

A Representation Free Quantum Stochastic Calculus

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We develop a representation free stochastic calculus based on three inequalities (semimartingale inequality, scalar forward derivative inequality, scalar conditional variance inequality). We prove that our scheme includes all the previously developed stochastic calculi and some new examples. The abstract theory is applied to prove a Boson Levy martingale representation theorem in bounded form and a general existence, uniqueness, and unitarity theorem for quantum stochastic differential equations. © 1992 Academic Press, Inc.

0. INTRODUCTION

Stochastic calculus is a powerful tool in classical probability theory. Recently various kinds of stochastic calculi have been introduced in quantum probability [18, 12, 10, 21]. The common features of these calculi are:

— One starts from a representation of the canonical commutation (or anticommutation) relations (CCR or CAR) over a space of the form $L^2(\mathbf{R}_+, dt; \mathcal{H})$ (where \mathcal{H} is a given Hilbert space).

— One introduces a family of operator valued measures on \mathbf{R}_+ , related to this representation, e.g., expressed in terms of creation or annihilation or number or field operators.

— One shows that it is possible to develop a theory of stochastic integration with respect to these operator valued measures sufficiently rich to allow one to solve some nontrivial stochastic differential equations.

— One tries to use the above theory to prove an operator form of the Itô formula (usually stated in the form that the weak or strong product of two stochastic integrals is a sum of stochastic integrals).

The basic application of the theory is the construction of unitary Markovian cocycles (in the sense of [1]) as solutions of quantum stochastic differential equations, which provides, via the quantum Feynman-Kac perturbation scheme, a new nontrivial generalization of the Schrödinger equation as well as a dilation in the sense of Kümmerer [24]. This stage of development of the theory has been reached up to now only by the operator approaches of [18] for the Boson Fock space on $L^2(\mathbf{R}_+)$ and subsequently of [10] for the Fermion case, of [19] for the universal invariant case, and of [22] for some more general quasi-free case (cf. also the kernel approach in [21, 27]).

The vector approach of [11, 12] has the advantage of underlining the analogy between the classical and the quantum theory of stochastic integration, but pays it with the lack of a satisfactory Itô formula. On the other hand, the Hudson-Parthasarathy approach can produce such a formula by making heavy use of very specific properties of the particular representation chosen and of the operators involved, such as for example: the continuous tensor product structure, the factorization property of the coherent vectors, the explicit form of the action of the creation, annihilation or number process on the coherent or number vectors,.... This implies, not only that one has to change techniques every time one changes representation, but also that one loses the contact with the classical stochastic integral, because these properties do not play any role in the definition of the classical stochastic integral. In [2] the program of constructing a theory of stochastic integration which could unify the classical as well as the several different quantum theories developed up to now, was formulated.

The basic motivation of this program, independent of the esthetic motivation, related to the unification of the existing theories, was provided by the theory of quantum noise [3, 4] in which the following two problems arise:

(i) To develop a stochastic calculus for general quantum filtrations with sufficiently strong “chaoticity properties” (cf. [3, Sect. 2]). In particular we do not want our theory to be limited to continuous tensor products (or graded tensor products) nor to expected filtrations, nor to filtrations commuting or anticommuting with the past.

(ii) To classify the basic “quantum noises”, i.e., to look for some class of “basic quantum noises” which could play, in the quantum case, the role of “fundamental archetypes” played, in the classical case, by the Wiener and the Poisson process.

The solution of Problem (i) above is necessary if we want to apply stochastic calculus to quantum field theory, where several important examples of non factorizable representations arise. The solution of problem (ii) is related to the development of sufficiently powerful “representation theorems” (cf. [20, 23] for the classical case and [7, 9, 16] for the quantum case) whose final goal should be to produce a full classification of all the canonical forms of the quantum noises. It is clear that, if we want to reconstruct the properties of a quantum noise from some moment and martingale requirements, we cannot start from stochastic calculus based on some particular property of the representation, but on the contrary, the form of the representation should be deduced from the statistical assumptions.

The realization of this program had to undergo several preliminary steps: from the pioneering attempt of [2, 5, 6], to the individuation of the Dellacherie–Letta–Protter approach to stochastic integration as the natural bridge between the classical and the quantum theory [5, 15], the development of a satisfactory notion of brackets (mutual quadratic variation) and the consequent “stochastic integration independent Itô tables” [8, 9].

The present paper constitutes an attempt to bring this program to a conclusion by producing a theory of stochastic integration which:

- (1) includes all the known examples.
- (2) is applicable to examples to which the previous theories were not.
- (3) allows us to prove an Itô formula without the assumption that the increments of the basic integrators commute (or anticommute) with the past.

The basic ideas of our approach are the following: In Section (2) we introduce an abstract notion of operator semimartingale and we single out a class of semimartingales with particularly good properties (the integrators of scalar type). Intuitively a semimartingale is good if the stochastic integrals of sufficiently regular processes are still semimartingales. We single out two easily verifiable conditions which guarantee that a semimartingale is an integrator of scalar type:

- (i) the scalar forward derivative inequality (3.9).
- (ii) the scalar conditional variance inequality (Definition (3.5)).

We fix then a family of integrators of scalar type, called the “basic integrators,” and all the stochastic integrals we consider are meant with respect to this family of integrators.

The inequalities (i) and (ii) depend on the domain on which the basic integrators are considered. The known examples of basic integrators are creation, annihilation, and number processes with respect to a given quasi-free (gaussian) representation of the CCR or of the CAR. The domains on which these integrators are usually considered are the linear span of the coherent vectors [18] or the n -particle vectors [10] or the invariant domain of [8] which combines both the previous domains. In all these domains the inequalities (i) and (ii) are easily verified by direct calculation (on the domain of coherent vectors they assume a particularly simple form). The verification of these inequalities is purely computational and we do not include the lengthy but elementary calculations involved. In Section 5 we prove an existence and uniqueness theorem for stochastic differential equations, driven by integrators of scalar type, with bounded coefficients adapted to the initial space.

In Section 6 we recall from [8, 9] the notion of Meyer bracket (or mutual quadratic variation) of two semimartingales, and in Section 7 we prove a weak Itô formula for two integrators of scalar type admitting a weak Meyer bracket.

In order to prove the unitarity condition for the solution of a stochastic differential equation (Section 9), we need two additional conditions:

(iii) a ρ -commutation condition (cf. Definition (6.2)),

(iv) the Itô table of the basic integrators (in the sense of (6.5)) has scalar structure coefficients.

Again these two conditions are verified in all the known examples. Moreover it is an easy exercise (using the results of [8, 15]) to prove that the free Euclidean Boson field satisfies all the conditions (i), ..., (iv). This provides in particular an example of a nonfactorizable space where stochastic integration can be developed. Even when restricted to the case when the 1-particle space is some $\Gamma(L_2(\mathbf{R}_+, dt; \mathcal{H}))$, the results of Sections 6 and 7 are stronger than the known ones, since they hold on the larger invariant domain of [8].

In Section 8 we show how the Hermite polynomials arise from iteration of quantum stochastic integrals satisfying a Boson commutation relation; this extends a well-known result in classical probability theory, but at the moment we have no Fermion analogue of this result.

In Section 9 we prove a Boson type Lévy theorem for a single semimartingale. The previous results in this direction could bypass the problem of the nonexistence of a representation free stochastic calculus either using a

formulation of the problem which broke the analogy with the classical case [7, 16] or by exploiting the very specific properties of the Fermion commutation relations [9]. In particular, the present technique allows us to obtain the CCR in the Weyl form, while in [6] only the simpler unbounded form of the CCR was obtained.

Finally, in Section 10 we prove that the example recently considered by Boukas, and which is not included in the previously considered examples can be easily included in our theory. This inclusions allows us, in particular, to solve some problems left open in Boukas approach.

1. SIMPLE STOCHASTIC INTEGRALS

Throughout the paper we shall use the following notation:

- \mathcal{H} is a complex separable Hilbert space,
- $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on \mathcal{H} ,
- \mathcal{D} is a total subset of \mathcal{H} ,
- $(\mathcal{A}_t)_{t \in \mathbf{R}_+}$ is an increasing family of W^* -algebras of operators on \mathcal{H} ,
- \mathcal{A} is a W^* -algebra of operators in \mathcal{H} such that $\mathcal{A}_t \subseteq \mathcal{A}$ for all $t \in \mathbf{R}_+$,
- \mathcal{A}'_t is the commutant of \mathcal{A}_t in $\mathcal{B}(\mathcal{H})$,
- for each vector $\xi \in \mathcal{D}$, we define $\mathcal{H}_t(\xi)$ to be the closure of the subspace $[\mathcal{A}_t \xi] = \{a\xi : a \in \mathcal{A}_t\}$,
- e^ξ_t is the orthogonal projection onto $\mathcal{H}_t(\xi)$,
- $\mathcal{L}(\mathcal{D}; \mathcal{H})$ is the vector space of all linear operators F with domain containing \mathcal{D} such that the adjoint operator F^* also has \mathcal{D} in its domain. So for $F \in \mathcal{L}(\mathcal{D}; \mathcal{H})$ we have that

$$\langle \eta, F\xi \rangle = \langle F^* \eta, \xi \rangle, \quad \text{for all } \xi, \eta \in \mathcal{D}. \quad (1.1)$$

EXAMPLES. The following choices of \mathcal{H} and \mathcal{D} give the most studied examples of quantum stochastic calculi. In the Boson Fock space quantum stochastic calculus developed in [18], \mathcal{H} is the Hilbert space $h \otimes \Gamma(L^2(\mathbf{R}_+))$, where h is a complex separable Hilbert space (the initial space) and $\Gamma(L^2(\mathbf{R}_+))$ the symmetric Fock space over $L^2(\mathbf{R}_+)$ moreover \mathcal{D} is the set of vectors $u \otimes \psi(f)$, where $u \in h$ and $\psi(f)$ is the exponential (or coherent) vector with test function f . In the Fermion Fock space quantum stochastic calculus developed in [10], \mathcal{H} is the tensor product of a complex separable Hilbert space h with the antisymmetric Fock space over $L^2(\mathbf{R}_+)$ and \mathcal{D} is the set of n -particle vectors.

DEFINITION 1.1. A random variable F is an element of $\mathcal{L}(\mathcal{D}, \mathcal{H})$.

DEFINITION 1.2. A stochastic process in \mathcal{H} , indexed by \mathbf{R}_+ , is a family $(F_t)_{t \geq 0}$ of random variables such that for each $\eta \in \mathcal{D}$ the map $t \in \mathbf{R}_+ \rightarrow F_t \eta$ is Borel measurable. Alternatively, a stochastic process indexed by \mathbf{R}_+ can be looked at as a map $t \in \mathbf{R}_+ \mapsto F(t) = F_t \in \mathcal{L}(\mathcal{D}; \mathcal{H})$ with the above mentioned measurability property. In this paper we shall only deal with processes indexed by (subintervals of) \mathbf{R}_+ .

We will use indifferently the notation F_t or $F(t)$. If each F_t is a bounded operator and moreover, for each $T < +\infty$

$$\sup_{t \in [0, T]} \|F_t\|_\infty < +\infty$$

then the process F is called *locally bounded*. If each F_t has the form

$$F_t = f(t) \cdot 1,$$

where f is a complex valued measurable function and 1 is the identity operator in $\mathcal{B}(\mathcal{H})$, then F is called a *scalar process*. Two processes will be considered equivalent if they coincide on \mathcal{D} . Due to (1.1) each stochastic process (F_t) uniquely defines the adjoint process (F_t^*) .

DEFINITION 1.3. For all $t \in \mathbf{R}_+$ we denote by \mathcal{D}'_t the linear span of $\mathcal{A}'_t \mathcal{D}$. We say that an operator F is *t-adapted* to \mathcal{A}_t if $D(F) = \mathcal{D}'_t$, $D(F^*) \supseteq \mathcal{D}'_t$ and if

$$Fa'_t \xi = a'_t F \xi \quad \text{and} \quad F^* a'_t \xi = a'_t F^* \xi \quad (1.2)$$

for all $a'_t \in \mathcal{A}'_t$ and $\xi \in \mathcal{D}$.

Clearly, if F is a t -adapted operator, then it is in $\mathcal{L}(\mathcal{D}, \mathcal{H})$ and the restriction $F^*|_{\mathcal{D}'_t}$ of F^* to \mathcal{D}'_t is again a t -adapted operator. Moreover, if F is a t -adapted operator and $s \leq t$, then $F|_{\mathcal{D}'_s}$ is an s -adapted operator.

The following proposition clarifies the notion of adaptedness.

PROPOSITION 1.4. If F is a t -adapted operator then the closure \bar{F} of F is affiliated with \mathcal{A}_t and there exists a sequence $(F^{(n)})_n$ in \mathcal{A}_s such that $(F^{(n)})_n$ and $(F^{(n)*})_n$ converge to F and F^* respectively strongly on \mathcal{D} . Conversely, if $(F^{(n)})_n$ and $(F^{(n)*})_n$ are sequences in \mathcal{A}_t strongly convergent on \mathcal{D} , then the operator defined by

$$D(F) = \{ \xi \in \mathcal{H} \mid (F^{(n)} \xi)_n \text{ and } (F^{(n)*} \xi)_n \text{ converge} \}$$

$$F \xi = \lim_{n \rightarrow \infty} F^{(n)} \xi$$

is a t -adapted operator.

Proof. Since F is a random variable, it is closable. Let \bar{F} denote the closure of F defined on the domain \mathcal{D}^- . We know that $\mathcal{A}'_{s\downarrow}\mathcal{D}$ is contained in $D(F)$. Consider $x \in \mathcal{D}^-$ and a sequence $(\xi_n)_n$ in the linear span of \mathcal{D} such that

$$\lim_{n \rightarrow \infty} \xi_n = x, \quad \lim_{n \rightarrow \infty} F\xi_n = \bar{F}x.$$

For all $a' \in \mathcal{A}'_{s\downarrow}$ the sequence $(a'\xi_n)_n$ converges strongly to $a'x$ therefore we have

$$\lim_{n \rightarrow \infty} Fa'\xi_n = \lim_{n \rightarrow \infty} a'F\xi_n = a'\bar{F}x.$$

So $a'x \in D(\bar{F})$ and $\bar{F}a'x = a'\bar{F}x$. This shows that \bar{F} is affiliated with $\mathcal{A}_{t\downarrow}$. The closure \bar{F} of F can be decomposed as the product $U|\bar{F}|$ of a partial isometry U in $\mathcal{A}_{t\downarrow}$ and the positive self-adjoint $|\bar{F}|$. Let $(E(\lambda))_{\lambda \geq 0}$ be the spectral family of $|\bar{F}|$; for all $\lambda \geq 0$, $E(\lambda)$ is an element of $\mathcal{A}_{s\downarrow}$. For each integer n put

$$F^{(n)} = U \int_0^n \lambda dE(\lambda).$$

Clearly $F(n)$ is an element of $\mathcal{A}_{t\downarrow}$, its adjoint is given by

$$F^{(n)*} = \int_0^n \lambda dE(\lambda)U^*$$

and $F^{(n)}$ converges strongly to F on the domain $\mathcal{D} \subseteq D(|\bar{F}|) = D(\bar{F})$. Moreover we know that $F^* = (\bar{F})^* = |\bar{F}|U^*$ so, for all $\xi \in \mathcal{D}$, we have $U^*\xi \in \mathcal{D}(|\bar{F}|)$ by the definition of random variable. Therefore $F^{(n)*}\xi$ converges strongly to $F^*\xi$. This completes the proof of the first part of the proposition. Conversely let $(F^{(n)})_n$ and F be as above. We show first that F is a random variable. Indeed $\mathcal{D} \subseteq D(F)$ and for any $\xi \in D(F)$ we have

$$\langle F\xi, \eta \rangle = \lim_{n \rightarrow \infty} \langle F^{(n)}\xi, \eta \rangle = \lim_{n \rightarrow \infty} \langle \xi, F^{(n)*}\eta \rangle.$$

Hence η is in the domain of the adjoint F^* of F and $F^*\eta = \lim_{n \rightarrow \infty} F^{(n)*}\eta$. Now take $\xi \in \mathcal{D}$ and $a' \in \mathcal{A}'_{t\downarrow}$. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} F^{(n)}a'\xi &= \lim_{n \rightarrow \infty} a'F^{(n)}\xi = a'F\xi \\ \lim_{n \rightarrow \infty} F^{(n)*}a'\xi &= \lim_{n \rightarrow \infty} a'F^{(n)*}\xi = a'F^*\xi. \end{aligned}$$

This completes the proof. \blacksquare

A sequence (F_n) of t -adapted operators is said to *converge strongly on \mathcal{D}* if for any vector $\xi \in \mathcal{D}$ the sequences $(F^{(n)}\xi)_n$ and $(F^{(n)*}\xi)_n$ converge in norm in \mathcal{H} . Note that the sequences then also converge on \mathcal{D}'_t . Denote now by F the process given by $D(F_t) = \mathcal{D}'_t$ and $F\xi = \lim_n F^{(n)}\xi$ for $\xi \in \mathcal{D}'_t$. Then one easily checks that F is again t -adapted. Hence the strong limit on \mathcal{D}'_t of a sequence of t -adapted operators is a t -adapted operator.

DEFINITION 1.5. A stochastic process $(F_t)_{t \geq 0}$ is *adapted* to the filtration $(\mathcal{A}_{t\downarrow})_{t \geq 0}$ if F_t is t -adapted for all $t \geq 0$. We shall denote by \mathcal{S} the vector space of all *simple adapted* processes, i.e., those adapted processes (F_t) which can be written in the form

$$F_t = \sum_{k=1}^n \chi_{[t_k, t_{k+1})}(t) F_{t_k} \quad (1.3)$$

for some finite integer n and with $0 \leq t_0 < t_1 < \dots < t_{n+1} < \infty$.

Remark. Let, in the above notations, $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$, where (Ω, \mathcal{F}, P) is a probability space with a past filtration $(\mathcal{F}_{t\downarrow})$ and let $\mathcal{A}_{t\downarrow} = L^\infty(\Omega, \mathcal{F}_{t\downarrow}, P)$ acting by multiplication on \mathcal{H} . Let $\mathcal{D} = \mathcal{A} \cdot 1$ (1 is the constant function on Ω equal to 1) and let $e_{t\downarrow} = e_{t\downarrow}$. Then, since $e_{t\downarrow} \in \mathcal{A}'_{t\downarrow}$, (1.2) implies that

$$M_t e_{t\downarrow} = e_{t\downarrow} M_t \quad \text{on } \mathcal{D} \quad (1.4)$$

and, in the classical case, the notion of adaptedness introduced in Definition (1.3), is equivalent to the usual one.

In the following all the processes we shall consider will be adapted to the filtration $(\mathcal{A}_{t\downarrow})_{t \geq 0}$ therefore we call them simply “adapted.”

DEFINITION 1.6. An *additive process* is a family $M = (M(s, t))_{0 \leq s \leq t}$ of random variables such that:

- (i) for all $s \leq t$ in \mathbf{R}_+ , the operator $M(s, t)$ is a t -adapted operator
- (ii) for all r, s, t with $r < s < t$

$$M(r, t) = M(r, s) + M(s, t); \quad M(t, t) = 0 \quad (1.5)$$

on \mathcal{D} (and hence also on $\mathcal{A}'_{t\downarrow} \cdot \mathcal{D} = \mathcal{D}'_t$).

Remark. To every additive process $(M(s, t))_{0 \leq s \leq t}$ we associate the adapted process $M(t) = M(0, t)$ ($t > 0$). Conversely, to every adapted process (M_t) we associate the additive process $M(s, t) = M(t) - M(s)$. This correspondence characterizes the process $M(t)$ up to the random variable $M(0) = M_0$, called the *initial value* of the process.

DEFINITION 1.7. An additive process M is called *regular* if

$$\forall \xi \in \mathcal{D}, \forall s \leq t : \mathcal{H}_{r,1}(\xi) \subseteq D(\bar{M}^*(s, t)) \quad (1.6)$$

$$\forall s \leq t : M^*(s, t) \mathcal{D} \subseteq \mathcal{D}'_s, \quad (1.7)$$

where \bar{M} denotes the closure of M and \bar{M}^* stands for either \bar{M} and M^* .

LEMMA 1.8. Let M be a regular additive process and suppose that \mathcal{D}'_t is a core for $M(s, t)^*$ for all $s \leq t$. Define

$$M^+(s, t) = M(s, t)^*|_{\mathcal{D}'_t}. \quad (1.8)$$

Then M^+ is again a regular additive process.

Proof. Clearly, since $\mathcal{D} \subseteq D(M(s, t)^*)$ for all $s \leq t$, we obtain the additivity of M^+ from the additivity of M .

Now choose $s \leq t$, $\xi \in \mathcal{D}$ and $a'_t \in \mathcal{A}'_{t,1}$. Then, since $M(s, t)$ is t -adapted, we have

$$M^+(s, t) a'_t \xi = M(s, t)^* a'_t \xi = a'_t M(s, t)^* \xi = a'_t M^+(s, t) \xi. \quad (1.9)$$

Moreover, as $M^+(s, t) \subseteq M(s, t)^*$, we have $M^+(s, t)^* \supseteq M(s, t)$. Hence, $\mathcal{D}'_t \subseteq D(M^+(s, t)^*)$ and

$$M^+(s, t)^* a'_t \xi = M(s, t) a'_t \xi = a'_t M(s, t) \xi = a'_t M^+(s, t)^* \xi. \quad (1.10)$$

We conclude that $M^+(s, t)$ is t -adapted. Finally, as \mathcal{D}'_t is a core for $M(s, t)^*$, we know that $D(\overline{M^+(s, t)}) = D(M(s, t)^*) \supseteq \mathcal{H}_{s,1}(\xi)$ for all $\xi \in \mathcal{D}$. Since also $M^+(s, t) \mathcal{D} = M(s, t)^* \mathcal{D} \subseteq \mathcal{D}_s$, we obtain the regularity of M^+ . ■

LEMMA 1.9. Let M be a regular additive process and F an adapted process. Then for all $s \leq t$ we have that

$$F_s M(s, t) \quad \text{and} \quad \bar{M}(s, t) F_s|_{\mathcal{D}'_t}$$

are t -adapted processes.

Proof. It follows immediately from the definitions that $D(F_s M(s, t)) = \mathcal{D}'_t$ and that $F_s M(s, t) a'_t \xi = a'_t F_s M(s, t) \xi$ for $a'_t \in \mathcal{A}'_{t,1}$ and $\xi \in \mathcal{D}$.

Now let $\xi, \eta \in \mathcal{D}$ and $a'_t, b'_t \in \mathcal{A}'_{t,1}$. Then, using adaptedness and the fact that $b'_t \eta \in \mathcal{D}'_t \subseteq \mathcal{D}'_s \subseteq D(F_s^*)$ and $F_s^* \eta \in \mathcal{H}_{s,1}(\eta) \subseteq D(M(s, t)^*)$, we find that

$$\langle F_s M(s, t) a'_t \xi, b'_t \eta \rangle = \langle a'_t \xi, b'_t M(s, t)^* F_s^* \eta \rangle$$

which implies that $\mathcal{D}'_t \subseteq D((F_s M(s, t))^*)$ and that $(F_s M(s, t))^* b'_t \eta = b'_t M(s, t)^* F_s^* \eta = b'_t (F_s M(s, t))^* \eta$. This proves the t -adaptedness of $F_s M(s, t)$.

To show the t -adaptedness of $\bar{M}(s, t) F_s|_{\mathcal{D}'_t}$, first note that for $a'_t \in \mathcal{A}'_{t \downarrow}$ and $\xi \in \mathcal{D}$ we have that $a'_t \mathcal{H}_{s \downarrow}(\xi) \subseteq D(\bar{M}(s, t))$ and that for $\eta \in \mathcal{H}_{s \downarrow}(\xi)$ we have that $\bar{M}(s, t) a'_t \eta = a'_t \bar{M}(s, t) \eta = a'_t M(s, t) \eta$. Using this, we deduce that $\mathcal{D}'_t \subseteq D(\bar{M}(s, t) F_s)$ and $\bar{M}(s, t) F_s a'_t \xi = a'_t \bar{M}(s, t) F_s \xi$ for $a'_t \in \mathcal{A}'_{t \downarrow}$ and $\xi \in \mathcal{D}$.

Now let $\xi, \eta \in \mathcal{D}$ and $a'_t, b'_t \in \mathcal{A}'_{t \downarrow}$. Then, using adaptedness and the fact that $\mathcal{D}'_t \subseteq D(M(s, t)^*)$ and $M(s, t)^* \eta \in \mathcal{D}'_s \subseteq D(F_s^*)$, we find that

$$\langle \bar{M}(s, t) F_s a'_t \xi, b'_t \eta \rangle = \langle a'_t \xi, b'_t F_s^* M(s, t)^* \eta \rangle$$

which implies that $\mathcal{D}'_t \subseteq D((\bar{M}(s, t) F_s|_{\mathcal{D}'_t})^*)$ and that $(\bar{M}(s, t) F_s|_{\mathcal{D}'_t})^* b'_t \eta = b'_t F_s^* M(s, t)^* \eta = b'_t (\bar{M}(s, t) F_s|_{\mathcal{D}'_t})^* \eta$. This proves the t -adaptedness of $\bar{M}(s, t) F_s|_{\mathcal{D}'_t}$. ■

From this lemma it follows that one can define meaningfully the stochastic integral for simple adapted functions with respect to a regular additive process.

DEFINITION 1.10. Let M be a regular additive process and $F \in S$ a simple adapted process of the form (1.5). We define the *left stochastic integral* of F with respect to M over the interval $[0, t]$ as an operator on \mathcal{D}'_t by

$$\int_0^t dM_s F_s = \sum_{k=1}^n \bar{M}(t_k \wedge t, t_{k+1} \wedge t) F_k|_{\mathcal{D}'_t} \quad (1.11)$$

The *right stochastic integral* is given by

$$\int_0^t F_s dM_s = \sum_{k=1}^n F_{t_k} M(t_k \wedge t, t_{k+1} \wedge t). \quad (1.12)$$

Remark. From the additivity of M it follows easily that the left and right stochastic integrals are independent of the choice of the representation of F in the form (1.5).

THEOREM 1.11. Let M be a regular additive process and $F \in S$ a simple adapted process. Then $\int_0^t dM_s F_s$ and $\int_0^t F_s dM_s$ as defined above on \mathcal{D}'_t are t -adapted operators. Moreover the mappings

$$F \in S \mapsto \int_0^t dM_s F_s \in \mathcal{L}(\mathcal{D}, \mathcal{H}) \quad \text{and} \quad F \in S \mapsto \int_0^t F_s dM_s \in \mathcal{L}(\mathcal{D}, \mathcal{H})$$

are linear. Finally, if for all $s \leq t$ \mathcal{D}'_t is a core for $M(s, t)^*$, we have that

$$\begin{aligned} \left(\int_0^t dM_s F_s \right)^* \Big|_{\mathcal{D}'_t} &= \int_0^t F_s^+ dM_s^+ \\ \left(\int_0^t F_s dM_s \right)^* \Big|_{\mathcal{D}'_t} &= \int_0^t dM_s^+ F_s^+, \end{aligned}$$

where $F_s^+ = F_s^*|_{\mathcal{D}'_s}$ and $M^+(s, t) = (M(s, t))^*|_{\mathcal{D}'_t}$.

Proof. The result follows straightforwardly from Lemmas 1.9 and 1.10. We omit the details.

2. SEMIMARTINGALES AND INTEGRATORS

We shall denote $\mathcal{S}_{t\downarrow}^0$ the subspace of \mathcal{S} consisting of all simple process F such that

$$F_u = 0 \quad \text{if } u > t. \quad (2.1)$$

The following definition extends in a natural way the notion of semimartingale as introduced in [23] or [30].

DEFINITION 2.1. Let \mathcal{T}_1 be a topology on \mathcal{S} and \mathcal{T}_2 be a topology on $\mathcal{L}(\mathcal{D}; \mathcal{H})$. A regular additive process M is called a $(\mathcal{T}_1, \mathcal{T}_2)$ -semimartingale if for each $t \in \mathbf{R}_+$ the maps

$$F \in \mathcal{S}_{t\downarrow}^0 \rightarrow \int dM_s F_s \in \mathcal{L}(\mathcal{D}; \mathcal{H}) \quad (2.2)$$

$$F \in \mathcal{S}_{t\downarrow}^0 \rightarrow \int F_s dM_s \in \mathcal{L}(\mathcal{D}; \mathcal{H}) \quad (2.3)$$

are continuous with respect to the topologies $\mathcal{T}_1, \mathcal{T}_2$.

For any positive nonatomic measure μ on \mathbf{R}_+ , for any $\xi \in \mathcal{D}$, and for any stochastic process F we denote

$$\|F\|_{\xi, t, \mu}^2 = \int_0^t \|F_s \xi\|^2 d\mu(s). \quad (2.4)$$

The topologies \mathcal{T}_1 on \mathcal{S} one usually considers are induced by seminorms of the form (2.4) while the topologies \mathcal{T}_2 on $\mathcal{L}(\mathcal{D}; \mathcal{H})$ are those given by the strong or weak convergence on \mathcal{D} .

An additive regular process M such that for each $\xi \in \mathcal{D}$, there exists a positive locally finite nonatomic measure μ_ξ on \mathbf{R}_+ such that for each $t \geq 0$ the maps

$$F \in S \rightarrow \int_0^t dM_s F_s \cdot \xi; \quad F^+ \in S \rightarrow \int_0^t F_s^+ dM_s^+ \cdot \xi \quad (2.5)$$

are continuous from S with the $\|\cdot\|_{\xi, t, \mu_\xi}$ -seminorm to \mathcal{H} with the norm topology, has been called a *regular semimartingale* in [5] and there it was shown to include all the examples of "basic integrators" considered up to now in the literature. The regular semimartingale condition is equivalent to the existence, for each $\xi \in \mathcal{D}$ of a positive, locally finite, nonatomic measure μ_ξ and for each $t \geq 0$ of a constant $c_{t, \xi} > 0$ such that, for all elements F of S ,

$$\left\| \int_0^t dM_s F_s \cdot \xi \right\|^2 \leq c_{t, \xi} \cdot \int_0^t \|F_s \xi\|^2 d\mu_\xi(s) \quad (2.6)$$

$$\left\| \int_0^t F_s^+ dM_s^+ \cdot \xi \right\|^2 \leq c_{t, \xi} \cdot F_0^+ \|F_s^+ \xi\|^2 d\mu_\xi(s). \quad (2.7)$$

However, for the purposes of the present paper the class of regular semimartingales is too narrow because of its dependence on the domain \mathcal{D} . For example, the creation process on the Fock space on $L^2(\mathbf{R}_+)$ is a regular semimartingale on the (noninvariant) domain of coherent vectors (cf. [5]), but not on the (invariant) domain of the polynomial of the fields applied to the coherent vectors [8]. On this larger domain it satisfies the following condition: for each $\xi \in \mathcal{D}$ there exists a finite subset $J(\xi) \subseteq \mathcal{D}$ such that for each simple process F and for each $0 < T < +\infty$ one has

$$\left\| \int_0^T dM_r F_r \cdot \xi \right\|^2 \leq c_{T, \xi} \int_0^T d\mu_\xi(r) \sum_{\eta \in J(\xi)} \|F_r \eta\|^2 \quad (2.8)$$

$$\left\| \int_0^T F_r^+ dM_r^+ \cdot \xi \right\|^2 \leq c_{T, \xi} \int_0^T d\mu_\xi(r) \sum_{\eta \in J(\xi)} \|F_r^+ \eta\|^2. \quad (2.9)$$

Moreover, for each $\eta \in J(\xi)$, one has

$$J(\eta) \subseteq J(\xi). \quad (2.10)$$

We can always suppose that $\xi \in J(\xi)$ and, because of (2.9), we can suppose that, for each $\eta \in J(\xi)$, we have also

$$c_{T, \eta} \leq c_{T, \xi}, \quad \mu_\eta \leq \mu_\xi. \quad (2.11)$$

The same condition is satisfied by the annihilation and the number processes on the same space. As discussed in Section 6 if we want that at least the simple processes with respect to a certain family $(M^\alpha)_\alpha$ of basic integrators form an algebra, then the domain \mathcal{D} has to be invariant under the action of all the $M^\alpha(s, t)$. Under this requirement, if we want to include the simplest examples, then the semimartingale inequalities (2.6), (2.7) must be replaced by the more general conditions (2.8), (2.9). These considerations motivate the following:

DEFINITION 2.2. An additive regular process M is called an *integrator of scalar type* if, for each, $\xi \in \mathcal{D}$ there exists a finite set $J(\xi) \subseteq \mathcal{D}$ such that the conditions (2.8), (2.9), (2.10) (hence also (2.11)) are satisfied.

In the following for each $\xi \in \mathcal{D}$ and $t \geq 0$ we shall fix a choice of the non-atomic measure μ_ξ and of the constant $c_{t,\xi} > 0$ so that the inequalities (2.11) are satisfied and, from now on, these symbols will be referred to this choice.

Remarks. (1) In this paper we are interested only in one-sided (left or right) stochastic integrals. The above conditions, however, could be generalized to deal with some two sided stochastic integrals.

(2) In [17] it has been shown that the basic techniques and ideas of the present paper can be applied to include in our theory the stochastic integration with respect to the free noise developed in [32]. The increments of the free noises satisfy a generalized ρ -commutation relation (cf. Definition (6.2)) which implies some slight modifications in the proof of unitarity conditions.

(3) Taking in (2.8), (2.9) the process F to be the characteristic function of an interval $(a, b) \subseteq (0, t)$ times the identity operator, we deduce in particular that

$$\max \{ \|(M_b - M_a)\xi\|^2, \|(M_b^+ - M_a^+)\xi\|^2 \} \leq c_{t,\xi} \mu_\xi(a, b) \sum_{\eta \in J(\xi)} \|\eta\|^2 \quad (2.12)$$

which, because of our assumption of the nonatomicity of μ_ξ , implies that every integrator of scalar type is strongly* continuous on \mathcal{D} .

The conditions (2.8), (2.9) imply that, if $F^{(n)}$ is a sequence in S and F is a stochastic process such that $F^{(n)} \rightarrow F$ in the $\|\cdot\|_{\eta,t,\mu_\xi}$ -seminorm for all $\eta \in J(\xi)$, i.e., for each $\eta \in \mathcal{D}$ and $t \in \mathbf{R}_+$,

$$\lim_{n \rightarrow \infty} \int_0^t \|F_s^{(n)} \eta - F_s \eta\|^2 d\mu_\xi(s) = \lim_{n \rightarrow \infty} \int_0^t \|F_s^{(n)+} \eta - F_s^+ \eta\|^2 d\mu_\xi(s) = 0 \quad (2.13)$$

then the sequences

$$\int_0^t dM_s F_s^{(n)} \xi, \quad \int_0^t F_s^{(n)+} dM_s^+ \xi$$

are Cauchy in \mathcal{H} and their limits are the same for any sequence satisfying (2.13). Thus we can define

$$\int_0^t dM_s F_s \xi := \lim_{n \rightarrow \infty} \int_0^t dM_s F_s^{(n)} \xi; \quad \int_0^t F_s^+ dM_s^+ \xi := \lim_{n \rightarrow \infty} \int_0^t F_s^{(n)+} dM_s^+ \xi.$$

Moreover we have

$$\begin{aligned} \left\| \int_0^t dM_s F_s \xi \right\|^2 &\leq c_{t,\xi} \int_0^t \sum_{\eta \in J(\xi)} \|F_s \xi\|^2 d\mu_\eta(s) \\ \left\| \int_0^t F_s^+ dM_s^+ \xi \right\|^2 &\leq c_{t,\xi} \int_0^t \sum_{\eta \in J(\xi)} \|F_s^+ \xi\|^2 d\mu_\eta(s). \end{aligned} \quad (2.14)$$

Let M be an integrator of scalar type and let us denote $L_{\text{loc}}^2(\mathbf{R}_+; dM)$ the space of all the adapted processes F with the topology given by the seminorms $\|\cdot\|_{\eta,t,\mu_\xi}$ such that, for all $\xi \in \mathcal{D}$, $\eta \in J(\xi)$ and all $0 \leq t < +\infty$

$$\int_0^t (\|F_s \eta\|^2 + \|F_s^+ \eta\|^2) d\mu_\xi(s) < \infty, \quad (2.15)$$

where μ_ξ is as in Definition (2.2).

THEOREM 2.3. *Let M be a integrator of scalar type and suppose that, for any $\xi \in \mathcal{D}$, the measures μ_ξ in the inequalities (2.8), (2.9) are absolutely continuous with respect to the Lebesgue measure. Then the stochastic integral with respect to M can be extended by continuity from S to $L_{\text{loc}}^2(\mathbf{R}_+, dM)$ and the inequalities (2.14) hold.*

Proof. We have only to show that S is dense in $L_{\text{loc}}^2(\mathbf{R}_+; dM)$ for the topology given by the seminorms (2.15). Suppose first that F and F^+ are strongly continuous on \mathcal{D} and consider then the sequence of elements of S

$$F_t^{(n)} = \sum_k \chi_{[k/n, (k+1)/n)}(t) F_{k/n}.$$

Then $F^{(n)}$ converges to F in $L_{\text{loc}}^2(\mathbf{R}_+; dM)$. In fact for all $\xi \in \mathcal{D}$, all $t \geq 0$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that, if

$$|r - s| < \delta; \quad 0 \leq r, s \leq t$$

then

$$\|F_r \xi - F_s \xi\| < \varepsilon/t, \quad \|F_r^+ \xi - F_s^+ \xi\| < \varepsilon/t.$$

Thus for all n such that $1/n < \delta$ we have

$$\|F_s^{(n)} \xi - F_s \xi\| < \varepsilon/t, \quad \|F_s^{(n)+} \xi - F_s^+ \xi\| < \varepsilon/t$$

from which the required convergence property easily follows. Now let $(F_t)_{t \geq 0}$ be an adapted process and let $(\phi_n)_{n \geq 1}$ be the sequence of positive measurable functions

$$\phi_n(t) = n\chi_{[0, 1/n)}(t).$$

Let us consider the processes

$$F_s^{(n)} = \int_0^s \phi_n(u) F_{s-u} du$$

which is strongly continuous on \mathcal{D} and adapted. Then, for all $\eta \in J(\xi)$,

$$\|F_s^{(n)} \eta - F_s \eta\|^2 = \left\| \int_0^t \phi_n(u) [F_{s-u} \eta - F_s \eta] du \right\|^2$$

and therefore

$$\begin{aligned} \int_0^t \|F_s^{(n)} \eta - F_s \eta\|^2 d\mu_\xi(s) &\leq \int_0^t d\mu_\xi(s) \int_0^s \phi_n(s-u) \|F_u \eta - F_s \eta\|^2 du \\ &= \int_0^t d\mu_\xi(s) \int_0^s \phi_n(v) \|F_{s-v} \eta - F_s \eta\|^2 dv \end{aligned}$$

by Fubini's theorem this is equal to

$$\begin{aligned} &= \int_0^t dv \phi_n(v) \int_u^t \|F_{s-u} \eta - F_s \eta\|^2 d\mu_\xi(s) \\ &= n \int_0^{1/n} du \int_u^t \|F_{s-u} \eta - F_s \eta\|^2 d\mu_\xi(s) \\ &\leq n \int_0^{1/n} du \int_0^t \|F_{s-u} \xi - F_s \xi\|^2 d\mu_\xi(s) \end{aligned}$$

which tends to zero as $n \rightarrow \infty$ by the absolute continuity of μ_ξ . Since F is arbitrary, the same argument applies to F^+ , and this ends the proof. ■

Note that the proof of the fact that any process F , strongly continuous on \mathcal{D} , is in $L^2(\mathbf{R}_+, dM)$ and is integrable with respect to M does not depend on the absolute continuity of μ_ξ with respect to the Lebesgue measure. Moreover, for $F \in L^2(\mathbf{R}_+; dM)$, we have defined by the above approximation method right and left stochastic integrals as operators on \mathcal{D}' ,

PROPOSITION 2.4. *Let M be as in Theorem (2.3). For all $F \in L^2_{\text{loc}}(\mathbf{R}_+; dM)$*

(i) *the maps $(s, t) \rightarrow \int_s^t dM_r F_r; (s, t) \rightarrow \int_s^t F_r dM_r$ are additive adapted processes strongly continuous on \mathcal{D} .*

(ii) *If $F, G \in L^2_{\text{loc}}(\mathbf{R}, dM)$ then $F + G \in L^2_{\text{loc}}(\mathbf{R}, dM)$ and*

$$\begin{aligned} \int_0^t dM_s (F_s + G_s) &= \int_0^t dM_s F_s + \int_0^t dM_s G_s \\ \int_0^t (F_s + G_s) dM_s &= \int_0^t F_s dM_s + \int_0^t G_s dM_s \end{aligned}$$

on the domain \mathcal{D}' .

Proof. It is easy to see, from the definition of regular additive process that (i) and (ii) hold for all $F \in S$. The extension to $F \in L^2_{\text{loc}}(\mathbf{R}; dM)$ follows approximating F with a sequence of elements of S .

It is clear from the definition that any finite linear combination of integrators is still an integrator. A scalar, locally bounded, additive process (i.e., a process of the form $M(s, t) = \sigma(s, t) \cdot 1$, with σ a positive, real valued, locally bounded measure) is clearly an integrator. If $F \in L^2_{\text{loc}}(\mathbf{R}_+; dM)$, its stochastic integral might not be an integrator (not even a regular process). The following is a simple and easily applicable sufficient condition which assures that an additive process is an integrator of scalar type.

PROPOSITION 2.5. *Let M be a regular additive process with the following properties:*

(i) *for all $\xi, \eta \in \mathcal{D}$ there exists finite subsets $J(\xi, \eta), J(\xi)$ of \mathcal{D} such that $J(\xi, \xi) \subseteq J(\xi)$ and, for all $\xi_1, \xi_2 \in J(\xi, \eta)$ we have*

$$J(\xi_1) \subseteq J(\xi), \quad J(\xi_2) \subseteq J(\eta) \quad (2.16)$$

(ii) *for all $\xi, \eta \in \mathcal{D}$ and all $\xi_1, \xi_2 \in J(\xi, \eta)$ there exist locally bounded non atomic measures ν_{ξ_1, ξ_2} and σ_{ξ_1, ξ_2} on \mathbf{R}_+ such that, for all $0 \leq s < t$ and any pair of s -adapted operators F, G one has*

$$|\langle M(s, t) F \xi_1, G \xi_2 \rangle| \leq \sum_{\xi_1, \xi_2 \in J(\xi, \eta)} \nu_{\xi_1, \xi_2}(s, t) |\langle F \xi_1, G \xi_2 \rangle| \quad (2.17)$$

$$\begin{aligned} &\max\{|\langle M(s, t) F \xi_1, M(s, t) G \xi_2 \rangle|, |\langle FM^+(s, t) \xi_1, GM(s, t)^+ \xi_2 \rangle|\} \\ &\leq \sum_{\xi_1, \xi_2 \in J(\xi, \eta)} \sigma_{\xi_1, \xi_2}(s, t) |\langle F \xi_1, G \xi_2 \rangle|. \end{aligned} \quad (2.18)$$

Then M is an integrator of scalar type.

Remark. It will be clear from Section 3 that a sufficient condition for (2.17) is that M has a scalar forward derivative and for (2.18) that M has a scalar conditional variance on \mathcal{D} . This justifies the term “integrator of scalar type.”

Proof. We only have to check the semimartingale inequalities (2.8), (2.9). To this goal we introduce a notation, which shall be used throughout the paper. If t and dt are positive numbers, for any additive process M , we write

$$dM(t) := dM_t := M(t, t + dt). \quad (2.19)$$

With these notations, if F_1, F_2 are simple processes, for each $t, dt \geq 0$, with dt small enough so that both F_1, F_2 are constant in any interval of width dt not containing points of discontinuity, we define

$$dN_j(t) := dM(t) F_j(t) = M(t, t + dt) F_j(t); \quad j = 1, 2.$$

Then, for any pair of vectors $\xi_1, \xi_2 \in J(\xi, \eta)$, we have the identity

$$\begin{aligned} d\langle N_1(t) \xi_1, N_2(t) \xi_2 \rangle &= \langle dN_1(t) \xi_1, N_2(t) \xi_2 \rangle + \langle N_1(t) \xi_1, dN_2(t) \xi_2 \rangle \\ &\quad + \langle dN_1(t) \xi_1, dN_2(t) \xi_2 \rangle \\ &= \langle dM_t F_1(t) \xi_1, N_2(t) \xi_2 \rangle + \langle N_1(t) \xi_1, dM_t F_2(t) \xi_2 \rangle \\ &\quad + \langle dM_t F_1(t) \xi_1, dM_t F_2(t) \xi_2 \rangle. \end{aligned} \quad (2.20)$$

Applying (2.17) and (2.18) to (2.20) we obtain

$$\begin{aligned} |d\langle N_1(t) \xi_1, N_2(t) \xi_2 \rangle| &\leq \sum_{\xi_a, \xi_b \in J(\xi_1, \xi_2)} [dv_{\xi_a, \xi_b}(t) (|\langle F_1(t) \xi_a, N_2(t) \xi_b \rangle| \\ &\quad + |\langle N_1(t) \xi_a, F_2(t) \xi_b \rangle|) \\ &\quad + d\sigma_{\xi_a, \xi_b}(t) |\langle F_1(t) \xi_a, F_2(t) \xi_b \rangle|]. \end{aligned} \quad (2.21)$$

Choosing $\xi_1 = \xi_2 \in J(\xi)$, and $F_1 = F_2 = F$, so that $N_1 = N_2 = N$, (2.21) becomes

$$d\|N(t) \xi_1\|^2 \leq \sum_{\xi_a, \xi_b \in J(\xi_1)} d\mu_\xi(t) (|\langle F(t) \xi_a, N(t) \xi_b \rangle| + \|F(t) \xi_a\|^2)$$

with

$$\mu_\xi = 2 \sum_{\xi_a, \xi_b \in J(\xi)} v_{\xi_a, \xi_b} + \sigma_{\xi_a, \xi_b}.$$

Summing on $\xi_1 \in J(\xi)$ this inequality and denoting

$$\varphi(t) := \left(\sum_{\xi_a \in J(\xi)} \|N(t) \xi_a\|^2 \right)^{1/2}, \quad \theta(t) := \left(\sum_{\xi_a \in J(\xi)} \|F(s) \xi_a\|^2 \right)^{1/2}$$

one obtains the inequality

$$\varphi^2(t) \leq c \int_0^t d\mu_\xi(s) \varphi(s) \theta(s) + c \int_0^t d\mu_\xi(s) \theta^2(s),$$

where $c = c(\xi)$ is an easily estimated constant. From this the following estimate on $\tilde{\varphi}(t) := \sup_{s \in [0, t]} \varphi(s)$,

$$\begin{aligned} \tilde{\varphi}^2(t) &\leq c \tilde{\varphi}(t) \cdot \int_0^t d\mu_\xi(s) \theta(s) + c \int_0^t d\mu_\xi(s) \theta^2(s) \\ &\leq \frac{1}{2} \tilde{\varphi}^2(t) + \frac{1}{2} c^2 \left(\int_0^t d\mu_\xi(s) \theta(s) \right) + c \int_0^t d\mu_\xi(s) \theta^2(s), \end{aligned}$$

i.e.,

$$\tilde{\varphi}^2(t) \leq (c^2 \mu_\xi(0, t) + 2c) \int_0^t d\mu_\xi(s) \theta^2(s).$$

This gives the inequality (2.7). In a similar way one proves (2.8). ■

PROPOSITION 2.6. *Let M be an integrator of scalar type and let f be a continuous function. Then*

$$N_t := \int_0^t f(s) dM_s$$

is an integrator of scalar type. Moreover, for all $\xi \in \mathcal{D}$

$$J^N(\xi) = J^M(\xi); \quad \mu_\xi^N = \mu_\xi^M; \quad c_{t, \xi}^N = \|f\chi_{(0, t)}\|_\infty. \quad (2.22)$$

Proof. If $f = \sum_j f(t_j) \chi_{(t_j, t_{j+1})}$ is a step function and F a simple adapted process, then

$$\begin{aligned} \left\| \int_0^t dN_r F_r \xi \right\|^2 &= \left\| \sum_j \int_{t_j}^{t_{j+1}} dM_r f(t_j) F_r \xi \right\|^2 = \left\| \int_0^t dM_r (f(r) F_r) \xi \right\|^2 \\ &\leq c_{t, \xi} \int_0^t d\mu_\xi(s) \sum_{\eta \in J(\xi)} \|f(s) F_s \eta\|^2 \\ &\leq c_{t, \xi} \|f\chi_{(0, t)}\|_\infty \int_0^t d\mu_\xi(s) \sum_{\eta \in J(\xi)} \|F_s \eta\|^2. \end{aligned}$$

The inequality (2.14) allows to complete the proof by density arguments. In a similar way one proves the inequality for the adjoint process N^+ .

3. FORWARD DERIVATIVES AND MARTINGALES

An important case when condition (2.17) of Lemma (2.5) is satisfied is when the process associated to M admits a scalar forward derivative.

DEFINITION 3.1. A regular additive process M admits a (locally bounded) *forward derivative* if for any $\xi_1, \xi_2 \in \mathcal{D}$ there exist a positive locally bounded nonatomic measure ϱ_{ξ_1, ξ_2} such that, for all adapted processes F, G and for each $s \geq 0$ the complex valued measure

$$(a, b) \subseteq [s, \infty) \rightarrow \langle F_s \xi, M(a, b) G_s \xi \rangle \quad (3.1)$$

is absolutely continuous with respect to ϱ_{ξ_1, ξ_2} for all $s \in R_+$ and an adapted bounded operator valued process $D_+^{\xi_1, \xi_2} M$ such that

$$\lim_{\varepsilon \downarrow 0} \frac{\langle F_t \xi_1, M(t, t + \varepsilon) G_t \xi_2 \rangle}{\varrho_{\xi_1, \xi_2}(t, t + \varepsilon)} = \langle F_t \xi_1, D_+^{\xi_1, \xi_2} M(t) G_t \xi_2 \rangle. \quad (3.2)$$

If $\xi_1 = \xi_2 = \xi$, we simply write $D_+^\xi M(t)$. The additive adapted process $\int_s^t D_+^{\xi_1, \xi_2} M(r) d\varrho_{\xi_1, \xi_2}(r)$, where the integral is meant weakly on the domain of all the vectors of the form $F_s \xi$ for some adapted process F and some $\xi \in \mathcal{D}$, is also called the (ξ_1, ξ_2) -*drift* (or the (ξ_1, ξ_2) -*bounded variation part*) of M .

The fundamental theorem of calculus states that, if a function f has a continuous derivative then the increments of the function f are the integrals of its derivative. The fundamental theorem of stochastic calculus states that, if a process F has a continuous forward derivative, then the increments of F are the integrals of its forward derivative plus a martingale.

DEFINITION 3.2. A regular additive process M is called a (ξ_1, ξ_2) -*martingale* if, for all $0 \leq s \leq t \leq u < +\infty$, and all adapted processes F, G one has

$$\langle \bar{M}(t, u) F_s \xi_1, G_s \xi_2 \rangle = 0. \quad (3.3)$$

If $\xi_1 = \xi_2 = \xi$, then we call M a ξ -martingale.

THEOREM 3.3. Let M be a regular additive process such that for all $\xi_1, \xi_2 \in \mathcal{D}$ and for all $t \geq 0$ the forward derivative $D_+^{\xi_1, \xi_2} M(t)$ exists and is

strongly continuous on \mathcal{D} . Then there exists a (ξ_1, ξ_2) -martingale M^{ξ_1, ξ_2} such that

$$M(s, t) = \int_s^t D_+^{\xi_1, \xi_2} M(r) d\rho_{\xi_1, \xi_2} + M^{\xi_1, \xi_2}(s, t). \quad (3.4)$$

Proof. Let F, G be adapted processes and $0 \leq s \leq t$. The real measures

$$(a, b) \rightarrow \operatorname{Re} \langle F_s \xi_1, \bar{M}(a, b) G_s \xi_2 \rangle, \quad (a, b) \rightarrow \operatorname{Im} \langle F_s \xi_1, \bar{M}(a, b) G_s \xi_2 \rangle$$

are absolutely continuous with respect to ρ_{ξ_1, ξ_2} . By [23, Theorem (4.42)] the existence of the limit (3.2) implies that for any interval $[s, t] \subseteq \mathbf{R}$ one has

$$\begin{aligned} \langle \bar{M}(s, t) F_s \xi_1, G_s \xi_2 \rangle &= \int_s^t \langle D_+^{\xi_1, \xi_2} M(r) F_s \xi_1, G_s \xi_2 \rangle d\rho_{\xi_1, \xi_2}(r) \\ &= \left\langle \int_s^t D_+^{\xi_1, \xi_2} M(r) d\rho_{\xi_1, \xi_2}(r) F_s \xi_1, G_s \xi_2 \right\rangle. \end{aligned}$$

Therefore the additive process

$$M^{\xi_1, \xi_2}(s, t) := M(s, t) - \int_s^t D_+^{\xi_1, \xi_2} M(r) d\rho_{\xi_1, \xi_2}(r)$$

has the desired properties.

PROPOSITION 3.4. *Let $\xi_1, \xi_2 \in \mathcal{D}$ and let M be a (ξ_1, ξ_2) -martingale. Then for all $s > 0$ and for all adapted processes F, G the maps*

$$\begin{aligned} (t, u] \subseteq (s, +\infty] &\rightarrow \langle \bar{M}(t, u) F_s \xi_1, \bar{M}(t, u) G_s \xi_2 \rangle \\ (t, u] \subseteq (s, +\infty] &\rightarrow \langle F_s M^+(t, u) \xi_1, G_s M^+(t, u) \xi_2 \rangle \end{aligned} \quad (3.5)$$

are complex measures. If M, M^+ are strongly continuous, then these measures are nonatomic.

Proof. Clear.

DEFINITION 3.5. If there exists an additive adapted process denoted by $\langle\langle M^+, M \rangle\rangle$ and a *-linear map ρ mapping adapted processes into themselves such that, for all $\xi_1, \xi_2 \in \mathcal{D}$ and all $s < t < u$

$$\langle \bar{M}(t, u) F_s \xi_1, \bar{M}(t, u) G_s \xi_2 \rangle = \langle F_s \xi_1, \langle\langle M^+, M \rangle\rangle(t, u) G_s \xi_2 \rangle$$

and

$$\langle F_s M^+(t, u) \xi_1, G_s M^+(t, u) \xi_2 \rangle = \langle \rho(F_s) \xi_1, \langle\langle M^+, M \rangle\rangle^*(t, u) \rho(G_s) \xi_2 \rangle.$$

The additive process $\langle\langle M^+, M \rangle\rangle$ is called the *oblique* (or *Watanabe*) *bracket* of M or simply the *conditional covariance* of M . If the oblique bracket of M is a scalar process, then M is said to have a *scalar conditional variance*.

Remark. Note that, if M has a scalar conditional variance σ_{ξ_1, ξ_2} then the inequality (2.18) is satisfied.

COROLLARY 3.6. *Any additive process with a continuous forward derivative and a locally bounded conditional variance is an integrator in the sense of Definition (1.3).*

Proof. Such a process satisfies all the conditions of Proposition (2.5).

Additive processes with scalar forward derivatives on \mathcal{D} , i.e., for which, for any $\xi_1, \xi_2 \in \mathcal{D}$

$$\frac{D^{\xi_1, \xi_2} M(t)}{D\rho_{\xi_1, \xi_2}} = \varphi_{\xi_1, \xi_2}(t) \quad (3.6)$$

for some function φ_{ξ_1, ξ_2} in $L^1_{\text{loc}}(\mathbf{R}, d\rho_{\xi_1, \xi_2})$ are important for the applications. Denoting ν_{ξ_1, ξ_2} the measure

$$\nu_{\xi_1, \xi_2}(s, t) = \int_s^t \varphi_{\xi_1, \xi_2}(r) d\rho_{\xi_1, \xi_2}(r) \quad (3.7)$$

the relation (3.4) implies that

$$\langle F_s \xi_1, M(a, b) G_s \xi_2 \rangle = \nu_{\xi_1, \xi_2}(s, t) \langle F_s \xi_1, G_s \xi_2 \rangle \quad (3.8)$$

for all adapted processes F, G , strongly continuous on \mathcal{D} . From this it follows that any additive process M with a scalar forward derivative enjoys the following property:

For all $\xi, \eta \in \mathcal{D}$ there exists a measure $\nu_{\xi, \eta}$ on \mathbf{R}_+ finite on bounded intervals such that, for all adapted processes A, B, C, D strongly continuous on \mathcal{D} and all $s < t$

$$\begin{aligned} & |\langle A(s) \xi, \bar{M}(s, t) B(s) \eta \rangle - \langle C(s) \xi, \bar{M}(s, t) D(s) \eta \rangle| \\ & \leq \nu_{\xi, \eta}(s, t) |\langle A(s) \xi, B(s) \eta \rangle - \langle C(s) \xi, D(s) \eta \rangle|. \end{aligned} \quad (3.9)$$

Condition (3.9) will be called the *scalar forward derivative inequality*. It will play an important role in the proof of the unitarity conditions (cf. Theorem 9.2).

4. THE $o(dt)$ -NOTATION

Let \mathbf{R}_{\leq}^2 denote the set

$$\mathbf{R}_{\leq}^2 := \{(s, t) \in \mathbf{R}^2 : s \leq t\} \quad (4.1)$$

DEFINITION 4.1. Let $\varphi: \mathbf{R}_{\leq}^2 \rightarrow \mathbf{C}$ be a function. We say that φ is of order $o(dt)$ if for every bounded interval (s, t) in \mathbf{R}_+

$$\lim_{|P(s, t)| \rightarrow 0} \sum_{s=t_1 < \dots < t_k = t} \varphi(t_j, t_{j+1}) = 0. \quad (4.2)$$

If $\varphi, \psi: \mathbf{R}_{\leq}^2 \rightarrow \mathbf{C}$ are two functions such that $\varphi - \psi$ is of order $o(dt)$, we write

$$d\varphi \equiv d\psi \quad (4.3)$$

or also

$$\varphi(s, t) \equiv \psi(s, t).$$

A map $\xi: \mathbf{R}_+ \rightarrow \mathcal{H}$ is said to be *adapted* if there exists $\eta \in \mathcal{D}$ such that $\xi(s) \in \mathcal{H}_{s \downarrow}(\eta)$ for all $s \in \mathbf{R}_+$.

Let $\mathcal{F}_1, \mathcal{F}_2$ be two families of \mathcal{H} -valued adapted functions and let $F: \mathbf{R}_{\leq}^2 \rightarrow \mathcal{L}(\mathcal{D}, \mathcal{H})$ be a strongly measurable map. We say that F is of order $o(dt)$ weakly on $(\mathcal{F}_1, \mathcal{F}_2)$ (respectively strongly on \mathcal{F}_1) if for each $\xi_1 \in \mathcal{F}_1, \xi_2 \in \mathcal{F}_2$, the scalar map

$$(s, t) \mapsto \langle \xi_1(s), F(s, t) \xi_2(s) \rangle \quad \text{respectively } (s, t) \mapsto \|F(s, t) \xi_1(s)\|$$

is well defined and is of order $o(dt)$ in the sense of Definition (4.1)

We shall denote \mathcal{PD} the family of \mathcal{H} -valued functions

$$\mathcal{PD} := \{s \mapsto A_s \xi : A \text{ is an adapted strongly continuous process; } \xi \in \mathcal{D}\}, \quad (4.4)$$

\mathcal{D} itself is identified in the obvious way to a subset of \mathcal{PD} . If $F, G: \mathbf{R}_{\leq}^2 \rightarrow \mathcal{L}(\mathcal{D}, \mathcal{H})$ are maps such that $F - G$ is of order $o(dt)$ weakly on $(\mathcal{F}_1, \mathcal{F}_2)$ (resp. strongly on \mathcal{F}_1). We write

$$dF \equiv dG \quad \text{weakly on } (\mathcal{F}_1, \mathcal{F}_2) \text{ (resp. strongly on } \mathcal{F}_1). \quad (4.5)$$

Sometimes we also shall use the notation

$$F(s, t) \equiv G(s, t). \quad (4.6)$$

PROPOSITION 4.2. *Suppose that M is an integrator of scalar type. Then, defining for any adapted process F , strongly continuous on \mathcal{D}*

$$Y_t = Y_0 + \int_0^t dM_s F_s \quad (4.7)$$

one has

$$dY_t \equiv dM_t F_t \quad \text{strongly on } \mathcal{D}. \quad (4.8)$$

Proof. For any $0 < t, dt \leq T < +\infty$ using the integrator of scalar type inequality one has

$$\begin{aligned} & \left\| \int_t^{t+dt} dM_s F_s \xi - dM_t F_t \xi \right\|^2 \\ &= \left\| \int_t^{t+dt} dM_s [F_s - F_t] \xi \right\|^2 \\ &\leq c_{T,\xi} \sum_{\eta \in J(\xi)} \int_t^{t+dt} d\mu_\xi(s) \| [F_s - F_t] \eta \|^2 \\ &\leq c_{T,\xi} |J(\xi)| \sup_{s \in [t, t+dt]} \sup_{\eta \in J(\xi)} \| [F_s - F_t] \eta \|^2 \mu_\xi(t, t+dt) \end{aligned}$$

and the result follows from the strong continuity of F on \mathcal{D} .

5. STOCHASTIC DIFFERENTIAL EQUATIONS

Our goal in this section is to prove an existence and uniqueness theorem for stochastic differential equations sufficiently powerful:

- (i) to include all the known existence and uniqueness theorems for quantum stochastic differential equations with bounded coefficients,
- (ii) to prove an existence uniqueness and unitarity theorem in our more general framework,
- (iii) to include some of the equations deduced in [3, 4].

Some of our proofs are based on essentially new techniques which have led us to consider equations which are of more general type than those considered up to now (cf., for example, the proof of the unitarity condition in Theorem 9.2).

In this section I denotes a finite set, $|I|$ the number of its elements and $(M^{\alpha\beta})_{\alpha\beta \in I \times I}$ a set of integrators of scalar type. Summation over repeated greek indices will be understood. The coefficients of the stochastic differential equations we consider come from families $(F_\alpha(t))_{t \in \mathbf{R}_+, \alpha \in I}$,

$(G_\beta(t))_{t \in \mathbf{R}_+, \beta \in I}$ of locally bounded adapted processes, leaving the domain \mathcal{D} invariant, with the following property: for all $\alpha, \beta \in I$, and all adapted process H , integrable with respect to the $M^{\alpha\beta}$, and all continuous functions u, v on \mathbf{R}_+ satisfying $u(s) \leq s, v(s) \leq s$ for all $s \in \mathbf{R}_+$, the family of operators on \mathcal{H}

$$s \mapsto F_\alpha(u(s)) H(s) G_\beta(v(s)) \quad (5.1)$$

is an adapted process integrable with respect to $M^{\alpha\beta}$. This is a technical assumption we need in order for our stochastic differential equations to make sense. It is verified in all the examples considered up to now (for example, when $F_\alpha(t) = F_\alpha f_\alpha(t)$ ($t \in \mathbf{R}_+$) $G_\beta(t) = G_\beta g_\beta(t)$ ($t \in \mathbf{R}_+$), where F_α, G_β are constant operators in \mathcal{A}_{01} leaving the domain \mathcal{D} invariant and f_α, g_β bounded measurable functions).

Let $K_T = \sup_{s \in [0, T]} \max_{\alpha, \beta \in I} \|F_\alpha(s)\|_\infty^2$. We want to solve the stochastic differential equation

$$Y(t) = Y_0 + \int_0^t dM^{\alpha\beta}(s) F_\alpha(s) Y(s) G_\beta(s),$$

where Y_0 is an element of \mathcal{A}_{01} ; we will use also the notation

$$\begin{aligned} dY(t) &= dM^{\alpha\beta}(t) F_\alpha(t) Y(t) G_\beta(t) \\ Y(0) &= Y_0. \end{aligned} \quad (5.2)$$

For all $\xi \in \mathcal{D}$ let

$$c_{t, \xi} = \max_{\alpha\beta \in I \times I, \eta \in J(\xi)} c_{t, \alpha\beta, \eta}, \quad \mu_\xi = \sum_{\alpha\beta \in I \times I, \eta \in J(\xi)} \mu_\eta^{\alpha\beta}, \quad J(\xi) = \bigcup_{\alpha\beta \in I \times I} J_{\alpha\beta}(\xi),$$

where $c_{t, \alpha\beta, \xi}$, $\mu_\xi^{\alpha\beta}$ and $J_{\alpha\beta}(\xi)$ are the constant, the measure, and the subset of \mathcal{D} corresponding to $M^{\alpha\beta}$ in Definition 2.2.

THEOREM 5.1. *If the $M^{\alpha\beta}$ are integrators of scalar type and the processes F_α, G_β satisfy (5.1), then, for all $Y_0 \in \mathcal{A}_{01}$, there exists a solution of (5.2).*

Proof. Define by induction $Y^{(0)}(t) = Y_0$, $Y^{(0)+}(t) = Y_0^+$, and

$$Y^{(n+1)}(t) = \int_0^t dM_s^{\alpha\beta} F_\alpha(s) Y^{(n)}(s) G_\beta(s).$$

The sequence is well defined. In fact $Y^{(0)}$ is an additive adapted process strongly continuous on \mathcal{D} . Suppose $Y^{(n)}$ is an additive adapted process. Because of (5.1), for all $\alpha, \beta \in I$, $(F_\alpha(t) Y^{(n)}(t) G_\beta(t))_{t \in \mathbf{R}_+}$ is an adapted process integrable with respect to $M^{\alpha\beta}$, then by Proposition 2.4 (i), $Y^{(n+1)}$ is an additive adapted process strongly continuous on \mathcal{D} . It follows by

induction that $Y^{(n)}$ is an additive adapted process strongly continuous on \mathcal{D} for all $n \in \mathbb{N}$.

We will prove now the basic estimate

$$\|Y^{(n)}(t) \xi\|^2 \leq \max_{\eta \in J(\xi)} \|\eta\|^2 \cdot \|Y_0\|^2 \cdot |J(\xi)|^n K_T^n c_{T,\xi}^n (\mu_\xi(0, t))^n \frac{1}{n!}. \quad (5.3)$$

For all $n \in \mathbb{N}$ and all $t \leq T$, using the integrator of scalar type inequality (2.8), we have

$$\begin{aligned} \|Y^{(n)}(t) \xi\|^2 &= \left\| \int_0^t dM_{t_1}^{\alpha\beta} F_\alpha(t_1) Y^{(n-1)}(t_1) G_\beta(t_1) \xi \right\|^2 \\ &\leq c_{T,\xi} K_T \sum_{\xi_1 \in J(\xi)} \int_0^t \mu_\xi(dt_1) \sup_{\beta_1 \in I} \|Y^{(n-1)}(t_1) G_{\beta_1}(t_1) \xi_1\|^2 \\ &= c_{T,\xi} K_T \sum_{\xi_1 \in J(\xi)} \int_0^t \mu_\xi(dt_1) \\ &\quad \times \sup_{\beta_1 \in I} \left\| \left(\int_0^{t_1} dM_{t_2}^{\alpha_2\beta_2} F_{\alpha_2}(t_2) Y^{(n-2)}(t_2) G_{\beta_2}(t_2) \right) G_{\beta_1}(t_1) \xi_1 \right\|^2 \end{aligned}$$

because of (5.1) and the fact that $G_{\beta_1}(t_1)$ leaves \mathcal{D} invariant for all $t_1 \in [0, T]$

$$\begin{aligned} &= c_{T,\xi} K_T \sum_{\xi_1 \in J(\xi)} \int_0^t \mu_\xi(dt_1) \\ &\quad \times \sup_{\beta_1 \in I} \left\| \int_0^{t_1} dM_{t_2}^{\alpha_2\beta_2} (F_{\alpha_2}(t_2) Y^{(n-2)}(t_2) G_{\beta_2}(t_2) G_{\beta_1}(t_1)) \xi_1 \right\|^2. \end{aligned}$$

Using again the inequality (2.8)

$$\begin{aligned} &\leq c_{T,\xi}^2 K_T^2 |J(\xi)| \sum_{\xi_2 \in J(\xi)} \int_0^t \mu_\xi(dt_1) \int_0^{t_1} \mu_\xi(dt_2) \\ &\quad \times \sup_{\beta_1, \beta_2 \in I} \|Y^{(n-2)}(t_2) G_{\beta_2}(t_2) G_{\beta_1}(t_1) \xi_2\|^2. \end{aligned}$$

An n -fold iteration of the same arguments gives us the estimate

$$\begin{aligned} \|Y^{(n)}(t) \xi\|^2 &\leq c_{T,\xi}^n K_T^n |J(\xi)|^{n-1} \sum_{\xi_n \in J(\xi)} \int_0^t \mu_\xi(dt_1) \int_0^{t_1} \mu_\xi(dt_2) \dots \int_0^{t_{n-1}} \mu_\xi(dt_n) \\ &\quad \sup_{\beta_1, \beta_2, \dots, \beta_n \in I} \|Y_0 G_{\beta_n}(t_n) \dots G_{\beta_2}(t_2) G_{\beta_1}(t_1) \xi_n\|^2 \\ &\leq \max_{\eta \in J(\xi)} \|\eta\|^2 \cdot \|Y_0\|^2 \cdot |J(\xi)|^n K_T^n c_{T,\xi}^n (\mu_\xi(0, t))^n \frac{1}{n!}. \quad (5.4) \end{aligned}$$

The same estimate holds for $\|Y^{(n)+}(t)\xi\|^2$; therefore the series $\sum_{n=0}^{\infty} Y^{(n)}$, $\sum_{n=0}^{\infty} Y^{(n)+}$ converge in the strong topology on \mathscr{D} uniformly on bounded intervals of \mathbf{R}_+ . This implies that they define adapted processes Y , Y^+ strongly continuous on \mathscr{D} . We show now that Y is a solution of (5.2). By the integrator of scalar type estimate we have, for all $n \in \mathbf{N}$,

$$\begin{aligned} & \left\| \int_0^t dM_s^{\alpha\beta} F_\alpha(s) Y(s) G_\beta(s) \xi - \int_0^t dM_s^{\alpha\beta} F_\alpha(s) \sum_{k=0}^n Y^{(k)}(s) G_\beta(s) \xi \right\|^2 \\ & \leq c_{T,\xi} K_T \sum_{\eta \in J(\xi)} \int_0^t \sum_{k=n+1}^{\infty} \sup_{\beta \in I} \|Y^{(k)}(s) G_\beta(s) \eta\|^2 \mu_\xi(ds). \end{aligned} \quad (5.5)$$

The same argument that led us to (5.4) can be used to prove that, for all $s \in [0, T]$, $k \in \mathbf{N}$, $\beta \in I$, $\eta \in J(\xi)$

$$\|Y^{(k)}(s) G_\beta(s) \eta\|^2 \leq \max_{\eta \in J(\xi)} \|\eta\|^2 \cdot \|Y_0\|^2 \cdot |J(\xi)|^k K_T^{k+1} c_{T,\xi}^k (\mu_\xi(0, T))^k \frac{1}{k!}.$$

Then the series $\sum_{k=0}^{\infty} \sup_{\beta \in I} \|Y^{(k)}(s) G_\beta(s) \eta\|^2$ converges in the strong topology on \mathscr{D} uniformly on bounded intervals of \mathbf{R}_+ . It follows from Lebesgue's theorem that the right-hand side of (5.5) converges to zero as n goes to infinity and therefore

$$\lim_{n \rightarrow \infty} \int_0^t dM_s^{\alpha\beta} \left(F_\alpha(s) \sum_{k=0}^n Y(s) G_\beta(s) \right) = \int_0^t dM_s^{\alpha\beta} F_\alpha(s) Y(s) G_\beta(s)$$

strongly on \mathscr{D} . This, together with the identity

$$\sum_{k=0}^{n+1} Y^{(k)}(t) \xi = Y_0 \xi + \int_0^t dM_s^{\alpha\beta} F_\alpha(s) \sum_{k=0}^n Y^{(k)}(s) G_\beta(s) \xi$$

implies that Y verifies the stochastic differential equation (5.2).

The following propositions give two uniqueness results.

PROPOSITION 5.2. *The stochastic differential equation (5.2) has a unique locally bounded solution.*

Proof. Clearly it will be sufficient to prove that all bounded processes $(Z(t))_{t \in \mathbf{R}_+}$ satisfying the stochastic differential equation

$$Z(t) = \int_0^t dM_s^{\alpha\beta} F_\alpha(s) Z(s) G_\beta(s)$$

must be zero. In fact, for all $\xi \in \mathcal{D}$, applying the integrator of scalar type inequality (2.8) we have

$$\|Z(t) \xi\|^2 \leq c_{T,\xi} K_T \sum_{\eta \in J(\xi)} \int_0^t \mu_\xi(ds) \sup_{\beta \in I} \|Z(s) G_\beta(s) \eta\|^2$$

applying again (2.8) to the integral in the right-hand side $n-1$ times and computing the iterated integral as we did to prove the estimate (5.3) we obtain

$$\|Z(t) \xi\|^2 \leq \sup_{s \leq t} \|Z(s)\|_\infty^2 \max_{\eta \in J(\xi)} \|\xi\|^2 \cdot |J(\xi)|^n K_T^n c_{T,\xi}^n (\mu_\xi(0, t))^n \frac{1}{n!}.$$

Since this is true for all $n \in \mathbb{N}$ it follows that $Z(t) \xi = 0$ for all $t \in [0, T]$.

PROPOSITION 5.3. *Suppose that the integrators of scalar type $M^{\alpha\beta}$ verify the condition (2.17). Then the solution of (5.2) is unique.*

Proof. We prove that all adapted processes $(Z(t))_{t \in \mathbb{R}_+}$ satisfying (5.2) with $Z(0) = 0$ must be zero. In fact, for all $\xi, \eta \in \mathcal{D}$, applying (2.17) we have

$$|\langle \eta, Z(t) \xi \rangle| \leq \sum_{\eta_1, \xi_1 \in J(\xi, \eta)} \int_0^t v_{\eta\xi}(ds) \sup_{\alpha_1, \beta_1 \in I} |\langle F_{\alpha_1}^+(s) \eta_1, Z(s) G_{\beta_1}(s) \xi_1 \rangle|,$$

where $v_{\eta\xi}$ denotes the measure $\sum_{\eta_1, \xi_1 \in J(\xi, \eta)} |v_{\eta_1 \xi_1}|$. The same argument repeated n -times gives the inequality

$$|\langle \eta, Z(t) \xi \rangle| \leq \frac{1}{n!} (c'_T \mu_\xi(0, t))^n$$

for all $t \in [0, T]$, where c' is an easily computable constant and $n \in \mathbb{N}$. Then $\langle \eta, Z(t) \xi \rangle = 0$ for all $t \in [0, T]$; since T is arbitrary $Z(t) = 0$ for all $t \in \mathbb{R}_+$.

In the proof of the unitarity conditions for the Fermion case we shall need to solve an equation of the form

$$\begin{aligned} dY(t) &= dM^{\alpha\beta}(t) F_\alpha(t) \rho_{\alpha\beta}(Y(t)) G_\beta(t) \\ Y(0) &= Y_0, \end{aligned} \tag{5.6}$$

where $\rho_{\alpha\beta}$ are automorphisms of \mathcal{A} .

THEOREM 5.4. *Let the $M^{\alpha\beta}$ be as in Theorem 5.1 and suppose that for each $\alpha, \beta, \gamma, \delta$ there exists a complex number $u_{\alpha\beta}^{\gamma\delta}$ with $|u_{\alpha\beta}^{\gamma\delta}| \leq 1$ such that, for every interval (s, t)*

$$\rho_{\alpha\beta}(M^{\gamma\delta}(s, t)) = u_{\alpha\beta}^{\gamma\delta} M^{\gamma\delta}(s, t). \tag{5.7}$$

Then (5.6) has a solution and, under the same conditions of Propositions 5.2 and 5.3 the solution is unique.

Proof. Using condition (5.7) one can reproduce the first part of the proof of Theorem 5.1 with simple modifications due to the presence of the $\rho_{\alpha\beta}$'s. Using the fact that

$$\|\rho_{\alpha\beta}(X)\|_{\infty} \leq \|X\|_{\infty}$$

for any $X \in \mathcal{B}(\mathcal{H})$, one sees that the inequality (5.4) holds also in this case. This is sufficient to prove the existence and uniqueness result.

6. ALGEBRAS OF STOCHASTIC INTEGRALS

Itô's formula is a rule which allows us to write a (sufficiently regular) function of a finite set of stochastic integrals, as a sum of stochastic integrals. A consequence of Itô's formula is that the linear space generated by the stochastic integrals with "smooth" coefficients relative to a set of basic integrators is in fact an algebra. In this section we briefly discuss the analogue situation in the present context.

Let I be a set, and let $\mathfrak{I} = \{M^{\alpha} : \alpha \in I\}$ be a self-adjoint family of regular integrators of scalar type. We would like to associate to \mathfrak{I} a vector space $\mathcal{P}(\mathfrak{I})$ of adapted processes with the following properties:

(I) All the elements of $\mathcal{P}(\mathfrak{I})$ are sums of stochastic integrals with respect to integrators of the family \mathfrak{I} .

(II) $\mathcal{P}(\mathfrak{I})$ contains all the stochastic integrals of simple locally bounded adapted processes with respect to the elements of \mathfrak{I} (thus, in particular all the processes $M_t^{\alpha} = M^{\alpha}(0, t)$).

(III) $\mathcal{P}(\mathfrak{I})$ is closed under multiplication in the sense that, for each $X, Y \in \mathcal{P}(\mathfrak{I})$ and for each $t \geq 0$, $(X \cdot Y)(t) := X_t Y_t$ is a stochastic process in the sense of Definition (1.2) and belongs to $\mathcal{P}(\mathfrak{I})$.

(IV) $\mathcal{P}(\mathfrak{I})$ is minimal with respect to the above properties.

From (II) it follows in particular that \mathcal{D} must be an invariant domain for all the operators $M_{\alpha}(s, t)$ ($\alpha \in I$, $(s, t) \subseteq \mathbf{R}$ -bounded). In this section we shall assume that this is the case and we shall restrict the term "process" to those processes which leave the domain \mathcal{D} invariant. These processes form a *-algebra denoted \mathcal{P} .

PROPOSITION 6.1. *Let $\{M^{\alpha} : \alpha \in I\}$ be a self-adjoint family of regular integrators of scalar type and let us suppose that for all $\alpha, \beta \in I$, all $\xi \in \mathcal{D}$, all $s, t \in \mathbf{R}_+$, $s < t$ and all adapted process F*

$$D(M^{\beta}(s, t)) \supseteq M^{\alpha}(s, t) F_s \xi. \quad (6.0)$$

Then the limit

$$\llbracket M^\beta, M^\alpha \rrbracket (s, t) F_s \xi := \lim_{|P| \rightarrow 0} \sum M^\beta(t_{k-1}, t_k) M^\alpha(t_{k-1}, t_k) F_s \xi \quad (6.1)$$

exists in norm for any $\xi \in \mathcal{D}$ and any adapted process F , and is an additive adapted process $\llbracket M^\beta, M^\alpha \rrbracket$ satisfying the following equality

$$\begin{aligned} & M^\beta(s, t) M^\alpha(s, t) F_s \xi \\ &= \left\{ \int_s^t dM^\beta(r) M^\alpha(s, r) + \int_s^t dM^\alpha(r) M^\beta(s, r) + \llbracket M^\beta, M^\alpha \rrbracket (s, t) \right\} F_s \xi. \end{aligned} \quad (6.2)$$

Proof. Summing the algebraic identity

$$d(M^\beta M^\alpha)(t) = dM_t^\beta \cdot M_t^\alpha + M_t^\beta \cdot dM_t^\alpha + dM_t^\beta \cdot dM_t^\alpha \quad (6.3)$$

over all the intervals of a partition $P = \{t_k\}_{k=0}^n$ of $[s, t]$ we find

$$\begin{aligned} M^\beta(s, t) M^\alpha(s, t) F_s \xi &= \sum_{k=1}^n M^\beta(s, t_{k-1}) M^\alpha(t_{k-1}, t_k) F_s \xi \\ &\quad + \sum_{k=1}^n M^\beta(t_{k-1}, t_k) M^\alpha(s, t_{k-1}) F_s \xi \\ &\quad + \sum_{k=1}^n M^\beta(t_{k-1}, t_k) M^\alpha(t_{k-1}, t_k) F_s \xi. \end{aligned}$$

By Remark 3 after Definition 2.2 the functions $r \in [s, t] \rightarrow M^\alpha(s, r) F_s \xi$ ($\alpha \in I$) are continuous for any $\xi \in \mathcal{D}$ so that, by Theorem 2.3, as $|P| \rightarrow 0$, the first and second sum converge respectively to

$$\int_0^t dM^\alpha(r) M^\beta(s, r) F_s \xi, \quad \int_s^t dM^\beta(r) M^\alpha(s, r) F_s \xi. \quad (6.4)$$

Therefore also the third sum converges and this proves (6.1). It is clear that (6.2) holds

The additive process $\llbracket M^\beta, M^\alpha \rrbracket$ is called the *bracket* of M^β and M^α (or the square bracket, or the Meyer bracket, or the mutual quadratic variation (cf. [8])).

Thus, if we want the product $M^\beta(s, t) M^\alpha(s, t)$ to be expressible as a linear combination of stochastic integrals with respect to the M^γ_s , we must have that

$$\llbracket M^\beta, M^\alpha \rrbracket (s, t) = \int_s^t c_{\gamma}^{\beta\alpha}(r) dM^\gamma(r), \quad (6.5)$$

where summation over repeated indices is understood and the $c_{\gamma}^{\beta\alpha}$ are adapted processes which, for any pair $\alpha, \beta \in I$, are zero for all but a finite number of indices $\gamma \in I$. Thus condition (6.5) is a necessary condition for the solution of our problem. Such a condition is called an Itô table for the set of basic integrators \mathfrak{I} and the processes $c_{\gamma}^{\beta\alpha}(s)$ are called the *structure processes* of the Itô table. If the $c_{\gamma}^{\beta\alpha}$ are locally bounded continuous scalar functions, then, from Proposition 2.6, it follows that $\llbracket M^{\beta}, M^{\alpha} \rrbracket$ is also an integrator of scalar type.

Without any further assumptions, not much more can be said, since we cannot control terms of the form

$$M^{\alpha_1}(s, t) \cdot F_s \cdot M^{\alpha_2}(s, t) \cdot G_s \quad (6.6)$$

with F, G locally bounded adapted processes. However, if we assume that the integrators M^{α} are such that their increments in the future of any times s , ρ -commute with any operator adapted to the past of s , i.e. (all commutators being meant on \mathcal{D}),

$$M^{\alpha}(s, t) F_s = \rho(F_s) M^{\alpha}(s, t); \quad \forall \alpha \in I, \forall s < t \quad (6.7)$$

$$M^{\alpha}(t, u) M^{\beta}(r, s) = \rho_{\alpha}(M^{\beta}(r, s)) M^{\alpha}(t, u); \quad \forall \alpha, \beta \in I; r < s < t < u \quad (6.8)$$

then expressions like (6.6) become tractable. If $\rho_{\alpha}(M^{\beta}) = (M^{\beta})$, then we have the commutation of M^{α}, M^{β} (Boson case); if $\rho_{\alpha}(M^{\beta}) = -(M^{\beta})$, then we have the anticommutation of M^{α}, M^{β} (Fermion case).

DEFINITION 6.2. The integrators M^{α} are said to satisfy a *ρ -commutation relation* if, for all $\alpha \in I$, there exists an automorphism ρ_{α} of \mathcal{P} with the following properties:

$$\rho_{\alpha}^2 = \text{id} \quad (6.9)$$

ρ_{α} maps adapted processes into adapted processes, and

$$F_s \xi \in \mathcal{D}(M^{\alpha}(s, t)), \quad M^{\alpha}(s, t) \xi \in \mathcal{D}(\rho_{\alpha}(F_s)) \quad (6.10)$$

and

$$M^{\alpha}(s, t) F_s \xi = \rho_{\alpha}(F_s) M^{\alpha}(s, t) \xi, \quad F_s M^{\alpha}(s, t) \xi = M^{\alpha}(s, t) \rho_{\alpha}(F_s) \xi$$

for every $\xi \in \mathcal{D}$, $s < t$, and any adapted process F .

7. THE WEAK ITÔ FORMULA

LEMMA 7.1. *Let M, N be integrators of scalar type and H, K be strongly continuous adapted processes. Define*

$$X_t = \int_s^t dM_s H_s; \quad Y_t = \int_s^t dN_s K_s. \quad (7.1)$$

Then, in the notations (2.19) and (4.5)

$$\langle dX_t \xi, dY_t \eta \rangle \equiv \langle dM_t H_t \xi, dN_t K_t \eta \rangle. \quad (7.2)$$

Proof. For each $T > 0$ and each $t \in [0, T]$ one has

$$\begin{aligned} & \langle dX_t \xi, dY_t \eta \rangle - \langle dM_t H_t \xi, dN_t K_t \eta \rangle \\ &= \langle [dX_t - dM_t H_t] \xi, dY_t \eta \rangle + \langle dM_t H_t \xi, [dY_t - dN_t K_t] \eta \rangle \\ &= \left\langle \int_t^{t+dt} dM_s [H_s - H_t] \xi, dY_t \eta \right\rangle + \left\langle dM_t H_t \xi, \int_t^{t+dt} dN_s [K_s - K_t] \eta \right\rangle. \end{aligned} \quad (7.3)$$

Now

$$\begin{aligned} & \left| \left\langle \int_t^{t+dt} dM_s [H_s - H_t] \xi, dY_t \eta \right\rangle \right| \\ & \leq \left\| \int_t^{t+dt} dM_s [H_s - H_t] \xi \right\| \cdot \left\| \int_t^{t+dt} dN_s K_s \eta \right\| \end{aligned}$$

and by the integrator of scalar type inequality this is majorized by

$$\begin{aligned} & c_{T,\xi} \left(\sum_{\xi_1 \in J(\xi)} \int_t^{t+dt} d\mu_\xi^M(s) \| [H_s - H_t] \xi_1 \|^2 \right)^{1/2} \\ & \cdot \left(\sum_{\eta_1 \in J(\eta)} \int_t^{t+dt} d\mu_\eta^N(s) \| K_s \eta_1 \|^2 \right)^{1/2} \\ & \leq |J(\xi)| \cdot |J(\eta)| \max_{\xi_1 \in J(\xi), \eta_1 \in J(\eta)} \sup_{s \in [t, t+dt]} \| [H_s - H_t] \xi_1 \| \\ & \cdot \| K_s \eta_1 \| \mu_{M,N}(t, t+dt), \end{aligned}$$

where

$$\mu_{M,N} := \frac{1}{2}(\mu_\xi^M + \mu_\eta^N).$$

Since a similar estimate holds for the second term in the right-hand side of (7.3), one finds the majorization

$$\begin{aligned} & |\langle dX_t \xi, dY_t \eta \rangle - \langle dM_t H_t \xi, dN_t H_t \eta \rangle| \\ & \leq c(M, N, \xi, \eta, T, H, K) \Delta_T(dt) \mu_{M, N}(t, t + dt) \end{aligned}$$

with Δ_T satisfying

$$\lim_{dt \rightarrow 0} \sup_{[t, t + dt] \subseteq [0, T]} \Delta_T(dt) = 0.$$

Clearly this inequality implies (7.2).

DEFINITION 7.2. An integrator R is called the *weak Itô product* of two integrators M, N if

$$\langle dM_t^+ H_t \xi, dN_t K_t \eta \rangle \equiv \langle H_t \xi, dR_t K_t \eta \rangle \quad (7.4)$$

for any $\xi, \eta \in \mathcal{D}$ and any pair H, K of strongly continuous adapted processes.

Coherently with the notation introduced in Section 4, the relation (7.4) will be written in the symbolic notation

$$dM dN \equiv dR \quad \text{weakly on } (\mathcal{PD}, \mathcal{PD}). \quad (7.5)$$

If for any $\xi \in \mathcal{D}$ and any strongly continuous adapted process K one has

$$dN_t K_t \xi \subseteq D(dM_t) \quad (7.6)$$

then the relation (7.5) is not symbolic and it is equivalent to (7.4). The following is the *weak Itô formula for left stochastic integrals*.

THEOREM 7.3. Let M, N be integrators which satisfy the scalar forward derivative inequality (3.9) and whose weak Itô product is an integrator R . For any pair of strongly continuous adapted processes H, K , let

$$X_t = \int_0^t dM_r^+ H_r; \quad Y_t = \int_0^t dN_r K_r. \quad (7.7)$$

Then for any $\xi, \eta \in \mathcal{D}$

$$d\langle X_t \xi, Y_t \eta \rangle \equiv \langle H_t \xi, dM_t Y_t \eta \rangle + \langle X_t \xi, dN_t \eta \rangle + \langle H_t \xi, dR_t K_t \eta \rangle. \quad (7.8)$$

Proof. From the algebraic identity

$$d\langle X_t, \xi, Y_t, \eta \rangle = \langle dX_t, \xi, Y_t, \eta \rangle + \langle X_t, \xi, dY_t, \eta \rangle + \langle dX_t, \xi, dY_t, \eta \rangle$$

and the scalar forward derivative inequality (3.9) we obtain (Proposition 4.2)

$$d\langle X_t, \xi, Y_t, \eta \rangle \equiv \langle dM_t^+ H_t, \xi, Y_t, \eta \rangle + \langle X_t, \xi, dN_t K_t, \eta \rangle + \langle dX_t, \xi, dY_t, \eta \rangle. \quad (7.9)$$

Moreover from Lemma 7.1 we know that

$$\langle dX_t, \xi, dY_t, \eta \rangle \equiv \langle dM_t^+ H_t, \xi, dN_t K_t, \eta \rangle.$$

Hence (7.8) follows from (7.4).

8. THE HERMITE POLYNOMIALS

As a first application of the weak Itô formula we extend a known result on Hermite polynomials (cf., e.g., [29]). We suppose that the set I has only two elements, denoted 0, 1 and we denote M^0 and M^1 by A and M , respectively. We also assume that, in the notations of Section 4,

$$[M, M] = A; \quad [A, M] = [A, M] = 0; \quad [A, A] = 0 \quad (8.1)$$

weakly on $(\mathcal{PD}, \mathcal{PD})$. Then the inductive sequence

$$H_0(A, M) = 1 \quad (8.2)$$

$$H_{n+1}(A, M)(t) = \int_0^t dM(s) H_n(A, M)(s)$$

is well defined strongly on \mathcal{D} .

THEOREM 8.1. *Let M and A satisfy the conditions (8.1), (8.2) and assume that every polynomial in $A(t)$ and $M(t)$ is a process and moreover that*

(i) *M commutes with its own past, i.e.,*

$$[M_t, dM_t] = 0; \quad \forall t \in \mathbf{R}_+. \quad (8.3)$$

Then A_t and M_t commute for each $t \in \mathbf{R}_+$. If, moreover

(ii) *The domain \mathcal{D} is invariant under the action of A_t , M_t for any $t \in \mathbf{R}_+$ then $H_n(t) = H_n(A_t, M_t)$ is the n th Hermite polynomial in M_t with parameter A_t .*

Then, for all $n \geq 1$ we have

$$H_{n+1}(A, M) = \frac{M}{n+1} H_n(A, M) - \frac{A}{n+1} H_{n-1}(A, M). \quad (8.4)$$

Proof. For each $t \in \mathbf{R}_+$

$$dA_t \cdot M_t \equiv dM_t \cdot dM_t \cdot M_t = M_t \cdot dM_t \cdot dM_t \equiv M_t \cdot dA_t.$$

Moreover, because of (8.1)

$$d(A_t M_t) \equiv dA_t \cdot M_t + A_t \cdot dM_t; \quad d(M_t A_t) \equiv dM_t \cdot A_t + M_t \cdot dA_t$$

it follows that

$$d(A_t M_t - M_t A_t) \equiv 0$$

which implies the commutativity of A_t and M_t . Since A_t and M_t commute, $H_n(A_t, M_t)$ is well defined on \mathcal{D} by assumption (ii). From (8.1), (8.2) and the weak Itô formula for left stochastic integrals it follows that, for any $n \geq 2$, $t \in \mathbf{R}_+$ and $\xi, \eta \in \mathcal{D}$, one has

$$\begin{aligned} & \frac{1}{n+1} d\langle M_t^+ \xi, H_n \eta \rangle - \frac{1}{n+1} d\langle A_t^+ \xi, H_{n-1} \eta \rangle \\ & \equiv \frac{1}{n+1} [\langle \xi, dM_t \cdot H_n(t) \rangle + \langle M_t^+ \xi, dM_t H_{n-1}(t) \eta \rangle \\ & \quad + \langle \xi, dA_t H_{n-1}(t) \eta \rangle - \langle \xi, dA_t H_{n-1}(t) \eta \rangle - \langle A_t^+ \xi, dM H_{n-2}(t) \eta \rangle] \\ & = \frac{1}{n+1} \langle \xi, dM_t H_n(t) \eta \rangle \\ & \quad + \frac{n}{n+1} \left[\frac{1}{n} \langle M_t^+ \xi, dM_t H_{n-1}(t) \eta \rangle - \frac{1}{n} \langle A_t^+ \xi, dM_t H_{n-2}(t) \eta \rangle \right]. \end{aligned} \quad (8.5)$$

Now, for $n=1$ (8.4) is true since $H_0=1$, $H_1=M$. Moreover, since dM commutes with its own past

$$dH_2 \equiv dM \cdot M = \frac{1}{2}(dM \cdot M + M \cdot dM) \equiv \frac{1}{2}d[M^2 - A]$$

hence

$$\frac{1}{2}(MH_1 - AH_0) = \frac{1}{2}(M^2 - A) = H_2$$

so that (8.4) holds for $n = 2$. Therefore because of assumption (8.1) and of the identity (8.5)

$$\frac{1}{2}d\langle M_t^+ \xi, H_1(t) \eta \rangle - \langle A_t^+ \xi, H_0(t) \eta \rangle \equiv \langle \xi, dM_t H_1(t) \eta \rangle \equiv \langle \xi, dH_2(t) \eta \rangle.$$

Assume, by induction, that $H_n(t)$ is a polynomial in A_t and M_t and that

$$\frac{1}{n} \langle M_t^+ \xi, dM_t H_{n-1}(t) \eta \rangle - \frac{1}{n} \langle A_t^+ \xi, dM_t H_{n-2}(t) \eta \rangle \equiv \langle \xi, dH_n(t) \eta \rangle. \quad (8.6)$$

Then from (8.6) it follows that the right-hand side of (8.5) is equivalent to

$$\langle \xi, dM_t H_n(t) \eta \rangle \equiv \langle \xi, dH_{n+1}(t) \eta \rangle$$

by Proposition 4.2. By the induction assumption $H_n(t)$ and $H_{n-1}(t)$ are polynomials in A_t and M_t hence, by (8.5), it follows that, on \mathcal{D}

$$\frac{1}{n+1} d(M_t H_n(t) - A_t H_{n-1}(t)) \equiv dH_{n+1}(t)$$

which, due to (8.2) is equivalent to (8.4).

9. THE UNITARITY CONDITIONS

As an application of the weak Itô formula (7.8) we will give necessary and sufficient conditions on the coefficients of the stochastic differential equation

$$\begin{aligned} dX(t) &= dM^\alpha(t) F_\alpha(t) X(t) \\ X(0) &= X_0 \end{aligned} \quad (9.1)$$

which guarantee that its solution is a unitary operator on \mathcal{H} . In this section we shall assume that the set of basic integrators is self adjoint, i.e., that for every index α there is a unique index α^+ such that $(M^\alpha)^+ = M^{\alpha^+}$. Note that, by definition $(\alpha^+)^+ = \alpha$.

Let us first note the following obvious criterion:

LEMMA 9.1. *Let $(F_\alpha)_{\alpha \in I}$ and let X_0 be an element of \mathcal{A}_{01} and X be the solution of the stochastic differential equation (9.1). The following conditions are equivalent:*

- (a) *for all $t \in \mathbf{R}_+$, $X(t)$ can be extended to a unitary operator on \mathcal{H} .*

(b) X_0 is a unitary operator on \mathcal{H} and, for all $\eta, \xi \in \mathcal{D}$ and $t \in \mathbf{R}_+$

$$\begin{aligned} & \langle X^+(t) F_\gamma^+(t) dM_t^{\gamma+} \eta, X^+(t) \xi \rangle + \langle X^+(t) \eta, X^+(t) F_\gamma^+(t) dM_t^{\gamma+} \xi \rangle \\ & + \langle X^+(t) F_\beta^+(t) dM_t^{\beta+} \eta, X^+(t) F_\alpha^+(t) dM_t^{\alpha+} \xi \rangle \equiv 0 \end{aligned} \quad (9.2)$$

$$\begin{aligned} & \langle dM_t^{\gamma+} F_\gamma X(t) \eta, X(t) \xi \rangle + \langle X(t) \eta, dM_t^\gamma F_\gamma(t) X(t) \xi \rangle \\ & + \langle dM_t^\beta F_\beta X(t) \eta, dM_t^\alpha F_\alpha(t) X(t) \xi \rangle \equiv 0. \end{aligned} \quad (9.3)$$

Proof. Applying the weak Itô formula (Theorem (7.3)) we find that (9.2) is equivalent to the condition $\langle X^+(t) \eta, X^+(t) \xi \rangle = \langle X_0^+ \eta, X_0^+ \xi \rangle$ and (9.3) is equivalent to the condition $\langle X(t) \eta, X(t) \xi \rangle = \langle X_0 \eta, X_0 \xi \rangle$ for all $t \in \mathbf{R}_+$ and all $\eta, \xi \in \mathcal{D}$. Then the equivalence of items (a) and (b) is obvious.

We can give more easily verifiable conditions when the integrators of scalar type M^α are *linearly independent* in the sense that the equality

$$\int_0^t dM_s^\alpha G_\alpha(s) = 0$$

for all family $(G_\alpha(t))_{t \in \mathbf{R}_+, \alpha \in I}$ of adapted processes and for all $t \in \mathbf{R}_+$ implies that $G_\alpha = 0$ for all $\alpha \in I$.

THEOREM 9.2. *Let $(F_\alpha(t))_{t \in \mathbf{R}_+, \alpha \in I}$ and $(X(t))_{t \in \mathbf{R}_+}$ be as in Proposition 9.1. Assume that the brackets $\llbracket M^\beta, M^\alpha \rrbracket$ exist and are as in (6.5). Suppose that the integrators of scalar type M^α are linearly independent and satisfy a ρ -commutation relation. Then the following conditions are equivalent:*

(a) *for all $t \in \mathbf{R}_+$, $X(t)$ can be extended to a unitary operator on \mathcal{H} .*

(b) *X_0 is a unitary operator on \mathcal{H} and, for all $t \in \mathbf{R}_+$ and $\gamma \in I$*

$$F_\gamma(t) + \rho_\gamma(F_\gamma^+(t)) + c_\gamma^{\beta\alpha+}(t) \rho_{\alpha+}(F_\beta(t) F_\alpha^+(t)) = 0 \quad (9.4)$$

$$F_\gamma(t) + \rho_\gamma(F_\gamma^+(t)) + c_\gamma^{\beta+ \alpha}(t) \rho_{\beta+} \rho_\alpha(F_\beta^+(t)) F_\alpha(t) = 0. \quad (9.5)$$

Proof. (a) \Rightarrow (b) Let us suppose that, for all $t \in \mathbf{R}_+$, $X(t)$ can be extended to a unitary operator on \mathcal{H} . Then (9.2) can be written in the form

$$\int_0^t dM_s^\gamma (F_\gamma(s) + \rho_\gamma(F_\gamma^+(s)) + c_\gamma^{\beta\alpha+}(t) \rho_{\alpha+}(F_\beta(s) F_\alpha^+(s))) = 0$$

for all $t \in \mathbf{R}_+$, and (9.4) follows from the linear independence of the M^γ .

Since X is strongly continuous on \mathcal{D} and unitary operator-valued it is strongly continuous on \mathcal{H} ; then the stochastic processes

$$\rho_\gamma(X^+ F_\gamma^+) X, \quad \rho_\gamma(X^+) F_\gamma X, \quad \rho_\alpha \rho_{\beta^+}(X^+ F_\beta^+) F_\alpha X$$

are integrable with respect to all the M^γ , therefore (9.3) can be written in the form

$$\begin{aligned} \int_0^t dM_s^\gamma \rho_\gamma(X^+(s)) (F_\gamma(s) + \rho_\gamma(F_\gamma^+(s))) \\ + c_\gamma^{\beta^+ \alpha}(s) \rho_\alpha \rho_{\beta^+}(X^+(s) F_\beta^+(s)) F_\alpha(s)) X(s) = 0. \end{aligned}$$

Then we obtain (9.5) from the linear independence of the M^γ and the unitarity of $X(s)$.

(b) \Rightarrow (a) From (9.5) condition (9.3) follows and then we have $\langle X(t)\eta, X(t)\xi \rangle = \langle \eta, \xi \rangle$ for all $\eta, \xi \in \mathcal{D}$ and so $X(t)$ is an isometry for all $t \in \mathbf{R}_+$. Since X is strongly continuous on \mathcal{H} , the adapted processes

$$F_\gamma X X^+, \quad \rho_\gamma(X X^+) \rho_\gamma(F_\gamma^+), \quad \rho_{\alpha^+}(F_\beta) \rho_{\alpha^+}(X X^+) \rho_{\alpha^+}(F_\alpha^+)$$

are integrable with respect to all the M^γ . Therefore the process $Y = X X^+$ is a solution of the stochastic differential equation

$$\begin{aligned} dY = dM^\gamma(F_\gamma Y + \rho_\gamma(Y) \rho_\gamma(F_\gamma^+) + c_\gamma^{\beta^+ \alpha} \rho_{\alpha^+}(F_\beta) \rho_{\alpha^+}(Y) \rho_{\alpha^+}(F_\alpha)) \\ Y(0) = 1. \end{aligned}$$

This is an equation of the type we considered in Section 5 and we know from Theorem 5.4 that its solution must be unique. $Y(t) = 1$ for all $t \in \mathbf{R}_+$ is a solution because of (9.4), hence $X(t) X^+(t) = I$ and $X^+(t)$ is an isometry for all $t \in \mathbf{R}_+$. This completes the proof of the unitarity of X .

10. THE LEVY THEOREM

When the Meyer brackets are scalar nonatomic measures a fourth moment condition (cf. the inequality (10.2) below) holds. In the classical case this condition implies the continuity of the trajectories. In [8] it has been shown that, in the classical case, this condition is equivalent to the continuity of the trajectories for semimartingales and it has been proposed to assume this condition as the definition of the notion of continuity of the trajectories for a quantum process.

PROPOSITION 10.1. *Let M^α, M^β be integrators of scalar type such that*

$$[[M^{\beta^+}, M^\alpha]](s, t) = \sigma^{\beta, \alpha}(s, t) \quad (10.1)$$

for some \mathbf{C} -valued nonatomic measure with locally bounded variation $\sigma^{\beta, \alpha}$ (defined on \mathbf{R}_+). Then for all $0 \leq s < r < T < +\infty$ and $\xi \in \mathcal{D}$ one has

$$\|M^\beta(s, t) M^\alpha(s, t) \xi\|^2 \leq 3 \max\{c_{T, \xi}, 1\} [(|\sigma^{\beta, \alpha}| + \mu_\xi^\alpha + \mu_\xi^\beta)(s, t)]^2. \quad (10.2)$$

Proof. For all $s, t \in \mathbf{R}_+$ with $s < t$ we have, using (6.2),

$$\begin{aligned} & \|M^\beta(s, t) M^\alpha(s, t) \xi\|^2 \\ & \leq 3 \sum_{\eta \in J(\xi)} \left(\|\eta\|^2 |\sigma^{\beta, \alpha}(s, t)|^2 \right. \\ & \quad \left. + \left\| \int_s^t dM^\beta(r) M^\alpha(s, r) \eta \right\|^2 + \left\| \int_s^t dM^\alpha(r) M^\beta(s, r) \eta \right\|^2 \right) \\ & \leq 3 \max\{c_{T, \xi}, 1\} \sum_{\eta \in J(\xi)} \left(\|\eta\|^2 |\sigma^{\beta, \alpha}(s, t)|^2 \right. \\ & \quad \left. + \int_s^t \|M^\alpha(s, r) \eta\|^2 \mu_\xi^\beta(dr) + \int_s^t \|M^\beta(s, r) \eta\|^2 \mu_\xi^\alpha(dr) \right). \end{aligned} \quad (10.3)$$

The conclusion now follows from (2.12).

DEFINITION 10.2. Let M be an integrator of scalar type satisfying the condition (6.0) and let us denote

$$dM^0 = dt; \quad dM^1 = dM; \quad dM^2 = dM^+. \quad (10.4)$$

The pair M, M^+ is called a *Levy pair* if the Meyer brackets $\llbracket M^{\beta+} M^\alpha \rrbracket$ ($\beta, \alpha = 1, 2$) exist and are a \mathbf{C} -valued nonatomic measures which are absolutely continuous with respect to the Lebesgue measure. The Levy pair is called of *Boson type* if

$$[M^\alpha(I), M^\beta(J)] = 0 \quad \text{for all } I, J \subset \mathbf{R}_+ \text{ with } I \cap J = \emptyset. \quad (10.5)$$

For a Levy pair M, M^+ we define the bracket matrix

$$d(\llbracket M^{\beta+}, M^\alpha \rrbracket(t)) =: \sigma(t) dt =: \begin{pmatrix} \sigma^{11}(t) & \sigma^{12}(t) \\ \sigma^{21}(t) & \sigma^{22}(t) \end{pmatrix} dt. \quad (10.6)$$

An important consequence of (10.6) is that

$$dM_t^{\beta+} \cdot dM_t^\alpha \equiv \sigma^{\beta, \alpha}(t) dt \quad \text{strongly on } \mathcal{PD}. \quad (10.7)$$

The matrix valued function $t \mapsto \sigma(t)$ is positive definite in the sense that, if we define for all complex valued continuous functions f, g

$$(f, g)_\sigma := (f, \bar{f}) \cdot \sigma \cdot \begin{pmatrix} \bar{g} \\ g \end{pmatrix} \quad (10.8)$$

then for all $f, (f, f)_\sigma \geq 0$. Let $C_c(\mathbf{R}_+; \mathbf{C})$ denote the space of complex valued continuous functions on \mathbf{R}_+ with compact support and let \Im denote the symplectic form on $C_c(\mathbf{R}_+; \mathbf{C})$ defined by

$$\Im(f, g)_\sigma(t) := \frac{1}{2i} [(f, g)_\sigma(t) - (g, f)_\sigma(t)]. \quad (10.9)$$

THEOREM 10.3. *Let M, M^+ be a Levy pair of Boson type. The solutions of the stochastic differential equations*

$$dU_f(t) \equiv i(f(t) dM^+(t) + f^+(t) dM(t) - \frac{1}{2}(f, f)_\sigma(t) dt) U_f(t) \quad (10.10)$$

$$dU_f^+(t) \equiv iU_f^+(t)(-f(t) dM^+(t) - f^+(t) dM(t) - \frac{1}{2}(f, f)_\sigma(t) dt) \quad (10.11)$$

(with the initial condition $U_f(0) = U_f^+(0) = 1$) define a unitary representation

$$(\mathcal{H}, \{U_f | f \in C_c(\mathbf{R}_+; \mathbf{C})\})$$

of the CCR algebra over $C_c(\mathbf{R}_+; \mathbf{C})$ associated with the possibly degenerate real bilinear symplectic form (cf. [25])

$$(f, g) \rightarrow \int_0^\infty \Im(f, g)_\sigma(t) dt.$$

Moreover the one-parameter unitary group $(U_t)_{t \in \mathbf{R}}$ is strongly continuous on \mathcal{H} and its infinitesimal generator $B(f)$ has the form

$$B(f) = \int_0^\infty (f(s) dM^+(s) + f^+(s) dM(s)) \quad \text{on } \mathcal{D}. \quad (10.12)$$

Proof. We divide the proof into several steps. First note that, due to Theorem 5.1, the stochastic differential equations

$$U_f(t) = 1 + \int_0^t (if(s) dM^+(s) + if^+(s) dM(s) - \frac{1}{2}(f, f)_\sigma(s) ds) U_f(s) \quad (10.13)$$

$$U_f^+(t) = 1 + \int_0^t (-if^+(s) dM(s) - if(s) dM^+(s) - \frac{1}{2}(f, f)_\sigma(s) ds) U_f^+(s) \quad (10.14)$$

have unique solutions on the domain \mathcal{D} satisfying

$$\langle U_f(t) \xi, \eta \rangle = \langle \xi, U_f^+(t) \eta \rangle$$

for all $\xi, \eta \in \mathcal{D}$. Moreover, the unitarity conditions of Theorem 9.2 are immediately verified in this case. Therefore $U_f(t)$ can be extended to a unitary operator on \mathcal{H} (that we still denote $U_f(t)$).

Now note that, because of assumption (10.5), one has $[M^\alpha(s, t), U_f(s)] = 0$ for all $0 < s < t$ (this follows immediately from the expression of $U_f(t)$ in terms of the iterated series). Using this and the fact that $U_f(t)$ is a bounded operator for all f, t and $s \mapsto U_f(s)$ is strongly continuous on \mathcal{D} from the weak Itô formula and (9.6) we have, for all $t \in \mathbf{R}_+$, $f, g \in C(\mathbf{R}_+; \mathbf{C})$, $\xi, \eta \in \mathcal{D}$

$$\begin{aligned} d\langle \eta, U_f(t) U_g(t) \xi \rangle &= d\langle U_f^+(t) \eta, U_g(t) \xi \rangle \\ &= \langle U_f^+(t) \eta, [ig(t) dM_t^+ + ig^+(t) dM_t - \tfrac{1}{2}(g, g)_\sigma(t) dt] U_g(t) \xi \rangle \\ &\quad + \langle [-if^+(t) dM_t - if(t) dM_t^+ - \tfrac{1}{2}(f, f)_\sigma(t) dt] U_f^+(t) \eta, U_g(t) \xi \rangle \\ &\quad - \langle U_f^+(t) \eta, (f, g)_\sigma(t) U_g(t) \xi \rangle dt \\ &= \langle U_f^+(t) \eta, (i(f+g)^+ dM_t + i(f+g) dM_t^+) U_g(t) \xi \rangle \\ &\quad - \langle U_f^+(t) \eta, U_g(t) \xi \rangle (\tfrac{1}{2}(g, g)_\sigma(t) + (f, g)_\sigma(t) + \tfrac{1}{2}(f, f)_\sigma(t)) dt, \end{aligned}$$

i.e., the process $X(t) = U_f(t) U_g(t)$ satisfies the stochastic differential equation

$$\begin{aligned} X(t) &= I + \int_0^t (i(f+g)(s) dM^+(s) + i(f+g)^+(s) dM(s) \\ &\quad + [-\tfrac{1}{2}(f, f)_\sigma(s) - \tfrac{1}{2}(g, g)_\sigma(s) - (f, g)_\sigma(s)] ds) X(s). \end{aligned}$$

Let

$$Y(t) = \exp \left(-i \int_0^t \Im(f, g)_\sigma(s) ds \right) U_{f+g}(t).$$

$Y(t)$ clearly satisfies the same stochastic differential equation as $X(t)$, hence, by the uniqueness theorem

$$U_f(t) U_g(t) = \exp \left(-i \Im \int_0^t (f, g)_\sigma(s) ds \right) U_{f+g}(t). \quad (10.15)$$

Since f has compact support the limit as $t \uparrow \infty$ of $U_g(t)$ exists in a trivial way. We shall denote this limit by U_f .

LEMMA 10.4. $\varepsilon \rightarrow U_{\varepsilon f}$ is a one-parameter strongly continuous group of unitary operators on \mathcal{H} .

Proof. The group law follows from (10.15). For all $\varepsilon \in \mathbf{R}$, $U_{\varepsilon f}$ satisfies the stochastic differential equation

$$U_{\varepsilon f} = I + \varepsilon \int_0^\infty (if(s) dM^+(s) + if^+(s) dM(s)) U_{\varepsilon f}(s) + \frac{\varepsilon^2}{2} \int_0^\infty (f, f)_\sigma(s) U_{\varepsilon f}(s) ds$$

hence, for all $\xi \in \mathcal{D}$, by the integrator of scalar type inequality (2.7)

$$\|U_{\varepsilon f} \xi - \xi\|^2 \leq 2c_f \left(\sum_{\eta \in J(\xi)} \|\eta\|^2 \right) \varepsilon \left[\int_0^\infty |f(s)|^2 \mu_\xi(ds) + \frac{\varepsilon}{2} \left| \int_0^\infty (f, f)_\sigma(s) ds \right|^2 \right],$$

where c_f is a constant depending only on f . Therefore

$$\lim_{\varepsilon \rightarrow 0} U_{\varepsilon f} \xi = \xi.$$

And the thesis follows since a one-parameter group U_t of unitary operators, strongly continuous on a total set is strongly continuous on the whole space.

PROPOSITION 10.5. *Let $B(f)$ denote the infinitesimal generator of $\varepsilon \rightarrow U_{\varepsilon f}$. Then, on the domain \mathcal{D} , $B(f)$ coincides with*

$$\int_0^\infty (if(s) dM^+(s) + if^+(s) dM(s)).$$

Proof. For all $\xi \in \mathcal{D}$ we have

$$\begin{aligned} & \left\{ \frac{1}{\varepsilon} [U_{\varepsilon f} - I] - \int_0^\infty (if^+(s) dM(s) + if(s) dM^+(s)) \right\} \xi \\ &= \int_0^\infty (if^+(s) dM(s) + if(s) dM^+(s)) [U_{\varepsilon f}(s) - I] \xi \\ &+ \frac{\varepsilon}{2} \int_0^\infty (f, f)_\sigma(s) U_{\varepsilon f}(s) ds \xi \end{aligned}$$

therefore, by the integrator of scalar type inequality (2.7),

$$\begin{aligned} & \left\| \left\{ \frac{U_{\varepsilon f} - I}{\varepsilon} - \int_0^\infty (if^+ dM + if dM^+) \right\} \xi \right\|^2 \\ & \leq 2c_f \sum_{\eta \in J(\xi)} \int_0^\infty |f(s)|^2 \|U_{\varepsilon f} \eta - \eta\|^2 \mu_\xi(ds) + \frac{\varepsilon^2}{2} \left| \int_0^\infty (f, f)_\sigma(s) ds \right|^2 \|\eta\|^2. \end{aligned}$$

Since $\lim_{\varepsilon \rightarrow 0} U_{\varepsilon f} \eta = \eta$ and, for all $\varepsilon \in \mathbf{R}$,

$$\|U_{\varepsilon f} \eta - \eta\|^2 \leq 2\|U_{\varepsilon f} \eta\|^2 + 2\|\eta\|^2 = 4\|\eta\|^2$$

it follows from Lebesgue's dominated convergence theorem that

$$\lim_{\varepsilon \rightarrow 0} \frac{U_{\varepsilon f} \xi - \xi}{\varepsilon} = \int_0^\infty (if^+ dM + if dM^+) \xi$$

and this proves the thesis.

Now let us suppose that the integrators of scalar type M, M^+ are martingales with respect to a unit vector $\Phi \in \mathcal{D}$, in the sense of Definition 3.2, that is, for all $s, t \in \mathbf{R}_+$ with $s < t$ and all adapted processes H, K we have $\langle K(s) \Phi, M^\#(s, t) H(s) \Phi \rangle = 0$.

LEMMA 10.6. *If M, M^+ are Φ -martingales then*

$$\langle \Phi, U_f \Phi \rangle = \exp \left(-\frac{1}{2} \int_0^\infty (f, f)_\sigma(s) ds \right). \quad (10.16)$$

Proof. U_f satisfies the stochastic differential equation

$$U_f(t) = 1 + \int_0^t (if^+(s) dM(s) + if(s) dM^+(s)) U_f(s) - \frac{1}{2} \int_0^t (f, f)_\sigma(s) U_f(s) ds$$

and then $\langle \Phi, U_f(t) \Phi \rangle$ satisfies the ordinary differential equation

$$\langle \Phi, U_f(t) \Phi \rangle = 1 - \frac{1}{2} \int_0^t (f, f)_\sigma(s) \langle \Phi, U_f(s) \Phi \rangle ds$$

whose only solution is (10.16)

We will denote by $L^2(\mathbf{R}_+, \sigma)$ the real Hilbert space obtained by completion of the vector space $\mathcal{C}_c(\mathbf{R}_+, \mathbf{C})$ with respect to the real prescalar product

$$\langle f, g \rangle_\sigma = \int_0^\infty (f, g)_\sigma(t) dt.$$

Let Φ_σ denote the vacuum vector and $W_f (f \in L^2(\mathbf{R}_+, \sigma))$, the Weyl operator on the Fock space $\Gamma(L^2(\mathbf{R}_+, \sigma))$ with test function f .

COROLLARY 10.7. *Let (M, M^+) be a Levy pair that is a martingale with respect to a unit vector Φ and let \mathcal{K} be the closure of the linear span of the set $\{U_f \Phi | f \in \mathcal{C}_c(\mathbf{R}_+, \mathbf{C})\}$. The map*

$$U_f \Phi \rightarrow W_f \Phi_\sigma$$

extends to a unitary isomorphism of \mathcal{K} onto $\Gamma(L^2(\mathbf{R}_+, \sigma))$.

Proof. From the above discussion.

Let (M, M^+) be a Levy pair with covariance matrix σ and a, b be two functions in $\mathcal{C}(\mathbf{R}_+, \mathbf{C})$ such that

$$|a(t)|^2 - |b(t)|^2 \neq 0$$

for all $t \in \mathbf{R}_+$ so that the matrix

$$\begin{pmatrix} a(t) & b(t) \\ \bar{b}(t) & \bar{a}(t) \end{pmatrix}$$

is invertible. Let

$$B(t) = \int_0^t (a(s) dM(s) + b(s) dM^+(s))$$

$$B^+(t) = \int_0^t (\bar{a}(s) dM^+(s) + \bar{b}(s) dM(s)).$$

A simple computation using the weak Itô formula shows that (B, B^+) is a Levy pair with covariance matrix

$$\begin{pmatrix} a(t) & b(t) \\ \bar{b}(t) & \bar{a}(t) \end{pmatrix} \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} \bar{a}(t) & b(t) \\ \bar{b}(t) & a(t) \end{pmatrix}.$$

We say that (B, B^+) is obtained from (M, M^+) by a *random time change*. Clearly random time changes are in one-to-one correspondence with the elements of the complex symplectic group of order two $Sp(2, \mathbf{C})$ whose entries are continuous complex valued functions.

Remark. If M is a classical brownian motion with variance $\sigma > 0$ (i.e., for all $s < t$, $M_t - M_s$ is independent of \mathcal{F}_s and has law $N(0, \sigma(t-s))$) then the time change that makes M a standard (i.e., with $\sigma = 1$) brownian motion is $t \rightarrow t/\sqrt{\sigma}$ so that $(M_{t/\sqrt{\sigma}})_{t \geq 0}$ is a standard brownian motion. By the self-similarity property of the classical brownian motion this is equivalent to the multiplication of the random variables M_t by $\sqrt{\sigma}$. This justifies our definition of random time change.

The action of $Sp(2, \mathbf{C})$ on positive definite 2×2 matrices has four types of orbits that are classified in [8]. These induce equivalence classes on Levy pairs.

DEFINITION 10.8. Two Levy pairs $(M^{(i)}, M^{(i)+})$ in Hilbert spaces $\mathcal{H}^{(i)}$ ($i = 1, 2$) that are martingales with respect to unit vectors $\Phi^{(i)}$ are

isomorphic if there exists a unitary operator $U: \mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(2)}$ with $U(\Phi^{(1)}) = \Phi^{(2)}$ such that the Levy pair in $\mathcal{H}^{(2)}(UM^{(1)}U^+, UM^{(1)+}U^+)$ can be obtained from $(M^{(2)}, M^{(2)+})$ by a random time change.

We can now prove the following theorem.

THEOREM 10.9. *Let (M, M^+) be a Levy pair in a Hilbert space \mathcal{H} that is a martingale with respect to a unit vector Φ such that, for all $t \in \mathbf{R}_+$, the covariance matrices $\sigma(t)$ are in the same orbit \mathcal{O} . Then (M, M^+) is isomorphic to:*

(i) *the classical real brownian motion of \mathcal{O} is the orbit of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,*

(ii) *the classical complex brownian motion if \mathcal{O} is the orbit of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,*

(iii) *the Fock brownian motion on $L^2(\mathbf{R}_+)$ if \mathcal{O} is the orbit of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,*

(iv) *the universally invariant brownian motion on $L^2(\mathbf{R}_+)$ with parameter $1/\lambda$ ($0 < \lambda < 1$) if \mathcal{O} is the orbit of the matrix $\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$.*

Proof. Using (10.7) we can suppose that (M, M^+) is a Levy pair in $\Gamma(L^2(\mathbf{R}_+, \sigma))$ that is a martingale with respect to the unit vector Φ_σ . The time continuity of the entries of the elements of $Sp(2, \mathbf{C})$ that reduce the matrices $\sigma(t)$ in the orbit \mathcal{O} to their canonical form follows from straightforward computations (cf. [8, Proof of (6.6)]). In case (iv) the matrices

$$\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}, \quad \begin{pmatrix} 1/\lambda+1 & 0 \\ 0 & 1/\lambda-1 \end{pmatrix}$$

are in the same orbit (take $a(t) = 1/\sqrt{\lambda}$, $b(t) = 0$); the symplectic form associated with the second is

$$(f, g) = \frac{1}{\lambda} (2 \operatorname{Re} f \bar{g}) + 2i \operatorname{Im} f \bar{g}.$$

Therefore M, M^+ is isomorphic to the universally invariant brownian motion on $L^2(\mathbf{R}_+)$ with parameter $1/\lambda$.

11. STOCHASTIC CALCULUS ON THE FINITE DIFFERENCE ALGEBRA

The following is another example of a Hilbert space in which a quantum stochastic calculus can be developed. It was introduced by A. Boukas in his

Ph.D. Thesis [13]. We show that our theory includes also this example and moreover it allows us to fill some gaps which were left open in [13].

Let S be the set of all real simple functions $f: (0, \infty) \rightarrow (-1; 1)$, i.e., the set of functions that can be written in the form

$$f = \sum_{i=1}^n a_i \chi_{A_i},$$

where $n \in \mathbf{N}$, $a_i \in \mathbf{R}$ with $|a_i| < 1$ and $(A_i)_{i=1}^n$ is a family of disjoint sets of $(0, \infty)$ of finite Lebesgue measure.

Let \mathcal{H} be the Hilbert space obtained by completion of the vector space generated by the family

$$\mathcal{D} = \{\psi(f) \mid f \in S\}$$

endowed with the pre-scalar product

$$\langle \psi(g), \psi(f) \rangle = \exp \left(- \int_0^\infty \log(1 - g(s)f(s)) ds \right).$$

The space $\mathcal{H}_{(s,t)}$ ($s, t \in \mathbf{R}$, $s < t$) is defined as the completion of the vector space generated by elements of \mathcal{D} corresponding to test functions f with support contained in (s, t) .

It is to be noted that we have the following tensor product factorizations.

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{(0,t)} \otimes \mathcal{H}_{(t,\infty)} = \mathcal{H}_{(0,s)} \otimes \mathcal{H}_{(s,t)} \otimes \mathcal{H}_{(t,\infty)} \\ \psi(f) &= \psi(f|_{[0,t]}) \otimes \psi(f|_{[t,\infty)}) = \psi(f|_{[0,s]}) \otimes \psi(f|_{[s,t]}) \otimes \psi(f|_{[t,\infty)}). \end{aligned}$$

We will denote by $\mathcal{A}_t = B(\mathcal{H}_{(0,t)}) 1_{\mathcal{H}_{(0,t)}}$ the algebra of operators up to time t .

Let us define the operators on \mathcal{H} (see [13, Definition (1.4)]).

$$\begin{aligned} Q(g) \psi(f) &= \frac{d}{d\varepsilon} [\psi(f + \varepsilon g) + \psi(e^{\varepsilon g} f)] \Big|_{\varepsilon=0} \\ P(g) \psi(f) &= \left[\int_0^\infty g f dt + Q(fg) \right] \psi(f) \\ T(g) \psi(f) &= \left[P(g) + Q(g) + \int_0^\infty g dt \right] \psi(f). \end{aligned}$$

For all $u, v, g \in S$ we have [13, (1.2.1)]

$$\begin{aligned} \langle \psi(v), Q(g) \psi(u) \rangle &= \langle P(g) \psi(v), \psi(u) \rangle \\ \langle \psi(v), T(g) \psi(u) \rangle &= \langle T(g) \psi(v), \psi(u) \rangle \end{aligned}$$

and

$$[P(v), Q(u)] = T(vu)$$

$$[P(v), T(u)] = T(vu)$$

$$[T(v), Q(u)] = T(vu).$$

We will also write $Q_g(s, t)$, $P_g(s, t)$, and $T_g(s, t)$ to denote respectively $Q(g_{t\downarrow}) - Q(g_{s\downarrow})$, $P(g_{t\downarrow}) - P(g_{s\downarrow})$, and $T(g_{t\downarrow}) - T(g_{s\downarrow})$. It is easy to see that $(Q_g(s, t))_{0 \leq s \leq t}$, $(P_g(s, t))_{0 \leq s \leq t}$, and $(T_g(s, t))_{0 \leq s \leq t}$ are additive adapted process.

We can prove also the integrator of scalar type estimates:

PROPOSITION 11.1. *Let $f, g \in h$ and F be a simple adapted process written in the form*

$$F_t = \sum_{j=1}^n F_{s_j} \chi_{(s_j, s_{j+1}]},$$

where

$$0 \leq s_1 < s_2 < \cdots < s_{n+1} < +\infty.$$

We have then

$$\begin{aligned} \left\| \int_0^t dQ_g(s) F(s) \psi(f) \right\|^2 &\leq (1 + \sqrt{2}) \int_0^t \|F(s) \psi(f)\|^2 \frac{|g(s)|^2}{(1 - f^2(s))^2} ds \\ \left\| \int_0^t dP_g(s) F(s) \psi(f) \right\|^2 &\leq (1 + \sqrt{2}) \int_0^t \|F(s) \psi(f)\|^2 \frac{|g(s)|^2 |f(s)|^2}{(1 - f^2(s))^2} ds \\ \left\| \int_0^t dT_g(s) F(s) \psi(f) \right\|^2 &\leq (1 + \sqrt{2}) \int_0^t \|F(s) \psi(f)\|^2 \frac{|g(s)|^2 (1 + f(s))^2}{(1 - f^2(s))^2} ds. \end{aligned}$$

Proof. We will prove only the first inequality. The proof of the other ones is the same. Let us suppose, for simplicity, $s_{n+1} = t$.

$$\begin{aligned} &\left\| \int_0^t dQ_g(s) F(s) \psi(f) \right\|^2 \\ &= \left\| \sum_{j=1}^n Q_g(s_j, s_{j+1}) F(s_j) \psi(f) \right\|^2 \\ &= \sum_{j=1}^n \|Q_g(s_j, s_{j+1}) F(s_j) \psi(f)\|^2 \\ &\quad + 2\operatorname{Re} \sum_{1 \leq k < j \leq n} \langle Q_g(s_k, s_{k+1}) F(s_k) \psi(f), Q_g(s_j, s_{j+1}) F(s_j) \psi(f) \rangle. \end{aligned}$$

Due to the tensor product factorization of \mathcal{H} for all j we have

$$Q_g(s_j, s_{j+1}) F(s_j) \psi(f) = F(s_j) \psi(f_{s_j}) \otimes Q_g(s_j, s_{j+1}) \psi(f_{[s_j]}).$$

Then the first sum can be rewritten as

$$\sum_{j=1}^n \|F(s_j) \psi(f_{s_j})\|^2 \|Q_g(s_j, s_{j+1}) \psi(f_{[s_j]})\|^2$$

and, applying [13, Proposition (1.3)(i)]

$$\sum_{j=1}^n \|F(s_j) \psi(f)\|^2 \left\{ \left(\int_{s_j}^{s_{j+1}} \frac{f(s) g(s)}{1-f^2(s)} ds \right)^2 + \int_{s_j}^{s_{j+1}} \frac{|g(s)|^2}{(1-f^2(s))^2} ds \right\}.$$

Then we consider the second sum

$$\begin{aligned} 2 \operatorname{Re} \sum_{1 \leq k < j \leq n} \langle Q_g(s_k, s_{k+1}) F(s_k) \psi(f_{s_j}), F(s_j) \psi(f_{s_j}) \rangle \\ \times \langle \psi(f_{[s_j]}), Q_g(s_j, s_{j+1}) \psi(f_{[s_j]}) \rangle. \end{aligned}$$

Applying [13, Proposition (1.2) (i)] this is equal to

$$\begin{aligned} 2 \operatorname{Re} \sum_{1 \leq k < j \leq n} \langle Q_g(s_k, s_{k+1}) F(s_k) \psi(f), F(s_j) \psi(f) \rangle \int_{s_j}^{s_{j+1}} \frac{g(s) f(s)}{(1-f^2(s))} ds \\ = 2 \operatorname{Re} \sum_{1 \leq j \leq n} \left\langle \int_0^{s_j} dQ_g(s) F(s) \psi(f), F(s_j) \psi(f) \right\rangle \int_{s_j}^{s_{j+1}} \frac{g(s) f(s)}{(1-f^2(s))} ds \end{aligned}$$

and the absolute value of this sum can be majorized by

$$2 \sum_{j=1}^n \|F(s_j) \psi(f)\| \left\| \int_0^{s_j} dQ_g(s) F(s) \psi(f) \right\| \int_{s_j}^{s_{j+1}} \frac{|g(s)| |f(s)|}{(1-f^2(s))} ds.$$

Let

$$\begin{aligned} R(t) &= \sup_{s \leq t} \left\| \int_0^s dQ_g(r) F(r) \psi(f) \right\| \\ a(t) &= \left(\int_0^t \|F(s) \psi(f)\|^2 \frac{|g(s)|^2}{(1-f^2(s))^2} ds \right)^{1/2} \\ \delta &= \sup_{1 \leq j \leq n} |s_{j+1} - s_j|. \end{aligned}$$

Majorizing $|f|$ by 1 and applying the Schwartz inequality we obtain

$$R^2(t) \leq 2R(t) a(t) + (1 + \delta) a^2(t)$$

that yields

$$R(t) \leq (1 + \sqrt{2 + \delta}) a(t).$$

The first inequality follows taking partition finer than the given one and letting δ go to zero.

So $(P_g(s, t))_{0 \leq s \leq t}$, $(Q_g(s, t))_{0 \leq s \leq t}$, and $(T_g(s, t))_{0 \leq s \leq t}$ are semimartingales in the sense of our definition and we can solve stochastic differential equations driven by them. However the quadratic variation of two of these processes is not a process like $(P_g(t))_{t \geq 0}$, $(Q_g(t))_{t \geq 0}$, or $(T_g(t))_{t \geq 0}$ nor a scalar process and so we cannot give a weak Itô formula and unitarity conditions for the solutions of stochastic differential equations.

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