

# STOCHASTIC GOLDEN RULE FOR A SYSTEM INTERACTING WITH A FERMİ FIELD

L. ACCARDI

Centro Vito Volterra  
Università di Roma "Tor Vergata"  
Via Columbia, 2, 00133, Roma, Italy  
E-mail: accardi@volterra.mat.uniroma2.it

R.A. ROSCHIN

Steklov Mathematical Institute  
Russian Academy of Sciences  
Gubkin St. 8, 117966, GSP-1, Moscow, Russia  
E-mail: roschin@mi.ras.ru

I.V. VOLOVICH

Steklov Mathematical Institute  
Russian Academy of Sciences  
Gubkin St. 8, 117966, GSP-1, Moscow, Russia  
E-mail: volovich@mi.ras.ru

We consider the (causally) normally ordered form of the quantum white noise equation for a Fermi white noise. We find a new form of the normally ordered equation for some class of Hamiltonians and we obtain the inner Langevin equation for such Hamiltonians.

## 1. Introduction

The stochastic limit approach is a powerful method that allows to study the quantum dynamics of an open system. This method proved to be very efficient in study of various systems, interacting with Bose fields.<sup>1</sup> In the present paper we concentrate on the Fermi case.

In this case the superselection rules restrict the class of fundamental Fermi Hamiltonians. This restriction is discussed in Sec. 2.

The main goal of the stochastic limit approach is to study the quantum dynamics of open systems. Such a dynamics is given by an evolution operator  $U(t)$ , which satisfies the evolution equation in the interaction rep-

resentation:

$$\partial_t U(t) = -i\lambda V_{int}(t)U(t), \quad U(0) = 1. \quad (1)$$

Here  $V_{int}(t)$  is the evolution of the interaction Hamiltonian under the free evolution.

However, even in the simplest non-trivial cases, there are no exact solutions of Eq. (1). Therefore, we have to use some approximations.

One can express the solution of Eq. (1) using the iterated series and Feynman diagrams. The Feynman diagrams are a graphical representation of the combinatorial procedure, which is called "normal ordering". Usually, only a few terms of these series can be explicitly calculated. The stochastic limit approach provides another, more efficient and elegant way of solving Eq. (1).

Consider the evolution of an open quantum system in the rescaled time:  $t \rightarrow t/\lambda^2$ . In the stochastic limit approach we are interested in the limits of the rescaled evolution operator and of the rescaled interaction Hamiltonian:<sup>a</sup>

$$\lim_{\lambda \rightarrow 0} U(\lambda, \frac{t}{\lambda^2}) = U_t, \quad \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} V_{int}(\frac{t}{\lambda^2}) = h_t. \quad (2)$$

In many physically important cases, one can prove that  $U_t$  satisfy the (Bose) quantum white noise equation:

$$\partial_t U_t = -ih_t U_t, \quad U_0 = 1, \quad (3)$$

where the limit of the interaction Hamiltonian,  $h_t$ , can be expressed in terms of (Bose) quantum white noises. Although Eq. (3) seems to be very similar to its predecessor, Eq. (1), it can be explicitly solved. Again, we are going to find the normally ordered form of Eq. (3). It is called *causal normal order* to emphasize the fact that the commutation relations used to bring to normal order the terms of the iterated series, only have a meaning for time ordered products of creation and annihilation operators. The causally normally ordered form of Eq. (3) can be found explicitly.

The technique that allows to find and bring to causal normal order Eq. (3) is called the *stochastic golden rule*.

The aim of the present paper is to develop the stochastic golden rule for the cases when the interaction in (1) is of dipole type (cf. (8) below) and driven by a Fermi field.

<sup>a</sup>The exact sense of these limits is the subject of Statement 1.

## 2. Hamiltonian of the model

Let  $\mathcal{H}$  be a Hilbert space of the form

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R \quad (4)$$

where  $\mathcal{H}_S$  is a finite dimensional Hilbert space, and  $\mathcal{H}_R$  is an infinite dimensional Hilbert space. Suppose  $H$  is a Hamiltonian of the form

$$H = H_S \otimes 1 + 1 \otimes H_R + \lambda V_{int}, \quad (5)$$

where  $\lambda$  is a positive real parameter. We denote the first two terms of this Hamiltonian by  $H_0$ , and refer  $H_0$  as "free Hamiltonian".

Then, the pair  $(\mathcal{H}, H)$  describes an *open quantum system*. In other words, an open quantum system is a "small", discrete spectrum quantum system, interacting with some infinite dimensional environment, or field.

The simplest model that gives the Fermi white noise equation in the stochastic limit is a two-level system, interacting with a Fermi field.

Let us introduce this model, using the notation of Eqs. (4,5). Let the system Hilbert space be  $\mathcal{H}_S = \mathbb{C}^2$ . Denote the elements of the orthonormal basis for this space by  $|i\rangle$ ,  $i = 1, 2$ .

Let the environment Hilbert space be the Fermi (antisymmetric) Fock space

$$\mathcal{H}_R = \bigoplus_{n=0}^{\infty} (L_2(\mathbb{R}^3))^{\otimes_{\text{asym}} n}. \quad (6)$$

Denote the Fermi creation and annihilation operators in this space by  $a_k^\dagger$  and  $a_k$ ,  $k \in \mathbb{R}^3$ , respectively. They satisfy anticommutation relations:<sup>b</sup>

$$\{a_k^\dagger, a_{k'}\} = \delta(k - k'), \quad \{a_k, a_{k'}\} = 0.$$

Let the free Hamiltonian be:

$$H_0 = H_S + H_R = E|1\rangle\langle 1| \otimes 1 + 1 \otimes \int \omega(k) a_k^\dagger a_k dk. \quad (7)$$

Here  $E$  is a positive real number,  $\omega(k) = k^2 + m^2$ ,  $m \geq 0$ . Let the interaction Hamiltonian be:

$$V_{int} = \int (g(k) D^\dagger \otimes a_k + \overline{g(k)} a_k^\dagger \otimes D) dk, \quad (8)$$

<sup>b</sup>In the following,  $\{, \}$  denote anticommutator,  $\{x, y\} = xy + yx$ , and  $[, ]$  denote commutator,  $[x, y] = xy - yx$ .

where  $D$  is a bounded operator, acting on the system space, and  $g(k)$  is a smooth complex function with finite support.

In the following, we omit the tensor product in our formulae.

The superselection rules (see Ref. <sup>2</sup>) require that fundamental Hamiltonians should be even in the Fermi operators.<sup>c</sup> If the Hamiltonian is even, then  $D$  should be odd in the Fermi operators. Hence,  $D$  and  $a, a^\dagger$  should anticommute.<sup>d</sup> We will see that this case leads to a stochastic golden rule similar to the one known in the Bose case.

Effective (for example, non-relativistic) Hamiltonians may be odd in Fermi operators, in these cases  $D$  and  $a, a^\dagger$  may commute. An example of such Hamiltonian was considered in Ref. 4. If  $D$  and  $a, a^\dagger$  commute, then we get the new kind of the stochastic golden rule.

We will study both cases; first the "commutative", then the "anti-commutative" one.

As mentioned in the introduction, there are two main components in the stochastic golden rule: the existence of the limit and the causal normal order form of the white noise equation.

The first part, the existence of the limit, is quite similar for Bose and Fermi cases. Both cases were studied in Ref. 1. Here we just formulate the result for Fermi case.

**Statement 1.** The limit (2) of the evolution operator in the model defined by Eqs. (6-8), exists in the sense of correlators. Moreover, the limit of the rescaled evolution operator  $U_t$  satisfies the white noise equation

$$\partial_t U_t = -i h_t U_t, \quad (9)$$

with the white noise Hamiltonian of the form

$$h_t = D^\dagger b_t + b_t^\dagger D, \quad (10)$$

where  $b_t, b_t^\dagger$  are Fermi Fock white noise creation and annihilation operators. This means that they are operator valued distributions, acting on the Fock space  $\Gamma = \mathcal{F}(L^2(\mathbb{R}))$  and satisfying the following relations:

$$\{b_t^\dagger, b_{t'}\} = \gamma_- \delta_+(t - t'); \quad \{b_t, b_{t'}\} = 0. \quad (11)$$

where  $\delta_+(t)$  is the causal  $\delta$ -function (see Ref. 1) and

$$\gamma_- = \int_{-\infty}^0 d\sigma \int dk |g(k)|^2 e^{i\sigma(\omega(k) - E)}. \quad (12)$$

<sup>c</sup>An operator is called even, if it commutes with the parity operator, and odd, if it anticommutes. The parity operator in this case can be defined as in Sec. 3.

<sup>d</sup>An obvious generalization of Eqs. (4-8) is needed to include this case.



The relation between  $D$  and  $b, b^\dagger$  is the same as the relation between  $D$  and  $a, a^\dagger$ : they either commute, or anti-commute.

### 3. The "Commutative" case

The causally normally ordered form of the Fermi white noise equation and the Langevin equation in the "commutative" case are the main results of the present paper. Let us formulate these results as a theorem.

First of all, let us give some definitions.

Consider a monomial of the form:

$$b_{t_1}^{\epsilon_1} b_{t_2}^{\epsilon_2} \dots b_{t_n}^{\epsilon_n},$$

were,  $b_t^{\epsilon_i}$  denote either  $b_t$ , or  $b_t^\dagger$ . Denote the vacuum state in  $\Gamma$  by  $\Psi_0$ . There exists a unique operator  $\Theta$ , called *parity* operator, such that for any monomial

$$\Theta b_{t_1}^{\epsilon_1} b_{t_2}^{\epsilon_2} \dots b_{t_n}^{\epsilon_n} \Psi_0 = (-1)^n b_{t_1}^{\epsilon_1} b_{t_2}^{\epsilon_2} \dots b_{t_n}^{\epsilon_n} \Psi_0. \quad (13)$$

Note, that  $\Theta^2 = 1$ , hence  $\Theta^{-1} = \Theta = \Theta^*$ .

If  $A$  is an operator, then the map:  $A \rightarrow \Theta A \Theta^{-1} = \Theta A \Theta$  is a \*-automorphism. We will use the notation:

$$\hat{A} := \Theta A \Theta. \quad (14)$$

**Theorem 2.** Keep the notation and the assumptions of Statement 1. Suppose the operator  $D$  is such that

$$[D, b_t] = [D, b_t^\dagger] = 0. \quad (15)$$

and define  $\hat{U}_t \hat{U} := \Theta U \Theta$ .

Then,

(1) The following relations are satisfied:

$$\begin{aligned} b_t U_t &= \hat{U}_t b_t - i\gamma_- D U_t & (16.1) \\ U_t^* b_t^\dagger &= b_t^\dagger \hat{U}_t^* + i\gamma_-^* U_t^* D^\dagger & (16.1^*) \\ b_t U_t^* &= \hat{U}_t^* b_t + i\gamma_- U_t^* D^\dagger & (16.2) \\ U_t b_t^\dagger &= b_t^\dagger \hat{U}_t - i\gamma_-^* D^\dagger U_t & (16.2^*) \\ b_t \hat{U}_t &= U_t b_t + i\gamma_- D \hat{U}_t & (16.3) \\ \hat{U}_t^* b_t^\dagger &= b_t^\dagger U_t^* - i\gamma_-^* \hat{U}_t^* D^\dagger & (16.3^*) \\ b_t \hat{U}_t^* &= U_t^* b_t - i\gamma_- \hat{U}_t^* D & (16.4) \\ \hat{U}_t b_t^\dagger &= b_t^\dagger U_t + i\gamma_-^* D^\dagger \hat{U}_t & (16.4^*) \end{aligned} \quad (16)$$

(2) The causally normally ordered form of (9) is the system

$$\begin{cases} \partial_t U_t = -i (D^\dagger \hat{U}_t b_t + D b_t^\dagger U_t) - \gamma_- D^\dagger D U_t \\ \partial_t \hat{U}_t = i (D^\dagger U_t b_t + D b_t^\dagger \hat{U}_t) - \gamma_- D^\dagger D \hat{U}_t \end{cases} \quad (17)$$

with the initial conditions  $U_t = \hat{U}_t = 1$ .

(3) Let  $X$  be an operator on the system space. Consider the following matrices:

$$\mathcal{J}(X) = (\mathcal{J}_{ij}(X)(t)) := \begin{pmatrix} U_t^* X U_t & U_t^* X \hat{U}_t \\ \hat{U}_t^* X U_t & \hat{U}_t^* X \hat{U}_t \end{pmatrix}, \quad i, j = 1, 2. \quad (18)$$

$$T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (19)$$

Then,  $(\mathcal{J}_{ij}(X))$  satisfies the following inner Langevin equation:

$$\begin{aligned} \partial_t (\mathcal{J}(X)) &= -\gamma_- (\mathcal{J}(X D^\dagger D)) - \gamma_-^* (\mathcal{J}(D^\dagger D X)) \\ &+ \begin{pmatrix} (\gamma_- + \gamma_-^*) \mathcal{J}_{11}(D^\dagger X D) & (-\gamma_- + \gamma_-^*) \mathcal{J}_{12}(D^\dagger X D) \\ (-\gamma_-^* + \gamma_-) \mathcal{J}_{21}(D^\dagger X D) & (\gamma_- + \gamma_-^*) \mathcal{J}_{22}(D^\dagger X D) \end{pmatrix} \\ &+ i b_t^\dagger S T (\mathcal{J}_{ij}(D X)) - i b_t^\dagger T (\mathcal{J}_{ij}(X D)) S \\ &+ i S (\mathcal{J}_{ij}(D^\dagger X)) T b_t - i (\mathcal{J}_{ij}(X D^\dagger)) T S b_t. \end{aligned} \quad (20)$$

(4) The master equation for the partial expectation  $\langle U_t^* X U_t \rangle_0 = \bar{X}$  is:

$$\partial_t \bar{X} = -\gamma_- \bar{X} D^\dagger D - \gamma_-^* D^\dagger D \bar{X} + 2(\Re \gamma_-) D^\dagger \bar{X} D. \quad (21)$$

The proof of Theorem 2 is given in the next section.

**Remark.** The causally normally ordered form of the Fermi white noise equation is a *pair* of equations for  $U$  and  $\hat{U}$ . This is the new feature of the Fermi case.

## 4. Proof of Theorem 2.

### 4.1. Quasi-commutation rules for $b_t$ and $U_t$ .

Let us express the evolution operator as

$$U_t = \lim_{N \rightarrow \infty} U_t^{(N)},$$

$$U_t^{(N)} = \sum_{n=0}^N (-i)^n \int_0^\tau dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \left( \prod_{j=1}^n (D^\dagger b_{t_j} + D b_{t_j}^\dagger) \right). \quad (22)$$

The *time-consecutive principle* (see Ref. 2) states that expanding the product  $b_t U_\tau$  according to (22), any term of the form

$$b_t b_{t_1}^{\epsilon_1} b_{t_2}^{\epsilon_2} \dots b_{t_n}^{\epsilon_n}, \quad (23)$$

with  $t \geq t_1 \geq t_2 \geq \dots \geq t_n$ ,  $\epsilon_i \in \{, \dagger\}$ , is equal to

$$(-1)^n b_{t_1}^{\epsilon_1} b_{t_2}^{\epsilon_2} \dots b_{t_n}^{\epsilon_n} b_t + \{b_t, b_{t_1}^{\epsilon_1}\} b_{t_2}^{\epsilon_2} \dots b_{t_n}^{\epsilon_n},$$

where the anti-commutation relation for the fields  $b_t$  is given by (11). Using this, we obtain:

$$\begin{aligned} b_t U_\tau^{(N)} &= \sum_{n=1}^N (-i)^n \int_0^\tau dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n (-1)^n \left( \prod_{j=1}^n (D^\dagger b_{t_j} + D b_{t_j}^\dagger) \right) b_t \\ &\quad + (-i)(-i)^{n-1} D \{b_t, b_{t_1}^\dagger\} \left( \prod_{j=2}^n (D^\dagger b_{t_j} + D b_{t_j}^\dagger) \right) \\ &= \widehat{U}_\tau^{(N)} b_t - i D \gamma_- \chi_{[0,\tau]}(t) U_\tau^{(N-1)}. \end{aligned}$$

Here  $\chi_{[0,r]}$  is the characteristic function of the given interval  $[0, r]$ . In the limit  $N \rightarrow \infty$  we obtain Eq. (16.1)

$$b_t U_t = \widehat{U}_t b_t - i \gamma_- D U_t.$$

For  $b_t U_t^*$  we obtain:

$$\begin{aligned} b_t U_t^{(N)*} &= \sum_{n=1}^N (i)^n \int_0^\tau dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n (-1)^n \left( \prod_{j=n}^1 (D^\dagger b_{t_j} + D b_{t_j}^\dagger) \right) b_t \\ &\quad + (i)(i)^{n-1} \left( \prod_{j=n}^2 (D^\dagger b_{t_j} + D b_{t_j}^\dagger) \right) D \{b_t, b_{t_1}^\dagger\} \\ &= \widehat{U}_\tau^{(N)*} b_t + i \gamma_- \chi_{[0,\tau]}(t) U_\tau^{(N-1)*} D \end{aligned}$$

and in the limit we get Eq. (16.2):

$$b_t U_t^* = \widehat{U}_t^* b_t + i \gamma_- U_t^* D.$$

Let  $A, B$  be some operators. Since  $\Theta^2 = 1$ , observe that

$$P(AB) = \Theta AB \Theta = \Theta A \Theta^2 B \Theta = P(A)P(B). \quad (24)$$

and

$$(PA)^* = P(A^*). \quad (25)$$

From the general formula (13) we get

$$P b_t = -b_t, \quad P b_t^\dagger = -b_t^\dagger. \quad (26)$$

Let us apply  $P$  operator to  $b_t \widehat{U}_t$  and  $b_t \widehat{U}_t^*$ , and use Eqs. (16.1, 16.2). We have:

$$\begin{aligned} b_t \widehat{U}_t &= -P(b_t U_t) = -P(\widehat{U}_t b_t - i \gamma_- D U_t) \\ &= -(-U_t b_t - i \gamma_- D \widehat{U}_t) = U_t b_t + i \gamma_- D \widehat{U}_t. \end{aligned}$$

Hence, we get Eq. (16.3):

$$b_t \widehat{U}_t = U_t b_t + i \gamma_- D \widehat{U}_t.$$

$$\begin{aligned} b_t \widehat{U}_t^* &= -P(b_t U_t^*) = -P(\widehat{U}_t^* b_t + i \gamma_- U_t^* D) \\ &= -(-U_t^* b_t + i \gamma_- \widehat{U}_t^* D) = U_t^* b_t - i \gamma_- \widehat{U}_t^* D \end{aligned}$$

Hence, we get Eq. (16.4):

$$b_t \widehat{U}_t^* = U_t^* b_t - i \gamma_- \widehat{U}_t^* D.$$

Eqs. (16.1\*)-(16.4\*) are adjoint of Eqs. (16.1)-(16.4).

In Appendix A we prove the relations (16) using the integral form of the evolution equation, as done in Ref. 1 for bosons.

#### 4.2. Normally ordered equation for $U_t$ .

Let us rewrite the evolution equation (9), using (16.1).

$$\partial_t U_t = -i(D^\dagger b_t + D b_t^\dagger) U_t = -i(D^\dagger \widehat{U}_t b_t + D b_t^\dagger \widehat{U}_t) - \gamma_- D^\dagger D U_t. \quad (27)$$

Eq. (27) is not close because it involves both  $U_t$  and  $\widehat{U}_t$ .<sup>e</sup> Applying the  $P$  operator to it, and using (24-26), we obtain:

$$\partial_t \widehat{U}_t = i(D^\dagger U_t b_t + D b_t^\dagger \widehat{U}_t) - \gamma_- D^\dagger D \widehat{U}_t. \quad (28)$$

The system of Eqs. (27, 28) is closed and causally normally ordered in the sense that all the  $b_t^\dagger$  operators are on the left hand side and all the  $b_t$  are on the right hand side of the  $U_t$  system.

<sup>e</sup>The operators  $U$  and  $\widehat{U}$  are dependent, hence if one substitutes the definition of  $\widehat{U}$  into (27), then the result will be closed. R.R. is grateful to Prof. Y.G.Lu for pointing this out.



### 4.3. Inner Langevin equation.

The inner Langevin equation is a result of a direct computation. The idea of this computation is to apply Eqs. (27,28) to express  $\partial_t U_t$ , then to use Eq. (16) to find the causally normally ordered form of the terms.

Let us compute  $\partial_t \mathcal{J}_{11}(X)(t)$ :

$$\begin{aligned}\partial_t \mathcal{J}_{11}(X)(t) &= \partial_t (U_t^* X U_t) = \partial_t U_t^* X U_t + U_t^* X \partial_t U_t \\ &= \left( i(b_t^\dagger \hat{U}_t^* D + U_t^* D^\dagger b_t) - \gamma_-^* U_t^* D^\dagger D \right) X U_t \\ &\quad + U_t^* X \left( -i(D^\dagger \hat{U}_t b_t + b_t^\dagger D U_t) - \gamma_- D^\dagger D U_t \right) \\ &= i b_t^\dagger \hat{U}_t^* D X U_t + i U_t^* D^\dagger X b_t U_t - \gamma_-^* U_t^* D^\dagger D X U_t \\ &\quad - i U_t^* X D^\dagger \hat{U}_t b_t - i U_t^* b_t^\dagger X D U_t - \gamma_- U_t^* X D^\dagger D U_t\end{aligned}$$

Using

$$i U_t^* D^\dagger X b_t U_t = i U_t^* D^\dagger X \hat{U}_t b_t + \gamma_- U_t^* D^\dagger X D U_t$$

and

$$-i U_t^* b_t^\dagger X D U_t = -i b_t^\dagger \hat{U}_t^* X D U_t + \gamma_-^* U_t^* D^\dagger X D U_t$$

we obtain:

$$\begin{aligned}\partial_t \mathcal{J}_{11}(X)(t) &= -\gamma_- \mathcal{J}_{11}(X D^\dagger D) - \gamma_-^* \mathcal{J}_{11}(D^\dagger D X) + (\gamma_- + \gamma_-^*) \mathcal{J}_{11}(D^\dagger X D) \\ &\quad + i b_t^\dagger (\mathcal{J}_{21}(D X) - \mathcal{J}_{21}(X D)) + i (\mathcal{J}_{12}(D^\dagger X) - \mathcal{J}_{12}(X D^\dagger)) b_t. \quad (29)\end{aligned}$$

Let us compute  $\partial_t \mathcal{J}_{22}(X)(t)$ :

$$\begin{aligned}\partial_t \mathcal{J}_{22}(X)(t) &= \partial_t (\hat{U}_t^* X \hat{U}_t) = \partial_t \hat{U}_t^* X \hat{U}_t + \hat{U}_t^* X \partial_t \hat{U}_t \\ &= \left( -i(b_t^\dagger \hat{U}_t^* D + \hat{U}_t^* b_t D^\dagger) - \gamma_-^* \hat{U}_t^* D^\dagger D \right) X \hat{U}_t \\ &\quad + \hat{U}_t^* X \left( i(D^\dagger \hat{U}_t b_t + D b_t^\dagger \hat{U}_t) - \gamma_- D^\dagger D \hat{U}_t \right) \\ &= -i b_t^\dagger \hat{U}_t^* D X \hat{U}_t - i \hat{U}_t^* D^\dagger X b_t \hat{U}_t - \gamma_-^* \hat{U}_t^* D^\dagger D X \hat{U}_t \\ &\quad + i \hat{U}_t^* X D^\dagger \hat{U}_t b_t + i \hat{U}_t^* b_t^\dagger X D \hat{U}_t - \gamma_- \hat{U}_t^* X D^\dagger D \hat{U}_t\end{aligned}$$

Using

$$-i \hat{U}_t^* D^\dagger X b_t \hat{U}_t = -i \hat{U}_t^* D^\dagger X U_t b_t + \gamma_- \hat{U}_t^* D^\dagger X D \hat{U}_t$$

and

$$i \hat{U}_t^* b_t^\dagger X D \hat{U}_t = i b_t^\dagger U_t^* X D \hat{U}_t + \gamma_-^* U_t^* D^\dagger X D \hat{U}_t$$

we obtain:

$$\begin{aligned}\partial_t \mathcal{J}_{22}(X)(t) &= -\gamma_- \mathcal{J}_{22}(X D^\dagger D) - \gamma_-^* \mathcal{J}_{22}(D^\dagger D X) + (\gamma_- + \gamma_-^*) \mathcal{J}_{22}(D^\dagger X D) \\ &\quad + i b_t^\dagger (\mathcal{J}_{12}(X D) - \mathcal{J}_{12}(D X)) + i (\mathcal{J}_{21}(X D^\dagger) - \mathcal{J}_{21}(D^\dagger X)) b_t. \quad (30)\end{aligned}$$

Let us compute  $\partial_t \mathcal{J}_{12}(X)(t)$ :

$$\begin{aligned}\partial_t \mathcal{J}_{12}(X)(t) &= \partial_t U_t^* X \hat{U}_t + U_t^* X \partial_t \hat{U}_t \\ &= \left( i(b_t^\dagger \hat{U}_t^* D + U_t^* D^\dagger b_t) - \gamma_-^* U_t^* D^\dagger D \right) X \hat{U}_t \\ &\quad + U_t^* X \left( i(D^\dagger U_t b_t + D b_t^\dagger \hat{U}_t) - \gamma_- D^\dagger D \hat{U}_t \right) \\ &= i b_t^\dagger \hat{U}_t^* D X \hat{U}_t + i U_t^* D^\dagger b_t X \hat{U}_t - \gamma_-^* U_t^* D^\dagger D X \hat{U}_t \\ &\quad + i U_t^* X D^\dagger U_t b_t + i U_t^* X D b_t^\dagger \hat{U}_t - \gamma_- U_t^* X D^\dagger D \hat{U}_t\end{aligned}$$

Using

$$i U_t^* D^\dagger X b_t \hat{U}_t = i U_t^* D^\dagger X U_t b_t - \gamma_- i U_t^* D^\dagger X D \hat{U}_t$$

and

$$i U_t^* b_t^\dagger X D \hat{U}_t = i b_t^\dagger \hat{U}_t^* X D \hat{U}_t + \gamma_-^* U_t^* D^\dagger X D \hat{U}_t$$

we obtain:

$$\begin{aligned}\partial_t \mathcal{J}_{12}(X)(t) &= -\gamma_- \mathcal{J}_{12}(X D^\dagger D) - \gamma_-^* \mathcal{J}_{12}(D^\dagger D X) + (-\gamma_- + \gamma_-^*) \mathcal{J}_{22}(D^\dagger X D) \\ &\quad + i b_t^\dagger (\mathcal{J}_{22}(D X) + \mathcal{J}_{22}(X D)) + i (\mathcal{J}_{11}(D^\dagger X) + \mathcal{J}_{11}(X D^\dagger)) b_t. \quad (31)\end{aligned}$$

The computation of  $\partial_t \mathcal{J}_{12}(X)(t)$  is similar and therefore omitted.

Combining all terms in the matrix equation, we find:

$$\begin{aligned}\partial_t (\mathcal{J}_{ij}(X)(t)) &= -\gamma_- (\mathcal{J}_{ij}(X D^\dagger D)) - \gamma_-^* (\mathcal{J}_{ij}(D^\dagger D X)) \\ &\quad + \begin{pmatrix} (\gamma_- + \gamma_-^*) \mathcal{J}_{11}(D^\dagger X D) & (-\gamma_- + \gamma_-^*) \mathcal{J}_{12}(D^\dagger X D) \\ (-\gamma_-^* + \gamma_-) \mathcal{J}_{21}(D^\dagger X D) & (\gamma_- + \gamma_-^*) \mathcal{J}_{22}(D^\dagger X D) \end{pmatrix} \\ &\quad + i b_t^\dagger \begin{pmatrix} -\mathcal{J}_{21}(X D) + \mathcal{J}_{21}(D X) & \mathcal{J}_{22}(X D) + \mathcal{J}_{22}(D X) \\ -\mathcal{J}_{11}(X D) - \mathcal{J}_{11}(D X) & \mathcal{J}_{12}(X D) - \mathcal{J}_{12}(D X) \end{pmatrix} \\ &\quad + i \begin{pmatrix} -\mathcal{J}_{12}(X D^\dagger) + \mathcal{J}_{21}(D^\dagger X) & \mathcal{J}_{11}(X D^\dagger) + \mathcal{J}_{11}(D^\dagger X) \\ -\mathcal{J}_{22}(X D^\dagger) - \mathcal{J}_{22}(D^\dagger X) & \mathcal{J}_{21}(X D^\dagger) - \mathcal{J}_{21}(D^\dagger X) \end{pmatrix} b_t. \quad (32)\end{aligned}$$

Using the matrices  $T$  and  $S$ , defined in (19), one can rewrite the last two terms as (in the notation (18))

$$i b_t^\dagger (S T \mathcal{J}(D X) - T \mathcal{J}(X D) S) + i (S \mathcal{J}(D^\dagger X) T - \mathcal{J}(X D^\dagger) T S) b_t. \quad (33)$$

this can be checked directly by matrix multiplication.

#### 4.4. Canonical form of the Langevin equation.

We would like to represent the following inner Langevin equation:

$$\begin{aligned} \partial_t (\mathcal{J}_{ij}(X)) = & -\gamma_- (\mathcal{J}_{ij}(XD^\dagger D)) - \gamma_-^* (\mathcal{J}_{ij}(D^\dagger DX)) \\ & + \begin{pmatrix} (\gamma_- + \gamma_-^*) \mathcal{J}_{11}(D^\dagger XD) & (-\gamma_- + \gamma_-^*) \mathcal{J}_{12}(D^\dagger XD) \\ (-\gamma_-^* + \gamma_-) \mathcal{J}_{21}(D^\dagger XD) & (\gamma_- + \gamma_-^*) \mathcal{J}_{22}(D^\dagger XD) \end{pmatrix} \\ & + ib_t^\dagger ST (\mathcal{J}_{ij}(DX)) - ib_t^\dagger T (\mathcal{J}_{ij}(XD)) S \\ & + iS (\mathcal{J}_{ij}(D^\dagger X)) T b_t - i (\mathcal{J}_{ij}(XD^\dagger)) T S b_t \end{aligned} \quad (34)$$

in the canonical form

$$\partial_t (\mathcal{J}_{ij}(X)) = \sum_{kl=1,2} \left( \mathcal{J}_{kl}(\theta_{kl ij}^0(X)) + b_t^\dagger \mathcal{J}_{kl}(\theta_{kl ij}^+(X)) + \mathcal{J}_{kl}(\theta_{kl ij}^-(X)) b_t \right). \quad (35)$$

Let us find structure maps  $\theta^0, \theta^\pm$ .

The map  $\mathcal{J}_{ij}$  is linear, hence  $\theta^0, \theta^\pm$  can be obtained by rewriting the matrix multiplications in (34) in the index form:  $(a_{ij})(b_{jl}) = (\sum_j a_{ij} b_{jl})$ .

Thus,

$$\theta_{ijkl}^0(X) = \delta_{ki} \delta_{jl} (-\gamma_- XD^\dagger D - \gamma_-^* D^\dagger DX + l_{ij} D^\dagger XD),$$

where

$$l_{ij} = \begin{pmatrix} (\gamma_- + \gamma_-^*) (-\gamma_- + \gamma_-^*) \\ (\gamma_- - \gamma_-^*) (\gamma_- + \gamma_-^*) \end{pmatrix}$$

and

$$\theta_{ijkl}^+(X) = i (m_{ik} \delta_{jl} DX - T_{ik} S_{lj} XD).$$

$$\theta_{ijkl}^-(X) = i (S_{ik} T_{lj} D^\dagger X - \delta_{ik} n_{lj} XD^\dagger),$$

where

$$m_{ij} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (m_{ij}) = ST,$$

$$n_{ij} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (n_{ij}) = TS,$$

$$T = (T_{ij}) = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$S = (S_{ij}) = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

#### 4.5. The Master equation.

The master equation is the vacuum expectation of the inner Langevin equation (32). The last two terms, containing  $b_t$  and  $b_t^\dagger$ , vanish.

It is enough to take vacuum expectation only for one matrix element. The most interesting is  $\mathcal{J}_{11}(X)$ . Denote by  $\bar{X}$  the vacuum expectation  $\langle \mathcal{J}_{11}(X) \rangle_0$ . We obtain:

$$\partial_t \bar{X} = -\gamma_- \bar{X} D^\dagger D - \gamma_- D^\dagger D \bar{X} + 2\Re \gamma_- D^\dagger \bar{X} D. \quad (36)$$

#### 5. "Anti-commutative" case

**Theorem 3.** Keep the notation and assumptions of Statement 1. Suppose the operator  $D$  is such that

$$\{D, b_t\} = \{D, b_t^\dagger\} = 0. \quad (37)$$

Then, the following relation is satisfied

$$b_t U_t = U_t b_t + i\gamma_- D U_t. \quad (38)$$

*Proof.* Let us express the evolution operator as:

$$U_t = \lim_{N \rightarrow \infty} U_t^{(N)}$$

$$U_\tau^{(N)} = \sum_{n=0}^N (-i)^n \int_0^\tau dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \left( \prod_{j=1}^n (D^\dagger b_{t_j} + b_{t_j}^\dagger D) \right)$$

Using the time-consecutive principle (see (23)), we obtain:

$$\begin{aligned} b_t U_\tau^{(N)} &= \\ &= \sum_{n=1}^N (-i)^n \int_0^\tau dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n (-1)^{2n} \left( \prod_{j=1}^n (D^\dagger b_{t_j} + b_{t_j}^\dagger D) \right) b_t \\ &\quad + (-1)(-i)(-i)^{n-1} D \{b_t, b_{t_1}^\dagger\} \left( \prod_{j=2}^n (D^\dagger b_{t_j} + b_{t_j}^\dagger D) \right) \\ &= U_\tau^{(N)} b_t + i D \gamma_- \chi_{[0, \tau]}(t) U_\tau^{(N-1)} \end{aligned} \quad (39)$$

Taking the limit  $N \rightarrow \infty$  of Eq. (39) we obtain:

$$b_t U_t = U_t b_t + i\gamma_- D U_t$$

which is (38). The theorem is proved.

The relation (38) between  $b$  and  $U$  is similar to the Bose case.<sup>1</sup> In the case of Bose quantum white noise, the causally normally ordered form of the white noise equation (3) can be obtained using *only* the relation between Bose white noise and the evolution operator.

Using the proof for the Bose case, one can easily prove that replacing Bose by Fermi quantum white noise operators, the causally normally ordered form of the white noise equations is the same.

## 6. Conclusions

We studied the Fermi quantum white noise equations with a dipole-type interaction Hamiltonian (10). In the "commutative" case we obtain a new form of causally normally ordered white noise equation (17) and of inner Langevin equation (20). In the "anti-commutative" case we found the commutation relation (38). From Eq. (38) we get that in the "anti-commutative" case the causally normally ordered form of a Fermi white noise equations is the same as in the Bose case.

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## Appendix A. Proof of the relations (16) using integral equation

The evolution operator  $U_t$  satisfies the following integral equation

$$U_t = 1 - i \int_0^t dt' (Db_{t'}^\dagger + D^\dagger b_{t'}) U_{t'}.$$

$\hat{U}_t$  satisfies

$$\hat{U}_t = 1 + i \int_0^t dt' (Db_{t'}^\dagger + D^\dagger b_{t'}) \hat{U}_{t'}.$$

Consider the iterated series  $U^{(N)}$  for the solution of the integral equation with the initial condition

$$U^{(0)} = 1$$

and relation

$$U_t^{(N)} = 1 - i \int_0^t dt' (Db_{t'}^\dagger + D^\dagger b_{t'}) U_{t'}^{(N-1)}$$

and the same series for  $\hat{U}$  with the initial condition  $\hat{U}_t^{(0)} = 1$ . The limit of the series (which exists under our assumptions) is the solution of the integral equation

$$\lim_{N \rightarrow \infty} U_t^{(N)} = U_t.$$

We want to prove that for  $t \geq \tau$

$$b_t U_\tau = \hat{U}_\tau b_t - i\gamma_- D\chi_{[0,\tau]}(t) U_\tau. \quad (\text{A.1})$$

Let us prove the following relation for the iterated series (for  $t \geq \tau$ ):

$$b_t U_\tau^{(N)} = \hat{U}_\tau^{(N)} b_t - i\gamma_- D\chi_{[0,\tau]}(t) U_\tau^{(N-1)}.$$

This equation clearly holds for  $N = 1$ . Suppose it holds for all  $N \leq M-1$ . Let us proof it for  $N = M$ :

$$\begin{aligned} b_t U_\tau^{(M)} &= b_t \left( 1 - i \int_0^\tau dt' (Db_{t'}^\dagger + D^\dagger b_{t'}) U_{t'}^{(M-1)} \right) \\ &= b_t - i \int_0^\tau dt' b_t (Db_{t'}^\dagger + D^\dagger b_{t'}) U_{t'}^{(M-1)} \\ &= b_t + i \int_0^\tau dt' (Db_{t'}^\dagger + D^\dagger b_{t'}) b_t U_{t'}^{(M-1)} - i \int_0^\tau dt' D \{b_t, b_{t'}^\dagger\} U_{t'}^{(M-1)} \\ &= b_t + i \int_0^\tau dt' (Db_{t'}^\dagger + D^\dagger b_{t'}) \left( \hat{U}_{t'}^{(M-1)} b_t - i\gamma_- D\chi_{[0,t']}(t) U_{t'}^{(M-2)} \right) \\ &\quad - i\gamma_- \int_0^\tau dt' D\delta(t-t') U_{t'}^{(M-1)} \\ &= \left( 1 + i \int_0^\tau dt' (Db_{t'}^\dagger + D^\dagger b_{t'}) U_{t'}^{(M-1)} \right) b_t + \int_0^\tau dt' \chi_{[0,t']}(t) (\dots) \\ &\quad - i\gamma_- D\chi_{[0,\tau]}(t) U_\tau^{(M-1)} \\ &= \hat{U}_\tau^{(M)} b_t + 0 - i\gamma_- D\chi_{[0,\tau]}(t) U_\tau^{(M-1)}. \quad (\text{A.2}) \end{aligned}$$

The second term is equal to 0, because  $t \geq \tau$ .

Taking the limit  $N \rightarrow \infty$ , we obtain (A.1). Substituting  $t$  for  $\tau$  in (A.1), we get (16.1).

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