On Square Roots of Measures.

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Introduction.

In the present work is studied an operational calculus on « square roots of measures » (**) on a measurable space $(\Omega, \mathscr{B})$, in which these ones are interpreted as vectors in a complex Hilbert space $\mathscr{H}(\Omega, \mathscr{B})$. The structure of the space $\mathscr{H}(\Omega, \mathscr{B})$ is completely determined by the fact that this space is isomorphic to the inductive limit of the family $\{L^2_{\mathbf{C}}(\Omega, \mathscr{B}, m)\}$ indexed by the set of all positive measures on $(\Omega, \mathscr{B})$, partially ordered by absolute continuity and with the natural immersions

$$J_{m,n}: f \in L^2_{\mathbf{C}}(\Omega, \mathscr{B}, m) \to f \cdot \sqrt{\frac{dm}{dn}} \in L^2_{\mathbf{C}}(\Omega, \mathscr{B}, n)$$

$(m < n)$ (cfr. Sect. **2**; cor. (2.8)).

Nevertheless, the construction of $\mathscr{H}(\Omega, \mathscr{B})$ considered here is purely algebraic and allows a simple description of the single elements of this inductive limit, which are identified with square roots of measures.

The consideration of the space $\mathscr{H}(\Omega, \mathscr{B})$ arises naturally in the case when Ω is a function space interpreted, for instance, as the phase space of a system with an infinite number of degrees of freedom. In this case, in fact, does not exist a « natural » measure on Ω and, moreover, the simplest groups of transformations in general do not preserve the equivalence class of a measure on Ω. Therefore the study of a dynamical system on Ω leads to the consideration of one-parameter families of inequivalent measures, which makes, in general, impossible the use of a single space $L^2_{\mathbf{C}}(\Omega, \mathscr{B}, q)$ as state space for such dynamical systems.

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(**) Unless explicitly stated the contrary « measure » will mean real bounded measure throughout the paper.

Moreover, the fact that automorphisms on $(\Omega, \mathscr{B})$ act as Markoff operators, in the sense of [3], on $\mathscr{M}_R(\Omega, \mathscr{B})$—the Banach space of measures on $(\Omega, \mathscr{B})$—and as unitary operators on the Hilbert space of square roots of measures (cfr. Sect. **2**) and the equality

$$\langle\sqrt{m}, f\cdot\sqrt{m}\rangle = \int_\Omega f\,\mathrm{d}m\,,$$

which holds for an arbitrary measure m on Ω, suggest that the formalism of « square roots of measures » could be used to interpret the results of SEGAL [4] and NELSON [5] in a context more general than that of Gaussian measures.

This formalism is applied here to obtain a classification of those measures on product spaces which are « well behaved » (cfr. Sect. **4**) with respect to product measures in terms of incomplete components of an infinite direct product of Hilbert spaces. In case of product measures the equivalence relation which induces this classification turns out to be the measure-theoretical formulation of the equivalence relation among C_0-families introduced by VON NEUMANN in [1] (Def. (3.3.2.)). In this memoir (cfr. [1], pag. 326) VON NEUMANN expresses his program of applying the theory of infinite direct product of Hilbert spaces to the study of measures on infinite-product spaces. Therefore the classification obtained in the present work can be considered as a way of making explicit and realizing this program.

Finally, thanks to the recent results of ARAKI [6] and CONNES [7], most of the results in this paper could be extended to the noncommutative case (*).

1. – Construction of the space $\mathscr{H}(\Omega, \mathscr{B})$.

Let $(\Omega, \mathscr{B})$ be a measurable space, denote $\mathscr{M}_R(\Omega, \mathscr{B})$ (resp. $\mathscr{M}_C(\Omega, \mathscr{B})$) the space of bounded real (respectively complex) measures on $(\Omega, \mathscr{B})$, which is a Banach space for the total variation norm. If $x, y \in \mathscr{M}_R(\Omega, \mathscr{B})$ we write y if x is absolutely continuous with respect to y; $x \perp y$ if x and y are orthogonal (*i.e.* if there exists disjoint sets $A, B \in \mathscr{B}$, such that $|x|(\Omega) = |x|(A)$; $|y|(B) = |y|(\Omega)$; $|x|$ denoting the measure « total variation » of x). If $x \in \mathscr{M}_R(\Omega, \mathscr{B})$ and $x = x^+ - x^-$ is the Jordan decomposition of x, $x^+ \perp x^-$; $|x| = x^+ + x^-$. The convex cone of positive measures in $\mathscr{M}_R(\Omega, \mathscr{B})$ will be denoted $\mathscr{M}_R^+(\Omega, \mathscr{B})$. If $m, n \in \mathscr{M}_R^+(\Omega, \mathscr{B})$, then there exists a unique positive bounded measure on $(\Omega, \mathscr{B})$, defined by

$$A \in \mathscr{B} \mapsto \int_A \sqrt{\frac{\mathrm{d}m}{\mathrm{d}q}\cdot\frac{\mathrm{d}n}{\mathrm{d}q}}\cdot \mathrm{d}q\,, \tag{1}$$

(*) The Author is grateful to Dr. A. CONNES for having called his attention on this circumstance during his lecture in Varenna.

where q is an arbitrary positive measure dominating m and n (*i.e.* $m < q$; $n < q$). The measure thus defined which is easily seen to be independent on q, will be denoted $\sqrt{m \cdot n}$.

Let $\mathscr{H}(\Omega, \mathscr{B})$ be the complex vector space spanned by the symbols $[x]$ where $x \in \mathscr{M}_R(\Omega, \mathscr{B})$; with the following relations among the generators

$$\text{(r.1)} \qquad [x] + [x'] - [x + x'], \qquad x \perp x';$$

$$\text{(r.2)} \qquad [\xi x] - \sqrt{\xi}\,[x], \qquad x \geqslant 0;\ \xi \in \mathbf{R}(\sqrt{-1} = i);$$

$$\text{(r.3)} \qquad [m] + [n] - [m + n + 2\sqrt{m \cdot n}], \qquad m, n \in \mathscr{M}_R^+(\Omega, \mathscr{B});$$

$$\text{(r.4)} \qquad [m] - [n] - [\chi_+(m + n - 2\sqrt{m \cdot n})] + [\chi_-(m + n - 2\sqrt{m \cdot n})],$$

where $m, n \in \mathscr{M}_R^+(\Omega, \mathscr{B})$, and χ_+, χ_- are the characteristic functions of a Jordan partition of Ω with respect to the measure $m - n$.

Denote E the free complex vector space spanned by the symbols $[x]$, and E_0 the subspace of E spanned by the expression (r.1), ..., (r.4). Define a sesquilinear form β, by continuation on $E \times E$ of the equality

$$\beta([x], [y]) = \varrho(x^+, y^+) + \varrho(x^-, y^-) - i\varrho(x^+, y^-) + i\varrho(x^-, y^+),$$

where $\varrho(m, n) = \sqrt{m \cdot n}(\Omega)$ denotes the Hellinger integral of m and n. A simple computation shows that E_0 is contained in the subspace of E

$$\{c \in E : \beta(c, c) = 0\}.$$

Lemma 1.1. Every element $c \in E$ determines uniquely two measures x, y such that

$$c \equiv [x^+] - [x] + i([y^+] - [y^-]) \qquad (\mathrm{mod}\ E_0).$$

Proof. Let $c \in E$. Then $c = \sum_{i \in F} \lambda_i [z_i]$, for a finite set F, and $z_i \in \mathscr{M}_R(\Omega, \mathscr{B})$. By iterate application of the relations (r.1), (r.2), (r.3), one finds four positive measures, m, n, m', n', such that

$$c \equiv [m] - [n] + i([m'] - [n']) \qquad (\mathrm{mod}\ E_0),$$

and the desired decomposition is obtained from this one by application of Equality (r.4). To prove uniqueness, let

$$c \equiv [x_\nu^+] - [x_\nu^-] + i([y_\nu^+] - [y_\nu^-]) \qquad (\mathrm{mod}\ E_0),$$

be two decompositions of c, mod E_0, according to the first statement of the

Lemma. Then the expression

$$([x_1^+]+[x_2^-])-([x_1^-]+[x_2^+])+i([y_1^+]+[y_2^-])-i([u_1^-]+[y_2^+])$$

belongs to E_0, hence E_0 contains also

$$c'=[z_1]-[z_2]+i([z_3]-[z_4])\,,$$

where $z_1=x_1^+ + x_2^- + 2\sqrt{x_1^+\cdot x_2^-}$ and z_i $(2\leqslant i\leqslant 4)$ are defined in a similar way. By the remark preceding the Lemma

$$0=\beta(c',c')=\|z_1\|+\|z_2\|+\|z_3\|+\|z_4\|-2\varrho(z_1,z_2)-2\varrho(z_3,z_4)\,.$$

From the equality

$$\|z_1\|+\|z_2\|-2\varrho(z_1,z_2)=\int\limits_{\Omega}\left(\sqrt{\frac{\mathrm{d}z_1}{\mathrm{d}q}}-\sqrt{\frac{\mathrm{d}z_2}{\mathrm{d}q}}\right)^2\mathrm{d}q$$

$(z_1< q;\, z_2< q)$ it follows that the expression on the left-hand side is positive, and null if and only if $z_1=z_2$. That is, if and only if

$$x_1^+ + x_2^- + 2\sqrt{x_1^+\cdot x_2^-}=x_1^- + x_2^+ + 2\sqrt{x_1^-\cdot x_2^+}\,.$$

The last equality implies $x_1^+=x_2^+$; $x_1^-=x_2^-$. The same reasoning applied to z_3 and z_4 yields $y_1^+=y_2^+$; $y_1^-=y_2^-$. Therefore the proof is completed.

Corollary 1.2. $E_0=\{c\in E:\beta(c,c)=0\}$.

Proof. One has only to prove that $\beta(c,c)=0$ implies $c\in E_0$. Let

$$c\equiv[x^+]-[x^-]+i([y^+]-[y^-])\qquad(\mathrm{mod}\ E_0)\,,$$

be the decomposition of c, mod E_0, according to Lemma 1.1. Then

$$0=\beta(c,c)=\|x\|+\|y\|\,,$$

and the uniqueness of the decomposition implies $c\in E_0$.

Denote $\pi: E\to\mathscr{H}(\Omega,\mathscr{B})=E/E_0$, the canonical projection. The scalar product $\langle\pi(l),\pi(l')\rangle=\beta(l,l')$; $l,l'\in E$, defines on $\mathscr{H}(\Omega,\mathscr{B})$ the structure of a separated pre-Hilbert space.

Lemma 1.3. The uniform structure induced on $\mathscr{M}_R(\Omega,\mathscr{B})$ by the map

(1.3.1) $$(x,y)\in\mathscr{M}_R(\Omega,\mathscr{B})\times\mathscr{M}_R(\Omega,\mathscr{B})\mapsto\max\{\|x^+-y^+\|;\ \|x^--y^-\|\}\,,$$

is isomorphic to the uniform structure induced by the norm.

Proof. It is not difficult to verify that the sets

$$U(\varepsilon)=\{(x, y)\in\mathscr{M}_R\times\mathscr{M}_R:\max\{\|x^+-y^+\|;\ \|x^--y^-\|\}<\varepsilon\}$$

for $\varepsilon>0$, are a base of a uniform structure on $\mathscr{M}_R(\Omega,\mathscr{B})$. One has to prove that the identity map on $\mathscr{M}_R(\Omega,\mathscr{B})$ is uniformly continuous from the above-defined uniform structure to the usual one and conversely. Clearly

$$\max\{\|x^+-y^+\|,\ \|x^--y^-\|\}\leqslant\varepsilon$$

implies $\|x-y\|\leqslant 2\varepsilon$; thus the uniform structure defined by the map (1.3.1) is finer than the one defined by the norm. Conversely, suppose $\|x-y\|\leqslant\varepsilon$. For any two measures u, v, denote $u(v)$ (respectively $u(v)^\perp$) the component of u absolutely continuous (respectively Orthogonal) with respect to v. From the equality

$$\|x-y\|=\|x^+(y^+)-y^+(x^+(y^+))\|+\|x^-(y^+)+y^+((x^+(x^+))^\perp\|+$$
$$+\|x^-(y^-)-y^-(x^-(y^-))\|+\|x^+(y^-)+y^-(x^-(y^-))^\perp\|+\|x^+(y^+)^\perp\|+\|x^-(y^-)^\perp\|$$

and the initial hypothesis it follows that each of the summands is less than ε, hence

$$\|x^+-y^+\|=\|x^+(y^+)-y^+(x^+(y^+))\|+\|y^+(x^+(y^+))\|^\perp+\|x^+(y^+)^\perp\|\leqslant\varepsilon\,.$$

Analogously $\|x^--y^-\|\leqslant\varepsilon$, hence the uniformity defined by the (1.3.1) is less fine than the usual one, and this ends the proof.

Now, one easily verifies that the sets

$$V(\varepsilon)=\{(x, y)\in\mathscr{M}_R\times\mathscr{M}_R:\|x\|+\|y\|\ \psi-2[\varrho(x^+, y^+)+\varrho(x^-, y^-)]\leqslant\varepsilon\}$$

for $\varepsilon>0$, are neighborhoods of the diagonal in a base of uniform structure on $\mathscr{M}_R(\Omega,\mathscr{B})$. We shall call this uniform structure the ϱ-uniformity.

Lemma 1.4. The ϱ-uniformity is isomorphic to the norm uniformity on each bounded subset of $\mathscr{M}_R(\Omega,\mathscr{B})$.

Proof. Because of the preceding Lemma, it will be sufficient to prove that the ϱ-uniformity if isomorphic, on each bounded set, to the uniformity defined by the map (1.3.1). Since, for positive measures m, n one has

$$\|m\|+\|n\|-2\varrho(m, n)=\int_\Omega\left(\sqrt{\frac{dm}{dq}}-\sqrt{\frac{dn}{dq}}\right)^2dq\leqslant\int_\Omega\left|\frac{dm}{dq}-\frac{dn}{dq}\right|dq\,,$$

it follows that the norm uniformity is less fine than the ϱ-uniformity. Consider now two positive measures m, n such that $m < n$; then

$$\|m-n\| = \int_\Omega \left|1-\frac{dm}{dn}\right| dn \leqslant \left\{\int_\Omega \left|1-\sqrt{\frac{dm}{dq}}\right|^2 dq\right\}^{\frac{1}{2}} \cdot \left\{\int_\Omega \left|1+\sqrt{\frac{dm}{dq}}\right|^2 dn\right\}^{\frac{1}{2}} \leqslant$$

$$\leqslant c(\|m\|;\|n\|)\cdot\sqrt{\|m\|+\|n\|-2\varrho(m,n)}\,,$$

where $c(\|m\|;\|n\|) = \sqrt{\|m\|} + \sqrt{\|n\|}$.

Thus if m, n are arbitrary positive measures belonging to a bounded subset of $\mathscr{M}_R(\Omega, \mathscr{B})$, there exists a constant λ, such that

$$\|m-n\| \leqslant \lambda\cdot\sqrt{\|m^<\| + \|n\| - 2\varrho(m,n) + \|m^\perp\|}\,,$$

where $m = m^< + m^\perp$, is the orthogonal decomposition of m with respect to n. Thus, if

$$\|m\| + \|n\| - 2\varrho(m,n) \leqslant \varepsilon < 1\,,$$

one has $\|m-n\| \leqslant (\lambda+1)\sqrt{\varepsilon}$, where the constant λ depends only on the bounded set to whom m and n belong. Hence the identity map is uniformly continuous from the ϱ-uniformity to the norm-uniformity, on each bounded set. Therefore the Lemma is proved.

Lemma 1.5. The ϱ-uniformity has the same Cauchy sequences as the norm-uniformity.

Proof. Since the norm-uniformity is less fine than the ϱ-uniformity (cfr. Lemma 1.4), and equivalent to it on each bounded set, it will be sufficient to prove that a Cauchy sequence for the ϱ-uniformity is bounded.

Let (X_n) be a ϱ-Cauchy sequence, let $\sup_n \|X_n\| = \infty$. Assume first that each x_ν is positive. Since

$$\|x_\mu\| + \|x_\nu\| - 2\varrho(x_\mu, x_\nu) \geqslant \left(\sqrt{\|x_\mu\|} - \sqrt{\|x_\nu\|}\right)^2$$

for any μ and K, one can find a $\nu > \mu$ such that

$$\|x_\mu\| + \|x_\nu\| - 2\varrho(x_\mu; x_\nu) > K\,,$$

and this contradicts the fact that (x_ν) is a ϱ-Cauchy sequence. Thus a ϱ-Cauchy sequence of positive measures is bounded but the sequence (X_ν) is a ϱ-Cauchy sequence if and only if (x_ν^+) and (x_ν^-) are such ones. Thus any ϱ-Cauchy sequence is bounded, and this ends the proof.

Proposition 1.6. The pre-Hilbert space $H = \pi(E)$ is a Hilbert space.

Proof. Let $(\pi(l_n))$ be a Cauchy sequence in H, then for each n, there exists a unique couple of measures x_n, y_n such that $x_n^+ - x_n^-$, (respectively $y_n^+ - y_n^-$) being the Jordan decomposition of x_n, (respectively y_n), one has

$$\pi(l_n) = \pi[x_n^+] - \pi[x_n^-] + i(\pi[y_n^+] - \pi[y_n^-]) ,$$

and since

$$\|\pi(l_n)\|^2 = \|\pi[x_n^+] - \pi[x_n^-]\|^2 + \|\pi[y_n^+] - \pi(y_n^-]\|^2 ,$$

one can limit oneself to the case $y_n = 0$, for each n. If q is any positive measure dominating all the x_n, then $(\pi[l_n])$ is a Cauchy sequence in H if and only if the sequence $(\sqrt{\mathrm{d}x_n^+/\mathrm{d}q} - \sqrt{\mathrm{d}x_n^-/\mathrm{d}q})$ is Cauchy in $L^2(\Omega, \mathscr{B}, q)$. If f is a limit for this sequence, define x by $x^+ = (f^+)^2 \cdot q$, $x^- = (f^-)^2 \cdot q$. Then $\pi[l_n] \to \pi[x] \in H$.

Definition 1.7. The mapping

$$J : \pi[x^+] - \pi[x^-] + i(\pi[y^+] - \pi[y^-]) \to \pi[x^+] - \pi[x^-] - i(\pi[y^-] - \pi[y^+]) ,$$

will be called the canonical involution in $\mathscr{H}(\Omega, \mathscr{B})$. Let us denote $\mathscr{H}_r(\Omega, \mathscr{B})$ the fixed space of J (*i.e.* $J(h) = h$, for every $h \in \mathscr{H}_r(\Omega, \mathscr{B})$). $\mathscr{H}_r(\Omega, \mathscr{B})$ is the *real* subspace of $\mathscr{H}(\Omega, \mathscr{B})$ spanned by the vectors of the type $\pi[m]$, where m is a positive measure on $(\Omega, \mathscr{B})$. We shall denote $\mathscr{H}_r^+(\Omega, \mathscr{B})$ the set of elements in $\mathscr{H}(\Omega, \mathscr{B})$, of the form $\pi[m]$, with $m \in \mathscr{M}_R^+(\Omega, \mathscr{B})$.

Theorem 1. The mapping $\varkappa: \mathscr{M}_R(\Omega, \mathscr{B}) \to \mathscr{H}(\Omega, \mathscr{B})$ defined by the equality $\varkappa(x) = \pi[x]$, induces an isomorphism of uniform structures from $\mathscr{M}_R(\Omega, \mathscr{B})$ onto its image which ejoys the following properties:

(i.1) $\quad \varkappa(x) = \varkappa(x^+) + i\varkappa(x^-)$;

(i.2) $\quad \varkappa(\xi x) = \sqrt{\xi} \cdot \varkappa(x)$, $\qquad x > 0; \quad \xi \in \mathbf{R} \quad (\sqrt{-1} = i)$;

(i.3) $\quad \|\varkappa(x)\|^2 = \|x\|$;

(i.4) $\quad \varkappa(m) + \varkappa(n) = \varkappa(m + n + 2\sqrt{m \cdot n})$, $\qquad m, n \in \mathscr{M}_R^+(\Omega, \mathscr{B})$;

(i.5) $\quad \varkappa(m) - \varkappa(n) = \varkappa((m + n - 2\sqrt{m \cdot n}) \cdot \chi_+) - \varkappa((m + n - 2\sqrt{m \cdot n}) \cdot \chi_-)$,

where $m, n \in \mathscr{M}_R^+(\Omega, \mathscr{B})$, and χ_+, χ_- are the characteristic functions of a Jordan partition for the measure $m - n$.

Moreover $\varkappa(\mathscr{M}_R(\Omega, \mathscr{B})) = \mathscr{H}_r^+(\Omega, \mathscr{B}) + i\mathscr{H}_r^+(\Omega, \mathscr{B})$ and $\mathscr{H}_r^+(\Omega, \mathscr{B})$ is a generating cone for $\mathscr{H}_r(\Omega, \mathscr{B})$.

Proof. Assertions (i.1), (i.2), (i.4), (i.5), follow from the relations (r.1), ..., (r.4). Assertion (i.3) follows from the equality

$$\|\varkappa(x)\|^2 = \varrho(x^+, x^+) + \varrho(x^-, x^-) = \|x\| .$$

Identity (i.1) implies that $\varkappa(\mathcal{M}_R) = \mathcal{H}_r^+ + i\mathcal{H}_r^+$; equalities (i.4), (i.5) and the fact that $\mathcal{H}_r(\Omega, \mathcal{B})$ is spanned by the $\varkappa(m)$, $m \in \mathcal{M}_R^+$ imply that $\mathcal{H}_r^+(\Omega, \mathcal{B})$ is a generating cone.

The equality

$$\|\varkappa(x) - \varkappa(y)\|^2 = \|x\| + \|y\| - 2\{\varrho(x^+, y^+) + \varrho(x^-, y^-)\}$$

and Lemmata 1.4, 1.5, imply that $\varkappa$ is an isomorphism of uniform structures.

Remark. The properties listed in Theorem I justify the interpretation of the $\varkappa(x)$ as the « square root » of the real bounded measure x on $(\Omega, \mathcal{B})$. For this reason, in the following, we will often use the notation $\varkappa(x) = \sqrt{x}$.

Theorem II. The mapping α of $\mathcal{M}_R(\Omega, \mathcal{B})$ onto the Hilbert space $\mathcal{H}_r(\Omega, \mathcal{B})$ defined by the equality $\alpha(x) = \varkappa(x^+) - x(x^-)$ is an isomorphism of uniform structures with the following properties:

(i.1) $\quad \|\alpha(x)\|^2 = \|x\|$;

(i.2) $\quad \alpha(\lambda x) = \sqrt{\lambda} \cdot \alpha(x)\,, \qquad \lambda \in \mathbf{R}^+$;

(i.3) $\quad \alpha(-m) = -\alpha(m)\,, \qquad m \in \mathcal{M}_R^+(\Omega, \mathcal{B})$;

(i.4) $\quad m, n \in \mathcal{M}_R^+(\Omega, \mathcal{B}) \Rightarrow \langle \alpha(m), \alpha(n)\rangle = 0\,,$ if and only if $m \perp n$;

(i.5) $\quad \alpha(m) + \alpha(n) = \alpha(m + n + 2\sqrt{m \cdot n})\,, \qquad m, n \in \mathcal{M}(\Omega, \mathcal{B})$;

(i.6) $\quad \alpha(m) - \alpha(n) = \alpha(\chi_{E^+} \cdot (m + n - 3\sqrt{mn})) - \alpha(\chi_{E^-} \cdot (m + n - 2\sqrt{mn}))\,,$

where $m, n \in \mathcal{M}_R^+(\Omega, \mathcal{B})$ and $\{E^+, E^-)$ is a Jordan partition of Ω, relative to the measure $m - n$.

Proof. $\alpha^{-1}(\mathcal{H}_r(\Omega, \mathcal{B})) = \mathcal{M}_R(\Omega, \mathcal{B})$, and α is an homeomorphism because of Theorem I. Moreover $\|\alpha(x)\|^2 = \varrho(x^+, x^+) + \varrho(x^-, x^-) = \|x\|$ and this establishes (i.1). The equality $\langle \alpha(m), \alpha(n)\rangle = \varrho(m, n)$ implies (i.4); and the remaining properties follow immediately from the corresponding ones in Theorem I. The properties (i.1), ..., (i.6) characterize the space $\mathcal{H}_r(\Omega, \mathcal{B})$ in the sense specified by the following Theorem.

Theorem III. Let H be a real Hilbert space and let $\beta: \mathcal{M}_R(\Omega, \mathcal{B}) \to H$ be an homeomorphism which enjoys properties i.1), ..., i.6) of Theorem II. Then H is unitarily isomorphic to $\mathcal{H}_r(\Omega, \mathcal{B})$.

Proof. Put $u = \beta\alpha^{-1}$. Clearly u is a norm-preserving homeomorphism. From i.2), i.3) homogeneity follows, and additivity is a consequence of i.5), (i.6). Exchanging u with u^{-1} one deduces that u and u^{-1} are, isometric isomorphisms, hence u is unitary and this ends the proof.

Corollary 1.8. There exists an isomorphism of uniform structures: $\gamma : \mathscr{M}_C(\Omega, \mathscr{B}) \to \mathscr{H}(\Omega, \mathscr{B})$, defined by the property

$$\gamma(x + iy) = \alpha(x) + i\alpha(y) .$$

2. – Action of aut$(\Omega, \mathscr{B})$ on $\mathscr{H}(\Omega, \mathscr{B})$.

Denote aut$(\Omega, \mathscr{B})$ the group of one-one mappings of Ω into itself which are measurable together with their inverses. Then aut$(\Omega, \mathscr{B})$ has a natural action on $\mathscr{M}_R(\Omega, \mathscr{B})$ (and $\mathscr{M}_C(\Omega, \mathscr{B})$), defined by

$$\hat{T}x = x \circ T^{-1} , \qquad x \in \mathscr{M}_C(\Omega, \mathscr{B}) , \qquad T \in \text{aut}(\Omega, \mathscr{B}) . \tag{2.1}$$

The operator $\hat{T}$ thus defined has the following properties:

M1) $\hat{T}$ is linear,

M2) $\hat{T}$ preserves positivity,

M3) $\|\hat{T}x\| = \|x\|$,

M4) $\hat{T}$ preserves absolute continuity. (*i.e.* $x < y \Leftrightarrow \hat{T}x < \hat{T}y$).

Following the notations of [3], any operator which satisfies these four conditions will be called a « Markov operator ».

Let now $\gamma : \mathscr{M}_C(\Omega, \mathscr{B}) \to \mathscr{H}(\Omega, \mathscr{B})$ be the map considered in Corollary 1.8; and

$$\tilde{T} \circ \gamma = \gamma \circ \hat{T} . \tag{2.2}$$

Proposition 2.1. The map $T \in \text{aut}(\Omega, \mathscr{B}) \to \tilde{T}$ defined by formula (2.2) is a unitary representation of aut $(\Omega, \mathscr{B})$ on the Hilbert space $\mathscr{H}(\Omega, \mathscr{B})$.

Proof. It is clear, from i.2), i.3) of Theorem II that $\tilde{T}(\lambda h) = \lambda \cdot \tilde{T}(h)$, $\lambda \in \mathbf{R}$ for $h \in \mathscr{H}_r(\Omega, \mathscr{B})$. Moreover applying i.4) and i.5) of Theorem II one finds

$$\tilde{T}[\alpha(m) \pm \alpha(n)] = \alpha[\hat{T}m + \hat{T}n \pm 2\hat{T}\sqrt{mn}] ,$$

thus the additivity of $\tilde{T}$ follows from the equality

$$\hat{T}\sqrt{m \cdot n} = \sqrt{(\hat{T}m) \cdot (\hat{T}n)} .$$

Let now $m, n \in \mathscr{M}_R^+(\Omega, \mathscr{B})$, then if $m, n < q$

$$\langle \tilde{T}\alpha(m), \tilde{T}\alpha(n) \rangle = \int_\Omega \sqrt{\frac{dm}{dq}(\overset{-1}{T}\omega) \cdot \frac{dn}{dq}(\overset{-1}{T}\omega)}\, dq(\overset{-1}{T}\omega) = \varrho(m, n) = \langle \alpha(m), \alpha(n) \rangle .$$

Finally, if $x \in \mathscr{M}_C(\Omega, \mathscr{B})$, and $S, T \in \mathrm{aut}\,(\Omega, \mathscr{B})$

$$(\widetilde{ST})(\gamma x) = \gamma(\hat{S}\hat{T}x) = \tilde{S}\tilde{T}(\gamma x)\,,$$

and clearly $\widetilde{id}_\Omega = id_{\mathscr{H}}$. Therefore, the map $T \to \tilde{T}$ is a unitary representation, and this ends the proof.

According to the notation introduced in the preceding Section, we shall write $\sqrt{\hat{T}x} = \sqrt{\hat{T}} \cdot \sqrt{x}$, and we will use the notation $\sqrt{\hat{T}}$, for $\tilde{T}$.

Similarly, if M denotes the action by multiplication of $L_R^\infty(\Omega, \mathscr{B})$-algebra of real bounded, measurable functions on $(\Omega, \mathscr{B})$, the action

$$\tilde{M}(f) \circ \gamma = \gamma \circ M(f)\,,$$

of $L_R^\infty(\Omega, \mathscr{B})$ on $\mathscr{H}(\Omega, \mathscr{B})$ is defined.

Proposition 2.2. The following equalities hold:

(i.1) $\qquad \tilde{M}(f) = \tilde{M}(f^-) + i\tilde{M}(f^-)$;

(i.2) $\qquad \tilde{M}(f \cdot g) = \tilde{M}(f) \cdot \tilde{M}(g)$;

(i.3) $\qquad \tilde{M}(f) + \tilde{M}(g) = \tilde{M}((\sqrt{f} + \sqrt{g})^2)\,, \qquad f, g \in L_+^\infty(\Omega, \mathscr{B})$;

(i.4) $\qquad \tilde{M}(f) - \tilde{M}(g) = \tilde{M}[\chi_+ \cdot (\sqrt{f} - \sqrt{g})^2] - \tilde{M}[\chi_- \cdot (\sqrt{f} - \sqrt{g})^2]\,,$

where $f, g \in L_+^\infty(\Omega, \mathscr{B})$ and χ_+, χ_- are the characteristic functions of the supports of $(f-g)$ and $(f-g)$ respectively;

(i.5) $\qquad \|\tilde{M}(f)\|^2 = \|f\|_\infty\,.$

Proof. Equality i.2) is clear. The remaining ones follow immediately from the corresponding ones in Theorem I.

Writing $\tilde{M}(f) = \sqrt{f}$, one has $\sqrt{f \cdot x} = \sqrt{f} \cdot \sqrt{x}$ and the following equality holds:

$$(2.3) \qquad \langle \sqrt{m}, \sqrt{f} \cdot \sqrt{m} \rangle = \int_\Omega f\, dm\,, \qquad m \in \mathscr{M}_R^+, f \in L_+^\infty\,.$$

We shall denote $\mathscr{A} = \mathscr{A}(\Omega, \mathscr{B})$ (respectively $\mathscr{A}_r$) the complex (respectively real) algebra spanned by the $\tilde{M}(f)$, with $f \in L_+^\infty(\Omega, \mathscr{B})$, which is isomorphic to $L_C^\infty(\mathscr{B}, \mathscr{B})$.

Definition 2.3. For every x in $\mathscr{M}_R(\Omega, \mathscr{B})$ denote $\mathscr{H}_x(\Omega, \mathscr{B})$ the norm closure, in $\mathscr{H}$, of $\mathscr{A}[\sqrt{x}]$.

Lemma 2.4. For every $x \in \mathcal{M}_R(\Omega, \mathcal{B})$ one has

$$\text{(i.1)} \qquad \mathcal{H}_x(\Omega, \mathcal{B}) = \gamma(\mathcal{M}_x(\Omega, \mathcal{B})) ,$$

where $\mathcal{M}_x(\Omega, \mathcal{B})$ denotes the subspace of the measures absolutely continuous with respect to x;

$$\text{(i.2)} \qquad \mathcal{H}_x(\Omega, \mathcal{B}) \approx L^2(\mathcal{A}; \omega_{\sqrt{x}}) ,$$

where $L^2(\mathcal{A}, \omega_{\sqrt{x}})$ is the Hilbert space obtained from $\mathcal{A}$ and its positive functional $\omega_{\sqrt{x}}: a \in \mathcal{A} \to \langle \sqrt{x}, a \cdot \sqrt{x} \rangle$ by means of the Gel'fand-Naimark-Segal construction.

(i.3) there exists a canonical isomorphism $\mathcal{H}_x(\Omega, \mathcal{B}) \xrightarrow{\psi_x} L^2_C(\Omega, \mathcal{B}, |x|)$ such that

$$\psi_x(\sqrt{x}) = 1 \qquad \text{and} \qquad \psi_x(\alpha(y)) = \sqrt{\frac{dy^+}{d|x|}} - \sqrt{\frac{dy^-}{d|x|}} .$$

Proof. (i.1): Since $\gamma^{-1}(\mathcal{A} \cdot \sqrt{x}) = L^\infty_C(\Omega, \mathcal{B}) \cdot x$, and since the closure of the latter space in $\mathcal{M}_C$ is $\mathcal{M}_x$, the equality $\gamma(\mathcal{M}_x) = \mathcal{H}_x$ is a consequence of the fact that γ is a homeomorphism.

(i.2) is just the definition of $(\mathcal{H}_x(\Omega, \mathcal{B})$. Finally (i.3) follows from (i.2) and equality (2.3).

Proposition 2.5. Let $x, y \in \mathcal{M}_R(\Omega, \mathcal{B})$. Then $x \perp y$ if and only if $\mathcal{H}_x(\Omega, \mathcal{B})$ is orthogonal to $\mathcal{H}_y(\Omega, \mathcal{B})$; $x < y$ if and only if $\mathcal{H}_x(\Omega, \mathcal{B}) \subseteq \mathcal{H}_y(\Omega, \mathcal{B})$.

Proof. Suppose $\mathcal{H}_x \perp \mathcal{H}_y$; then for every $f \in L^\infty_+(\Omega, \mathcal{B})$, $\int_\Omega f \, d\sqrt{xy} = 0$ (for positive x, y). Thus $x \perp y$; for general x, y the assertion follows by linearity. Since the converse is obvious, the first assertion of the Lemma is proved. The second one follows from (i.1) of the preceding Lemma.

The group aut $(\Omega, \mathcal{B})$ acts on $L^\infty_C(\Omega, \mathcal{B})$ by means of

$$\text{(2.4)} \qquad \hat{\hat{T}}(f) = f \circ \overset{-1}{T} , \qquad f \in L^\infty_C(\Omega, \mathcal{B}) , \qquad T \in \text{aut}\,(\Omega, \mathcal{B}) .$$

Equality (2.4) defines an automorphism of the C^*-algebra $L^\infty_C(\Omega, \mathcal{B})$. Moreover the following equality holds:

$$\text{(2.5)} \qquad \hat{T} \circ M(f) = M(\hat{\hat{T}} f) \circ \hat{T} ,$$

which connects the natural actions of L^∞ and aut on $\mathcal{M}_C(\Omega, \mathcal{B})$, with the action of aut on L^∞, defined by (2.4).

Corollary (2.6). For every $x \in \mathcal{M}_R(\Omega, \mathcal{B})$, and $T \in \operatorname{aut}(\Omega, \mathcal{B})$,

$$\mathcal{H}_{\hat{T}x}(\Omega, \mathcal{B}) = \sqrt{\hat{T}}\,[\mathcal{H}_x(\Omega, \mathcal{B})]\,.$$

In particular the space $\mathcal{H}_x$ is invariant for the unitary operator $\sqrt{\hat{T}}$, if and only if the measure x is quasi-invariant for the automorphism T.

Proof. From (2.5) follows $\sqrt{\hat{T}}\,(\mathcal{A}\cdot\sqrt{x}) = \mathcal{A}\sqrt{\hat{T}x}$, which proves the first assertion. The second one follows from Proposition 2.5. Let for any $x \in \mathcal{M}_R(\Omega, \mathcal{B})$, $\mathcal{G}_x(\Omega, \mathcal{B})$ be the subgroup of the $\sqrt{\hat{T}}$ which leaves $\mathcal{H}_x(\Omega, \mathcal{B})$ invariant. From Corollary 2.6 it follows that

$$\mathcal{G}_x(\Omega, \mathcal{B}) = \{\sqrt{\hat{T}} : T \in \operatorname{aut}(\Omega, \mathcal{B}),\ \hat{T}x \sim x\}\,.$$

Corollary 2.7. $\mathcal{A}_x \otimes^{(\mathcal{H}_x)} \mathcal{G}_x \approx \mathcal{B}(\mathcal{H}_x)$.

Proof. Clearly $\mathcal{G}_x$ acts ergodically on $\mathcal{A}_x$.

Corollary. 2.8. $\mathcal{H}(\Omega, \mathcal{B}) \approx \underrightarrow{\lim}_x\, L^2_C(\Omega, \mathcal{B}, x)$.

3. – The isomorphism $\mathcal{H}_r \sigma^{\mathcal{A}} \mathcal{H}_r \approx \mathcal{M}_R(\Omega, \mathcal{B})$.

In the previous Sections the complex Hilbert space $\mathcal{H}(\Omega, \mathcal{B})$ has been identified with the « square root » of the real Banach space $\mathcal{M}_R(\Omega, \mathcal{B})$. In the present Section the converse problem is studied: that is, to express the Banach space $\mathcal{M}_R(\Omega, \mathcal{B})$ as the « square of its square root ».

Let $x, y \in \mathcal{M}_R(\Omega, \mathcal{B})$, $f \in L^\infty_R(\Omega, \mathcal{B})$. From the equality

$$\langle\sqrt{f}\cdot\sqrt{x};\sqrt{y}\rangle = \int_\Omega \sqrt{f}\cdot\{\sqrt{x^+y^+} + \sqrt{x^-y^-}\} + i\int_\Omega \sqrt{f}\cdot\{\sqrt{x^-y^+} - \sqrt{x^+y^-}\}$$

it follows that the map $\sqrt{f} \mapsto \langle\sqrt{f}\cdot\sqrt{x}, \sqrt{y}\rangle$ defines a complex measure on $L^\infty_R(\mathcal{B}, \hat{T})$, therefore the equality

$$\bar{\mu}(\sqrt{x};\sqrt{y}) = (\sqrt{x^+y^+} + \sqrt{x^-y^-}) + i(\sqrt{x^-y^+} - \sqrt{x^+y^-}) \tag{3.1}$$

determines a sesquilinear map from $\mathcal{H}\times\mathcal{H}$ into $\mathcal{M}_C(\Omega, \mathcal{B})$ such that

$$\|\bar{\mu}(\sqrt{x};\sqrt{y})\| \leqslant \sqrt{\|x\|\cdot\|y\|} = \|\sqrt{x}\|\cdot\|\sqrt{y}\|\,.$$

Thus the restriction of $\bar{\mu}$ to $\mathcal{H}_r\times\mathcal{H}_r$ extends to a (real) linear map from $\mathcal{H}_r \otimes \mathcal{H}_r$ to $\mathcal{M}_R(\Omega, \mathcal{B})$ which will be denoted μ. The generic element of $\mathcal{H}_r$ has the form $\alpha(x)$ (cfr. Theorem II) and

$$\mu(\alpha(x)\otimes\alpha(y)) = (\sqrt{x^+y^+} + \sqrt{x^-y^-}) - (\sqrt{x^-y^+} + \sqrt{x^+y^-})\,. \tag{3.2}$$

Lemma 3.1. Denote $\hat{\alpha}(f) = \sqrt{f^+} - \sqrt{f^-}$. Then $\ker \mu$ contains the following expressions:

$$\alpha(x) \otimes \alpha(y)\,, \qquad x \perp y; \tag{i.1}$$

$$[\hat{\alpha}(f)^2 \cdot \alpha(x)] \otimes \alpha(y) - [\hat{\alpha}(f) \cdot \alpha(x)] \otimes [\hat{\alpha}(f) \cdot \alpha(y)]\,. \qquad f \geqslant 0; \tag{i.2}$$

Proof. Both assertions follow immediately from equality (3.2).

Denote now by $\mathcal{O}_r$ the closed subspace of $\mathcal{H}_r \otimes \mathcal{H}_r$ spanned by expressions (i.1), (i.2).

Lemma 3.2. Let $\mathcal{H}_r \otimes^{(0)} \mathcal{H}_r$ denote the algebraic tensor product of $\mathcal{H}_r$ by itself. Then any vector h in $\mathcal{H}_r \mathcal{H}^{(0)} \mathcal{H}_r$ can be written in the form

$$h \equiv \sqrt{u^+} \otimes \sqrt{u^+} - \sqrt{u^-} \otimes \sqrt{u^-} \qquad (\mathrm{mod}\ \mathcal{O}_r)\,,$$

where u^+, u^- are positive measures mutually orthogonal.

Proof. Let $h = \sum_{i \in F} \lambda_i \sqrt{x_i} \otimes \sqrt{y_i}$ (F a finite set) be an expression for $h \in \mathcal{H}_2 \otimes^{(0)} \mathcal{H}_r$. Clearly one can suppose the x_i, y_i positive, and $x_i < y_i$, modulo $\mathcal{O}_r$. Now, if f, g, f', g' are positive functions in $L^1(\Omega, \mathcal{B}, y)$, the inequality

$$\|\sqrt{fy} \otimes \sqrt{gy} - \sqrt{f'y} \otimes \sqrt{g'y}\| \leqslant c_1 \sqrt{\|g\|_{L^1(y)} \cdot \|f - f'\|_{L^1(y)}} + c_2 \sqrt{\|f'\|_{L^1(y)} \cdot \|g - g'\|_{L^1(y)}}$$

(c_1, c_2 are constants depending only on y), it follows that

$$\sqrt{x_i} \otimes \sqrt{y_i} - \sqrt{z_i} \otimes \sqrt{z_i}\,, \qquad z_i = \sqrt{\frac{\mathrm{d}x_i}{\mathrm{d}y_i}} \cdot y_i\,,$$

can be uniformly approximated by expressions of the type (i.2) of Lemma 3.1, hence it lies in $\mathcal{O}_r$. This yields

$$h = \sum_{i \in F} \lambda_i \sqrt{z_i} \otimes \sqrt{z_i} \qquad (\mathrm{mod}\ \mathcal{O}_r)\,,$$

where $z_i = (\sqrt{\mathrm{d}x_i/\mathrm{d}y_i}) \cdot y_i$ and the λ_i can be assumed to be equal to ± 1.

We now apply an analogous of the Schmidt orthogonalization process: consider the sum $\lambda_1 \sqrt{z_1} \otimes \sqrt{z_1} + \lambda_2 \sqrt{z_2} \otimes \sqrt{z_2}$. If $z_2 = z_{2,1} + z_{2,2}$ is the Hahn-Lebesgue decomposition of z_2 with respect to z_1 (that is $z_{2,1} \perp z_{2,2}$, $z_{2,1} < z_1$, $z_{2,2} \perp z_1$), it is clear that the sum is equivalent mod $\mathcal{O}_r$ to

$$\lambda_1 \cdot \sqrt{z_1} \otimes \sqrt{z_1} + \lambda_2 \sqrt{z_{2,1}} \otimes \sqrt{z_{2,1}} + \lambda_2 \sqrt{z_{2,2}} \otimes \sqrt{z_{2,2}}\,. \tag{3.2.1}$$

Now, if $x < y$, and x, y are both positive measures, one easily verifies the equalities

$$(3.2.2)\quad \begin{cases} \sqrt{x}\otimes\sqrt{x}+\sqrt{y}\otimes\sqrt{y}=\sqrt{x+y}\otimes\sqrt{x+y} & (\mathrm{mod}\ \mathcal{O}_r)\,, \\ \sqrt{x}\otimes\sqrt{x}-\sqrt{y}\otimes\sqrt{y}= & \\ \quad =\sqrt{(x-y)^+}\otimes\sqrt{(x-y)^+}-\sqrt{(x-y)^-}\otimes\sqrt{(x-y)^-} & (\mathrm{mod}\ \mathcal{O}_r)\,. \end{cases}$$

From these equalities, applying inductively the decomposition (3.2.1), one obtains

$$h=\sum_{i\in F^+}\sqrt{u_i}\otimes\sqrt{u_i}-\sum_{J\in F^-}\sqrt{u_J}\otimes\sqrt{u_J}\qquad(\mathrm{mod}\ \mathcal{O}_r)\,,$$

where the u_i are mutually orthogonal. Therefore

$$h=\sqrt{\sum_{i\in F^+}u_i}\otimes\sqrt{\sum_{i\in E^+}u_i}-\sqrt{\sum_{i\in F^-}u_i}\otimes\sqrt{\sum_{i\in F^-}u_i}\,,$$

and the thesis follows by putting

$$u^+=\sum_{i\in F^+}u_i\,,\qquad u^-=\sum_{i\in F^-}u_J\,.$$

Remark that in general $\sqrt{u^+}\otimes\sqrt{u^+}-\sqrt{u^-}\otimes\sqrt{u^-}$ is not the orthogonal projection of h with respect to $\mathcal{O}_r$.

Lemma 3.3. For any $h\in\mathcal{H}_r\otimes^{(0)}\mathcal{H}_2$ the representation described in the Lemma above is unique.

Proof. Let $v=v^+-v^-\in\mathcal{M}_R(\Omega,\mathcal{B})$ be another measure which satisfies the condition of Lemma 3.2. Then

$$\sqrt{u^+}\otimes\sqrt{u^+}-\sqrt{u^-}\otimes\sqrt{u^-}-\sqrt{v^+}\otimes\sqrt{v^+}+\sqrt{v^-}\otimes\sqrt{v^-}\in\mathcal{O}_r\,.$$

Consider $v^+=v^+_++v^+_-+v^+_0$ (respectively $v^-=v^-_-+v^-_++v^-_0$) the orthogonal decomposition of v^+ (respectively v^-) which respect to u^+ and u^- (*i.e.* $v^+_+<u^+$; $v^+_-<u^-$; $v^+_0\perp|u|$; and analogously for v^-). Then, by iterate application of the equalities (i.1), (i.2), one can reduce the above expression to

$$\begin{aligned}\eta=\sqrt{(u^++v^-_+-v^+_+)^++(u^-+v^+_--v^-_-)^-+v^+_0}\otimes&\\ \otimes\sqrt{(u^++v^-_+-v^+_+)^++(u^-+v^+_--v^-_-)^-+v^+_0}+&\\ +\sqrt{(u^++v^-_+-v^+_+)^-+(u^-+v^+_--v^-_-)^++v^-_0}\otimes&\\ \otimes\sqrt{(u^++v^-_+-v^+_+)^-+(u^-+v^+_--v^-_-)^++v^-_0}\,,&\end{aligned}$$

where $\eta \in \mathcal{O}_r$. Hence it follows

$$u^+ + v_+^- - v_+^+ = u^- + v_-^+ - v_-^- = v_0^+ = v_0^- = 0\,,$$

hence, from the definition of the v_-^+

$$v_+^- = v_-^+ = 0\,, \qquad v_+^+ = u^+\,, \qquad v_-^- = u^-\,,$$

that is $u = v$; and this proves the Lemma.

Lemma 3.4. $\mathcal{O}_r = \ker \mu$.

Proof. By construction $\mathcal{O}_r \subseteq \ker \mu$. We shall prove that it is dense therein. Let $h \in \ker \mu$ and $1 > \varepsilon > 0$. There is an $h' \in \mathscr{H}_r \otimes^{(0)} \mathscr{H}_r$ such that $\|h - h'\| \leqslant \varepsilon$. If

$$h' = \sqrt{u^+} \otimes \sqrt{u^+} - \sqrt{u^-} \otimes \sqrt{u^-} \qquad (\operatorname{mod} \mathcal{O}_r)\,,$$

is the decomposition of h' described in Lemma 3.2, one has

$$\|u^+\| + \|u^-\| \leqslant \|\mu(h - h')\| \leqslant \|h - h'\| \leqslant \varepsilon\,.$$

Hence there is an $\eta \in \mathcal{O}_r$ such that $\|h - \eta\| \leqslant 2\sqrt{\varepsilon}$, and therefore $\mathcal{O}_r$ is dense in $\ker \mu$.

Lemma 3.5. The space $\mathscr{H}_r \otimes \mathscr{H}_r / \mathcal{O}_r$ is isomorphic, as a normed space, to $\mathscr{M}_R(\Omega, \mathscr{B})$.

Proof. The mapping from $\mathscr{H}_r \otimes \mathscr{H}_r / \mathcal{O}_r$ to $\mathscr{M}_R(\Omega, \mathscr{B})$ naturally induced by μ is a contraction and equality (3.2) implies that it is surjective. If $h \in \mathscr{H}_r \otimes \mathscr{H}_r$, then there exists $u = u^+ - u^- \in \mathscr{M}_R(\Omega, \mathscr{B})$ such that

$$h - \sqrt{u^+} \otimes \sqrt{u^+} - \sqrt{u^-} \otimes \sqrt{u^-} \in \mathcal{O}_r$$

(as one easily verifies by approximating h with elements of the algebraic tensor product $\mathscr{H}_r \otimes^{(0)} \mathscr{H}_r$). Hence

$$\operatorname{Inf}\{\|h + \eta\| : \eta \in \mathcal{O}_r\} \leqslant \|u\|\,.$$

Thus if $\pi : \mathscr{H}_r \otimes \mathscr{H}_r \to \mathscr{H}_r \otimes \mathscr{H}_r / \mathcal{O}_r$ is the canonical projection

$$\|\pi(h)\| \leqslant \|\mu(h)\| = \|u\|\,,$$

hence $\|\pi(h)\| = \|\mu(\pi(h))\|$, and this ends the Proof.

Definitions 3.6. Let H be an Hilbert space and A a C^*-algebra of operators on H. We shall denote $H\,\sigma^A\,H$, the normed space $H\otimes H/\mathcal{O}(A)$, where $\mathcal{O}(A)$ is the closed subspace of $H\otimes H$ spanned by the expressions

i) $h\otimes K\,,\quad h\perp K\,,\quad h, K\in H;$

ii) $(a^*ah)\otimes K-(ah)\otimes(aK),\quad a\in A,\ h, k\in H.$

We sum up the above considerations in the following:

Theorem IV. There is a natural isomorphism of Banach spaces

$$\mathcal{M}_R(\Omega,\mathcal{B})\approx\mathcal{H}_r(\Omega,\mathcal{B})\,\sigma^{\mathcal{A}_r}\,\mathcal{H}_r(\Omega,\mathcal{B})\,.$$

4. – On the classification of functional measures.

In Sect. **3** has been studied the mapping $\mu\colon\mathcal{H}\otimes\mathcal{H}\to\mathcal{M}_C$ which arises from the equality

$$\langle\sqrt{x};\ |f|\cdot\sqrt{x}\rangle=\int_\Omega |f|\,\mathrm{d}|x|\,. \tag{4.1}$$

In the present paragraph the corresponding equality will be considered in the case when Ω is a function space. To this aim, let us consider a family $(\Omega_i,\mathcal{B}_i)_{i\in I}$ of measurable spaces, and denote, for each $i\in I$, $\mathcal{H}(\Omega_i,\mathcal{B}_i)$ the complex Hilbert space constructed as in Sect. **1**. The infinite direct product of the family $\{\mathcal{H}(\Omega_i,\mathcal{B}_i)\}$ can be thus formed, and it splits, according to [1], into its « incomplete components » $\bigotimes_{i\in I}^{\mathfrak{G}}\mathcal{H}(\Omega_i,\mathcal{B}_i)$ relative to the equivalence classes of C_0-families.

For $i\in I$ let $\mathcal{A}^{(i)}$, $\mathcal{A}^{(i)}_r$ denote the Abelian algebras built in Sect. **2**. It is known that, for any finite set $F\subseteq I$, and $a_i\in\mathcal{A}^{(i)}$ there exists a unique operator $\bigotimes_{i\in F}a_i$ defined on $\bigotimes_{i\in I}\mathcal{H}(\Omega_i,\mathcal{B}_i)$ by the equality

$$\Big(\bigotimes_{i\in F}a_i\Big)\cdot\Big(\bigotimes_{i\in I}\sqrt{x_i}\Big)=\Big(\bigotimes_{i\in F}a_i\sqrt{x_i}\Big)\otimes\Big(\bigotimes_{i\in I-F}\sqrt{x_i}\Big)\,.$$

If $a_i=\hat{\alpha}_i(f_i)$, then

$$\Big\|\Big(\bigotimes_{i\in F}a_i\Big)\cdot\Big(\bigotimes_{i\in I}\sqrt{x_i}\Big)\Big\|^2=\int_{\prod_{i\in I}\Omega_i}\Big(\prod_{i\in F}|f_i|\Big)\cdot\mathrm{d}\prod_{i\in I}|x_i|\,,$$

where the measure $\prod_{i\in I}|x_i|$ is well defined because $\sum_{i\in I}|1-\|\sqrt{x_i}\||<\infty$.

Introducing the notation $(\Omega,\mathcal{B})=\prod_{i\in I}(\Omega_i,\mathcal{B}_i)$, and making use of equality (4.1), one deduces

$$\Big\|\Big(\bigotimes_{i\in F}\hat{\alpha}_i(f_i)\Big)\cdot\Big(\bigotimes_{i\in I}\alpha_i(x_i)\Big)\Big\|=\Big\|\hat{\alpha}\Big(\prod_{i\in F}|f_i|\Big)\cdot\alpha\Big(\prod_{i\in I}|x_i|\Big)\Big\|\,.$$

If the x_i are positive measures and the f_i are positive functions, the above equality becomes simply

$$\left\|\left(\bigotimes_{i\in F}\sqrt{f_i}\right)\cdot\left(\bigotimes_{i\in I}\sqrt{x_i}\right)\right\| = \left\|\sqrt{\prod_{i\in F} f_i}\cdot\sqrt{\prod_{i\in I} x_i}\right\|.$$

Thus one is naturally led to think these last two equalities as arising from a correspondence between $\mathscr{H}(\Omega, \mathscr{B})$ and $\bigotimes_{i\in I}\mathscr{H}(\Omega_i, \mathscr{B}_i)$ which extends the map

$$\bigotimes_{i\in I}\sqrt{m_i} \mapsto \sqrt{\prod_{i\in I} m_i} \tag{4.2}$$

defined in the case when all the m_i are positive.

Let us begin the study of the extension of the map (4.2) by restricting our attention to the incomplete direct product $\bigotimes_{i\in I}^{(\otimes\sqrt{m})}\mathscr{H}_r(\Omega_1, \mathscr{B}_i)$ in the case where almost all the m_i (*i.e.* all but a finite number) are positive measures. From now one, a C_0-family in $\otimes\mathscr{H}_r(\Omega_i, \mathscr{B}_i)$ which enjoys this property will be called an « essentially positive C_0-family ».

From [1] (Lemma (3.3.7)) it follows that we can suppose each m_i to be a probability measure.

Lemma 4.1. Let $(\alpha_i(x_i))$ be a C_0-family in $\bigotimes_{i\in I}\mathscr{H}_r(\Omega_i, \mathscr{B}_i)$. Then $(\alpha_i(x_i))$ is equivalent (in the sense of [1]) to an essentially positive C_3-family, if and only if $(\sqrt{x_i^+})_{i\in I}$, is a C_0-family. In such a case one has

$$(\alpha_i(x_i)) \approx (\sqrt{x_i^+}) \quad\text{and}\quad \sum_{i\in I} x_i^-(\Omega_i) < \infty .$$

Proof. Sufficiency. Recall that, by definition $\alpha_i(x_i) = \sqrt{x_i^+} - \sqrt{x_i^-}$; thus if the conditions stated in the Lemma are satisfied, clearly $(\alpha_i(x_i))$ is a C_0-family, and

$$\sum_{i\in I}|1-\langle a_i(x_i), \sqrt{x_i^+}\rangle| = \sum_{i\in I}|1-\|\sqrt{x_i^+}\|^2| < \infty,$$

which implies $(\alpha_i(x_i)) \approx (\sqrt{x_i^+})$.

Necessity. Suppose that $(\alpha_i(x_i))$ is a C_0-family equivalent to the essentially positive C_0-family $(\sqrt{m_i})$. Suppose first that, for almost all $i \in I$, $\|\alpha_i(x_i)\| = 1$. Then from the inequality $\varrho(x_i^+, m_i) \leqslant \sqrt{x_i^+(\Omega_i)}$ it follows

$$\sum_{i\in I}\varrho(m_i, x_i^-) \leqslant \sum_{i\in I}[(1-\varrho(m_i, x_i^+)) + \varrho(m_i, x_i^-)] = \sum_{i\in I}|1-\langle\alpha_i(x_i), \sqrt{m_i}\rangle| < \infty,$$

and therefore

$$\sum_{i\in I}|1-\|\sqrt{x^+_i}\|| \leqslant \sum_{i\in I}[1-\varrho(m_i, x_i^+)] < \infty .$$

These two inequalities imply that $(\sqrt{x_i^+})$ is a C_0-family equivalent to $(\sqrt{m_i})$. By transitivity $(\sqrt{x_i^+})$ is equivalent to $(\alpha_i(x_i))$, and this implies $\sum_{i\in I} x_i^-(\Omega_i) < \infty$. Now, if $\|\alpha_i(x_i)\| \neq 1$, for infinite i, equivalence implies that there is a finite set $F_0 \subseteq I$, such that $\prod_{i\in I-F_0} \langle \alpha_i(x_i); \sqrt{m_i}\rangle > 0$. Then applying Lemmata (3.3.6) and (3.3.7) of [1], one deduces

$$\sum_{i\in I} \left| 1 - \left\langle \sqrt{m_i}; \frac{a_i(x_i)}{\|a_i(x_i)\|} \right\rangle \right| < \infty ,$$

and the above result implies that $(\sqrt{x_i^+}/\|\alpha_i(x_i)\|)$ is a C_0-family equivalent to $(\sqrt{m_i})$.

Therefore, also $(\sqrt{x_i^+})$ is a C_0-family, and

$$\left\langle \bigotimes_{i\in F-F_0} \sqrt{m_i}; \left(\bigotimes_{\alpha\in I-F_0} z_\alpha\right)\cdot\left(\bigotimes_{i\in I-F_0} \sqrt{x_i^+}\right)\right\rangle = \left(\bigotimes_{\alpha\in I-F_0} z_\alpha\right)\cdot\left\langle \bigotimes_{i\in I-F_0} \sqrt{m_i};\ \bigotimes_{i\in I-F_0} \sqrt{x_i^+}\right\rangle > 0$$

(where we have put $z_i = 1/\|\alpha_i(x_i)\|$), but this relation implies $(\sqrt{x_i^+}) \approx (\sqrt{m_i})$. Finally, since both $(\alpha_i(x_i))$ and $(\sqrt{x_i^+})$ are C_0-families, one must have $\sum_{i\in I} x_i^-(\Omega_i) < \infty$. Equivalence implies that $(\sqrt{x_i^+}) \approx (\alpha_i(x_i))$, and therefore the proof is completed.

Denote now

$$\mathscr{H}_r(\Omega, \mathscr{B}) = \mathscr{H}_r\left(\prod_{i\in I} (\Omega_i, \mathscr{B}_i)\right), \qquad \alpha: \mathscr{M}_R(\Omega, \mathscr{B}) \to \mathscr{H}_r(\Omega, \mathscr{B})$$

the real Hilbert space and the map defined in Theorem II. Consider the map

$$u_0: \left(\bigotimes_{i\in F} \alpha_i(x_i)\right) \otimes \left(\bigotimes_{i\in I-F} \sqrt{m_i}\right) \to \alpha\left(\left(\prod_{i\in F} x_i\right)\cdot\left(\prod_{i\in I-F} m_i\right)\right),$$

which is defined on a total subset of $\bigotimes_{i\in I}^{(\sqrt{m_i})} \mathscr{H}_r(\Omega_i, \mathscr{B}_i)$ and takes values in $\mathscr{H}(\Omega, \mathscr{B})$.

Lemma 4.2. The map u_0 admits a unique extension u into a unitary transformation from $\bigotimes_{i\in I}^{(\sqrt{m_i})} \mathscr{H}_r(\Omega_i, \mathscr{B}_i)$ onto a closed subspace of $\mathscr{H}(\Omega, \mathscr{B})$. The unitary transformation U is characterized by the property that the image by U of any essentially positive C_0-family $(\alpha_i(x_i))$ in $\bigotimes_{i\in I}^{(\sqrt{m_i})} \mathscr{H}_r(\Omega_i, \mathscr{B}_i)$ is the image $\alpha\left(\prod_{i\in I} x_i\right)$ of the product measure $\prod_{i\in I} x_i$:

Proof. Suppose F and F' contain a single index i (respectively i'). Then

$$\left\langle \alpha\left(x_i\cdot\prod_{I-\{i\}} m_J\right); \alpha\left(y_i\cdot\prod_{I-\{i'\}} m_J\right)\right\rangle =$$
$$= \left\langle \sqrt{x_i^+}\cdot\prod_{I-\{i\}} m_J; \sqrt{y_i^+}\cdot\prod_{I-\{i'\}} m_J\right\rangle - \left\langle \sqrt{x_i^-}\cdot\prod_{I-\{i\}} m_J; \sqrt{y_i^+}\cdot\prod_{I-\{i'\}} m_J\right\rangle -$$
$$- \left\langle \sqrt{x_i^+}\cdot\prod_{I-\{i\}} m_J; \sqrt{y_i^-}\cdot\prod_{I-\{i'\}} m_J\right\rangle + \left\langle \sqrt{x_i^-}\cdot\prod_{I-\{i\}} m_J; \sqrt{y_i^-}\cdot\prod_{I-\{i'\}} m_J\right\rangle,$$

since $\varrho\left(\prod_{i\in I}\mu_i;\prod_{i\in I}\nu_i\right)=\prod_{i\in I}\varrho(\mu_i;\nu_i)$ (cfr. [2]), this expression is equal to

$$\varrho(x_i^+,m_i)\cdot\varrho(m_i,y_i^+)-\varrho(x_i^-,m_i)\cdot\varrho(m_{i'},y_{i'}^+)-$$
$$-\varrho(x_i^+,m_i)\cdot\varrho(y_i^-,m_i)+\varrho(x_i^-,m_i)\cdot\varrho(y_{i'}^-,m_{i'})=$$
$$=\left\langle\alpha_i(x_i)\cdot\bigotimes_{I-\{i\}}\sqrt{m_i};\ \alpha_i(y_i)\cdot\bigotimes_{I-\{i'\}}\sqrt{m_i}\right\rangle.$$

Thus, by induction on the number of elements of F and F', one deduces that u_0 preserves scalar products between C_0-families differing only in a finite number of indices from $\bigotimes_{i\in I}\sqrt{m_i}$. It is well known (cfr. [1]; Lemma (4.1.2)) that these C_0-families are total in $\bigotimes_{i\in I}^{\sqrt{m_i}}\mathscr{H}(\Omega_i,\mathscr{B}_i)$, thus it remains to prove that the linear extension of U_0 on the space spanned (algebraically) by such C_0-families, is well defined. That is, again using induction on the set of indices one reduces to verify that

$$\alpha_i(z_i)\cdot\bigotimes_{J\neq i}\sqrt{m_J}=[\alpha_i(x_i)\pm\alpha_i(y_i)]\cdot\bigotimes_{J\neq i}\sqrt{m'_J},$$

where x_i, y_i are positive, $z_i=x_i\pm y_i$ and almost all $m'_J=m_J$. And this relation easily follows from Theorem II.

Now, if $\bigotimes_{i\in I}\alpha_i(x_i)$ is an arbitrary essentially positive C_3-family equivalent to $\bigotimes_{i\in I}\sqrt{m_i}$, then

$$\lim_F\left(\bigotimes_{i\in F}\alpha_i(x_i)\right)\cdot\left(\bigotimes_{i\in I-F}\sqrt{m_i}\right)=\bigotimes_{i\in I}\alpha_i(x_i),$$
$$\lim_F\alpha\left(\prod_{i\in F}x_i\cdot\prod_{i\in I-F}m_i\right)=\alpha\left(\prod_{i\in I}x_i\right),$$

(both limits are meant for the filter of finite parts of I).

Therefore, by continuity $u\left(\bigotimes_{i\in I}\alpha_i(x_i)\right)=\alpha\left(\prod_{i\in I}x_i\right)$. Conversely, if u' is a unitary map from $\bigotimes^{(\sqrt{m_i})}\mathscr{H}_r(\Omega_i,\mathscr{B}_i)$ into $\mathscr{H}_r(\Omega,\mathscr{B})$, such that the above equality holds, then u' coincides with u on a total set, hence $u'=u$. The Lemma is proved.

We shall denote $\mathscr{K}\left(\prod_{i\in I}m_i\right)$ the image of $\bigotimes_{i\in I}^{(\sqrt{m_i})}\mathscr{H}_r(\Omega_i,\mathscr{B}_i)$ by the map u, defined in Lemma 4.2. Now we investigate the possibility of giving an intrinsic characterization of the space $\mathscr{K}\left(\prod_{i\in I}m_i\right)$. Remark that the last assertion of the above Lemma shows that the map u, hence also $\mathscr{K}\left(\prod_{i\in I}m_i\right)$, does not depend on the C_0-family $(\sqrt{m_i})$, but only on the set of essentially positive C_0-families contained in its equivalence class.

Consider the family $\mathscr{P}$ of all *product* measures on the measurable space $\prod_{i\in I}(\Omega_i,\mathscr{B}_i)$. Introduce in $\mathscr{P}$ the following relation:

Definition 4.3. Two product measures $\prod_{i\in I} x_i$ and $\prod_{i\in I} y_i$ are called « asymptotically equivalent », if there exists a finite subset $F\subseteq I$, such that the two measures $\prod_{i\in I-F} x_i$ and $\prod_{i\in I-F} u_i$, are not orthogonal. One easily verifies that effectively the above one is an equivalence relation. Hence the set $\mathscr{P}$ is partitioned into equivalence classes $\overline{\mathfrak{G}}$, by this relation. For each such equivalence class $\overline{\mathfrak{G}}$, define the space $\mathscr{K}(\overline{\mathfrak{G}})$ as the subspace of $\mathscr{H}_r\left(\prod_{i\in I}(\Omega_i, \mathscr{B}_i)\right)$, spanned by all the product measures in the equivalence class $\overline{\mathfrak{G}}$.

Lemma 4.4. The subspaces $\mathscr{K}(\overline{\mathfrak{G}})$ of $\mathscr{H}_r\left(\prod_{i\in I}(\Omega_i, \mathscr{B}_i)\right)$ corresponding to different equivalence classes $\overline{\mathfrak{G}}$, are mutually orthogonal.

Proof. It is an easy consequence of Kakutani's theorem formulated for the case of infinite products of a family of nonnormalized measures and of Therem II. Consider now two product measures $\prod_{i\in I} x_i, \prod_{i\in I} y_i$. A necessary condition for the families (x_i) and (y_i) to define product measures in that almost all measures in them be positive. And the two product measures are asymptotically equivalent, if and only if

$$\sum_{i\in I-F} |1-\varrho(x_i, y_i)| < \infty ,$$

where the set F is supposed chosen in such a way that for all indices J in its complement x_J and y_J are positive. Therefore the asymptotic equivalence of the two product measures is the same as the equivalence of the essentially positive C_0-families $\bigotimes_{i\in I} \alpha_i(x_i)$ and $\bigotimes_{i\in I} \alpha_i(y_i)$.

Thus a class $\overline{\mathfrak{G}}$ of asymptotically equivalent product measures finally determines an equivalence class $\mathfrak{G}^+$ of C_0-families which contains one (hence infinite) essentially positive C_0-family. Conversely, each such class of C_0-families is determined in the way described above by a class of asymptotically equivalent product measures. It is an easy consequence of Lemma 4.1 that if an esuivalence class $\mathfrak{G}$ of C_0-families contains one essentially positive C_0-family, then it cannot contain any C_0-family $\bigotimes_{i\in I} \alpha_i(x_i)$, with an infinite number of x_i strictly negative.

Therefore there exists a continuum of classes $\mathfrak{G}$ which do not contain any essentially positive C_0-family.

We now sum up our results in the following Theorem.

Theorem V. There is a one-to-one correspondence between equivalence classes $\mathfrak{G}^+$ of C_0-families in $\bigotimes_{i\in I} \mathscr{H}_r(\Omega_i, \mathscr{B}_i)$, which contain one essentially popositive C_0-family, and classes $\overline{\mathfrak{G}}$ of asymptotically equivalent *product* measures on $\prod_{i\in I}(\Omega_i, B_i)$. This correspondence is established by associating to each essentially positive C_0-family $\bigotimes_{i\in I} \alpha_i(x_i)$ in $\mathfrak{G}^+$, the equivalence class $\overline{\mathfrak{G}}$ of the product measure $\prod_{i\in I} x_i$.

Moreover there is a unitary isomorphism $u_{\mathfrak{G}^+}$ between $\bigotimes_{i\in I}^{\mathfrak{G}^+} \mathcal{H}_r(\Omega_i, \mathcal{B}_i)$ and $\mathcal{K}(\overline{\mathfrak{G}})$, which is characterized by the property

$$u_{\mathfrak{G}^+}\Big(\bigotimes_{i\in I} \alpha_i(x_i)\Big) = \alpha\Big(\prod_{i\in I} x_i\Big)$$

for any essentially positive C_0-family $\bigotimes \alpha_i(x_i) \in \mathfrak{G}^+$.

Proof. The first assertion has been proved in the considerations before the Theorem. The second one follows from Lemma 4.2.

We now study in a closer way the subspace $\mathcal{K}(\overline{\mathfrak{G}})$. By definition this is the closed subspace spanned by the $\alpha\Big(\prod_{i\in I} x_i\Big)$, with $\prod_{i\in I} x_i$ lying in one and the same class $\overline{\mathfrak{G}}$ of asymptotic equivalence.

Lemma 4.5. Each algebra $L^\infty_R(\Omega_i, \mathcal{B}_i)$ leaves $\mathcal{K}(\overline{\mathfrak{G}})$ invariant in its natural action $\hat{\alpha}_i$ on it.

Proof. Clear.

Denote now $\mathcal{A}_\otimes$ the norm clusure of the algebra spanned by the $\hat{\alpha}_i(L^\infty_R(\Omega_i, \mathcal{B}^i))$. By the preceding Lemma $\mathcal{A}_\otimes[\mathcal{K}(\overline{\mathfrak{G}})] \subseteq \mathcal{K}(\overline{\mathfrak{G}})$, and since $\mathcal{A}_\otimes$ is contained in $\mathcal{A}_r\Big(\prod_{i\in I} (\Omega_i, \mathcal{B}_i)\Big)$, one has also that $\mathcal{A}_\otimes\Big[\mathcal{H}_{\prod_{i\in I} x_i}\Big] \subseteq \mathcal{H}_{\prod_{i\in I} x_i}$ for every product measure $\prod_{i\in I} x_i$.

Lemma 4.6. For each $\prod_{i\in I} x_i$, $\mathcal{A}_\otimes \alpha\Big[\prod_{i\in I} x_i\Big]$ is dense in $\mathcal{H}_{\prod_{i\in I} x_i}$.

Proof. It will be sufficient to prove that in the canonical isomorphism $\mathcal{H}_{\prod_{i\in I} x_i} \to L^2_C\Big(\prod_{i\in I} (\Omega_i, \mathcal{B}_i, x_i)\Big)$ (cfr. Lemma 2.4), the set $\mathcal{A}_\otimes \alpha\Big[\prod_{i\in I} x_i\Big]$ goes into a dense subset. But in the isomorphism above $\alpha\Big(\prod_{i\in I} x_i\Big)$ goes into the function 1; and an element $\hat{\alpha}_i(f_i)$ goes into the function $\sqrt{f_i^+} - \sqrt{f_i^-}$ (cfr. Lemma 2.4). Thus it is sufficient to prove that the algebra $\otimes L^\infty(\Omega_i, \mathcal{B}_i)$ is dense in $L^\infty\Big(\prod_{i\in I} (\Omega_i, \mathcal{B}_i, x_i)\Big)$, or, equivalently, in $L^2_R\Big(\prod_{i\in I} (\Omega_i, \mathcal{B}_i, x_i)\Big)$. But each set in $\prod_{i\in I} \mathcal{B}_i$ is limit in $\Big(\prod_{i\in I} x_i\Big)$-measure, of disjoint unions of cylindrical sets. Thus each characteristic function in $L^\infty_R\Big(\prod_{i\in I} (\Omega_i, \mathcal{B}_i, x_i)\Big)$ is limit in measure and, because of boundedness, in $L^\infty_R\Big(\prod_{i\in I} x_i\Big)$ of elements in $\bigotimes_{i\in I} L^\infty_R(\Omega_i, \mathcal{B}_i)$.

Therefore the proof is completed.

Corollary 4.7. For every $\prod_{i\in I} x_i \in \mathfrak{G}$, $\mathcal{H}_{\prod_{i\in I} x_i} \subseteq \mathcal{K}(\overline{\mathfrak{G}})$.

$\mathcal{H}_{\prod_{i\in I} x_i}$ contains all the measures Ψ on the product $\prod_{i\in I} (\Omega_i, \mathcal{B}_i)$, which are equivalent to $\prod_{i\in I} x_i$, and, in general, it is composed exactly by those $\alpha(\Phi)$, with $\Phi < \prod_{i\in I} x_i$. From the relation $\mathcal{H}_{\prod_{i\in I} x_i} \subseteq \mathcal{K}(\overline{\mathfrak{G}})$ for every $\prod_{i\in I} x_i \in \overline{\mathfrak{G}}$ clearly follows that

$$\mathcal{K}(\overline{\mathfrak{G}}) = \overline{\bigcup_{\prod_{i\in I} x_i \in \mathfrak{G}} \mathcal{K}_{\prod_{i\in I} x_i}}$$

where the expression on the right-hand side denotes the closed linear subspace of $\mathcal{H}_r\left(\prod_{i\in I}(\Omega_i, \mathcal{B}_i)\right)$ spanned by the $\mathcal{H}_{\prod_{i\in I} x_i}$ with $\prod_{i\in I} x_i$ in $\overline{\mathfrak{G}}$.

Denote now $\mathcal{K} = \oplus \mathcal{K}(\overline{\mathfrak{G}})$ the subspace of $\mathcal{H}_r\left(\prod_{i\in I}(\Omega_i, \mathcal{B}_i)\right)$ obtained as direct sum of the spaces $\mathcal{K}(\overline{\mathfrak{G}})$, when $\overline{\mathfrak{G}}$ runs rhrough all the classes of asymptotic equivalence of *product* measures. Denote $\mathcal{K}^\perp$ the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}_r\left(\prod_{i\in I}(\Omega_i, \mathcal{B}_i)\right)$.

Lemma 4.8. A vector $\alpha(\Psi)$ is in $\mathcal{K}^\perp$, if and only if the measure Ψ is orthogonal to *all* product measures on $\prod_{i\in I}(\Omega_i, \mathcal{B}_i)$.

Proof. If Ψ is orthogonal to all product measures, it follows from Theorem II that $\alpha(\Psi) \in \mathcal{K}^\perp$. Conversely, let $\alpha(\Psi) \in \mathcal{K}^\perp$. Suppose, by contradition, that Ψ is not orthogonal to all product measures. Then there exists a product measure Φ such that $\Psi = \Psi_a + \Psi_0$, and $\Psi_a < \Phi$, $\Psi_0 \perp \Phi$. But then Corollary 4.7 im plies that $\alpha(\Psi_a) \in \mathcal{K}(\overline{\mathfrak{G}})$, where $\overline{\mathfrak{G}}$ is the class of asymptotic equivalence of Φ. Finally, since $\Psi_a \perp \Psi_0$, one has $\alpha(\Psi) = \alpha(\Psi_a) + \alpha(\Psi_0)$. Hence $\alpha(\Psi_a)$ is the projection of $\alpha(\Psi)$ on $\mathcal{K}(\mathcal{G}) \subseteq \mathcal{K}$, and our assumptions imply $\alpha(\Psi_a) = 0$, hence $\Psi_a = 0$. This means $\Psi \perp \Phi$ which contradicts the assumption. We deduce that, if $\alpha(\Psi) \in \mathcal{K}$, Ψ is orthogonal to all product measures on $\prod_{i\in I}(\Omega_i, \mathcal{B}_i)$ and this ends the Proof.

Denote now $\mathcal{V}$ the subspace of $\bigotimes_{i\in I} \mathcal{H}_r(\Omega_i, \mathcal{B}_i)$ obtained as direct sum of the incomplete components $\bigotimes_{i\in I}^{\mathfrak{G}^-} \mathcal{H}_r(\Omega_i, \mathcal{B}_i)$, relative to equivalence classes $\mathfrak{G}^-$ of C_0-families who do not contain any essentially positive C_0-family. Using the above notations, we now sum up the results in this paragraph:

Theorem VI. Let $(\Omega_i, \mathcal{B}_i)_{i\in I}$ be a family of measurable spaces, then there is an exact sequence

$$0 \to \mathcal{V} \xrightarrow{J} \bigotimes_{i\in I} \mathcal{H}_r(\Omega_i, \mathcal{B}_i) \xrightarrow{u} \mathcal{H}_r\left(\prod_{i\in I}(\Omega_i, \mathcal{B}_i)\right) \xrightarrow{P} \mathcal{K}^\perp \to 0\,,$$

where J and P denote the natural injection and projection respectively and u is the partial isometry implemented by the unitary isomorphisms

$$u_{\mathfrak{G}^+}\colon \otimes^{\mathfrak{G}^+} \mathcal{H}_r(\Omega_i, \mathcal{B}_i) \to \mathcal{K}(\overline{\mathfrak{G}}^+)\,,$$

by means of the equality

$$u = \sum_{\mathfrak{G}^+} u_{\mathfrak{G}^+} P_{\mathfrak{G}^+}\,,$$

where the sum is extended to all equivalence classes $\mathfrak{G}^+$ of C_0-families in $\bigotimes_{i\in J} \mathscr{H}_r(\Omega_i, \mathscr{B}_i)$ who contain an essentially positive family.

Proof. It follows from Theorem VI and proposition 4.8.

Remark. The space $\mathscr{V}$ is the subspace of $\bigotimes_{i\in I} \mathscr{H}_r(\Omega_i, \mathscr{B}_i)$, spanned by those C_0-families $(\alpha_i(x_i))$ such that for infinite indices i, x_i is nonpositive and $\sum_{i\in I} x_i^-(\Omega_i) = \infty$. The map u_0 of Theorem V does not extend to such C_0-families because $\prod_{i\in I} x_i$ cannot define a measure on $\prod_{i\in I} (\Omega_i, \mathscr{B}_i)$. Nevertheless such a C_0-family still determines a measure on... by means of the quadratic expression

$$\left\langle \bigotimes_{i\in I} \alpha_i(x_i); f\cdot\left(\bigotimes_{i\in I} \alpha_i(x_i)\right)\right\rangle = \int_{\prod_{i\in I}\Omega_i} \mathrm{d}\prod_{i\in I} |x_i| \qquad (f \geqslant 0)\,.$$

In this sense one can interpret $\bigotimes_{i\in I} \alpha_i(x_i)$ as the square root of a «virtual measure» on $\prod_{i\in I} (\Omega_i, \mathscr{B}_i)$.

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