a two-sided Hilbert module.

Definition 2.1. Let $\mathcal{B}$ be a unital $*$-algebra, and let S be a distinguished *generating* subset of self-adjoint elements in $\mathcal{B}$ containing $\mathbf{1}$. By $P(S)$, we denote the convex $\mathcal{B}$-cone generated by S (i.e., the set of all sums of elements of the form a^*ba with $b \in S$, $a \in \mathcal{B}$). We say the elements from $P(S)$ are S-*positive.*

A pre-Hilbert $\mathcal{B}$-module is a $\mathcal{B}$-$\mathcal{B}$-module E (where $\mathbf{1}x = x\mathbf{1} = x$) with a sesquilinear inner product $\langle \cdot, \cdot \rangle \colon E \times E \to \mathcal{B}$, fulfilling the following requirements:

[i] $\langle x, x \rangle = 0 \implies x = 0$ (*definiteness*);
[ii] $\langle x, yb \rangle = \langle x, y \rangle b$ (*right* $\mathcal{B}$-*linearity*);
[iii] $\langle x, by \rangle = \langle b^*x, y \rangle$ ($*$-*involution property*)

and the *positivity* condition, which assumes that for any choices of $b \in S$ and any of finitely many $x_i \in E$ there exist finitely many $b_k \in S$ and $b_{ki} \in \mathcal{B}$ such that

$$\langle x_i, bx_j \rangle = \sum_k b_{ki}^* b_k b_{kj}.$$

If the definiteness condition [i] is missing, then we speak of a *semi-inner product* and a *semi-Hilbert module.*

A mapping a on a semi-Hilbert $\mathcal{B}$-module E is *adjointable*, if there exists a mapping a^* on E such that $\langle x, ay \rangle = \langle a^* x, y \rangle$. By $\mathcal{L}^r(E)$ and $\mathcal{L}^a(E)$ we denote the spaces of right $\mathcal{B}$-linear and adjointable mappings on E respectively. Notice that on a pre-Hilbert module the adjoint is unique and $\mathcal{L}^a(E) \subset \mathcal{L}^r(E)$.

Since $\mathbf{1} \in S$, the inner product is S*-positive* (i.e. $\langle x, x \rangle \in P(S)$), and since S consists only of self-adjoint elements, the inner product is *symmetric* (i.e., $\langle x, y \rangle = \langle y, x \rangle^*$) and *left anti-linear* (i.e., $\langle xb, y \rangle = b^* \langle x, y \rangle$). It suffices to check the positivity on a subset of E, which generates E as a right module (see [3]); or to prove the existence of $b_k \in P(S)$.

Observation 2.2. Any semi-Hilbert module over a commutative algebra $\mathcal{B}$ has the trivial left module structure over $\mathcal{B}$, where the right and left multiplications are just the same. We denote this trivial left multiplication by $b^r \colon x \mapsto xb$ in order to distinguish it from a possible non-trivial left multiplication.

Let E and F be semi-Hilbert $\mathcal{B}$-modules. Their tensor product over $\mathcal{B}$ expressed by the formula $E \odot F = E \otimes F / \{xb \otimes y - x \otimes by\}$ becomes a semi-Hilbert $\mathcal{B}$-module if we introduce the inner product $\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle$. In the general case, $E \odot F$ and $F \odot E$ may be quite different objects (see for Example [7, Example 6.7]). Usually, the mapping $x \odot y \mapsto y \odot x$ is ill-defined. Therefore, it is not always possible to construct a symmetric Fock module over an arbitrary one-particle sector E. The tensor sign $\odot$ is "transparent" for algebra elements, i.e., $x \odot by = xb \odot y$. Any operator $a \in \mathcal{L}^a(E)$ gives rise to a well-defined operator $a \odot \mathrm{id} \in \mathcal{L}^a(E \odot F)$. This *embedding* is, however, not necessarily injective. For operators a on F, the situation is not so pleasant. In general, we can define $\mathrm{id} \odot a$ if a is bilinear, but this embedding need not to be injective.

Further, we will study semi-Hilbert modules. For several reasons, it is desirable to have a strictly positive inner product. For instance, contrary to a semi-inner product, an inner product guarantees the uniqueness of adjoint operators. We introduce a quotienting procedure which allows to construct a pre-Hilbert module from the given semi-Hilbert module, if on $\mathcal{B}$ there exists a separating set S^* of positive functionals which is compatible with the positivity structure determined by S. We say that a functional φ on $\mathcal{B}$ is S*-positive*, if $\varphi(b) \geq 0$ for all $b \in P(S)$. Let S^* be some set of S-positive functionals. We say that S^* *separates the points* (or S^* is *separating*), if the condition $\varphi(b) = 0$ for all $\varphi \in S^*$ implies $b = 0$. If S^* is a separating set of S-positive functionals on $\mathcal{B}$, then the set $\mathcal{N} = \{x \in E \colon \langle x, x \rangle = 0\}$ is a two-sided $\mathcal{B}$-submodule of E. Moreover, by definition, the quotient module $E_0 = E/\mathcal{N}$ inherits the pre-Hilbert $\mathcal{B}$-module structure $\langle x + \mathcal{N}, y + \mathcal{N} \rangle = \langle x, y \rangle$.

Before coming to Fock modules, we describe a construction which relates our algebraic definition of positivity with the "concrete" positivity of operators on pre-Hilbert spaces. Let π be a representation of $\mathcal{B}$ on a pre-Hilbert space G. In other words, G is a $\mathcal{B}$-$\mathbb{C}$-module. By equipping $\mathbb{C}$ with a convex $\mathbb{C}$-cone structure generated by $\mathbf{1}$ as a positive element, and by naturally extending our definition of pre-Hilbert modules to two-sided modules over different algebras in an obvious way, we can ask whether G is a pre-Hilbert module with its natural inner product. For this, it is necessary and sufficient to have the inequality $\langle g, \pi(b) g \rangle \geq 0$ for all $g \in G$ and all $b \in S$ (see [3]). If π has this property, then we say it is S*-positive.* In particular, π is S-positive if it sends elements in S to the sum of elements of the form $b^* b$ $(b \in \mathcal{L}^a(G))$. This property holds if and only if $\langle g, \pi(b) g \rangle \geq 0$ for all $g \in G$ and all $b \in S$ (see [3]).

Now let π be S-positive, and let E be a pre-Hilbert $\mathcal{B}$-module. Notice that $\mathrm{id}_{\mathbb{C}}$ constitutes a separating set of positive functionals on $\mathbb{C}$. The above tensor product construction goes through as before and we obtain a pre-Hilbert $\mathcal{B}$-$\mathbb{C}$-module $H = E \odot G$. In other words, H is a pre-Hilbert space with a representation $\rho(b) = b \odot \mathrm{id}$ for $\mathcal{B}$. Actually, with the same definition, ρ extends to a representation of $\mathcal{L}^a(E)$ on H. Additionally, we may interpret elements $x \in E$ as mappings $L_x \colon g \mapsto x \odot g$ in $\mathcal{L}^a(G, H)$ in $\mathcal{L}^a(G, H)$ with the adjoint $L_x^* \colon y \odot g \mapsto \pi(\langle x, y \rangle) g$. Of course, $L_{bxb'} = \rho(b) L_x \pi(b')$. Observe that $x \mapsto L_x$ is one-to-one if π is faithful. In this case the state ρ is also faithful.

Definition 2.3. A *full Fock module* over the two-sided pre-Hilbert $\mathcal{B}$-module E is the two-sided pre-Hilbert $\mathcal{B}$-module $\mathcal{F}(E) = \bigoplus_{n=0}^{\infty} E^{\odot n}$, where $E^{\odot 0} = \mathcal{B}$ with the inner product $\langle b, b'\rangle = b^*b'$, and the direct sum is algebraic. We denote the unit of $E^{\odot 0}$ by ω in order to distinguish it from the right or left multiplication by $\mathbf{1} \in \mathcal{B}$.

On $\mathcal{F}(E)$, we define for each $x \in E$ the *creation operator* $\ell^*(x)$ by setting

$$\ell^*(x)x_n \odot \cdots \odot x_1 = x \odot x_n \odot \cdots \odot x_1, \qquad \ell^*(x)\omega = x,$$

and its adjoint *annihilation operator*

$$\ell(x)x_n \odot \cdots \odot x_1 = \langle x, x_n\rangle x_{n-1} \odot \cdots \odot x_1, \qquad \ell(x)\omega = 0.$$

For each bilinear operator a on E, we define the operator $\lambda(a)$ on $\mathcal{F}(E)$ by setting

$$\lambda(a)x_n \odot \cdots \odot x_1 = ax_n \odot x_{n-1} \odot \cdots \odot x_1 + x_n \odot ax_{n-1} \odot \cdots \odot x_1 + \cdots + x_n \odot x_{n-1} \odot \cdots \odot ax_1$$

and $\lambda(a)\omega = 0$.

Fock modules were first considered in [8, 9]. The first formal definition of full Fock modules in the framework of Hilbert C^*-modules was given in [10, 11]. Here we use the extension to the framework of $*$-algebras as given in [3].

3. The algebra of number operators of $L^\infty(\mathbb{R})$

Consider the commutative $*$-algebra of number operators on the symmetric Fock space $\Gamma(L^2(\mathbb{R}))$ to elements of $L^\infty(\mathbb{R})$. As for the full Fock module, we use the algebraic definition of symmetric Fock space and regard $\Gamma(L^2(\mathbb{R}))$ as a subspace of $\mathcal{F}(L^2(\mathbb{R}))$.

Definition 3.1. On the full Fock space $\mathcal{F}(L^2(\mathbb{R}))$ with vacuum vector denoted by Ω, we define a projection P by setting

$$Pf_n \otimes \cdots \otimes f_1 \overset{def}{=} \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(n)} \otimes \cdots \otimes f_{\sigma(1)}$$

and $P\Omega = \Omega$. The *symmetric Fock space* $\Gamma(L^2(\mathbb{R}))$ coincides with $P\mathcal{F}(L^2(\mathbb{R}))$.

For $a \in L^\infty(\mathbb{R})$, we define the number operator $N_a = P\lambda(a)P$. By

$$\mathcal{N} = \operatorname{alg}\{N_a \ (a \in L^\infty(\mathbb{R}))\}$$

we denote the unital algebra generated by all N_a.

Clearly, $N_a^* = N_{a^*}$ and $N_a N_{a'} = N_{a'} N_a$, so that $\mathcal{N}$ is a commutative $*$-algebra. From the multiple polarization formula

$$\frac{1}{2^n} \sum_{\varepsilon_n, \ldots, \varepsilon_1 = \pm 1} \varepsilon_n \cdots \varepsilon_1 (\varepsilon_n f_n + \cdots + \varepsilon_1 f_1)^{\otimes n} = \sum_{\sigma \in S_n} f_{\sigma(n)} \otimes \cdots \otimes f_{\sigma(1)}$$

we see that the vectors Ω and $f^{\otimes n}$ $(f \in L^2(\mathbb{R}), n \in \mathbb{N})$ form a total subset of $\Gamma(L^2(\mathbb{R}))$. Note also that $N_a = P\lambda(a) = \lambda(a)P$. This follows easily from $P\lambda(a)P = \lambda(a)P$ and its adjoint.

Let I_1 and I_2 be two disjoint measurable subsets of $\mathbb{R}$. It is noteworthy that the well-known factorization formula

$$\overline{\Gamma(L^2(S_1)) \otimes \Gamma(L^2(S_2))} \cong \overline{\Gamma(L^2(S_1 \cup S_2))}$$

restricts to our algebraic domain, i.e.,

$$\Gamma(L^2(S_1)) \otimes \Gamma(L^2(S_2)) \cong \Gamma(L^2(S_1 \cup S_2))$$

(cf. the proof of Theorem 5.1). With this identification, we have $N_{\chi_{S_1}} = N_{\chi_{S_1}} \otimes \mathrm{id}$ and $N_{\chi_{S_2}} = \mathrm{id} \otimes N_{\chi_{S_2}}$. Similar statements are true for a factorization into more than two disjoint subsets.

Since $N_a\Omega = 0$ for any $a \in L^\infty(\mathbb{R})$, the vacuum state $\varphi_\Omega(\cdot) = \langle \Omega, \cdot\,\Omega\rangle$ is a character for $\mathcal{N}$. Its kernel consists of the span of all monomials with at least one factor N_a, and its GNS-pre-Hilbert space is just $\mathbb{C}\Omega$.

As the subset of $\mathcal{B}$ defining positivity on $\mathcal{B}$, we choose

$$S = \left\{ N_{\chi_{I_1}} \ldots N_{\chi_{I_n}} : I_i \text{ bounded intervals on } \mathbb{R} \ (n \in \mathbb{N}_0,\ i = 1, \ldots, n) \right\}.$$

Proposition 3.2. *The defining representation* id *of* $\mathcal{N}$ *on* $\Gamma(L^2(\mathbb{R}))$ *is* S*-positive.*

Proof. It is sufficient to show that N_{χ_I} is of the form $\sum\limits_i b_i^* b_i$, where the b_i are taken (for all I) from the commutative subalgebra of $\mathcal{L}^a\big(\mathcal{F}(L^2(\mathbb{R}))\big)$. Let $\lambda_i^n(a)$ $(n \in \mathbb{N}, 1 \le i \le n)$ be the representation of $L^\infty(\mathbb{R})$ which acts on the ith component of the n-particle sector of $\mathcal{F}(L^2(\mathbb{R}))$. Then

$$N_{\chi_I} = \sum_{1 \le i \le n < \infty} \lambda_i^n(\chi_I)* = \sum_{1 \le i \le n < \infty} \lambda_i^n(\chi_I)^* \lambda_i^n(\chi_I). \quad \square$$

In the next section we define a representation of $\mathcal{N}$ by assigning to each N_a an operator and an extension as an algebra homomorphism. The goal of the remainder of the present section is to show that this is possible, at least if we restrict to the subalgebra $\mathfrak{S}(\mathbb{R})$ of step functions, which is dense in $L^\infty(\mathbb{R})$ in a suitable weak topology.

Denote by N the *number operator* on $\mathcal{F}(L^2(\mathbb{R}))$ which sends $F \in E^{\otimes n}$ to nF. As $PN = NP$ we use the same symbol for the number operator on $\Gamma(L^2(\mathbb{R}))$. Then, clearly, $\operatorname{alg}\{N\}$ is isomorphic to the algebra of polynomials in one self-adjoint indeterminate variable. Moreover, for each measurable non-null-set $S \subset \mathbb{R}$, the algebra $\operatorname{alg}\{N_{\chi_S}\}$ is isomorphic to $\operatorname{alg}\{N\}$. Therefore, for any self-adjoint element a in a $*$-algebra $\mathcal{A}$ the mapping $N \mapsto a$ extends to a homomorphism $\operatorname{alg}\{N\} \to \mathcal{A}$.

Let $\mathfrak{t} = (t_0, \dots, t_m)$ be a $m+1$-tuple with $t_0 < \dots < t_m$. Then by using the factorization

$$\Gamma(L^2(t_0, t_m)) = \Gamma(L^2(t_0, t_1)) \otimes \cdots \otimes \Gamma(L^2(t_{m-1}, t_m))$$

we find

$$\mathcal{N}_\mathfrak{t} = \operatorname{alg}\{N_{\chi_{[t_{k-1}, t_k]}}\ (k = 1, \dots, m)\} = \operatorname{alg}\{N_{\chi_{[t_0, t_1]}}\} \otimes \cdots \otimes \operatorname{alg}\{N_{\chi_{[t_{m-1}, t_m]}}\}.$$

Therefore, any involutive mapping

$$T_\mathfrak{t} \colon \mathfrak{S}_\mathfrak{t}(\mathbb{R}) = \operatorname{span}\{\chi_{[t_{k-1}, t_k]}\ (k = 1, \dots, m)\} \to \mathcal{A}$$

with commutative range defines a unique homomorphism $\rho_\mathfrak{t} \colon \mathcal{N}_\mathfrak{t} \to \mathcal{A}$ satisfying $\rho_\mathfrak{t}(N_a) = T_\mathfrak{t}(a)$.

Now we are ready to prove the universal property of the algebra $\mathcal{N}_\mathfrak{S} := \bigcup_\mathfrak{t} \mathcal{N}_\mathfrak{t}$, which shows that $\mathcal{N}_\mathfrak{S}$ is a symmetric tensor algebra over the involutive vector space $\mathfrak{S}(\mathbb{R})$.

Theorem 3.3. *Let* $T \colon \mathfrak{S}(\mathbb{R}) \to \mathcal{A}$ *be an involutive mapping with commutative range. Then there exists a unique homomorphism* $\rho \colon \mathcal{N}_\mathfrak{S} \to \mathcal{A}$ *fulfilling the condition* $\rho(N_a) = T(a)$.

Proof. It suffices to remark that $\mathcal{N}_\mathfrak{S}$ is the inductive limit of $\mathcal{N}_\mathfrak{t}$ over the set of all tuples $\mathfrak{t}$ directed increasingly by the natural inclusion of tuples. Denoting by $\beta_{\mathfrak{t}\mathfrak{s}}$ the canonical embedding $\mathcal{N}_\mathfrak{s} \to \mathcal{N}_\mathfrak{t}$ $(\mathfrak{s} \le \mathfrak{t})$ we easily check that $\rho_\mathfrak{t} \circ \beta_{\mathfrak{t}\mathfrak{s}} = \rho_\mathfrak{s}$. In other words, the family $\rho_\mathfrak{t}$ has the unique extension as the homomorphism of ρ to all of $\mathcal{N}_\mathfrak{S}$. $\square$

4. Realization of the square of white noise

The idea underlying the realization of relations (1.1a) and (1.1b) on a symmetric Fock module is to take the right-hand side of (1.1a) as the definition of an $\mathcal{N}$-valued inner product on the module E generated by the elements $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and then to define the left multiplication by elements of $\mathcal{N}$ so that the generating elements f fulfill Eq. (1.1b). However, the direct attempt to use the inner product determined by Eq. (1.1a) fails. Therefore, we start with the linear ansatz (see (4.1)) and later adjust the constants in a suitable way.

In view of Theorem 3.3, for the time being, we restrict to elements in $\mathcal{N}_\mathfrak{S}$. By (1.1a), this makes it necessary also to restrict to elements $f \in \mathfrak{S}(\mathbb{R})$.

On $\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_\mathfrak{S}$ with its natural right $\mathcal{N}_\mathfrak{S}$-module structure, for arbitrary positive constants β, and γ we define the sesquilinear mapping $\langle \cdot, \cdot \rangle$ by setting

$$\langle f \otimes \mathbf{1}, g \otimes \mathbf{1} \rangle = M_{\overline{f}g}, \qquad \text{where} \quad M_a = \beta \operatorname{Tr} a + \gamma N_a, \tag{4.1}$$

and by right linear and left anti-linear extensions.

We define the left action of M_a by setting

$$M_a(f \otimes \mathbf{1}) \stackrel{def}{=} f \otimes M_a + \alpha a f \otimes \mathbf{1}$$

and using right linear extension to all elements of $\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}}$. Here α is an arbitrary real constant. Note that the scalar term in M_a does not change this commutation relation. Therefore, N_a fulfills the same commutation relations with α replaced by α/γ. By (1.1b), this fraction should be equal to 2.

By definition, multiplication by M_a from the left is a right linear mapping on $\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}}$. One easily checks that $M_{a^*} = M_a^*$ is an adjoint with respect to the sesquilinear mapping (4.1). By Theorem 3.3, this left action extends to a left action of all elements of $\mathcal{N}_{\mathfrak{S}}$.

Proposition 4.1. *The mapping* (4.1) *is a semi-inner product so that* $\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}}$ *is a semi-Hilbert* $\mathcal{N}_{\mathfrak{S}}$ *-module.*

Proof. We must check the positivity condition only, because the remaining properties are obvious. We remarked already that it is sufficient to check the positivity for elements of the form $\chi_{I_i} \otimes \mathbf{1}$, because these elements generate $\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}}$ as a right module. Additionally, we may assume that $I_i \cap I_j = \varnothing$ for $i \neq j$. Then $\langle \chi_{I_i} \otimes \mathbf{1}, b(\chi_{I_j} \otimes \mathbf{1}) \rangle = 0$ for $i \neq j$, whatever $b \in \mathcal{N}_{\mathfrak{S}}$ may be. Now let b be in S. We may assume (possibly after suitably modifying the set $\{I_i\}$) that b has the form $\prod_i N_{\chi_{I_i}}^{n_i}$, where $n_i \in \mathbb{N}_0$. Note that

$$N_{\chi_{I_i}}(\chi_{I_j} \otimes \mathbf{1}) = (\chi_{I_j} \otimes \mathbf{1}) N_{\chi_{I_i}}$$

for $i \neq j$ and

$$N_{\chi_I}^n(\chi_I \otimes \mathbf{1}) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\alpha}{\gamma}\right)^{(n-k)} (\chi_I \otimes N_{\chi_I}^k).$$

The proof is completed by induction. This implies that

$$\langle \chi_{I_i} \otimes \mathbf{1}, b(\chi_{I_j} \otimes \mathbf{1}) \rangle = \delta_{ij} M_{\chi_{I_i}} \sum_{k=0}^{n_i} \binom{n_i}{k} \left(\frac{\alpha}{\gamma}\right)^{(n_i-k)} N_{\chi_{I_i}}^k \prod_{\ell \neq i} N_{\chi_{I_\ell}}^{n_\ell}.$$

Set $b_k = \langle \chi_{I_k} \otimes \mathbf{1}, b(\chi_{I_k} \otimes \mathbf{1}) \rangle$ and $b_{ki} = \delta_{ki} \mathbf{1}$. Then

$$\langle \chi_{I_i} \otimes \mathbf{1}, b(\chi_{I_j} \otimes \mathbf{1}) \rangle = \sum_k b_{ki}^* b_k b_{kj},$$

where $b_k \in P(S)$. $\square$

We may filter out the length-zero elements so that

$$E := \mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}} / \mathcal{N}_{\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}}}$$

is a two-sided pre-Hilbert $\mathcal{N}_{\mathfrak{S}}$-module. Set

$$f \otimes b + \mathcal{N}_{\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}}} = fb.$$

Clearly we have

$$M_a f = f M_a + \alpha a f. \tag{4.2}$$

On the generating subset $f_n \odot \ldots \odot f_1$ of $E^{\odot n}$, by repeated application of Eq. (4.2) we find

$$M_a f_n \odot \cdots \odot f_1 = f_n \odot \cdots \odot f_1 M_a + \alpha \lambda(a) f_n \odot \cdots \odot f_1.$$

Therefore, on the full Fock module $\mathcal{F}(E)$ we have the relation

$$M_a = M_a^r + \alpha \lambda(a), \tag{4.3}$$

where M_a^r denotes the multiplication by M_a from the right in the sense of observation 2.2.

Now we try to define the symmetric Fock module over E similarly to Definition 3.1. The basis for the symmetrization is a flip which exchanges the order of factors in the tensor $f \otimes g$. As we already remarked, we may not hope to define flips on $E \odot E$ by just sending $x \odot y$ to $y \odot x$ for all $x, y \in E$. We may, however, hope to succeed, if we, as was done in [7] for *centered* modules, define such a flip only on the x, y that come from a suitable generating subset of E. In order to build the general flips on $E^{\odot n}$, we also must be sure that the flip is a bilinear operation.

Proposition 4.2. *The mapping*

$$\tau \colon f \odot g \mapsto g \odot f, \qquad f, g \in \mathfrak{S}(\mathbb{R}) \subset E,$$

extends to a unique bilinear unitary (i.e., inner product preserving and surjective) isomorphism $E \odot E \to E \odot E$.

Proof. We find the relation

$$\begin{aligned}\langle f \odot g, f' \odot g' \rangle &= \langle g, \langle f, f' \rangle g' \rangle = \langle g, M_{\overline{f}f'} g' \rangle = \langle g, g' M_{\overline{f}f'} + \alpha \overline{f} f' g' \rangle \\ &= M_{\overline{f}f'} M_{\overline{g}g'} + \alpha M_{\overline{gf}f'g'} = M_{\overline{g}g'} M_{\overline{f}f'} + \alpha M_{\overline{fg}g'f'} = \langle g \odot f, g' \odot f' \rangle.\end{aligned}$$

The elements $f \odot g$ form a (right) generating subset of $E \odot E$. Therefore, τ extends as a well-defined isometric mapping to $E \odot E$. Clearly, this extension is surjective so that τ is unitary.

It remains to show that τ is bilinear. Again it suffices to show that this property holds on the generating subset and, of course, to prove it only for the generators M_a. We find

$$\begin{aligned}\tau(M_a f \odot g) &= \tau(f \odot g M_a + \alpha(af \odot g + f \odot ag)) \\ &= g \odot f M_a + \alpha(g \odot af + ag \odot f) = M_a g \odot f = M_a \tau(f \odot g). \quad \square\end{aligned}$$

Now we are in a position to define the symmetric Fock module $\Gamma(E)$ precisely as in Definition 3.1. The preceding proposition also shows that $PM_a = M_a P$, i.e., P is a bilinear projection. Again, we have

$$P\lambda(a) = \lambda(a)P = P\lambda(a)P.$$

Consequently, the relation (4.3) remains true also on the symmetric Fock module. In the sequel, we do not distinguish between $\lambda(a)$ and its restriction to $\Gamma(E)$. In both cases we denote the number operator by $N := \lambda(\mathbf{1})$. Since $\lambda(a)$ is bilinear, so is N and, of course, $NP = PN$. Eq. (3.1) also implies that the symmetric tensors form a generating subset.

For $x \in E$, we define the creation operator on $\Gamma(E)$ as $a^*(x) = \sqrt{N} P \ell^*(x)$. Clearly, $x \mapsto a^*(x)$ is a bilinear mapping, because $x \mapsto \ell^*(x)$ is. We find the following commutation relation

$$M_a a^*(f) = a^*(M_a f) = a^*(f M_a + \alpha a f) = a^*(f) M_a + \alpha a^*(af).$$

Of course, $a^*(x)$ has an adjoint, namely, $a(x) = \ell(x) P \sqrt{N}$.

Now we restrict our attention to the creators $a^*(f)$ and annihilators $a(f)$ of elements f in $\mathfrak{S}(\mathbb{R}) \subset E$. Their actions on symmetric tensors $g^{\odot n}$ $(g \in \mathfrak{S}(\mathbb{R}))$ have the form

$$a^*(f) g^{\odot n} = \frac{1}{\sqrt{n+1}} \sum_{i=0}^{n} g^{\odot i} \odot f \odot g^{\odot (n-i)}, \qquad a(f) g^{\odot n} = \sqrt{n} M_{\overline{f}g} g^{\odot (n-1)}.$$

Clearly, $a^*(f) a^*(g) = a^*(g) a^*(f)$. However, nothing like this is true for $a^*(x)$ and $a^*(y)$ for more general elements in $x, y \in E$.

For the CCR, we have to compute the products $a(f) a^*(f')$ and $a^*(f') a(f)$. We find

$$\begin{aligned}a(f) a^*(f') g^{\odot n} &= \frac{1}{\sqrt{n+1}} a(f) \sum_{i=0}^{n} g^{\odot i} \odot f \odot g^{\odot (n-i)} = M_{\overline{f}f'} g^{\odot n} + M_{\overline{f}g} \sum_{i=0}^{n-1} g^{\odot i} \odot f' \odot g^{\odot (n-1-i)}, \\ a^*(f') a(f) g^{\odot n} &= \sqrt{n} a^*(f') M_{\overline{f}g} g^{\odot (n-1)} = \sqrt{n} \big(M_{\overline{f}g} a^*(f') - \alpha a^*(\overline{f} g f') \big) g^{\odot (n-1)} \\ &= M_{\overline{f}g} \sum_{i=0}^{n-1} g^{\odot i} \odot f' \odot g^{\odot (n-1-i)} - \alpha \lambda(\overline{f} f') g^{\odot n}.\end{aligned}$$

Taking the difference, the sums over i disappear. Taking into account the fact that $g^{\odot n}$ is arbitrary and using Eq. (4.3), we find

$$[a(f), a^*(f')] = M_{\overline{f}f'} + \alpha\lambda(\overline{f}f') = 2M_{\overline{f}f'} - M^r_{\overline{f}f'} = \beta \operatorname{Tr}(\overline{f}f') + 2\gamma N_{\overline{f}f'} - \gamma N^r_{\overline{f}f'}.$$

In other words, by putting $\beta = 2c$ and $\gamma = 2$, we have realized (1.1a) by the operators $a^*(f)$, however, only modulo the right multiplication by a certain element of $\mathcal{N}_{\mathfrak{S}}$. Notice that this is independent of the choice of α. Putting $\alpha = 4$, we also realize Eq. (1.1b).

So we have to do two things. Firstly, we must get rid of the contributions of N^r_a in the above relation. Secondly, in order to compare with the construction from [1], we must interpret our construction in terms of pre-Hilbert spaces. Both goals can be achieved at once by the following construction.

We consider the tensor product $H = \Gamma(E) \odot \mathbb{C}\Omega$ of $\Gamma(E)$ with the pre-Hilbert $\mathcal{N}_{\mathfrak{S}} - \mathbb{C}$-module $\mathbb{C}\Omega$ which is the pre-Hilbert space endowed with the GNS-representation of the vacuum state φ_Ω on $\mathcal{N}_{\mathfrak{S}}$. This tensor product is admissible, because by Proposition 3.2, the defining representation of $\mathcal{N}$ on $\Gamma(L^2(\mathbb{R}))$ is S-positive, and therefore, any subrepresentation on an invariant subspace is S-positive too. Thus, H is a pre-Hilbert space and carries the representation of $\mathcal{L}^a(\Gamma(E))$. In this representation all operators N^r_a are represented by 0. Indeed, N^r_a commutes with everything, so that we put it on the right, and obtain

$$N^r_a g^{\odot n} \odot \Omega = g^{\odot n} \odot N^r_a \Omega = 0.$$

By B^+_f, we denote the image of $a^*(f)$ in $\mathcal{L}^a(H)$. The image of N_a coincides with the image of $4\lambda(a)$. We denote both by the same symbol N_a. By $\Phi = \omega \odot \Omega$ we denote the vacuum vector in H.

Theorem 4.3. *The operators* $B^+_f, N_a \in \mathcal{L}^a(H)$ $(f, a \in \mathfrak{S}(\mathbb{R}))$ *fulfill the relations* (1.1a) *and* (1.1b), *and* $[B^+_f, B^+_g] = [N_a, N_{a'}] = 0$. *Moreover, the linear hull of the vectors* ${B^+_f}^n \Phi$ $(f \in \mathfrak{S}(\mathbb{R}), n \in \mathbb{N}_0)$ *is dense in* H.

Remark 4.4. It follows that H is precisely a pre-Hilbert space as constructed in [1]. However, in [1], the inner product on the total set of vectors was defined *a priori* and it was quite tedious to show that it is positive. Here the positivity and also the well-definedness of the representation are readily fulfilled.

Remark 4.5. Putting $H_n = \operatorname{span}\{{B^+_f}^n \Phi\ (f \in \mathfrak{S}(\mathbb{R}))\}$, we see that $H = \bigoplus_{n=0}^\infty H_n$ is an interacting Fock space with creation operators B^+_f as introduced in [12] in the notation of [3].

Theorem 4.6. *The realization of the relations* (1.1a) *and* (1.1b) *extends to elements* $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $a \in L^\infty(\mathbb{R})$ *as a representation by operators on* $\bigoplus_{n=0}^\infty \overline{H_n}$.

Proof. We extend the definition of the operators B^+_f and N_a formally to $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $a \in L^\infty(\mathbb{R})$, regarding them as operators on vectors of the form ${B^+_f}^n \Phi$ $(f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ n \in \mathbb{N}_0)$. We define the inner product of such vectors by continuous extension of the inner product of those vectors, where $f \in \mathfrak{S}$ in the weak-$*$ topology of $L^\infty(\mathbb{R})$ Such an extension clearly is well defined and unique. The positivity of this inner product is inherited from approximation by the inner products, and the well-definiteness of our operators follows, because all the operators have formal adjoints. □

5. The associated product system

Let $I \subset \mathbb{R}$ be a finite union of intervals. Denote by H_I the subspace of H spanned by the vectors of the form ${B^+_f}^n \Phi$ $(f \in \mathfrak{S}(I), n \in \mathbb{N}_0)$. In particular, for $0 \le t \le \infty$ set $H_t := H_{[0,t)}$. This means that $H_0 = H_\varnothing = \mathbb{C}\Phi$. Notice that H_I is independent of whether the intervals in I are open, half-open, or closed.

Denote by $I + t$ the time shifted set I. Denote by f_t the time shifted function $f_t(s) = f(s - t)$. Obviously, by sending ${B^+_f}^n \Phi$ to ${B^+_{f_t}}^n \Phi$ we define an isomorphism $H_I \to H_{I+t}$.

Observe that by relation (1.1a), the operators B_f and B^+_g corresponding to functions $f \in \mathfrak{S}(I)$ and $g \in \mathfrak{S}(\mathbb{R} \setminus I)$ commute. Define $\mathcal{N}_I := \operatorname{alg}\{N_a\ (a \in \mathfrak{S}(I))\}$. Then by the relation (1.1b) the elements of $\mathcal{N}_I$ also commute with all B_g corresponding to functions $g \in \mathfrak{S}(\mathbb{R} \setminus I)$.

Theorem 5.1. *Let $I, J \subset \mathbb{R}$ be finite unions of intervals such that $I \cap J = \varnothing$. Then*

$$U_{IJ} : B_f^{+n} B_g^{+m} \Phi \mapsto B_f^{+n} \Phi \otimes B_g^{+m} \Phi, \qquad f \in H_I, \quad g \in H_J,$$

extends as an isomorphism $H_{I \cup J} \to H_I \otimes H_J$. The composition of these isomorphisms is associative in the sense that

$$(U_{IJ} \otimes \mathrm{id}) \circ U_{(I \cup J)K} = (\mathrm{id} \otimes U_{JK}) \circ U_{I(J \cup K)}.$$

Proof. Of course, the vectors $B_f^{+n} B_g^{+m} \Phi$ are total in $H_{I \cup J}$ and the vectors $B_f^{+n} \Phi \otimes B_g^{+m} \Phi$ are total in $H_I \otimes H_J$. Thus, it suffices to justify isometry. We have

$$\left\langle B_f^{+n} B_g^{+m} \Phi, B_{f'}^{+n'} B_{g'}^{+m'} \Phi \right\rangle = \left\langle B_g^{+m} \Phi, B_f^n B_{f'}^{+n'} B_{g'}^{+m'} \Phi \right\rangle.$$

Without loss of generality, we assume that $n \geq n'$. Then we have

$$B_f^n B_{f'}^{+n'} = B_f^{n-n'} B_f^{n'} B_{f'}^{+n'} = B_f^{n-n'} \sum_{k=0}^{n'} b_k B_{f'}^{+k} B_f^k,$$

where $b_k \in \mathcal{N}_I$. Since B_f commutes with B_g^+ and $B_f \Phi = 0$, the only nonzero contribution comes from $B_f^{n-n'} b_0$. On the other hand, also b_0 commutes with B_g^+ and $b_0 \Phi = \Phi \varphi_\Omega(b_0) = \Phi \langle \Phi, b_0 \Phi \rangle$. Therefore, also $B_f^{n-n'}$ commutes with B_g^+ and comes to act directly on Φ and gives 0, unless $n = n'$. Hence the only nonzero contributions appear for $n = n'$ and $m = m'$. We obtain

$$\left\langle B_f^{+n} B_g^{+m} \Phi, B_{f'}^{+n} B_{g'}^{+m} \Phi \right\rangle = \left\langle B_g^{+m} \Phi, B_{g'}^{+m} \Phi \right\rangle \langle \Phi, b_0 \Phi \rangle = \left\langle B_g^{+m} \Phi, B_{g'}^{+m} \Phi \right\rangle \left\langle B_f^{+n} \Phi, B_{f'}^{+n} \Phi \right\rangle,$$

as desired. $\square$

Corollary 5.2. *We have*

$$H_s \otimes H_t \cong H_{[0,s)+t} \otimes H_t \cong H_{s+t}.$$

The isomorphisms

$$U_{st} : H_s \otimes H_t \to H_{s+t}$$

act associatively. In other words, the spaces H_t form a tensor product system of pre-Hilbert spaces in the sense of [13].

The definition given in [13] is purely algebraic. It is quite interesting to ask whether the $\overline{H_t}$ form a tensor product system in the stronger sense of Arveson [14], who introduced the notion of a *product system.* In particular, to this end all $\overline{H_t}$ must be separable infinite-dimensional spaces and some measurability conditions must be checked. We proceed in a more indirect way.

If $\overline{H_t}$ is an Arveson system, then naturally the question of what type it has arises. Type I product systems are precisely those which come from a symmetric Fock space. The characteristic property of the symmetric Fock space among other Arveson systems is that it is spanned by exponential vectors. In other words, we must find all families $\psi_t \in \overline{H_t}$ (so-called *units*) which compose under tensor product like exponential vectors to indicator functions, i.e., $\psi_s \otimes \psi_t = \psi_{s+t}$.

Good candidates for exponential vectors are

$$\psi_\rho(t) = \sum_{n=0}^{\infty} \frac{B_{\rho\chi_t}^{+\;n} \Phi}{n!} = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} B_{\chi_t}^{+n} \Phi,$$

where $\rho \in \mathbb{C}$ and $\chi_t := \chi_{[0,t]}$. If we replace B^+ by the creators on the ordinary symmetric Fock space, we obtain precisely the usual exponential vector $\psi(\rho\chi_t)$. Whenever $\psi_{\rho_0}(t)$ exists, it is an analytic vector-valued function of ρ with $|\rho| < |\rho_0|$. It is not difficult to check that whenever $\psi_\rho(s)$ and $\psi_\rho(t)$ exist, then also $\psi_\rho(s+t)$ exists and equals $\psi_\rho(s) \otimes \psi_\rho(t)$ (see Corollary 5.2). Moreover, since $\psi_\rho(t)$ is an analytic function in variable ρ, it is differentiable. It follows that

$$B_{\chi_t}^{+n} \Phi = \left. \frac{d^n}{d\rho^n} \right|_{\rho=0} \psi_\rho(t)$$

is in the closed linear span of $\psi_\rho(t)$ $(|\rho| < |\rho_0|)$. Therefore, if for each $t > 0$ there exists $\rho_0 > 0$ such that $\psi_{\rho_0}(t)$ exists, then the vectors $\psi_\rho(t)$ and their time shifts form a total subset of $\overline{H_\infty}$.

Lemma 5.3. *If* $|\rho| < 1/2$, *then* $\psi_\rho(t)$ *exists. Moreover,*

$$\langle \psi_\rho(t), \psi_\sigma(t)\rangle = e^{-(ct/2)\ln(1-4\overline{\rho}\sigma)}, \tag{5.1}$$

where the function

$$\varkappa\colon (\rho,\sigma) \mapsto -\frac{c}{2}\ln(1-4\overline{\rho}\sigma)$$

is a positive definite kernel on $U_{1/2}(0) \times U_{1/2}(0)$.

Proof. First, we show that the left-hand side of (5.1) exists in the simpler case $\sigma = \rho$. This establishes the existence of $\psi_\rho(t)$.

Set $f = \rho\chi_t$. Then $(\overline{f}f)f = |\rho|^2 f$. This yields the commutation relation $N_{\overline{f}f}B_f^+ = B_f^+ N_{\overline{f}f} + 2|\rho|^2 B_f^+$. Moreover, $2c\,\mathrm{Tr}(\overline{f}f) = 2c|\rho|^2 t$. We have

$$\begin{aligned}
B_f {B_f^+}^n &= B_f^+ B_f {B_f^+}^{n-1} + (2c|\rho|^2 t + 4N_{\overline{f}f}){B_f^+}^{n-1}\\
&= B_f^+ B_f {B_f^+}^{n-1} + {B_f^+}^{n-1}\big(2c|\rho|^2 t + 8|\rho|^2(n-1)\big) + {B_f^+}^{n-1} 4N_{\overline{f}f}\\
&= {B_f^+}^n B_f + n{B_f^+}^{n-1} 4N_{\overline{f}f} + {B_f^+}^{n-1} 2|\rho|^2\big((ct + 4(n-1)) + (ct + 4(n-2)) + \cdots + (ct+0)\big)\\
&= {B_f^+}^n B_f + n{B_f^+}^{n-1} 4N_{\overline{f}f} + {B_f^+}^{n-1} 2n|\rho|^2(ct + 2(n-1)).
\end{aligned}$$

If we apply this operator to the vacuum Φ, the first two summands disappear. We find the recursion formula

$$\frac{\langle {B_f^+}^n\Phi, {B_f^+}^n\Phi\rangle}{(n!)^2} = 4|\rho|^2\left(\frac{ct}{2n} + \frac{n-1}{n}\right)\frac{\langle {B_f^+}^{n-1}\Phi, {B_f^+}^{n-1}\Phi\rangle}{((n-1)!)^2}.$$

It is clear that the series

$$\sum_{n=0}^{\infty} \frac{\langle {B_f^+}^n\Phi, {B_f^+}^n\Phi\rangle}{(n!)^2}$$

converges if and only if $4|\rho|^2 < 1$ or $|\rho| < 1/2$.

For fixed $\rho \in U_{\frac{1}{2}}(0)$, the function $\langle \psi_\rho(t), \psi_\rho(t)\rangle$ is the uniform limit of entire functions in the variable t and, therefore, it is an entire function on t as well. In particular, since $\psi_\rho(s+t) = \psi_\rho(s) \otimes \psi_\rho(t)$, there must exist a number $\varkappa \in \mathbb{R}$ (actually, in $\mathbb{R}_+$, because $\langle \psi_\rho(t), \psi_\rho(t)\rangle \geq 1$) such that

$$\langle \psi_\rho(t), \psi_\rho(t)\rangle = e^{\varkappa t}.$$

We find $\varkappa$ by differentiating at $t = 0$. The only contribution to the product

$$\left.\frac{d}{dt}\right|_{t=0} 4|\rho|^2\left(\frac{ct}{2n} + \frac{n-1}{n}\right)\cdots 4|\rho|^2\left(\frac{ct}{2} + 0\right)$$

comes by the *Leibniz rule*, if we differentiate the last factor and put $t = 0$ in the remaining ones. We find

$$\left.\frac{d}{dt}\right|_{t=0} \langle \psi_\rho(t), \psi_\rho(t)\rangle = \sum_{n=1}^{\infty} (4|\rho|^2)^n \frac{1}{n}\frac{c}{2} = -\frac{c}{2}\ln(1 - 4|\rho|^2).$$

The remaining statements follow essentially by the same computations, replacing $|\rho|^2$ by $\overline{\rho}\sigma$. Clearly, $\overline{\rho}\sigma$ is a positive definite kernel. Then by the *Schur lemma*, the function $\varkappa(\rho,\sigma)$ is also positive definite as the limit of positive linear combinations of powers of $\overline{\rho}\sigma$. □

Remark 5.4. The function $\varkappa$ is nothing but a covariance function of the product system in the sense of Arveson [14], which is defined on the set of all units, restricted to the set of special units $\psi_\rho(t)$. In the set of all units, we must also take into account the multiples e^{ct} of our units. The covariance function on this two-parameter set is only a conditionally positive kernel.

Set

$$v_\rho = \sqrt{\frac{c}{2}}\left(2\rho, \frac{(2\rho)^2}{\sqrt{2}}, \dots, \frac{(2\rho)^n}{\sqrt{n}}, \dots\right) \in \ell^2.$$

Then $\langle v_\rho, v_\sigma \rangle = -(c/2)\ln(1-4\overline{\rho}\sigma)$ and the vectors v_ρ are total in ℓ^2. In other words, the Kolmogorov decomposition for the covariance function is the pair $(\ell^2, \rho \mapsto v_\rho)$. The following theorem is a simple corollary of Lemma 5.3.

Theorem 5.5. *There is a unique time shift invariant isomorphism*

$$\overline{H_\infty} \to \overline{\Gamma(L^2(\mathbb{R}_+, \ell^2))}$$

such that

$$\psi_\rho(t) \mapsto \psi(v_\rho \chi_t).$$

Consequently, $\overline{H_t}$ *is a type* I *Arveson product system.*

Remark 5.6. By defining E_I as the submodule of E generated by $\mathfrak{S}(I)$, we obtain the relation

$$\Gamma(E_{I\cup J}) = \Gamma(E_I) \odot \Gamma(E_J)$$

for disjoint I and J precisely as in the proof of Theorem 5.1. The above introduced isomorphism sends $a^*(f)^n a^*(g)^m \omega$ to $a^*(f)^n \omega \odot a^*(g)^m \omega$, and the argument is the same with the exception that, which makes it even simpler, $b\omega = \omega b$. Clearly, by setting $E_t = E_{[0,t)}$, we find a tensor product system $\{\Gamma(E_t)\}$ of pre-Hilbert $\mathcal{N}_{\mathfrak{S}} - \mathcal{N}_{\mathfrak{S}}$-modules in the sense of [13].

6. Connections with the finite difference algebra

After the rescaling $c \to 2$ and $\rho \to \rho/2$, the right-hand side of Eq. (5.1) extended in the obvious way from indicator functions to step functions, is the kernel used by Boukas [15, 16] to define the representation space for Feinsilver's *finite difference algebra* [4]. Therefore, the Boukas space and ours coincide.

Once established that the representation spaces coincide, it is natural to ask whether the algebra of the square of white noise contains elements fulfilling the relations of the finite difference algebra. Indeed, setting $c = 2$ and defining

$$Q_f = \frac{1}{2}(B_f^+ + N_f), \qquad P_f = \frac{1}{2}(B_{\overline{f}} + N_f), \qquad T_f = \mathbf{1}\operatorname{Tr} f + P_f + Q_f \tag{6.1}$$

for $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we obtain

$$[P_f, Q_g] = [T_f, Q_g] = [P_f, T_g] = T_{fg}. \tag{6.2}$$

In the particular case $f = \overline{f} \in \mathfrak{S}$, these are precisely the relations of the finite difference algebra. In fact, the operators Q_f, P_f and T_f are precisely those which were found by Boukas. However, it is not clear whether the relations $T_f = \mathbf{1}\operatorname{Tr} f + P_f + Q_f$ already follows from (6.2), or they are independent. In the second case, the Boukas representation has no chance to be faithful.

In all cases, the operators Q_f, P_f and T_f are not sufficient to recover B_f, B_f^+ and N_f. We can only recover the operators $B_f^+ - B_f$ and $B_f^+ + B_f + 2N_f$. Whereas the algebra of square of white noise is generated by creation, annihilation, and number operators, the representation of the finite difference algebra is generated by certain linear combinations of such operators.

The second author is supported by Deutsche Forschungsgemeinschaft.

References

1. L. Accardi, Y. G. Lu, and I. V. Volovich, *White Noise Approach to Classical and Quantum Stochastic Calculi*, Preprint, Rome, to appear in the lecture notes of the Volterra International School of the same title, held in Trento (1999).
2. P. Sniady, *Quadratic Bosonic and Free White Noise*, Preprint (1999).
3. L. Accardi and M. Skeide, *Interacting Fock Space Versus Full Fock Module*, Preprint, Rome, 1998, Revised in January 2000.
4. P. J. Feinsilver, "Discrete analogues of the Heisenberg–Weyl algebra," *Mh. Math.*, **104**, 89–108 (1987).
5. A. Boukas, "Stochastic calculus on the finite-difference Fock space," in: *Quantum Probability and Related Topics* (AccardiL., editor), Vol. VI, World Scientific (1991).
6. K. R. Parthasarathy and K. B. Sinha, "Unification of quantum noise processes in Fock spaces," in: *Quantum Probability and Related Topics* (Accardi L., editor), Vol. VI, World Scientific (1991), pp. 371–384.
7. M. Skeide, "Hilbert modules in quantum electrodynamics and quantum probability," *Comm. Math. Phys.*, **192**, 569–604 (1998).
8. L. Accardi and Y. G. Lu, "From the weak coupling limit to a new type of quantum stochastic calculus," in: *Quantum Probability and Related Topics* (Accardi L., editor), Vol. VII, World Scientific (1992), pp. 1–14.
9. L. Accardi and Y. G. Lu, "The Wigner semi-circle law in quantum electrodynamics," *Comm. Math. Phys.*, **180**, 605–632 (1996).
10. M. V. Pimsner, "A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$," in: *Free Probability Theory* (Voiculescu D. V., editor), Vol. 12, Fields Institute Communications (1997), pp. 189–212.
11. R. Speicher, *Combinatorial Theory of the Free Product with Amalgamation and Operator-valued Free Probability Theory*, Vol. 627, Mem. Amer. Math. Soc, Amer. Math. Soc., Providence, R.I. (1998).
12. L. Accardi, Y. G. Lu, and I. V. Volovich, *Interacting Fock Spaces and Hilbert Module Extensions of the Heisenberg Commutation Relations*, IIAS Publications, Kyoto (1997).
13. B. V. R. Bhat and M. Skeide, *Tensor Product Systems of Hilbert Modules and Dilations of Completely Positive Semigroups*, Preprint, Cottbus, Revised version, Rome, Submitted to Infinite- Dimensional Analysis, Quantum Probability and Related Topics, 1999.
14. W. Arveson, *Continuous Analogues of Fock Space*, Vol. 409, Mem. Amer. Math. Soc, Amer. Math. Soc., Providence, R.I. (1989).
15. A. Boukas, *Quantum Stochastic Analysis: a Non-Brownian Case*, PhD thesis, Southern Illinois University (1988).
16. A. Boukas, "An example of quantum exponential process," *Mh. Math.*, **112**, 209–215 (1991).

(L. Accardi) Center V. Volterra, Università degli Studi di Roma "Tor Vergata"
E-mail: accardi@volterra.mat.uniroma2.it
(M. Skeide) Lehrstuhl für Wahrscheinlichkeitstheorie und Statistik,
Brandenburgische Technische Universität Cottbus
E-mail: skeide@math.tu-cottbus.de