The Hidden Matching-Structure of the Composition of Strips: a Polyhedral Perspective

Yuri Faenza * Gianpaolo Oriolo † Gautier Stauffer ‡

Abstract

Stable set problems subsume matching problems since a matching is a stable set in a so-called line graph but stable set problems are hard in general while matching can be solved efficiently [11]. However, there are some classes of graphs where the stable set problem can be solved efficiently. A famous class is that of claw-free graphs; in fact, in 1980 Minty [19, 20] gave the first polynomial time algorithm for finding a maximum weighted stable set (MWSS) in a claw-free graph. One of the reasons why stable set in claw-free graphs can be solved efficiently is because the so called augmenting path theorem [4] for matching generalizes to claw-free graphs [5] (this is what Minty is using). We believe that another core reason is structural and that there is a intrinsic matching structure in claw-free graphs. Indeed, recently Chudnovsky and Seymour [8] shed some light on this by proposing a decomposition theorem for claw-free graphs where they describe how to compose all claw-free graphs from building blocks. Interestingly the composition operation they defined seems to have nice consequences for the stable set problem that go much beyond claw-free graphs. Actually in a recent paper [21] Oriolo, Pietropaoli and Stauffer have revealed how one can use the structure of this composition to solve the stable set problem for composed graphs in polynomial time by reduction to matching. In this paper we are now going to reveal the nice polyhedral counterpart of this composition procedure, i.e. how one can use the structure of this composition to describe the stable set polytope from the matching one and, more importantly, how one can use it to separate over the stable set polytope in polynomial time. We will then apply those general results back to where they originated from: stable set in claw-free graphs, to show that the stable set polytope can be reduced to understanding the polytope in very basic structures (for most of which it is already known). In particular for a general claw-free graph G, we show two integral extended formulation for $STAB(G)$ and a procedure to separate in polynomial time over $STAB(G)$; moreover, we provide a complete characterization of $STAB(G)$ when G is any claw-free graph with stability number at least 4 having neither homogeneous pairs nor 1-joins. We believe that the missing bricks towards the characterization of the stable set polytope of claw-free graphs are more technical than fundamentals; in particular, we have a characterization for most of the building bricks of the Chudnovsky-Seymour decomposition result and we are therefore very confident it is only a question of time before we solve the remaining case.

*Dipartimento di Ingegneria dell'Impresa, Università di Roma "Tor Vergata". Email:faenza@disp.uniroma2.it

†Dipartimento di Ingegneria dell'Impresa, Università di Roma "Tor Vergata". Email:oriolo@disp.uniroma2.it

‡Institut de Mathèmatiques de Bordeaux, Universite de Bordeaux 1. Email:gstauffer@math.u-bordeaux1.fr

0.1 Preliminaries

Chudnovsky and Seymour [7] introduced a composition operation in order to define a decomposition result for claw-free graphs. This composition procedure is general and applies to non-claw-free graphs as well. We borrow a couple of definitions from their work (even if our definitions are slightly different, and indeed closer to those in [8]). A *strip* (G, a, b) is a graph (not necessarily connected) with two designated simplicial vertices a and b that are non-adjacent to each other (a vertex is *simplicial* if its neighborhood is a clique).

Given two vertex-disjoint strips (G_1, a_1, b_1) and (G_2, a_2, b_2), we define the *gluing* of those two strips as the union of $G_1 \setminus \{a_1, b_1\}$ and $G_2 \setminus \{a_2, b_2\}$ together with all edges between $N_{G_1}(a_1)$ and $N_{G_2}(a_2)$ and all edges between $N_{G_1}(b_1)$ and $N_{G_2}(b_2)$. Note also that gluing (G_1, a_1, b_1) and (G_2, b_2, a_2) would not result in the same graph but still we denote the gluing of (G_1, a_1, b_1) and (G_2, a_2, b_2) by $G_1 + G_2$ because a_1, b_1, a_2, b_2 will always be clear from the context. The gluing can be generalized to more strips by introducing a composition operation.

Definition 1. *Let G^0 be a disjoint union of cliques with $2k$ vertices and $(G_1, a_1, b_1), \ldots, (G_k, a_k, b_k)$ k vertex disjoint strips. Let ϕ be a one-to-one mapping from $\{a_1, ..., a_k, b_1, ..., b_k\}$ to $V(G^0)$. For all $i = 1, ..., k$, define G_i as the gluing of (G_i, a_i, b_i) with $(G^{i-1}, \phi(a_i), \phi(b_i))$. The graph G^k is the* composition of the strips (G_i, a_i, b_i), $i = 1, ..., k$ w.r.t. (G^0, ϕ).

In the paper we always denote by G^0 the disjoint union of cliques with $2k$ vertices involved in some composition of k strips. Also, for $i = 1, \ldots, k$, we let $A_i = N_{G_i}(a_i)$, $B_i = N_{G_i}(b_i)$. We often abuse notation and we talk about a strip G_i as the graph $G_i[V(G_i) \setminus \{a_i, b_i\}]$ and about (G_i, a_i, b_i) as the whole graph G_i i.e. including vertices a_i and b_i. We now introduce two crucial strips.

The *trivial strip* (g, a, b) is the strip with vertex set $V(g) := \{a,\, w^1,\, w,\, w^2,\, b\}$ and edge set $E(g) := \{aw^1,\, aw,\, bw^2,\, bw,\, w^1w_2,\, w^1w,\, w^2w\}$. The *generalized trivial strip* (g^*, a, b) has vertex set $V = \{a,\, u,\, w,\, w^1,\, w^2,\, v,\, b\}$ and edge set $E = \{au,\, uw,\, uw^1,\, ww^1,\, ww^2,\, w^1w^2,\, vw,\, vw^2,\, bv\}$.

Throughout the paper, we often replace a strip (G_i, a_i, b_i), involved in some composition, with the (generalized) trivial strip. In these cases, we however keep the designated simplicial vertices of (G_i, a_i, b_i) and let therefore the trivial strip be (g_i, a_i, b_i) (with vertices a_i, w_i^1, $w_i, \ldots$). Of course, in these cases, we also keep the same mapping ϕ. Finally, we let $[k]$ denote the set $\{1, 2, \ldots, k\}$.

1 Warm up: a first extended formulation

Let G be the composition of the strips (G_i, a_i, b_i), $i \in [k]$, w.r.t. a pair (G^0, ϕ). As it is shown in [21], a MWSS in G can be efficiently computed, provided that we are able to solve the MWSS problem on each strip. We sketch the argument in the following, the reader should refer to [21] for more details.

Suppose that we are given a weight function $w : V(G) \mapsto \mathbb{Q}$. The main observation is that, in order to compute a MWSS of G, for each strip we are interested in four *crucial* stable sets of $G_i \setminus \{a_i, b_i\}$: a MWSS $S_i^{\bar{A}\bar{B}}$ that picks no vertex from both A_i and B_i; a MWSS $S_i^{\bar{A}}$ that picks no vertex from A_i; a MWSS $S_i^{\bar{B}}$ that picks no vertex from B_i; a MWSS $S_i^{\emptyset}$ that has no restrictions.

One may therefore replace each strip (G_i, a_i, b_i) with the trivial strip (g_i, a_i, b_i), and give each non-simplicial vertex of the strip a suitable weight. For sake of clarity, in this section, we refer to the non simplicial vertices of g_i respectively as $e_i^{\emptyset} \equiv w, e_i^{\bar{A}} \equiv w^1, e_i^{\bar{B}} \equiv w^2$. The weights are: $w(e_i^{\emptyset}) = w(S_i^{\emptyset}) - w(S_i^{\bar{A}\bar{B}})$, $w(e_i^{\bar{A}}) = w(S_i^{\bar{A}}) - w(S_i^{\bar{A}\bar{B}})$ and $w(e_i^{\bar{B}}) = w(S_i^{\bar{B}}) - w(S_i^{\bar{A}\bar{B}})$.

Let H be the composition of the trivial strips (g_i, a_i, b_i), $i \in [k]$, w.r.t. (G^0, ϕ). It can be shown that H is a line graph, and that every MWSS of H corresponds to a MWSS of G. We have therefore:

Theorem 2. *[21] Let G be the composition of the strips (G_i, a_i, b_i), $i = 1, ..., k$ w.r.t. a graph G^0. Suppose that the* MWSS *problem can be solved in time $O(p_i(n))$ for the different strips $i = 1, ..., k$ (n being the number of vertices of G). Then the* MWSS *problem on G can be solved in time $O(\sum_{i=1,...,k} p_i(n) + match(n))$, where $match(n)$ is the time required to solve a matching problem on a graph with n vertices. If $p_i(n)$ is polynomial for each i, then the* MWSS *problem can be solved in polynomial time.*

As we show in the following, Theorem 2 has a "polyhedral" counterpart; namely, we are able to derive an extended linear description $EF_1(G)$ of $STAB(G)$, provided that we have a (possibly extended) linear description of $STAB(G_i)$, for each strip (G_i, a_i, b_i). Moreover, if the separation problem over each $STAB(G_i)$ can be solved in polynomial time, the separation problem over $EF_1(G)$ can also be solved in polynomial time.

We start with a few notations. For each strip (G_i, a_i, b_i), $i \in [k]$, let $n_i + 2$ be the number of vertices in the strip; the number of vertices of G is therefore $n = \sum_{i=1..k} n_i$. Also the number of vertices of H is $3k$.

We let $y : V(H) \mapsto \mathbb{R}_+$ be a vector with one component for each vertex in H. We extend it with a vector $y(e^{\bar{A}\bar{B}}) : [k] \mapsto \mathbb{R}_+$ with one component for each strip. Recall that, by construction, $V(H)$ can be partitioned into k classes, each corresponding to some strip. Each class, in its turn, consists of the vertices $e_i^{\emptyset}, e_i^{\bar{A}}, e_i^{\bar{B}}$. Also since H is a line graph, a linear description $Ay \leq b$ of its stable set polytope is thus available ([10], see also [25]). Note that the latter constraints do not involve vector $y(e^{\bar{A}\bar{B}})$.

We now move to vertices of G. Let $z : V(G) \mapsto \mathbb{R}_+$ be a vector with one component for each vertex in G. Observe that also the vertices of G can be partitioned into k classes, each corresponding to some strip. We associate to each vertex v of G, that is coming from the strip G_i, four more variables (copies), namely: $x_i^{\emptyset}(v), x_i^{\bar{A}}(v), x_i^{\bar{B}}(v), x_i^{\bar{A}\bar{B}}(v)$. The rationale is the following: $x_i^{\bar{A}}(v)$ is a copy of $z(v)$ that is "active" if $y(e_i^{\bar{A}}) = 1$, i.e. if we are considering stable sets of G_i that take no vertex in A_i etc.

Now assume that we are given a linear description $D_i z_i \leq f_i$ of the stable set polytope P_i of the strip $G_i \setminus \{a_i, b_i\}$ (note that by now we are assuming that this description is given in the original space $\mathbb{R}^{n_i}$, we shall generalize this later). Then a linear description of the convex hull of the stable sets of $G_i \setminus \{a_i, b_i\}$ that take no vertex from A_i is the following: $\{x_i^{\bar{A}} \in \mathbb{R}^{n_i} : D_i x_i^{\bar{A}} \leq f_i; x_i^{\bar{A}}(A_i) = 0\}$.

With the same argument, we can characterize the convex hull of the stable sets of the graph $G_i \setminus \{a_i, b_i\}$, that takes no vertex from B_i etc. With a slight abuse of notation, for each $X = \{\emptyset, \bar{A}, \bar{B}, \bar{A}\bar{B}\}$, we refer to the corresponding system of inequalities as $D_i^X x_i^X \leq f_i^X$. Consider the following polytope:

$$
\begin{aligned}
EF_1(G) = \{(z, x, y) \in \mathbb{R}_+^{n+4n+4k} :& \\
& Ay \leq b && (1)\\
& z(v) = x_i^{\emptyset}(v) + x_i^{\bar{A}}(v) + x_i^{\bar{B}}(v) + x_i^{\bar{A}\bar{B}}(v) \quad && \forall v \in V \text{ and } i : v \in V(G_i) \quad (2)\\
& y(e_i^{\emptyset}) + y(e_i^{\bar{A}}) + y(e_i^{\bar{B}}) + y(e_i^{\bar{A}\bar{B}}) = 1 && \text{for all } i \in [k] \quad (3)\\
& D_i^X x_i^X \leq y(e_i^X) \cdot f_i^X && \forall i \in [k] \text{ and } X \in \{\emptyset, \bar{A}, \bar{B}, \bar{A}\bar{B}\}\} \quad (4)
\end{aligned}
$$

As we show in the following, $EF_1(G)$ is an extended linear description of STAB(G). First, we recall the following well-known result (see e.g. [25]).

Theorem 3. *A polyhedron Q is integral if and only if $max_{x \in Q} cx$ is an integer for each integral c such that the maximum is attained.*

Theorem 4. *$EF_1(G)$ is an extended integral formulation of STAB(G).*

It is easy to check that $(\bar{z}, \bar{x}, \bar{y})$ is a integral point of $EF_1(G)$ only if $\bar{z}$ is the characteristic vector of a stable set of G, and conversely the characteristic vector $\bar{z}$ of each stable set of G can be extended to an integral point $(\bar{z}, \bar{x}, \bar{y}) \in EF_1(G)$ (i.e. $EF_1(G)$ is a formulation of STAB(G)). Thus in order to prove Theorem 4 we are left to show the integrality of the polytope. Note that the polytope in the $y-$variables defined by constraints (1) and (3) is non-empty and integral, since it is a matching polytope plus some equations that univocally define the vector $y(e^{\bar{A}\bar{B}})$. Moreover, all its variables are non-negative, since for each $i \in [k]$, vertices $e_i^{\emptyset}$, $e_i^{\bar{A}}$, $e_i^{\bar{B}}$ form a clique in H. Thus, if the polytope in the $(x, y)-$space defined by (1), (3), and (4) is integral, then $EF_1(G)$ is also integral, since variables z are defined via constraints (2) as the sum of $x-$variables. The integrality of $EF_1(G)$ is then an immediate consequence of the next lemma.

Lemma 5. *Let $n, q, p \in \mathbb{N}$, and $P = \{y \in \mathbb{R}^n_+ : Ay \leq b\}$ be an integer non-empty polyhedron for some $A \in \mathbb{Z}^{p\times n}$, $b \in Z^p$. Moreover, for $i \in [q]$, let $n_i, p_i \in \mathbb{N}$, and $P^i = \{x^i \in \mathbb{R}^{n_i} : A^i x^i \leq b^i\}$ be an integer non-empty polyhedron for some $A^i \in \mathbb{Z}^{p_i\times n_i}$, $b^i \in Z^{p_i}$. Last, let $\phi : [q] \to [n]$. Then the polyhedron $Q = \{(x, y) \in \mathbb{R}^{N+n} : y \in P \text{ and } A^i x^i \leq y_{\phi(i)} b^i \text{ for each } i \in [q]\}$ is integral, where we set $N = \sum_{i=1}^q n_i$ and $x = \{x^1, \ldots, x^q\}^T$.*

Proof. For each $i \in [q]$ and $\alpha \geq 0$, let $P^i_\alpha := \{x^i \in \mathbb{R}^{n_i} : A^i x^i \leq \alpha b^i\}$. We start with two claims whose simple proofs we skip.

Claim 6. *Let $B \in Z^{n\times q}$ and $c, d \in \mathbb{Z}^q$ for some $n, q \in \mathbb{N}$. For $\alpha \in \mathbb{Q}_+$, let R_α be the polyhedron $\{x \in \mathbb{R}^n : Bx \leq \alpha d\}$ and suppose that $\tilde{x}$ is an optimal solution to $\max_{x\in R_1} cx$. Then for each $\alpha \in \mathbb{Q}_+$, R_α is non-empty and $\alpha\tilde{x}$ is an optimal solution to $\max_{x\in R_\alpha} cx$.*

Claim 7. *The projection of Q over the $y-$space coincides with P.*

Let $(u, w) \in \mathbb{R}^N \times \mathbb{R}^n$ be an integral cost function such that $max_{(x,y)\in Q} ux + wy$ is attained. Then, for each $i \in [q]$, $\max u^i x^i$ is obtained in some vertex $\bar{x}^i$ that we can assume integral by hypothesis. Then:

$$\max_{(x,y)\in Q} ux + wy = \max_{y\in P} (wy + \max_{x:x^i\in P^i_{y_{\phi(i)}} \text{ for } i\in[q]} \sum_{i=1}^q u^i x^i) = \max_{y\in P} (wy + \sum_{i=1}^q \max_{x^i\in P^i_{y_{\phi(i)}}} u^i x^i) =$$

$$= \max_{y\in P} (wy + \sum_{i=1}^q u^i(y_{\phi(i)}\bar{x}^i)) = \max_{y\in P} (\sum_{j=1}^n y_j \cdot (w_j + \sum_{i:\phi(i)=j} u^i \bar{x}^i)) = \max_{y\in P} (\bar{w}y)$$

the first equality holds by Claim 7, the second by the fact that the polyhedra P^i live in different spaces, and the third by Claim 6. Also, for each $j \in [q]$, $\bar{w}_j = w_j + \sum_{i:\phi(i)=j} u^i x^i$ is a sum of integers and thus an integer itself. Given the integrality of P and Theorem 3, the statement holds. □

The next lemma, whose simple proof we skip, addresses the separation problem over Q.

Lemma 8. *Let P, P^i for $i = 1, \ldots, k$ be as in the hypothesis of Lemma 5. If the separation problem over P and each of the P^i can be solved in polynomial time, then the separation problem over Q can be solved in polynomial time.*

Corollary 9. *Let G be the composition of strips (G_i, a_i, b_i) for $i = 1, \ldots k$. If for each i the separation problem over P_i can be solved in polynomial time, then the separation problem over $EF_1(G)$ can be solved in polynomial time.*

Remark. *It is straightforward to extend Theorem 4 and Lemma 9 to the case where for some strip (G_i, a_i, b_i) we have a description of the stable set polytope P_i of graph $G_i \setminus \{a_i, b_i\}$ in an extended space rather than in the original one. We defer details to the journal version of this paper.*

We conclude with a couple of comments. Lemma 5 can be interpreted as a "polyhedral combination" of polyhedra and, in this sense, generalizes Balas' union of polyhedra [1, 2]. One could indeed repeat the argument used in this section in order to derive an extended formulation for other polytopes which admit a decomposable structure similar to $STAB(G)$ (a similar approach has been proposed by Kaibel and Loos [17]). Although being simple, the extended formulation $EF_1(G)$ "splits" the variable corresponding to each node of G into four new ones; it is not clear therefore how to derive from this extended characterization a complete description or a separation routine in the original space. In the next section, we show a formulation requiring a better knowledge of each strip but giving more insight on the polytope in the original space.

2 Another extended formulation

For the sake of simplicity, in this section we assume that $\phi(a_i) \cup \phi(b_i)$ is a stable set of G_0, that $A_i, B_i \neq \emptyset$, and that the strip is connected. In all these cases, we have a 1-join in the graph G and we can use Chvátal clique separator theorem [9] to show that those strips can be treated independently. We are interested in understanding when a point $x \in \mathbb{R}^{|V|}$ lies in $STAB(G)$. By definition, $x \in STAB(G)$ if and only if it is a convex combination of stable sets of G. In particular $x_{|G_i}$ should be in $STAB(G_i)$ for all i, thus from now on we assume this is true.

Definition 10. *Let $G = (V, E)$ be a graph and $A, B \subseteq V$ two cliques of G. We denote by $STAB(G, t, A, B)$:=$\{x \in STAB(G) : \exists x_1, ..., x_S$ stable sets of G and $\lambda_1, ..., \lambda_S \geq 0$, such that $x = \sum_s \lambda_s x^s$, $\sum_s \lambda_s = 1$ and $\displaystyle\sum_{s:x^s(A)=1\&x^s(B)=1} \lambda_s = t\}$.*

For each strip (G_i, a_i, b_i) and point $x_{|G_i}$, let $\underline{t_i}$ (resp. $\overline{t_i}$) be the minimum (resp. maximum) $t \in [0, 1]$ such that $x_{|G_i} \in STAB(G_i, t, A_i, B_i)$. Observe that, by convexity, for all $t_i \in [\underline{t_i}, \overline{t_i}]$, $x_{|G_i} \in STAB(G_i, t_i, A_i, B_i)$. [NB: $\underline{t_i}$ and $\overline{t_i}$ are function of $x_{|G_i}$ and the corresponding $x_{|G_i}$ will always be clear from the context]. We then consider the gluing $G_i + g_i$ of (G_i, a_i, b_i) with the trivial strip (g_i, a_i, b_i). For any $0 \leq t_i \leq 1$, we extend the point x on $V(g_i)$ as follows: $x(w_i^1) = x(B_i) - t_i$, $x(w_i) = 1 + t_i - x(A_i) - x(B_i)$, $x(w_i^2) = x(A_i) - t_i$ (the choice of t_i will be clear from the context).

Lemma 11. *A point $x_{|G_i}$ lies in $STAB(G_i, t_i, A_i, B_i)$ if and only if $x_{|G_i+g_i}$ lies in $STAB(G_i + g_i)$.*

Proof. Necessity: Let x^s, $s = 1, ..., S$ be the stable sets of G_i and $\lambda_1, ..., \lambda_S \geq 0$, such that $\sum_s \lambda_s = 1$, $x_{|G_i} = \sum_s \lambda_s x^s$ and $\sum_{s:x^s(A_i)=1\&x^s(B_i)=1} \lambda_s = t_i$. We deduce that $\sum_{s:x^s(A_i)=1\&x^s(B_i)=0} \lambda_s = x(A_i) - t_i$, $\sum_{s:x^s(B_i)=1\&x^s(A_i)=0} \lambda_s = x(B_i) - t_i$ and $\sum_{s:x^s(A_i)=0\&x^s(B_i)=0} \lambda_s = 1 + t_i - x(A_i) - x(B_i)$. Now we simply extend each stable set x^s on g_i picking w_i if $x^s(A_i) = 0$ and $x^s(B_i) = 0$, picking w_i^1 if $x^s(A_i) = 0$ and $x^s(B_i) = 1$, picking w_i^2 if $x^s(A_i) = 1$ and $x^s(B_i) = 0$ and picking none if $x^s(A_i) = 1$ and $x^s(B_i) = 1$. Then $x_{|G_i+g_i} = \sum_s \lambda_s x^s_{|G_i+g_i}$, proving that $x_{|G_i+g_i} \in STAB(G_i + g_i)$.

Sufficiency: If $x_{|G_i+g_i} \in STAB(G_i + g_i)$, then let x^s, $s = 1, ..., S$ be the stable sets of $G_i + g_i$ and $\lambda_1, ..., \lambda_S \geq 0$, such that $\sum_s \lambda_s = 1$ and $x_{|G_i+g_i} = \sum_s \lambda_s x^s$. We use the restriction of x^s to G_i to prove that $x_{|G_i}$ lies in $STAB(G_i, t_i, A_i, B_i)$. Hence we need to prove that $\delta = \sum_{s:x^s(A_i)=1\&x^s(B_i)=1} \lambda_s$ is equal to t_i. Since the stable sets x^s intersecting A_i or B_i cannot pick w, there are at most $1-(x(A_i)-t_i)-(x(B_i)-t_i)-t_i = 1+\delta-x(A_i)-x(B_i)$ picking w but since exactly $1+t_i-x(A_i)-x(B_i)$ are picking w, it follows that $\delta \geq t_i$. Vice-versa, the stable sets picking w, w^1 or w^2 cannot intersect both A_i and B_i, thus $\delta \leq 1 - (x(A_i) - t_i) - (x(B_i) - t_i) - (1 + t_i - x(A_i) - x(B_i)) = t_i$. □

It follows that $\underline{t_i}$ and $\overline{t_i}$ are the minimum and maximum values of t_i for which $x_{|G_i+g_i}$ lies in $STAB(G_i+g_i)$. We now replace each strip (G_i, a_i, b_i) with the generalized trivial strip (g_i^*, a_i, b_i) and let H be the composition of the strips (g_i^*, a_i, b_i) with respect to G^0, ϕ. Let us denote $f: [0,1]^{2k} \mapsto \mathbb{R}^{|V(H)|}$ the function mapping $y = f(\underline{t_i}, \overline{t_i}, i = 1, .., k)$ as follows: for each i, $y(u_i) := x(A_i)$, $y(v_i) := x(B_i)$, $y(w_i) := 1 + \underline{t_i} - x(A_i) - x(B_i)$, $y(w_i^1) := x(B_i) - \frac{\overline{t_i}+\underline{t_i}}{2}$ and $y(w_i^2) := x(A_i) - \frac{\overline{t_i}+\underline{t_i}}{2}$. The next definition is crucial with respect to $STAB(G)$.

Definition 12. *We say that $x_{|G_i}$'s are compatible if and only if $y \in STAB(H)$.*

Lemma 13. *A point $x \in \mathbb{R}^{|V(G)|}$ lies in $STAB(G)$ if and only if $x_{|G_i} \in STAB(G_i)$ for all i and $x_{|G_i}$'s are compatible with respect to the composition.*

Proof. We assume that $x_{|G_i} \in STAB(G_i)$ for all i, otherwise $x \notin STAB(G)$. Thus intervals $[\underline{t_i}, \overline{t_i}]$ are non empty. We iterate on the number of strips (in any order). We consider strip (G_i, a_i, b_i) for some i. Note that G can be expressed as the gluing of (G_i, a_i, b_i) with some strip $(\bar{G}_i, \bar{u}_i, \bar{v}_i)$. We extend x on $\bar{u}_i, \bar{v}_i$ by setting $x_{\bar{u}_i} := x(A_i)$ and $x_{\bar{v}_i} := x(B_i)$.

Claim. *x can be expressed as a convex combination of stable sets of G if and only if there exists t_i such that $x_{|G_i} \in STAB(G_i, t_i, A_i, B_i)$ and $x_{|(\bar{G}_i, \bar{u}_i, \bar{v}_i)} \in STAB((\bar{G}_i, \bar{u}_i, \bar{v}_i), t_i, \bar{u}_i, \bar{v}_i)$.*

Necessity. Let x^s, $s = 1, ..., S$ be the stable sets of G and $\lambda_1, ..., \lambda_S \geq 0$, such that $\sum_s \lambda_s = 1$, $x = \sum_s \lambda_s x^s$. Let $t_i := \sum_{s:x^s(A_i)=1 \& x^s(B_i)=1} \lambda_s$. Then $x_{|G_i} = \sum_s \lambda_s x^s_{|G_i}$ and thus $x_{|G_i} \in STAB(G_i, t_i, A_i, B_i)$. Extend x^s on $\bar{u}_i$ and $\bar{v}_i$ by setting $x_s(\bar{u}_i) := x_s(A_i)$ and $x_s(\bar{v}_i) := x_s(B_i)$, then $x_{|(\bar{G}_i, \bar{u}_i, \bar{v}_i)} = \sum_s \lambda_s x^s_{|(\bar{G}_i, \bar{u}_i, \bar{v}_i)}$ and thus $x_{|(\bar{G}_i, \bar{u}_i, \bar{v}_i)} \in STAB((\bar{G}_i, \bar{u}_i, \bar{v}_i), t_i, \bar{u}_i, \bar{v}_i)$.

Sufficiency. Let x^s, $s = 1, ..., S_1$ be the stable sets of G_i and $\lambda_1, ..., \lambda_{S_1} \geq 0$, such that $\sum_s \lambda_s = 1$, $x_{|G_i} = \sum_s \lambda_s x^s$. Let y^s, $s = 1, ..., S_2$ be the stable sets of $(\bar{G}_i, \bar{u}_i, \bar{v}_i)$ and $\mu_1, ..., \mu_{S_2} \geq 0$, such that $\sum_s \mu_s = 1$, $x_{(|\bar{G}_i, \bar{u}_i, \bar{v}_i)} = \sum_s \mu_s y^s$. Consider indices $X \subseteq \{1, ..., S_1\}$ of the stable sets $\{x^s, s = 1, .., S_1\}$ that satisfy $x^s(A_i) = 0$ and $x^s(B_i) = 1$ and indices $Y \subseteq \{1, ..., S_2\}$ of the stable sets in $\{y^s, s = 1, .., S_2\}$ that satisfy $y^s(\bar{u}_i) = 0$ and $y^s(\bar{v}_i) = 1$. For each $s \in X, t \in Y$, $x^s \cup y^t \backslash \{\bar{v}_i\}$ are stable sets in G. We are looking for multipliers $\gamma_{s,t} \geq 0$ for each of those stables sets such that $\sum_t \gamma_{s,t} = \mu_t$ for all $t \in Y$ and $\sum_s \gamma_{s,t} = \lambda_s$ for all $s \in X$. This can be modeled as a flow problem whose feasibility is guaranteed by $\sum_t \mu_t = x(B_i) - t_i = x(\bar{v}_i) - t_i = \sum_s \lambda_s$. Similarly we recombine stable sets with $x^s(A_i) = y^s(\bar{u}_i) = 1$ and $x^s(B_i) = y^s(\bar{v}_i) = 1$; stable sets such that $x^s(A_i) = y^s(\bar{u}_i) = 0$ and $x^s(B_i) = y^s(\bar{v}_i) = 1$; stable sets such that $x^s(A_i) = x^s(B_i) = y^s(\bar{u}_i) = y^s(\bar{v}_i) = 0$. It is easy to check that the corresponding convex combination generate x. □

We let $\bar{G}_i + g_i^*$ be the gluing of $(\bar{G}_i, \bar{u}_i, \bar{v}_i)$ with (g_i^*, a_i, b_i). Note that this graph is isomorphic to $(\bar{G}_i, \bar{u}_i, \bar{v}_i) + g_i$, where we identified vertices $\bar{u}_i, \bar{v}_i$ of the latter with vertices u_i, v_i of the former. Let us extend x on g_i^* by setting $x(u_i) := x(A_i)$, $x(v_i) := x(B_i)$, $x(w_i) := 1 + t_i - x(A_i) - x(B_i)$, $x(w_i^1) := x(B_i) - t_i$ and $x(w_i^2) := x(A_i) - t_i$. From Lemma 11 and from what argued above, we know that $x \in STAB(G)$ if and only if $\exists t_i \in [\underline{t_i}, \overline{t_i}]$ such that $x_{|\bar{G}_i + g_i^*} \in STAB(\bar{G}_i + g_i^*)$. Let us now consider the point $(x_{|\bar{G}_i}, y_{|g_i^*})$.

Claim. *$\exists t_i \in [\underline{t_i}, \overline{t_i}]$ such that $x_{|\bar{G}_i + g_i^*} \in STAB(\bar{G}_i + g_i^*)$ if $(x_{|\bar{G}_i}, y_{g_i^*}) \in STAB(\bar{G}_i + g_i^*)$.*

Necessity. If $x_{|\bar{G}_i + g_i^*} \in STAB(\bar{G}_i + g_i^*)$, then redistributing $t_i - \underline{t_i}$ of the stable sets picking w_i into $\frac{t_i - \underline{t_i}}{2}$ picking w_i^1 and $\frac{t_i - \underline{t_i}}{2}$ picking w_i^2 and then removing w_i^1 from $\overline{t_i} - t_i$ stable set picking w^1 and w^2 from $\overline{t_i} - t_i$ stable set picking w^2, we get the feasibility of $(x_{|\bar{G}_i}, y_{g_i^*})$.

Sufficiency. Goes along the same line as sufficiency for Lemma 11 □

We iterate this argument starting from G and substituting one strip at a time with (g_i^*, a_i, b_i). Let H_i the graph obtained at the $i-$th step: by the previous arguments, $x \in STAB(G)$ if and only if $z \in STAB(H_i)$, where $z = x(v)$ for all $v \in G_{i+1}, G_{i+2}, \ldots, G_k$, and $z = y_{g_j^*}(v)$ for all $v \in g_1^*, \ldots, g_i^\star$. At the end of the procedure we obtain the required statement. □

Note that the compatibility of $x_{|G_i}$'s only relies on the intervals $[\underline{t_i}, \overline{t_i}]$ (which are only function of G_i and $x_|G_i$) and on the composition. Thus, if we substitute G_i with any other strip G'_i and if we consider the graph G' obtained from the composition with respect to (G_0, ϕ) and a point $x' \in \mathbb{R}^{|V(G')|}$ such that $[\underline{t'_i}, \bar{t'_i}]$ are the set of feasible t'_i's for $x'_{|G'_i}$ with $[\underline{t'_i}, \bar{t'_i}] = [\underline{t_i}, \overline{t_i}]$, then x' is a convex combination of stable sets of G' if and only if x is a convex combination of stable sets of G. Therefore, H somehow encapsulates the "structure" of the composition.

We now understand that all we need to assert the compatibility of different $x_{|G_i}$ are the intervals $[\underline{t_i}, \overline{t_i}]$ and the feasibility of y for $STAB(H)$. Thus if we can compute efficiently those intervals, we can also assert easily if a point x lies in $STAB(G)$ or not.

Suppose that we have a linear description of $STAB(G_i + g_i)$ in some extended space, i.e. $STAB(G_i + g_i) = \{z \in R^{|V(G_i+g_i)|} : \exists \zeta \in R^m \text{ such that } C^i z + D^i \zeta \leq f_i\}$ (possibly $m = 0$ and not necessarily compact). Substituting z with $x_{|G_i+g_i}$, we have that $x_{|G_i+g_i} \in STAB(G_i + g_i)$ if and only if $\exists \zeta : C^i_{|G_i} x_{|G_i} + C^i_{|g_i} x_{|g_i} + D^i \zeta \leq f_i$. Observe that since $x_{|g_i}$ is an affine function of t_i and $x_{|A_i}$, $x_{|B_i}$, we have a linear system of inequalities in $x_{|G_i}$, t_i and ζ and thus the bounds on t_i can be expressed as affine functions of x_{G_i} and ζ i.e we can rewrite the system in the form: $\{(x_{|G_i}, \zeta, t_i) : \underline{\nu}_i^j x_{|G_i} + \underline{\mu}_i^j \zeta + \underline{\rho}_i^j \leq t_i \text{ for } j = 1, ..., \underline{J_i};\ t_i \leq \bar{\nu}_i^j x_{|G_i} + \bar{\mu}_i^j \zeta + \bar{\rho}_i^j \text{ for } j = 1, ..., \bar{J}_i\ ;\ M_i x_{|G_i} + N_i \zeta \leq \beta_i\}$. Note that $\{\underline{\nu}_i^j x_{|G_i} + \underline{\mu}_i^j \zeta + \underline{\rho}_i^j \leq \bar{\nu}_i^k x_{|G_i} + \bar{\mu}_i^k \zeta + \bar{\rho}_i^k \text{ for } j = 1, ..., \underline{J_i},\ k = 1, ..., \bar{J}_i\ ;\ M_i x_{|G_i} + N_i \zeta \leq \beta_i\}$ is the projection of the t_i variable (using Fourier-Motzkin procedure) and thus it defines a extended representation of $STAB(G_i)$ ($x_{|G_i}$ is in $STAB(G_i)$ if and only if there exists t_i such that $x_{|G_i+g_i} \in STAB(G_i + g_i)$). The following result is thus a straightforward corollary of Lemma 13.

Corollary 14. *A point x lies in $STAB(G)$ iff there exist $0 \leq t_i^- \leq t_i^+ \leq 1$ and $\zeta \in R^m$ such that*

- *(i) (a) $\underline{\nu}_i^j x_{|G_i} + \underline{\mu}_i^j \zeta + \underline{\rho}_i^j \leq t_i^-$ for all i, for all $j = 1, ..., \underline{J_i}$*
- *(i) (b) $t_i^+ \leq \bar{\nu}_i^j x_{|G_i} + \bar{\mu}_i^j \zeta + \bar{\rho}_i^j$ for all i, for all $j = 1, ..., \bar{J}_i$*
- *(i) (c) $t_i^- \leq t_i^+$ for all i*
- *(ii) $x_{|G_i} \in STAB(G_i)$ for all i*
- *(iii) $y = f(t_i^-, t_i^+, i = 1, .., k) \in STAB(H)$*

Observe now that H is a line graph and thus a complete description of $STAB(H)$ is available. Also $f(t_i^-, t_i^+, i = 1, .., k)$ is an affine function of $t_i^-, t_i^+, i = 1, .., k$, thus the system above is a linear extended formulation of $STAB(G)$ (we can plug in $M_i x_{|G_i} + N_i \zeta \leq \beta_i$ instead of $x \in STAB(G_i)$).

Corollary 15. *The system defined in Corollary 14 is a linear extended formulation for $STAB(G)$.*

Remark. *A similar notion of compatibility was used by Chudnovsky and Seymour (see [27]). They define $[\underline{t_i}, \overline{t_i}]$ for the gluing of 2 strips and then use two different gadgets to assert the compatibility (paths of length 2 and 3 with suitable weights). The compatibility of the composition of many strips is then derived by induction. While this was enough for their purposes, this inductive argument does not scale polynomially to provide a polynomial time separation procedure as we are going to describe now. We want to stress however on the fact that our result was inspired from theirs.*

3 Separation and projection

Based on the extended formulation provided in Corollary 15, we are now going to define a simple polynomial time separation procedure for $STAB(G)$ that involves only linear programming and a separation procedure for the matching polytope. We will assume for the separation procedure that, for each i, the linear extended description of $STAB(G_i + g_i)$ we used in the previous section is a compact one (i.e. it has a polynomial number of variables and constraints).

Remark. *We can relax this to a separation procedure for $STAB(G_i + g_i)$ but we defer this result to the journal version of this paper.*

The separation procedure

Given a point $x^* \in R^{|V(G)|}$, we first check feasibility of $x^*_{|G_i} \in STAB(G_i)$ for all $i = 1, ..., k$ through the compact systems. If for some i, $x^*_{|G_i} \notin STAB(G_i)$, we generate a separating hyperplane through the dual of the compact system representing $STAB(G_i + g_i)$ (see for instance [3]). Else we compute for each $i = 1, ..., k$ the values $\underline{t_i}, \overline{t_i}$ corresponding to x^*, that we denote by $\underline{t_i}^*, \overline{t_i}^*$. Those values can be computed through the minimization or maximization of t_i over the compact representation of $STAB(G_i + g_i)$ and thus we get $\underline{t_i}^*$ and $\overline{t_i}^*$ via linear programming. We also use linear programming duality to get certificates of optimality for $\underline{t_i}^*$ and $\overline{t_i}^*$ i.e. multipliers $\underline{\mu_i}^*$ and $\bar{\mu_i}^*$. Using $\underline{\mu_i}^*$ and $\bar{\mu_i}^*$ we can generate two valid inequalities for $STAB(G_i + g_i)$: the first one of the form $c_1 x_{|G_i} + \delta_1 \leq t_i$ tight at $(x^*_{|G_i}, \underline{t_i}^*)$ and a second one of the form $t_i \leq c_2 x_{|G_i} + \delta_2$ tight at $(x^*_{|G_i}, \overline{t_i}^*)$.

Now, by Lemma 13, we know that we only have to check if $y = f(\underline{t_i}^*, \overline{t_i}^*, i = 1, .., k) \in STAB(H) = MATCH(R_H)$. We check feasibility of y via a separation procedure for matching, e.g. [22]. If y is feasible then $x \in STAB(G)$ and if not, we use the violated inequality from matching to generate a violated inequality in the original space by substituting $\underline{t_i}^*$ and $\overline{t_i}^*$ with $c_1 x_{|G_i} + \delta_1$ and $c_2 x_{|G_i} + \delta_2$ respectively.

The inequality we generate through this procedure is clearly violated by x^*. In order to prove that our separation procedure is well defined, we need to prove that the corresponding inequality is valid for $STAB(G)$. The rest of this section is dedicated to this, and to prove that all the inequalities we need for $STAB(G)$ are obtained by such a substitution. This will prove the following:

Lemma 16. *We can separate over $STAB(G)$ in polynomial time using only linear programming and an algorithm for separating over the matching polytope if we have a compact representation of $STAB(G_i + g_i)$ for each i.*

In order to prove the result, we need to understand how we can generate the facets of $STAB(G)$. We will study this by projecting the extended formulation from Corollary 14 using Fourier-Motzkin procedure. Since the purpose now is not tractability but feasibility, we will assume for this purpose that we have a representation of $STAB(G_i + g_i)$ in the original space i.e. $\zeta = 0$ in Corollary 14. We have to understand the form of the different inequalities involving t_i^- and t_i^+ to understand how they will be composed in the Fourier-Motzkin procedure.

Let us first study the inequality coming from the matching polytope $STAB(H)$. We focus on the facet that are neither non negativity inequalities nor clique inequalities because it is easy to check that the cliques of g_i^* and the non negativity inequalities in g_i^* are also contained in the clique and non negativity constraints associated with $G_i + g_i$ or correspond to the inequality $t_i^- \leq t_i^+$.

Thus they are already in the system (i) and we treat them without loss of generality as such (i.e. not as facets of the matching polytope of H).

To show that our separation procedure is well defined, it is enough to prove that the facets of the matching polytope of H at $y = f(t_i^-, t_i^+, i = 1,..,k)$ involving the variables t_i^- or t_i^+ are of the form $\sum_i l_i^- t_i^- + l_i^+ t_i^+ \leq r$ with $l_i^- > 0$ and $l_i^+ = 0$ or $l_i^- = 0$ and $l_i^+ < 0$. Indeed in this case, we can remove the variables t_i^- and t_i^+ one at a time using Fourier-Motzkin elimination procedure. Indeed observe that in that case, the inequalities generated by the Fourier-Motzkin procedure while removing t_i^- come from combining inequalities [in corollary 14] from (i) (a) and from (iii) of the form $t_i^- \leq ...$ or from (i) (a) and (i)(c) $t_i^- \leq t_i^+$. Observe that the combination with the later one will not bring much since this will introduce new inequalities $... \leq t_i^+$ that can only be removed later on using the upper bound on t_i^+ (but then as already observed earlier, those inequalities are valid for $STAB(G_i)$). Thus, while removing t_i^- we are only interested in combining inequalities from (i) of the form $... \leq t_i^-$ with inequalities from (iii) of the form $t_i^- \leq ....$ This precisely correspond to substituting t_i^- the lhs of (i) into (iii). We can argue similarly for t_i^+. [NB: The constraints $0 \leq t_i^-$ and $t_i^+ \leq 1$ are already in (i) and are treated as such].

The following lemma proves that apart from the clique of g_i^* and G_0, all the facets of the $STAB(H)$ are of the desired form. We postpone its proof to the appendix.

Lemma 17. *Let $\sum_v a_v y_v \leq b_v$ be a facet of $STAB(H)$ that is neither a non negativity inequality nor a clique inequality of H and let $i \in [k]$ be such that not all $a_{u_i}, a_{v_i}, a_{w_i}, a_{w_i^1} a_{w_i^2}$ are equal to 0. Then $a_{u_i} = a_{v_i} = a_{w_i} = \lambda > 0$ and either $a_{w_i^1} = a_{w_i^2} = 0$ or $a_{w_i^1} = a_{w_i^2} = \lambda$.*

The Fourier-Motzkin procedure gives us already a way to characterize all the valid inequalities we need for $STAB(G)$ i.e. we need to substitute the value of t_i^- by $c_1 x_{|G_i} + \delta_1$ and t_i^+ by $c_2 x_{|G_i} + \delta_2$ into the facet inequalities of the matching polytope of H, for all $c_1 x_{|G_i} + \delta_1 \leq t_i$ and $t_i \leq c_2 x_{|G_i} + \delta_2$ facets of $STAB(G_i + g_i)$. But we will now analyze precisely the kind of inequalities we get through this procedure to derive a sharper characterization of $STAB(G)$. From now on, we assume that we have a description of all facets of $STAB(G_i + g_i)$ for all i.

Let us focus on a facets of $STAB(H)$ that are neither non-negativity nor clique inequalities. It follows from matching theory that these facets correspond to odd set inequalities, therefore they are rank. The corresponding (rank) inequality reads from Lemma 17:

$$(1) \quad \sum_{i \in \mathcal{I}} (y_{u_i} + y_{v_i} + y_{w_i} + \lambda_i (y_{w_i^1} + y_{w_i^2})) \leq r$$

with $\mathcal{I} \subseteq \{1, ..., k\}$ and $\lambda_i \in \{0, 1\}$. Let us now consider a facet of the graph $G_i + g_i$ that is neither a non-negativity nor a clique inequality. The proof of Lemma 17 can be trivially extended to $STAB(G_i + g_i)$ to show that all such facets will have the same coefficient γ_i on all vertices of g_i or only w_i will be in the support of the facet. It follows that the facet reads:

$$(2) \quad \beta_i x_{|G_i} + \gamma_i (x_{w_i} + \mu_i (x_{w_i^1} + x_{w_i^2})) \leq \rho_i$$

with $\beta_i \in \mathbb{R}^{|V(G_i)|}, \gamma_i \geq 0$ and $\mu_i \in \{0, 1\}$. Observe that in the extended formulation, if $\lambda_i = 0$, $y_{u_i} + y_{v_i} + y_{w_i} + \lambda_i (y_{w_i^1} + y_{w_i^2})$ reads $1 + t_i^-$ and if $\lambda_i = 1$, $y_{u_i} + y_{v_i} + y_{w_i} + \lambda_i (y_{w_i^1} + y_{w_i^2})$ reads $1 - t_i^+ + x(A_i) + x(B_i)$. Similarly if $\mu_i = 1$, $(x_{w_i} + \mu_i (x_{w_i^1} + x_{w_i^2}))$ reads $1 - t_i^-$ and if $\mu_i = 0$, $(x_{w_i} + \mu_i (x_{w_i^1} + x_{w_i^2}))$ reads $1 + t_i^+ - x(A_i) - x(B_i)$. Therefore in the Fourier-Motzkin procedure, when removing t_i^- and t_i^+, we will only combine inequalities for which $\lambda_i + \mu_i = 1$. Thus, from (1), we will generate all inequalities of the form

$$(3)\quad \sum_{i\in\mathcal{I}} \frac{\beta_i}{\gamma_i} x_{|G_i} \le r + \sum_{i\in\mathcal{I}} (\frac{\rho_i}{\gamma_i} - 2)$$

where for all $i \in \mathcal{I}$, an inequality (2) such that $\lambda_i + \mu_i = 1$ is substitute.

Let $\mathcal{F} = \{K_1, K_2, ..., K_{2n+1}\}$ an odd set of cliques of G. Let $T \subseteq V$ be the set of vertices which are covered by at least two cliques of $\mathcal{F}$. The inequality $\sum_{v\in T} x(v) \le n$ is a valid inequality for $STAB(G)$ and an inequality of this type is called an Edmonds' inequality. As we shall prove in the next Lemma, the procedure we have defined gives a complete characterization of $STAB(G)$ from combining the facets of all $STAB(G_i + g_i)$'s and Edmonds' inequalities for a collection of auxiliary graphs. Even though the procedure described above to combine those facets is derived from Fourier-Motzkin elimination, the way we have stated it only relies on the "pure" form of the inequalities (1) for $STAB(H)$, for some auxiliary graph H, and (2) for $STAB(G_i + g_i)$'s. It is thus worth recalling it in this term: from any facet of type (1) of $STAB(H)$ and any facets of type (2) of each $STAB(G_i + g_i)$) such that $\lambda_i + \mu_i = 1$, we can combine a valid inequality (3) and doing it for all suitable auxiliary graphs yield the complete characterization.

Lemma 18. *$STAB(G)$ can be described by non negativity and clique inequalities, inequalities describing $STAB(G_i)$, for all i, and inequalities obtained as follows: 1) Consider any partition of $[k]$ into $\mathcal{I}_1$, $\mathcal{I}_2$ and define the graph $\mathcal{H}(\mathcal{I}_1, \mathcal{I}_2)$ that is the composition of $(G_i[A_i \cup B_i], a_i, b_i)$ for all $i \in \mathcal{I}_1$ and (g_i^*, a_i, b_i) for all $i \in \mathcal{I}_2$. $\mathcal{H}(\mathcal{I}_1, \mathcal{I}_2)$ admits only Edmonds inequalities as facets; 2) Consider any facet of $STAB(\mathcal{H}(\mathcal{I}_1, \mathcal{I}_2))$ of the form $\sum_{i\in\mathcal{I}_1} \gamma_i x_{|A_i\cup B_i} + \sum_{i\in\mathcal{I}_2}(y_{u_i} + y_{v_i} + y_{w_i} + \lambda_i(y_{w_i^1} + y_{w_i^2})) \le r$ and apply the procedure explained above to the strips in $\mathcal{I}_1$.*

Proof. Sketch. $\mathcal{H}(\mathcal{I}_1, \mathcal{I}_2)$ admits only Edmonds inequalities as facets because $(G_i[\{a_i, b_i\} \cup A_i \cup B_i], a_i, b_i)$ is a "special" fuzzy linear interval strip and one can show that the inequalities are obtained by lifting matching inequalities to Edmonds' inequalities. We skip the technical details. We have proved that using Fourier-Motzkin with non negativity and non clique facets of $STAB(G_i + g_i)$, we get all inequalities of the form (2). We need to show now what we would get from substituting the clique inequalities and non negativity constraint in $STAB(G_i + g_i)$. Those inequalities correspond to : $t_i^- \ge 0$, $t_i^+ \le x(A_i), X(B_i)$, $t_i^- \ge X(A_i) + X(B_i) - 1$. The inequality we would get are valid inequalities involving only $x_{|A_i \cup B_i}$ for strip i. And thus they would be generated by the procedure when $i \in \mathcal{I}_1$ for some $\mathcal{I}_1$. $\square$

Corollary 19. *The characterization of $STAB(G)$ reduces to characterizing $STAB(G_i + g_i)$ for each i. In particular, if $STAB(G_i + g_i)$ is rank-perfect, then $STAB(G)$ his rank-perfect.*

The last result shows an alternative proof that the facets of $STAB(G)$, when G is the composition of fuzzy linear interval strips, are either non negativity, or clique, or Edmonds' inequalities.

4 Application to SSP of claw-free graphs

We now apply our previous (general) results to where they originated from: stable sets in claw-free graphs. In the following, we assume the reader to be familiar with the topic, and therefore some arguments are just sketched. We start with a decomposition theorem by Chudnovsky and Seymour.

Theorem 20. *[7] For every claw-free graph G with $\alpha(G) \ge 4$, if G does not admit a 1-join and there is no homogeneous pair of cliques in G, then either G is a circular interval graph, or G is a composition of linear interval strips, XX-strips, and antihat strips.*

We will also use another decomposition theorem, less detailed, but algorithmic.

Theorem 21. *[16] Every claw-free graph G with $\alpha(G) \geq 4$, is either a distance claw-free graph, or the composition of distance simplicial strips and 5-wheel strips. Moreover, in time $O(n^4)$ we can recognize whether G belongs to the former or the latter class, and, in case, find the strips in the decomposition (as well as the graph G^0).*

XX-strips and antihat strips are defined later. A *5-wheel* strip is a strip (G_i, a_i, b_i) with an induced 5-wheel and stability number at most three. A *distance simplicial* graph [16] is a connected graph such that there exists v with $\alpha(N_j(v)) = 1$ for each j. A *distance claw-free* [24] graph is a connected graph such that, for every v and every j, $\alpha(N_j(v)) \leq 2$.

Extended formulations. The technique developed in Sections 1 and 2 can be applied to obtain two extended formulations for the stable set polytope of claw-free graphs. We suppose in the following that $\alpha(G) \geq 4$, otherwise $STAB(G)$ can be expressed as the projection of the convex hull of the $O(n^3)$ stable sets of G. If the graph is distance claw-free, then we can use a compact extended formulation due to Pulleyblank and Shepherd [24]. It is easy to see that, if (G, a, b) is a distance simplicial strip, then both G and $G + g$ are distance claw-free graphs. In order to apply Theorem 4, we need a complete description of the stable set polytope of each strip (possibly in some extended space). $STAB(G_i)$ admits a compact extended formulation both if (G_i, a_i, b_i) is distance simplicial or if (G_i, a_i, b_i) is a $5-$wheel strip (since it has stability number at most 3). In order to apply Corollary 15, we conversely need for each strip a complete description of the stable set of $G_i + g$ (again possibly extended). If G_i is distance simplicial, then again we can use the compact extended formulations for the stable set of a distance claw-free graph; if it is a $5-$wheel strip, we express $STAB(G)$ as the convex hull of $O(n^4)$ stable sets. Thus in both case we can obtain an extended formulation for the stable set polytope of claw-free graphs, where the only non-compact system is the one describing the matching polyhedron. Thus, not surprisingly, a compact formulation for the matching polytope would imply a compact one for the stable set polytope of claw-free graphs.

Polynomial time separation Let $G(V, E)$ be claw-free and fix $x \in \mathbb{R}^{|V|}$. We assume that $\alpha(G) \geq 4$. Otherwise we again express $STAB(G)$ as the convex hull of $O(n^3)$ stable sets and generate a separating hyperplane, if any, through the dual of the resulting compact system. From Theorem 21 we check in polynomial time whether G is a distance claw-free graph, or the composition of distance claw-free strips and 5-wheel strips. In the former case, we use the compact extended formulation of $STAB(G)$ mentioned above. In the latter case, we rely on Lemma 16. We therefore just need an algorithm for separating over the matching polytope [22] and for each strip (G_i, a_i, b_i) a compact representation of $STAB(G_i + g_i)$, when G_i is either a distance simplicial graph, or a graph with an induced 5-wheel and $\alpha(G_i) \leq 3$. Both cases have already been considered above.

Polyhedral characterization when $\alpha(G) \geq 4$, no homogenous pairs and no 1-join. Let $G(V, E)$ be claw-free with $\alpha(G) \geq 4$, no homogenous pairs and no 1-join. From Theorem 20, G is either a circular interval graph, or the a composition of linear interval strips, XX-strips, and antihat strips. In the former case, a linear description of $STAB(G)$ is given in [12]. In the latter case, we rely on Lemma 18. We therefore just need a linear representation of $STAB(G + g)$, when G is either a linear interval strip, or a XX-strip, or an antihat strip. The composition of a linear interval strip with the trivial strip g is a circular interval graph, and has been considered above.

Let (G, a, b) be the XX strip. Since a XX strip has a bounded number of vertices (up to 13), we derived a computer based description of $STAB(G + g)$. Details, as well as the definition of the XX strip are in the Appendix. Let (G, a, b) be an antihat strip. In this case, $\alpha(G + g) \leq 3$. Thus, we use the well known result [6] asserting that each facet inducing inequality of $STAB(G)$, G claw-free, can be described by a system of roots (affinely independent stable sets tight for the inequality) $S_1, \ldots, S_n$ such that $|S_i| \in \{p, p + 1\}$ for some p. Since a complete description of $STAB(G)$, when $\alpha(G) \leq 2$, has been given by Cook (see [18]) and Shepherd [26] and this can be trivially extended to describe the facets described by a system of roots of size 1 and 2 in claw-free graphs (see the arguments in the alternative proof by [18]), the problem of describing $STAB(G+g)$ reduces to characterizing the facets described by a system of roots are of size either two or three. Detail on the analysis of those facets, as well as the definition of the antihat strip, are in the Appendix.

We close with a couple of remarks. We recently became aware that a linear description of $STAB(G)$, for G claw-free with $\alpha(G) \geq 4$, no homogenous pairs and no 1-join, has also been obtained by Galluccio, Gentile and Ventura [14].

Chudnovsky and Seymour recently proved another much more detailed decomposition theorem for a *general* claw-free graph G [8]. Building upon this theorem and Lemma 18, a characterization of $STAB(G)$ would follow from a linear representation of $STAB(G_i+g_i)$, for each strip G_i involved in the decomposition and a description of all other building bricks (such as fuzzy circular interval graphs). So far we have been able to derive this characterization for all strips in the decomposition that have $\alpha \geq 4$. Recalling again that all facets described by a system of roots of size 1 and 2 can be derived from Cook (see [18]), thus the problem of providing a complete description for $STAB(G)$ for *any* claw-free graph reduces to that of describing the facet inducing inequalities that can described by a system of roots of size either two or three. In general, this can be quite difficult (e.g. it can have arbitrary many coefficient, see [23]). However for the class of claw-free graphs with $\alpha \geq 4$, the strips with $\alpha \leq 3$ seems to be friendlier from the polyhedral point of view and we are ruling out the different cases one by one. Our main remaining issue concerns a strip that resembles – but, unfortunately, is not exactly! – an antihat strip. However we strongly believe that ruling out each remaining cases is more a question of finding the right technical statements to prove that the inequalities are in a class we already know than discovering new fundamentals properties.

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5 Appendix

5.1 Proof of Lemma 17

Proof. for ease of notation, we denote $u = u_i, v = v_i, \ldots$. Let K_1 be the clique $u \cup (n(u) \setminus V(g_i^*))$, K_2 the clique $\{u, w, w^1\}$, K_3 the clique $\{w, w^1, w^2\}$, K_4 the clique $\{v, w, w^2\}$ and K_5 the clique $v \cup (n(v) \setminus V(g_i^*))$. We will repeatedly use the following fact: since the facet is not a clique, for each clique K, there must exist a root of the facet missing K.

Assume first that $a_{w^1}, a_{w^2} \neq 0$. K_2 is missed implies that $a_{w^1} \leq a_{w^2}$ (the root missing K_2 has to pick w^2 otherwise we can add w^1 to the root an violate the facet but then we can swap w^1 and w^2) and similarly $a_w \leq a_{w^2}$. Vice versa, K_4 is missed implies that $a_{w^1} \geq a_{w^2}$. Therefore both values are equal to say $\lambda > 0$. w is in a root of the facets (otherwise the inequality is $x_w \geq 0$). Thus $a_w \geq a_{w^1}, a_{w^2} = \lambda$ (again by swapping w with w^1 or w^2). We know at the current stage that $a_w = a_{w^1} = a_{w^2} = \lambda$. By similar arguments, K_3 is missed implies that $a_u \geq a_{w^1} = \lambda$ and $a_v \geq a_{w^2} = \lambda$ (the roots that miss K_3 pick u and v otherwise we can add w^1 or w^2). Now finally, the root missing K_1 must pick w or w^1 otherwise we add u to the root and we violate the inequality (we already now that $a_u \geq \lambda > 0$). Thus $a_u \leq \lambda$ because otherwise, swapping the vertex in $\{w, w^1\}$ with u would yield a stable set violating the inequality. Similarly, $a_v \leq \lambda$. And the result follows.

Let us assume now that wlog $a_{w^1} = 0$. Then because there is a root missing K_4, $a_{w^2} = 0$ too. Moreover the facet is also a facet of the support graph i.e. $G[V(G) \setminus \{w^1, w^2\}]$. We are now going to argue on the support graph. Suppose $a_u > 0$. There must exists a root missing K_1 but then it has to take w (otherwise we add u). Thus $a_w \geq a_u$. Similarly $a_w \geq a_v$. There is a root missing edge $\{u, w\}$, otherwise it is a clique inequality : this root picks v otherwise we can add w and violate the inequality. But this implies that $a_w \leq a_v$ (by swapping v, w). Similarly $a_w = a_u$ and the results follows. We are left to check the case where $a_u = 0$. But then since there must exist a root missing edge (w, v), $a_w = 0$ (otherwise you either add w to the root if it does not take u or just swap u and v if it does). K_5 is missed by some root, thus either w is in this root and then swapping w and v implies that $a_v = 0$ or w is not in but then we can add v and thus again $a_v = 0$: both cases yield contradiction with $a_v, a_u, a_w, a_{w^1}, a_{w^2}$ not all 0. □

5.2 XX-strip

Consider the graph $G(V, E)$ with $V = \{v_1, \ldots, v_{13}\}$ and the following adjacencies: $v_1 - \cdots - v_6$ is a hole in G of length 6. v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5; v_9 is adjacent to v_6, v_1 v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to v_3, v_4, v_6, v_1, v_9, v_{10}; v_{12} is adjacent to v_2, v_3, v_5, v_6, v_9, v_{10}; and v_{13} is adjacent to v_1, v_2, v_4, v_5, v_7, v_8. Let $X \subseteq \{v_{11}, v_{12}, v_{13}\}$; then the strip $(G \setminus X; v_7, v_8)$ is an *XX-strip*. We derived a computer based description of $STAB(G + g)$ using Komei Fukuda's CCD. Inequalities are listed below (variables corresponding to vertices w_i, w_1^i, w_2^i of the trivial strip are denoted respectively by x_{14}, x_{15}, and x_{16}).

$$
\begin{array}{rcll}
x_1 + x_4 + x_{11} + x_{13} + x_{14} + x_{15} + x_{16} & \leq & 2 & (1) \\
x_2 + x_3 + x_4 + x_{13} + x_{14} + x_{15} + x_{16} & \leq & 2 & (2) \\
x_2 + x_5 + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} & \leq & 2 & (3) \\
x_1 + x_2 + x_4 + x_5 + x_9 + x_{10} + x_{13} + x_{14} & \leq & 2 & (4) \\
x_1 + x_5 + x_6 + x_{13} + x_{14} + x_{15} + x_{16} & \leq & 2 & (5) \\
x_1 + x_4 + x_5 + x_6 + x_9 + x_{10} + x_{11} + x_{13} + x_{14} & \leq & 2 & (6) \\
x_1 + x_2 + x_5 + x_6 + x_9 + x_{10} + x_{12} + x_{13} + x_{14} & \leq & 2 & (7) \\
x_2 + x_3 + x_4 + x_5 + x_9 + x_{10} + x_{12} + x_{13} + x_{14} & \leq & 2 & (8) \\
x_1 + x_2 + x_3 + x_4 + x_9 + x_{10} + x_{11} + x_{13} + x_{14} & \leq & 2 & (9) \\
x_1 + x_2 + x_3 + x_6 + 2x_9 + x_{10} + x_{11} + x_{12} & \leq & 2 & (10) \\
x_3 + x_4 + x_5 + x_6 + x_9 + 2x_{10} + x_{11} + x_{12} & \leq & 2 & (11) \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + 2x_9 + 2x_{10} + x_{11} + x_{12} + x_{13} + x_{14} & \leq & 3 & (12) \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} & \leq & 3 & (13)
\end{array}
$$

Note that $(1, 2, 3, 4, 5)$ are Edmonds's inequality with support a circular interval graph, $(6, 7, 8, 9)$ are Edmonds's inequality with support a fuzzy circular interval graph, $(10, 11)$ are lifted 5-wheel inequalities, (12) is the gear inequality (see [13]), where (13) has a full support.

5.3 Antihat-strip

Let $n \geq 0$. Let $A = \{0, a1, \ldots, a_n\}$; $B = \{b_0, b_1, \ldots b_n\}$ and $C = \{c_1, \ldots, c_n\}$ be three cliques, pairwise disjoint. Let J be the graph with vertex set $A \cup B \cup C$ and with adjacency as follows: for $0 \leq i \leq j \leq n$, let a_i, b_j be adjacent if and only if $i = j > 0$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. Let $X \subset (A \cup B \cup C)$ with $a_0, b_0 \notin X$; then the strip $(G \setminus X; a_0, b_0)$ is called an *antihat strip.*

We say that a graph $G(V,E)$ is a $3-$*clique* if $A, B, C, \{a,b\}$ is a partition of V, where $a,b \notin A \cup B \cup C$, A, B, C are cliques, $N(a) = A$, and $N(b) = B$, and $\alpha(G \setminus \{a,b\}) \leq 2$. One can easily check that if (G,a,b) is an antihat strip, then G is a three clique. In the following, we give a linear description of $STAB(G+g)$, when G is a 3-clique. We need a few preliminaries.

Definition 22. *A graph G is a* combination of odd antiholes *if G can be built up from odd antiholes by iterating sequential lifting and complete join operations, i.e.,*

- *G is a lifting of $H_1 \times H_2 \times \ldots \times H_q$;*
- *for each $j = 1,\ldots,q$, H_j is either an odd antihole or a combination of odd antiholes.*

As it was shown by Cook (see [18]) and Shepherd [26], one can characterize all facets of STAB(G) if $\alpha(G) = 2$:

Theorem 23. *The facets of the stable set polytope of a graph $G = (V,E)$ with $\alpha(G) = 2$ are trivial constraints and inequalities*

$$2\sum_{v\in P} x_v + \sum_{v\in Q} x_v \leq 2 \tag{5}$$

for each pair (P,Q) such that P is a clique, Q is a combination of odd antiholes, P is totally joined to Q, and P and Q are maximal with respect to these properties.

We will build upon on the following result of Galluccio and Sassano to characterize the facets of claw-free graphs described by a system of n roots of size 1 and 2.

Lemma 24. *[15] Let $G(V,E)$ be a graph with $\alpha(G) = 2$. If $x(Q) \leq 2$ is facet-defining for $STAB(G)$, then $Q = V$ and G is a combination of odd anti-hole.*

Corollary 25. *Let $G(V,E)$ be a claw-free graph, and $\lambda x \leq \lambda_0$ define a facet F of $STAB(G)$ that can be described by a system of roots of size 1 or 2, with at least a root of size 1 and a root of size 2. Then $\lambda x \leq \lambda_0$ is of the form (5).*

Proof. Let $\lambda x \leq \lambda_0$ be a facet-defining inequality of STAB(G) which satisfies the hypothesis of the statement. This implies that there exists at least a vertex w such that $\lambda_w = \lambda_0$ and a vertex w' such that $\lambda_{w'} < \lambda_0$, and note that w is complete to all other vertices of the support S of the inequality. Denote by P all vertices v with $\lambda_v = \lambda_w$ and by Q the other vertices in the support (note that Q is non-empty, since it contains w'). It follows that P is a clique, and it is complete to Q . Moreover, since G is claw-free, $\alpha(Q) \leq 2$. Then $\sum_{v\in Q} \lambda_v x_v \leq \lambda_0$ is facet-defining for STAB(G') for $G' = G[Q]$ with $\alpha(G') = 2$ and all of its root have size 2, i.e. it is a multiple of $\sum_{v\in Q} x_v \leq 2$. From Lemma 24, Q is a composition of odd antiholes. Thus the inequality $2\sum_{i\in P} x_i + \sum_{i\in Q} x_i \leq 2$ is valid for STAB(G) and tight for all roots, i.e. is a multiple of $\lambda x \leq \lambda_0$. □

Lemma 26. *Let G be a $3-$clique graph. Then $STAB(G+g)$ is completely defined by the following inequalities:*

(i) Trivial and clique inequalities.

(ii) $2\sum_{v\in P} x_v + \sum_{v\in Q} x_v \leq 2$ for each pair (P,Q) such that P is a clique, Q is a combination of odd antiholes, P is totally joined to Q, and P and Q are maximal with respect to these properties.

(iii) $\sum_{v\in \bar{A}\cup\bar{B}\cup\bar{C}\cup\{c,d,e\}} x_v \leq 2$ for all $\bar{A} \subseteq A$, $\bar{B} \subseteq B$ and $\bar{C} \subseteq C$ such that: $\bar{C}$ is complete to $\bar{A} \cup \bar{B}$; $\bar{A}, \bar{B}, \bar{C}$ are maximal with this property.

Proof. Let G be a $3-$clique graph, and recall $\alpha(G+g) \leq 3$ holds. Fix a facet-defining inequality $\lambda x \leq \lambda_0$ that is not a trivial inequality and consider the subgraph $\bar{G}$ of $G+g$ induced by the support S of this inequality. We already observed (cfr. Section 4) that $\lambda x \leq \lambda_0$ belongs to at least one of the sets $S_{\{1\}}, S_{\{1,2\}}, S_{\{2\}}, S_{\{2,3\}}, S_{\{3\}}$, where for each $K \subset [3]$, S_K is the set of facets described by a system of n roots whose size belong to K.

For $K = \{1\}$ (resp. $K = \{2\}$), n roots lie on the hyperplane $\sum_{v \in S} x_v = 1$ (resp $\sum_{v \in S} x_v = 2$). Thus, there is no stable set of size 3 in $\bar{G}$, since it would violate the inequality. Then by Theorem 23 all facet defining inequalities are of the kind (i) or (ii). To settle the case $K = \{3\}$, note that each stable set of size 3 in G takes a vertex from C. Thus, this facet correspond to the clique inequality $x(C) \leq 1$. We are left with the case the facet belong to $S_{\{1,2\}} \setminus (S_{\{1\}} \cup S_{\{2\}})$ or $S_{\{2,3\}} \setminus (S_{\{2\}} \cup S_{\{3\}})$. The facet of $S_{\{1,2\}} \setminus (S_{\{1\}} \cup S_{\{2\}})$ correspond to facets (ii) because of Corollary 25. Thus we are left with the case of n affinely independent root of size 2 and 3. We already argued (cfr. Section 3) that one of the following happens: either no vertex in $\{w, w^1, w^2\}$ belongs to the support, or only w belong to the support, or they all belong to the support and have the same coefficient. If one of the first two cases holds, then $\alpha(\bar{G}) \leq 2$, contradicting the existence of a root of size 3. Thus, let $\{w, w^1, w^2\}$ be in the support and have the same coefficient. Let C_1 be the subset of $C \cap S$ that is complete to $S \cap (A \cup B)$, and $C_2 = (C \cap S) \setminus C_1$. We shall prove that the facet-defining inequality is a positive multiple of $\sum_{v \in S \setminus C_2} x_v \leq 2$. Note indeed that this inequality is valid and tight for each stable set of size 3 in $G+g$ (since they all pick a vertex from C_2), it is valid for all stable sets of size 2, and tight for the subset of those that do not take a vertex in C_2. Thus we are left to show that stable sets $\{u, v\}$ with $u \in C_2$ are not roots of the facet. Note that either $v \in A$, or $v \in B$, or $v \in \{w, w^1, w^2\}$. If $v \in A$ (resp. B) I can add d (resp. w) to the stable set and increase λx; thus, this stable set cannot be a root. So we can assume $v \in \{w, w^1, w^2\}$, and since $\lambda_w = \lambda_{w^1} = \lambda_{w^2}$, if (u, v) is a root, so are (u, w^1), (u, w^2). Since each vertex of C_2 has a non-adjacent in $S \cap (A \cup B)$, we can assume w.l.o.g. that $(u, a) \notin E$ for some $a \in A \cap S$. Thus $\{u, a, w^2\}$ is a feasible stable set that violates the inequality, i.e. a contradiction. $\square$