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A Fermion Levy Theorem

L. ACCARDI

Centro Metematico V. Volterra, Dipartimento di Matematica, Università di Roma II, Italy

AND

J. QUAEGEBEUR

Department of Mathematics, Faculty of Sciences, Katholieke Universiteit Leuven, Belgium

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The notion of *quantum process with continuous trajectories* is defined in terms of mutual quadratic variations and it is proved that for classical stochastic processes, this notion of continuity of trajectories coincides with the usual one. Our main result is that any continuous trajectory difference martingale M which is a Grassmann measure with scalar non-atomic brackets is isomorphic to a Fermion white noise (mean zero Fermi–Gaussian family) whose covariance coincides with the brackets of M. This is a fermion version of the Levy representation theorem for classical Brownian motion. © 1992 Academic Press, Inc.

1. INTRODUCTION

According to M. Emery and P. A. Meyer [12], "One of the great theorems of probability theory is Levy's result that any martingale (X_t) such that $X_0 = 0$ and

$$d[X, X] = dt$$

has the same law as Brownian motion... ." In this note we prove a Fermion generalization of this Levy theorem. In fact our result generalizes Levy's theorem also in another direction, namely we do not require the index set of our process to be a subset of the real line, but we allow it to be an arbitrary measurable space. This gives in particular, a Levy theorem for fields—a result which seems to be new even in classical probability. The natural tool for this generalization is the quantum probabilistic extension

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of the theory of mutual quadratic variations (or square brackets) proposed in [8]. The possibility of a quantum Levy theorem was conjectured by one of us during the 1984 Heidelberg Conference on Quantum Probability and some basic ideas of the proof (in the Boson and cyclic-separating case) are described in [2]. In [9] a supersymmetric quantum Levy theorem is proved in the context of the quantum independent increment processes. In [3] a Boson–Fermion theorem of Levy type is proved. In the latter paper, in order to bypass the problem posed by the non-existence (at that time) of a stochastic calculus for general quantum semimartingales, the problem was studied in the context of the so-called "Levy fields" whose classical analogue are the exponential semi-martingales associated to the initially given Levy martingale via the solution of a stochastic differential equation. Currently such a general quantum stochastic integration theory has been developed and this has allowed us to prove the Levy theorem in its original formulation (i.e., directly in terms of a single quantum martingale) and also in the Boson case [6]. All the above mentioned results make essential use of the fourth moments condition introduced in the present paper. Here we prove that, in the classical case and in the conditions of the above mentioned papers, this four moment condition is in fact equivalent to the continuity of the trajectories. From a quantum probabilistic point of view, the most serious limitation of all these theorems (including the present one) is the assumption that the increments of the martingale commute with the past (or its equivalent in the multidimensional case). This assumption restricts the discussion to a narrow class of quantum noises. On the other hand the above mentioned quantum Levy theorems from one side and the recently proved quantum invariance principles [4, 5] from the other side, prove that, within the class of quantum noises which

(i) have increments commuting with the past

(ii) have continuous trajectories (cf. Section 5),

(iii) have brackets of constant scalar type (cf. Section 6 for this notion),

both in the Boson and the Fermion case, one has only four canonical forms. In the Boson case two are classical stochastic processes (the real and the complex Brownian motion) and the other two are quantum processes (the Fock Brownian motion and the one parameter family of the universal invariant Brownian motions).

In Section 2 we introduce our notations. Section 3 shows the connection between Grassman measures and Clifford algebras. Section 4 contains the proof of the Fermion Levy theorem. In Section 5 the relations between square brackets, oblique brackets, and fourth moments conditions are discussed. In Section 6 a natural action of the complex symplectic group of order two (and more generally of the current group on **R** associated to it) on the stochastic processes is introduced and the four canonical forms mentioned above are deduced as invariants of this action.

2. Notations

Throughout this paper $\mathscr{A}$ will denote a topological $*$-algebra (always with unit, by definition); $(T, \mathscr{T})$ a measurable space; and $\mathscr{T}_0 \subseteq \mathscr{T}$ a subfamily of $\mathscr{T}$, closed under finite unions, intersections, and relative complements. For an $\mathscr{A}$-valued measure M on $\mathscr{T}_0$, we write

$$M_1 = M; \qquad M_2 = M^*, \tag{2.1}$$

where M^* is the $\mathscr{A}$-valued measure M on $\mathscr{T}_0$ defined by

$$M^*(I) = M(I)^*, \qquad I \in \mathscr{T}_0. \tag{2.2}$$

Often we shall also write M_I instead of $M(I)$.

If M, N are two $\mathscr{A}$-valued measures on $\mathscr{T}_0$ their *mutual quadratic variation*—or *bracket*— is defined by

$$[[M, N]](I) = \lim_{(I_k) \in \mathscr{P}(I)} \sum_{I_k} M(I_k)\, N(I_k), \tag{2.3}$$

where $[[M, N]]$ is defined if the limit (2.3) exists in the topology of $\mathscr{A}$ for each $I \in \mathscr{T}_0$.

As in [8] we use the double square brackets notation in order to prevent confusion with commutators.

For each $I \in \mathscr{T}$, $\mathscr{P}(I)$ denotes the family of all the finite partitions (I_k) of I. Since $\mathscr{P}(I)$ is an increasing net for the partial order induced by the usual refinement relation among partitions, if X_π is a family of elements of $\mathscr{A}$ indexed by $\mathscr{P}(I)$, the expression

$$\lim_{\mathscr{P}(I)} X_\pi = X \tag{2.4}$$

has a natural meaning. With this definition of brackets it is clear that, if the brackets $[[M, N]]$ of two $\mathscr{A}$-valued measures M, N exist, they are necessarily an $\mathscr{A}$-valued measure on $\mathscr{T}_0$. As explained in [8], one can adopt a weaker notion of convergence in the definition of brackets. In that case, however, the fact that for two general measures M, N, their bracket is still a measure does not follow from the definition and should be assumed. The symbol $\{\cdot, \cdot\}$ will denote the anticommutator

$$\{a, b\} = ab + ba, \qquad a, b \in \mathscr{A} \tag{2.5}$$

and for an arbitrary operator $X \in \mathscr{A}$ we shall use the notations

$$X^{\varepsilon} = X \text{ if } \varepsilon = 1 \qquad X^{\varepsilon} = X^{*} \text{ if } \varepsilon = 2 \tag{2.6}$$

and

$$\operatorname{Re} X = \frac{1}{2}(X + X^{*}); \qquad \operatorname{Im} X = \frac{1}{2i}(X - X^{*}). \tag{2.7}$$

As in [9] we use the double square bracket notation [[ ]] in order to prevent confusion with commutators.

The multiples of the identity in $\mathscr{A}$ will be called *scalar operators.*

By a real pre-Hilbert space we mean a real vector space endowed with a, possibly complex valued and possibly degenerate, scalar product. Given a real pre-Hilbert space H, a *Clifford algebra* over H is a pair $\{\mathscr{C}\ C\}$ where $\mathscr{C}$ is a $*$-algebra and

$$C: f \in H \to C(f) = C_f \in \mathscr{C} \tag{2.8}$$

is a real linear map satisfying

$$C(f)^{*} = C(f), \qquad \forall f \in H \tag{2.9}$$

$$C(f) \cdot C(g) + C(g) \cdot C(f) =: \{C(f), C(g)\} = 2\langle f, g \rangle, \qquad f, g \in H. \tag{2.10}$$

Moreover the set $\{C(f): f \in H\}$ is a set of algebraic generators of $\mathscr{C}$.

Notice that (2.9) and (2.10) imply that a necessary condition for the existence of a Clifford algebra on a real pre-Hilbert space H is that the scalar product on H is real valued. This condition is also sufficient. More precisely, given such an H there exists a, unique up to isomorphism, Clifford algebra over H, denoted $\{\mathscr{C}(H)\ C\}$ such that if $\{\mathscr{C}'\ C'\}$ is any other Clifford algebra over H, then the map

$$C(f) \in \mathscr{C}(H) \mapsto C'(f) \in \mathscr{C}' \tag{2.11}$$

extends to an homomorphism of $\mathscr{C}(H)$ onto $\mathscr{C}'$. The pair $\{\mathscr{C}(H)\ C\}$ is called the Clifford algebra over H and it can be shown that on it there exists a unique C^*-norm. The completion of $\mathscr{C}(H)$ under this norm (still denoted $\mathscr{C}(H)$ when no confusion can arise) is called the Clifford C^*-algebra and if $\{\mathscr{C}'\ C'\}$ is a pair such that: (i) $\mathscr{C}'$ is a C^*-algebra; (ii) $C': H \to \mathscr{C}'$ is a map satisfying (2.8), (2.9) and such that the algebra spanned by the $C'(f)$ $(f \in H)$ is dense in $\mathscr{C}'$ then the map (2.11) extends to an homomorphism of C^*-algebras. These properties will be referred to as the universal property of the Clifford algebra (resp. C^*-algebra).

Now let H be a real pre-Hilbert space with a complex structure defined by an operator $i: H \to H$ satisfying $i^2 = -1$ and with a complex valued scalar product (in Section 3) we shall see that to every difference martingale with a scalar conditional variance one can canonically associate such a space).

By a *real structure* on H we mean a real pre-Hilbert subspace $H_0 \subseteq H$ such that:

(i) The restriction of the scalar product of H on H_0 is real valued.

(ii) $H_0 + iH_0 = H$.

If H is a complex Hilbert space and H_0 defines a real structure on H and if the scalar product on H is sesquilinear, then $H_0 \cap iH_0 = \{0\}$ because of (i) and, for the same reason, H_0 is orthogonal to the space iH_0 for the pre-scalar product on H given by the real part of the initial scalar product on H if and only if H is a complex Hilbert space.

If H is a pre-Hilbert space as above then on the real Hilbert space (with real valued scalar product) $\{H, \mathrm{Re}\langle \cdots \rangle_H\}$ the Clifford algebra $\{\mathscr{C}(H), C\}$ is well defined. Moreover, for each real structure H_0 on H we can define the operators

$$a(f_0) = C(f_0) + iC(if_0); \qquad a^+(f_0) = C(f_0) - iC(if_0) \tag{2.12a}$$

$$a(if_0) = a^+(f_0) = C(f_0) - iC(if_0) \tag{2.12b}$$

$(f_0 \in H_0)$ and extend the maps $a^\pm(\cdot)$ by real linearity. The resulting maps $a^\pm(\cdot)\colon H \to \mathscr{C}(H)$ satisfy the *Canonical Anticommutation Relations* (CAR)

$$\{a(f), a^+(g)\} = 2\langle f, g\rangle \qquad \forall f, g \in H \cong H_0 \oplus iH_0. \tag{2.13}$$

The operators $C(f)$ are called the *Segal fields* and are intrinsically defined; the operators $a^\pm(f)$ are called the *creation* and *annihilation* operators, respectively, and depend on the choice of a real structure H_0 on the Hilbert space H. Given such a structure the algebra $\mathscr{C}(H)$ can be identified, by the real linearity of C, with the complex polynomial algebra in the non-commuting variables $a^\pm(f_0)$ with $f_0 \in H_0$. This remark will be used in the proof of Theorem (4.4). For more informations on Clifford algebras and their role in quantum field theory, we refer to [13, 14].

Now we introduce the notion of Gaussian state and, even if our main result concerns exclusively the Fermion–Gaussian states, we give this definition in full generality.

DEFINITION (2.1). Let H be a set, $\mathscr{A}$ a $*$-algebra, and φ a state on $\mathscr{A}$. A self-adjoint family of operators $B = \{b(f)\colon f \in H\} \subseteq \mathscr{A}$, indexed by H, is called a mean zero *Gaussian family* with respect to φ if the following identities hold:

$$\varphi(b(f_1) \cdots b(f_n)) = 0 \tag{2.14}$$

if n is odd and

$$\varphi(b(f_1)\cdot\ \cdots\ \cdot b(f_{2n}))=\sum_{j_1,\ldots,j_{2n}}\varepsilon_n(j_1,\ldots,j_{2n})\cdot\varphi((b(f_{j_1})\cdot b(f_{j_2}))$$
$$\cdot\ \cdots\ \cdot\varphi(b(f_{j_{2n-1}})\cdot b(f_{j_{2n}})),\tag{2.15}$$

where the sum is taken over all the $(j_1, ..., j_{2n})$ from 1 to $2n$ such that $j_1 < j_2 \cdots j_{2n-1} < j_{2n}$ and $j_1 < j_3 < \cdots < j_{2n-1}$ and where, for each natural integer n, $(j_1, ..., j_{2n})$ is a permutation over n symbols and

$$\varepsilon_n(j_1,\ldots,j_{2n})=\begin{cases}+1 & \text{for each } n \text{ and each } (j_1,\ldots,j_{2n})\\ \operatorname{sgn}(j_1,\ldots,j_{2n}) & \text{for each } n \text{ and each } (j_1,\ldots,j_{2n}).\end{cases}$$

In the former case, we speak of a *Boson–Gaussian family*, in the latter case a *Fermion–Gaussian family.* In particular, if the polynomial algebra generated by the $b(f)$ $(f\in H)$ (or its closure in a topology for which φ is continuous) coincides with $\mathscr{A}$ then φ is called a *mean zero Gaussian state on* $\mathscr{A}$.

The real bilinear form on H

$$\varphi(b(f)\cdot b(g))=: q(f, g)$$

is called the *covariance* (or correlation function or two-point function) of the state φ. A Fermi–Gaussian φ state with covariance given by

$$q(f, g)=\varphi(b(f)\cdot b(g))=\operatorname{Re}\langle f, g\rangle+i\operatorname{Im}\langle f, Qg\rangle,\tag{2.16}$$

where Q is a bounded self-adjoint real linear operator on H satisfying

$$\|Q\|\leqslant 1\tag{2.17}$$

is called a regular Fermi–Gaussian state on H. Since in this paper we shall deal only with regular Fermi–Gaussian states we shall call them simply Gaussian.

A simple consequence of (2.14), (2.15) is that if H is a real vector space, then if $\{b(f): f\in H\}\subseteq\mathscr{A}$ is a Gaussian family with respect to a state φ on $\mathscr{A}$ and the map $f\mapsto b(f)$ is real linear, then:

(i) The restriction of φ on the polynomial algebra generated by the $b(f)$ $(f\in H)$ is completely determined by the correlation function $\varphi(b(f)\cdot b(g))$.

(ii) If $\{D(f): f\in H\}\subseteq\mathscr{A}$ is another family with the property that each $D(f)$ is a linear combination of the operators $b(g)$, then also

$\{D(f): f \in H\}$ is a mean zero Gaussian family with respect to the same state and with the same covariance (cf. the following definition).

In particular if $\{\mathscr{C}(H)\ C\}$ is the Clifford algebra over a real pre-Hilbert space H, a *gauge invariant* mean zero Gaussian state φ on $\mathscr{C}(H)$ with covariance Q is a state φ on $\mathscr{C}(H)$ such that the family $\{C(f): f \in H\}$ is a mean zero Fermi–Gaussian family with respect to φ with covariance Q. It is known that for every real linear self-adjoint Q on H satisfying (2.17) there exists a gauge invariant mean zero Gaussian state φ on $\mathscr{C}(H)$ with covariance Q.

The following theorem is not essential for what follows, but we include it since the fact that the Gaussian states are intrinsically related to the canonical commutation and anticommutation relations has some interest of its own. This relation has been deduced by Giri and von Waldenfels [13] and von Waldenfels [19] in connection with the quantum central limit theorems. The proof given here is direct and exploits only the Gaussianity of the state.

THEOREM (2.2). *In the notations of Definition* (2.1) *denote, for* $a, b \in \mathscr{A}$, *for any natural integer* n, *and* $j_1, ..., j_n \leqslant n$ *natural integers,*

$$[a, b]_\varepsilon = \begin{cases} ab - ba, & \text{if} \quad \varepsilon_n(j_1, ..., j_n) = 1 \\ ab + ba, & \text{if} \quad \varepsilon_n(j_1, ..., j_n) = \operatorname{sgn}(j_1, ..., j_n) \end{cases} \tag{2.18}$$

and let $\{\mathscr{H}, \pi, \Phi\}$ *be the GNS representation of the polynomial* $*$*-algebra* $\mathscr{P}(B)$ *generated by the set* B. *Then for any pair* $b_1, b_2 \in B$ *one has*

$$[\pi(b_1), \pi(b_2)]_\varepsilon = q(b_1, b_2) \mp q(b_2, b_1), \tag{2.19}$$

where on the right hand side of (2.19), *the sign* $-$ *holds in the Boson case, the sign* $+$ *in the Fermion case.*

Proof. Let q be the covariance of the Gaussian state. We have to show that for each $p \in \mathbf{N}$ and for each $b_1, ..., b_{2p}, b', b'' \in B$ (with some of the b_j's possibly equal among themselves), one has

$$\begin{aligned} &\varphi(b_1 \cdot \ \cdots \ \cdot b_j \cdot [b', b'']_\varepsilon \cdot b_{j+1} \cdot \ \cdots \ \cdot b_{2p}) \\ &\quad = \sigma(b', b'') \cdot \varphi(b_1 \cdot \ \cdots \ \cdot b_j \cdot b_{j+1} \cdot \ \cdots \ \cdot b_{2p}), \end{aligned} \tag{2.20}$$

where $\sigma(b', b'')$ is defined by

$$\varphi([b', b'']_\varepsilon) = q(b', b'') - q(b'', b') =: \sigma(b', b''). \tag{2.21}$$

To this goal, notice that the left hand side of the expression (2.20) is equal to

$$\begin{aligned}
&\varphi(b_1 \cdot \cdots \cdot b_j \cdot b_{j+1} \cdot \cdots \cdot b_{2p+2}) \\
&= \sum_{\substack{(S_1, ..., S_{p+1}) \in \mathscr{P}_{2p+2,2} \\ S_h \neq (j+1, j+2);\, h = 1, ..., p+1}} \varepsilon(S_1, ..., S_{p+1}) \cdot q(b_{S_1}) \cdot \cdots \cdot q(b_{S_{p+1}}) \\
&\quad + \sum_{(S_1, ..., S_p) \in \mathscr{P}_{2p,2}} \varepsilon(S_1, ..., b', b'', ..., S_p) \cdot q(b_{S_1}) \\
&\quad \cdot \cdots \cdot q(b_{S_j}) \cdot \cdots \cdot q(b', b'') \cdot q(b_{S_{j+1}}) \cdot \cdots \cdot q(b_{S_p}) \qquad (2.22)
\end{aligned}$$

while

$$\begin{aligned}
&\varphi(c_1 \cdot \cdots \cdot c_{j+1} \cdot c_j \cdot \cdots \cdot c_{2p+2}) \\
&= \sum_{\substack{(S_1, ..., S_{p+1}) \in \mathscr{P}_{2p+2,2} \\ S_h \neq (j+1, j+2);\, h = 1, ..., p+1}} \varepsilon(S_1, ..., S_{p+1}) \cdot q(c_{S_1}) \cdot \cdots \cdot q(c_{S_{p+1}}) \\
&\quad + \sum_{(S_1, ..., S_p) \in \mathscr{P}_{2p,2}} \varepsilon(S_1, ..., b'', b', ..., S_p) \\
&\quad \cdot q(b_{S_1}) \cdot \cdots \cdot q(b_{S_j}) \cdot \cdots \cdot q(b'', b') \cdot q(b_{S_{j+1}}) \cdot \cdots \cdot q(b_{S_p}). \qquad (2.23)
\end{aligned}$$

Now notice that the first terms in the right hand sides of (2.22) and (2.23) are equal because in both cases the pairs (h', j), $(h'', j+1)$ can appear in the sum if and only if both h' and h'' are less or equal than $j-1$, hence h' and h'' can be exchanged giving rise to another term which is still in the sum. In the Boson case this term will have the same sign as the previous one. In the Fermion case, since the permutation differs by the previous one only for one exchange, the two terms will appear with opposite signs. In particular, in the Fermion case,

$$\varepsilon(S_1, ..., S_j, b', b'', S_{j+1}, ..., S_{2p}) = -\varepsilon(S_1, ..., S_j, b'', b', ..., S_{j+1}, ..., S_{2p}). \qquad (2.24)$$

In both case we can suppose (up to the exchange of b'' with b') that for all permutations

$$(S_1, ..., S_j, S_{j+1}, ..., S_{2p})$$

one has

$$\varepsilon(S_1, ..., S_j, b', b'', S_{j+1}, ..., S_{2p}) = \varepsilon(S_1, ..., S_j, S_{j+1}, ..., S_{2p}).$$

This implies that

$$\varphi(b_1\cdot\ \cdots\ \cdot b_j\cdot[b',b'']_\varepsilon\cdot b_{j+1}\cdot\ \cdots\ \cdot b_{2p})$$
$$=\sum_{(S_1,\dots,S_p)\in\mathscr{P}_{2p,2}}\varepsilon(S_1,\dots,S_p)\cdot q(b_{S_1})\cdot\ \cdots\ \cdot q(b_{S_j})\cdot\ \cdots$$
$$\cdot[q(b',b'')\mp q(b'',b')]\cdot q(b_{S_{j+1}})\cdot\ \cdots\ \cdot q(b_{S_p}) \tag{2.25}$$

and this proves (2.20).

3. Grassmann Measures and Clifford Algebras

Heuristically a Grassmann measure on $\mathscr{T}_0$ is an operator valued measure on $\mathscr{T}_0$ with the property that operators associated to disjoint sets anti-commute. In this section we show that the algebra spanned by the range of a Grassmann measure with a scalar quadratic variation is canonically isomorphic to the Clifford algebra over a pre-Hilbert space uniquely determined by the brackets of the measure.

Definition (3.1). A measure $M\colon\mathscr{T}_0\to A$ is called a *Grassmann measure* if, in the notation (2.6)

$$\{M^{\varepsilon_1}(I_1),M^{\varepsilon_2}(I_2)\}=0;\qquad\forall\varepsilon_1,\varepsilon_2=1,2;\ \forall I_1,I_2\in\mathscr{T}_0;\ \text{such that } I_1\cap I_2=0. \tag{3.1}$$

Lemma (3.2). *Let M be a Grassman measure. Then the brackets $[[M,M]]$ and $[[M^*,M^*]]$ exist and for each $I\subseteq T$ one has*

$$M(I)^2=\tfrac{1}{2}\{M_I,M_I\}=[[M,M]](I) \tag{3.2}$$

$$M^*(I)^2=\tfrac{1}{2}\{M_I^*,M_I^*\}=[[M^*,M^*]](I). \tag{3.3}$$

Moreover, if either of the brackets

$$[[M,M^*]]\qquad\textit{or}\qquad[[M^*,M]] \tag{3.4}$$

exists, then the other one exists and for each $I\subseteq T$ one has

$$\{M^*(I),M(I)\}=[[M^*,M]](I)+[[M,M^*]](I). \tag{3.5}$$

Proof. Let $I\subseteq T$ and let $I_1,\dots,I_n$ be a partition of I by elements of $\mathscr{T}_0$. Then using (3.1) one finds

$$\{M^*(I),M(I)\}=\sum_j M^*(I_j)\,M(I_j)+\sum_j M(I_j)\,M^*(I_j).$$

Hence if either of the brackets (3.4) exists, the other one exists too and (3.5) holds. The identities (3.2), (3.3) follow by a similar argument since $X^2 = (1/2)\{X, X\}$.

COROLLARY (3.3). *If M is a Grassman measure on $\mathscr{T}_0$ such that one of the brackets* (3.4) *exists then for each $I, J \subseteq \mathscr{T}_0$*

$$\{M_i^*(I), M_j(J)\} = [[M_i^*, M_j]](I \cap J) + [[M_j^*, M_i]](I \cap J). \quad (3.6)$$

From now on we shall use the notation

$$[[M_i^*, M_j]] = \sigma_{ij}, \qquad i, j = 1, 2 \quad (3.7)$$

so that

$$[[M^*, M]] = \sigma_{11}; \qquad [[M, M^*]] = \sigma_{22};$$
$$[[M^*, M^*]] = \sigma_{12}; \qquad [[M, M]] = \sigma_{21}.$$

Thus, by definition $\sigma_{ij}: \mathscr{T}_0 \to \mathscr{A}$ is a finitely additive measure and with this notation,

$$\{M_i^*(I), M_j(J)\} = 2 \operatorname{Re} \sigma_{ij}(I \cap J). \quad (3.8)$$

The matrix (σ_{ij}), whose coefficients are $\mathscr{A}$-valued measures, will be denoted σ. For $J \in \mathscr{T}_0$ define the (field) measures

$$B(\chi_J) = \frac{1}{2}[M(J) + M^*(J)] \quad (3.9)$$

$$B(i\chi_J) = \frac{1}{2i}[M(J) - M^*(J)]. \quad (3.10)$$

A more precise notation for $B(\chi_J)$, $B(i\chi_J)$ would have been respectively $B(J)$, $B_i(J)$. Our choice is aimed at evidentiating the interpretation of these measures as field operators on the real pre-Hilbert space of complex valued step functions on T (cf. Corollary (3.5)).

For each step function $f: T \to \mathbf{C}$ of the form

$$f = \sum_{j \in F} f_j \chi_{I_j}; \qquad f_j \in \mathbf{C},\ I_j \in \mathscr{T}_0; \qquad j \in F\text{—a finite set} \quad (3.11)$$

we define the field operator $B(f)$ by real linearity, i.e.,

$$B(f) = \sum_{j \in F} (\operatorname{Re} f_j)\, B(\chi_{I_j}) + \sum_{j \in F} (\operatorname{Im} f_j)\, B(i\chi_{I_j}). \quad (3.12)$$

An elementary computation using (3.8) shows that, in the notations (3.7), for each $I, J \in \mathcal{T}_0$ one has

$$\{B(\chi_I), B(\chi_J)\} = \tfrac{1}{2} \sum_{j \in F} \sigma_{ij}(I \cap J) \tag{3.13}$$

$$\{B(\chi_I), B(i\chi_J)\} = \operatorname{Im} \sigma_{12}(I \cap J) \tag{3.14}$$

$$\{B(i\chi_I), B(i\chi_J)\} = \tfrac{1}{2}(\sigma_{11} + \sigma_{22})(I \cap J) - \operatorname{Re} \sigma_{12}(I \cap J). \tag{3.15}$$

Whence one deduces that if f, g are given by

$$f = \sum_{j \in F} f_j \chi_{I_j}; \qquad g = \sum_{j \in F} g_j \chi_{I_j} \tag{3.16}$$

then $B(f)$ and $B(g)$ satisfy the anti-commutation relations

$$\{B(f), B(g)\} = \sum_{k \in F} (\operatorname{Re} f_k, \operatorname{Im} f_k)$$

$$\cdot \begin{pmatrix} \tfrac{1}{2}(\sigma_{11} + \sigma_{22})_k + \operatorname{Re}(\sigma_{12})_k & \operatorname{Im}(\sigma_{12})_k \\ \operatorname{Im}(\sigma_{12})_k & \tfrac{1}{2}(\sigma_{11} + \sigma_{22})_k - \operatorname{Re}(\sigma_{12})_k \end{pmatrix} \cdot \begin{pmatrix} \operatorname{Re} g_k \\ \operatorname{Im} g_k \end{pmatrix}$$

$$= \frac{1}{2} \sum_{k \in F} (\bar{f}_k, f_k) \cdot \begin{pmatrix} \tfrac{1}{2}(\sigma_{11} + \sigma_{22})_k & (\sigma_{12})_k \\ (\sigma_{21})_k & \tfrac{1}{2}(\sigma_{11} + \sigma_{22})_k \end{pmatrix} \cdot \begin{pmatrix} g_k \\ \bar{g}_k \end{pmatrix}, \tag{3.17}$$

where we used the notation

$$(\sigma_{ij})_k = \sigma_{ij}(I_k). \tag{3.18}$$

Introducing for the right hand side of (3.17) the shorthand notation

$$2(f \mid g)_\sigma \tag{3.19}$$

we obtain in conclusion

$$\{B(f), B(g)\} = 2(f \mid g)_\sigma. \tag{3.20}$$

In particular (3.17) implies that for all f of the form (3.11), $(f|f)_\sigma$ is a positive operator.

Now suppose that M has scalar brackets, meaning by this that the brackets σ_{ij} defined by (3.7) are scalar measures (here and in the following we shall identify the elements of $\mathbf{C}$ with the scalar multiples of the identity in $\mathcal{A}$).

Then the vector space of all the complex valued functions on of the form (3.11), endowed with the (possibly degenerate) scalar product defined by (3.17), is a real pre-Hilbert space which we shall denote $H_0(\sigma)$. We shall also denote $H_0(\sigma)$ as the algebraic quotient of $H_0(\sigma)$ by the subspace of zero norm elements and $\{\mathscr{C}_0(\sigma), C\}$ the Clifford algebra over $H_0(\sigma)$. In particular, the same symbol will denote a function of the form (3.11) and its class modulo the zero norm elements. The completion of $H_0(\sigma)$, denoted $H(\sigma)$, can be identified to the classes of complex functions f on $(T, \mathscr{T})$ such that

$$(f | g)_\sigma = \frac{1}{4} \int (\bar{f}(t), f(t)) \cdot \begin{pmatrix} \frac{1}{2}(d\sigma_{11}(t) + d\sigma_{22}(t)) & (d\sigma_{12})(t) \\ (d\sigma_{21})(t) & \frac{1}{2}(d\sigma_{11}(t) + d\sigma_{22}(t)) \end{pmatrix} \cdot \begin{pmatrix} f(t) \\ \bar{f}(t) \end{pmatrix} < \infty. \tag{3.21}$$

The scalar product in $H(\sigma)$ will be still denoted $(\cdot|\cdot)_\sigma$ and the norm will be denoted $\|\cdot\|_\sigma$. In the following we shall denote $\mathscr{M}_0$ as the $*$-algebra spanned by the range of M. It is clear that $\mathscr{M}_0$ coincides with the complex algebra spanned by the $B(f)$ $(f \in H_0(\sigma))$.

THEOREM (3.4). *If the Grassman measure M has scalar brackets and if the algebra $\mathscr{A}$ has no self-adjoint nilpotent elements, then the map*

$$b(f) \in \mathscr{C}_0 \mapsto B(f) \in \mathscr{M}_0, \qquad f \in H_0(\sigma) \tag{3.22}$$

extends to a $*$-isomorphism of $\mathscr{C}_0$ with $\mathscr{M}_0$.

Proof. The map (3.22) is well defined because if $\|f\|_\sigma = 0$ then

$$0 = \|f\|^2 = \tfrac{1}{2}\{B(f), B(f)\} = B(f)^2 \in \mathscr{A}$$

hence $B(f) = 0$ since $\mathscr{A}$ has no self-adjoint nilpotents. By the universal property of the Clifford algebra this map extends to an isomorphism of $\mathscr{C}$ onto $\mathscr{M}_0$.

COROLLARY (3.5). *Let $\mathscr{A}$ be a C^*-algebra, then the norm closure of $\mathscr{M}_0$ in $\mathscr{A}$ is isomorphic to the Clifford C^*-algebra over $H(\sigma)$.*

Proof. If (f_n) is a sequence in $H_0(\sigma)$ converging to $f \in H(\sigma)$ then, since

$$\|B(f_m) - B(f_n)\|^2 = \|B(f_m - f_n)\|^2 = \|f_m - f_n\|_\sigma^2$$

it follows that $B(f_n)$ converges in norm to an element of $\mathscr{A}$ which we denote $B(f)$. If $B(g)$ is another such an element, then by continuity $\{B(f), B(g)\} = (f \mid g)_\sigma$. Whence the thesis follows from the universality of the Clifford C^*-algebra.

4. The Fermion Levy Theorem

In this section we show that if an $\mathscr{A}$-valued Grassman measure M on $\mathscr{T}_0$, beyond having scalar brackets, is also a difference martingale for a projective family of conditional expectations, then the restriction on the algebra $\mathscr{M}_0$, generated by the range of M, of any state φ on $\mathscr{A}$, compatible with these conditional expectations, induces on $\mathscr{C}(H_0(\sigma)) \cong \mathscr{M}_0$ a Gaussian state whose covariance is uniquely determined by σ.

We shall assume that, for each $I \in \mathscr{T}_0$ there exists a set $I^p \subseteq T$, called *the past of* I such that

$$I \cap I^p = \varnothing$$

$$I \subseteq J \Rightarrow I^p \supseteq J^p.$$

Definition (4.0). A (finite or infinite) sequence (I_n) of elements of $\mathscr{T}_0$ is called *time ordered* if

$$j < k \Rightarrow \bigcup_{h=1}^{j} I_h \subseteq I_k^p.$$

Remark (1). The example we have in mind with this definition is $T = \mathbf{R}_+$; $\mathscr{T}_0 = \{(s, t]: 0 \leqslant s \leqslant t < \infty\}$, and if $I = (s, t]$ then $I^p = [0, s]$. This choice gives the usual time ordering. Our motivation for the greater generality has been the attempt to apply our results to a relativistic context. However, in this attempt we met serious difficulties in finding a "past function" well behaved with respect to partitions.

In this paper by a *conditional expectation* we shall mean a map $E: \mathscr{A} \to \mathscr{A}$ satisfying

$$E^2 = E. \tag{4.1}$$

Notice that the condition $E(1) = 1$ is not required. We shall assume that, for each $I \in \mathscr{T}_0$ there exists a conditional expectation

$$E_{I^p}: \mathscr{A} \to \mathscr{A} \tag{4.2}$$

which is continuous in the topology of $\mathscr{A}$ and satisfies the following conditions:

(i) (Projectivity) For each $I, J \in \mathscr{T}_0$ such that $I \subseteq J$

$$E_{I^p} \cdot E_{J^p} = E_{J^p} \cdot E_{I^p} = E_{J^p}. \tag{4.3}$$

(ii) For every $I \in \mathscr{T}_0$ and for every pair of polynomials P_{I^p}, Q_{I^p} in the non-commuting variables

$$\{M(J_1)\, M(J_2)^+ : J_1, J_2 \subseteq I^p\} \tag{4.4a}$$

one has

$$E_{I^p}(P_{I^p} \cdot a \cdot Q_{I^p}) = P_{I^p} \cdot E_{I^p}(a) \cdot Q_{I^p}, \qquad \forall a \in \mathscr{A}. \tag{4.4b}$$

This condition implies in particular that the polynomial algebra in the variables (4.4a) is contained in the fixed points of E_{I^p}. From now on we shall fix a family $\{E_{I^p} : I \in \mathscr{T}_0\}$ of conditional expectations on $\mathscr{A}$

DEFINITION (4.1). An (E_{I^p})-difference martingale, or simply a *difference martingale* is a measure $M: \mathscr{T}_0 \to \mathscr{A}$ such that for each $I \in \mathscr{T}_0$ one has

$$E_{I^p}(M_i(I)) = 0, \qquad i = 1, 2. \tag{4.5}$$

Notice that, in view of (4.3), this implies that for each $I, J \in \mathscr{T}_0$ satisfying $J \subseteq I$ one has

$$E_{I^p}(M_i(J)) = 0, \qquad i = 1, 2. \tag{4.6}$$

LEMMA (4.2). *Let $M: \mathscr{T}_0 \to \mathscr{A}$ be an (E_{I^p})-difference martingale such that the bracket $[[M, M]]$ exists. Then for each $I, I_1, I_2 \in \mathscr{T}_0$ such that $I_1, I_2 \subseteq I$ one has*

$$\begin{aligned} E_{I^p}(M_i^+(I_1) \cdot M_j(I_2)) &= E_{I^p}([[M_i^+, M_j]](I_1 \cap I_2)) \\ &= E_{I^p}(\sigma_{ij}(I_1 \cap I_2)) \qquad (i, j = 1, 2). \end{aligned} \tag{4.7}$$

Proof. Let I, I_1, I_2 be as in the formulation of the lemma. Remark that, for $I_1 \cap I_2 = \varnothing$

$$E_{I^p}(M_i^+(I_1) \cdot M_j(I_2)) = 0. \tag{4.8}$$

In fact since $I_1 \subseteq I$ one has $I^p \subseteq I_1^p$, and therefore, since also $I_2 \subseteq I_1^p$ one has

$$\begin{aligned} E_{I^p}(M_i^+(I_1) \cdot M_j(I_2)) &= E_{I^p} \circ E_{I_1^p}(M_i^+(I_1) \cdot M_j(I_2)) \\ &= E_{I^p}(E_{I_1^p}(M_i^+(I_1)) \cdot M_j(I_2)) = 0. \end{aligned}$$

Now let $J=I_1=I_2$ and let $\{J_1, ..., J_k\}$ be a partition of J. Then by (4.8)

$$E_{IP}(M_i^+(J)\cdot M_j(J))=E_{IP}\left(\sum_{h=1}^{k} M_i^+(J_h)\cdot M_j(J_h)\right)$$

and by the continuity of E_{IP} we conclude

$$E_{IP}(M_i^+(J)\cdot M_j(J))=E_{IP}([[M_i^+, M_j]](J)). \tag{4.9}$$

In conclusion, for I_1, I_2 as in the theorem one has, using (4.8) and (4.9),

$$\begin{aligned} E_{IP}(M_i^+(I_1\cdot M_j(I_2)) &= E_{IP}(M_i^+(I_1\cap I_2)\, M_j(I_1\cap I_2)) \\ &= E_{IP}([[M_i^+, M_j]](I_1\cap I_2)). \end{aligned}$$

In particular, if M has scalar brackets then, in the notation (3.7) one has

$$E_{IP}(M_i^+(I_1)\cdot M_j(I_2))=[[M_i^+, M_j]](I_1\cap I_2)=\sigma_{ij}(I_1\cap I_2).$$

Now notice that on the real pre-Hilbert space $H_0(\sigma)$ the multiplication by i defines a natural complex structure and that the space $H_{0,re}(\sigma)$ of the real valued step functions is such that the restriction of the scalar product of $H_0(\sigma)$ on $H_{0,re}(\sigma)$ is real valued and one has the natural decomposition

$$H_0(\sigma)=H_{0,re}(\sigma)\oplus iH_{0,re}(\sigma)$$

hence the creation and annihilation operators $a^{\pm}(f)$, with f in H, can be defined as in (2.12) and $\mathscr{C}(H_0(\sigma))$ can be identified to the polynomial algebra in the noncommuting variables $a^{\pm}(\chi_I)$ with $I\in\mathscr{T}_0$.

In the following we shall assume that $\mathscr{A}$ has no self-adjoint nilpotents; we shall denote $u\colon \mathscr{C}\to\mathscr{M}_0\subseteq\mathscr{A}$ as the isomorphism defined in Theorem (3.4).

Using the identification of $\mathscr{C}(H_0(\sigma))$ with the polynomial algebra in the noncommuting variables $a^{\pm}(\chi_I)$ with $I\in\mathscr{T}_0$, and additivity, we can further assume that $\mathscr{C}(H_0(\sigma))$ is spanned by the products of the form

$$a^{(j_1)}(\chi_{I_1})\cdot\ \cdots\ \cdot a^{(j_n)}(\chi_{I_n})$$

with (in the notation (2.6)) $n\in\mathbf{N}$, $j_1, ..., j_n=1, 2$, and $I_1, ..., I_n\in\mathscr{T}_0$ and where for any pair $j, k=1, ..., n$, the sets I_j, I_k are either disjoint or coincident. Finally, by repeated use of the CAR, we can suppose that $\mathscr{C}(H_0(\sigma))$ is spanned by the products of the form

$$a(\chi_{I_1})^{(h_1)}\cdot a^+(\chi_{I_1})^{(k_1)}\cdot\ \cdots\ \cdot a(\chi_{I_n})^{(k_n)}\cdot a^+(\chi_{I_n})^{(k_n)} \tag{4.10}$$

with $I_1, ..., I_n\in\mathscr{T}_0$ mutually disjoint.

THEOREM (4.3). *Let* (E_{I^p}) $(I \in \mathscr{T}_0)$ *be a projective family of conditional expectations with the properties described at the beginning of this section and let* M *be an* (E_{I^p})*-difference martingale which is a Grassmann measure with scalar non-atomic brackets* $(\sigma_{ij})(\cdot)$ $(i, j = 1, 2)$. *Let* φ *be any state on* $\mathscr{A}$, *compatible with the family* (E_{I^p}) *of conditional expectations, and let* φ_σ *be the Gaussian state on* $\{\mathscr{C}(H_0(\sigma)), C\}$ *with mean zero and covariance uniquely determined by*

$$\varphi_\sigma(a^{(i)+}(\chi_I) \cdot a^{(j)}(\chi_J)) = \sigma_{ij}(I \cap J), \qquad I, J \in \mathscr{T}_0,\ i, j = 1, 2. \tag{4.11}$$

Assume moreover that $\mathscr{C}(H_0(\sigma))$ *is spanned by the products of the form* (4.10) *with the additional condition that the sets* $I_1, ..., I_n \in \mathscr{T}_0$ *are time ordered in the sense of Definition* (4.0). *Then*

$$\varphi_\sigma = \varphi \circ u. \tag{4.12}$$

Remark (1). The thesis of the theorem can be rephrased by saying that the family $\{M(I), M(I)^* : I \in \mathscr{T}_0\}$ is a mean zero Fermi–Gaussian family with respect to the state φ with covariance given by the identity operator (in the space $H_0(\sigma)$).

Proof. Since $u(a(\chi_I)) = M(I)$ for each $I \in \mathscr{T}_0$, by our assumptions it will be sufficient to prove the identity

$$\begin{aligned}\varphi_\sigma(a(\chi_{I_1})^{h_1} \cdot a^+(\chi_{I_1})^{k_1} \cdot \cdots \cdot a(\chi_{I_n})^{k_n} \cdot a^+(\chi_{I_n})^{k_n}) \\ = \varphi(M^+(I_1)^{h_1} \cdot M(I_1)^{k_1} \cdot \cdots \cdot M^+(I_n)^{h_n} \cdot M(I_n)^{k_n})\end{aligned} \tag{4.13}$$

for all $n \in \mathbf{N}, ..., j_n$ and

$$I_1, ..., I_n \in \mathscr{T}_0$$

mutually disjoint and time ordered, and for all $h_1, k_1, ..., h_n, k_n = 0, 1$ such that $h_j + k_j \geqslant 1$. To this goal let us first consider the right hand side of (4.14). Since the I_k are time ordered, the compatibility of φ with the $E_{I_n^p}$ implies that this is equal to

$$\begin{aligned}\varphi(M^+(I_1)^{h_1} \cdot M(I_1)^{k_1} \cdot \cdots \cdot M^+(I_{n-1})^{h_{n-1}} \\ \cdot M(I_{n-1})^{k_{n-1}} \cdot E_{I_n^p}[M^+(I_n)^{h_n} \cdot M(I_n)^{k_n}])\end{aligned}$$

and this is zero if $h_n + k_n = 1$ since M is a difference martingale, while if $h_n + k_n = 2$ it is equal to

$$\varphi(M^+(I_1)^{h_1} \cdot M(I_1)^{k_1} \cdot \cdots \cdot M^+(I_{n-1})^{h_{n-1}} \cdot M(I_{n-1})^{k_{n-1}}) \cdot \sigma_{11}(I_n) \tag{4.14}$$

because of Lemma (4.2) and the assumption that σ is scalar. By iteration of this argument we see that the right hand side of (4.14) is zero if for some $j=1, \dots, n$ one has $h_j+k_j=1$ and it is equal to

$$\sigma_{11}(I_1)\cdot \cdots \cdot \sigma_{11}(I_n) \tag{4.15}$$

if $h_j+k_j=2$ for all $j=1, \dots, n$. Now let us consider the left hand side of (4.14). Because of Gaussianity this is equal to zero if

$$\sum_j h_j+\sum_j k_j \tag{4.16}$$

is odd. If the sum (4.16) is equal to $2p$ then the left hand side of (4.14) has the form

$$\varphi_\sigma(a^{(\varepsilon_{r_1})}(\chi_{I_{r_1}})\cdot \cdots \cdot a^{(\varepsilon_{r_{2p}})}(\chi_{I_{r_{2p}}})) \tag{4.17}$$

with $\varepsilon_{r_j}=1, 2$ and $r_1, \dots, r_{2p}$ varying in the set $1, \dots, n$ and corresponding to those of the indices $h_1, k_1, \dots, h_m, k_m$ which are not equal to zero. Because of (2.14), (4.17) has the form

$$\sum_{(i_1, j_1; \dots; i_p, j_p)} \operatorname{sgn}(i_1, j_1; \dots; i_p, j_p)\, \varphi_\sigma(a^{(\varepsilon_{r_{i_1}})}(\chi_{I_{r_{i_1}}})\cdot a^{(\varepsilon_{r_{j_1}})}(\chi_{I_{r_{j_1}}}))$$
$$\cdot \cdots \cdot \varphi_\sigma(a^{(\varepsilon_{r_{i_p}})}(\chi_{I_{r_{i_p}}})\cdot a^{(\varepsilon_{r_{j_p}})}(\chi_{I_{r_{j_p}}})), \tag{4.18}$$

where $(i_1, j_1; \dots; i_p, j_p)$ is a permutation of the indices $1, \dots, 2p$ such that

$$i_1<i_2< \cdots <i_p; \qquad i_\alpha<j_\alpha,\ \alpha=1, \dots, p.$$

Because of the assumption (4.11), (4.18) is equal to

$$\sum_{(i_1, j_1; \dots; i_p, j_p)} \operatorname{sgn}(i_1, j_1; \dots; i_p, j_p)\, \varphi_\sigma(a^{(\varepsilon^+_{r_{i_1}})(\chi I_{r_{i_1}})}\cdot a^{(\varepsilon^+_{r_{j_1}})(\chi I_{r_{j_1}})})$$
$$\cdot \cdots \cdot \varphi_\sigma(a^{(\varepsilon^+_{r_{i_p}})}(\chi_{I_{r_{i_p}}})\cdot a^{(\varepsilon^+_{r_{j_p}})}(\chi_{I_{r_{j_p}}})), \tag{4.19}$$

where ε^+ denotes the conjugate index of ε (i.e., $\varepsilon^+=2$ if $\varepsilon=1$, $\varepsilon^+=1$ if $\varepsilon=2$). But the I_r are either disjoint or coincident and the only case of coincidence can occur in correspondence of a pair $a^+(\chi_{I_r})\cdot a(\chi_{I_r})$ whose φ_σ-expectation value is $\sigma_{11}(I_r)$ because of (4.10) and (4.11). Thus, if for some $j_0\in\{1, \dots, n\}$ one has $h_{j_0}+k_{j_0}=1$, then among the intersections $I_{r_i}\cap I_{r_j}$ there will always be an empty one and (4.19) will be zero. If this is not the case then $p=n$ $(r_1, \dots, r_{2p})=(1, 1, 2, 2, \dots, n, n)$ and in the sum (4.19) the only surviving term corresponds to the identity permutation which gives the expression (4.15). Thus the theorem is proved.

5. Continuity of the Trajectories

In this section we clarify the role played by the continuity of the trajectories in the proof of the Fermi–Levy theorem. Throughout this section, we shall suppose that the multiplication and the involution are continuous in the topology of $\mathscr{A}$. The discussion which follows is valid for any two $\mathscr{A}$-valued measures X, Y, or T, and assumes neither any commutation relation between the values of these measures on disjoint intervals nor any martingale type property of them.

We shall keep the notations and assumptions of the previous sections, with the exception that in this and the following section we shall deal only with the case in which the index set T is a sub-interval of $\mathbf{R}$ with the Borel σ-algebra, $\mathscr{T}_0$ is the family of bounded sub-intervals of T of the form $(s, t]$, and the past function is defined by

$$(s, t] \subseteq T \mapsto (-\infty, s] \cap T = (s, t]^p. \tag{5.1}$$

Instead of $E_{(s,t]^p}$ we shall use the notation $E_{s]}$. Hence, particularizing to the present situation the notions introduced at the beginning of Section 4, we see that the family $(E_{s]})$ of conditional expectations is projective, i.e.,

$$s \leqslant t \Rightarrow E_{s]} \cdot E_{t]} = E_{s]} \tag{5.2}$$

and, if $P_{t]}\ d_{t]}$ are any two polynomials in the non-commutative variables

$$\{X(a, b),\ Y(a', b'),\ a \leqslant b,\ a' \leqslant b',\ b, b' \leqslant t \in T\}$$

then for any $\mathscr{A}$

$$E_{t]}(P_{t]} \cdot a \cdot Q_{t]}) = P_{t]} \cdot E_{t]}(a) \cdot Q_{t]}. \tag{5.3}$$

Moreover each $E_{t]}$ is a continuous map from $\mathscr{A}$ to $\mathscr{A}$. If $\mathscr{P}(s, t)$ is a finite partition of the interval $(s, t]$ in subintervals, we denote $|\mathscr{P}(s, t)|$ as the maximum length of the intervals $I \in \mathscr{P}(s, t)$. Throughout this section the symbols X and Y will denote two $\mathscr{A}$-valued finitely additive measures on T.

Definition (5.1). If the limit

$$\lim_{|\mathscr{P}(s,t)| \to 0} \sum_{(a,b] \in \mathscr{P}(s,t)} E_{a]}(X(a, b) \cdot Y(a, b)) \tag{5.4}$$

exists in $\mathscr{A}$, then we denote it by

$$\langle\langle X, Y \rangle\rangle(s, t) \tag{5.5}$$

and call it the *oblique* (or *Watanabe*) *brackets* of X and Y (as opposed to the *square* (or *Meyer*) *brackets* given by (2.3)). Notice that this notion depends on the choice of the projective family $(E_{t]})$. Sometimes, to underline this dependence we will use the notations

$$\langle\langle X, Y\rangle\rangle_{(E_{t]})}(s, t) \qquad \text{or} \qquad \langle\langle X, Y\rangle\rangle_{\varphi}(s, t),$$

where φ is any state on $\mathscr{A}$ compatible with the family of conditional expectations $(E_{t]})$.

DEFINITION (5.2). An operator $a \in \mathscr{A}$ is called of *initial type* with respect to the family of conditional expectations $(E_{t]})$ if for each $t \in T$

$$E_{t]}(a) = a. \tag{5.6}$$

PROPOSITIONS (5.3). *Suppose that* $\langle\langle X, Y\rangle\rangle$ *exists. Then*:

(i) *The map* $(s, t] \subseteq T \mapsto \langle\langle X, Y\rangle\rangle(s, t)$ *is a finitely additive measure.*

(ii) *If also* $[[X, Y]]$ *exists, then for each* $(s, t] \subseteq T$, *one has*

$$E_{s]}([[X, Y]](s, t)) = E_{s]}(\langle\langle X, Y\rangle\rangle(s, t)). \tag{5.7}$$

(iii) *If both* $[[X, Y]]$ *and* $\langle\langle X, Y\rangle\rangle$ *exist and are of initial type, then*

$$[[X, Y]] = \langle\langle X, Y\rangle\rangle. \tag{5.8}$$

(iv) *If* X *and* Y *are* $(E_{t]})$*-difference martingales then*

$$E_{s]}(X(s, t) \cdot Y(s, t)) = E_{s]}(\langle\langle X, Y\rangle\rangle(s, t)). \tag{5.9}$$

Proof. All the properties follow easily from the definition.

LEMMA (5.4). *For any pair of* $\mathscr{A}$*-valued measures* X, Y *with the property that both brackets* $[[X, Y]]$ *and* $\langle\langle X, Y\rangle\rangle$ *exists in* $\mathscr{A}$ *and moreover, for each* $(s, t] \subseteq T$, *the quantity* $E_{s]}(X(s, t) \cdot Y(s, t))$ *is of initial type one has, for any interval* $(s, t] \subseteq T$,

$$\begin{aligned} &E_{s]}(|[[X, Y]](s, t] - \langle\langle X, Y\rangle\rangle(s, t]|^2) \\ &\quad = \lim_{|\mathscr{P}(s,t)| \to 0} E_{s]} \sum_{(t_j, t_{j+1}] \in \mathscr{P}(s,t)} \\ &\qquad \cdot [[E_{t_j]}(|dX(t_j) \cdot dY(t_j)|^2) - |E_{t_j]}(dX(t_j) \cdot dY(t_j))|^2]) \end{aligned} \tag{5.10}$$

(in the sense that the limit on the right hand side exists and is equal to the left hand side).

Proof. Let $\mathscr{P}(s,t)=\{t_1<\cdots<t_m\}$ be any partition of $(s,t]$ and consider the difference

$$E_{s]}\left[\left|\sum_{t_j} dX(t_j)\cdot dY(t_j)\right| - \left|\sum_{t_j}(E_{t_j]}(dX(t_j)\cdot dY(t_j)))\right|^2\right]. \tag{5.11}$$

Since multiplication, $E_{s]}$ and $*$ are continuous in the topology of $\mathscr{A}$, it follows that the expression (5.11) tends to

$$E_{s]}(|[[X,Y]](s,t] - \langle\langle X,Y\rangle\rangle(s,t]|^2) \tag{5.12}$$

as $|\mathscr{P}(s,t)|\to 0$. On the other hand

$$\begin{aligned}
&E_{s]}\left(\left|\sum_{t_j}(dX(t_j)\cdot dY(t_j))\right|^2\right)\\
&\quad=\sum_{t_j} E_{s]}(|(dX(t_j)\cdot dY(t_j))|^2)\\
&\quad\quad+2\operatorname{Re}\sum_{t_i<t_j} E_{s]}(|(dX(t_j)^+\cdot dY(t_j)^+\cdot dX(t_k)\cdot dY(t_k))|^2)\\
&\quad=\sum_{t_j} E_{s]}(|(dX(t_j)\cdot dY(t_j))|^2)\\
&\quad\quad+2\operatorname{Re}\sum_{t_i<t_j} E_{s]}(dX(t_j)^+\cdot dY(t_j)^+\cdot E_{t_j]}[|dX(t_k)\,dX(t_j)\cdot dY(t_j)|^2])\\
&\quad\quad+E_{s]}\left(\left|\sum_{t_j} E_{t_j]}[dX(t_j)\cdot dY(t_j)]\right|^2\right)\\
&\quad\quad-2\operatorname{Re}\sum_{t_i,t_j} E_{s]}([dX(t_j)\cdot dY(t_j)]^+\cdot E_{t_k]}[dX(t_k)\cdot dY(t_k)]).
\end{aligned} \tag{5.13}$$

Using the scalar type assumption for the third term and (5.13), for the first one, (5.13) becomes

$$\begin{aligned}
&E_{s]}\sum_{t_j}[E_{t_j]}(|dX(t_j)\cdot dY(t_j)|^2)\,|dX(t_j)\cdot dY(t_j)|^2]\\
&\quad\quad+E_{s]}\left(\left|\sum_{t_j} E_{t_j]}[dX(t_j)\cdot dY(t_j)]\right|^2\right)\\
&\quad\quad-2\operatorname{Re}\sum_{t_i,t_j} E_{s]}([dX(t_j)\cdot dY(t_j)]^+\cdot E_{t_k]}[dX(t_k)\cdot dY(t_k)])\\
&\quad=E_{s]}\sum_{t_j}[E_{t_j]}(|dX(t_j)\cdot dY(t_j)|^2) - |(E_{t_j]}(dX(t_j)\cdot dY(t_j)))|^2]
\end{aligned} \tag{5.14}$$

and from this (5.10) immediately follows.

Recall that a family $\mathscr{S}_0$ of states on $\mathscr{A}$ is called separating if for each $a \in \mathscr{A}$

$$\varphi(|a|^2) = 0 \qquad \forall \varphi \in \mathscr{S}_0 \Leftrightarrow a = 0.$$

COROLLARY (5.5). *Let X, Y be finitely additive $\mathscr{A}$-valued measures such that both brackets $[[X, Y]]$, $\langle\langle X, Y \rangle\rangle$ exist in $\mathscr{A}$. Suppose that there exists a separating family $\mathscr{S}_0$ of states on $\mathscr{A}$ with the following properties:*

(i) *For each $\varphi \in \mathscr{S}_0$ there exists a projective family $(E_{t]}) = (E^{\varphi}_{t]})$ of conditional expectations such that $E_{s]}(X(s, t) \cdot Y(s, t))$ is of initial type for any $(s, t] \subseteq T$.*

(ii) *For each $\varphi \in \mathscr{S}_0$ and for each $(s, t] \subseteq T$ one has*

$$\lim_{|\mathscr{P}(s, t)| \to 0} \sum_{(t_j, t_{j+1}] \in \mathscr{P}(s, t)} |E_{t_j]}(dX(t_j) \cdot dY(t_j))|^2 = 0.$$

(iii) *For each $\varphi \in \mathscr{S}_0$ the bracket $\langle\langle X, Y \rangle\rangle_{\varphi}$ is independent on φ.*

Then the following statements are equivalent:

$$[[X, Y]] = \langle\langle X, Y \rangle\rangle \tag{5.15}$$

and

$$\lim_{|\mathscr{P}(s, t)| \to 0} \sum_{(t_j, t_{j+1}] \in \mathscr{P}(s, t)} \varphi(|dX(t_j) \cdot dY(t_j)|^2) = 0. \tag{5.16}$$

Proof. Applying Lemma (5.4) to each family $(E^{\varphi}_{t]})$ $(\varphi \in \mathscr{S}_0)$ and using the compatibility of φ with $E^{\varphi}_{t]}$, we see that (5.16) is equivalent to

$$\varphi(|[[X, Y]](s, t) - \langle\langle X, Y \rangle\rangle (s, t)|^2) = 0 \tag{5.17}$$

for each $\varphi \in \mathscr{S}_0$ and $(s, t] \subseteq T$. The statement then follows since $\mathscr{S}_0$ is separating.

Remark (1). The calculations in Section (5.1) of [5] show that the choices

$$X(s, t) = A(\chi_{[s,t]}); \qquad Y(s, t) = A^+(\chi_{[s,t]})$$

(all the terms are referred to as the Fock space over $L^2(\mathbf{R}_+)$) provide a simple example of a situation in which all the conditions (i), (ii), (iii) are satisfied. By adding an initial space one can construct new examples from this one. A simple case in which condition (ii) of Corollary (5.5) is verified occurs when there exists a real valued non-atomic measure ν_T on T, bounded on bounded intervals and a locally ν_T-integrable $\mathscr{A}$-valued function F satisfying, for each $(s, t] \subseteq T$,

$$|E_{s]}(X(s, t) \cdot Y(s, t))|^2 \leqslant F(s) \cdot \nu(s, t)^2.$$

Examples in which this condition is satisfied are easily constructed by considering polynomials in the A and A^+ processes in the Fock or universal invariant representation of the CCR over $L^2(\mathbf{R}_+)$.

In order to prove that, in the case of classical stochastic processes, the conditions in Corollary (5.5) are equivalent to the continuity of the trajectories, let us consider the case in which $\mathscr{A} = L^\infty(\Omega, \mathscr{F}, P)$ with the topology of convergence P-almost everywhere (for some probability space $(\Omega, \mathscr{F}, P)$) and $X = Y =$ a real valued semimartingale such that $\langle\langle X, X\rangle\rangle$ exists. In this case

$$E_{t]}(dX(t)^2) = d\langle\langle X, X\rangle\rangle(t) + o(dt) \tag{5.18}$$

hence the limit in condition (ii) of Corollary (5.5) is the sum of the squares of the jumps of $\langle\langle X, X\rangle\rangle$ in $(s, t]$. Condition (ii) of Corollary (5.5) thus means, in this case, that $\langle\langle X, X\rangle\rangle$ is non-atomic, i.e., continuous. On the other hand, taking for φ the dP-integral which is clearly separating for $\mathscr{A}$, condition (5.15), i.e.,

$$[[X, X]] = \langle\langle X, X\rangle\rangle \qquad P-\text{a.e.} \tag{5.19}$$

is surely equivalent to the continuity of the trajectories of X if $\langle\langle X, X\rangle\rangle$ is continuous, because it is a general fact that the sum of the jumps of $[\![X, X]\!]$ in $(s, t]$ is equal to the sum of the squares of the jumps of X in $(s, t]$ (cf. [1, (44.1), p. 234] or [2, (31.13), p. 114]).

In general the relation between $[\![X, X]\!]$ and $\langle\langle X, X\rangle\rangle$ is the following (cf. [1] or [2]): $\langle\langle X, X\rangle\rangle$ is the only predictable process Y such that $[\![X, X]\!] - Y$ is a martingale.

If X is complex valued, by considering separately the real and the imaginary part we arrive at the conclusion that (using the notation (2.6)) if the brackets $[[X^{\varepsilon_1}, X^{\varepsilon_2}]]$ $\langle\langle X^{\varepsilon_1}, X^{\varepsilon_2}\rangle\rangle$ $(\varepsilon_j = 1, 2)$ exist, then X has continuous trajectories if and only if the measures $\langle\langle X^{\varepsilon_1}, X^{\varepsilon_2}\rangle\rangle$ are non-atomic (i.e., the Meyer brackets are continuous and the equality

$$[[X^{\varepsilon_1}, X^{\varepsilon_2}]] = \langle\langle X^{\varepsilon_1}, X^{\varepsilon_2}\rangle\rangle, \qquad \varepsilon_1, \varepsilon_2 = 1, 2 \tag{5.20}$$

holds. Furthermore, if X is a continuous trajectory semimartingale, then the brackets $[[X, X]]$ exist and are continuous. This equivalence naturally suggests the following definition:

DEFINITION (5.6). An $\mathscr{A}$-valued process such that the brackets $[[X^{\varepsilon_1}, X^{\varepsilon_2}]]$ $\langle\langle X^{\varepsilon_1}, X^{\varepsilon_2}\rangle\rangle$ $(\varepsilon_j = 1, 2)$ exist is said to have *continuous trajectories* if the $\mathscr{A}$-valued measures $\langle\langle X^{\varepsilon_1}, X^{\varepsilon_2}\rangle\rangle$ are non-atomic and the equalities (5.20) hold.

Remark (1). The continuity of $\langle\langle X^{\varepsilon_1}, X^{\varepsilon_2}\rangle\rangle$ is a necessary hypothesis. The martingale

$$M_t = \begin{cases} 0 & \text{if } \ t < 1 \\ \pm 1 & \text{if } \ t = 1, \text{ with probability } 1/2 \end{cases}$$

and constant if $t \geqslant 1$ is clearly discontinuous, but

$$[[M, M]](0, t) = \langle\langle M, M\rangle\rangle(0, t) = \begin{cases} 0 & \text{if } \ t < 1 \\ 1 & \text{if } \ t \geqslant 1. \end{cases}$$

REMARK (2). Notice that, in the conditions of Corollary (5.5) and with $X = Y$ or $Y = X^*$, condition (5.16), which is a condition on the fourth mixed moments of X and X^* becomes equivalent to the continuity of the trajectories of X in the sense of Definition (5.6), at least in the case in which the measures $\langle\langle Y^{\varepsilon_1}, Y^{\varepsilon_2}\rangle\rangle$ are scalar valued (which is the case considered in [2, 3]).

6. RANDOM TIME CHANGES AND CANONICAL FORMS

In this section $\mathscr{A}$ will denote a topological $*$-algebra and T an interval in $\mathbf{R}$. An $\mathscr{A}$-valued measure M on T satisfying

$$M(s, t) \in \mathbf{C} \cdot 1_{\mathscr{A}}, \qquad (s, t] \subseteq T$$

will be called a *scalar measure*. An $\mathscr{A}$-valued function f on T satisfying

$$f(t) \in \mathbf{C} \cdot l_{\mathscr{A}}, \qquad t \in T$$

will be called a *scalar function*. A *step function* on $\mathscr{A}$ is an $\mathscr{A}$-valued function which is piecewise constant, continuous from the left, and assumes finitely many values on every bounded interval. Now let us recall from [6] the notion of associative Ito algebra.

DEFINITION (6.1). An *Ito algebra* over $\mathscr{A}$ is a complex linear space $\mathscr{I}$ of $\mathscr{A}$-valued finitely additive measures on T such that:

(i) For each pair of measures $M, N \in \mathscr{I}$, the mutual quadratic variation $[[M, N]]$ of M and N exists and belongs to $\mathscr{I}$.

(ii) The map $(M, N) \in \mathscr{I} \times \mathscr{I} \mapsto [[M, N]]$ is associative.

(iii) The set $\mathscr{I}$ is self-adjoint, i.e., for each $M \in \mathscr{I}$, the $\mathscr{A}$-valued measure on T

$$M^*(s, t) = M(s, t)^* \tag{6.1}$$

also belongs to $\mathscr{I}$.

(iv) For each $M, N \in \mathscr{I}$ one has

$$[[M, N]]^* = [[N^*, M^*]]. \tag{6.2}$$

In this section $\mathscr{I}$ will denote a fixed associative Ito algebra whose elements will be called the *integrators* or the *stochastic differentials* and $\mathscr{F}$ will denote a topological $*$-algebra of measurable $\mathscr{A}$-valued functions on T, whose elements will be called the *integrands*. We assume that $\mathscr{F}$ contains a dense sub-algebra of step functions, denoted $\mathscr{S}$, and we shall use the same symbol $*$ for the involutions in $\mathscr{A}$, in $\mathscr{I}$, and in $\mathscr{F}$.

DEFINITION (6.2). A *stochastic integration* over $\{\mathscr{F}\ \mathscr{S}\}$ with respect to $\mathscr{I}$ is a structure of $\mathscr{F} - *$-bimodule on $\mathscr{I}$ with right and left actions denoted respectively,

$$(f, M) \in \mathscr{F} \times \mathscr{I} \mapsto M \circ f \in \mathscr{I} \tag{6.3a}$$

$$(f, M) \in \mathscr{F} \times \mathscr{I} \mapsto f \circ M \in \mathscr{I} \tag{6.3b}$$

subject to the following conditions:

(i) If

$$f(t) = \sum_j f(t_j)\, \chi_{(t_j, t_{j+1}]} \in \mathscr{S}$$

is a step function, then

$$(f \circ M)(s, t) = \sum_j f(t_j) \cdot M(t_j, t_{j+1}) \tag{6.4a}$$

$$(M \circ f)(s, t) = \sum_j M(t_j, t_{j+1}) \cdot f(t_j). \tag{6.4b}$$

(ii) For each $M \in \mathscr{I}$, the maps

$$f \in \mathscr{F} \mapsto f \circ M; \qquad f \in \mathscr{F} \mapsto M \circ f$$

are continuous.

Notice that, by the definition of a bimodule [10], for all $f, g \in \mathscr{F}$ and $M \in \mathscr{I}$, the following properties hold:

$$(f \circ M) \circ g = f \circ (M \circ g) \tag{6.5a}$$

$$(\lambda \cdot 1_{\mathscr{F}}) \circ M = M \circ (\lambda \cdot 1_{\mathscr{F}}) = \lambda \cdot M. \tag{6.5b}$$

Notice that, by the definition of an $\mathscr{I} - *$-bimodule the right and left action of $\mathscr{F}$ on $\mathscr{I}$ must be compatible with the multiplication and the involution in $\mathscr{I}$, i.e., one must have

$$[[f \circ M, N \circ g]] = f \circ [[M, N]] \circ g \tag{6.6}$$

$$(f \circ M)^* = M^* \circ f^*. \tag{6.7}$$

Sometimes we use the notations

$$(f \circ M)(s, t) = \int_s^t f\, dM; \qquad (M \circ f)(s, t) = \int_s^t dM\, f.$$

Following the terminology introduced in [2] we introduce the

Definition (6.3). A stochastic integration over $\mathscr{F}$ with respect to $\mathscr{I}$ is said to satisfy a *ρ-communication relation* if for all M in $\mathscr{I}$, there exists a $*$-automorphism

$$\rho_M : \mathscr{F} \to \mathscr{F}$$

such that

$$\rho_M^2 = \iota_{\mathscr{F}}, \tag{6.8}$$

where $\iota_{\mathscr{F}}$ denotes the identity map on $\mathscr{F}$ and

$$a \circ M = M \circ \rho_M(a) \qquad \forall a \in \mathscr{F}. \tag{6.9}$$

From now on we assume that it is given a stochastic integration over $\mathscr{F}$ with respect to S which satisfies a ρ-commutation relation.

Definition (6.4). Let $M \in \mathscr{I}$ and let the pair (B, B^*) be obtained from the pair (M, M^*) via the transformation

$$B = M \circ a^* + M^* \circ b \tag{6.10}$$

$$B^* = b^* \circ M + a \circ M^* = M \circ \rho(b)^* + M^* \circ \rho(b) \tag{6.11}$$

which we write in matrix form

$$(B, B^*) = (M, M^*) \circ \begin{pmatrix} a^* & \rho(b)^* \\ b & \rho(a) \end{pmatrix} \tag{6.12}$$

with $a, b \in \mathscr{F}$ and $\rho = \rho_M$. If the transformation (6.12) is invertible, then we say that the pair (B, B^*) has been obtained from the pair (M, M^*) by a *random time change* with coefficients $a, b \in \mathscr{F}$.

Notice that, if a and b are constant scalars, the associated random time change is an invertible linear map in the linear space spanned by M and M^* which commutes with the natural involution in this space. If M is an annihilation operator, such a map (more precisely, the one obtained by it normalizing to one the determinant) is called, in the physical literature, a *Bogoliubov transformation.*

To every pair (M, M^*) we associate its bracket matrix

$$\left[\left[\begin{pmatrix} M^* \\ M \end{pmatrix}, (M, M^*)\right]\right] = \begin{pmatrix} [[M^*, M]] & [[M^*, M^*]] \\ [[M, M]] & [[M, M^*]] \end{pmatrix}. \tag{6.13}$$

For brevity, in the following, the right hand side of (6.13) will be denoted

$$\sigma = \sigma(M) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}. \tag{6.14}$$

Notice that, if the pair (B, B^*) is related to (M, M^*) by the random time change (6.12), then the corresponding bracket matrices are related by

$$\begin{pmatrix} [[B^*, B]] & [[B^*, B^*]] \\ [[B, B]] & [[B, B^*]] \end{pmatrix}$$

$$= \begin{pmatrix} a & b^* \\ \rho(b) & \rho(a)^* \end{pmatrix} \circ \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \circ \begin{pmatrix} a^* & \rho(b)^* \\ b & \rho(a) \end{pmatrix}. \tag{6.15}$$

DEFINITION (6.5). We say that $M \in \mathscr{I}$ has *brackets of scalar type* if there exists a random time change which transforms the pair (M, M^*) into a new pair (B, B^*) whose bracket matrix (6.5) is a 2×2-matrix (necessarily of positive type) with coefficients in the complex valued measures on T. If moreover the coefficients σ_{ij} $(j = 1, 2)$ of σ have the form

$$\sigma_{ij}(s, t) = \sigma_{ij}^0 \cdot v(s, t), \qquad i, j = 1, 2, \tag{6.16}$$

where $(\sigma_{ij}^0) = \sigma^0$ is a complex valued 2×2 matrix (of positive) type and v is a positive real valued measure, then we say that M has *brackets of constant scalar type.* If $M \in \mathscr{I}$ has brackets of constant scalar type then we can assume, up to a random time change, that its bracket matrix has the form (6.16), where $(\sigma^0_{ij}) = \sigma^0$ is a constant complex valued matrix. The random time changes with coefficients in $\mathbf{C}$, identified to a subalgebra of $\mathscr{F}$ will then be in one-to-one correspondence with the complex 2×2 matrices of the form

$$\begin{pmatrix} \bar{a} & \bar{b} \\ b & a \end{pmatrix} \tag{6.17}$$

with

$$|a|^2 - |b|^2 \neq 0,$$

i.e., with the elements of $\mathbf{R} \times Sp(2, \mathbf{C})$ where $Sp(2, \mathbf{C})$ denotes the complex symplectic group of order 2. Correspondingly, the matrix σ^0 associated to (M, M^*) will vary under the action

$$\sigma \mapsto g^* \cdot \sigma \cdot g, \qquad g \in \mathbf{R} \times Sp(2, \mathbf{C}). \tag{6.18}$$

The orbits of the 2×2 complex matrices under the action (6.18) are easily classified by the following lemma. This provides a complete list of the canonical forms of the bracket matrices of those pairs (M, M^*) of $\mathscr{A}$-valued measures with brackets of constant scalar type (and therefore of their equivalence classes modulo random time changes).

LEMMA (6.6). *The orbits of the action* (6.18) *of* $\mathbf{R} \times Sp(2, \mathbf{C})$ *on the non-zero* 2×2 *complex positive semi-definite matrices, are the following*:

(i) *The orbit, denoted* C, *of the matrix*

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{6.19}$$

(ii) *For each* λ *in* $[0, 1]$, *the orbit, denoted* O_λ, *of the matrix*

$$\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}. \tag{6.20}$$

Moreover for $\lambda \neq \lambda'$, O_λ *and* O'_λ *are disjoint and* $\sigma = (\sigma_{i,j})$ *belongs to the orbit* C *if and only if*

$$\tfrac{1}{2}(\sigma_{1,1} + \sigma_{2,2}) = |\sigma_{1,2}| \tag{6.21}$$

while it belongs to the orbit O_λ $(\lambda \in [0, 1])$, *if and only if*

$$|\sigma_{1,2}| \neq \tfrac{1}{2}(\sigma_{1,1} + \sigma_{2,2}) \tag{6.22}$$

$$\lambda = \tfrac{1}{2}(\sigma_{1,1} - \sigma_{2,2}) \tag{6.23}$$

(*up to orbit equivalence one can always assume that* $\sigma_{1,1} \geqslant \sigma_{2,2}$).

Proof. For σ as above and any real number ϕ, one has

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} \cdot \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} e^{2i\phi} \\ \sigma_{2,1} e^{-2i\phi} & \sigma_{2,2} \end{pmatrix}. \tag{6.24}$$

Thus, up to equivalence, we can assume that $\sigma_{1,2}$ is real. In this case the matrix $(\sigma_{i,j})$ can be written in the form

$$\begin{pmatrix} \alpha+\gamma & \beta \\ \beta & \alpha-\gamma \end{pmatrix} \tag{6.25}$$

with α, β, γ given respectively by

$$\alpha = \tfrac{1}{2}(\sigma_{1,1} + \sigma_{2,2}), \qquad \beta = |\sigma_{1,2}|, \qquad \gamma = \tfrac{1}{2}(\sigma_{1,1} - \sigma_{2,2}) \tag{6.26}$$

and satisfying

$$\alpha^2 - \gamma^2 \geqslant \beta^2. \tag{6.27}$$

From the explicit form of $(a, b)^+ \sigma(a, b)$ one immediately verifies that a matrix of the form (6.25) is equivalent to a diagonal matrix if and only if there exist two complex numbers a, b such that

$$|a|^2 - |b|^2 \neq 0 \tag{6.28}$$

$$0 = \beta(a^2 + b^2) + 2\alpha ab = \beta(a+b)^2 + 2(\alpha - \beta)\, ab. \tag{6.29}$$

If $\alpha = \beta \neq 0$, then (6.29) contradicts (6.28) and σ cannot be equal to any diagonal (nonzero) matrix. The case $\alpha = \beta = 0$ is excluded by (6.27) and by the assumption that $\sigma \neq 0$. Moreover, because of (6.27), the condition $\alpha = \beta$ implies that $\gamma = 0$, hence in this case σ is a multiple of the matrix (6.19).

If $\alpha \neq \beta$, then Eqs. (6.28), (6.29) can always be solved in a, b. Therefore, in this case the matrix (6.25) is equivalent to a diagonal matrix which, up to equivalence, can always be written in the form (6.20). Conversely, under the action of $\mathbf{R} \times Sp(2, \mathbf{C})$, a matrix of the form (6.20) is transformed into a matrix of the form (6.25) with α and β arbitrary (satisfying (6.27)) and $\gamma = \lambda$. Thus the number λ completely characterizes the orbit. Again using (6.26), condition (6.23) follows. Finally the positivity of (6.20) implies that $|\lambda| \leqslant 1$ and this ends the proof.

Finally, let us see how the scalar product (3.21), the commutators (3.17), and the correlation functions $\varphi(B_f \cdot B_g)$ look like in the various canonical forms.

To this goal let us denote $\langle \cdots \rangle$ as the usual scalar product on $L^2(\mathbf{R}, dt; \mathbf{C}) = L^2(\mathbf{R})$. In case of the orbit C, i.e., for σ of the form (6.19), we find

$$\{B_f, B_g\} = 2\langle \operatorname{Re} f, \operatorname{Re} g\rangle$$
$$\varphi(B_f \cdot B_g) = \langle \operatorname{Re} f, \operatorname{Re} g\rangle.$$

In the case of the orbits O_λ $(0 \leqslant \lambda \leqslant 1)$, i.e., of the form (6.20), one finds

$$\{B_f, B_g\} = \operatorname{Re}\langle f, g\rangle$$
$$\varphi(B_f \cdot B_g) = \tfrac{1}{2}[\operatorname{Re}\langle f, g\rangle + i\lambda \operatorname{Im}\langle f, g)\rangle].$$

The four canonical forms listed in the Introduction correspond to the orbits C, O_0, O_1 and O_λ with $0 < \lambda < 1$.

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References

1. L. Accardi, A. Frigerio, and J. Lewis, Quantum stochastic processes, *Publ. Res. Inst. Math. Sci.* **18** (1982), 97–133.
2. L. Accardi and K. R. Parthasarathy, Stochastic calculus on local algebras, *in* "Quantum Probability and Applications" (L. Accardi and W. von Waldenfels, Eds.), Lecture Notes in Math., Vol. 1136, Springer-Verlag, New York/Berlin.
3. L. Accardi and K. R. Parthasarathy, A martingale characterization of canonical commutation and anticommutation relations, *J. Funct. Anal.* **77** (1988), 211–231.
4. L. Accardi and A. Bach, The harmonic oscillator as quantum central limit of Bernoulli processes, *Probab. Theory Related Fields*, in press.

5. L. ACCARDI AND A. BACH, Central limits of squeezing operators, *in* "Quantum Probability and Applications IV," Springer-Verlag, New York/Berlin, Lect. Notes in Math., Vol. 396.
6. L. ACCARDI AND F. FAGNOLA, Quaegebeur quantum stochastic calculus, *J. Funct. Anal.* **104** (1992), 149–197.
7. L. ACCARDI, A. FRIGERIO, AND Y. G. LU, The weak coupling limit as a quantum functional central limit, *Comm. Math. Phys.* **131** (1990), 537–570.
8. L. ACCARDI AND J. QUAEGEBEUR, Ito algebras of Gaussian quantum fields, *J. Funct. Anal.* **85** (1988), 213–263.
9. L. ACCARDI, M. SCHURMANN, AND W. VON WALDENFELS, Quantum independent increment processes on superalgebras, *Math. Z.* **198** (1988), 451–477.
10. N. BOURBAKI, Eléments de mathématique, VI, 2, *in* "Algèbre Linéaire," Hermann, Paris, 1962.
11. C. DELLACHERIE AND P.-A. MEYER, "Probability and Potential B," North-Holland, Amsterdam.
12. M. EMERY AND P. A. MEYER, A new example of chaotic representation, preprint, Strasbourg, February 1988.
13. N. GIRI AND W. VON WALDENFELS, An algebraic version of the central limit theorem, *Z. Wahrsch. Verw. Gebiete* **42** (1978), 129–134.
14. L. GROSS, Existence and uniqueness of physical ground states, *J. Funct. Anal.* **10** (1972), 52–109.
15. L. GROSS, On the formula of Matthews and Salam, *J. Funct. Anal.* **25** (1977), 162–209.
16. R. L. HUDSON AND J. M. LINDSAY, Use of non-Fock quantum Brownian motion and a quantum martingale representation theorem, *in* "Quantum Probability and Applications, II" (L. Accardi and W. von Waldenfels, Eds.), Lecture Notes in Math., Vol. 1136, Springer-Verlag, New York/Berlin.
17. G. LETTA, Martingales et intégration stochastique, *Ann. Scuola Norm. Sup.*
18. D. SHALE AND W. F. STINESPRING, States on the Clifford algebra, *Ann. Math.* **80** (1964), 365–381.
19. W. VON WALDENFELS, An algebraic central limit theorem in the anticommuting case, *Z. Wahrsch. Verw. Gebiete* **42** (1978), 135–140.