Control of quantum stochastic diferential equations

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Contents

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Abstract. We review the basic features of the quantum stochastic calculus. Iteration schemes for the computation of the matrix elements of solutions of unitary quantum stochastic evolutions and associated quantum flows are provided along with a basic error analysis of the convergence of the iteration schemes. The application of quantum stochastic calculus to the solution of the quantum version of the quadratic cost control problem is described.

1. Quantum Stochastic Calculus

Let $B_t = \{B_t(\omega)/\,\omega \in \Omega\}$, $t \geq 0$, be one-dimensional Brownian motion. Integration with respect to B_t was defined by Itô in [28]. A basic result of the theory is that stochastic integral equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s \tag{1.1}$$

can be interpreted as stochastic differential equations of the form

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \tag{1.2}$$

where differentials are handled with the use of Itô's formula

$$(dB_t)^2 = dt, \quad dB_t\, dt = dt\, dB_t = (dt)^2 = 0 \tag{1.3}$$

In [27], Hudson and Parthasarathy obtained a Fock space representation of Brownian motion and Poisson process.

Definition 1. *The Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ over $L^2(\mathbb{R}_+, \mathcal{C})$ is the Hilbert space completion of the linear span of the exponential vectors $\psi(f)$ under the inner product*

$$< \psi(f), \psi(g) >= e^{<f,g>} \tag{1.4}$$

where $f, g \in L^2(\mathbb{R}_+, \mathcal{C})$ and $< f, g >= \int_0^{+\infty} \bar{f}(s)\, g(s)\, ds$ where, here and in what follows, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. .

The annihilation, creation and conservation operators $A(f)$, $A^{\dagger}(f)$ and $\Lambda(F)$ respectively, are defined on the exponential vectors $\psi(g)$ of Γ as follows.

Definition 2.

$$A_t\psi(g) = \int_0^t g(s)\,ds\;\psi(g) \tag{1.5}$$

$$A_t^{\dagger}\psi(g) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\psi(g+\epsilon\chi_{[0,t]}) \tag{1.6}$$

$$\Lambda_t\psi(g) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}\,\psi(e^{\epsilon\chi_{[0,t]}}g) \tag{1.7}$$

The basic quantum stochastic differentials dA_t, $dA_t^{\dagger}$, and $d\Lambda_t$ are defined as follows.

Definition 3.

$$dA_t = A_{t+dt} - A_t \tag{1.8}$$

$$dA_t^{\dagger} = A_{t+dt}^{\dagger} - A_t^{\dagger} \tag{1.9}$$

$$d\Lambda_t = \Lambda_{t+dt} - \Lambda_t \tag{1.10}$$

The fundamental result which connects classical with quantum stochastics is that the processes B_t and P_t defined by

$$B_t = A_t + A_t^{\dagger} \tag{1.11}$$

and

$$P_t = \Lambda_t + \sqrt{\lambda}(A_t + A_t^{\dagger}) + \lambda t \tag{1.12}$$

are identified, through their statistical properties e.g their vacuum characteristic functionals

$$< \psi(0), e^{i\,s\,B_t}\,\psi(0) >= e^{-\frac{s^2}{2}\,t} \tag{1.13}$$

and

$$< \psi(0), e^{i\,s\,P_t}\,\psi(0) >= e^{\lambda\left(e^{i\,s}-1\right)\,t} \tag{1.14}$$

with Brownian motion and Poisson process of intensity λ respectively.

Hudson and Parthasarathy defined stochastic integration with respect to the noise differentials of Definition 3 and obtained the Itô multiplication table

$\cdot$	$dA_t^\dagger$	$d\Lambda_t$	dA_t	dt
$dA_t^\dagger$	0	0	0	0
$d\Lambda_t$	$dA_t^\dagger$	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

Within the framework of Hudson-Parthasarathy Quantum Stochastic Calculus, classical quantum mechanical evolution equations take the form

$$\begin{aligned} dU_t &= -\left(\left(iH + \frac{1}{2}\,L^* L\right)\,dt + L^*\,W\,dA_t - L\,dA_t^\dagger + (1-W)\;d\Lambda_t\right) U_t \\ U_0 &= 1 \end{aligned} \tag{1.15}$$

where, for each $t \geq 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$ of a system Hilbert space $\mathcal{H}$ and the noise (or reservoir) Fock space Γ. Here H, L, W are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, with W unitary and H self-adjoint. Notice that for $L = W = -1$ equation (1.15) reduces to a classical SDE of the form (1.2). Here and in what follows we identify time-independent, bounded, system space operators X with their ampliation $X \otimes 1$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}_+, \mathcal{C}))$.

The quantum stochastic differential equation satisfied by the quantum flow

$$j_t(X) = U_t^*\,X\,U_t \tag{1.16}$$

where X is a bounded system space operator, is

$$\begin{aligned} dj_t(X) &= j_t\left(i\;[H,X] - \frac{1}{2}\,(L^*LX + XL^*L - 2L^*XL)\right)\;dt \\ &+\; j_t\,([L^*,X]\;W)\;dA_t + j_t\,(W^*\;[X,L])\;dA_t^\dagger + j_t\,(W^*\,X\,W - X)\;d\Lambda_t \\ j_0(X) &= X, \quad t \in [0,T] \end{aligned} \tag{1.17}$$

The commutation relations associated with the operator processes A_t, $A_t^\dagger$ are the Canonical (or Heisenberg) Commutation Relations (CCR), namely

$$\left[A_t, A_t^\dagger\right] = t\,I \tag{1.18}$$

Classical and quantum stochastic calculi were unified by Accardi, Lu, and Volovich in [17] within the framework of the white noise theory of

T. Hida. Denoting the basic white noise functionals by a_t and $a_t^\dagger$, they showed that the stochastic differentials of the Hudson-Partasarathy processes of [27] can be written as

$$dA_t = a_t\, dt \tag{1.19}$$

$$dA_t^\dagger = a_t^\dagger\, dt \tag{1.20}$$

$$d\Lambda_t = a_t a_t^\dagger\, dt \tag{1.21}$$

and Hudson-Partasarathy stochastic differential equations are reduced to white noise equations. This unification started a whole new theory corresponding to quantum stochastic processes given by powers of the white noise functionals. The results for the first such nonlinear extension, the square of white noise, can be summarized as follows.

Let $U(sl(2;\mathbb{R}))$ denote the universal enveloping algebra of $sl(2;\mathbb{R})$ with generators $B^\dagger$, M, B^- satisfying the commutation relations

$$\left[B^-, B^\dagger\right] = M \quad , \quad \left[M, B^\dagger\right] = 2B^\dagger \quad , \quad [M, B^-] = -2B^- \tag{1.22}$$

with involution

$$(B^-)^* = B^\dagger \quad , \quad M^* = M \tag{1.23}$$

After renormalization (cf. [17]), the square of white noise stochastic differentials

$$dB_t^- = {a_t}^2\, dt \tag{1.24}$$

$$dB_t^\dagger = {a_t^\dagger}^2\, dt \tag{1.25}$$

$$d\Lambda_t = a_t a_t^\dagger\, dt \tag{1.26}$$

can be defined on a Fock space. It was proved by Accardi-Skeide in [20] that the Fock space suitable for representing the square of white noise processes is the Finite Difference Fock space developed by Boukas-Feinsilver in [22] based on the Finite Difference Lie algebra of Feinsilver (cf. [26]).

The unitarity of solutions problem for “square of white noise” evolutions was open for several years. Preliminary work was done by Accardi, Hida, Boukas, and Kuo in [1], [4], [5],[13], [15]. In [8] Accardi and Boukas used the Boson Fock space representation of the square of white noise processes obtained by Accardi-Frantz-Skeide in [14], to show that square of white noise unitary evolutions satisfy quantum SDE of the type

$$
\begin{aligned}
dU_t &= \left(\left(-\frac{1}{2}(l(D_-)|D_-)+iH\right)dt + d\mathcal{A}_t(D_-) + d\mathcal{A}_t^\dagger(-l(W)D_-) + d\mathcal{L}_t(W-I)\right)U_t \\
U_0 &= 1
\end{aligned}
$$

formulated on the module $\mathcal{B}(\mathcal{H}_S)\otimes\Gamma(\mathcal{K})$, where $\mathcal{H}_S$ is a system Hilbert space, $\mathcal{K}=l_2(\mathbb{N})$ and $\Gamma(\mathcal{K})$ denotes the Fock space over $\mathcal{K}$ (see [14] for notation and details).

Applications of quantum stochastic calculus to the control of quantum evolution and Langevin equations (quantum flows) can be found in [7], [9], [10], [21], [23], [24], [25].

2. Matrix Elements and Iteration Schemes

The fundamental theorems of the Hudson-Partasarathy quantum stochastic calculus give formulas for expressing the matrix elements of quantum stochastic integrals in terms of ordinary Riemann-Lebesgue integrals.

Theorem 1. *Let*

$$M(t)=\int_0^t E(s)\,d\Lambda(s)+F(s)\,dA(s)+G(s)\,dA^\dagger(s)+H(s)\,ds \tag{2.1}$$

where E, F, G, H are (in general) time dependent adapted processes. Let also $u\otimes\psi(f)$ and $v\otimes\psi(g)$ be in the exponential domain of $\mathcal{H}\otimes\Gamma$. Then

$$< u\otimes\psi(f), M(t)\,v\otimes\psi(g) >= \tag{2.2}$$
$$\int_0^t < u\otimes\psi(f), \big(\bar{f}(s)\,g(s)\,E(s)+g(s)\,F(s)+\bar{f}(s)\,G(s)+H(s)\big)\,v\otimes\psi(g) > \,ds$$

Proof. See theorem 4.1 of [27]

□

Theorem 2. *Let*

$$M(t)=\int_0^t E(s)\,d\Lambda(s)+F(s)\,dA(s)+G(s)\,dA^\dagger(s)+H(s)\,ds \tag{2.3}$$

and

$$M'(t)=\int_0^t E'(s)\,d\Lambda(s)+F'(s)\,dA(s)+G'(s)\,dA^\dagger(s)+H'(s)\,ds \tag{2.4}$$

where E, F, G, H, E', F', G', H' are (in general) time dependent adapted processes. Let also $u\otimes\psi(f)$ and $v\otimes\psi(g)$ be in the exponential domain of $\mathcal{H}\otimes\Gamma$. Then

$$(2.5)\qquad < M(t)\, u \otimes \psi(f), M'(t)\, v \otimes \psi(g) >=$$
$$\int_0^t \{< M(s)\, u \otimes \psi(f), \left(\bar{f}(s)\, g(s) E'(s) + g(s)\, F'(s) + \bar{f}(s)\, G'(s) + H'(s)\right) v \otimes \psi(g) >$$
$$+ < (\bar{g}(s)\, f(s)\, E(s) + f(s)\, F(s) + \bar{g}(s)\, G(s) + H(s))\, u \otimes \psi(f), M'(s)\, v \otimes \psi(g) >$$
$$+ < (f(s) E(s) + G(s))\, u \otimes \psi(f), (g(s) E'(s) + G'(s))\, v \otimes \psi(g) >\}\, ds$$

Proof. See theorem 4.3 of [27]

□

We are interested in defining iteration schemes which can be used to compute the matrix elements

$$(2.6)\quad < u \otimes \psi(f), U_t\, v \otimes \psi(g) >, \;\; < u \otimes \psi(f), j_t(X)\, v \otimes \psi(g) >$$

and the corresponding probability amplitudes

$$(2.7)\quad |< u \otimes \psi(f), U_t\, v \otimes \psi(g) > |^2, \;\; | < u \otimes \psi(f), j_t(X)\, v \otimes \psi(g) > |^2$$

related to the quantum flow (1.16) and the Hudson-Parthasarathy stochastic differential equation

$$(2.8)\qquad dU_t = \left(K\, dt + B\, dA_t + C\, dA_t^\dagger + D\, d\Lambda_t\right)\, U_t$$

with initial condition

$$(2.9)\qquad U_0 = I$$

where $t \in [0, T]$ for some $T > 0$, and K, B, C, D are bounded system space operators of the form appearing in (1.15), i.e

$$(2.10)\qquad K = -\left(iH + \frac{1}{2}\, L^* L\right)$$
$$(2.11)\qquad B = -L^*\, W$$
$$(2.12)\qquad C = L$$
$$(2.13)\qquad D = W - 1$$

Equations (2.8) and (2.9) have the integral form

$$(2.14)\quad U_t = I + \int_0^t \, K\, U_s\, ds + B\, U_s\, dA_s + C\, U_s\, dA_s^\dagger + D\, U_s\, d\Lambda_s$$

defined (cf. [27], Proposition 7.1) as the $[0,T]$-uniform limit of the sequence $U_n = \{U_{n,t} \,/\, t \geq 0\}$ defined recursively on the exponential domain of $\mathcal{H} \otimes \Gamma$ by

$$U_{0,t} = I \tag{2.15}$$

and, for $n \geq 1$,

$$U_{n,t} = I + \int_0^t K\, U_{n-1,s}\, ds + B\, U_{n-1,s}\, dA_s + C\, U_{n-1,s}\, dA_s^\dagger + D\, U_{n-1,s}\, \Lambda_s \tag{2.16}$$

By Theorem 1, the matrix elements of (2.16) are given, for $n \geq 1$, by the recursion scheme

$$\begin{aligned} &< u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) > = < u \otimes \psi(f), v \otimes \psi(g) > \\ &+ \int_0^t \{\overline{f}(s) g(s) < u \otimes \psi(f), D\, U_{n-1,s}\, v \otimes \psi(g) > + g(s) < u \otimes \psi(f), B\, U_{n-1,s}\, v \otimes \psi(g) > \\ &+ \overline{f}(s) < u \otimes \psi(f), C\, U_{n-1,s}\, v \otimes \psi(g) > + < u \otimes \psi(f), K\, U_{n-1,s}\, v \otimes \psi(g) > \}\, ds \end{aligned} \tag{2.17}$$

Letting

$$u_{D^*} = D^* u \tag{2.18}$$
$$u_{B^*} = B^* u \tag{2.19}$$
$$u_{C^*} = C^* u \tag{2.20}$$
$$u_{K^*} = K^* u \tag{2.21}$$

we can rewrite iteration scheme (2.17) as

Iteration Scheme 1. *(Unitary Evolutions)*

$$\begin{aligned} &< u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) > = < u \otimes \psi(f), v \otimes \psi(g) > \\ &+ \int_0^t \{\overline{f}(s) g(s) < u_{D^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) > + g(s) < u_{B^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) > \\ &+ \overline{f}(s) < u_{C^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) > + < u_{K^*} \otimes \psi(f), U_{n-1,s}\, v \otimes \psi(g) > \}\, ds \end{aligned} \tag{2.22}$$

with

$$< u \otimes \psi(f), U_{0,t}\, v \otimes \psi(g) > = < u \otimes \psi(f), v \otimes \psi(g) > \tag{2.23}$$

The limit form of (2.22) as $n \to +\infty$ is

$$(2.24) \quad < u\otimes\psi(f), U_t\, v\otimes\psi(g) > = < u\otimes\psi(f), v\otimes\psi(g) >$$
$$+\int_0^t \{\overline{f}(s)g(s) < u_{D^*}\otimes\psi(f), U_s\, v\otimes\psi(g) > + g(s) < u_{B^*}\otimes\psi(f), U_s\, v\otimes\psi(g) >$$
$$+\overline{f}(s) < u_{C^*}\otimes\psi(f), U_s\, v\otimes\psi(g) > + < u_{K^*}\otimes\psi(f), U_s\, v\otimes\psi(g) >\}\, ds$$

which, by subtraction of the cases $t = t_n$ and $t = t_{n-1}$ where $t_{n-1} \leq t_n$, implies

$$(2.25) \quad < u\otimes\psi(f), U_{t_n}\, v\otimes\psi(g) > - < u\otimes\psi(f), U_{t_{n-1}}\, v\otimes\psi(g) > =$$
$$\int_{t_{n-1}}^{t_n} \{\overline{f}(s)g(s) < u_{D^*}\otimes\psi(f), U_s\, v\otimes\psi(g) > + g(s) < u_{B^*}\otimes\psi(f), U_s\, v\otimes\psi(g) >$$
$$+\overline{f}(s) < u_{C^*}\otimes\psi(f), U_s\, v\otimes\psi(g) > + < u_{K^*}\otimes\psi(f), U_s\, v\otimes\psi(g) >\}\, ds$$

or

Iteration Scheme 2. *(Time Iteration of Unitary Evolutions)*

$$(2.26) \quad < u\otimes\psi(f), U_{t_n}\, v\otimes\psi(g) > = < u\otimes\psi(f), U_{t_{n-1}}\, v\otimes\psi(g) > +$$
$$\int_{t_{n-1}}^{t_n} \{\overline{f}(s)g(s) < u_{D^*}\otimes\psi(f), U_s\, v\otimes\psi(g) > + g(s) < u_{B^*}\otimes\psi(f), U_s\, v\otimes\psi(g) >$$
$$+\overline{f}(s) < u_{C^*}\otimes\psi(f), U_s\, v\otimes\psi(g) > + < u_{K^*}\otimes\psi(f), U_s\, v\otimes\psi(g) >\}\, ds$$

The integral form of the quantum flow equation (1.17) is

$$(2.27) \quad j_t(X) = X + \int_0^t j_s(\hat{K})\, ds + j_s(\hat{B})\, dA_s + j_s(\hat{C})\, dA_s^\dagger + j_s(\hat{D})\, d\Lambda_s$$

where

$$(2.28) \quad \hat{K} = i[H, X] - \frac{1}{2}(L^*LX + XL^*L - 2L^*XL)$$
$$(2.29) \quad \hat{B} = [L^*, X]\, W$$
$$(2.30) \quad \hat{C} = W^*\, [X, L]$$
$$(2.31) \quad \hat{D} = W^*\, X\, W - X$$

The corresponding iteration scheme is

$$(2.32) \quad j_{0,t}(X) = X$$

and for $n \geq 1$

$$(2.33) \quad j_{n,t}(X) = X + \int_0^t j_{n-1,s}(\hat{K})\, ds + j_{n-1,s}(\hat{B})\, dA_s + j_{n-1,s}(\hat{C})\, dA_s^\dagger + j_{n-1,s}(\hat{D})\, d\Lambda_s$$

The matrix element form of the iteration scheme (2.32) and (2.33) is

$$(2.34) \quad < u \otimes \psi(f), j_{n,t}(X)\, v \otimes \psi(g) > = < u \otimes \psi(f), X\, v \otimes \psi(g) > +$$
$$< u \otimes \psi(f), \left(\int_0^t j_{n-1,s}(\hat{K})\, ds + j_{n-1,s}(\hat{B})\, dA_s + j_{n-1,s}(\hat{C})\, dA_s^\dagger + j_{n-1,s}(\hat{D})\, d\Lambda_s \right) v \otimes \psi(g) >$$

which by Theorem 1 yields

Iteration Scheme 3. *(General Quantum Flows)*

$$(2.35) \quad < u \otimes \psi(f), j_{n,t}(X)\, v \otimes \psi(g) > = < u \otimes \psi(f), X\, v \otimes \psi(g) >$$
$$+ \int_0^t \{ \overline{f}(s) g(s) < u \otimes \psi(f), j_{n-1,s}(\hat{D})\, v \otimes \psi(g) > + g(s) < u \otimes \psi(f), j_{n-1,s}(\hat{B})\, v \otimes \psi(g) >$$
$$+ \overline{f}(s) < u \otimes \psi(f), j_{n-1,s}(\hat{C})\, v \otimes \psi(g) > + < u \otimes \psi(f), j_{n-1,s}(\hat{K})\, v \otimes \psi(g) > \} \, ds$$

Notice that for $n = 1$ (2.35) becomes

$$(2.36) \quad < u \otimes \psi(f), j_{1,t}(X)\, v \otimes \psi(g) >$$
$$= < u \otimes \psi(f), X\, v \otimes \psi(g) > + < u \otimes \psi(f), \left(\int_0^t \hat{K}\, ds + \hat{B}\, dA_s + \hat{C}\, dA_s^\dagger + \hat{D}\, d\Lambda_s \right) v \otimes$$
$$= < u \otimes \psi(f), X\, v \otimes \psi(g) > + \int_0^t \{ \overline{f}(s) g(s) < u \otimes \psi(f), \hat{D}\, v \otimes \psi(g) > + g(s) < u \otimes \psi(f), \hat{B}$$
$$+ \overline{f}(s) < u \otimes \psi(f), \hat{C}\, v \otimes \psi(g) > + < u \otimes \psi(f), \hat{K}\, v \otimes \psi(g) > \} \, ds$$
$$= < u \otimes \psi(f), X\, v \otimes \psi(g) > + \int_0^t \{ \overline{f}(s) g(s) < u_{\hat{D}} \otimes \psi(f), v \otimes \psi(g) > + g(s) < u_{\hat{B}} \otimes \psi(f),$$
$$+ \overline{f}(s) < u_{\hat{C}} \otimes \psi(f), v \otimes \psi(g) > + < u_{\hat{K}} \otimes \psi(f), v \otimes \psi(g) > \} \, ds$$

and so, letting $u_{X^*} = X^* u$, we have

$$(2.37) \quad < u \otimes \psi(f), j_{1,t}(X)\, v \otimes \psi(g) > = < u_{X^*} \otimes \psi(f), v \otimes \psi(g) > +$$
$$\int_0^t \overline{f}(s) g(s)\, ds < u_{\hat{D}} \otimes \psi(f), v \otimes \psi(g) > + \int_0^t g(s)\, ds < u_{\hat{B}} \otimes \psi(f), v \otimes \psi(g) >$$
$$+ \int_0^t \overline{f}(s)\, ds < u_{\hat{C}} \otimes \psi(f), v \otimes \psi(g) > + \int_0^t ds < u_{\hat{K}} \otimes \psi(f), v \otimes \psi(g) >$$

The general theory of quantum flows, in the context of Hudson-Partasarathy calculus, can be found in [29]. We now consider flows $\{ j_t(X) \,/\, t \geq 0 \}$ of the standard quantum mechanical form

$$(2.38) \qquad j_t(X) = U_t^* \, X \, U_t$$

where U_t is , for each $t \geq 0$, a unitary operator.

Proposition 1. *Let X be a bounded system space operator, let U_t and $U_{n,t}$ be for each $t \in [0, T]$ and $n \geq 1$ as in (2.14) and (2.16) respectively, and let U_t^* and $U_{n,t}^*$ be their adjoints. If*

$$j_t(X) = U_t^* \, X \, U_t \tag{2.39}$$

and

$$j_{n,t}(X) = U_{n,t}^* \, X \, U_{n,t} \tag{2.40}$$

then

$$\lim_{n\to\infty} < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > = < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > \tag{2.41}$$

for all $u \otimes \psi(f)$ and $v \otimes \psi(g)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$. Convergence is uniform on $[0,T]$.

Proof.

$$\begin{aligned}
& | < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > - < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > | \\
= \;& | < u \otimes \psi(f), (j_t(X) - j_{n,t}(X)) \; v \otimes \psi(g) > | \\
= \;& | < u \otimes \psi(f), \left(U_t^* \, X \, U_t - U_{n,t}^* \, X \, U_{n,t}\right) \; v \otimes \psi(g) > | \\
= \;& | < u \otimes \psi(f), \left((U_t^* - U_{n,t}^*) \, X \, U_t + U_{n,t}^* \, X \, (U_t - U_{n,t})\right) \; v \otimes \psi(g) > | \\
\leq \;& | < u \otimes \psi(f), (U_t^* - U_{n,t}^*) \, X \, U_t \, v \otimes \psi(g) > | + | < u \otimes \psi(f), U_{n,t}^* \, X \, (U_t - U_{n,t}) \, v \otimes \psi(g) \\
= \;& | < (U_t - U_{n,t}) \, u \otimes \psi(f), X \, U_t \, v \otimes \psi(g) > | + | < U_{n,t} \, u \otimes \psi(f), X \, (U_t - U_{n,t}) \, v \otimes \psi(g) \\
\leq \;& \|(U_t - U_{n,t}) \, u \otimes \psi(f)\| \, \|X\| \, \|U_t\| \, \|v \otimes \psi(g)\| + \|U_{n,t} \, u \otimes \psi(f)\| \, \|X\| \, \|(U_t - U_{n,t}) \, v \otimes \psi(\\
\leq \;& \|(U_t - U_{n,t}) \, u \otimes \psi(f)\| \, \|X\| \, \|v \otimes \psi(g)\| + \|U_{n,t} \, u \otimes \psi(f)\| \, \|X\| \, \|(U_t - U_{n,t}) \, v \otimes \psi(g)\|
\end{aligned}$$

since $\|U_t\| = 1$. Since $U_{n,t}$ converges to U_t on the exponential domain of $\mathcal{H} \otimes \Gamma$ uniformly with respect to t and $\|U_{n,t} \, u \otimes \psi(f)\|$ is bounded, it follows that

$$| < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > - < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > | \to 0$$

as $n \to +\infty$.

$\square$

The iteration scheme for the matrix element associated with (2.40) is obtained, with the use of Theorems 1 and 2 as follows:

$$\begin{aligned}
& < u \otimes \psi(f), U_{n,t}^* \, X \, U_{k,t} \, v \otimes \psi(g) > = < U_{n,t} \, u \otimes \psi(f), X \, U_{k,t} \, v \otimes \psi(g) > \\
& = < \left(I + \int_0^t \, K \, U_{n-1,s} \, ds + B \, U_{n-1,s} \, dA_s + C \, U_{n-1,s} \, dA_s^\dagger + D \, U_{n-1,s} \, d\Lambda_s\right) \, u \otimes \psi(f), \\
& X \, \left(I + \int_0^t \, K \, U_{k-1,s} \, ds + B \, U_{k-1,s} \, dA_s + C \, U_{k-1,s} \, dA_s^\dagger + D \, U_{k-1,s} \, d\Lambda_s\right) \, v \otimes \psi(g) >
\end{aligned} \tag{2.42}$$

$$=< u \otimes \psi(f), (X\, v) \otimes \psi(g) > +$$

$$+ < u \otimes \psi(f), \left(\int_0^t X\, K\, U_{k-1,s}\, ds + X\, B\, U_{k-1,s}\, dA_s + X\, C\, U_{k-1,s}\, dA_s^\dagger + X\, D\, U_{k-1,s}\, d\Lambda_s \right) v \otimes$$

$$+ < \left(\int_0^t K\, U_{n-1,s}\, ds + B\, U_{n-1,s}\, dA_s + C\, U_{n-1,s}\, dA_s^\dagger + D\, U_{n-1,s}\, d\Lambda_s \right) u \otimes \psi(f), (X\, v) \otimes \psi$$

$$+ < \left(\int_0^t K\, U_{n-1,s}\, ds + B\, U_{n-1,s}\, dA_s + C\, U_{n-1,s}\, dA_s^\dagger + D\, U_{n-1,s}\, d\Lambda_s \right) u \otimes \psi(f),$$

$$\left(\int_0^t X\, K\, U_{k-1,s}\, ds + X\, B\, U_{k-1,s}\, dA_s + X\, C\, U_{k-1,s}\, dA_s^\dagger + X\, D\, U_{k-1,s}\, d\Lambda_s \right) v \otimes \psi(g)$$

$$=< u \otimes \psi(f), (X\, v) \otimes \psi(g) >$$

$$+ \int_0^t < u \otimes \psi(f), \left(\bar{f}(s)\, g(s)\, X\, D + g(s)\, X\, B + \bar{f}(s)\, X\, C + X\, K \right) U_{k-1,s}\, v \otimes \psi(g) >$$

$$+ \int_0^t < (\bar{g}(s)\, f(s)\, X^*\, D + f(s)\, X^*\, B + \bar{g}(s)\, X^*\, C + X^*\, K)\, U_{k-1,s}\, u \otimes \psi(f), v \otimes \psi(g) >$$

$$+ \int_0^t \{ < (U_{n,s} - 1)\, u \otimes \psi(f), \left(\bar{f}(s)\, g(s)\, X\, D + g(s)\, X\, B + \bar{f}(s)\, X\, C + X\, K \right) U_{k-1,s}\, v \otimes \psi$$

$$+ < (\bar{g}(s)\, f(s)\, D + f(s)\, B + \bar{g}(s)\, C + K)\, U_{n-1,s}\, u \otimes \psi(f), X\ (U_{k,s} - 1)\ v \otimes \psi(g) >$$

$$+ < (f(s)\, D + C)\, U_{n-1,s}\, u \otimes \psi(f), (g(s)\, X\, D + X\, C)\, U_{k-1,s}\, v \otimes \psi(g) > \}\, ds$$

and using (2.16) we obtain

Iteration Scheme 4. *(Quantum Mechanical Flows) For $n, k \geq 1$*

$$< u \otimes \psi(f), U_{n,t}^*\, X\, U_{k,t}\, v \otimes \psi(g) >=< u \otimes \psi(f), (X\, v) \otimes \psi(g) > \tag{2.43}$$

$$+ \int_0^t \{ \bar{f}(s)\, g(s)\ < u \otimes \psi(f), U_{n,s}^*\, X\, D\, U_{k-1,s}\, v \otimes \psi(g) > + g(s)\ < u \otimes \psi(f), U_{n,s}^*\, X\, B\, U_{k-}$$

$$+ \bar{f}(s)\ < u \otimes \psi(f), U_{n,s}^*\, X\, C\, U_{k-1,s}\, v \otimes \psi(g) > + < u \otimes \psi(f), U_{n,s}^*\, X\, K\, U_{k-1,s}\, v \otimes$$

$$\bar{f}(s)\, g(s)\ < u \otimes \psi(f), U_{n-1,s}^*\, D^*\, X\, U_{k,s}\, v \otimes \psi(g) > + \bar{f}(s)\ < u \otimes \psi(f), U_{n-1,s}^*\, B^*\, X\, U_{k,}$$

$$+ g(s)\ < u \otimes \psi(f), U_{n-1,s}^*\, C^*\, X\, U_{k,s}\, v \otimes \psi(g) > + < u \otimes \psi(f), U_{n-1,s}^*\, K^*\, X\, U_{k,s}\, v \otimes$$

$$+ \bar{f}(s)\, g(s)\ < u \otimes \psi(f), U_{n-1,s}^*\, D^*\, X\, D\, U_{k-1,s}\, v \otimes \psi(g) > + \bar{f}(s)\ < u \otimes \psi(f), U_{n-1,s}^*\, D^*\, X\, C$$

$$+ g(s)\ < u \otimes \psi(f), U_{n-1,s}^*\, C^*\, X\, D\, U_{k-1,s}\, v \otimes \psi(g) > + < u \otimes \psi(f), U_{n-1,s}^*\, C^*\, X\, C\, U_{k-1,s}$$

Letting $n = k$ in (2.43) we obtain the value of the matrix element

$$< u \otimes \psi(f), j_{n,t}(X)\, v \otimes \psi(g) >$$

Notice that for $n = 0$ or $k = 0$ (2.43) reduces to (2.22).

3. Error Analysis

Proposition 2. *Let $\epsilon > 0$, and let U_t and $U_{n,t}$, where $0 \leq t \leq T < +\infty$, be defined respectively by (2.14) and (2.16). Then for all $u \otimes \psi(f)$*

and $v \otimes \psi(g)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$, with g locally bounded and $u, v \neq 0$

$$(3.1) \quad |< u \otimes \psi(f), U_t\, v \otimes \psi(g) > - < u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) > | < \epsilon$$

for all $t \in [0, T]$ provided that

$$(3.2) \qquad \sqrt{\frac{\lambda^{n+1}\, T^{n+1}}{(n+1)!}} < \frac{\epsilon}{\|u\|\, \|v\|\, e^{\frac{\|f\|^2}{2}}\, e^{\frac{\|g\|^2}{2}}}$$

where $\epsilon > 0$ is the required degree of accuracy and

$$\begin{array}{lrcl}
(3.3) & \|f\|^2 & = & \int_0^{+\infty} |f(s)|^2\, ds \\
(3.4) & \|g\|^2 & = & \int_0^{+\infty} |g(s)|^2\, ds \\
(3.5) & \lambda & = & 6\, \alpha(T)^2\, e^T\, M \\
(3.6) & M & = & \max\left(\|K\|, \|B\|, \|C\|, \|D\|\right) \\
(3.7) & \alpha(T) & = & \sup_{0 \leq s \leq T}\, \max\left(|g(s)|^2, |g(s)|, 1\right)
\end{array}$$

Proof.

$$\begin{gathered}
| < u \otimes \psi(f), U_t\, v \otimes \psi(g) > - < u \otimes \psi(f), U_{n,t}\, v \otimes \psi(g) > |^2 \\
= | < u \otimes \psi(f), (U_t - U_{n,t})\, v \otimes \psi(g) > |^2 \leq \|u \otimes \psi(f)\|^2\, \|(U_t - U_{n,t})\, v \otimes \psi(g)\|^2 \\
= \|u \otimes \psi(f)\|^2\, \|\{\int_0^t K\, (U_{s_1} - U_{n-1,s_1})\, ds_1 + B\, (U_{s_1} - U_{n-1,s_1})\, dA_{s_1} + C\, (U_{s_1} - U_{n-1,s_1})\, dA_{s_1}^\dagger \\
+ D\, (U_{s_1} - U_{n-1,s_1})\, d\Lambda_{s_1}\}\, v \otimes \psi(g)\|^2
\end{gathered}$$

which by Corollary 1 and Theorem 4.4 of [27] is

$$\begin{aligned}
&\leq \; 6\,\alpha(T)^2\,\|u\otimes\psi(f)\|^2 \int_0^T e^{t-s_1}\,\{\|K\,(U_{s_1}-U_{n-1,s_1})\,v\otimes\psi(g)\|^2\\
&+\; \|B\,(U_{s_1}-U_{n-1,s_1})\,v\otimes\psi(g)\|^2+\|C\,(U_{s_1}-U_{n-1,s_1})\,v\otimes\psi(g)\|^2\\
&+\; \|D\,(U_{s_1}-U_{n-1,s_1})\,v\otimes\psi(g)\|^2\}\,ds_1\\
&\leq \; 6\,\alpha(T)^2\,(\max\{\|K\|,\|B\|,\|C\|,\|D\|\})^2\,\|u\otimes\psi(f)\|^2 \int_0^T e^{t-s_1}\,\|(U_{s_1}-U_{n-1,s_1})\,v\otimes\psi(g)\|\\
&= \; \lambda\|u\otimes\psi(f)\|^2 \int_0^T \|(U_{s_1}-U_{n-1,s_1})\,v\otimes\psi(g)\|^2\,ds_1\\
&\leq \; \lambda^2\|u\otimes\psi(f)\|^2 \int_0^T\int_0^{s_1} \|(U_{s_2}-U_{n-2,s_2})\,v\otimes\psi(g)\|^2\,ds_2\,ds_1\\
&\leq \; \lambda^3\|u\otimes\psi(f)\|^2 \int_0^T\int_0^{s_1}\int_0^{s_2} \|(U_{s_3}-U_{n-3,s_3})\,v\otimes\psi(g)\|^2\,ds_3\,ds_2\,ds_1\\
&\vdots\\
&\leq \; \lambda^n\|u\otimes\psi(f)\|^2 \int_0^T\int_0^{s_1}\int_0^{s_2}\ldots\int_0^{s_n-1} \|(U_{s_n}-U_{0,s_n})\,v\otimes\psi(g)\|^2\,ds_n\ldots ds_3\,ds_2\,ds_1
\end{aligned}$$

which using

$$\begin{aligned}
U_{s_n} &= 1+\int_0^{s_n} K\,U_{s_{n+1}}\,ds_{n+1}+B\,U_{s_{n+1}}\,dA_{s_{n+1}}+C\,U_{s_{n+1}}\,dA^{\dagger}_{s_{n+1}}+D\,U_{s_{n+1}}\,d\Lambda_{s_{n+1}}\\
U_{0,s_n} &= 1
\end{aligned}$$

and the unitarity of $U_{s_{n+1}}$, becomes

$$\begin{aligned}
&\leq \; \lambda^{n+1}\|u\otimes\psi(f)\|^2 \int_0^T\int_0^{s_1}\int_0^{s_2}\ldots\int_0^{s_n-1}\int_0^{s_n} \|U_{s_{n+1}}\,v\otimes\psi(g)\|^2\,ds_{n+1}\,ds_n\ldots ds_3\,ds_2\,ds_1\\
&= \; \lambda^{n+1}\|u\otimes\psi(f)\|^2 \int_0^T\int_0^{s_1}\int_0^{s_2}\ldots\int_0^{s_n-1}\int_0^{s_n} \|v\otimes\psi(g)\|^2\,ds_{n+1}\,ds_n\ldots ds_3\,ds_2\,ds_1\\
&= \; \|u\otimes\psi(f)\|^2\,\|v\otimes\psi(g)\|^2\,\lambda^{n+1} \int_0^T\int_0^{s_1}\int_0^{s_2}\ldots\int_0^{s_n-1}\int_0^{s_n} ds_{n+1}\,ds_n\ldots ds_3\,ds_2\,ds_1\\
&= \; \|u\otimes\psi(f)\|^2\,\|v\otimes\psi(g)\|^2\,\lambda^{n+1}\,\frac{T^{n+1}}{(n+1)!}\\
&= \; \|u\|^2\,\|v\|^2\,e^{\|f\|^2}\,e^{\|g\|^2}\,\lambda^{n+1}\,\frac{T^{n+1}}{(n+1)!}
\end{aligned}$$

which is less than ϵ^2 provided that (3.2) is satisfied.

□

Corollary 1. *In the notation of Proposition 2*

$$| \, | < u \otimes \psi(f), U_t \, v \otimes \psi(g) > | - | < u \otimes \psi(f), U_{n,t} \, v \otimes \psi(g) > | \quad | < \epsilon \tag{3.8}$$

for all $t \in [0,T]$ and u, v, f, g with $u, v \neq 0$, provided that

$$\sqrt{\tfrac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}} < \frac{\epsilon}{\|u\| \, \|v\| \, e^{\frac{\|f\|^2}{2}} \, e^{\frac{\|g\|^2}{2}}} \tag{3.9}$$

Proof. The proof follows by applying the triangle inequality to (3.1). □

Proposition 3. *In the notation of Proposition 1*

$$| < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > - < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > | < \epsilon \tag{3.10}$$

for all $t \in [0,T]$ and u, v, f, g with $u, v \neq 0$, provided that

$$\sqrt{\tfrac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}} + 1 < \frac{\epsilon^{1/2}}{\|u\|^{1/2} \, \|v\|^{1/2} \, e^{\frac{\|f\|^2}{4}} \, e^{\frac{\|g\|^2}{4}} \, \|X\|^{1/2}} \tag{3.11}$$

Proof.

$$\begin{aligned} & | < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > - < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > | \\ \leq \; & \|(U_t - U_{n,t}) \, u \otimes \psi(f)\| \, \|X\| \, \|v \otimes \psi(g)\| \\ + \; & (\|(U_t - U_{n,t}) \, u \otimes \psi(f)\| + \|u \otimes \psi(f)\|) \, \|X\| \, \|(U_t - U_{n,t}) \, v \otimes \psi(g)\| \end{aligned}$$

and using, as in the proof of Proposition 2,

$$\|(U_t - U_{n,t}) \, a \otimes \psi(b)\|^2 \leq \|a \otimes \psi(b)\|^2 \, \tfrac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}$$

we obtain

$$\begin{aligned} & | < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > - < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > | \\ \leq \; & \|u \otimes \psi(f)\| \, \|v \otimes \psi(g)\| \, \|X\| \left(2 \sqrt{\frac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}} + \frac{\lambda^{n+1} \, T^{n+1}}{(n+1)!} \right) \\ \leq \; & \left(\sqrt{\frac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}} + 1 \right)^2 \|u\| \, \|v\| \, e^{\frac{\|f\|^2}{2}} \, e^{\frac{\|g\|^2}{2}} \, \|X\| < \epsilon \end{aligned}$$

from which (3.11) follows. □

Corollary 2. *In the notation of Proposition 3*

$$| \, | < u \otimes \psi(f), j_{n,t}(X) \, v \otimes \psi(g) > | - | < u \otimes \psi(f), j_t(X) \, v \otimes \psi(g) > | \, | < \epsilon \tag{3.12}$$

for all $t \in [0, T]$ *and* u, v, f, g *with* $u, v \neq 0$, *provided that*

$$\sqrt{\frac{\lambda^{n+1} \, T^{n+1}}{(n+1)!}} + 1 < \frac{\epsilon^{1/2}}{\|u\|^{1/2} \, \|v\|^{1/2} \, e^{\frac{\|f\|^2}{4}} \, e^{\frac{\|g\|^2}{4}} \, \|X\|^{1/2}} \tag{3.13}$$

Proof. The proof follows by applying the triangle inequality to (3.10). □

4. Appplications to quantum stochastic control

The quadratic cost control problem of classical stochastic control theory was extended to the quantum stochastic framework by L. Accardi and A. Boukas in [7], [9], [10], [21], [23], [24], [25].

In the case of first order white noise it was shown that if $U = \{U_t \, / \, t \geq 0\}$ is a stochastic process satisfying on a finite interval $[0, T]$ the quantum stochastic differential equation

$$dU_t = (F \, U_t + u_t) \, dt + \Psi \, U_t \, dA_t + \Phi \, U_t \, dA_t^\dagger + Z \, U_t \, d\Lambda_t, \quad U_0 = 1 \tag{4.1}$$

then the performance functional

$$Q_{\xi,T}(u) = \int_0^T [< U_t \, \xi, X^2 \, U_t \, \xi > + < u_t \, \xi, u_t \, \xi >] \, dt - < u_T \, \xi, U_T \, \xi > \tag{4.2}$$

satisfies

$$\min Q_{\xi,T}(u) = < \xi, \Pi \xi > \tag{4.3}$$

where the minimum is taken over all processes of the form $u_t = -\Pi \, U_t$, ξ is an arbitrary vector in the exponential domain of the tensor product of the system Hilbert space and the BosonFock space over $L^2([0, +\infty), \mathbb{C})$, and Π is the solution of the Algebraic Riccati Equation

$$\Pi \, F + F^* \Pi + \Phi^* \Pi \Phi - \Pi^2 + X^2 = 0 \tag{4.4}$$

with the additional conditions

$$\Pi \, \Psi + \Phi^* \, \Pi + \Phi^* \, \Pi \, Z = 0 \tag{4.5}$$

$$\Pi \, Z + Z^* \, \Pi + Z^* \, \Pi \, Z = 0 \tag{4.6}$$

Using this we have proved (ref. [9], [10]) that if X is a bounded self-adjoint system operator such that the pair $(i\,H,\,X)$ is stabilizable, then the quadratic performance functional

$$J_{\xi,T}(L,W)=\int_0^T [\,\|j_t(X)\,\xi\|^2+\frac{1}{4}\|j_t(L^*L)\,\xi\|^2\,]\,dt+\frac{1}{2}\|j_T(L)\,\xi\|^2 \tag{4.7}$$

associated with the quantum stochastic flow $\{j_t(X)=U_t^*\,X\,U_t\,/\,t\geq 0\}$ satisfying

$$dj_t(X)=j_t(i[H,X]-\tfrac{1}{2}(L^*LX+XL^*L-2L^*XL))\,dt \tag{4.8}$$
$$+j_t([L^*,X]\,W)\,dA_t+j_t(W^*\,[X,L])\,dA_t^\dagger++j_t(W^*\,X\,W-X)\,d\Lambda_t,\quad j_0(X)=X$$

where $U=\{U_t\,/\,t\geq 0\}$ is the solution of

$$dU_t=-((iH)+\frac{1}{2}\,L^*L)\,dt+L^*\,W\,dA_t-L\,dA_t^\dagger+(1-W)\,d\Lambda_t)\,U_t,\quad ,U_0=1, \tag{4.9}$$

is minimized by choosing

$$L=\sqrt{2}\,\Pi^{1/2}\,W_1 \tag{4.10}$$
$$W=W_2 \tag{4.11}$$

where Π is the solution of the Algebraic Riccati Equation

$$i\,[H,\Pi]+\Pi^2+X^2=0 \tag{4.12}$$

and W_1, W_2 are bounded unitary system operators commuting with Π. Moreover

$$\min_{L,W} J_{\xi,T}(L,W)=<\xi,\Pi\,\xi> \tag{4.13}$$

In the case of quantum stochastic differential equations driven by the square of white noise processes, we have shown (re. [10]) that if X is a bounded self-adjoint system operator such that the pair $(i\,H,\,X)$ is stabilizable then the performance functional

$$J_{\xi,T}(D_-,W)=\int_0^T [\,\|j_t(X)\,\xi\|^2+\frac{1}{4}\|j_t((D_-^*|D_-^*))\,\xi\|^2\,]\,dt+\frac{1}{2}<\xi,j_T((D_-^*|D_-^*))\,\xi> \tag{4.14}$$

associated with the quantum flow $\{j_t(X)=U_t^*\,X\,U_t\,/\,t\geq 0\}$, where $U=\{U_t\,/\,t\geq 0\}$ is the solution of the quantum stochastic differential equation

$$\begin{aligned} dU_t &= ((-\frac{1}{2}(D_-^*|D_-^*) + iH)\,dt + d\mathcal{A}_t(D_-) + d\mathcal{A}_t^\dagger(-r(W)D_-^*) + d\mathcal{L}_t(W-I)) \quad (4.15)\\ U_0 &= 1 \end{aligned}$$

is minimized by choosing

$$D_- = \sum_n D_{-,n} \otimes e_n \tag{4.16}$$

and

$$W = \sum_{\alpha,\beta,\gamma} W_{\alpha,\beta,\gamma} \otimes \rho^+(B^{+^\alpha} M^\beta B^{-\gamma}) \tag{4.17}$$

so that

$$\frac{1}{2}(D_-^*|D_-^*) = (\frac{1}{2}\sum_n D_{-,n}\, D_{-,n}^*) \otimes 1 = \Pi \tag{4.18}$$

and

$$[D_{-,n}, D_{-,m}] = [D_{-,n}, D_{-,m}^*] = [D_{-,n}, W_{\alpha,\beta,\gamma}] = [D_{-,n}, W_{\alpha,\beta,\gamma}^*] = 0 \tag{4.19}$$

for all $n, m, \alpha, \beta, \gamma$, where Π is the positive self-adjoint solution of the Algebraic Riccati equation

$$i\,[H, \Pi] + \Pi^2 + X^2 = 0 \tag{4.20}$$

Moreover

$$\min_{D_-, W} J_{\xi,T}(D_-, W) = <\xi, \Pi\,\xi> \tag{4.21}$$

References

[1] L. Accardi, A. Boukas, *Unitarity conditions for stochastic differential equations driven by nonlinear quantum noise*, Random Operators and Stochastic Equations, 10 (1) (2002) 1–12.

[2] L. Accardi, A. Boukas, *Stochastic evolutions driven by non-linear quantum noise*, Probability and Mathematical Statistics, 22 (1) (2002) 141–154.

[3] L. Accardi, A. Boukas, *Stochastic evolutions driven by non-linear quantum noise II*, Russian Journal of Mathematical Physics, 8 (4) (2001) 401 – 413.

[4] L. Accardi, A. Boukas, *Square of white noise unitary evolutions on Boson Fock space*, International conference on stochastic analysis in honor of Paul Kree, Hammamet, Tunisie, October 22–27, 2001.

[5] L. Accardi, A. Boukas, *Unitarity conditions for the renormalized square of white noise*, Trends in Contemporary Infinite Dimensional Analysis and Quantum Probability, Natural and Mathematical Sciences Series 3, 7–36, Italian School of East Asian Studies, Kyoto, Japan (2000).

[6] L. Accardi, A. Boukas, *The semi-martingale property of the square of white noise integrators*, Stochastic Partial Differential Equations and Applications, pp. 1–19, eds g. Da Prato and L. Tubaro, Marcel Dekker, Inc. (2002).

[7] L. Accardi, A. Boukas, *Control of elementary quantum flows*, Proceedings of the 5th IFAC symposium on nonlinear control systems, July 4-6, 2001, St. Petersburg, Russia".

[8] L. Accardi, A. Boukas, *The unitarity conditions for the square of white noise*, Dimensional Anal. Quantum Probab. Related Topics, 6 (2) (2003) 1–26.

[9] L. Accardi, A. Boukas, *Quadratic control of quantum processes*, Russian Journal of Mathematical Physics (2002).

[10] L. Accardi, A. Boukas, *Control of quantum Langevin equations*, Open Systems and Information Dynamics, 9 (2002) 1–15.

[11] L. Accardi, A. Boukas, *Quantum stochastic Weyl operators*, to appear in the IEEE Proceedings of the International Conference "Physics and Control" (PhysCon 2003), August 20-22, 2003, St. Petersburg, Russia.

[12] L. Accardi, A. Boukas, *Higher powers of quantum white noise*, to appear (2003).

[13] L. Accardi, A. Boukas, H.H. Kuo, *On the unitarity of stochastic evolutions driven by the square of white noise*, Infinite Dimensional Analysis, Quantum Probability, and Related Topics, 4 (4) (2001) 1–10.

[14] L. Accardi, U. Franz, M. Skeide, *Renormalized squares of white noise and non-gaussian noises as Levy processes on real Lie algebras*, Comm. Math. Phys. 228 (1) (2002) 123–150.

[15] L. Accardi, T. Hida, H.H Kuo, *The Itô table of the square of white noise*, Infinite Dimensional Analysis, Quantum Probability, and Related Topics, 4 (2001) 267–275.

[16] L. Accardi, Y.G Lu, I.V Volovich, *Quantum theory and its stochastic limit* , Springer 2002.

[17] L. Accardi, Y.G Lu, I.V Volovich, *White noise approach to classical and quantum stochastic calculi*, Lecture Notes of the Volterra International School of the same title, Trento, Italy, 1999, Volterra Center preprint 375.

[18] L. Accardi, Y.G Lu, I.V Volovich, *Non-linear extensions of classical and quantum stochastic calculus and essentially infinite dimensional analysis*, Volterra Center preprint no.268, 1996.

[19] L. Accardi, N. Obata, *Towards a non-linear extension of stochastic calculus*, Publications of the Research Institute for Mathematical Sciences, Kyoto, RIMS Kokyuroku 957 (1996) 1–15 Obata N. (ed).

[20] L. Accardi, M. Skeide, *On the relation of the square of white noise and the finite difference algebra*, Infinite Dimensional Analysis, Quantum Probability, and Related Topics, 3 (2000) 185–189.

[21] Boukas A.,*Linear Quantum Stochastic Control*, Quantum Probability and related topics, 105-111, QP -PQ IX, World Scientific Publishing, River Edge NJ,1994.

[22] Boukas A., *Quantum stochastic analysis: a non Brownian case*, Ph.D Thesis, Southern Illinois University, 1998.

[23] Boukas A., *Application of Operator Stochastic Calculus to an Optimal Control problem*, Mat. Zametki 53 (5) (1993) 48–56 Russian. Translation in Math. Notes 53 (5–6) (1993) 489–494, MR 96a 81070.

[24] Boukas A., *Operator valued stochastic control in Fock space with applications to noise filtering and orbit tracking*, Journal of Probability and Mathematical Statistics, 16 (1) (1994).

[25] Boukas A., *Stochastic Control of operator-valued processes in Boson Fock space*, Russian Journal of Mathematical Physics 4 (2) (1996) 139–150. MR 97j 81178

[26] P. J. Feinsilver, *Discrete analogues of the Heisenberg-Weyl algebra*, Monatsh. Math., 104 (1987) 89–108.

[27] R. L. Hudson, K. R. Parthasarathy, *Quantum Ito's formula and stochastic evolutions*, Commun. Math. Phys. 93 (1984) 301–323 .

[28] Ito K., *On stochastic differential equations*, Memoirs Amer. Math. Soc. 4 (1951).

[29] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhauser Boston Inc., 1992.

[30] S. Wolfram: The Mathematica Book, 4th ed., Wolfram Media/Cambridge University Press (1999)