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A Martingale Characterization of Canonical Commutation and Anticommutation Relations

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Using a martingale condition and some restrictions on moments up to fourth order the characterisation problem of boson, fermion, and classical brownian motions is studied from a unified point of view entirely within the framework of elementary operator theory. Global commutation and anticommutation rules turn out to be consequences of corresponding commutation and anticommutation rules between past and future observables.

1. INTRODUCTION

Suppose $(\Omega, \mathscr{F}, P)$ is a probability space with an increasing filtration $\{\mathscr{F}_t, t \geqslant 0\}$ of subalgebras of $\mathscr{F}$ where $\mathscr{F}_0$ is trivial. Let $\{x(t, \omega),\ t \geqslant 0,\ \omega \in \Omega\}$ be a real valued stochastic process with continuous sample paths and satisfying the following conditions: (i) $x(t, \cdot)$ is a martingale with respect to the filtration $\{\mathscr{F}_t\}$; (ii) $\mathbb{E}x(t, \cdot) = 0$, $\mathbb{E}(\{x(t, \cdot) - x(s, \cdot)\}^2 \mid \mathscr{F}_s) = t - s$ for all $0 \leqslant s < t < \infty$. Then it is a classical result of P. Levy (cf. [3, 4]) that $x(t, \omega)$ is a standard brownian motion. Owing to a well-known result of A. N. Kolmogorov the continuity of sample paths can be ensured by a fourth moment condition of the form

$$\mathbb{E}(x(t, \cdot) - x(s, \cdot))^4 \leqslant C |t - s|^{1+\delta} \qquad \text{for all} \quad s, t \geqslant 0,$$

where $C > 0$, $\delta > 0$ are some constants. Thus the characterisation of brownian motion can be based on a martingale property and some moment

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conditions alone. Furthermore, such a result can be expressed entirely in terms of a commuting family of selfadjoint operators $X(f)$ in $L_2(P)$ where f varies over all real valued square integrable functions on $\mathbb{R}_+ = [0, \infty)$, $X(f)$ is the operator of multiplication by the random variable $\int_0^\infty f(t)\, x(dt)$ and the integral is defined in the mean square stochastic sense of Ito and Doob [3]. Even though continuous trajectories do not make sense in the context of quantum probability theory the martingale and moment conditions admit an obvious translation. In this paper we drop the commutativity hypothesis, impose a martingale condition and some restrictions on moments up to fourth order, and explore the characterisation problem of boson, fermion, and classical brownian motions from a unified point of view entirely within the framework of elementary operator theory. An interesting feature of this investigation is that global canonical commutation and anticommutation relations turn out to be consequences of the martingale condition, with some restrictions on moments up to fourth order and corresponding commutation and anticommutation rules between past and future observables.

2. Levy Fields

Let $\mathscr{H}$ be a complex separable Hilbert space and let $\phi \in \mathscr{H}$ be a fixed unit vector. To each f in the complex Hilbert space $\mathscr{k} = L_2(\mathbb{R}_+)$ let there be associated a selfadjoint operator $X(f)$ in $\mathscr{H}$ such that ϕ is in the domain of $X(f_1)\, X(f_2) \cdots X(f_n)$ for all $f_j \in \mathscr{k}$, $1 \leqslant j \leqslant n$, $n = 1, 2...$ For any set $E \subset \mathbb{R}_+$ we denote by χ_E its indicator function and write

$$f_{t]} = \chi_{[0,\, t]} f, \qquad f_{[t} = \chi_{[t,\, \infty)} f, \qquad f \in \mathscr{k}$$

with the convention $f_{0]} = 0$, $f_{\infty]} = f$. Let $\mathscr{M}$, $\mathscr{M}_{t]}$ denote respectively the linear manifolds in $\mathscr{H}$ generated by $\{\phi, X(f_1) \cdots X(f_n)\phi, f_j \in \mathscr{k}, 1 \leqslant j \leqslant n, n = 1, 2, ...\}$, $\{\phi, X(f_{t]}^{(1)}) \cdots X(f_{t]}^{(n)})\phi, f^{(j)} \in \mathscr{k}, 1 \leqslant j \leqslant n, n = 1, 2, ...\}$. We say that the family $X = \{X(f), f \in \mathscr{k}\}$ of selfadjoint operators is a *Levy field* with *cyclic vector* ϕ and *covariance kernel* K if the following conditions hold:

(a) $\mathscr{M}$ is dense in $\mathscr{H}$;

(b) $X(0) = 0$ and the map $(f_1, f_2, ..., f_n) \to X(f_1)\, X(f_2) \cdots X(f_n)\phi$ is real multilinear on $\mathscr{k}^n$;

(c) the map $t \to X(f_{1]}^{(1)})\, X(f_{t]}^{(2)}) \cdots X(f_{t]}^{(n)})\phi$ is strongly continuous in the closed interval $[0, \infty]$;

(d) $\langle u, X(f_{[t})v \rangle = 0$ for all $u, v \in \mathscr{M}_{t]}$, $t \geqslant 0$, $f \in \mathscr{k}$;

(e) there exists a complex valued strongly continuous functional $K(\cdot, \cdot)$ on $\mathscr{k} \times \mathscr{k}$ such that

$$\langle u, X(f_{[t}) X(g_{[t})v\rangle = \langle u, v\rangle K(f_{[t}, g_{[t})$$

for all $u, v \in \mathscr{M}_{t]}$, $t \geqslant 0$, $f, g \in \mathscr{k}$.

Condition (d) expresses the martingale property in the language of operators. Condition (e) is a direct translation of the second moment or, equivalently, the covariance condition imposed in Levy's characterisation of classical brownian motion. If $\mathscr{H}$ is the Hilbert space of all square integrable functions on the probability space of standard brownian motion and $X(f)$ denotes the selfadjoint operator of multiplication by $\int_0^\infty \operatorname{Re} f\, dw$, w being the sample path, then $X = \{X(f), f \in \mathscr{k}\}$ is a Levy field with cyclic vector 1 and covariance kernel $K(f, g) = \int_0^\infty (\operatorname{Re} f)(\operatorname{Re} g)\, dt$. We emphasise that the main feature of this example is the commutativity of all the operators $X(f)$, $f \in \mathscr{k}$.

As a noncommutative example we may consider the case when $\mathscr{H}$ is the boson (fermion) Fock space over $\mathscr{k}$ and $X(f) = a(f) + a^\dagger(f)$ where $a(f)$ and $a^\dagger(f)$ are the boson (fermion) annihilation and creation operators in $\mathscr{H}$ associated with f. Then X is a Levy field with the Fock vacuum vector as the cyclic vector and covariance kernel $K(f, g) = \int_0^\infty \bar{f} g\, dt$.

One of the interesting problems of quantum martingale theory is the classification of Levy fields up to equivalence defined in the following obvious way. Two Levy fields X_i in Hilbert space $\mathscr{H}_i$, with cyclic vector ϕ_i, and the same covariance kernel K on $\mathscr{k} \times \mathscr{k}$, $i = 1, 2$ are said to be *equivalent* if there exists a unitary operator $U: \mathscr{H}_1 \to \mathscr{H}_2$ such that $U\phi_1 = \phi_2$, $UX_1(f_1) X_1(f_2) \cdots X_1(f_n)\, \phi_1 = X_2(f_1) \cdots X_2(f_n)\, \phi_2$ for all $f_j \in \mathscr{k}$, $1 \leqslant j \leqslant n$, $n = 1, 2, \ldots$.

In the present section we define the notion of a certain stochastic integral in a routine manner and prove the existence of a family of "coherent vectors." This shows that there must exist a natural isomorphism between $\mathscr{H}$ and a suitable Fock space. However, without additional hypotheses on the Levy field, which we shall examine in the next section, we are unable to prove the totality of coherent vectors in $\mathscr{H}$ and hence to establish the desired isomorphism of $\mathscr{H}$ with a suitable Fock space.

In the following propositions we work with a fixed Levy field X satisfying conditions (a)–(e). We denote by $\mathscr{H}_{t]}$ the closure $\bar{\mathscr{M}}_{t]}$ of the linear manifold $\mathscr{M}_{t]}$ with the understanding $\mathscr{H}_{0]} = \mathbb{C}\phi$, $\mathscr{H}_{\infty]} = \mathscr{H}$, and $\mathscr{M}_{0]} = \mathbb{C}$, $\mathscr{M}_{\infty]} = \mathscr{M}$. For any operator T on $\mathscr{H}$ we denote by $D(T)$ its domain.

PROPOSITION 2.1. *For all $t \geqslant 0$, $f, g \in \mathscr{k}$, $u, v \in \mathscr{H}_{t]}$ the following properties hold*:

(i) $\mathscr{H}_{t]} \subset D(X(f_{[t}))$,

(ii) $\langle u, X(f_{[t})v\rangle = 0$,

(iii) $\langle u, X(f_{[t})\, X(g_{[t})v\rangle = \langle u, v\rangle\, K(f_{[t}, g_{[t})$.

Proof. For any $u \in \mathscr{H}_{t]}$ choose $u_n \in \mathscr{M}_{t]}$ such that $\|u_n - u\| \to 0$ as $n \to \infty$. By property (e) of the Levy field

$$\|X(f_{[t})(u_m - u_n)\|^2 = \langle u_m - u_n, X(f_{[t})^2(u_m - u_n)\rangle$$
$$= \|u_m - u_n\|^2\, K(f_{[t}, f_{[t}).$$

Since $X(f_{[t})$ is selfadjoint and hence closed it follows that $u \in D(X(f_{[t]}))$ and $X(f_{[t})u = \lim_{n\to\infty} X(f_{[t})\, u_n$. This implies (i). Now (ii) and (iii) follow immediately from (d) and (e), respectively. ∎

PROPOSITION 2.2. *There exists a complex 2×2 nonnegative definite matrix valued function* $\left(\begin{smallmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{smallmatrix}\right)$ *in* $\mathbb{R}_+$ *which is bounded and satisfies the relation*

$$K(f, g) = \int_0^\infty \rho(f, g, t)\, dt, \tag{2.1}$$

where

$$\rho(f, g, \cdot) = (f_1, f_2)\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\begin{pmatrix}g_1\\ g_2\end{pmatrix}, \tag{2.2}$$

$f = f_1 + if_2$, $g = g_1 + ig_2$ being respectively the decompositions of f and g into their real and imaginary parts.

Proof. Let $f, g \in \mathscr{k}$ be fixed. By properties (b) and (d) of the Levy field we have

$$K(f, g_{t]}) = \langle \phi, \{X(f_{t]}) + X(f_{[t})\}\, X(g_{t]})\phi\rangle$$
$$= \langle \phi, X(f_{t]})\, X(g_{t]})\phi\rangle$$
$$= K(f_{t]}, g_{t]}).$$

On the other hand

$$K(f_{t]}, g) = \langle \phi, X(f_{t]})\{X(g_{t]} + X(g_{[t})\}\phi\rangle$$
$$= K(f_{t]}, g_{t]}) + \langle X(f_{t]})\phi, X(g_{[t})\phi\rangle$$
$$= K(f_{t]}, g_{t]}).$$

Thus

$$K(f, g_{t]}) = K(f_{t]}, g) \tag{2.3}$$

Let $\mathscr{k}_{\mathbb{R}}$ denote $\mathscr{k}$ when considered as a real Hilbert space so that $\mathscr{k}_{\mathbb{R}} = L_2^0(\mathbb{R}_+) \oplus L_2^0(\mathbb{R}^+)$ where $L_2^0(\mathbb{R}_+)$ is the Hilbert space of real square integrable functions on $\mathbb{R}_+$. Then in view of property (b) of the Levy field $\operatorname{Re} K(f, g)$ and $\operatorname{Im} K(f, g)$ are bounded bilinear forms on $\mathscr{k}_{\mathbb{R}}$. Hence there exist bounded operators A, B on $\mathscr{k}_{\mathbb{R}}$ such that

$$\langle f, Ag \rangle_0 = \operatorname{Re} K(f, g), \qquad \langle f, Bg \rangle_0 = \operatorname{Im} K(f, g),$$

where 0 indicates the inner product in $\mathscr{k}_{\mathbb{R}}$. Equation (2.3) implies that A and B commute with multiplication by $\chi_{[0,t]}$ for every t. Since $K(f, g) = \overline{K(g, f)}$ it follows that A is selfadjoint and B is skewadjoint. Thus A and B are respectively multiplications by real 2×2 symmetric and skew-symmetric matrix valued functions when $\mathscr{k}_{\mathbb{R}}$ is viewed as the space of $\mathbb{R}^2$-valued square integrable functions on $\mathbb{R}_+$. This implies that $K(f, g)$ has the form (2.1) where $\rho(f, g, \cdot)$ is given by (2.2). The boundedness of A and B implies that $((\rho_{ij}))$ is bounded on $\mathbb{R}_+$. Since $K(f, f) \geqslant 0$ for all f it follows that $((\rho_{ij}(t)))$ is nonnegative definite almost everywhere. ∎

DEFINITION 2.3. A map $\xi: \mathbb{R}_+ \to \mathscr{H}$ is said to be *adapted* to the Levy field X if $\xi(t) \in \mathscr{H}_{t]}$ for every t.

Let $\mathscr{A}(X)$ denote the linear space of all strongly continuous maps from $\mathbb{R}_+$ into $\mathscr{H}$ adapted to X. For any $f \in \mathscr{k}$ let

$$X_f(a, b) = X(f_{b]}) - X(f_{a]}), \qquad 0 \leqslant a \leqslant b \leqslant \infty.$$

To any partition $\mathscr{P}$ of an interval $[a, b]$ into $t_0 = a < t_1 < t_2 < \cdots < t_n < b = t_{n+1}$, $\xi \in \mathscr{A}(X)$, and $f \in \mathscr{k}$, we can, in view of Proposition 2.1, associate the Riemann sum

$$S(\mathscr{P}, f, \xi) = \sum_{j=0}^{n} X_f(t_j, t_{j+1})\, \xi(t_j). \tag{2.4}$$

We write

$$\delta(\mathscr{P}) = \max_{0 \leqslant j \leqslant n} |t_{j+1} - t_j|.$$

PROPOSITION 2.4. *Let $f, g \in \mathscr{k}$, $\xi, \eta \in \mathscr{A}(X)$. Suppose $\mathscr{P}$, $\mathscr{P}'$ are partitions of intervals $[a, b]$, $[a', b']$, respectively, into $t_0 = a < t_1 < \cdots < t_n < b = t_{n+1}$, $t'_0 = a' < t'_0 < \cdots < t'_{n'} < b' = t'_{n+1}$. Then*

(i) $\langle \phi, S(\mathscr{P}, f, \xi) \rangle = 0$,

(ii) $\langle S(\mathscr{P}, f, \xi), S(\mathscr{P}, g, \eta) \rangle = \sum_{j=0}^{n} \langle \xi(t_j), \eta(t_j) \rangle \int_{t_j}^{t_{j+1}} \rho(f, g, s)\, ds$,

(iii) $\langle S(\mathscr{P}, f, \xi), S(\mathscr{P}', g, \eta) \rangle = 0$ *if* $b \leqslant a'$

where $\rho(f, g, \cdot)$ is defined by (2.2) *in Proposition* 2.2.

Proof. (i) follows immediately from (ii) in Proposition 2.1. To prove (ii) consider any four points $s_1 < s_2 \leqslant s_3 < s_4$ in $\mathbb{R}_+$. The proof of Proposition 2.1 shows that $X_f(s_1, s_2)\,\xi(s_1) \in \mathscr{H}_{s_2]}$. By (ii) in Proposition 2.1

$$\langle X_f(s_1, s_2)\,\xi(s_1), X_g(s_3, s_4)\,\eta(s_3)\rangle = 0. \tag{2.5}$$

This together with (iii) in Proposition 2.1 implies that

$$\begin{aligned}
&\langle S(\mathscr{P}, f, \xi), S(\mathscr{P}, g, \eta)\rangle \\
&\quad = \sum_{j=0}^{n} \langle X_f(t_j, t_{j+1})\,\xi(t_j), X_g(t_j, t_{j+1})\,\eta(t_j)\rangle \\
&\quad = \sum_{j=0}^{n} \langle \xi(t_j), \eta(t_j)\rangle\, K(f\chi_{[t_j, t_{j+1}]}, g\chi_{[t_j, t_{j+1}]}).
\end{aligned}$$

Now an application of Proposition 2.2 yields (ii). Finally (iii) follows immediately from (2.5). ∎

COROLLARY 2.5. *Let $f \in \mathscr{k}$, $\xi \in \mathscr{A}(X)$. Then for any finite interval $[a, b] \subset \mathbb{R}_+$*

$$\lim_{\delta(\mathscr{P}) \to 0} S(\mathscr{P}, f, \xi) = \int_a^b X_f(ds)\,\xi(s), \text{ say,}$$

exists in the strong sense as $\mathscr{P}$ varies over finite partitions of $[a,b]$.

Proof. This follows from the isometry properties (ii) and (iii) of Riemann sums in Proposition 2.4 by the same kind of arguments that are employed in the definition of mean square stochastic integrals in the sense of Ito and Doob [3]. ∎

PROPOSITION 2.6. *Let $f, g \in \mathscr{k}$, $\xi, \eta \in \mathscr{A}(X)$. Then the following properties hold:*

(i) $\left\langle \phi, \int_0^t X_f(ds)\,\xi(s)\right\rangle = 0,$

(ii) $\left\langle \int_a^b X_f(ds\,\xi(s), \int_c^d X_g(ds)\,\eta(s)\right\rangle = 0$ *if* $0 \leqslant a < b \leqslant c < d,$

(iii) $\left\langle \int_a^b X_f(ds)\,\xi(s), \int_a^b X_g(ds)\,\eta(s)\right\rangle = \int_a^b \rho(f, g, s)\langle \xi(s), \eta(s)\rangle\, ds,$

(iv) $\int_a^b X_f(ds)\,\xi(s) + \int_b^c X_f(ds)\,\xi(s) = \int_a^c X_f(ds)\,\xi(s)$ *if* $a < b < c,$

(v) *as a function of t,* $\int_0^t X_f(ds)\,\xi(s)$ *is an element of* $\mathscr{A}(X)$,

(vi) $\int_a^b X_f(ds)\,\xi(s)$ *is real linear in* f *and linear in* ξ.

Proof. All the properties (i)–(vi) follow immediately from Corollary 2.5, Proposition 2.4, and the definition of a Levy field. ∎

PROPOSITION 2.7. *For any* $f_1, f_2, \ldots \in \mathscr{k}$ *define the adapted processes* $\{\eta(f_1, f_2, \ldots, f_n, t)\}$ *inductively by*

$$\eta(f_1, t) = \int_0^t X_{f_1}(ds)\phi = X_{f_1}(0, t)\phi,$$

$$\eta(f_1, \ldots, f_{n+1}, t) = \int_0^t X_{f_1}(ds)\,\eta(f_2, f_3, \ldots, f_{n+1}, s), \qquad n = 1, 2, \ldots. \tag{2.6}$$

Then

(i) $\langle \phi, \eta(f_1, f_2, \ldots, f_n, t)\rangle = 0$ *for all* $t \geqslant 0$, $n = 1, 2, \ldots$

(ii) $\langle \eta(f_1, f_2, \ldots, f_m, t), \eta(g_1, g_2, \ldots, g_n, s)\rangle = 0$ *if* $m \neq n$,

(iii) $\langle \eta(f_1, f_2, \ldots, f_n, t), \eta(g_1, g_2, \ldots, g_n, s)\rangle$

$$= \int_{0 < t_n < t_{n-1} < \cdots < t_1 < t \wedge s} \prod_{i=1}^{n} \rho(f_i, g_i, t_i)\, dt_1\, dt_2 \cdots dt_n,$$

where $t \wedge s$ *denotes the minimum of* t *and* s *and* $\rho(f, g, \cdot)$ *is defined by Proposition* 2.2.

Proof. Owing to (v) in Proposition 2.6, the functionals $\eta(f_1, \ldots, f_n, t)$ are defined as elements of $\mathscr{A}(X)$. (i) follows from (i) in Proposition 2.6. By (ii), (iii), and (iv) in the same proposition we have

$$\langle \eta(f_1, f_2, \ldots, f_m, t), \eta(g_1, g_2, \ldots, g_n, s)\rangle$$
$$= \int_0^{t \wedge s} \rho(f_1, g_1, t_1) < \eta(f_2, f_3, \ldots, f_m, t_1), \eta(g_2, g_3, \ldots, g_n, t_1) > dt_1.$$

Now (ii) and (iii) follow by induction on (m, n). ∎

Remark. It is clear that the vectors $\eta(f_1, f_2, \ldots, f_n, t)$ defined in Proposition 2.7 behave like the n-particle vectors in a Fock space. It is a conjecture that all such vectors together with ϕ span $\mathscr{H}_{t]}$ as n varies over the set of natural numbers and the f_j's vary over $\mathscr{k}$.

PROPOSITION 2.8. *Let*

$$\eta_n(f, t) = \eta(f, f, ..., f, t), \qquad n = 1, 2, ...,$$

where the element f is repeated n-fold in (2.6). *Then the infinite series*

$$\phi + \eta_1(f, t) + \cdots + \eta_n(f, t) + \cdots = \psi_f(t), \text{ say}, \tag{2.7}$$

converges strongly and uniformly in t. Furthermore

$$\langle \psi_f(t), \psi_g(s) \rangle = \exp K(f_{t]}, g_{s]}) \tag{2.8}$$

for all $0 \leqslant s \leqslant t < \infty$, $f, g \in \mathscr{k}$.

Proof. From (iii) in Proposition 2.7 we have

$$\|\eta_n(f, t)\|^2 = \int_{0 < t_n < t_{n-1} < \cdots < t_1 < t} \prod_{i=1}^{n} \rho(f, f, t_i)\, dt_1\, dt_2 \cdots dt_n.$$

By (2.1) and (2.2) we obtain

$$\|\eta_n(f, t)\|^2 = (n!)^{-1} K(f_{t]}, f_{t]})^n \leqslant (n!)^{-1} K(f, f)^n.$$

This shows that (2.7) converges strongly and uniformly. By Proposition 2.7

$$\langle \eta_m(f, t), \eta_n(g, s) \rangle = 0 \qquad \text{if} \quad m \neq n$$

and

$$\begin{aligned} \langle \eta_n(f, t), \eta_n(g, s) \rangle & \int_{0 < t_n < t_{n-1} < \cdots < t_1 < t \wedge s} \prod_{i=1}^{n} \rho(f, g, t_i)\, dt_1\, dt_2 \cdots dt_n \\ &= (n!)^{-1} K(f_{t \wedge s]}, g_{t \wedge s]})^n \\ &= (n!)^{-1} K(f_{t]}, g_{s]})^n. \end{aligned}$$

These relations imply (2.8). ∎

COROLLARY 2.9. *Let* $\psi_f(t)$ *be defined by* (2.7). *Then there exists a vector* $\psi(f)$ *such that*

$$\lim_{t \to \infty} \|\psi_f(t) - \psi(f)\| = 0 \qquad \text{for each} \quad f \in \mathscr{k}.$$

Furthermore,

$$\langle \psi(f), \psi(g) \rangle = \exp K(f, g) \qquad \text{for all} \quad f, g \in \mathscr{k}.$$

Proof. This is immediate from (2.8) and the continuity of the covariance kernel K. ∎

Remark. The vector $\psi(f)$ may be called the *intrinsic coherent vector* associated with f for the Levy field X. The conjecture in the remark after the proof of Proposition 2.7 can be reformulated as follows: is the family $\{\psi(f), f \in \mathscr{h}\}$ of intrinsic coherent vectors total in $\mathscr{H}$?

Under some additional conditions to be discussed in the next section the answer will be in the affirmative.

3. Levy Fields and Commutation Relations

Let ε be a fixed constant denoting ± 1. We shall now consider a Levy field X over $\mathscr{h} = L_2(\mathbb{R}_+)$ with cyclic vector ϕ, covariance kernel K, and satisfying the additional condition

$$\{X(f_{t]})\, X(f_{[t}) + \varepsilon X(f_{[t})\, X(f_{t]})\} = 0 \tag{3.1}$$

for all $u \in \mathscr{M}$, $f \in \mathscr{h}$, $t \geqslant 0$ in the notations of Section 2. We call X a *Levy boson* or *fermion* field according to whether $\varepsilon = -1$ or 1.

We now introduce a fourth order conditional moment condition on X which is inspired by the discussion in Section 1. To this end, for any $0 \leqslant s < t < \infty$, $f, g \in \mathscr{h}$, we write

$$\theta(f, g, s, t) = \sup_{\substack{u \in \mathscr{M}_{s]} \\ \|u\| = 1}} \|\{X_f(s, t)\, X_g(s, t) - K(f\chi_{[s,t]}, g\chi_{[s,t]})\} u\|^2.$$

For any partition $\mathscr{P}$ of any finite interval of the form $[0, t]$ by $0 = t_0 < t_1 < t_2 < \cdots < t_n < t = t_{n+1}$ let

$$V(f, g, \mathscr{P}) = \sum_{j=0}^{n} \theta(f, g, t_j, t_{j+1}).$$

We say that a Levy field is *smooth* if

$$\lim_{\delta(\mathscr{P}) \to 0} V(f, g, \mathscr{P}) = 0 \tag{3.2}$$

for every $t > 0$, $\delta(\mathscr{P})$ denoting $\max_{0 \leqslant j \leqslant n} (t_{j+1} - t_j)$.

The next proposition gives a more easily verifiable sufficient condition for the smoothness of a Levy field which is modeled after the classical Kolmogorov's criterion for the continuity of trajectories of a stochastic process.

Proposition 3.1. *Let X be a Levy field. Suppose there exist two families* *of nonnegative Radon measures* $\{\mu_{f,g}, f, g \in \mathscr{k}\}$, $\{\nu_{f,g}, f, g \in \mathscr{k}\}$ *such that each* $\nu_{f,g}$ *is absolutely continuous and*

$$\begin{aligned}\|\{X_f(s,t)\,X_g(s,t) - K(f\chi_{[s,t]}, g\chi_{[s,t]})\}u\|^2 \\ \leqslant \|u\|^2\,\mu_{f,g}((s,t])\,\nu_{f,g}((s,t])\end{aligned} \tag{3.3}$$

for all $u \in \mathscr{M}_{s]}$, $0 \leqslant s < t < \infty$, $f, g \in \mathscr{k}$. *Then X is smooth.*

Proof. Condition (3.3) implies

$$\theta(f, g, t_j, t_{j+1}) \leqslant \mu_{f,g}((t_j, t_{j+1}])\,\nu_{f,g}((t_j, t_{j+1}])$$

and hence

$$V(f, g, \mathscr{P}) \leqslant \mu_{f,g}((0,t])\max_j \nu_{f,g}((t_j, t_{j+1}]) \tag{3.4}$$

for any partition $\mathscr{P}$ of $[0,t]$ by $0 = t_0 < t_1 < \cdots < t_n < t = t_{n+1}$. The absolute continuity of $\nu_{f,g}$ implies that the right hand side of (3.4) tends to 0 as $\delta(\mathscr{P}) \to 0$. ∎

Remark. Suppose $X(f)$ is multiplication by $\int_0^\infty (\operatorname{Re} f)\,dw$ in $L_2(P)$ where P is the probability measure of standard brownian motion and w denotes the brownian path. Then X, as mentioned in Section 2, is a Levy field with $K(f,g) = \int_0^\infty (\operatorname{Re} f)(\operatorname{Re} g)\,dt$ as covariance kernel and cyclic vector 1, the constant function. For any $u \in \mathscr{M}_{s]}$

$$\begin{aligned}&\|\{X_f(s,t)\,X_g(s,t) - K(f\chi_{[s,t]}, g\chi_{[s,t]})\}u\|^2 \\ &\quad = \|u\|^2\left\{\left(\int_s^t \operatorname{Re} f\right)^2\left(\int_s^t \operatorname{Re} g\right)^2 + \int_s^t (\operatorname{Re} f)(\operatorname{Re} g)\right\} \\ &\quad \leqslant 2\|u\|^2\int_s^t (\operatorname{Re} f)^2\int_s^t (\operatorname{Re} g)^2,\end{aligned}$$

where we have adopted the convention of denoting the Lebesgue integral of a function f over the interval $[a,b]$ by $\int_a^b f$. Thus (3.3) holds with $\mu_{f,g}((s,t]) = 2\int_s^t (\operatorname{Re} f)^2$ and $\nu_{f,g}((s,t]) = \int_s^t (\operatorname{Re} g)^2$.

Condition (3.3) also obtains in the cases when $X(f) = a(f) + a^\dagger(f)$ where $a(f)$ and $a^\dagger(f)$ are respectively the annihilation and creation operators of a boson or fermion field over $\mathscr{k}$ which is either free or quasifree.

PROPOSITION 3.2. *Let X be a Levy boson or fermion field which is also smooth. Let* $f, g \in \mathscr{k}$ *be fixed. Suppose* $\xi\colon \mathbb{R}_+ \to \mathscr{H}$ *is a strongly continuous* *map such that* $\xi(t) \in \mathcal{M}_{t]}$ *for every* t *and the map* $t \to X_f(0, t)\, \xi(t)$ *is also strongly continuous. Let*

$$\eta(t) = \int_0^t X_g(ds)\, \xi(s), \qquad t \geqslant 0.$$

Then $\eta(t) \in D(X_f(0, t))$ *and*

$$X_f(0, t)\, \eta(t) = \int_0^t X_f(ds)\, \eta(s) - \varepsilon \int_0^t X_g(ds)\, X_f(0, s)\, \xi(s)$$

$$+ \int_0^t \rho(f, g, s)\, \xi(s)\, ds,$$

where $\rho(f, g, \cdot)$ *is defined by Proposition* 2.2.

Proof. Consider an arbitrary partition $\mathscr{P}$ of $[0, t]$ by $0 = t_0 < t_1 < \cdots < t_n < t = t_{n+1}$. From the properties (a)–(e) of the Levy field we obtain

$$\begin{aligned}
&X_f(0; t) \sum_{j=0}^{n} X_g(t_j, t_{j+1})\, \xi(t_j)\\
&= \sum_{j=0}^{n} X_f(0, t_j)\, X_g(t_j, t_{j+1})\, \xi(t_j)\\
&\quad + \sum_{j=0}^{n} \left\{ X_f(t_j, t_{j+1})\, X_g(t_j, t_{j+1}) - \int_{t_j}^{t_{j+1}} \rho(f, g, s)\, ds \right\} \xi(t_j)\\
&\quad + \sum_{j=0}^{n} \left\{ \int_{t_j}^{t_{j+1}} \rho(f, g, s)\, ds \right\} \xi(t_j)\\
&\quad + \sum_{j=0}^{n} X_f(t_{j+1}, t)\, X_g(t_j, t_{j+1})\, \xi(t_j)\\
&= S_1 + S_2 + S_3 + S_4, \text{ say,}
\end{aligned} \tag{3.5}$$

where $S_i = S_i(\mathscr{P})$ denote the ith sum on the right hand side of (3.5), $1 \leqslant i \leqslant 4$. Clearly,

$$\lim_{\delta(\mathscr{P}) \to 0} S_3(\mathscr{P}) = \int_0^t \rho(f, g, s)\, \xi(s)\, ds. \tag{3.6}$$

From (3.1) we get

$$S_1(\mathscr{P}) = -\varepsilon \sum_{j=0}^{n} X_g(t_j, t_{j+1})\, X_f(0, t_j)\, \xi(t_j).$$

Since by hypothesis $X_f(0, \cdot)\, \xi(\cdot) \in \mathscr{A}(X)$ Corollary 2.5 implies

$$\lim_{\delta(\mathscr{P}) \to 0} S_1(\mathscr{P}) = -\varepsilon \int_0^t X_g(ds)\, X_f(0, s)\, \xi(s). \tag{3.7}$$

In order to analyse $S_2(\mathscr{P})$ we first observe that in view of property (e) of the Levy field and Proposition 2.2

$$\left\langle \left\{ X_f(t_j, t_{j+1})\, X_g(t_j, t_{j+1}) - \int_{t_j}^{t_{j+1}} \rho(f, g, s)\, ds \right\} \xi(t_j), \right.$$
$$\left. \left\{ X_f(t_k, t_{k+1})\, X_g(t_k, t_{k+1}) - \int_{t_k}^{t_{k+1}} \rho(f, g, s)\, ds \right\} \xi(t_k) \right\rangle = 0$$

whenever $j \neq k$. Thus

$$\begin{aligned} \|S_2(\mathscr{P})\|^2 &= \sum_{j=0}^{n} \left\| \left\{ X_f(t_j, t_{j+1})\, X_g(t_j, t_{j+1}) \right.\right. \\ &\qquad \left.\left. - \int_{t_j}^{t_{j+1}} \rho(f, g, s)\, ds \right\} \xi(t_j) \right\|^2 \\ &\leqslant \left\{ \sup_{0 \leqslant s \leqslant t} \|\xi(s)\|^2 \right\} V(f, g, \mathscr{P}) \end{aligned}$$

which, in view of (3.2), tends to 0 as $\delta(\mathscr{P}) \to 0$. Thus

$$\lim_{\delta(\mathscr{P}) \to 0} S_2(\mathscr{P}) = 0. \tag{3.8}$$

We have

$$\begin{aligned} S_4(\mathscr{P}) &= \sum_{k > j}\sum X_f(t_k, t_{k+1})\, X_g(t_j, t_{j+1})\, \xi(t_j) \\ &= \sum_k X_f(t_k, t_{k+1}) \left\{ \sum_{j \leqslant k-1} X_g(t_j, t_{j+1})\, \xi(t_j) \right\} \\ &= \sum_k X_f(t_k, t_{k+1}) \int_0^{t_k} X_g(ds)\, \xi(s) \\ &\quad + \sum_k X_f(t_k, t_{k+1}) \left\{ \sum_{j \leqslant k-1} \left(X_g(t_j, t_{j+1})\, \xi(t_j) - \int_{t_j}^{t_{j+1}} X_g(ds)\, \xi(s) \right) \right\}. \end{aligned}$$

As $\delta(\mathscr{P}) \to 0$ the first sum on the extreme right hand side above converges to $\int_0^t X_f(ds)\, \eta(s)$. Putting

$$\omega_\xi(\delta) = \sup_{\substack{0 \leqslant a \leqslant b \leqslant t \\ |b-a| < \delta}} \|\xi(a) - \xi(b)\|^2 \tag{3.9}$$

we obtain from Propositions 2.4 and 2.6

$$\left\| \sum_k X_f(t_k, t_{k+1}) \left\{ \sum_{j \leqslant k-1} \left(X_g(t_j, t_{j+1}) \, \xi(t_j) - \int_{t_j}^{t_{j+1}} X_g(ds) \, \xi(s) \right) \right\} \right\|^2$$

$$= \sum_k \int_{t_k}^{t_{k+1}} \rho(f, f, s) \left\| \sum_{j \leqslant k-1} \int_{t_j}^{t_{j+1}} X_g(ds)(\xi(t_j) - \xi(s)) \right\|^2 ds$$

$$= \sum_{k>j} \sum \int_{t_k}^{t_{k+1}} \rho(f, f, s) \, ds \int_{t_j}^{t_{j+1}} \rho(g, g, s') \, ds' \, \omega_\xi(\delta(\mathscr{P}))$$

$$\leqslant \omega_\xi(\delta(\mathscr{P})) \int_0^t \rho(f, f, s) \, ds \int_0^t \rho(g, g, s) \, ds.$$

Since ξ is strongly continuous $\omega_\xi(\delta(\mathscr{P})) \to 0$ as $\delta(\mathscr{P}) \to 0$ and hence

$$\lim_{\delta(\mathscr{P}) \to 0} S_4(\mathscr{P}) = \int_0^t X_f(ds) \, \eta(s). \tag{3.10}$$

Combining (3.6)–(3.8) and (3.10) we conclude

$$\lim_{\delta(\mathscr{P}) \to 0} X_f(0, t) \sum_{j=0}^{n} X_g(t_j, t_{j+1}) \, \xi(t_j)$$

$$= -\varepsilon \int_0^t X_g(ds) \, X_f(0, s) \, \xi(s) + \int_0^t X_f(ds) \, \eta(s)$$

$$+ \int_0^t \rho(f, g, s) \, \xi(s) \, ds.$$

By definition

$$\lim_{\delta(\mathscr{P}) \to 0} \sum_{j=0}^{n} X_g(t_j, t_{j+1}) \, \xi(t_j) = \int_0^t X_g(ds) \, \xi(s).$$

Since $X_f(0, t) = X(f_{t]})$ is selfadjoint and hence closed the required result follows. ∎

PROPOSITION 3.3. *Let X be a Levy field. Suppose $\xi: \mathbb{R}_+ \to \mathscr{H}$ is a strongly continuous map such that $\xi(s) \in \mathscr{M}_{s]}$ for every s and the map $t \to X_f(0, t) \, \xi(t)$ is strongly continuous. Let $\eta(t) = \int_0^t \xi(s) \, ds$. Then $\eta \in \mathscr{A}(X)$, $\eta(t) \in D(X_f(0, t))$, and*

$$X_f(0, t) \, \eta(t) = \int_0^t X_f(ds) \, \eta(s) + \int_0^t X_f(0, s) \, \xi(s) \, ds. \tag{3.11}$$

Proof. Let $\mathscr{P}$ be as in the proof of Proposition 3.2. Then

$$\begin{aligned} X_f(0, t) \sum_{j=0}^{n} (t_{j+1} - t_j)\, \xi(t_j) &= \sum_{j=0}^{n} (t_{j+1} - t_j)\, X_f(0, t_j)\, \xi(t_j) \\ &\quad + \sum_{j=0}^{n} (t_{j+1} - t_j)\, X_f(t_j, t_{j+1})\, \xi(t_j) \\ &\quad + \sum_{j=0}^{n} (t_{j+1} - t_j)\, X_f(t_{j+1}, t)\, \xi(t_j) \\ &= S_1 + S_2 + S_3, \text{ say,} \end{aligned} \tag{3.12}$$

where $S_i = S_i(\mathscr{P})$ denotes the ith sum on the right hand side of (3.12) for $i = 1, 2, 3$. The strong continuity of $X_f(0, \cdot)\, \xi(\cdot)$ implies that

$$\lim_{\delta(\mathscr{P}) \to 0} S_1(\mathscr{P}) = \int_0^t X_f(0, s)\, \xi(s)\, ds. \tag{3.13}$$

By conditions (d) and (e) for a Levy field (cf. Section 2) and Proposition 2.2

$$\begin{aligned} \|S_2(\mathscr{P})\|^2 &= \sum_{j=0}^{n} (t_{j+1} - t_j)^2\, \|\xi(t_j)\|^2 \int_{t_j}^{t_{j+1}} \rho(f, f, s)\, ds \\ &\leqslant \Big(\sup_{0 \leqslant s \leqslant t} \|\xi(s)\|^2 \Big)\, \delta(\mathscr{P})^2 \int_0^t \rho(f, f, s)\, ds. \end{aligned}$$

Then

$$\lim_{\delta(\mathscr{P}) \to 0} S_2(\mathscr{P}) = 0. \tag{3.14}$$

Coming to S_3 we have

$$\begin{aligned} S_3(\mathscr{P}) &= \sum_{k > j} \sum (t_{j+1} - t_j)\, X_f(t_k, t_{k+1})\, \xi(t_j) \\ &= \sum_k X_f(t_k, t_{k+1}) \left\{ \sum_{j \leqslant k-1} \left((t_{j+1} - t_j)\, \xi(t_j) - \int_{t_j}^{t_{j+1}} \xi(s)\, ds \right) \right\} \\ &\quad + \sum_k X_f(t_k, t_{k+1})\, \eta(t_k). \end{aligned}$$

The second term on the right hand side above converges to $\int_0^t X_f(ds)\, \eta(s)$ as $\delta(\mathscr{P}) \to 0$. By Proposition 2.4 the square of the norm of the first sum is equal to

$$\sum_k \int_{t_k}^{t_{k+1}} \rho(f, f, s) \left\| \sum_{j \leqslant k-1} \int_{t_j}^{t_{j+1}} [\xi(t_j) - \xi(s')]\, ds' \right\|^2 ds$$
$$\leqslant t \int_0^t \rho(f, f, s)\, ds\, \omega_\xi(\delta(\mathscr{P})),$$

where $\omega_\xi(\delta)$ is defined by (3.9). Thus

$$\lim_{\delta(\mathscr{P}) \to 0} S_3(\mathscr{P}) = \int_0^t X_f(ds)\, \eta(s). \tag{3.15}$$

Combining (3.12)–(3.15) and using the closure property of the operator $X_f(0, t)$ we obtain (3.11). ∎

If $\xi, \eta_j, \zeta \in \mathscr{A}(X)$, $1 \leqslant j \leqslant n$ and

$$\xi(t) = \xi(0) + \sum_j \int_0^t X_{f_j}(ds)\, \eta_j(s) + \int_0^t \zeta(s)\, ds, \qquad t \geqslant 0, f_j \in \mathscr{k}$$

then we write

$$d\xi = \sum_j X_{f_j}(dt)\, \eta_j(t) + \zeta(t)\, dt.$$

PROPOSITION 3.4. *For any $f_j \in \mathscr{k}$, $1 \leqslant j \leqslant n$, let*

$$\xi(f_1, f_2, ..., f_n, t) = X_{f_1}(0, t)\, X_{f_2}(0, t) \cdots X_{f_n}(0, t)\phi, \tag{3.16}$$

where the right hand side is interpreted as ϕ when $n = 0$. If X is a Levy boson or fermion field which is also smooth then

$$\begin{aligned} &d\xi(f_1, f_2, ..., f_n, t) \\ &\quad = \sum_{i=1}^n (-\varepsilon)^{i-1} X_{f_i}(dt)\, \xi(f_1, ..., \hat{f}_i, ..., f_n, t) \\ &\qquad + \sum_{1 \leqslant i < j \leqslant n} (-\varepsilon)^{i+j-1} \rho(f_i, f_j, t)\, \xi(f_1, ..., \hat{f}_i, ..., \hat{f}_j, ..., f_n, t), \end{aligned} \tag{3.17}$$

where $\wedge$ over a letter implies omission of such a term, and ρ is defined by Proposition 2.2.

Proof. Let $n = 2$. In Proposition 3.2 put $f = f_1$, $g = f_2$, $\xi(t) \equiv \phi$. Then we get

$$d\xi(f_1, f_2, t) = X_{f_1}(dt)\, \xi(f_2, t) - \varepsilon X_{f_2}(dt)\, \xi(f_1, t) + \rho(f_1, f_2, t)\phi\, dt$$

which proves (3.17) for $n=2$. Suppose (3.17) has been established for all $m \leqslant n$. Since

$$\xi(f, f_1, \ldots, f_n, t) = X_f(0, t)\, \xi(f_1, \ldots, f_n, t)$$

and $\xi(f_1, \ldots, f_n, \cdot) \in \mathscr{A}(X)$, by condition (c) for a Levy field we obtain from Proposition 3.2

$$\begin{aligned} d\left[X_f(0, t) \int_0^t X_{f_i}(ds)\, \xi(f_1, \ldots, \hat{f}_i, \ldots, f_n, s)\right] &= X_f(dt) \int_0^t X_{f_i}(ds)\, \xi(f_1, \ldots, \hat{f}_i, \ldots, f_n, s) \\ &\quad - \varepsilon X_{f_i}(dt)\, \xi(f, f_1, \ldots, \hat{f}_i, \ldots, f_n, t) \\ &\quad + \rho(f, f_i, t)\, \xi(\hat{f}, f_1, \ldots, \hat{f}_i, \ldots, f_n, t)\, dt. \end{aligned} \tag{3.18}$$

By Proposition 3.3

$$\begin{aligned} d\left\{X_f(0, t) \int_0^t \rho(f_i, f_j, s)\, \xi(f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_n, s)\, ds\right\} &= X_f(dt) \int_0^t \rho(f_i, f_j, s)\, \xi(f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_n, s)\, ds \\ &\quad + \rho(f_i, f_j, t)\, \xi(f, f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_n, t)\, dt, \qquad i<j. \end{aligned} \tag{3.19}$$

It is to be noted that (3.19) is obtained from Proposition 3.3 even though $\rho(f_i, f_j, t)$ need not be continuous in t. Multiplying both sides of (3.18) by $(-\varepsilon)^{i-1}$, both sides of (3.19) by $(-\varepsilon)^{i+j-1}$, and adding up we get from the induction hypothesis

$$\begin{aligned} d\xi(f, f_1, \ldots, f_n, t) &= X_f(dt)\, \xi(\hat{f}, f_1, \ldots, f_n, t) \\ &\quad + \sum_i (-\varepsilon)^i X_{f_i}(dt)\, \xi(f, f_1, \ldots, \hat{f}_i, \ldots, f_n, t) \\ &\quad + \sum_i \rho(f, f_i, t)(-\varepsilon)^{i-1} \xi(\hat{f}, f_1, \ldots, \hat{f}_i, \ldots, f_n, t)\, dt \\ &\quad + \sum_{i<j} \rho(f_i, f_j, t)(-\varepsilon)^{i+j-1} \xi(f, f_1, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_n, t)\, dt. \end{aligned}$$

If we rename $(f, f_1, \ldots, f_n)$ as $(f_1, f_2, \ldots, f_{n+1})$ the above relation is the same as (3.17) with n replaced by $n+1$. ∎

PROPOSITION 3.5. *Let X be a smooth Levy boson or fermion field with cyclic vector ϕ and covariance kernel K. Let*

$$E(f_1, f_2, ..., f_n, t) = \langle \phi, \xi(f_1, ..., f_n, t)\rangle, \qquad f_j \in \mathscr{k},\ 1 \leqslant j \leqslant n, \tag{3.19}$$

where $\xi(f_1, ..., f_n, \cdot)$ is defined by (3.16). *Then*

$$\begin{aligned} &E(f_1, ..., f_m, f, g, g_1, ..., g_n, t) + \varepsilon E(f_1, ..., f_m, g, f, g_1, ..., g_n, t) \\ &\quad = \{K(f, g, t) + \varepsilon K(g, f, t)\}\, E(f_1, ..., f_m, g_1, ..., g_n, t) \end{aligned} \tag{3.20}$$

for all $f_i, f, g, g_j \in \mathscr{k}$, $m = 0, 1, 2, ..., n = 0, 1, 2, ...,$ where $K(f, g, t) = K(f_{t]}, g_{t]})$.

Proof. By condition (e) for a Levy field

$$E(f, g, t) = \langle \phi, X_f(0, t)\, X_g(0, t)\phi\rangle = K(f, g, t)\langle \phi, \phi\rangle$$

which is the same as (3.20) when $m = 0$, $n = 0$. Now we use induction on the pair (m, n). By Propositions 2.6 and 3.4

$$\begin{aligned} &\frac{dE}{dt}(f_1, ..., f_m, f, g, g_1, ..., g_n, t) \\ &\qquad \sum_{1 \leqslant i < j \leqslant m} (-\varepsilon)^{i+j-1} \rho(f_i, f_j, t)\, E(f_1, ..., \hat{f}_i, ..., \hat{f}_m, f, g, g_1, ..., g_n, t) \\ &\qquad + \sum_{1 \leqslant i < j \leqslant n} (-\varepsilon)^{i+j-1} \rho(g_i, g_j, t) \\ &\qquad \times E(f_1, ..., f_m, f, g, g_1, ..., \hat{g}_i, ..., \hat{g}_j, ..., g_n, t) \\ &\qquad + \sum_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} (-\varepsilon)^{m+i+j-1} \rho(f_i, g_j, t) \\ &\qquad \times E(f_1, ..., \hat{f}_i, ..., f_m, f, g, g_1, ..., \hat{g}_j, ..., g_n, t) \\ &\qquad + \sum_{1 \leqslant i \leqslant m} (-\varepsilon)^{i+m} \rho(f_i, f, t)\, E(f_1, ..., \hat{f}_i, ..., f_m, g, g_1, ..., g_n, t) \\ &\qquad + \sum_{1 \leqslant i \leqslant m} (-\varepsilon)^{i+m+1} \rho(f_i, g, t)\, E(f_1, ..., \hat{f}_i, ..., f_m, f, g_1, ..., g_n, t) \\ &\qquad + \sum_{1 \leqslant j \leqslant n} (-\varepsilon)^{j} \rho(f, g_j, t)\, E(f_1, ..., f_m, g, g_1, ..., \hat{g}_j, ..., g_n, t) \\ &\qquad + \sum_{1 \leqslant j \leqslant n} (-\varepsilon)^{j-1} \rho(g, g_j, t)\, E(f_1, ..., f_m, f, g_1, ..., \hat{g}_j, ..., g_n, t) \\ &\qquad + \rho(f, g, t)\, E(f_1, ..., f_m, g_1, ..., g_n, t). \end{aligned} \tag{3.21}$$

Suppose (3.20) has been established for m', n' whenever $(m', n') < (m, n)$ where $<$ means $m' \leqslant m$, $n' \leqslant n$ with strict inequality in one of them. Then interchanging f, g in (3.21), multiplying both sides of the resulting equation by ε, and adding to (3.21), we obtain from the induction hypothesis

$$\begin{aligned}
&\frac{d}{dt}\{E(f_1, ..., f_m, f, g, g_1, ..., g_n, t) + \varepsilon E(f_1, ..., f_m, g, f, g_1, ..., g_n, t)\}\\
&\quad = (K(f, g, t) + \varepsilon K(g, f, t)) \Bigg\{ \sum_{1 \leqslant i < j \leqslant m} (-\varepsilon)^{i+j-1} \rho(f_i, f_j, t)\\
&\qquad \times E(f_1, ..., \hat{f}_i, ..., \hat{f}_j, ..., f_m, g_1, ..., g_n, t)\\
&\qquad + \sum_{1 \leqslant i < j \leqslant n} (-\varepsilon)^{i+j-1} \rho(g_i, g_j, t)\\
&\qquad \times E(f_1, ..., f_m, g_1, ..., \hat{g}_i, ..., \hat{g}_j, ..., g_n, t)\\
&\qquad + \sum_{\substack{1 \leqslant i \leqslant m\\ 1 \leqslant j \leqslant n}} (-\varepsilon)^{m+i+j-1} \rho(f_i, g_j, t)\\
&\qquad \times E(f_1, ..., \hat{f}_i, ..., f_m, g_1, ..., \hat{g}_j, ..., g_n, t)\Bigg\}\\
&\qquad + (\rho(f, g, t) + \varepsilon \rho(g, f, t))\, E(f_1, ..., f_m, g_1, ..., g_n, t)\\
&\quad = \frac{d}{dt}\{(K(f, g, t) + \varepsilon K(g, f, t))\, E(f_1, ..., f_m, g_1, ..., g_n, t)\}.
\end{aligned}$$

This implies (3.20) for (m, n). ∎

THEOREM 3.6. *Let $\{X(f), f \in \mathscr{k}\}$ be a smooth Levy boson or fermion field of operators in $\mathscr{H}$ with cyclic vector ϕ and covariance kernel $K(f, g)$. Suppose $\mathscr{M}$ is the linear manifold generated by all vectors of the form $X(f_1)\, X(f_2) \cdots X(f_n)\phi$, $f_j \in \mathscr{k}$, $n = 1, 2, ...,$ and ϕ. Then*

$$X(f)\, X(g) + \varepsilon X(g)\, X(f) = K(f, g) + \varepsilon K(g, f)$$

on the domain $\mathscr{M}$ for all f, $g \in \mathscr{k}$, where $\varepsilon = -1$ or 1 according to whether X is a Levy boson or fermion field.

Proof. By condition (c) for a Levy field

$$\lim_{t \to \infty} E(f_1, ..., f_n, t) = \langle \phi, X(f_1) \cdots X(f_n)\phi \rangle = E(f_1, ..., f_n), \text{ say.}$$

By Proposition 3.5

$$\begin{aligned}
&\langle X(f_m)\cdots X(f_1)\phi, \{X(f)\,X(g)+\varepsilon X(g)\,X(f)\}\,X(g_1)\cdots X(g_n)\phi\rangle \\
&\quad = E(f_1, ..., f_m, f, g, g_1, ..., g_n)+\varepsilon E(f_1, ..., f_m, g, f, g_1, ..., g_n) \\
&\quad = \lim_{t\to\infty}\{E(f_1, ..., f_m, f, g, g_1, ..., g_n, t)+\varepsilon E(f_1, ..., f_m, g, f, g_1, ..., g_n, t)\} \\
&\quad = \lim_{t\to\infty}\{K(f, g, t)+\varepsilon K(g, f, t)\}\,E(f_1, ..., f_m, g_1, ..., g_n, t) \\
&\quad = \{K(f, g)+\varepsilon K(g, f)\}\langle X(f_m)\cdots X(f_1)\phi, X(g_1)\cdots X(g_n)\phi\rangle. \quad \blacksquare
\end{aligned}$$

THEOREM 3.7. *Let X, X' be two smooth Levy boson (fermion) fields in $\mathscr{H}$, $\mathscr{H}'$, respectively, with cyclic vectors ϕ, ϕ' and the same covariance kernel K. Then they are equivalent.*

Proof. Let

$$E^{\#}(f_1, f_2, ..., f_n, t)=\langle \phi^{\#}, X^{\#}_{f_1}(0, t)\,X^{\#}_{f_2}(0, t)\cdots X^{\#}_{f_n}(0, t)\,\phi^{\#}\rangle$$

for all $f_1, ..., f_n$, $n=1, 2, ...$, where $\#$ indicates two equations with or without the prime $'$. From Proposition 3.4 it is clear that $E^{\#}$ obeys the same set of differential equations

$$\frac{dE^{*}}{dt}(f_1, ..., f_n, t)=\sum(-\varepsilon)^{i+j-1}\rho(f_i, f_j, t)\,E^{\#}(f_1, ..., \hat{f}_i, ..., \hat{f}_j, ..., f_n, t)$$

and

$$E(f_1, f_2, t)=E'(f_1, f_2, t)=K(f_1, f_2, t),$$

where ρ is defined by Proposition 2.2 and $K(f_1, f_2, t)$ is as in Proposition 3.5. Since $E^{\#}(f_1, ..., f_n, 0)=0$ it follows that

$$E(f_1, f_2, ..., f_n, t)=E'(f_1, f_2, ..., f_n, t).$$

Letting $t\to\infty$ and using condition (c) for Levy fields we get

$$\langle\phi, X(f_1)\cdots X(f_n)\phi\rangle=\langle\phi', X'(f_1)\cdots X'(f_n)\phi'\rangle$$

for all $n=1, 2, ..., f_j\in \mathscr{k}$. The selfadjointness of the operators $X^{\#}(f)$ implies that the correspondence $\phi\to\phi'$, $X(f_1)\,X(f_2)\cdots X(f_n)\,\phi\to X'(f_1)\,X'(f_2)\cdots X'(f_n)\,\phi'$ for all $n=1, 2, ..., f_j\in \mathscr{k}$ extends to a unitary isomorphism from $\mathscr{H}$ onto $\mathscr{H}'$. $\blacksquare$

Remark. Theorem 3.17 shows that a smooth Levy boson or fermion field is determined up to equivalence by its covariance kernel K or, equivalently, the 2×2 matrix valued function $((\rho_{ij}))$ occurring in Proposition 2.2. We call $((\rho_{ij}))$ the *covariance density* of the Levy field. In cases when $((\rho_{ij}(t)))$ has constant rank almost everywhere we shall construct concrete models of Levy fields using Fock spaces.

Case 1. Let $\rho_{11}(t)\,\rho_{22}(t) - |\rho_{12}(t)|^2 > 0$ a.e. t. Define

$$p_{\pm}(t) = \tfrac{1}{2}(1 \pm (\rho_{11}(t)\,\rho_{22}(t) - [\operatorname{Re}\rho_{12}(t)]^2)^{-1/2} \operatorname{Im}\rho_{12}(t)\},$$

$$S_+ = p_+^{1/2} \begin{pmatrix} \rho_{11} & \operatorname{Re}\rho_{12} \\ \operatorname{Re}\rho_{12} & \rho_{22} \end{pmatrix}^{1/2}, \tag{3.22}$$

$$S_- = p_-^{1/2} \begin{pmatrix} \rho_{11} & -\operatorname{Re}\rho_{12} \\ -\operatorname{Re}\rho_{12} & \rho_{22} \end{pmatrix}^{1/2}. \tag{3.23}$$

Define real linear maps $S_\pm$ on $\mathscr{k}$ by putting

$$\begin{pmatrix} \operatorname{Re} S_\pm f \\ \operatorname{Im} S_\pm f \end{pmatrix} = S_\pm \begin{pmatrix} \operatorname{Re} f \\ \operatorname{Im} f \end{pmatrix},$$

where on the right hand side $S_\pm$ are interpreted as the 2×2 matrices (3.22), (3.23).

Let $\Gamma(\mathscr{k})$ denote the boson Fock space over $\mathscr{k}$ with vacuum vector ϕ_0. Put $\mathscr{H} = \Gamma(\mathscr{k}) \otimes \Gamma(\mathscr{k})$, $\phi = \phi_0 \otimes \phi_0$, and

$$X^-(f) = \{a(S_+ f) + a^\dagger(S_+ f)\} \otimes 1 + 1 \otimes \{a(S_- \bar{f}) + a^\dagger(S_- \bar{f})\},$$

where $a(f)$, $a^\dagger(f)$ are the annihilation and creation operators associated with f in $\Gamma(\mathscr{k})$. Then $\{\tilde{X}^-(f), f \in \mathscr{k}\}$ is a smooth Levy boson field with cyclic vector ϕ and convariance density $((\rho_{ij}))$, where $\sim$ indicates closure.

In $\mathscr{k}$ define the unitary operators R_s for each $s \geqslant 0$ by putting

$$\begin{aligned} (R_s f) &= -f(t) && \text{if} \quad t \leqslant s, \\ &= f(t) && \text{if} \quad t > s. \end{aligned}$$

Let $J(s) = \Gamma(R_s)$ denote the second quantization of R_s acting in $\Gamma(\mathscr{k})$. Define

$$X^+(f) = \int_0^\infty J(s) \otimes J(s)\, X_f^-(ds),$$

where $X_f^-([0, t]) = X^-(f\chi_{[0,t]})$ and the right hand side is a quantum stochastic integral in the sense of [1]. It follows from the results of [2, 5] that $\{\tilde{X}^+(f), f \in \mathscr{k}\}$ is a smooth Levy fermion field with cyclic vector ϕ and convariance density $((\rho_{ij}))$.

It is to be noted that when $\operatorname{Im} \rho_{12}(t) \equiv 0$, X^- reduces to a classical complex Gaussian field.

Case 2. Let $\operatorname{rank}((\rho_{ij}(t)) = 1$ a.e. Then we can express

$$((\rho_{ij}(t))) = \begin{pmatrix} \rho_1(t) \\ \rho_2(t) \end{pmatrix} \overline{(\rho_1(t)}, \overline{\rho_2(t))},$$

where ρ_1, ρ_2 are some bounded complex valued Borel functions on $\mathbb{R}_+$. Following the notations in case 1 put $\mathscr{H} = \Gamma(\mathscr{k})$, and define

$$X^-(f) = a(\bar{\rho}_1 \operatorname{Re} f + \bar{\rho}_2 \operatorname{Im} f) + a^\dagger(\bar{\rho}_1 \operatorname{Re} f + \bar{\rho}_2 \operatorname{Im} f).$$

Then $\{\tilde{X}^-(f), f \in \mathscr{k}\rangle$ is a smooth Levy boson field in $\mathscr{H}$ with cyclic vector ϕ_0 and covariance density $((\rho_{ij}))$. When $\operatorname{Im} \rho_{12} = 0$ then ρ_1 and ρ_2 can be chosen to be real valued and X^- becomes a Gaussian field.

To construct the corresponding smooth Levy fermion field with covariance density $((\rho_{ij}))$ put

$$X^+(f) = \int_0^\infty J(s)\, X_f^-(ds).$$

Then the family $\{\tilde{X}^+(f), f \in \mathscr{k}\}$ has the required property

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