6. T. Hida, N. Obata and K. Saitô, *Nagoya Math. J.* **128** (1992) 65–93.
7. Z. Y. Huang, *Adv. Math.* **17** (1988) 360–378.
8. Z. Y. Huang, *Nagoya Math. J.* **129** (1993) 23–42.
9. R. L. Hudson and K. R. Parthasarathy, *Commun. Math. Phys.* **93** (1984) 301–325.
10. S. L. Luo, *Wick algebra of generalized operators involving quantum white noise*, *J. Operator Theory* **38** (1997) 367–378.
11. P. A. Meyer, *Distribution, noyaux, symbol d'apres Krée, Lect. Note Math., Vol. 1321* (1988) 467–476.
12. N. Obata, *J. Math. Soc. Japan* **45** (1993) 421–445.
13. K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus* (Birkhäuser, 1993).

Infinite Dimensional Analysis, Quantum Probability and Related Topics
Vol. 1, No. 3 (1998) 467–483

THE SEMI-CIRCLE DIAGRAMS IN THE STOCHASTIC LIMIT OF THE ANDERSON MODEL

LUIGI ACCARDI*

Graduate School of Polymathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan
and
Centro V. Volterra, Università di Roma Tor Vergata, via Tor Vergata, SNC 00133 Roma, Italy

YUN GANG LU†

Dipartimento di Matematica, Università di Bari, Campus Universitario, Via E. Orabona, N. 4, 70125 Bari, Italy

VIERI MASTROPIETRO

Dipartimento di Matematica, Università di Roma Tor Vergata and Centro V. Volterra

Received 8 October 1997
Revised 7 February 1998

We prove that, in the stochastic limit of the Anderson model only the non-crossing diagrams survive for the transition amplitude from the first excited state of the free Hamiltonian to the first excited state of the interacting Hamiltonian. This confirms a conjecture of Migdal (1958) and Abrikosov, Gorkov, Dzyaloshinski (1975). From this we deduce a closed (nonlinear) Schwinger–Dyson type equation for the limit transition amplitude whose solution can be found and gives the explicit dependence of this amplitude on the momentum of the excited state.

1. Introduction

The Anderson model was proposed in Ref. 1 to explain the finite conductivity of metals: it describes a system of fermions interacting with a δ-correlated classical Gaussian random field ϕ on $\mathbf{R}^d$, with $d \geq 3$, modeling the impurities of the metal. The Hamiltonian is:

$$H = H_0 + H_I = \int_\Lambda dx\ \tilde{\psi}^+(x)\left(\frac{\partial_x^2}{2m} - \mu\right)\tilde{\psi}^-(x) + \lambda\int_\Lambda dx\ \phi(x)\tilde{\psi}^+(x)\tilde{\psi}^-(x)\,, \quad (1.1)$$

*E-mail: accardi@math.nagoya-u.ac.jp
†E-mail: lu@pascal.dm.uniba.it

where λ is a real number (strength of the interaction), $\Lambda \subset \mathbf{R}^d$ is a square box of side L, $\mu = p_F^2/2m$ is the *chemical potential*, p_F is the *Fermi momentum*, m is the fermion mass, and $\tilde{\psi}^\varepsilon(x)$, $\varepsilon = \pm 1$ is the *fermionic field* with periodic boundary condition:

$$\tilde{\psi}^\varepsilon(x) = \frac{1}{L^{d/2}} \sum_k e^{i\varepsilon kx} a_k^\varepsilon \tag{1.2}$$

where $k = 2n\pi/L$, $n = (n_1, \ldots, n_d) \in \mathbf{Z}^d$, $\varepsilon \in \{1, -1\}$, $\{a_k^\varepsilon, a_{k'}^{-\varepsilon'}\} = a_k^\varepsilon a_{k'}^{-\varepsilon'} + a_{k'}^{-\varepsilon'} a_k^\varepsilon = \delta_{\varepsilon,\varepsilon'}\delta_{k,k'}$. In what follows it will be convenient to expand the random variables $\phi(x)$ in Fourier series

$$\phi(x) = \frac{1}{L^d} \sum_p \phi_p e^{-ipx} \tag{1.3}$$

leading to a discrete family of (complex valued) Gaussian random variables ϕ_p with $p \in (2n\pi/L)\mathbf{Z}^d$ and

$$E(\phi_k \phi_{k'}) = L^d \delta_{k,-k'}, \qquad \phi_k^* = \phi_{-k}. \tag{1.4}$$

In the notations introduced above, the free evolution is characterized by the following property:

$$\begin{aligned} \tilde{\psi}^\varepsilon(x,t) = e^{iH_0 t}\tilde{\psi}^\varepsilon(x) e^{-iH_0 t} &= \frac{1}{\sqrt{L^d}} \sum_k e^{i\varepsilon\left(kx + \left(\frac{|k|^2}{2m} - \mu\right)t\right)} a_k^\varepsilon \\ &= \frac{1}{\sqrt{L^d}} \sum_k e^{i\varepsilon(kx + \omega_k t)} a_k^\varepsilon, \end{aligned} \tag{1.5}$$

where $\omega_k = [|k|^2/2m - \mu]$. The Hamiltonian in interaction representation is

$$\lambda H_I(t) = \lambda e^{iH_0 t} H_I e^{-iH_0 t} = \lambda \int_\Lambda dx\ \phi(x) \tilde{\psi}^+(x,t) \tilde{\psi}^-(x,t). \tag{1.6}$$

It is convenient to regularize this Hamiltonian by introducing a cutoff and normal order:

$$\lambda H_I(t) = \lambda \int_\Lambda dx\ \phi(x) : \psi^+(x,t)\psi^-(x,t) : + \text{h.c.}, \tag{1.7}$$

where $\psi^\varepsilon(x,t)$ is the regularized version of $\tilde{\psi}^\varepsilon(x,t)$ given by

$$\psi^\varepsilon(x,t) = \frac{1}{\sqrt{L^d}} \sum_k e^{i\varepsilon(kx + \omega_k t)} g_k a_k^\varepsilon \tag{1.8}$$

g_k is a complex valued cutoff function to be specified in the following and $: \cdot :$ denotes normal order

$$: a_{k+p}^+ a_k : = a_{k+p}^+ a_k - \langle \Phi_F, a_{k+p,\sigma}^+ a_{k,\sigma} \Phi_F \rangle \tag{1.9}$$

with respect to the *ground state* of H_0

$$\Phi_F = \prod_{|k| \le p_F} a_k^+ |0\rangle. \tag{1.10}$$

The *evolution operator* at time T is defined in the usual way:

$$U_T = U_T^{(\lambda,L)} = 1 + \sum_{n=1}^{\infty} (-i)^n \lambda^n \int_0^T dt_1 \cdots \int_0^{t_{n-1}} dt_n\ H_I(t_1) \cdots H_I(t_n) \tag{1.11}$$

where for each finite L and T the series converges in norm with respect to the fermions and in L^2 with respect to the Gaussian field.

The first excited state $\psi_k^\varepsilon \Phi_F$ of the free Hamiltonian can be obtained by adding or subtracting a particle to the ground state of the free Hamiltonian. The corresponding interacting state is obtained imagining that the interaction is switched on at time $-\infty$ so that it is given by $U_{-\infty}\psi_k^\varepsilon \Phi_F$, where U_{-T} is given by Eq. (1.11) with $H_I(-t)$ replacing $H_I(t)$.

This leads to study the projection of the interacting state on the free one

$$\lim_{L,T\to\infty} E\left(\frac{\langle \Phi_F, \psi_k^{-\varepsilon} U_{-T} \psi_k^\varepsilon \Phi_F \rangle}{\langle \Phi_F, U_{-T}, \Phi_F \rangle}\right) \tag{1.12}$$

and from the known identity[2]

$$\frac{\langle \Phi_F, \psi_k^{-\varepsilon} U_{-T} \psi_k^\varepsilon \Phi_F \rangle}{\langle \Phi_F, U_{-T}, \Phi_F \rangle} = \langle \Phi_F, \psi_k^\varepsilon U_{-T} \psi_k^{-\varepsilon} \Phi_F \rangle_{\text{conn}},$$

where $\langle \cdot \rangle_{\text{conn}}$ denotes the expectation with respect to the connected diagrams, one can restrict one's attention to these diagrams. The determination of the limit (1.12) is the first and the most difficult step towards the determination of two-point correlation function

$$\lim_{L,T\to\infty} E\left(\frac{\langle \Phi_F, U_{-T} \psi_k^{-\varepsilon} \psi_k^\varepsilon U_T \Phi_F \rangle}{\langle \Phi_F, U_{-T} U_T \Phi_F \rangle}\right). \tag{1.13}$$

The phenomenon of Anderson localization has been related to an exponential decay of the Fourier transform of Eq. (1.15), as opposed to the power law of the free case. This exponential decay was proved in Ref. 5 for large λ or $d = 1$ as a consequence of the results of Refs. 6 and 7 and others on the Schrödinger equation with a random potential. However, it is not known what is the decay when λ is small or $d \geq 2$. In Ref. 2, following an idea of Ref. 9 for the boson case, it was shown that, if one neglects a suitable class of contributions in the perturbative expansion for Eq. (1.13), the so-called *crossing diagrams*, one obtains a closed equation Schwinger–Dyson for the limit (1.13) from which the exponential decay can be deduced. However, the above-mentioned authors did not specify under which physical conditions the crossing diagrams can be neglected with respect to the non-crossing ones.

A first attempt to clarify this point was done in Ref. 11 who replaced the original Hamiltonian (1.1) by a discrete mean field approximation in which a fictitious N-valued index was added to the fermions and the classical random field was replaced by a random matrix. In the large-N limit of this model only the non-crossing diagrams survive. This model was generalized by Ref. 10 who replaced the large random matrix by *free independent random variables* and obtained an equation of

Schwinger–Dyson type. However, this result is essentially equivalent to the original assumption that only the non-crossing diagrams survive; in fact, as shown by Voiculescu,[12] $N\times N$ Hermitian matrices with independent Gaussian entries become, in the large-N limit, free random variables whose n-point correlation functions are described precisely by the non-crossing (or half-planar) diagrams. Moreover, the physical meaning of this type of results is not clear because they depend in an essential way on the large-N limit of a fictitious index which is absent from the original Hamiltonian.

We propose a different approach to study the Anderson model, based on the so-called *stochastic limit* which in recent years has evolved into a very general method yielding useful insights in a variety of physical problems ranging from quantum optics to bosonization, to solid state or field theory models.[4]

The physical idea of the stochastic limit is that it is reasonable that even a small interaction can produce a relevant effect if the time T and the box size L are very large. This suggests one to study Eq. (1.12) in the limit $\lambda\to 0$, $L\to\infty$, $T\to\infty$. These three limits cannot be performed independently, otherwise one would obtain a trivial result. The detail about how the limit has to be performed in order to avoid trivialities is given by the second-order term in the expansion (1.11). One finds in fact that the only possibility that the limit of this term exists and is nontrivial is to take the limits in the following way:

$$L\to\infty,\ \lambda\to 0,\ T\to\infty,\quad \lambda^2 T\to \text{const.}$$

The above limit is equivalent to the *rescaling* $T\to T/\lambda^2$ followed by the limits

$$L\to\infty,\qquad \lambda\to 0\,. \tag{1.14}$$

The limits must be taken exactly in the order from left to right in (1.14) otherwise no limit exists. In conclusion, our goal is to study the limit

$$\lim_{\lambda\to 0}\lim_{L\to\infty} E(\langle\Phi_F,\psi(x)U_{T/\lambda^2}\psi^+(y)\Phi_F\rangle_{\text{conn}})\,. \tag{1.15}$$

We prove that only the non-crossing diagrams in the perturbative expansion for the transition amplitude contribute to the above limit. This result can be physically interpreted as that the non-crossing diagrams are the dominating ones when the time T is very long compared with the coupling λ (this is just the physically interesting regime). The reason why, in this limit, only the non-crossing diagrams survive will be explained in Theorem 2.1. Theorem 2.1 contains all the analytic informations needed to deduce the main result of this paper, i.e. Eq. (3.2). We do not give the full proof of the theorem, which is long and technical, but only illustrate, in Appendix B, its main idea for the case of the four-point function.

Theorem 2.1 shows in particular that, even if the limit diagrams of the correlation functions are only non-crossing, the corresponding probability distribution for the time averaged interacting Hamiltonian is not, like in the large-N limit, the Wigner semi-circle law, usually associated to these diagrams, but a nonlinear deformation of it.

The fact that we find a deformation of the free law is not surprising: one can expect that our limiting model is much closer to the original one than the $N\to\infty$, so a deviation from a mean field theory, taking into account the nonlinearity of the interaction, is expected.

Despite this nonlinearity, some properties of the free random variables still survive and this allows us to obtain (using a combinatorial argument described in Appendix A) the main result of this paper, i.e. Eq. (3.2). This is a closed equation for the transition probability amplitude which, even if similar to the Schwinger–Dyson equation obtained in Refs. 2 and 9 for the two-point correlation function by resummation of the non-crossing Feynman graphs and to the equation for the one-particle Green function in the $N\to\infty$ limit of the Anderson model, was not previously known in the literature.

This dominating role of the non-crossing diagrams is not specific to the present model but seems to be a universal phenomenon for interacting quantum fields. In fact it first appeared in the stochastic limit of QED without dipole approximation[3] (but in that case the nonlinear deformation of the semicircle law is completely different from the present one). As noticed in Ref. 8 this universality is related to the momentum conservation in Feynman graphs, a fact which also plays an important role in this paper.

The paper is organized as follows. In Sec. 2 we show that the time averaged interacting Hamiltonian becomes, in the limit, a quantum stochastic variable distributed according to a deformation of the Wigner law. In Sec. 3 we study the limiting evolution operator and we show that it obeys a remarkable closed equation. It is likely that from the structure of such equation it follows exponential decay for the Fourier transform of Eq. (1.12).

2. The Limit of the Connected Correlators

The first step of the stochastic limit approach was suggested by the first-order term of the iterated series (1.11) after the rescaling $T\to T/\lambda^2$. This is equal to the "time averaged" interacting Hamiltonian (with $S=0$):

$$B_\lambda(T,S):=\lambda\int_{S/\lambda^2}^{T/\lambda^2} dt\,\frac{1}{L^d}\sum_{k\neq k'} g(k)\bar g(k')\phi_{k'-k}a_k^+a_{k'}e^{it(\omega(k)-\omega(k'))}\,. \tag{2.1}$$

We show that:

Theorem 2.1. *For each $N\in\mathbf{N}$*

$$\lim_{\lambda\to\infty}\lim_{L\to\infty}\mathbf{E}\big(\langle\phi_F,a_{k_0}B_\lambda(T_1,S_1)\cdots B_\lambda(T_N,S_N)a_{k_0}^+\phi_F\rangle_{\text{conn}}\big)\,, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_{\text{conn}}$ means expectation with respect to the connected diagrams, always exists and is equal to zero if N is odd while, if $N=2n$, it is equal *to*

$$\sum_{\sigma\in\mathcal{S}_{1,2n}}\ \sum_{\{l_h,r_h\}_{h=1}^n\in\mathcal{P}(1,2n)}\ \prod_{h=1}^n\langle\chi_{[S_{\sigma(l_h)},T_{\sigma(l_h)}]},\chi_{[S_{\sigma(r_h)},T_{\sigma(r_h)}]}\rangle_{L^2(\mathbf{R})}\int dk_{l_1}\cdots dk_{l_n}$$

$$\chi_{B_{\mathrm{F}}^c}(k_0)\prod_{h=1}^n\langle k_{l_h},k_{r_h}\rangle F_\sigma(k_{l_h})F_\sigma(k_{r_h})\,, \tag{2.3}$$

where

(a) $\mathcal{P}(1,2n)$ *is the set of all non-crossing pair partitions* $\{l_1,r_1,\ldots,l_n,r_n\}$ *of the set* $\{1,\ldots,2n\}$,

(b) $\mathcal{S}_{1,2n}$ *are the permutations of the set* $\{1,\ldots,2n\}$,

(c) *the* k_{r_h} *are linear combinations of the* $\{k_{l_h}\}_{h=1}^n$ *determined in the following system of linear equations:*

$$k_0=k_{2n};\ k_{l_{h-1}}-k_{l_h}=k_{r_h}-k_{r_{h-1}}\,.$$

(d)

$$\langle k_l,k_r\rangle=\int_{-\infty}^{\infty}e^{i[\omega(k_l)-\omega(k_r)]u}g(k_l)^2g(k_r)^2\,du\,,$$

(e) *denoting* B_{F} *the Fermi sphere* $B_{\mathrm{F}}:=\{k,|k|\le p_{\mathrm{F}}\}$, $B_{\mathrm{F}}^c=\mathbf{R}^d\setminus B_{\mathrm{F}}$ *its set-theoretical complement, the function* $F_\sigma(k_{l_h})$ *is defined by:*

$$F_\sigma(k_{l_h}):=\begin{cases}\chi_{B_{\mathrm{F}}^c}(k_{l_h}) & \text{if } \sigma(l_h)<\sigma(l_h+1)\,,\\ \chi_{B_{\mathrm{F}}}(k_{l_h}) & \text{if } \sigma(l_h)>\sigma(l_h+1)\,,\end{cases} \tag{2.4}$$

$$F_\sigma(k_{r_h}):=\begin{cases}\chi_{B_{\mathrm{F}}^c}(k_{r_h}) & \text{if } \sigma(r_h)<\sigma(r_h+1)\,,\\ \chi_{B_{\mathrm{F}}}(k_{r_h}) & \text{if } \sigma(r_h)>\sigma(r_h+1)\,.\end{cases} \tag{2.5}$$

The proof of the above theorem for $n=4$ is in Appendix B, and a general proof be will published elsewhere. Note the analogy of the above result with a central limit theorem; the $B_\lambda(T,S)$ are integrated over an integral extended on an interval of amplitude $n=T\lambda^{-2}$ and normalized by $1/\sqrt{n}$. However, the distribution of the limiting fields is not Gaussian; there is not a sum over all the pair partition but only over the *non-crossing* pair partitions. This property characterizes the free variables, and in fact if in Eq. (2.3) we neglect

(a) the dependence on the pair partition of the momenta k_{r_h},

(b) the dependence on σ of the factors $\langle\chi_{[S_{\sigma(l_h)},T_{\sigma(l_h)}]},\chi_{[S_{\sigma(r_h)},T_{\sigma(r_h)}]}\rangle_{L^2(\mathbf{R})}$ and $F_\sigma(k_{l_h})F_\sigma(k_{r_h})$,

then the above theorem says that $B_\lambda(T,S)$ are, in the limit, *free* (in the sense of Ref. 12) random variables as their distribution is given by the Wigner semi-circle law. This result has to be compared with Ref. 10 in which in the mean field approximation, i.e. in the limit $N\to\infty$ it was shown that the interacting Hamiltonian is a free variable.

3. The Limit of the Connected Transition Amplitude

Theorem 3.1. *In the notations of Theorem 2.1 the following identity is valid at any order of the perturbative series:*

$$\lim_{\lambda\to0}\lim_{L\to\infty}E(\langle\Phi_{\mathrm{F}},a_{k_0}^{\varepsilon}U_{T/\lambda^2}a_{k_0}^{-\varepsilon}\Phi_{\mathrm{F}}\rangle_{\mathrm{conn}})$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\chi^\varepsilon(k_0)T^n\sum_{\{l_h,r_h\}_{h=1}^n\in\mathcal{P}(1,2n)}\ \sum_{\substack{\sigma\in\mathcal{S}_{1,2n}\\ \sigma(l_h)-\sigma(r_h)=\pm1\ \forall h\in\{1,\ldots,n\}}}$$

$$\int dk_{l_1}\cdots dk_{l_n}\prod_{h=1}^n\langle k_{l_h},k_{r_h}\rangle_{\varepsilon_h}F_\sigma(k_{l_h})F_\sigma(k_{r_h})\equiv G^\varepsilon(k_0)\,, \tag{3.1}$$

where $\varepsilon_h=1$ *if* $\sigma(l_h)-\sigma(r_h)=+1$ *and* $\varepsilon_h=-1$ *otherwise,*

$$\langle k_l,k_r\rangle_{-1}=\int_{-\infty}^{0}e^{i[\omega(k_l)-\omega(k_r)]u}g(k_l)^2g(k_r)^2\,du\,,$$

$$\langle k_l,k_r\rangle_{1}=\int_{0}^{\infty}e^{i[\omega(k_l)-\omega(k_r)]u}g(k_l)^2g(k_r)^2\,du\,,$$

and $\chi^\varepsilon(k_0)=\chi_{B_{\mathrm{F}}^c}(k_0)$ *if* $\varepsilon=1$ *and* $\chi^\varepsilon(k_0)=\chi_{B_{\mathrm{F}}}(k_0)$ *if* $\varepsilon=-1$.

The above theorem says that, in the perturbative expansion for the transition amplitude, only the non-crossing diagrams contribute in the limit.

From the explicit expression of Eq. (3.1) it follows that G^ε obeys a closed equation. In fact, the following holds:

Theorem 3.2. *In the same notations as in Theorem 3.1 one has:*

$$G^\varepsilon(k_0)=G_0^\varepsilon(k_0)+TG_0^\varepsilon(k_0)G^\varepsilon(k_0)\int_{-\infty}^{+\infty}du\,e^{-i\omega(k_0)u}g(k_0)^2$$

$$\times\int_{\mathbb{R}^d}g(k)^2e^{i\omega(k)u}[G^+(k)+G^-(k)]\,dk\,, \tag{3.2}$$

where $G_0^\varepsilon(k_0)=\chi^\varepsilon(k_0)$.

Proof. Let us denote $G^{\varepsilon,(n)}(k)$ the nth term of the series Eq. (3.1) so that

$$G^\varepsilon(k)=\sum_{n=0}^{\infty}G^{\varepsilon,(n)}(k)\,.$$

We use a well-known property of the set $\mathcal{P}(1,2n)$ of the non-crossing pair partitions (n.c.c.p.) namely:

$$\mathcal{P}(1,2n)=\bigcup_{m=1}^{n}\{(1,2m)\cup\mathcal{P}(2,2m-1)\cup\mathcal{P}(2m+1,2n)\}$$

with the convention that $\mathcal{P}(x,y)=0$ if $x\geq y$. From it, it follows that

$$\sum_{\{l_h,r_h\}_{h=1}^n\in\mathcal{P}(1,2n)} = \sum_{m=1}^n \sum_{\{l_h,r_h\}_{h=2}^{m-1}\in\mathcal{P}(2,2m-1)} \sum_{\{l_h,r_h\}_{h=m+1}^n\in\mathcal{P}(2m+1,2n)} .$$

Moreover, in Appendix A it is proved that, for a fixed a non-crossing pair partition $(l_1,r_1,\ldots,l_n,r_n)$ we can write

$$\sum_{\substack{\sigma\in S_{1,2n}\\ \sigma(l_h)-\sigma(r_h)=\pm1\ \forall h\in\{1,\ldots,n\}}} \prod_{h=1}^n F_\sigma(k_{l_h})F_\sigma(k_{r_h}) = \frac{n!}{(m-1)!(n-m)!}$$

$$\times\chi^\varepsilon(k_0)\Bigg[\chi_{B_{\rm F}}(k_{l_1}) \sum_{\substack{\sigma\in S_{2,2m-1}\\ \sigma(l_h)-\sigma(r_h)=\pm1\ \forall h\in\{2,\ldots,m-1\},\sigma(l_2)>\sigma(l_1)}} \prod_{h=2}^{m-1} F_\sigma(k_{l_h})F_\sigma(k_{r_h})$$

$$+\chi_{B_{\rm F}^c}(k_{l_1}) \sum_{\substack{\sigma\in S_{2,2m-1}\\ \sigma(l_h)-\sigma(r_h)=\pm1\ \forall h\in\{2,\ldots,m-1\},\sigma(l_2)<\sigma(l_1)}} \prod_{h=2}^{m-1} F_\sigma(k_{l_h})F_\sigma(k_{r_h})\Bigg]$$

$$\times \sum_{\substack{\sigma\in S_{2m+1,2n}\\ \sigma(l_h)-\sigma(r_h)=\pm1\ \forall h\in\{m+1,\ldots,n\}}} \prod_{h=m+1}^n F_\sigma(k_{l_h})F_\sigma(k_{r_h}) . \tag{3.3}$$

Furthermore, by the momentum conservation, in every connected component of the pair partition, the incoming momentum is equal to the outgoing one. In particular,

$$k_0=k_{r_1}=k_{r_n} .$$

Using the remarks above $G^{\varepsilon,(n)}(k_0)$ can be rewritten, for $n\geq1$, as:

$$\sum_{m=1}^n \frac{T^n}{(m-1)!(n-m)!}\chi^\varepsilon(k_0)\int_{-\infty}^{+\infty}du\int e^{i[\omega(k_{l_1})-\omega(k_0)]u}\,du\; g(k_0)^2 g(k_{l_1})^2\,dk_{l_1}$$

$$\times\Bigg[\chi_{B_{\rm F}}(k_{l_1})\int dk_{l_2}\cdots dk_{l_{m-1}} \sum_{\{l_h,r_h\}_{h=2}^{m-1}\in\mathcal{P}(2,2m-1)} \prod_{h=2}^{m-1} \sum_{\substack{\sigma\in S_{2,2m-1}\\ \sigma(l_h)-\sigma(r_h)=\pm1,\sigma(l_2)<\sigma(l_1)}}$$

$$\langle k_{l_h},k_{r_h}\rangle_{\varepsilon_h}F_\sigma(k_{l_h})F_\sigma(k_{r_h})$$

$$+\chi_{B_{\rm F}^c}(k_{l_1})\int dk_{l_2}\cdots dk_{l_{m-1}} \sum_{\{l_h,r_h\}_{h=m+1}^{n}\in\mathcal{P}(2,2m-1)} \prod_{h=2}^{m-1} \sum_{\substack{\sigma\in S_{2,2m-1}\\ \sigma(l_h)-\sigma(r_h)=\pm1,\sigma(l_2)>\sigma(l_1)}}$$

$$\langle k_{l_h},k_{r_h}\rangle_{\varepsilon_h}F_\sigma(k_{l_h})F_\sigma(k_{r_h})\Bigg]$$

$$\times\chi^\varepsilon(k_0)\int dk_{l_{m+1}}\cdots dk_{l_n} \sum_{\{l_h,r_h\}_{h=m+1}^n\in\mathcal{P}(2m+1,2n)} \prod_{h=m+1}^n$$

$$\sum_{\substack{\sigma\in S_{2m+1,2n}\\ \sigma(l_h)-\sigma(r_h)=\pm1}} \langle k_{l_h},k_{r_h}\rangle_{\varepsilon_h}F_\sigma(k_{l_h})F_\sigma(k_{r_h}) . \tag{3.4}$$

By associating the factor $(n-m)!$ to the $\mathcal{P}(2m+1,2n)$ sum we see that this term reproduces $G^{\varepsilon,(n-m)}(k_0)$. Similarly associating the factor $(m-1)!$ to the two $\mathcal{P}(2,2m-1)$ sums in square brackets, we see that these sums reproduce respectively $G^{+,(m-1)}(k_0)$ and $G^{-,(m-1)}(k_0)$. This shows that Eq. (3.4) can be written in the form

$$G^{\varepsilon,(n)}(k_0)=T\int_{-\infty}^{+\infty}du\int e^{i[\omega(k_{l_1})-\omega(k_0)]u}\,du\; g(k_0)^2g(k_{l_1})^2\,dk_{l_1}\,\chi^\varepsilon(k_0)$$

$$\times\sum_{n_1+n_2=n-1}[(G^{+,(n_1)}(k_{l_1})+G^{-,(n_1)}(k_{l_1})]G^{\varepsilon,(n_2)}(k_0)$$

and summing over n, Eq. (3.2) is found. □

Remark 3.1. Let $S(k,\omega)$ be the Fourier transform of the *Green function*

$$S(x-y,t-s)=\lim_{L,T\to\infty}E(\langle\Phi_{\rm F},\mathcal{T}\psi^+_{x,t}\psi^-_{y,s}U_TU_{-T}\Phi_{\rm F}\rangle)_{\rm conn}$$

if $\mathcal{T}$ is the time order product. The closed equation found in Ref. 2 is

$$S(k_0,\omega)=S_0(k_0,\omega)+S_0(k_0,\omega)S(k_0,\omega)\int_{-\infty}^{+\infty}dk\,u(k_0-k)S(k,\omega) ,$$

where $S_0(k,\omega)$ is the free Green function and $u(p)$ is a suitable cutoff function. The similarity with our Eq. (3.2) is striking. In fact equations of this type are a general feature of the non-crossing diagrams.

Appendix A. Proof of Eq. (3.3)

In this Appendix, we prove Eq. (3.3). Fix the injective maps

$$\begin{aligned} l&:h\in\{1,\ldots,n\}\to l_h\in\{1,\ldots,2n\} ,\\ r&:l\in\{1,\ldots,n\}\to r_h\in\{1,\ldots,2n\} ,\end{aligned} \tag{A.1}$$

in such a way that $\{(l_1,r_1),\ldots,(l_n,r_n)\}$ is a non-crossing pair partition of $\{1,\ldots,2n\}$.

For each subset of pairs $I\subseteq\{(l_1,r_1),\ldots,(l_n,r_n)\}$ define

$$I(\sigma,l):=\{l_h:(l_h,r_h)\in I\,;\ \sigma(l_h)<\sigma(l_h+1)\} .$$

Similarly we can define

$$I(\sigma,r):=\{r_h:(l_h,r_h)\in I\,;\quad \sigma(r_h)<\sigma(r_h+1)\} .$$

Definition A.1. Given a finite set I and a permutation σ on I, the *index of monotonicity* of σ is the map

$$j_I(\sigma, \cdot) : x \in I \to j(\sigma, x) \in \{1, 0\}$$

defined by:

$$j(\sigma, x) = 1 \Leftrightarrow \sigma(x) < \sigma(x+1) .$$

Introducing the convention (for any set A)

$$\chi_A(k)^\varepsilon = \begin{cases} 1, & \text{if } \varepsilon = 0 \\ \chi_A(k), & \text{if } \varepsilon = 1 \end{cases}$$

one can write the L.H.S. of Eq. (3.3) in the form

$$\sum_{\sigma \in \mathcal{S}'_{1,2n}} \left[\prod_{h=1}^{n} \chi_{B_F^c}(k_{l_h})^{j(\sigma,l_h)} \chi_{B_F}(k_{l_h})^{1-j(\sigma,l_h)} \right] \times \left[\prod_{h=1}^{n} \chi_{B_F^c}(k_{r_h})^{j(\sigma,r_h)} \chi_{B_F}(k_{r_h})^{1-j(\sigma,r_h)} \right] , \tag{A.2}$$

where

$$\sum_{\sigma \in \mathcal{S}'_{1,2n}} = \sum_{\substack{\sigma \in \mathcal{S}_{1,2n} \\ \sigma(l_h) - \sigma(r_h) = \pm 1 \ \forall h \in \{1,\ldots,n\}}} .$$

Now the set $\{l_1, r_1, \ldots, l_n, r_n\}$ coincides with $\{1, \ldots, 2n\}$. The sum (A.2) can be rewritten in the form

$$\sum_{\sigma \in \mathcal{S}'_{1,2n}} \prod_{\alpha=1}^{2n} \chi_{B_F^c}(k_\alpha)^{j(\sigma,\alpha)} \chi_{B_F}(k_\alpha)^{1-j(\sigma,\alpha)} . \tag{A.3}$$

Remark A.1. We see that the terms in the sum (A.3) depend on σ only through the function $j(\sigma, \cdot)$.

Lemma A.1. *Every $\sigma \in \mathcal{S}'_{1,2n}$ can be written uniquely:*

$$\sigma = \overline{\sigma} \sigma_m \sigma_{n-m} ,$$

where

(a) *$\overline{\sigma}$ is characterized by the property that there exist sets $F \subseteq \{1, \ldots, 2m\}$, $F' \subseteq \{2m+1, \ldots, 2n\}$ such that:*

$$\overline{\sigma} F = F' ; \quad \overline{\sigma} F' = F ; \quad \overline{\sigma}(x) = x \quad \text{for any} \quad x \notin F \cup F' \tag{A.4}$$

and $\overline{\sigma}$ is monotone on $F \cup F'$.

(b) *$\sigma_m(x) = x$ for $x \notin \{1, \ldots, 2m\}$.*

(c) *$\sigma_{n-m}(x) = x$ for $x \notin \{2m+1, \ldots, 2n\}$.*

The proof of the above lemma is trivial. Denote now

$$I = I_{\overline{\sigma}} := \overline{\sigma}^{-1}\{1, 2m\} ,$$

then

$$I^c = \overline{\sigma}^{-1}\{2m+1, \ldots, 2m\} .$$

Denote

$$\mathcal{S}'_I(\overline{\sigma}) = \overline{\sigma} \mathcal{S}'_{\{1,2m\}} \overline{\sigma}^{-1} , \qquad \mathcal{S}'_{I^c}(\overline{\sigma}) = \overline{\sigma} \mathcal{S}'_{\{2m+1,2n\}} \overline{\sigma}^{-1} \tag{A.5}$$

and notice that, if σ has the form (A.4),

$$\sigma = (\overline{\sigma} \sigma_m \sigma_{n-m} \overline{\sigma}^{-1}) \overline{\sigma} = \sigma_I \sigma_{I^c} \overline{\sigma} . \tag{A.6}$$

Introducing the notation

$$\phi(k_x, j(\sigma, x)) = \chi_{B_F^c}(k_x)^{j(\sigma,x)} \chi_{B_F}(k_x)^{1-j(\sigma,x)}$$

and using

$$\sigma_I \sigma_{I^c} \bar{\sigma} x = \sigma_I \bar{\sigma} x \quad \text{for} \quad x \in \{1, \ldots, 2m\}$$

and the analog identity for σ_{I^c}, the sum (A.2) can be written in the form

$$\sum_{\overline{\sigma}} \sum_{\mathcal{S}'_I(\bar{\sigma})} \sum_{\mathcal{S}'_{I^c}(\bar{\sigma})} \prod_{x \in \{1,\ldots,2n\}} \varphi(k_x, j(\sigma_I \sigma_{I^c} \overline{\sigma}, x))$$

$$= \sum_{\overline{\sigma}} \left(\sum_{\sigma_I \in \mathcal{S}'_I(\bar{\sigma})} \prod_{x \in \{1,\ldots,2m\}} \varphi(k_x, j(\sigma_I \overline{\sigma}, x) \right) \times \left(\sum_{\sigma_{I^c} \in \mathcal{S}'_{I^c}(\bar{\sigma})} \prod_{x \in \{2m+1,\ldots,2n\}} \varphi(k_x, j(\sigma_{I^c} \overline{\sigma}, x)) \right) . \tag{A.7}$$

Thus, if we prove that both the σ_I-sum and the σ_{I^c}-sum do not depend on $\overline{\sigma}$, the expression (A.7) shall be equal to

$$\binom{n}{m} \left(\sum_{\sigma \in \mathcal{S}'_{1,2m}} \prod_{x \in \{1,\ldots,2m\}} \varphi(k_x, j(\sigma, x)) \right) \times \left(\sum_{\sigma' \in \mathcal{S}'_{2m+1,2n}} \prod_{x \in \{2m+1,\ldots,2n\}} \varphi(k_x, j(\sigma', x)) \right) . \tag{A.8}$$

Lemma A.2. *For every $\overline{\sigma}$, the two sums*

$$\sum_{\sigma \in \mathcal{S}'(1,2m)} \prod_{x \in \{1,\ldots,2m\}} \varphi(k_x, j(\sigma, x)) \tag{A.9}$$

and

$$\sum_{\sigma_I \in \mathcal{S}'_I(\bar\sigma)} \prod_{x\in\{1,\dots,2m\}} \varphi(k_x, j(\sigma_I\bar\sigma, x)) \tag{A.10}$$

coincide.

Proof. It is sufficient to show that, given $\bar\sigma$, for each $\sigma_I \in \mathcal{S}'_I$ there exists $\sigma \in \mathcal{S}'_{2m}$ such that

$$j(\sigma_I\bar\sigma, x) = j(\sigma, x), \qquad \forall x \in \{1,\dots,2m\}.$$

Recall that, by definition

$$\bar\sigma\{1,\dots,2m\} = I_{\bar\sigma}$$

and let $\sigma_0 \in \mathcal{S}'_I$ be the unique permutation such that

$$\sigma_0 I = \sigma_0\bar\sigma\{1,\dots,2m\} =: I_0$$

is an ordered set, i.e. the ordered version of I. Clearly the sum (A.10) is equal to

$$\sum_{\sigma_I \in \mathcal{S}'_I(\bar\sigma)} \prod_{x\in\{1,\dots,2m\}} \varphi(k_x, j(\sigma_I\sigma_0\bar\sigma, x)). \tag{A.11}$$

The map

$$\beta := \sigma_0\bar\sigma : \{1,\dots,2m\} \to I_0$$

is the unique monotone mapping between $\{1,\dots,2m\}$ and the ordered version of I.

The sum (A.11) then becomes

$$\sum_{\sigma_I \in \mathcal{S}'_I(\bar\sigma)} \prod_{x\in\{1,\dots,2m\}} \varphi(k_x, j(\sigma_I\beta, x)). \tag{A.12}$$

Now denote

$$\sigma := \beta^{-1}\sigma_I\beta \tag{A.13}$$

and notice that

$$j(\sigma_I\beta, x) = 1 \Leftrightarrow \sigma_I(\beta(x)) < \sigma_I(\beta(x+1))$$

but β is monotone, so because of injectivity β^{-1} is also monotone, therefore

$$j(\sigma_I\beta, x) = 1 \Leftrightarrow \beta^{-1}\sigma_I\beta(x) \le \beta^{-1}\sigma_I\beta(x+1) \Leftrightarrow \sigma(x) < \sigma(x+1) \Leftrightarrow j(\sigma, x) = 1.$$

This implies that

$$j(\sigma_I\beta, x) = j(\sigma, x) = 1.$$

So we can write the sum (A.12) as

$$\sum_{\beta\sigma\beta^{-1}=\sigma_I\in\mathcal{S}'_{I(\bar\sigma)}} \prod_{x\in\{1,\dots,2m\}} \varphi(k_x, j(\sigma, x)).$$

It is now clear that the sum over all $\sigma_I \in \mathcal{S}'_I(\bar\sigma)$ in (A.9) is equivalent to the sum over all $\sigma \in \mathcal{S}'_{1,2m}$. This ends the proof of (A.8).

The proof of (3.3) follows noting that $k_0 = k_{r_m}$ and $k_{l_1} = k_{r_{m-1}}$ where (l_{m-1}, r_{m-1}) is the last pair partition "enclosed" in (l_1, r_1), so that the only dependence of the summand of (A.8) on $\sigma(l_1), \sigma(r_1)$ is that $\varphi(k_{l_1}) = \varphi(k_{r_{m-1}}) = \chi_{B^c_F}(k_{l_1})$ if $\sigma(l_2) < \sigma(l_1)$ and $\varphi(k_{l_1}) = \varphi(k_{r_{m-1}}) = \chi_{B_F}(k_{l_1})$ otherwise. □

Appendix B. The Four-Point Function

We do not prove here Theorem 2.1 in general, but we consider a particular case in order to explain in an intuitive way why in the limit only the non-crossing diagrams survive.

Let us consider

$$\lim_{\lambda\to 0}\lim_{L\to\infty} E(\langle \phi_F, a_k B_\lambda(T_1,S_1)B_\lambda(T_2,S_2)B_\lambda(T_3,S_3)B_\lambda(T_4,S_4)] a_k^+ \phi_F\rangle). \tag{B.1}$$

Using (2.6), (B.1) can be written more explicitly in the form:

$$\begin{aligned}
&\lambda^4 \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2 \int_{S_3/\lambda^2}^{T_3/\lambda^2} dt_3 \int_{S_4/\lambda^2}^{T_4/\lambda^2} dt_4 \int dx_1\, dx_2\, dx_3\, dx_4 \\
&\quad\times E(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))\langle \phi_F, a_k : \psi^+(x_1,t_1)\psi(x_1,t_1) :: \psi^+(x_2,t_2)\psi(x_2,t_2) : \\
&\quad\times : \psi^+(x_3,t_3)\psi(x_3,t_3) :: \psi^+(x_4,t_4)\psi(x_4,t_4) : a_k^+\rangle_{\mathrm{conn}}.
\end{aligned} \tag{B.2}$$

The above fermionic expectation is given by the sum of several terms. Let us select (for definiteness) the following one (the sum over σ in (2.3) comes from taking into account all such terms):

$$\begin{aligned}
&\langle a_k\psi^+(x_1,t_1)\rangle\langle\psi(x_1,t_1)\psi^+(x_2,t_2)\rangle\langle\psi(x_2,t_2)\psi^+(x_3,t_3)\rangle \\
&\quad\times \langle\psi(x_3,t_3)\psi^+(x_4,t_4)\rangle\langle\psi(x_4,t_4)a_k^+\rangle.
\end{aligned} \tag{B.3}$$

Since the expectation over the Gaussian variables in (B.2) is given by

$$\delta(x_1-x_4)\delta(x_2-x_3) + \delta(x_1-x_2)\delta(x_3-x_4) + \delta(x_1-x_3)\delta(x_2-x_4). \tag{B.4}$$

We have three terms which shall be studied separately (notice that the third term corresponds to a crossing diagram while the other ones are non-crossing).

The first term is

$$\begin{aligned}
&\lambda^4 \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2 \int_{S_3/\lambda^2}^{T_3/\lambda^2} dt_3 \int_{S_4/\lambda^2}^{T_4/\lambda^2} dt_4 \int dx_1\, dx_2 \langle a_k\psi^+(x_1,t_1)\rangle \\
&\quad\times \langle\psi(x_1,t_1)\psi^+(x_2,t_2)\rangle\langle\psi(x_2,t_2)\psi^+(x_2,t_3)\rangle\langle\psi(x_2,t_3)\psi^+(x_1,t_4)\rangle\langle\psi(x_1,t_4)a_k^+\rangle \\
&= \lambda^4\frac{1}{L^{2d}}\sum_{k',k''} \int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2 \int_{S_3/\lambda^2}^{T_3/\lambda^2} dt_3 \int_{S_4/\lambda^2}^{T_4/\lambda^2} dt_4 \\
&\quad\times e^{-i(\omega(k)-\omega(k'))(t_1-t_2)} e^{-i(\omega(k)-\omega(k''))(t_3-t_4)},
\end{aligned} \tag{B.5}$$

$$\chi(|k|\ge p_{\mathrm F})\chi(|k'|\ge p_{\mathrm F})\chi(|k''|\ge p_{\mathrm F})[g(k)g(k')g(k)g(k')][g(k)g(k'')g(k)g(k'')]$$

$$= \frac{1}{L^{2d}}\sum_{k',k''}\int_{S_1}^{T_1} d\tau_1 \int_{(S_2-\tau_1)/\lambda^2}^{(T_2-\tau_1)/\lambda^2} d\tau_2 \int_{S_3}^{T_3} d\tau_3 \int_{(S_4-t_3)/\lambda^2}^{(T_4-t_3)/\lambda^2} d\tau_4$$

$$\times\, e^{-i(\omega(k)-\omega(k'))\tau_2} e^{-i(\omega(k)-\omega(k''))\tau_4}$$

$$\times\, \chi(|k|\ge p_{\mathrm F})\chi(|k'|\ge p_{\mathrm F})\chi(|k''|\ge p_{\mathrm F})[g(k)g(k')g(k)g(k')]$$

$$\times\, [g(k)g(k'')g(k)g(k'')]\,.$$

In the limit $\lambda\to 0, L\to\infty$ the above expression becomes

$$\langle \chi_{[S_1,T_1]}, \chi_{[S_2,T_2]}\rangle_{L^2(\mathbf{R})}\langle \chi_{[S_3,T_3]}, \chi_{[S_4,T_4]}\rangle_{L^2(\mathbf{R})}$$

$$\times \int dk_1\, dk_2\, \chi_{B_{\mathrm F}^c}(k)\langle k_1,k\rangle\langle k_2,k\rangle \chi_{B_{\mathrm F}^c}(k_1)\chi_{B_{\mathrm F}^c}(k_2)\,.$$

The second term is

$$\lambda^4\frac{1}{L^{2d}}\sum_{k',k''}\int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2 \int_{S_3/\lambda^2}^{T_3/\lambda^2} dt_3 \int_{S_4/\lambda^2}^{T_4/\lambda^2} dt_4$$

$$\times\, e^{-i(\omega(k)-\omega(k'))(t_1-t_4)} e^{-i(\omega(k')-\omega(k''))(t_2-t_3)}\,,$$

$$\chi(|k|\le p_{\mathrm F})\chi(|k'|\le p_{\mathrm F})\chi(|k''|\le p_{\mathrm F})[g(k)g(k')g(k)g(k')][g(k)g(k'')g(k)g(k'')] \tag{B.6}$$

$$= \frac{1}{L^{2d}}\sum_{k',k''}\int_{S_1}^{T_1} d\tau_1 \int_{S_2}^{T_2} d\tau_2 \int_{(S_4-\tau_1)/\lambda^2}^{(T_4-t_1)/\lambda^2} d\tau_4 e^{-i(\omega(k)-\omega(k'))\tau_4}$$

$$\times \int_{(S_3-\tau_2)/\lambda^2}^{(T_3-t_2)/\lambda^2} e^{-i(\omega(k')-\omega(k''))\tau_3}\chi(|k|\ge p_{\mathrm F})\chi(|k'|\ge p_{\mathrm F})$$

$$\times\, \chi(|k''|\ge p_{\mathrm F})[g(k)g(k')g(k)g(k')][g(k')g(k'')g(k')g(k'')]\,.$$

In the limit $L\to\infty$ the sum over k',k'' becomes an integral which is bounded by $O[1/((\tau_4)^{d/2}(\tau_4-\tau_3)^{d/2})]$ so that by taking the limit $\lambda\to 0$, we have

$$\langle \chi_{[S_1,T_1]}, \chi_{[S_4,T_4]}\rangle_{L^2(\mathbf{R})}\langle \chi_{[S_2,T_2]}, \chi_{[S_3,T_3]}\rangle_{L^2(\mathbf{R})}$$

$$\times \int dk_1\, dk_2\, \chi_{B_{\mathrm F}^c}(k)\langle k_1,k\rangle\langle k_2,k_1\rangle \chi_{B_{\mathrm F}^c}(k_1)\chi_{B_{\mathrm F}^c}(k_2)\,.$$

Finally the third term, corresponding to the only crossing diagram in the four-point function, is

$$\lambda^4\frac{1}{L^{2d}}\sum_{k',k''}\int_{S_1/\lambda^2}^{T_1/\lambda^2} dt_1 \int_{S_2/\lambda^2}^{T_2/\lambda^2} dt_2 \int_{S_3/\lambda^2}^{T_3/\lambda^2} dt_3 \int_{S_4/\lambda^2}^{T_4/\lambda^2} dt_4$$

$$\times\, e^{-i(\omega(k)-\omega(k'))(t_1-t_2)} e^{-i(\omega(k'')-\omega(k-k'+k''))(t_3-t_4)} e^{-i(\omega(k)-\omega(k''))t_2} e^{-i(\omega(k)-\omega(k''))t_4}$$

$$\times\, \chi(|k|\ge p_{\mathrm F})\chi(|k'|\ge p_{\mathrm F})\chi(|k''|\ge p_{\mathrm F})[g(k)g(k')g(k)g(k')][g(k)g(k'')g(k)g(k'')]\,. \tag{B.7}$$

With the change of variables $\lambda^2 t_2=\tau_2, \lambda^2 t_4=\tau_4$, $t_1-\tau_2/\lambda^2=\tau_1$, $t_3-\tau_4/\lambda^2=\tau_3$ this becomes

$$= \frac{1}{L^{2d}}\sum_{k,k'}\int_{S_2}^{T_2} d\tau_2 \int_{S_4}^{T_4} d\tau_4 \int_{(S_1-\tau_2)/\lambda^2}^{(T_1-\tau_2)/\lambda^2} d\tau_1 \int_{(S_3-\tau_4)/\lambda^2}^{(T_3-\tau_4)/\lambda^2} d\tau_3$$

$$\times\, e^{-i(\omega(k)-\omega(k'))\tau_1} e^{-i(\omega(k'')-\omega(k-k'+k''))\tau_3} e^{-i(\omega(k)-\omega(k''))\tau_2/\lambda^2} e^{-i(\omega(k)-\omega(k''))\tau_4/\lambda^2}$$

$$\times\, \chi(|k|\le p_{\mathrm F})\chi(|k'|\le p_{\mathrm F})\chi(|k''|\le p_{\mathrm F})[g(k)g(k')g(k)g(k')][g(k)g(k'')g(k)g(k'')]\,,$$

and, after the limit $L\to\infty$, the resulting integral vanishes in the limit $\lambda\to 0$ by dominated convergence and the Riemann–Lebesgue lemma.

Note Added in Proof

If, as in the case of physical interest, $\omega(k)=|k|^2$ and $g(k)$ depends only on $|\,k\,|$, i.e. $g(k)=g(|k|)$, then the nonlinear Eq. (3.2) can be explicitly solved as follows. First, note that, for $\varepsilon=1,2$, $G^\varepsilon(k_0)=0$ if and only if $G_0^\varepsilon(k_0)=0$. The "if" part is clear because $G_0^\varepsilon(k_0)$ multiplies the right-hand side of (3.2). Conversely, if $G^\varepsilon(k_0)=0$, then (3.2) becomes $G_0^\varepsilon(k_0)=G^\varepsilon(k_0)=0$. Second, note that, by definition of G_0^ε, one has $G_0^+(k)+G_0^-(k)=1$ identically in k. Therefore, by the above remark

$$G_0^+(k)=1 \Leftrightarrow G_0^-(k)=0 \Leftrightarrow G^-(k)=0\,,$$

$$G_0^-(k)=1 \Leftrightarrow G_0^+(k)=0 \Leftrightarrow G^+(k)=0$$

for all k. Equivalently this means that, identically in k

$$G_0^+G^- = G_0^-G^+ \equiv 0\,.$$

It follows that, adding the two equations (3.2) (for $\varepsilon=\pm$), one obtains

$$\sum_{\varepsilon=\pm} G^\varepsilon(k_0) = \sum_{\varepsilon=\pm} G_0^\varepsilon(k_0) + T\sum_{\varepsilon=\pm} G_0^\varepsilon(k_0)\sum_{\varepsilon'=\pm} G^{\varepsilon'}(k_0)$$

$$\times \int_{\mathbb R} du \int_{\mathbb R^d} dk e^{i(\omega(k)-\omega(k_0))u}|g(k_0)g(k)|^2 \sum_{\varepsilon'=\pm} G^\varepsilon(k)\,. \tag{1}$$

Using again that $G_0^+(k) + G_0^-(k) \equiv 1$ for all k and introducing the notation

$$G(k) := G^+(k) + G^-(k)$$

we have

$$G(k_0) = 1 + TG(k_0) \int_{\mathbb{R}} du \int_{\mathbb{R}^d} dk \; e^{i(\omega(k)-\omega(k_0))u} |g(k_0)g(k)|^2 G(k) \,. \tag{2}$$

Under our assumptions on $\omega(k) = |k|^2$ and on $g(k) = g(|k|)$, the integral (2) can be explicitly evaluated giving

$$\int_{\mathbb{R}^d} dk |g(|k_0|)g(|k|)|^2 G(k) 2\pi\delta(|k|^2 - |k_0|^2) = 2\pi |g(r_0)|^4 \cdot r_0 \int_{S_{r_0}} G(r_0, \sigma) d\sigma \,.$$

Therefore (2) is equivalent to

$$G(k_0) = \frac{1}{1 - 2\pi T |g(r_0)g(r_0)|^2 \cdot r_0 \int_{S_{r_0}} G(r_0,\sigma)d\sigma} \,, \tag{3}$$

where S_{r_0} denotes the sphere centered in the origin and with radius r_0. The right-hand side of (4) tells us that $G(k_0)$ depends only on $|\, k_0 \,|$, so that

$$G(r_0) = \frac{1}{1 - 2\pi T |S_{r_0}| G(r_0) |g(r_0)|^4 r_0} \,, \tag{4}$$

where $|S_{r_0}|$ is the surface of the sphere S_{r_0}. Denoting

$$a := 2\pi T |S_{r_0}| \cdot |g(r_0)|^4 r_0$$

we see that $G(r_0)$ satisfies the equation

$$G^2(r_0)a - G(r_0) + 1 = 0 \tag{5}$$

which, if $4a \leq 1$ (and this can be realized either by taking T to be small or by taking r_0 to be large, because g is a Schwarz function) has solutions $G(r_0) = (1 \pm \sqrt{1-4a})/2a$. But (4) implies that $G \to 1$ as $a \to 0$ and therefore

$$G(r_0) = \frac{1 - \sqrt{1-4a}}{2a} = \frac{2}{1 + \sqrt{1-4a}} \tag{6}$$

which is smooth and tends to 1 as $r_0 \to \infty$.

References

1. P. W. Anderson, *Phys. Rev.* **109** (1958) 1492.
2. A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, 1975).
3. L. Accardi and Y. G. Lu, *the Wigner semi-circle law in quantum electrodynamics, Commun. Math. Phys.* **180** (1996) 605–632, Volterra preprint No. 126 (1992).
4. L. Accardi, Y. G. Lu and I. Volovich, *Quantum theory and its stochastic limit*, to appear.
5. M. Aizenman, *Graf localization bound for an electron gas*, preprint.
6. J. Frölich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy, Commun. Math. Phys.* **88** (1983) 151–184.

7. J. Frölich, T. Spencer, F. Martinelli and E. Scoppola, *Commun. Math. Phys.* **101** (1988) 21.
8. J. Gough, Volterra preprint (1996), The stochastic sector of interacting–free quantum field theory.
9. A. B. Migdal, *Sov. Phys. JETP.* **7** (1958) 996.
10. R. Speicher and P. Neu, *Spectra of Hamiltonians with generalized single-site dynamical disorder, Z. Phys.* **B95** (1994) 101–111; *Random matrix theory for CPA: Generalization of Wegner's n-orbital model*, Preprint 1994; *J. Stat. Phys.* **80** (1995) 7279–7307.
11. F. Wegner, *Phys. Rev.* **B19** (1979) 783.
12. D. Voiculescu, *Free noncommutative random variables, random matrices and the II_1 factors of free groups*, in *Quantum Probability and Related Fields* (1991), pp. 473–487.