Analytical and computational study of curvature depending functionals in image segmentation

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Preface

In the present thesis we study variational problems for image segmentation. The segmentation problem in computer vision theory consists in finding a decomposition of an image into homogeneous regions which represent meaningful parts of objects in the image.

If brightness of the image is mathematically represented by a real valued measurable function g, the segmentation problem consists in finding a pair (u, C) such that C is a family of curves that represent the boundaries of the regions where brightness is homogeneous, and u is a smooth approximation (i.e., a denoised version) of the image datum g in each region.

In the variational approach to image segmentation, given the datum g, we look for a pair (u, C) that minimizes a suitable energy functional. In the present thesis we consider a specific class of functionals which contain the integral of a function of curvature along the unknown set of curves C. The functionals are of the type

$$\mathcal{G}(u, C) = \int_{\Omega \setminus C} |\nabla u|^2 dx + \int_{\Omega} |u - g|^2 dx + \int_C [1 + \psi(k)] \, d\mathcal{H}^1 + \# P(C),$$

where Ω is the image domain, k is the curvature of the curves in C, ψ is a convex function, $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, $P(C)$ is the set of the endpoints of the curves in C, and $\#$ is the counting measure. The functional $\mathcal{G}$ is minimized over the families of curves C and functions $u \in W^{1,2}(\Omega \setminus C)$.

Minimization of $\mathcal{G}$ is a variational problem with free discontinuities, since the function u may have discontinuities along the unknown set C of curves.

The thesis is constituted by two parts.

In the first part we consider a general class of functions ψ of the curvature, which includes functions with linear growth at infinity. We prove the existence of minimizers of the functional $\mathcal{G}$ in a suitable class of families of curves constituted by curves having derivative of bounded variation.

In the second part we consider the specific function $\psi(k) = k^2$ and we consider an approximation result of the functional $\mathcal{G}$, previously proved in the literature, by means of Γ-convergence. The numerical minimization of the functional $\mathcal{G}$ is a challenging problem that cannot be solved by using straightforward either finite difference or finite element methods. Though Γ-convergence makes it possible to approximate the functional $\mathcal{G}$ by means of functionals without geometric terms, for instance defined on Sobolev spaces, the minimization of the functionals Γ-converging to $\mathcal{G}$ is still a difficult numerical problem, since such functionals contain terms of the type

$$\int_\Omega |\nabla s|^2 \left(\operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right)\right)^2 dx,$$

where s is a function that gives an approximate description of the set of curves C, and div denotes the divergence operator.

We consider modified approximating functionals that are numerically more tractable and we study the variational properties of such functionals. We prove existence of minimizers for the modified functionals and we study their convergence properties to the original Γ-converging functionals when suitable parameters of the functionals vary.

Then we derive the system of Euler equations of the modified functionals, we design an iterative numerical scheme based on finite differences for the solution of the Euler equations, and we discuss the outcome of some computer experiments on simple simulated images.

Chapter 1

Existence of minimizers of a curvature depending functional with linear growth at infinity

1.1 Introduction

In recent years much attention has been devoted to mathematical methods for image reconstruction. In mathematical models the grey level image of a scene is typically a real valued measurable function g defined on a plane domain Ω, and g measures the brightness of the scene. There are many mathematical approaches to image reconstruction problems. Particularly, such problems have been studied from the point of view of the Calculus of Variations minimizing suitable energy functionals.

An important variational problem has been proposed by Mumford and Shah [28] for the problem of image segmentation which consists in findinga decomposition of the image domain Ω into regions with relatively uniform brightness. Such regions correspond to meaningful parts of objects in a scene. Mumford and Shah look for minimizers of the following functional

$$G^{MS}(u,C)=\int_{\Omega\setminus C}|\nabla u|^2dx+\int_{\Omega\setminus C}|u-g|^2dx+\mathcal{H}^1(C)\,,$$

where $\Omega\subset\mathbb{R}^2$ is a bounded open set, $g\in L^\infty(\Omega)$ and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. The functional is minimized over pairs (u,C) such that $C\subset\overline{\Omega}$ is a closed set and $u\in C^1(\Omega\setminus C)$. The set C represents the boundaries of the regions of a segmentation, and the function u is a smooth approximation (i.e., denoised) in $\Omega\setminus C$ of the image datum g.

The existence of minimizers of G^{MS} has been proved by De Giorgi, Carriero and Leaci [19], and by Dal Maso, Morel and Solimini [26]. Mumford and Shah studied the properties of minimizers assuming that C is a finite set of nonintersecting $\mathcal{C}^2$ arcs, possibly with some endpoints on the boundary of Ω. They proved that the curves of any such minimizer C may meet each other only at triple points with 120^o angles, and that they may meet the boundary of Ω only perpendicularly.

Such constraints constitute a drawback of the model, since corners and junctions, which are important for pattern recognition, are distorted. Since the length measure is not sensitive to corners and junctions, curvature depending functionals may be considered [5, 23, 31, 32, 33, 29, 30]. Coscia [14] proposed a functional for segmentation, defined on families C of curves, that includes the integral of square curvature $\int_C k^2 d\mathcal{H}^1$ and the number of endpoints of the curves in C. The functional is

$$G(u,C) = \int_{\Omega\setminus C} |\nabla u|^2 dx + \int_\Omega |u-g|^2 dx + \mathcal{F}_0(C)\,,$$

where C is an admissible family of curves and $u \in W^{1,2}(\Omega \setminus C)$. A family C of curves is admissible if C is a finite family of curves of class $W^{2,2}$ which do not cross each other or themselves, except possibly with the same tangent vector. The functional $\mathcal{F}_0$ is defined by

$$\mathcal{F}_0(C) = \inf\{\mathcal{F}(\hat{C}) : [\hat{C}] = [C]\},$$

where $[C]$ denotes the image of a family of curves C, and

$$\mathcal{F}(C) = \sum_{\gamma\in C} \int_\gamma (\alpha_K k^2 + \alpha_L) d\mathcal{H}^1 + \alpha_P \# P,$$

where α_L, α_K, α_P are positive weights, k is the curvature of the curve $\gamma \in C$, P is the set of the endpoints of all the curves in C with the exception of the regular closed ones, and # denotes the counting measure. Note that the introduction of $\mathcal{F}_0$ permits us to deal with the image of a family of curves, independently of the particular parametrization of the curves. Admissible families of curves are introduced for semicontinuity reasons. Existence of minimizers of G has been proved in [14] by proving compactness and lower semicontinuity results in the class of admissible families C of curves with $u \in W^{1,2}(\Omega \setminus [C])$.

According to the variational model proposed by Coscia, corner points of curves in a family C are considered as endpoints of some curves. The energy functional penalizes the number of endpoints, but it is not sensitive to the amplitude of angles at corner points. It would be interesting to consider a functional that takes into account also the amplitude of angles at corner points and penalizes large variations of the direction of the tangent vector of the curves.

In the present work we consider a curvature depending functional with linear growth at infinity with respect to the modulus of curvature. The functional penalizes both endpoints of curves and amplitude of angles (changes of direction) at corner points and junctions which are not considered as endpoints.

We consider families of curves having derivatives with bounded variation, i.e., families $C = \{\gamma_i\}_i$ with γ_i of class $W^{1,1}$ and $\dot{\gamma}_i \in BV$. We follow a framework developed by Bellettini and Paolini [8]. For any curve $\gamma \in C$, parametrized with constant velocity, we define an argument function $\theta \in BV$ such that $\dot{\gamma} = (R\cos\theta, R\sin\theta)$ a.e., with $R > 0$. Then θ is the angle that the derivative $\dot{\gamma}$ of a curve forms with the positive direction of a fixed axis. In this way we identify the tangent vectors with points of the unit circle S^1 and we consider corners in terms of jumps of the corresponding angles. The definition of θ must be considered $\mod 2\pi$.

The functional that replaces $\int_\gamma k^2 d\mathcal{H}^1$ in the present work is

$$K(\gamma) = \int_\gamma \psi(\theta, \dot{\theta}) d\mathcal{H}^1 \,,$$

where $\psi : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ is a Borel function satisfying suitable convexity and growth conditions.

Note that if we choose $\psi(\theta, \dot{\theta}) = |\dot{\theta}|^2$ then we have the functional considered by Coscia. An important example is the function with linear growth at infinity considered by Mumford and Nitzberg ([29]) for applications to image segmentation:

$$\psi(\theta, \dot{\theta}) = \begin{cases} c_1 \dot{\theta}^2 & \text{if } \ |\dot{\theta}| < T \\ c_2 |\dot{\theta}| - c_3 & \text{if } \ |\dot{\theta}| \geq T, \end{cases}$$

with $T, c_1 > 0$, $c_1 T^2 = c_2 T - c_3$.

In the next sections we shall define precisely the functional K on argument functions and curves having derivative of bounded variation. The contribute to the energy of a corner point along a curve will be given by the jump of the argument function θ at such a point.

We consider the functional

$$\mathcal{G}(u, C) = \int_{\Omega \setminus C} |\nabla u|^2 dx + \int_\Omega |u - g|^2 dx + \mathcal{F}_0(C) \,,$$

where $\mathcal{F}_0(C) = \inf\{\mathcal{F}(\hat{C}) : [\hat{C}] = [C]\}$, with

$$\mathcal{F}(C) = \sum_{\gamma \in C} [\alpha_K K(\gamma) + \alpha_L L(\gamma)] + \alpha_P \# P(C) \,,$$

where $L(\gamma)$ is the length of γ.

We prove that $\mathcal{G}$ admits minimizers among all pairs (u, C), with C admissible family of curves and $u \in W^{1,2}(\Omega \setminus [C])$. Since for curves having derivative of bounded variation the tangent vector is not defined everywhere, we need to extend the class of admissible families of curves considered in [14]. In order to construct a suitable class of admissible families of curves having compactness properties, we generalize the notion of curves without crossings with respect to the curves of class $W^{2,2}$ considered by Coscia. This notion of family of curves, and the associated theorems proving their main properties and compactness results, constitute the main original contribution of this part of the thesis. In order to achieve such results new techniques concerning the properties of this type of curves have been developed with respect to the previous works by Bellettini, Paolini and Coscia. Such techniques and new theorems are discussed in Sections 1.5, 1.6 and 1.7.

1.2 Notations and preliminary definitions

We denote by $\#$ the counting measure, by $\mathcal{H}^h$ the h-dimensional Hausdorff measure in $\mathbb{R}^2$ for $h = 0, 1$ and by $|B|$ the Lebesgue measure of a Borel set $B \subseteq \mathbb{R}^2$. Given $z_0 \in \mathbb{R}^2$ and $r > 0$, we denote by $D_r(z_0)$ the open disk $D_r(z_0) = \{z \in \mathbb{R}^2 : |z - z_0| < r\}$. We will use standard notation for the Lebesgue and Sobolev spaces L^p and $W^{s,p}$.

We denote by $\mathcal{M}$ the class of all Lebesgue measurable subsets of $\mathbb{R}^2$. We identify $\mathcal{M}$ with a closed subset of $L^1(\mathbb{R}^2)$ by means of the map $E \longmapsto \chi_E$, where χ_E is the characteristic function of E, i.e., $\chi_E(z) = 1$ if $z \in E$, $\chi_E(z) = 0$ if $z \notin E$. The $L^1(\mathbb{R}^2)$-topology on $\mathcal{M}$ is, therefore, the topology on $\mathcal{M}$ induced by the distance $d(E_1, E_2) = |E_1 \bigtriangleup E_2|$, where $E_1, E_2 \in \mathcal{M}$ and $\bigtriangleup$ is the symmetric difference of sets.

For any subset D of $\mathbb{R}^2$, we denote by $\mathrm{int}(D)$ the interior of D, by $\overline{D}$ the closure of D, and by ∂D the topological boundary of D. Given two sets $A, B \subseteq \mathbb{R}^2$, by $A \subset\subset B$ we mean that $\overline{A}$ is a compact set contained in B. We indicate by I a bounded open interval of $\mathbb{R}$.

We denote by $\psi : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ a Borel function having the following properties:

(i) $\psi(\eta, \cdot)$ is convex on $\mathbb{R}$ for any $\eta \in \mathbb{R}$;

(ii) ψ is lower semicontinuous on $\mathbb{R} \times \mathbb{R}$;

(iii) $\psi(\cdot, 0)$ is bounded;

(iv) there exist two constants $A > 0$ and $B \geq 0$ such that

$$\psi(\eta, \xi) \geq A|\xi| - B \quad \forall (\eta, \xi) \in \mathbb{R} \times \mathbb{R};$$

(v) $\psi(\eta, \xi) = \psi(\eta + \pi, \xi)$ for any $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}$;

(vi) $\psi(\eta, \xi) = \psi(\eta, -\xi)$ for any $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}$.

1.2.1 The space $BV(I)$

If λ is a scalar or vector-valued Radon measure, its total variation will be denoted by $|\lambda|$. Let $B \subseteq I$ be a Borel set and $f : B \to \mathbb{R}$ be a Borel function; the integral of f on B with respect to λ will be indicated by $\int_B f \, d\lambda$.

If μ is a scalar Radon measure on I, we have a unique decomposition $\lambda = \lambda^a + \lambda^s$, where λ^a is absolutely continuous and λ^s is singular with respect to μ. The density of λ^a with respect to μ, which is a function belonging to L^1_μ, will be indicated by $\frac{d\lambda}{d\mu}$ and will be called the Radon-Nikodym derivative of λ with respect to μ. Then

$$\frac{d\lambda}{d\mu}(t) = \lim_{\rho \to 0^+} \frac{\lambda(]t-\rho, t+\rho[)}{\mu(]t-\rho, t+\rho[)} \qquad \text{for } \mu - a.e. \ t \in I,$$

and $\lambda(B) = \int_B \frac{d\lambda}{d\mu} d\mu + \lambda^s(B)$ for every Borel set $B \subseteq I$.

The space $BV(I)$ is defined as the space of all functions $f \in L^1(I)$ whose distributional derivative $\dot{f}$ is a Radon measure with bounded total variation in I. We say that $f = (f_1, f_2) : I \to \mathbb{R}^2$ belongs to $BV(I; \mathbb{R}^2)$ if $f_i \in BV(I)$ for $i = 1, 2$.

Given $f \in BV(I)$, we shall write

$$\dot{f} = \dot{f}^a \, dt + \dot{f}^s,$$

where $\dot{f}^a \in L^1(I)$ is the density of the absolutely continuous part of $\dot{f}$ with respect to the Lebesgue measure dt on I, and $\dot{f}^s$ is the singular part. We shall use the same notation whenever $f \in BV(I; \mathbb{R}^2)$.

If $f \in BV(I)$ we indicate by S_f the jump set of f, and we set

$$f(t-) = \mathrm{ap} - \liminf_{\tau \to t} f(\tau) \leq f(t+) = \mathrm{ap} - \limsup_{\tau \to t} f(\tau).$$

It is known that

$$S_f = \{t \in I : \, f(t-) < f(t+)\} = \{t \in I : \, |\dot{f}|(\{t\}) > 0\},$$

and that S_f is at most countable. If $f = (f_1, f_2) \in BV(I; \mathbb{R}^2)$, by S_f we mean $S_{f_1} \cup S_{f_2}$.

Moreover, if $B \subseteq I$ is a Borel set we have

$$\dot{f}^s(B) = \int_{B \cap S_f} (f(t+) - f(t-)) d\mathcal{H}^0(t) + C_f(B),$$

where C_f is the *Cantor part* of $\dot{f}$, and is a measure such that $C_f(\{t\}) = 0$ for any $t \in I$.

Let $f \in BV(I)$, and set $\tilde{f}(t) = (f(t-)+f(t+))/2$ for any $t \in I$; then $\tilde{f}$ is a function of bounded variation in the classical sense, $f = \tilde{f}$ almost everywhere in I, $\dot{f}^a$ coincides almost everywhere with the pointwise derivative of $\tilde{f}$, and

$$\{f(t-), f(t+)\} = \{\tilde{f}(t-), \tilde{f}(t+)\} \quad \text{for any } t \in I,$$

where

$$\tilde{f}(t-) = \lim_{\tau \to t^-} \tilde{f}(\tau), \qquad \tilde{f}(t+) = \lim_{\tau \to t^+} \tilde{f}(\tau).$$

We shall sometimes identify the function f with its representative $\tilde{f}$, which is defined pointwise everywhere on I.

We say that a sequence $\{f_h\}_h \subset BV(I;\mathbb{R}^2)$ is *weakly convergent in BV* to a function $f \in BV(I;\mathbb{R}^2)$ if $f_h \to f$ in $L^1(I)$ and $\dot{f}_h \rightharpoonup \dot{f}$ weakly in the sense of measure as $h \to +\infty$.

For more informations about functions of bounded varation see Ambrosio, Fusco and Pallara [2].

1.2.2 *BV* families of curves

Let $[a,b]$ be an interval of $\mathbb{R}$. In the sequel, we call *curve* any function $\gamma : [a,b] \to \mathbb{R}^2$ of class $W^{1,1}[a,b]$ such that $|\dot{\gamma}| \neq 0$ in $[a,b]$.

If the curve γ is not closed, γ is called an arc and the points $\gamma(a)$, $\gamma(b)$ are the *endpoints* of γ. The set

$$[\gamma] = \{\gamma(t) : \ t \in [a,b]\}$$

is the *trace* of γ. The length of γ will be denoted by $L(\gamma)$, and the arclength parameter will be denoted by s. A curve is *simple* if $\gamma(t_1) = \gamma(t_2)$ only if either $t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. A *closed curve* defined on an interval $[a,b]$ may be extended to a $(b-a)$-periodic $W^{1,1}_{loc}$ function on $\mathbb{R}$. A closed curve may be considered both as a curve having a single endpoint $\gamma(a) = \gamma(b)$, and as a curve without endpoints.

If $C = \{\gamma^i\}_i$ is a family of curves parametrized on disjoint intervals $[a_i, b_i]$, then with abuse of notation we write $C : S \to \mathbb{R}^2$, where $S = \cup_i [a_i, b_i]$. We denote by $[C]$ the *trace* of C, i.e., the union of the traces of the curves in C.

We say that the family C is *disjoint* if $[\gamma^i] \cap [\gamma^j] = \emptyset$ for any $\gamma^i, \gamma^j \in C$ such that $i \neq j$.

Let $C = \{\gamma^i\}_i$ be a family of curves. Each closed curve of C may be either without endpoints or with a single endpoint. We denote by $P(C)$ the set of the endpoints of all the curves in C.

Definition 1. *We say that a family of curves $C = \{\gamma_i\}_i$ is of class $\mathcal{B}$ if $\gamma_i \in W^{1,1}([a_i, b_i]; \mathbb{R}^2)$ and $\dot{\gamma}_i \in BV([a_i, b_i], \mathbb{R}^2)$ for any $i \in \mathbb{N}$.*

If $C : S \to \mathbb{R}^2$ is a family of curves of class $\mathcal{B}$, we write $C \in \mathcal{B}(S; \mathbb{R}^2)$.

Let γ be a closed curve without endpoints of class $\mathcal{B}$; if $B \subseteq [a, b]$ is a Borel set containing the point a or b (or both), by $|\dot{\gamma}|(B)$ we mean $|\dot{\gamma}|(B \cap]a, b[) + |\dot{\gamma}|(\{a\})$, where $|\dot{\gamma}|(\{a\})$ is defined for the periodic extension of γ.

Definition 2. *Let $\gamma \in \mathcal{B}([a, b]; \mathbb{R}^2)$ and let $\{\gamma_h\}_h \subset \mathcal{B}([a, b]; \mathbb{R}^2)$ be a sequence of curves. We say $\{\gamma_h\}_h$ is weakly convergent to γ if $\gamma_h \to \gamma$ uniformly, and $\dot{\gamma}_h \rightharpoonup \dot{\gamma}$ weakly in BV as $h \to \infty$.*

In the following $\Omega \subset \mathbb{R}^2$ denotes an open bounded set. We say that γ is a curve in Ω if $[\gamma] \subset \overline{\Omega}$. We say that $C = \{\gamma^i\}_i$ is a family of curves in Ω if $[C] \subset \overline{\Omega}$.

Definition 3. *Let C be a finite family of curves of class $\mathcal{B}$ in Ω, and let $\{C_h\}_h$ be a sequence of families of curves of class $\mathcal{B}$ in Ω. We say that the sequence of traces $\{[C_h]\}_h$ is weakly convergent to the trace of the family C, if the following conditions are satisfied:*

(i) *each of the families C_h contains a finite number m of curves $\{\gamma_h^1, ..., \gamma_h^m\}$ (m independent of h) such that for $i = 1, \ldots, m$ the sequence $\{\gamma_h^i\}_h$, reparametrized on a fixed interval, converges weakly to a curve γ_i of class $\mathcal{B}$;*

(ii) *there exists a finite number of points such that the maximum distance of the trace of the remaining curves of C_h (i.e., $[C_h \setminus \{\gamma_h^1, \ldots, \gamma_h^m\}]$) from this set of points goes to zero as $h \to \infty$;*

(iii) *if we set $C' = \{\gamma^1, ..., \gamma^m\}$, then $[C'] = [C]$.*

1.3 The argument functions

The content of this section is taken from Bellettini and Paolini ([8], Section 3). We define the angle between the derivative $\dot{\gamma}$ of a curve of class $\mathcal{B}$ (parametrized with constant velocity) and the positive direction of the x-axis.

Lemma 1. *Let* $f \in BV(]a,b[;\mathbb{R}^2)$ *be such that* $|f(t)| = R$ *a.e.* $t \in]a,b[$, *for a suitable* $R > 0$. *Then there exists a function* $\Theta :]a,b[\to \mathbb{R}$ *satisfing the following properties:*

(i) $\Theta \in BV(]a,b[)$ *and* $f(t) = (R\,cos\,\Theta(t), R\,sin\,\Theta(t))$ *a.e.* $t \in]a,b[$*;*

(ii) $S_\Theta = S_f$ *;*

(iii) $-\pi < \Theta(t+) - \Theta(t-) \leq \pi$ *a.e.* $t \in]a,b[$ *.*

Note that Θ is unique up to an addition of an integer multiple of 2π.

In our case, we consider $f = \dot{\gamma}$ for a curve $\gamma : [a,b] \to \mathbb{R}^2$ of class $\mathcal{B}$ parametrized with constant velocity, and we call the corresponding Θ on $]a,b[$ an *argument of* $\dot{\gamma}$.

If γ is a closed curve without endpoints, since γ is defined on $[a,b]$, we set

$$|\dot{\Theta}|([a,b]) = |\dot{\Theta}|(]a,b[) + |\Theta(a+) - \Theta(b-) - 2k\pi|,$$

where

$$\Theta(a+) = \text{ap} - \lim_{t\to a+} \Theta(t) = \lim_{t\to a+} \tilde{\Theta}(t), \quad \Theta(b-) = \text{ap} - \lim_{t\to b-} \Theta(t) = \lim_{t\to b-} \tilde{\Theta}(t),$$

and $k \in \mathbb{Z}$ is such that $-\pi < \Theta(a+) - \Theta(b-) - 2k\pi \leq \pi$. Then we call Θ an argument of $\dot{\gamma}$ on $[a,b]$.

Let γ be a curve of class $\mathcal{B}$ such that $|\dot{\gamma}(s)| = 1$ a.e. on $[0, L(\gamma)] = I$, and let Θ be an argument of $\dot{\gamma}$ on I. Let us compare the two measures $\ddot{\gamma}$ and $\dot{\Theta}$. Using the properties of BV functions, the assertion (i) of Lemma (1) and the uniqueness of the Lebesgue decomposition of a measure, one can show that

$$\ddot{\gamma}^a = (-\sin\,\Theta, \cos\,\Theta)\dot{\Theta}^a \quad \text{a.e. in } I,$$

therefore we have $|\ddot{\gamma}^a| = |\dot{\Theta}^a|$. While, if $B \subseteq I$ is a Borel set, then

$$\begin{aligned} \ddot{\gamma}^s(B) &= \int_{B\cap(I\setminus S_\Theta)} (-\sin\,\Theta, \cos\,\Theta)\,d\dot{\Theta}^s \\ &+ \sum_{s\in B\cap S_\Theta} (\cos\,\Theta(s+) - \cos\,\Theta(s-), \sin\,\Theta(s+) - \sin\,\Theta(s-))\delta_s, \end{aligned} \tag{1.1}$$

where δ_s denotes the Dirac distribution at point s. Hence, using the equality $2(1 - \cos\,\phi) = 4\,\sin^2(\phi/2)$, we obtain that for any $s \in S_\Theta$

$$\begin{aligned} &(|\ddot{\gamma}^s|(\{s\}))^2 = |(\cos\,\Theta(s+) - \cos\,\Theta(s-), \sin\,\Theta(s+) - \sin\,\Theta(s-))|^2 \\ &= 4\sin^2((\Theta(s+) - \Theta(s-))/2) \leq (\Theta(s+) - \Theta(s-))^2 = (|\dot{\Theta}^s|(\{s\}))^2, \end{aligned}$$

so that

$$|\ddot{\gamma}^s|(\{s\}) \le |\dot{\Theta}^s|(\{s\}).$$

Now, recalling that Θ is unique up to an addition of an integer multiple of 2π, we consider energy functionals defined on functions which are equivalent $(\mod 2\pi)$. We give the following definition.

Definition 4. *Let θ_1, $\theta_2 \in BV(]a,b[)$. We say that θ_1 and θ_2 are equivalent if for almost every $t \in]a,b[$ there exists $k(t) \in \mathbb{Z}$ such that $\theta_1(t) = \theta_2(t) + 2\,k(t)\,\pi$. We denote by $[\theta_1]$ the equivalence class of θ_1.*

Moreover, we denote by $BV(]a,b[;\mathbb{R}/2\pi) = \{[\theta] : \theta \in BV(]a,b[)\}$ the quotient space of $BV(]a,b[)$ respect to the equivalence relation above defined.

Note that if $\theta^* \in [\theta]$ then

$$|\dot{\theta}^*|(]a,b[) = |\dot{\theta}|(]a,b[) + 2n\pi \ge 0$$

for a suitable $n \in \mathbb{Z}$. Hence given $[\theta] \in BV(]a,b[;\mathbb{R}/2\pi)$ there exists a function $\Theta \in [\theta]$, that we shall call a minimal representative of $[\theta]$, such that

$$|\dot{\Theta}|(]a,b[) = \min\{|\dot{\theta}^*|(]a,b[) \,:\, \theta^* \in [\theta]\}.$$

We shall denote by $\mathcal{M}[\theta]$ the set of all minimal representatives Θ of $[\theta] \in BV(]a,b[;\mathbb{R}/2\pi)$ such that $\Theta(a+) \in [0,2\pi[$. We also have the following properties:

(a) if $\theta^* \in [\theta]$, $\theta^*(a+) \in [0,2\pi[$ and $|\theta^*(t+) - \theta^*(t-)| \le \pi$ for any $t \in]a,b[$, then $\theta^* \in \mathcal{M}[\theta]$;

(b) if Θ_1, $\Theta_2 \in \mathcal{M}[\theta]$ then

$$|\dot{\Theta}_1^s|(B) = |\dot{\Theta}_2^s|(B) \quad \text{for any Borel set } B \subseteq]a,b[;$$

(c) if θ_1, $\theta_2 \in [\theta]$ then

$$|\dot{\theta}_1^a| = |\dot{\theta}_2^a| \qquad \text{a.e. in }]a,b[.$$

We now give the definition of convergence of a sequence in $BV(]a,b[;\mathbb{R}/2\pi)$.

Definition 5. *Let $[\theta] \in BV(]a,b[;\mathbb{R}/2\pi)$, and let $\{[\theta_h]\}_h \subset BV(]a,b[;\mathbb{R}/2\pi)$. We say that $[\theta_h] \to [\theta]$ as $h \to +\infty$ if, for any $\theta^* \in [\theta]$ and any $h \in \mathbb{N}$, there exists $\theta_h \in [\theta_h]$ such that $\theta_h \to \theta^*$ in $L^1(]a,b[)$ as $h \to +\infty$.*

Note that if $[\theta] \in BV(]a,b[;\mathbb{R}/2\pi)$, a sequence $\{[\theta_h]\}_h \subset BV(]a,b[;\mathbb{R}/2\pi)$ converges to $[\theta]$ if and only if for any $\Theta \in \mathcal{M}[\theta]$ and any $h \in \mathbb{N}$, there exists $\theta_h^* \in [\theta_h]$ such that $\theta_h^* \to \Theta$ in $L^1(]a,b[)$. However, if $\Theta_h \in \mathcal{M}[\theta_h]$ the sequence $\{\Theta_h\}_h$ does not converge in general to a minimal representative of $[\theta]$ in $L^1(]a,b[)$.

We conclude this section with an useful result on convergence (for the proof see Bellettini and Paolini ([8], Proposition 4.1).

Theorem 1. *Let γ be a curve of class $\mathcal{B}$, and let $\{\gamma_h\}_h \subset \mathcal{B}$ be a sequence of curves such that $\dot{\gamma}_h \to \dot{\gamma}$ in $L^1([a,b];\mathbb{R}^2)$ as $h \to +\infty$. Assume that γ and each γ_h are parametrized with constant velocity on $[a,b]$. Let θ, $\theta_h \in BV([a,b])$ be arguments of $\dot{\gamma}$, $\dot{\gamma}_h$ respectively, for any $h \in \mathbb{N}$. Then $[\theta_h] \to [\theta]$ as $h \to +\infty$.*

1.4 The functional K on argument functions and curves of class $\mathcal{B}$

In this section we introduce energy functionals defined both on argument functions, and on families of curves of class $\mathcal{B}$.

Let $\psi : \mathbb{R} \times \mathbb{R} \to [0,+\infty]$ be a Borel function satisfying properties (i)-(vi) listed in Section 1.2. Denote by ψ_∞ the *recession function* of ψ with respect to the variable ξ, i.e.,

$$\psi_\infty(\eta,\xi) = \lim_{t\to 0^+} t\psi(\eta,\xi/t) \qquad \forall(\eta,\xi) \in \mathbb{R}\times\mathbb{R},$$

where the existence of the above limit is a conseguence of the convexity of ψ. Let $\theta \in BV(]a,b[;\mathbb{R}/2\pi)$, $\theta_1 \in [\theta]$ and $\Theta \in \mathcal{M}[\theta]$; for any open interval $I \subseteq]a,b[$ we define

$$K(\theta,I) = K^a(\theta,I) + K^s(\theta,I)\,,$$

where

$$\begin{aligned} K^a(\theta,I) &= \int_I \psi(\theta_1,\dot{\theta}_1^a)dt\,, \\ K^s(\theta,I) &= \int_{I\setminus S_\Theta} \psi_\infty(\Theta,1)d|\dot{\Theta}^s| + \sum_{s\in I\cap S_\Theta} \int_{\Theta(s-)}^{\Theta(s+)} \psi_\infty(\tau,1)d\tau\,. \end{aligned}$$

Note that by condition (v) of definition of ψ and by the properties of argument functions, the term $K^a(\theta,I)$ does not depend on the choice of $\theta_1 \in [\theta]$, and the term $K^s(\theta,I)$ does not depend on the choice of $\Theta \in \mathcal{M}[\theta]$. Note that if $\theta \in C^2([a,b])$, then $K(\theta,I) = K^a(\theta,I) = \int_I \psi(\theta,\dot{\theta})dt$.

Let now Θ be an argument of $\dot\gamma$, with $\gamma : [a, b] \to \mathbb{R}^2$ a curve of class $\mathcal{B}$ parametrized with constant velocity. If γ is a curve with endpoints we set

$$K^s(\Theta, [a, b]) = K^s(\Theta,]a, b[),$$

otherwise, if γ is a closed curve without endpoints we set

$$K^s(\Theta, [a, b]) = K^s(\Theta,]a, b[) + \left| \int_{\Theta(b-)}^{\Theta(a+)-2k\pi} \psi_\infty(\tau, 1) d\tau \right| ,$$

where $k \in \mathbb{Z}$ is such that $-\pi < \Theta(a+) - \Theta(b-) - 2k\pi \leq \pi$. Here the absolute value appears, since $\Theta(b-)$ is not necessarily less than $\Theta(a+) - 2k\pi$. Therefore K does not depend on the choice of θ, then we may write $K(\theta, [a, b]) = K(\gamma)$.

The following theorem, proved by Bellettini and Paolini ([8], Theorem 5.1), shows that the functional K is lower semicontinuous.

Theorem 2. *Let $I \subseteq]a, b[$ be an open interval. Let $[\theta] \in BV(I; \mathbb{R}/2\pi)$, and let $\{[\theta_h]\}_h \subset BV(I, \mathbb{R}/2\pi)$ be such that $[\theta_h] \to [\theta]$ as $h \to +\infty$. Then*

$$K(\theta, I) \leq \liminf_{h\to+\infty} K(\theta_h, I) .$$

The above theorem still holds when the functions θ are replaced by arguments of derivatives of curves of class $\mathcal{B}$ parametrized with constant velocity, and when the open interval $]a, b[$ is replaced by $[a, b]$.

Now we define the functional K for a family of curves $C = \{\gamma^i\}_i$ of class $\mathcal{B}$ parametrized by arclength. Let Θ^i be an argument of $\dot\gamma^i$ on $[0, L(\gamma^i)]$ for any i. We define $K(C) = K^a(C) + K^s(C)$ where

$$K^a(C) = \sum_{\gamma^i \in C} K^a(\Theta^i, [0, L(\gamma^i)]), \qquad K^s(C) = \sum_{\gamma^i \in C} K^s(\Theta^i, [0, L(\gamma^i)]).$$

If γ is a curve of class $\mathcal{B}$, in the following we set $K(\gamma) = K(\Theta, [0, L(\gamma)])$.

To conclude this section we prove the following useful lemma.

Lemma 2. *Let γ be a curve of class $\mathcal{B}$ parametrized by arclength, and let Θ be an argument of $\dot\gamma$. Then*

$$K(\gamma) + \text{Ł}(\gamma) \geq c|\dot\Theta|([0, L(\gamma)]) ,$$

where c is a positive constant independent of γ.

Proof. Using property (iv) of the function ψ, we have

$$\begin{aligned}\psi_\infty(\eta,\xi) &= \lim_{t\to 0^+} t\psi(\eta,\xi/t)\\ &\geq \lim_{t\to 0^+}(A|\xi|-Bt) = A|\xi|\,.\end{aligned}$$

Let $\mu=0$ if γ is a curve with endpoints, and $\mu=1$ if γ is a closed curve without endpoints. If $I=(0,L(\gamma))$, we have

$$\begin{aligned}K^a(\gamma) &= \int_I \psi(\Theta,\dot\Theta^a)ds \geq \int_I (A|\dot\Theta^a|-B)ds\\ &= A|\dot\Theta^a|(I)-B|I| = A|\dot\Theta^a|(I)-BL(\gamma)\,,\end{aligned}$$

$$\begin{aligned}K^s(\gamma) &= \int_{I\setminus S_\Theta}\psi_\infty(\Theta,1)d|\dot\Theta^s| + \sum_{s\in I\cap S_\Theta}\int_{\Theta(s-)}^{\Theta(s+)}\psi_\infty(\tau,1)d\tau\\ &+ \mu\left|\int_{\Theta(0-)}^{\Theta(L(\gamma)+)-2k\pi}\psi_\infty(\tau,1)d\tau\right|\\ &\geq A|\dot\Theta^s|(I\setminus S_\Theta) + A\sum_{s\in I\cap S_\Theta}|\Theta(s+)-\Theta(s-)|\\ &+ \mu A\,|\Theta(L(\gamma)+)-\Theta(0-)-2k\pi|\\ &= A|\dot\Theta^s|(I)+\mu A\,|\Theta(L(\gamma)+)-\Theta(0-)-2k\pi| = A|\dot\Theta^s|([0,L(\gamma)])\,.\end{aligned}$$

Collecting the above inequalities we find

$$\begin{aligned}K(\gamma) &= K^a(\gamma)+K^s(\gamma)\geq A|\dot\Theta^a|(I)-BL(\gamma)+A|\dot\Theta^s|([0,L(\gamma)])\\ &= A|\dot\Theta|([0,L(\gamma)])-BL(\gamma)\,,\end{aligned}$$

therefore

$$K(\gamma)+BL(\gamma)\geq A|\dot\Theta|([0,L(\gamma)])\,.$$

It follows

$$\begin{aligned}K(\gamma)+L(\gamma) &\geq K(\gamma)+\min\{1,\frac{1}{B}\}BL(\gamma)\geq \min\{1,\frac{1}{B}\}\,(K(\gamma)+BL(\gamma))\\ &\geq \min\{1,\frac{1}{B}\}\,A\,|\dot\Theta|([0,L(\gamma)])\,,\end{aligned}$$

1.5 Families of curves without crossings

In this section a new method is introduced in order to characterize the family of curves without crossings. Such a method is an original contribution of the present work with respect to the approach followed by Coscia [14] for $W^{2,2}$ curves.

Let γ be a simple closed curve such that $[\gamma] \subset \Omega$. Then, by using the Jordan curve theorem [26], the set $\Omega \setminus [\gamma]$ can be written uniquely as the disjoint union of two open sets Ω_I and Ω_O such that Ω_I is simply connected. The sets Ω_I and Ω_O are called, respectively, the inside and the outside of γ in Ω.

Definition 6. *Let γ be a simple closed curve such that $[\gamma] \subset \Omega$, and let Ω_I, Ω_O be the inside and the outside of γ in Ω. We say that Ω_I, Ω_O are the partition of Ω induced by γ.*

Definition 7. *Let γ be a curve such that $[\gamma] \subset \Omega$. We say that $P = \gamma(t_0)$ is a returning point of γ if for each $c > 0$ and each neighbourhood $(t_0 - c, t_0 + c)$ there exist $t_1 < t_0 < t_2$ such that $\gamma(t_1) = \gamma(t_2)$.*

We now look for a class of admissible families of curves on which to define our energy functional. For reasons of compactness, a family C of curves must have no crossing points, therefore the crossing points are penalized by forcing them to be considered as the endpoints of some curves.

If the curves are smooth, we can impose that there are no points $\gamma^i(t_1) = \gamma^j(t_2)$, with t_1 and t_2 interior to the domains of γ^i and γ^j, respectively, and with possibly $i = j$, such that the tangent vector $\dot{\gamma}^i(t_1)$ is not parallel to $\dot{\gamma}^j(t_1)$.

But in the case of curves of class $\mathcal{B}$ the tangent vector is not defined everywhere, therefore we need a different notion of curves without crossings.

Now, let γ^1, γ^2 be two curves in Ω; in order to define such a notion we need to split their set of intersection into two subsets. We denote $\mathcal{T} = [\gamma^1] \cap [\gamma^2]$,

$$\mathcal{T}_1 = \{P = \gamma^1(t) \in \mathcal{T} : \exists t_0 > 0 \,\text{such that}\, [\gamma^1_{|(t,t+t_0)}] \subset \mathcal{T} \text{or/and } [\gamma^1_{|(t-t_0,t)}] \subset \mathcal{T}\},$$

and finally $\mathcal{T}_0 = \mathcal{T} \setminus \mathcal{T}_1$.

1.5.1 Curves without crossings at points of $\mathcal{T}_0$

In this section we examine the set $\mathcal{T}_0$. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a curve, let $t_0 \in [a, b]$, and let $J \subset [a, b]$ be an open neighbourhood of t_0. We denote by $\gamma_{|J}$ the curve γ restricted to the interval J. We need following results.

Theorem 3. *Let* γ^1 *and* γ^2 *be two curves of class* $\mathcal{B}$ *in* Ω*, parametrized by arclength and such that*

$$K(\gamma^i) + L(\gamma^i) \leq H < +\infty, \quad i = 1, 2, \tag{1.2}$$

where H *is a positive constant. Let* $P \in \mathcal{T}_0$ *be a not returning point of* γ^1 *and* γ^2*, such that* $\{P\} = \gamma^1(s_1) = \gamma^2(s_2)$ *for some* $s_1 \in (0, L(\gamma^1))$ *and some* $s_2 \in (0, L(\gamma^2))$*. Then there exist two open neighbourhoods,* $J_1 \subset (0, L(\gamma^1))$ *of* s_1*, and* $J_2 \subset (0, L(\gamma^2))$ *of* s_2*, respectively, with* $[\gamma^1{}_{|J_1}] \subset \Omega$ *and* $[\gamma^2{}_{|J_2}] \subset \Omega$*, such that the curves* $\gamma^1{}_{|J_1}$ *and* $\gamma^2{}_{|J_2}$ *are simple arcs. Moreover, there exists an open neighbourhood* $J_* \subset J_1$ *of* s_1 *such that the distance function* $dist(P, \gamma^1(s))$ *is monotone increasing in* $J_* \cap \{s > s_1\}$ *and is monotone decreasing in* $J_* \cap \{s < s_1\}$*.*

The following example shows a situation which we have to take into account in the proof of the theorem.

Example. Let γ^1 and γ^2 be two curves with the following traces:

$$\begin{aligned} [\gamma^1] &= \{(0,y) : y \in [-1,0]\} \cup \{(x,0) : x \in [0,1]\} \\ &\quad \cup \{(\cos t, \sin t) : t \in [0, \pi/4]\} \cup \{(1/\sqrt{2}, y) : y \in [\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}]\}, \\ [\gamma^2] &= \{(x,x) : x \in [-1,1]\}. \end{aligned} \tag{1.3}$$

If $P = (0,0)$, the endpoints P_- and P_+ of γ^1 do not belong to the same arcwise connected component of $\{P_-, P_+\} \cup (\Omega \setminus ([\gamma^1] \cup [\gamma^2]))$. However, we may choose J_1 in such a way that

$$\{(1/\sqrt{2}, y) : y \in [\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}}]\} \cap [\gamma^1{}_{|J_1}] = \emptyset.$$

The theorem shows that it is always possible to find a neighbourhood J_1 with such a property. We prove two preliminary lemmata that will be used to prove Theorem 3.

Lemma 3. *Let* $f \in BV(I)$ *and let* $[a,b]$ *be an interval such that* $[a,b] \subset I$*. The following inequality holds:*

$$|\dot{f}|([a,b]) \geq |\tilde{f}(a) - f(b-)| + |f(b+) - f(b-)|.$$

Proof. Since we have

$$|\dot{f}|([a,b]) \geq |\dot{f}|(a) + |\dot{f}|((a,b)) + |\dot{f}(b)|$$

$$\geq |f(a+)-f(a-)| + |f(b-)-f(a+)| + |f(b+)-f(b-)|\,,$$

it follows

$$|\dot{f}|([a,b]) \geq |f(b-)-f(a+)| + |f(b+)-f(b-)|$$

and

$$|\dot{f}|([a,b]) \geq |f(b-)-f(a-)| + |f(b+)-f(b-)|$$

hence, we find

$$|\dot{f}|([a,b]) \geq$$

$$\geq \frac{|f(b-)-f(a+)| + |f(b+)-f(b-)| + |f(b-)-f(a-)| + |f(b+)-f(b-)|}{2}$$

$$\geq |f(b-) - \frac{f(a+)+f(a-)}{2}| + |f(b+)-f(b-)|\,.$$

□

Lemma 4. *Let* γ *be a curve of class* $\mathcal{B}$ *parametrized by arclength, and let* Θ *be an argument of* $\dot{\gamma}$*. Let* $s_1 \in (0, L(\gamma))$ *and* $P = \gamma(s_1)$*. If there exist* $s_0 \in (0, L(\gamma))$ *with* $s_0 > s_1$ *and* $Q = \gamma(s_0) \neq P$*, and* $\epsilon > 0$ *such that*

$$dist(P,Q) - dist(P,\gamma(s)) \geq 0 \quad \text{for } t \in (s_0-\epsilon, s_0+\epsilon),$$

then $|\dot{\Theta}|((s_1, s_0]) \geq \frac{\pi}{2}$.

Proof. The disk $D_R(P)$, where $R = dist(P,Q)$, is such that

$$[\gamma_{|(s_0-\epsilon,s_0+\epsilon)}] \subset \overline{D}_R(P), \qquad Q \in [\gamma] \cap \partial D_R(P).$$

Let us choose a system of coordinates in the plane with the origin at $\gamma(s_1)$ and such that $Q = (0,-R)$. We write $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ in the form

$$\gamma(s) = \int_{s_1}^{s} (\cos\tilde{\Theta}(\tau), \sin\tilde{\Theta}(\tau))d\tau + \gamma(s_1) = (\int_{s_1}^{s} \cos\tilde{\Theta}(\tau)d\tau, \int_{s_1}^{s} \sin\tilde{\Theta}(\tau)d\tau), \tag{1.4}$$

since $\gamma(s_1) = P \equiv (0,0)$. The argument Θ is such that

$$\lim_{s\to s_0+} \tilde{\Theta}(s) + 2n_1\pi \in [0,\pi], \tag{1.5}$$

for some $n_1 \in \mathbb{Z}$. Otherwise, we get $\lim_{s\to s_0+} \sin\tilde{\Theta}(s) < 0$, which implies

$$\gamma_2(s_0+\delta) - \gamma_2(s_0) = \int_{s_0}^{s_0+\delta} \sin\tilde{\Theta}(\tau)d\tau) < 0,$$

so that $\gamma_2(s_0+\delta) < -R$ for some positive $\delta < \epsilon$ small enough, which contradicts the condition $[\gamma_{|(s_0-\epsilon,s_0+\epsilon)}] \subset \overline{D}_R(P)$.

Analogously, we have

$$\lim_{s\to s_0-} \tilde{\Theta}(s) + 2n_2\pi \in [\pi, 2\pi], \tag{1.6}$$

for some $n_2 \in \mathbb{Z}$. Using Lemma 3, for any $s \in (s_1, s_0)$ we have

$$|\dot{\Theta}|([s,s_0]) \geq |\tilde{\Theta}(s) - \tilde{\Theta}(s_0-)| + |\Theta(s_0+) - \Theta(s_0-)|. \tag{1.7}$$

Let us suppose that

$$\Theta(s_0-) + 2n_2\pi = \lim_{s\to s_0-} \tilde{\Theta}(s) + 2n_2\pi \in [3\pi/2, 2\pi] \tag{1.8}$$

for some $n_2 \in \mathbb{Z}$. In order to prove that $|\dot{\Theta}|((s_1,s_0]) \geq \frac{\pi}{2}$, let us suppose that such an inequality is not satisfied. Then, from (1.7) it would follow

$$\frac{\pi}{2} > |\tilde{\Theta}(s) - \tilde{\Theta}(s_0-)| + |\Theta(s_0+) - \Theta(s_0-)|, \tag{1.9}$$

for any $s \in (s_1, s_0)$, from which using (1.5) and (1.8), we get

$$\Theta(s_0+) + 2n_1\pi = \lim_{s\to s_0+} \tilde{\Theta}(s) + 2n_1\pi \in [0, \pi/2), \tag{1.10}$$

with $n_1 = n_2 - 1$. Let us show that (1.8) and (1.9) imply

$$\tilde{\Theta}(s) + 2n_3(s)\pi \in (-\pi/2, \pi/2), \tag{1.11}$$

for some $n_3(s) \in \mathbb{Z}$ and for any $s \in (s_1, s_0)$. If (1.11) is not satisfied, then (1.8) and (1.9) imply

$$\tilde{\Theta}(s) + 2n_2\pi \in [-\pi, 3\pi/2], \tag{1.12}$$

from which, using (1.8) and (1.10) it follows

$$\lim_{s\to s_0+} \tilde{\Theta}(s) \geq \lim_{s\to s_0-} \tilde{\Theta}(s), \qquad \tilde{\Theta}(s_0-) \geq \tilde{\Theta}(s),$$

so that

$$|\tilde{\Theta}(s) - \tilde{\Theta}(s_0-)| + |\Theta(s_0+) - \Theta(s_0-)| = \Theta(s_0+) - \tilde{\Theta}(s).$$

Then, from (1.10) and (1.12) we get

$$\Theta(s_0+) - \tilde{\Theta}(s) \geq \frac{\pi}{2},$$

which contradicts (1.9), hence (1.11) is satisfied for any $s \in (s_1, s_0)$. Now, using (1.4) and (1.11) we find

$$\gamma_1(s_0) = \int_{s_1}^{s_0} \cos \tilde{\Theta}(\tau) d\tau > 0,$$

which yields a contradiction, since $Q = (\gamma_1(s_0), \gamma_2(s_0)) = (0, -R)$. Hence, if (1.8) holds true, the inequality $|\dot{\Theta}|((s_1, s_0]) \geq \frac{\pi}{2}$ is satisfied. By a symmetry argument such an inequality is satisfied also if

$$\lim_{s \to s_0-} \tilde{\Theta}(s) + 2n_2\pi \in [\pi, 3\pi/2),$$

for some $n_2 \in \mathbb{Z}$, and the proof of the lemma is achieved. $\square$

The following lemma has been proved by Bellettini and Paolini ([8], Lemma 8.2).

Lemma 5. *Let $\gamma \in \mathcal{B}$ be parametrized by arclength, and let Θ be an argument of $\dot{\gamma}$ on $[0, L(\gamma)]$. Let $0 \leq s_1 < s_2 \leq L(\gamma)$ be such that $\gamma(s_1) = \gamma(s_2)$. Then $|\dot{\Theta}|([s_1, s_2]) \geq \pi$.*

Proof of Theorem 3. In order to prove that there exist neighbourhoods J_1 and J_2 such that the curves $\gamma^1{}_{|J_1}$ and $\gamma^2{}_{|J_2}$ are simple arcs, we have to show that there exist neighbourhoods of the points s_1, s_2 such that P is not an accumulation point of self-intersection points of each curve, when the curve is restricted to the corresponding neighbourhood. If that happens, then, for any J_1 and J_2, the curves $\gamma^1{}_{|J_1}$ and $\gamma^2{}_{|J_2}$ would contain an infinite number of closed loops.

Lemma 5 shows that the contribution of each closed loop of γ^1 and γ^2 to the total variation of the argument of $\dot{\gamma}^1$ and $\dot{\gamma}_2$, respectively, is greater than or equal to π. Let such closed loops be defined on intervals $[a_i^h, b_i^h] \subset [0, L(\gamma^i)]$, with $h \in \mathbb{N}$ and $i = 1, 2$. Let Θ^i, $i = 1, 2$, denote the argument of $\dot{\gamma}^i$. Then, using (1.17), Lemma 2 and Lemma 5, there exists $C > 0$, independent of h, such that

$$H \geq K(\gamma^i) + L(\gamma^i) \geq C|\dot{\Theta}^i|([0, L(\gamma^i)]) \geq \sum_{h=1}^{m} C|\dot{\Theta}^i|([a_i^h, b_i^h]) \geq m\pi C\,,$$

for $i = 1, 2$ and any $m \in \mathbb{N}$. It follows that the number of closed loops is bounded by

$$m \leq \frac{H}{\pi C} < \infty.$$

Then there exist neighbourhoods of s_1 and s_2 such that the curves γ^1 and γ^2, when they are restricted to such neighbourhoods, have a finite number of points of self-intersections. It follows that there exist neighbourhoods J_1 and J_2 such that the curves ${\gamma^1}_{|J_1}$ and ${\gamma^2}_{|J_2}$ are simple arcs.

Since $P \notin \partial\Omega$ we may choose J_1 and J_2 in such a way that $[{\gamma^1}_{|J_1}] \cap \partial\Omega = \emptyset$ and $[{\gamma^2}_{|J_2}] \cap \partial\Omega = \emptyset$.

Let $J_1 = (a, b)$, so that $a < s_1 < b$, and let $P_- = \gamma^1(a)$, $P_+ = \gamma^1(b)$.

Let $\{a_n\}_n \subset (a, s_1)$ be an increasing sequence of points converging to s_1, and let $\{b_n\}_n \subset (s_1, b)$ be a decreasing sequence of points converging to s_1. Then we have

$$H \geq K({\gamma^1}_{|J_1}) + L({\gamma^1}_{|J_1}) \geq \sum_{n=1}^{\infty} \left[K({\gamma^1}_{|[a_n,a_{n+1})}) + L({\gamma^1}_{|(a_n,a_{n+1})}) \right],$$

$$H \geq K({\gamma^1}_{|J_1}) + L({\gamma^1}_{|J_1}) \geq \sum_{n=1}^{\infty} \left[K({\gamma^1}_{|(b_n,b_{n+1}]}) + L({\gamma^1}_{|(b_n,b_{n+1})}) \right].$$

It follows that

$$\begin{aligned} &\lim_{m\to+\infty} \sum_{n=m}^{\infty} \left[K({\gamma^1}_{|[a_n,a_{n+1})}) + L({\gamma^1}_{|(a_n,a_{n+1})}) \right] = \\ &\lim_{m\to+\infty} \left[K({\gamma^1}_{|[a_m,s_1)}) + L({\gamma^1}_{|(a_m,s_1)}) \right] = 0, \end{aligned} \tag{1.13}$$

and analogously,

$$\lim_{m\to+\infty} \left[K({\gamma^1}_{|(s_1,b_m]}) + L({\gamma^1}_{|(s_1,b_m)}) \right] = 0. \tag{1.14}$$

Using Lemma 2, and taking into account that the curves ${\gamma^1}_{|(a_m,s_1)}$ and ${\gamma^1}_{|(s_1,b_m)}$ are not closed, there exists a positive constant C, independent of m, such that

$$\begin{aligned} K({\gamma^1}_{|[a_m,s_1)}) + L({\gamma^1}_{|(a_m,s_1)}) &\geq C|\dot{\Theta}^1|([a_m, s_1)), \\ K({\gamma^1}_{|(s_1,b_m]}) + L({\gamma^1}_{|(s_1,b_m)}) &\geq C|\dot{\Theta}^1|((s_1, b_m]). \end{aligned}$$

Then, using (1.13) and (1.14), it follows that the sequences

$$\{|\dot{\Theta}^1|([a_m, s_1))\}_m, \quad \{|\dot{\Theta}^1|((s_1, b_m])\}_m \qquad \text{converge to 0 as } m \to +\infty. \tag{1.15}$$

Therefore, there exists an open neighbourhood J_* of s_1 such that the distance function $dist(P, \gamma^1(s))$ is monotone increasing in $J_* \cap \{s > s_1\}$.

Otherwise, for any open neighbourhood J_* of s_1, there exist a point $s_0 \in J_* \cap \{s > s_1\}$ and $\epsilon > 0$ small enough, such that

$$dist(P, Q) - dist(P, \gamma^1(s)) \geq 0 \quad \text{for } s \in (s_0 - \epsilon, s_0 + \epsilon),$$

where $Q = \gamma^1(s_0) \neq P$. Then, using Lemma 4, it would follow that

$$|\dot{\Theta}^1|((s_1, s_0]) \geq \pi/2,$$

but that contradicts (1.15). Analogously, J_* can be chosen in such a way that the distance function $dist(P, \gamma^1(s))$ is monotone decreasing in $J_* \cap \{s < s_1\}$. □

Remark 1. Theorem 3 holds true if $\gamma^1 = \gamma^2$, and $s_1 \neq s_2$. In this case we may choose $J_1 \cap J_2 = \emptyset$ and the proof is the same.

Remark 2. *The lemmas 4 and 5 show that a curve γ such that*

$$K(\gamma) + L(\gamma) \leq H < +\infty \tag{1.16}$$

where H is a positive constant, contains an finite number of returning points.

We may prove the following theorem.

Theorem 4. *Let γ^1 and γ^2 be two curves of class $\mathcal{B}$ in Ω, parametrized by arclength and such that*

$$K(\gamma^i) + L(\gamma^i) \leq H < +\infty, \quad i = 1, 2, \tag{1.17}$$

where H is a positive constant. Let $P \in \mathcal{T}_0$ be a returning point of γ^1 and γ^2, such that $\{P\} = \gamma^1(s_1) = \gamma^2(s_2)$ for some $s_1 \in (0, L(\gamma^1))$ and some $s_2 \in (0, L(\gamma^2))$.

Then there exist two open neighbourhoods, $J_1 \subset (0, L(\gamma^1))$ of s_1, and $J_2 \subset (0, L(\gamma^2))$ of s_2, respectively, with $[\gamma^1{}_{|J_1}] \subset \Omega$ and $[\gamma^2{}_{|J_2}] \subset \Omega$, such that the curves $\gamma^1{}_{|J_1 \cap \{s > s_1\}}$, $\gamma^1{}_{|J_1 \cap \{s < s_1\}}$, $\gamma^2{}_{|J_2 \cap \{s > s_2\}}$ and $\gamma^2{}_{|J_2 \cap \{s < s_2\}}$ are simple arcs.

Moreover, there exists an open neighbourhood $J_* \subset J_1$ of s_1 such that the distance function $dist(P, \gamma^1(s))$ is monotone increasing in $J_* \cap \{s > s_1\}$ and is monotone decreasing in $J_* \cap \{s < s_1\}$.

Proof. We may repeat the same arguments of the proof of the theorem 3. □

Now, we may define the families of curves without crossings in $\mathcal{T}_0$. The idea is to generalize the definition of the two sides of a smooth curve based on the normal unit vector, that in the case of curves of class $\mathcal{B}$ is not defined everywhere.

Let P be as in theorem 3, and let $t_- < s_1 < t_+$ be such that $\gamma^1_{|[t_-,t_+]}$ is simple arc with distance function monotone. We denote by $P^- = \gamma^1(t_-)$, $P^+ = \gamma^1(t_+)$. Let ϵ^* be such that $\bar{D}_{\epsilon^*}(P) \subset \Omega$. We consider the following set

$$T = \bar{D}_\epsilon(P)\,.$$

where $0 < \epsilon < \epsilon^*$, and let ϵ, t_-, t_+ be such that $P^-, P^+ \in \Gamma = \partial T$ maintaining the previous properties. Now Γ is a simple closed curve passing through P^-, P^+, and we may split it in two arcs Γ_a, Γ_b. We choose a direction on Γ, so that the path along Γ from $P^- = \gamma^1(t_-)$ to $P^+ = \gamma^1(t_+)$ is an arc denoted by Γ_a. While the path from $P^+ = \gamma^1(t_+)$ to $P^- = \gamma^1(t_-)$ is an arc denoted by Γ_b. Note that

$$[\Gamma^a] \cap [\Gamma^b] = \{P^-, P^+\}$$

and

$$[\Gamma_a] \cup [\gamma^1_{|[t_-,t_+]}] \quad , \qquad [\Gamma_b] \cup [\gamma^1_{|[t_-,t_+]}]\,,$$

are simple closed curves. Let Ω_a, Ω_b be their inside, respectively. Let $\sigma > 0$ be such that $\gamma^2_{|[s_2-\sigma,s_2+\sigma]}$ is contained in the union of such sets.

In the similar way we argue if P is a returning point. Let P and $J^1 \cap J^*$ be as in theorem 4, and let $t_- < s_1 < t_+$ be such that $\gamma^1_{|[t_-,s_1]}$, $\gamma^1_{|[s_1,t_+]}$ are simple arcs with distance function monotone. We denote by $P^- = \gamma^1(t_-)$, $P^+ = \gamma^1(t_+)$. We consider the set

$$T = \bar{D}_\epsilon(P)$$

where $\epsilon > 0$ is such that $T \subset \Omega$, and let ϵ, t_-, t_+ be such that $P^-, P^+ \in \Gamma = \partial T$ maintaining the previous properties.

Note that, possibly decreasing ϵ, $\gamma^1_{|[t_-,t_+]}$ is a sequence of closed simple curves and $P^- = P^+$. Let Ω_a be the union of the inside of such curves, and let Ω_b be

$$\mathcal{C}_T(\Omega_a \cup [\gamma^1_{[t_-,t_+]}])\,.$$

Note that Ω_a, Ω_b are open sets. Let $\Gamma = \partial T$, we split it in $\Gamma^a = \{P^-\}$ and $\Gamma^b = \Gamma \backslash \{P^-\}$. Note that

$$\partial\Omega^b = \Gamma^b \cup [\gamma^1_{[t_-,t_+]}]\,.$$

Naturally, the choice of Ω^a and Ω^b can be reversed. Finally, let $\sigma > 0$ be such that $\gamma^2_{|[s_2-\sigma,s_2+\sigma]}$ is contained in the union of such sets.

Definition 8. *Let $\gamma^1 : [a_1, b_1] \to \mathbb{R}^2$ and $\gamma^2 : [a_2, b_2] \to \mathbb{R}^2$ be two curves in Ω. Let $P = \gamma^1(s_1) = \gamma^2(s_2) \in \Omega$ be a point such that $P \in \mathcal{T}_0$. We say that γ^1 is admissible respect to γ^2 at the point P if either*

$$[\gamma^2_{|[s_2-\sigma, s_2+\sigma}] \cap \Omega_a = \emptyset,$$

or

$$[\gamma^2_{|[s_2-\sigma, s_2+\sigma]}] \cap \Omega_b = \emptyset.$$

1.5.2 Curves without crossings at points of $\mathcal{T}_1$

In this section, recalling the notations and definitions of the previous section, we study the set $\mathcal{T}_1$. We may say that $\mathcal{T}_1$ contain the restrictions of γ^1 and γ^2 to the intervals on which the two curves coincide.

We denote by $Q^- = \gamma^1(t_{q-}) = \gamma^2(s_{q-}), Q^+ = \gamma^1(t_{q+}) = \gamma^2(s_{q+})$ the endpoints of such a restriction, therefore $[\gamma^1_{|[t_{q-}, t_{q+}]}] \equiv [\gamma^2_{|[s_{q-}, s_{q+}]}]$.

We suppose Q^-, Q^+ are not returning points of γ^1. Let $t_- < t_{q-}, t_+ > t_{q+}$ be such that $\gamma^1_{|[t_-, t_{q-}]}$ and $\gamma^1_{|[t_{q+}, t_+]}$ are simple arcs with distance function monotone (see theorem3). We again denote by $P^- = \gamma^1(t_-)$, $P^+ = \gamma^1(t_+)$.

Now we denote the set of returning points contained on $\gamma^1_{|(t_{q-}, t_{q+})}$ by $\mathcal{R} = \{\gamma^1(\hat{t}_m) : m \in \hat{J})\}$. The remark 2 ensures us that $|\hat{J}| < \infty$, with possibly $\hat{J} = \emptyset$. If $\hat{J} \neq \emptyset$, let

$$r_1 = \min\{|\hat{t}_m - \hat{t}_{m'}| : m, m' \in \hat{J}\} > 0$$

while if $\hat{J} = \emptyset$ then $r_1 = \infty$. Then we define

$$r_0 = \min\{(t_{q-} - t_-), (t_+ - t_{q+})\} \quad r_* = \min\{r_0, r_1\}$$

$$\delta = \max\{r_* \geq r > 0 : |\dot{\Theta}^1|(t-r, t+r) < \frac{\pi}{2}\,, \ \ t \in [t_{q-}, t_{q+}] \setminus \{\bigcup_{m \in \hat{J}} [\hat{t}_m - r_*, \hat{t}_m + r_*]\}\}$$

and let $\epsilon^* > 0$ be such that $\bar{D}_{\epsilon^*}(\gamma^1(t)) \subset \Omega, \quad \forall t \in [t_-, t_+]$. The proof of Theorem 3 shows that γ^1 does not contain an infinite number of closed loops, then $\delta > 0$. We consider the following sets

$$T^- = \bar{D}_\epsilon(Q^-), \quad T^+ = \bar{D}_\epsilon(Q^+)\,.$$

where $0 < \epsilon < \epsilon^*$, and let ϵ, t_-, t_+ be such that $P^- \in \partial T^-, P^+ \in \partial T^+$ maintaining the previous properties. Possibly decreasing ϵ, we may define $R^- = \gamma^1(t_{R-}), R^+ = \gamma^1(t_{R+})$

$$t_{R-} = \max\{t \in [t_{q-}, t_{q-} + \delta] : \ \gamma^1(t) \in T^-\}$$

$$t_{R+} = \min\{t \in [t_{q+} - \delta, t_+] : \ \gamma^1(t) \in T^+\}$$

such that $t_{R-} \in \partial T$ and $t_{R+} \in \partial T^+$. $\Gamma^- = \partial T^-$ is a closed curve passing through P^-, R^-, and we may split it in two arcs Γ^-_a, Γ^-_b. We choose a direction on Γ^-, so that the path along Γ^- from $P^- = \gamma^1(t_-)$ to R^- is an arc denoted by Γ^-_a. While the path from R^- to $P^- = \gamma^1(t_-)$ is an arc denoted by Γ^-_b. Note that

$$[\Gamma^-_a] \cap [\Gamma^-_b] = \{P^-, \, R^-\}$$

and

$$[\Gamma^-_a] \cup [\gamma^1_{|[t_-,t_{R-}]}] \quad , \qquad [\Gamma^-_b] \cup [\gamma^1_{|[t_-,t_{R-}]}] \, ,$$

are simple closed curves. Let Ω^-_a, Ω^-_b be their inside, respectively. Now if Q^- (or Q^+) is a returning point, we define T^-, T^+ and $\Gamma^-_a, \Gamma^-_b, \Omega^-_a, \Omega^-_b$ as in previous section. Let R^-, R^+ be defined as above. Now we may define the sets

$$\Gamma^l = \partial D_\epsilon(\gamma^1(t_l)) \ , \qquad l = 0, 1, ..., n$$

such that, possibly decreasing ϵ, the following properties are satisfied

i) $t_1 = t_{R-}$, and $0 < t_{l+1} - t_l \leq \frac{\delta}{3} \quad l = 1, ..., n-1$;

ii) if $\hat{J} \neq \emptyset$, we impose that $\forall m \in \hat{J} \ \exists! \, l \in \{1, ..., n\}$ such that $\hat{t}_m = t_l$;

iii) let Γ^l, for $l = 1, ..., n$, be defined as in section1.5.1, $\Gamma^l \cap \Gamma^{l+1} \neq \emptyset$, for $l = 1, ..., n-1$, and

$$\Gamma^l \cap [\gamma^1_{|(t_{l-1},t_l]}] \neq \emptyset \quad , \qquad \Gamma^l \cap [\gamma^1_{|(t_l,t_{l+1}]}] \neq \emptyset \quad l = 2, ..., n-1$$

$$\Gamma^1 \cap [\gamma^1_{|(t_{q-},t_1]}] \neq \emptyset \quad , \qquad \Gamma^1 \cap [\gamma^1_{|(t_1,t_2]}] \neq \emptyset$$

iv) n is such that $\Gamma^n \cap [\gamma^1_{|(t_{n-1},t_n]}] \neq \emptyset$ and

$$\Gamma^n \cap \big(\Gamma \backslash \gamma^1(t_{R+})\big) \neq \emptyset \, , \qquad \Gamma^n \cap [\gamma^1_{|(t_{q-},t_+]}] = \emptyset$$

v) finally

$$\# \left(\Gamma^l \cap [\gamma^1_{|[t_{l-1},t_{l+1}]}] \right) = 2 \quad l = 1, ..., n-1$$

$$\# \left(\Gamma^n \cap [\gamma^1_{|[t_{n-1},t_{q+}]}] \right) = 2 \ .$$

Let $R_{l,-} = \gamma^1(t_{l-}), R_{l,+} = \gamma^1(t_{l+})$ be such that

$$t_{l-} = \max\{t \in [t_{l-1}, t_{l+1}] : \ \gamma^1(t) \in \partial D_\epsilon(\gamma^1(t_l))\}$$

$$t_{l+} = \min\{t \in [t_{l-1}, t_{l+1}] : \ \gamma^1(t) \in \partial D_\epsilon(\gamma^1(t_l))\} \quad l = 2, ..., n-1$$

and

$$\begin{aligned} t_{1-} &= \max\{t \in [t_{q-}, t_2] : \; \gamma^1(t) \in \partial D_\epsilon(\gamma^1(t_1))\} \\ t_{1+} &= \min\{t \in [t_{q-}, t_2] : \; \gamma^1(t) \in \partial D_\epsilon(\gamma^1(t_l))\} \\ t_{n-} &= \max\{t \in [t_n, t_{q+}] : \; \gamma^1(t) \in \partial D_\epsilon(\gamma^1(t_n))\} \\ t_{n+} &= \min\{t \in [t_{n-1}, t_{q+}] : \; \gamma^1(t) \in \partial D_\epsilon(\gamma^1(t_n))\}\,. \end{aligned}$$

As arguing for Γ^- , for each l we split the closed curve Γ^l using as endpoints $\{R_{l,-}, R_{l,+}\}$ and we define Ω^l_a, Ω^l_b. Hence, if $\gamma^1(t_l)$ is not a returning point, we obtain the arcs Γ^l_a, Γ^l_b, where Γ^l_a has empty intersection with Ω^{l-1}_b, and Γ^l_b has empty intersection with Ω^{l-1}_a. While if $\gamma^1(t_l)$ is a returning point then we split Γ^l in an arc and in the point $\gamma^1(t_{l+})$. We denote the arc by Γ^l_a if has not empty intersection with Ω^{l-1}_a (so $\Gamma^l_b = \gamma^1(t_{l+})$), or by Γ^l_b if has not empty intersection with Ω^{l-1}_b (so $\Gamma^l_a = \gamma^1(t_{l+})$).

In the same way, if Q^+ is not a returning point, we split the closed curve $\Gamma^+ = \partial T^+$ in Γ^+_a, Γ^+_b using as endpoints $\{R+,\, P^+\}$, where Γ^+_a has empty intersection with Ω^n_b, and Γ^+_b has empty intersection with Ω^n_a. Note that

$$[\Gamma^+_a] \cap [\Gamma^+_b] = \{P^+,\, \gamma^1(t_{R+})\}$$

and

$$[\Gamma^+_a] \cup [\gamma^1_{|[t_{R+}, t_+]}] \quad , \qquad [\Gamma^+_b] \cup [\gamma^1_{|[t_{R+}, t_+]}]\,,$$

are simple closed curves. Let Ω^+_a, Ω^+_b be their inside, respectively. Otherwise, if Q^+ is a returning point, we define Γ^+_a, Γ^+_b as above with $l-1 = n$. So we define Ω^+_a, Ω^+_b as in section 1.5.1.

Finally, let $\sigma > 0$ be such that $\gamma^2_{|[s_{q-}-\sigma, s_{q-}]}$ and $\gamma^2_{|[s_{q+}, s_{q+}+\sigma]}$ are contained in the union of such sets.

Definition 9. *Let $\gamma^1 : [a_1, b_1] \to \mathbb{R}^2$ and $\gamma^2 : [a_2, b_2] \to \mathbb{R}^2$ be two curves in Ω. We say that γ^1 is admissible respect to γ^2 at the endpoints of $[t_{q-}, t_{q+}]$ either if*

$$[\gamma^2_{|[s_{q-}-\sigma, s_{q-}]}] \cap \Omega^-_a = \emptyset \quad \textit{and} \quad [\gamma^2_{|[s_{q+}, s_{q+}+\sigma]}] \cap \Omega^+_a = \emptyset,$$

or if

$$[\gamma^2_{|[s_{q-}-\sigma, s_{q-}]}] \cap \Omega^-_b = \emptyset \quad \textit{and} \quad [\gamma^2_{|[s_{q+}, s_{q+}+\sigma]}] \cap \Omega^+_b = \emptyset.$$

Definition 10. *Let $\gamma^1 : [a_1, b_1] \to \mathbb{R}^2$ and $\gamma^2 : [a_2, b_2] \to \mathbb{R}^2$ be two curves in Ω. We say that γ^1 and γ^2 do not cross on $[t_{q-}, t_{q+}]$ if:*

- *i) γ^1 is admissible respect to itself at points of $\mathcal{T}_0$ contained in $[t_{R-}, t_{R+}] \backslash \hat{J}$, according to the definition 8;*

ii) if is satisfied i), we ask that γ^1 is admissible respect to itself at points of $\mathcal{T}_0$ contained in $[t_{R-}, t_{R+}] \cap \hat{J}$, according to the definition 8*;*

iii) if is satisfied ii), we ask that γ^1 is admissible respect to itself at the endpoints of continuous tracts of γ^1 contained in the set $\mathcal{T}_1$ and in $[t_{R-}, t_{R+}] \backslash \hat{J}$, according to the definition 9 *.*

iv) if is satisfied iii), we ask that γ^1 is admissible respect to γ^2 at the endpoints of $[t_{q-}, t_{q+}]$, according to the definition 9*;*

Finally, we may define the families of curves without crossings.

Definition 11. *We say that a family $C = \{\gamma^i\}_i$ of curves in Ω is without crossings if for any point $P \notin P(C) \cup \partial\Omega$ such that there exist $\gamma^i, \gamma^j \in C$ such that $P \in \mathcal{T}^{i,j} = [\gamma^i] \cap [\gamma^j]$, with possibly $i = j$, then*

i) γ^i and γ^j do not cross on each interval $I \subset [0, L(\gamma^i)]$ maximal respect to properties: $P \in [\gamma^i_{|I}]$ and

$$[\gamma^i_{|I}] \subset \mathcal{T}_1^{i,j} = \{\gamma^i(t) \in \mathcal{T}^{i,j} : \exists t_0 > 0 \ \text{ such that } \ [\gamma^i_{|(t,t+t_0)}] \subset \mathcal{T}^{i,j}$$

$$\text{or/and } \ [\gamma^i_{|(t-t_0,t)}] \subset \mathcal{T}^{i,j}\}\,;$$

ii) γ^i and γ^j do not cross at $P \in \mathcal{T}_0^{i,j} = \mathcal{T}^{i,j} \backslash \mathcal{T}_1^{i,j}$.

1.6 The admissible families of curves and the energy functional

Let $C = \{\gamma^i\}_{i \in \tilde{I}}$ be a family of curves of class $\mathcal{B}$ parametrized by arclength. Following [14], we define the functional

$$\mathcal{A}(C) = \sum_{\gamma^i \in C} (K(\gamma^i) + L(\gamma^i)) + \# P(C)\,.$$

Recalling the notations used in previous section, we denote

$$\mathcal{T}(C) = \bigcup_{i,j \in \tilde{I}} \mathcal{T}^{i,j}$$

$$\mathcal{T}_1(C) = \bigcup_{i,j \in \tilde{I}} \mathcal{T}_1^{i,j} \quad , \quad \mathcal{T}_0(C) = \bigcup_{i,j \in \tilde{I}} \mathcal{T}_0^{i,j}\,.$$

Then we define the admissible families of curves:

Definition 12. *We say that a family C of curves in Ω is admissible if the following properties hold:*

(i) *C is of class $\mathcal{B}$;*

(ii) *$\mathcal{A}(C) < +\infty$;*

(iii) *C is without crossings.*

The following proposition extends Proposition 3.6 of [14] to families of curves of class $\mathcal{B}$.

Proposition 1. *Let C be an admissible family of curves in Ω parametrized by arclength, and for any $\gamma \in C$ let Θ be an argument of $\dot{\gamma}$. Let $\omega \in \mathbb{R}$ be such that $0 < \omega < \pi$. Then the following properties hold:*

(i) *the total number of curves in C is finite;*

(ii) *let $H > 0$ be such that $\mathcal{A}(C) \leq H < +\infty$, then the total number of curves $\gamma \in C$ such that $|\dot{\Theta}|([0, L(\gamma)]) \geq \omega$ is bounded by a constant depending only on H and ω.*

Proof. We split the family C into two subfamilies denoted by C^+ and C^- as follows. Let $\gamma \in C$; we say that $\gamma \in C^-$ if $|\dot{\Theta}|([0, L(\gamma)]) < \omega$, and we set $C^+ = C \setminus C^-$.

Using Lemma 2, for any curve $\gamma \in C^+$, there exists a positive constant c independent of γ such that

$$K(\gamma) + L(\gamma) \geq c|\dot{\Theta}|([0, L(\gamma)]) \geq c\,\omega\,.$$

Then we get

$$H \geq \mathcal{A}(C) \geq \sum_{\gamma \in C^+} (K(\gamma) + L(\gamma)) \geq c\,\omega \# C^+\,,$$

from which it follows

$$\# C^+ \leq \frac{H}{c\,\omega}\,. \tag{1.18}$$

which yields statement (ii) of the proposition.

Let us now consider the curves in C^-: from Lemma 5 it follows that such curves cannot be closed. Hence, each curve $\gamma \in C^-$ joins two distinct points in $P(C)$. Since $\# P(C) \leq H$, if we set

$$\delta(C) = \min\{|P_i - P_j| \,:\, P_i, P_j \in P(C),\, P_i \neq P_j\}\,,$$

we have that $\delta(C)$ is a positive number. Then for any curve γ joining two distinct points $P_i \neq P_j$ in $P(C)$ we have

$$L(\gamma) \geq |P_i - P_j| \geq \delta(C)\,,$$

which implies

$$H \geq \mathcal{A}(C) \geq \sum_{\gamma \in C^-} L(\gamma) \geq \delta(C) \# C^-\,,$$

so that also C^- is finite and statement (i) of the proposition follows. □

We now define the energy functional. Given three positive numbers α_K, α_L and α_P, for any admissible family C of curves in Ω we define the functional

$$\mathcal{F}(C) = \sum_{\gamma \in C} [\alpha_K K(\gamma) + \alpha_L L(\gamma)] + \alpha_P \# P(C)\,,$$

and we set

$$\mathcal{F}_0(C) = \inf\{\mathcal{F}(C^*)\,:\,[C^*] = [C]\}\,.$$

The introduction of $\mathcal{F}_0$ allows us to deal with a set S in $\mathbb{R}^2$, image of a set of curves, independently of the particular representation of S and also of the parametrization of the curves.

Let now $g \in L^2(\Omega)$; for any admissible family C of curves in Ω, and any function $u \in W^{1,2}(\Omega)$, we define the functional

$$\mathcal{G}(u, C) = \int_\Omega |u - g|^2 dx + \int_{\Omega \setminus [C]} |\nabla u|^2 dx + \mathcal{F}_0(C)\,.$$

Note that the functional $\mathcal{G}$ depends on $[C]$ rather than on C.

1.7 Compactness and lower semicontinuity results

The proof of the compactness theorem for families of curves without crossings is much more complicated with respect to the $W^{2,2}$ case dealt with Coscia and required the development of a new method of proof.

In this section we obtain compactness and lower semicontinuity results for the functional $\mathcal{F}$ with respect to the weak convergence of sequences of traces of families of curves (Definition 3). This notion of convergence takes into account possible reparametrizations of the curves and the fact that curves may collapse into points. Then we prove the existence of minimizers of the functional $\mathcal{G}$.

We prove the following lemma, which will be useful to prove the coerciveness of functional $\mathcal{F}$ with respect to the weak convergence of traces of families of curves.

Lemma 6. *Let* $\gamma \in \mathcal{B}$ *be an arc parametrized by arclength, and let* Θ *be an argument of* $\dot{\gamma}$. *If* $|\dot{\Theta}|([0, L(\gamma)]) \leq \pi/3$, *then*

$$|\gamma(0) - \gamma(L(\gamma))| \leq L(\gamma) \leq 2|\gamma(0) - \gamma(L(\gamma))|. \tag{1.19}$$

Proof. Let I be such that $[0, L(\gamma)] \subset I$, and let us extend Θ on I as follows:

$$\Theta(s) = \begin{cases} \tilde{\Theta}(0+) & \text{if } s \in I \cap \{s < 0\} \\ \tilde{\Theta}(L(\gamma)-) & \text{if } s \in I \cap \{s > L(\gamma)\}. \end{cases}$$

Then $\Theta \in BV(I)$. Let $\{\rho_h\}_h$ be a sequence of symmetric mollifiers, and define $\alpha^h = \Theta * \rho_h$. Then, by the properties of symmetric mollifiers,

$$\lim_{h\to\infty} \alpha^h(s) = (\Theta(s+) + \Theta(s-))/2 = \tilde{\Theta}(s) \tag{1.20}$$

for any $s \in I$. We set

$$\gamma^h(s) = \int_0^s (\cos\alpha^h(\tau), \sin\alpha^h(\tau))d\tau + \gamma(0) \quad \forall s \in [0, L(\gamma)].$$

The curves γ^h are parametrized by arclength and, using (1.20), $\lim_{h\to\infty} \gamma^h = \gamma$ uniformly on $[0, L(\gamma)]$. Particularly,

$$\gamma^h(0) = \gamma(0),\ L(\gamma^h) = L(\gamma) \quad \forall h, \qquad \lim_{h\to\infty} \gamma^h(L(\gamma)) = \gamma(L(\gamma)). \tag{1.21}$$

Since, by construction, the extended function Θ satisfies

$$\Theta(0+) = \Theta(0-), \qquad \Theta(L(\gamma)+) = \Theta(L(\gamma)-),$$

using Proposition (1.15) of [12], we have

$$\lim_{h\to\infty} \int_{[0,L(\gamma)]} |\dot{\alpha}^h| ds = |\dot{\Theta}|([0, L(\gamma)]) \leq \pi/3. \tag{1.22}$$

In the following we argue as in [14] for the curves γ^h. Up to a rotation of coordinates depending on h, we may assume that the tangent unit vector of γ^h at the point $s = 0$ has components $(1, 0)$. Then, using (1.22), for any $\varepsilon > 0$ we have

$$|\alpha^h(s) - \alpha^h(0)| \leq \int_{[0,L(\gamma)]} |\dot{\alpha}^h| ds \leq \pi/3 + \varepsilon, \tag{1.23}$$

for any $s \in [0, L(\gamma)]$ and h large enough.

Then the tangent vector at γ^h has a positive horizontal component for any $s \in [0, L(\gamma)]$. Hence, with respect to the rotated coordinate system, the curve $\gamma^h = (\gamma_1^h, \gamma_2^h)$ is the graph of a function $u_h \in \mathcal{C}^\infty(a_h, b_h)$, with $a_h = \gamma_1^h(0)$ and $b_h = \gamma_1^h(L(\gamma))$. Since, using (1.23), $|u_h'(x)| \le \sqrt{3} + O(\varepsilon)$ for any $x \in (a_h, b_h)$, using (1.21), we have

$$\begin{aligned} L(\gamma) &= L(\gamma^h) = \int_{a_h}^{b_h} \sqrt{1 + |u_h'|^2}dx \le 2(1 + O(\varepsilon))(b_h - a_h) \\ &\le 2(1 + O(\varepsilon))|\gamma^h(0) - \gamma^h(L(\gamma))| = 2(1 + O(\varepsilon))|\gamma(0) - \gamma^h(L(\gamma))|. \end{aligned}$$

By letting h tend to infinity, ε tend to zero, and using (1.21) the second inequality of (1.19) follows. The first inequality is obvious. $\square$

Theorem 5. *Let $\{C_h\}_h$ be a sequence of admissible families of curves in Ω such that*

$$\mathcal{F}(C_h) \le H < +\infty \quad \text{for all } h.$$

Then there exist a subsequence $\{C_{h_k}\}_k$ and an admissible family C of curves in Ω such that $\{[C_{h_k}]\}_k$ converges weakly to $[C]$. Moreover,

$$\liminf_{h \to +\infty} \mathcal{F}(C_h) \ge \mathcal{F}(C).$$

Proof. *Step 1: compactness.* Up to the extraction of a subsequence we may assume that

$$\liminf_{h \to +\infty} \mathcal{F}(C_h) = \lim_{h \to +\infty} \mathcal{F}(C_h).$$

By the inequality

$$\mathcal{F}(C) \ge \alpha_0 \, \mathcal{A}(C), \qquad \alpha_0 = \min\{\alpha_K, \alpha_L, \alpha_P\} > 0\,,$$

we get $\mathcal{A}(C_h) \le H/\alpha_0$ for all h. Then, using Proposition 1, each family C_h consists of a finite number of curves.

For any h we split the family C_h into two subfamilies C_h^+ and C_h^- as in the proof of Proposition 1, with $\omega = \pi/3$. Using (1.18) we have

$$\# C_h^+ \le \frac{3H}{c\pi},$$

so that $\# C_h^+$ is bounded by a constant independent of h. Hence, up to the extraction of a subsequence, we may assume that $\# C_h^+ = m$ for all h. Then we set

$$C_h^+ = \{\gamma_h^1, \ldots, \gamma_h^m\}, \qquad \gamma_h^i : [0, L(\gamma_h^i)] \to \overline{\Omega} \quad i = 1, \ldots, m,$$

where all the curves γ_h^i are parametrized by arclength.

Moreover, since the length of every curve is bounded from above by H/α_L, up to the extraction of a further subsequence, we may assume that $L(\gamma_h^i) \to L^i$ as $h \to \infty$, for any $i = 1, \dots, m$. Since L^i may be equal to zero for some i, we further split the curves of C_h^+ into two groups:

(i) the curves such that $L(\gamma_h^i) \to 0$;

(ii) the curves such that $L^i > 0$.

Then, up to a subsequence, there exists a finite set $P_1 \subset \overline{\Omega}$ of points such that the maximum distance of the trace of the curves of the group (i) from P_1 goes to zero as $h \to \infty$.

Now we consider the curves of group (ii). We fix $i \in \{1, \dots, m\}$ such that $L^i > 0$, and we drop the index i from γ_h^i in order to avoid a cumbersome notation. We reparametrize the curves γ_h with constant velocity on the same interval $[0, L^i]$. Let θ_h be an argument of the curve γ_h reparametrized on $[0, L^i]$. Then, using the uniqueness of the Lebesgue decomposition of a measure, the functionals $K(\gamma_h)$ can be rewritten by means of the reparametrized curves in the following way (see Section 5 of [8]):

$$K^a(\gamma_h) = \frac{L(\gamma_h)}{L^i} \int_0^{L^i} \psi\left(\theta_h, \frac{L^i}{L(\gamma_h)} \dot{\theta}_h^a\right) dt, \tag{1.24}$$

$$K^s(\gamma_h) = K^s(\theta_h,]0, L^i[) + \mu \left| \int_{\theta_h(L^i-)}^{\theta_h(0+)-2k\pi} \psi_\infty(\tau, 1) d\tau \right|, \tag{1.25}$$

where $k \in \mathbb{Z}$ is such that $-\pi < \theta_h(0+) - \theta_h(L^i-) - 2k\pi \leq \pi$, and $\mu = 0$ if γ_h is a curve with endpoints, otherwise $\mu = 1$.

Since Ω is bounded, $\sup_{t \in [0, L^i]} |\gamma_h(t)|$ is uniformly bounded with respect to h. Moreover, we have

$$\int_0^{L^i} |\dot{\gamma}_h| dx = L(\gamma_h) \leq \frac{H}{\alpha_L}$$

for any h, so that the sequence $\{\gamma_h\}_h$ is uniformly bounded in $W^{1,1}([0, L^i])$.

Using the properties of the argument functions, for any h we have

$$|\ddot{\gamma}_h^a| = |\dot{\theta}_h^a| \quad \text{a.e. in } I = (0, L^i), \qquad |\ddot{\gamma}_h^s|\{t\} \leq |\dot{\theta}_h^s|\{t\} \quad \forall t \in S_{\theta_h}, \tag{1.26}$$

and, using (1.1), we have

$$\ddot{\gamma}_h^s(I \setminus S_{\theta_h}) = \int_{I \setminus S_{\theta_h}} (-\sin \theta_h, \cos \theta_h) d\dot{\theta}_h^s,$$

which implies

$$|\ddot{\gamma}_h^s|(I \setminus S_{\theta_h}) \leq |\dot{\theta}_h{}^s|(I \setminus S_{\theta_h}). \tag{1.27}$$

Using (1.26) and (1.27) we obtain

$$|\ddot{\gamma}_h|([(0, L^i]) \leq |\dot{\theta}_h|([(0, L^i]). \tag{1.28}$$

Using (1.24) and property (iv) of the function ψ (see Section 1.2), we get

$$K^a(\gamma_h) \geq \int_0^{L^i} (A|\dot{\theta}_h^a| - B)dt = A|\dot{\theta}_h^a|([0, L^i]) - BL^i,$$

from which, since $L(\gamma_h) \to L^i$ as $h \to \infty$, it follows for h large enough

$$\begin{aligned} |\dot{\theta}_h^a|([0, L^i]) &\leq \frac{1}{A}(K^a(\gamma_h) + BL(\gamma_h) + B\delta) \\ &\leq \frac{\max\{1, B\}}{A}(K^a(\gamma_h) + L(\gamma_h)) + \frac{B}{A}\delta \\ &\leq \frac{\max\{1, B\}}{A}H + \frac{B}{A}\delta, \end{aligned} \tag{1.29}$$

where δ is a positive constant independent of h.

Using (1.25) and arguing as in the proof of Lemma 2, we have

$$|\dot{\theta}_h^s|([0, L^i]) \leq \frac{1}{A}K^s(\gamma_h) \leq \frac{H}{A}. \tag{1.30}$$

Collecting (1.28), (1.29) and (1.30), we get

$$|\ddot{\gamma}_h|([(0, L^i]) \leq M, \tag{1.31}$$

where M is a positive constant independent of h. Since $\{\gamma_h\}_h$ is uniformly bounded in $W^{1,1}([0, L^i])$, using (1.31), it follows that there exist a subsequence $\{\gamma_{h_k}\}_k$ and a curve $\gamma \in \mathcal{B}$ such that $\{\gamma_{h_k}\}_k$ converges weakly to γ as $k \to +\infty$.

Then all the sequences $\{\gamma_h^i\}_h$ of the curves which do not have infinitesimal length (i.e., the curves belonging to the group (ii)), admit subsequences $\{\gamma_{h_k}^i\}_k$ weakly converging to curves γ^i of class $\mathcal{B}$. Particularly, the endpoints of the curves converge in $\mathbb{R}^2$ to the endpoints of the limit curves.

Since $\omega = \pi/3$, using Lemma 5, all closed curves of C_h belong to C_h^+. Hence C_h^- consists of arcs. Let us now consider the set of endpoints $P(C_h^-)$: since $\#P(C_h^-) \leq H/\alpha_P$, up to a subsequence we may assume that $\#P(C_h^-) = r$ independent of h, i.e.,

$$P(C_h^-) = \{P_h^1, \dots, P_h^r\},$$

and that each of the sequences $\{P_h^i\}_h$ converges to a limit P^i (not necessarily all distinct).

For any pair of limit points P^l, P^m such that $P^l \neq P^m$, and for h large enough, the length of the curves in C_h^- joining P_h^l and P_h^m is greater than $|P^l - P^m|/2$. Hence, the number of such curves is uniformly bounded with respect to h, therefore, up to a subsequence, we may assume that this number is a constant $n_{l,m}$ independent of h. Moreover we have

$$K(\gamma) \leq H/\alpha_K, \qquad L(\gamma) \leq H/\alpha_L$$

for any curve $\gamma \in C_h^-$ joining P_h^l and P_h^m. Then, arguing as for the curves of group (ii) in C_h^+, each of the $n_{l,m}$ sequences of curves admit a subsequence $\{\gamma_{h_k}^i\}_k$ weakly converging to a curve γ^i of class $\mathcal{B}$.

Finally, consider a couple of limit points $P^l = P^m$ with $l \neq m$: by Lemma 6 the length of the curves in C_h^- joining P_h^l and P_h^m is less than $2|P_h^l - P_h^m|$, which tends to zero as $h \to \infty$. Then, up to a subsequence, there exists a finite set $P_2 \subset \overline{\Omega}$ of points such that the maximum distance of the trace of such curves from P_2 goes to zero as $h \to \infty$.

Let now C be the set of curves obtained from the limits of curves in $C_{h_k}^+$, such that $L^i > 0$, and of curves in $C_{h_k}^-$, whose endpoints have distinct limits. Let P be the set of the limits of the endpoints of all the curves in C_{h_k}: we have $P(C) \subset P$.

We now prove that C is a family of curves without crossings. Let us suppose that this assertion is false. There are two possibilities: *I*) at least one crossing point belongs to $\mathcal{T}_1(C)$; *II*) at least one crossing point belongs to $\mathcal{T}_0(C)$. To semplify notations, we will study first the case I assuming true the case II.

Case I). Since in the case *II* will be proved that there are not crossing points in $\mathcal{T}_0(C)$, then there exists a crossing point in $\mathcal{T}_1(C)$ if and only if there exist two curves $\gamma^i, \gamma^j \in C$ (with possibly $i = j$) and two intervals $[t_{q-}, t_{q+}]$, $[s_{q-}, s_{q+}]$ such that

$$[\gamma^i_{|[t_{q-},t_{q+}]}] \equiv [\gamma^j_{|[s_{q-},s_{q+}]}],$$

and γ^i is not admissible respect to γ^j at endpoints of $[t_{q-}, t_{q+}]$. As in section 1.5.2, we define $\delta, \epsilon > 0$ and the open sets $\Omega_a^-, \Omega_b^-, \Omega_a^+, \Omega_b^+$ for the curve γ^i. Let $\sigma > 0$ be such that $\gamma^j_{|[s_{q-}-\sigma,s_{q-}]}$ and $\gamma^j_{|[s_{q+},s_{q+}+\sigma]}$ are contained in the union of such sets.

Let now $\{\gamma_{h_k}^i\}_k$, $\{\gamma_{h_k}^j\}_k$ be the sequences uniformly converging to γ^i, γ^j respectively. For k large enough, let I_k, J_k be the intervals such that

$$[\gamma^i_{h_k|I_k}] \equiv [\gamma^j_{h_k|J_k}],$$

and $I_k = [t^k_{q-}, t^k_{q+}] \to [t_{q-}, t_{q+}]$, $J_k = [s^k_{q-}, s^k_{q+}] \to [s_{q-}, s_{q+}]$. We define δ^k, ϵ^k as in section 1.5.2 for any k. Now, there are two possibilities: $i)$ $\{\delta^k\}_k$ converges to a positive constant, $ii)$ $\{\delta^k\}_k$, and then $\{\epsilon^k\}_k$, converges to zero.

Case I.i. The uniform convergence implies that $\delta^k \to \delta$, and there exists a sequence $\{\epsilon\}_k$, defined as in section 1.5.2, converging to ϵ.

Then we can define the open sets $\Omega^-_{a,k}, \Omega^-_{b,k}, \Omega^+_{a,k}, \Omega^+_{b,k}$ for the curve $\gamma^i_{h_k}$. Moreover, there exists a sequence $\{\sigma_k\}_k$ converging to σ, such that $\gamma^j_{h_k|[s^k_{q-}-\sigma_k, s^k_{q-}]}$ and $\gamma^j_{h_k|[s^k_{q+}, s^k_{q+}+\sigma_k]}$ are contained in the union of such open sets. Note that the sequences $\{\Omega^-_{a,k}\}_k, \{\Omega^-_{b,k}\}_k, \{\Omega^+_{a,k}\}_k, \{\Omega^+_{b,k}\}_k$ converge to the sets $\Omega^-_a, \Omega^-_b, \Omega^+_a, \Omega^+_b$, respectively.

Now, since γ^i is not admissible respect to γ^j at endpoints of $[t_{q-}, t_{q+}]$, we have

$$[\gamma^j_{|[s_{q-}-\sigma, s_{q-}]}] \cap \Omega^-_a \neq \emptyset \quad \text{and} \quad [\gamma^j_{|[s_{q+}, s_{q+}+\sigma]}] \cap \Omega^+_b \neq \emptyset, \tag{1.32}$$

or/and

$$[\gamma^j_{|[s_{q-}-\sigma, s_{q-}]}] \cap \Omega^-_b \neq \emptyset \quad \text{and} \quad [\gamma^j_{|[s_{q+}, s_{q+}+\sigma]}] \cap \Omega^+_a \neq \emptyset\,.$$

Let us suppose that the first situation happens. Then there exist two different points, Q and R, such that $Q \in [\gamma^j_{[s_{q-}-\sigma, s_{q-}]}] \cap \Omega^-_a$ and $R \in [\gamma^j_{[s_{q+}, s_{q+}+\sigma]}] \cap \Omega^+_b$. Since Ω^-_a and Ω^+_b are open of $\mathbb{R}^2$ there exist neighbourhoods $U_Q \subset \Omega^-_a$ of Q and $U_R \subset \Omega^+_b$ of R, respectively, such that $U_Q \cap U_R = \emptyset$.

Because of the uniform convergence of $\gamma^i_{h_k}$ to γ_i, for k large enough we have $U_Q \subset \Omega^-_{a,k}$ and $U_R \subset \Omega^+_{b,k}$. Since C_{h_k} is without crossings, then, either $[\gamma^j_{h_k|[s^k_{q-}-\sigma^k, s^k_{q-}]}] \cap U_Q = \emptyset$, or $[\gamma^j_{h_k|[s^k_{q+}, s^k_{q+}+\sigma_k]}] \cap U_R = \emptyset$.

Since $\gamma^j_{h_k}$ converges uniformly to γ^j as k tends to infinity and, using (2.9.1),both $[\gamma^j_{[s_{q-}-\sigma, s_{q-}]}] \cap U_Q \neq \emptyset$, and $[\gamma^j_{[s_{q+}, s_{q+}+\sigma]}] \cap U_R \neq \emptyset$, it follows that, for k large enough, both $[\gamma^j_{h_k|[s^k_{q-}-\sigma_k, s^k_{q-}]}] \cap U_Q \neq \emptyset$, and $[\gamma^j_{h_k|[s^k_{q+}, s^k_{q+}+\sigma_k]}] \cap U_R \neq \emptyset$. But this contradicts the hypotesys that C_{h_k} is without crossings, so that C is a family of curves without crossings.

Finally, it may happen that some of the curves γ of C intersect at an endpoint P^* of one of them: then to make C an admissible family, it is enough to divide each of these curves γ into two curves, one ending and the other beginning at P^*. Since the number of endpoints is finite, we may in a finite number of steps convert C into an admissible family with the same image. This process does not increase the set of endpoints, since P^* already belongs to $P(C)$, and the functional K may only lose a jump of angle at P^*. It follows that γ^i and γ^j do not cross at P.

Case I.ii). Let $P_{k-} = \gamma^i_k(t_{k-})$, $P_{k+} = \gamma^i_k(t_{k+})$ be defined as in Theorem 3. Note that if the arc $\gamma^i_{h_k|[t^k_{q-}, t^k_{q+}]}$ collapses, we resort to case II. But we need to define P_{k-}, P_{k+} that do not converge to Q_-, Q_+, respectively, to construct $\Omega^{\pm}_{a,k}$, $\Omega^{\pm}_{b,k}$ converging to the sets $\Omega^{\pm}_a$, $\Omega^{\pm}_b$ defined above.

We fix a parameter t^k_{q+} and let be $\gamma^i_k(t^k_{q+}) = P^k_i \to \gamma^i_{h_k}(t^k_{q+}) = Q^-$. Let $S_+ = \{Q_{r,k}\}_r \subset [\gamma^i_{h_k|(t \geq t^k_{q+})}]$ be the set of all nonsimple points of $\gamma^i_{h_k|(t \geq t^k_{q+})}$ such that $Q_{r,k} \to Q^+$. Note that we may have $\#S_+ = \infty$.

Let us suppose for instance that $P_{k-} = \gamma^i_k(t_{k-})$ converges to a point different from Q^-, while $P_{k-} = \gamma^i_k(t_{k-})$ converges to Q^+. By the definition of δ_k, if $\delta_k \to 0$, then in the arc of $\gamma^i_{h_k}(t)$ with $t \geq t^k_{q+}$ there exists at least a closed curve contained in $\gamma^i_{h_k}$, passing through a point of S_+, and collapsing to Q^+. Denote by γ^i_k, γ^j_k the curves $\gamma^i_{h_k}$, $\gamma^j_{h_k}$ respectively.

Remark 3. *Let us suppose that there does not exist at least a closed curve contained in γ^i_k, passing through a point of S_+, and collapsing to Q^+. Then if we move on γ^i_k from t^k_{q+} in the direction of increasing t, we have the following possibilities for $\{t > t^k_{q+}\}$:*

- *we reach $\gamma^i_k(L(\gamma^i_k))$ without meeting nonsimple points. Then we may choose $P_{k+} = \gamma^i_k(L(\gamma^i_k))$ not converging to Q_+, since Q_+ is not an endpoint;*

- *we reach a nonsimple point Q_k, which is not converging to Q_+, without meeting first points of S_+. Then we may choose $P_{k+} = Q_k$;*

- *we reach P^i_k along a closed curve that does not collapse. Then, if s^1_{k+} is such that*

  $$\gamma^i_k(s^1_{k+}) = P^i_k, \qquad s^1_{k+} > t^k_{q+},$$

  and for any $t \in (t^k_{q+}, s^1_{k+})$ we have $\gamma^i_k(t) \neq P^i_k$, we may choose $P_{k+} \in (t^k_{q+}, s^1_{k+})$;

- *we reach a point of $S_+ \backslash P^i_k$, from which a loop not collapsing to Q_+ then starts. Then, if s^1_{k-}, s^1_{k+} are such that*

  $$\gamma^i_k(s^1_{k-}) = \gamma^i_k(s^1_{k+}) = Q_{r,k}, \qquad s^1_{k-} < s^1_{k+}, r \in \{1, ..., M\},$$

  and for any $t \in (s^1_{k-}, s^1_{k+})$ we have $\gamma^i_k(t) \neq Q_{r,k}$, we may choose $P_{k+} \in (s^1_{k-}, s^1_{k+})$.

Therefore in all these cases $\{P^i_k\}_k$ does not converge to Q_+, so that we can resort to case I.i).

We suppose that there exists at least a closed curve in γ_k^i that collapses on Q_+. Using Lemma 5 we have

$$\#C_{k+} \leq \frac{H}{\pi},$$

where C_{k+} is the set of closed curves contained in γ_k^i and passing through the points of S_+. So $\#C_{k+}$ is bounded by a constant independent of k. Hence, up to the extraction of a subsequence, we may assume that $\#C_{k+} = N$ for all k. With similar arguments, we may assume that the number of curves in $C_{P_k^i}$ that collapse on Q_+ is R. Let $X = \{s_0^k, ..., s_d^k\}$ be an increasing set of parameters and let $I_X^k = [s_0^k, s_d^k] \subset [0, L(\gamma_k^i)]$ be the largest interval having the properties

$$\gamma_k^i(s_l^k) = P_k^i, \quad l = 0, \dots, d, \qquad \gamma^i_{k|I_X^k} \to Q_+ \tag{1.33}$$

for any k, and $d \leq R$. Therefore, by means of the definition of the interval I_X^k, starting from the point $P_k^i = \gamma_k^i(t_{q+}^k)$, we select the greatest number of consecutive closed curves collapsing to Q_+. Such curves cannot be simple. Note that $s_d^k \neq L(\gamma_k^i)$ since Q_+ is not an endpoint. Moreover, since I_X^k is the largest interval having the properties (1.33), for $t > s_d^k$ each closed curve in P_k^i that collapses to Q_+ has to contain an arc that does not converge to Q_+.

Argument A. Let us consider what happens for $t > s_d^k$. We have two cases:

1) we have an arc that does not collapse to Q_+, then the situation is the same as in Remark 3. Hence we may define

   $$J_{1,k} = (t_{k-}, s_0^k) \cup (s_d^k, t_{k+}),$$

   where t_{k+} is chosen as in Theorem 3.

2) There exists an arc collapsing to Q_+. Then we have the following cases:

   2.1) the arc is simple. Hence, if t_k is the endpoint of this arc, for $t > t_k$ we have a situation as in Remark 3. Therefore, we define J_k^i as in the case 1);

   2.2) the arc is not simple. Then it contains at least a point of S_+. We denote by $Q_{a_1,k} = \gamma_k^i(t_{a_1,k})$ the first point of S_+ after $P_k^i = \gamma_k^i(s_d^k)$, i.e., $t_{a_1,k} = \min\{t > s_d^k : \gamma_k^i(t) \in S_+\}$. Then we require that J_k^i is such that

   $$J_k^i \cap (t_{k-}, t_{a_1,k}] = (t_{k-}, s_0^k) \cup (s_d^k, t_{a_1,k}].$$

Note that in this way we eliminate the curves in I^k_X. Then we repeat for $Q_{a_1,k}$ the same arguments used for P^i_k. Therefore, we define at the point $Q_{a_1,k}$ a set having the same properties of X, and we denote such a set by X_{a_1}. Hence we still eliminate the largest number of consecutive closed curves collapsing to Q_+ which pass through $Q_{a_1,k}$.

End of Argument A

Now we repeat the *Argument A* whenever we meet points of S_+, after skipping the closed curves collapsing at points of S_+ that have been previously met. Let C be the largest number of these steps. Note that C can be infinite. For each step we repeat the *Argument A* and we add a new interval to J^i_k, so that we obtain

$$J^i_k \cap (t_{k-}, \sup\{t_{a_l,k} : l = 1, ..., C\}] = I_{1,k} = (t_{k-}, s^k_0) \cup (s^k_d, t_{a_1,k}) \cup (\cup_{l=1}^{C} (\max X_{a_l}, t_{a_{l+1},k}))\,,$$

where $C \leq \# S_+$. Now we have

$$|I_{1,k}| < L(\gamma^i_k) < H < \infty,$$

then there exists finite

$$\sup\{t_{a_l,k} : l = 1, ..., C\} = a_C < \infty,$$

and there exists a point $P_C = \gamma^i_k(a_C) \neq \gamma^i_k(L(\gamma^i_k))$. It follows that $\gamma^i_{k,[t^k_{q+},a_C]} \to Q_+$, and for $t > a_C$ we resort to the cases (1), (2.1) of *Argument A*. Then for $t > a_C$ we may define $P_{k+} = \gamma^i_k(t_{k+})$ as in Theorem 3, in such a way that P_{k+} does not converge to Q_+, and we define

$$J^i_k = I_{1,k} \cup [a_C, t_{k+})\,.$$

Let now $J_{1,k} = (t_{k-}, s^k_0) \cup (s^k_q, t_{k+})$. Note that $J_{1,k}$ is not an interval, but we can still construct the open sets Ω^-_a, Ω^-_b, Ω^+_a, Ω^+_b, since in the construction we use the trace of the curve, and we do not use the fact that the parameters are consecutive both for $t < t^k_{q-}$ and $t > t^k_{q+}$. Moreover, the functional is not evaluated at the point $\gamma^i_k(t^k_{q+})$, but at $\gamma^i_k(t^k_{q+}+)$, $\gamma^i_k(t^k_{q+}-)$. Hence it is possible to eliminate the curves which collapse, since the functional may have a finite contribution at $P^i_k = \gamma^i_k(t^k_{q+})$, but the contribution from $(t^k_{q+}+, t]$, $[t, t^k_{q+}-)$ tends to zero with t.

If γ^j_k has points both interior and exterior to the above closed curves, then we eliminate the arcs of γ^j_k interior to such closed curves, since they collapse on Q_+.

Let η^j_k be the curves constructed in such a way. Note that the eliminated arcs, because of admissibility, pass through P^i_k without yielding crossing points, therefore η^j_k are continuous curves. Then, γ^j_k does not cross γ^i_k in

P^i_k if and only if η^j_k and $\gamma^i_{k|_{J_{1,k}}}$, obtained according to the above construction, are without crossing for every X and I^k_X which satisfy (1.33).

We may use the same arguments if P_{k-} converges to Q_-, while P_{k+} converges to a different point from Q_+.

If both $\{P_{k-}\}_k$ and $\{P_{k+}\}_k$ converge to $Q_- = Q_+$ for any $J_{1,k}$, the only case in which we cannot use the above arguments is when P^i_k is interior to a closed collapsing curve. In this case our construction cannot be used, since such a curve is eliminated. However we do not need to examine such points, since the behaviour of γ^j_k is important only with respect to arcs of γ^i_k that do not converge to points.

Each P_{k-} and P_{k+} defined in such a way does not converge to Q_- and Q_+, respectively. Now, if we eliminate the closed collapsing curves contained in arc $\gamma^i_{h_k|[t^k_{q-},t^k_{q+}]}$, we can repeat the same argument of case *I.i.*

Case II). Then there exist a point $P \notin P(C) \cup \partial\Omega$, curves $\gamma^i, \gamma^j \in C$ (with possibly $i = j$), t_1 and t_2 such that $\gamma^i(t_1) = \gamma^j(t_2) = \{P\} \in \mathcal{T}_0(C)$.

As in Section 1.5.1, we define $\delta, \epsilon > 0$ and the disjoint open sets Ω_a, Ω_b for the curve γ^i. Let $\sigma > 0$ be such that $\gamma^j_{|[t_2-\sigma,t_2+\sigma]}$ is contained in the union of such sets. Let now $\{\gamma^i_{h_k}\}_k$, $\{\gamma^j_{h_k}\}_k$ be the sequences uniformly converging to γ^i, γ^j respectively. Let $\{P_k = \gamma^i(t^k_1) = \gamma^j(t^2_k)\}_k$ be the sequence converging to P. Now, for k large enough, either $P_k \in \mathcal{T}_1(C_{h_k})$ or $P_k \in \mathcal{T}_0(C_{h_k})$. In the first case, we argue as in case I considering that

$$[\gamma^i_{|[t_{q-},t_{q+}]}] = \{P\} \quad \text{and} \quad \Omega^-_a = \Omega^+_a \quad \Omega^-_b = \Omega^+_b \,. \tag{1.34}$$

In the second case, we argue as in case I using the properties (1.34) and the following properties

$$[\gamma^i_{|[t^k_{q-},t^k_{q+}]}] = \{P_k\} \quad \text{and} \quad \Omega^-_{a,k} = \Omega^+_{a,k} \quad \Omega^-_{b,k} = \Omega^+_{b,k} \,. \tag{1.35}$$

Therefore the admissibility of C is proved.

Step 2: lower semicontinuity. Using Theorem 2, the continuity of the functional $\int |\dot\gamma| dt$ with respect to the weak convergence of curves, and the fact that $\#P(C) \le \#P \le \#P(C_{h_k})$, the lower semicontinuity property follows:

$$\mathcal{F}(C) \le \lim_{k\to+\infty} \mathcal{F}(C_{h_k}) = \liminf_{h\to+\infty} \mathcal{F}(C_h).$$

□

Now we may prove an existence result for the functional $\mathcal{G}$.

Theorem 6. *The functional $\mathcal{G}$ has a minimizer in the class of pairs (u, C), with C admissible family of curves in Ω and $u \in W^{1,2}(\Omega \setminus [C])$.*

Proof. *Step 1: the functional $\mathcal{F}_0$ is lower semicontinuous.* The lower semicontinuity of $\mathcal{F}_0$ is a direct conseguence of Step 2 in the proof of Theorem 5. Indeed, let $\{[C_h]\}_h$ be a sequence of traces of admissible families of curves converging to the trace of an admissible curve $[C]$ such that

$$\liminf_{h\to+\infty} \mathcal{F}_0(C_h) < +\infty$$

(otherwise the result is trivial), and let us select a subsequence, still denoted by $\{[C_h]\}_h$, along which the liminf is a limit. Then, by the definition of $\mathcal{F}_0$ and the lower semicontinuity of $\mathcal{F}$, for each h we may consider a family C_h^* with $[C_h] = [C_h^*]$ and

$$\mathcal{F}_0(C_h) \leq \mathcal{F}(C_h^*) \leq \mathcal{F}_0(C_h) + \frac{1}{h}.$$

We may apply to $\{C_h^*\}_h$ the previous result and we obtain that (up to a subsequence) it converges to the trace of a family C^* (which satisfies $[C^*] = [C]$, since $[C_h^*] = [C_h] \to [C]$) such that

$$\mathcal{F}(C^*) \leq \liminf_{h\to+\infty} \mathcal{F}(C_h^*) = \lim_{h\to+\infty} \mathcal{F}_0(C_h).$$

But $\mathcal{F}_0(C) = \mathcal{F}_0(C^*) \leq \mathcal{F}(C^*)$, and the lower semicontinuity of $\mathcal{F}_0$ is proved.

Step 2: proof of the existence result. First of all, we select a subsequence $\{u_{h_k}, C_{h_k}\}_k$ such that

$$\liminf_{h\to+\infty} \mathcal{G}(u_h, C_h) = \lim_{k\to+\infty} \mathcal{G}(u_{h_k}, C_{h_k}) < +\infty\,.$$

We may assume that $\mathcal{F}(C_{h_k}) \leq H < +\infty$, where H is a positive constant independent of k. By the definition of $\mathcal{F}_0$ for each k we may consider an admissible family $C_{h_k}^*$ with $[C_{h_k}^*] = [C_{h_k}]$, and

$$\mathcal{F}_0(C_{h_k}) \leq \mathcal{F}(C_{h_k}^*) \leq \mathcal{F}_0(C_{h_k}) + 1/k, \qquad \mathcal{G}(u_{h_k}, C_{h_k}^*) = \mathcal{G}(u_{h_k}, C_{h_k}).$$

Since $\mathcal{F}(C_{h_k})$ is uniformly bounded with respect to k, by applying to $\{C_{h_k}\}_k$ the result of Step 1 in the proof of Theorem 5, there exist a subsequence, which we still denote by the same index h_k, and an admissible family C of curves in Ω such that $\{[C_{h_k}]\}_k$ converges to $[C]$.

Now we consider the corresponding subsequence $\{u_{h_k}\}_k$ and we call P the set of the limits of the endpoints of the curves in C_{h_k} and of the points P^i, limits of the curves that converge to a point (see Step 1 in the proof of Theorem 5). We have to prove that exist a subsequence $\{u_{h_k}\}_k$ and a function $u \in W^{1,2}(\Omega \setminus [C])$ such that $u_{h_k} \to u$ weakly in $W^{1,2}(\Omega^*)$ for every $\Omega^* \subset\subset \Omega \setminus ([C] \cup P)$ and $u_{h_k} \to u$ a.e. in Ω.

We now consider a sequence $\{\Omega_i\}_i$ of open sets $\Omega_i \subset\subset \Omega\backslash([C]\cup P)$ that invade $\Omega \setminus ([C] \cup P)$. For every i the distance $dist(\Omega_i, [C] \cup P)$ is positive, then for k sufficiently large $\Omega \cap [C_{h_k}] = \emptyset$ and $u_{h_k} \in W^{1,2}(\Omega_i)$. Moreover, since $\mathcal{G}(u_{h_k}, C_{h_k})$ is uniformly bounded with respect to k, we have

$$\begin{aligned}
&\int_{\Omega_i} (|u_{h_k}|^2 + |Du_{h_k}|^2)dx \\
&\leq \int_{\Omega\setminus[C_{h_k}]} (|u_{h_k}|^2 + |Du_{h_k}|^2)dx \\
&\leq \int_{\Omega\setminus[C_{h_k}]} (2|u_{h_k} - g|^2 + 2|g|^2 + |Du_{h_k}|^2)dx \leq c(H, ||g||_{L^2}),
\end{aligned}$$

where c is a constant independent of i and k. Then the sequence $\{u_{h_k}\}_k$ is equibounded in $W^{1,2}(\Omega_i)$, hence there exists a subsequence $\{u_{h_k}\}_k$ converging weakly in $W^{1,2}(\Omega_i)$ and a.e. in Ω_i to a function $u \in W^{1,2}(\Omega_i)$. By a diagonal argument we obtain a function u defined on $\Omega \setminus ([C] \cup P)$ and a subsequence $\{u_{h_k}\}_k$ converging to u weakly in $W^{1,2}(\Omega^*)$ for every $\Omega^* \subset\subset \Omega \setminus ([C] \cup P)$ and almost everywhere in Ω.

It remains to prove that $u \in W^{1,2}(\Omega \setminus [C])$ and the lower semicontinuity of $\mathcal{G}$. By the above inequality and the weak lower semicontinuity of the $W^{1,2}$ norm, we have

$$\begin{aligned}
\int_{\Omega_i} (|u|^2 + |Du|^2)dx &\leq \liminf_{k\to\infty} \int_{\Omega_i} (|u_{h_k}|^2 + |Du_{h_k}|^2)dx \\
&\leq c(H, ||g||_{L^2}),
\end{aligned}$$

for any i, then $u \in W^{1,2}(\Omega \setminus ([C] \cup P))$, i.e. $u \in W^{1,2}(\Omega \setminus [C])$, being P a finite set of points. In particular,

$$\int_{\Omega\setminus[C]} |Du|^2 dx \leq \liminf_{k\to\infty} \int_{\Omega\setminus[C_{h_k}]} |Du_{h_k}|^2 dx\,.$$

Moreover, using the weak convergence in $W^{1,2}(\Omega_i)$ of u_{h_k} to u we obtain that

$$\int_{\Omega_i} |u-g|^2 dx = \lim_{k\to\infty} \int_{\Omega_i} |u_{h_k} - g|^2 dx \leq \liminf_{k\to\infty} \int_{\Omega\setminus[C_{h_k}]} |u_{h_k} - g|^2 dx,$$

for every i, therefore

$$\int_{\Omega\setminus[C]} |u-g|^2 dx = \liminf_{k\to\infty} \int_{\Omega\setminus[C_{h_k}]} |u_{h_k} - g|^2 dx\,.$$

Collecting the above results and using the semicontinuity of $\mathcal{F}_0$, we have

$$\begin{aligned}\mathcal{G}(u,C) &\leq \liminf_{k\to\infty} \int_{\Omega\setminus[C_{h_k}]} |Du_{h_k}|^2 dx \\ &+ \liminf_{k\to\infty} \int_{\Omega\setminus[C_{h_k}]} |u_{h_k}-g|^2 dx + \liminf_{k\to\infty} \mathcal{F}_0(C_{h_k}) \\ &\leq \lim_{k\to\infty} \mathcal{G}(u_{h_k}, C_{h_k}) = \lim_{h\to\infty} \mathcal{G}(u_h, C_h)\,.\end{aligned}$$

The existence of minimizers for the functional $\mathcal{G}$ then follows from the compactness and lower semicontinuity results. $\square$

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Chapter 2

Variational approximation of a functional depending on squared curvature for numerical computation

2.1 Introduction

In mathematical image reconstruction theory, by image of a scene we mean a real valued measurable function g on a plane domain Ω, which measures the grey level, i.e., the brightness at each point of Ω. In general, this function g is discontinuous along the lines corresponding to sudden changes in the visible surfaces (e.g. edges of objects, shadows, different reflectances).

The image segmentation problem consists in finding a pair (u, C) such that C is a set of curves, which decompose the image into regions with relatively uniform brightness, and $u \in C^1(\Omega \setminus C)$ is a smooth approximation of $g \in L^\infty(\Omega)$ on each region. The set C will be understood as the union of the lines which give the schematic description of the image.

Many different approaches have been proposed to find u and C. Geman and Geman [22] proposed a discrete optimization method based on the minimization of an energy function. This idea was developed by Mumford and Shah, who proposed to look for a pair (u, C) which minimizes the following functional:

$$G^{MS}(u, C) = \int_{\Omega \setminus C} |\nabla u|^2 dx + \int_\Omega |u - g|^2 dx + \mathcal{H}^1(C),$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open set, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. The functional is minimized over the pairs (u, C) such that $C \subset \overline{\Omega}$ is a closed set, and $u \in C^1(\Omega \backslash C)$.

The existence of minimizers of G^{MS} has been proved independently by De Giorgi, Carriero and Leaci [19], and by Dal Maso, Morel and Solimini

[26], using the compactness and lower semicontinuity theorems of Ambrosio [1, 2].

Mumford and Shah studied the properties of minimizers (u, C) of G^{MS} assuming that C is a finite union of simple $\mathcal{C}^2$ curves meeting $\partial\Omega$ and meeting each other only at their endpoints. They proved that the vertices of C may only be:

(i) triple points where three curves meet with 120° angles;

(ii) points on the boundary of Ω where one curve meets $\partial\Omega$ perpendicularly;

(iii) points where a curve ends and meets nothing.

In this way corners and junctions with angles different from 120°, which are important to identify superimposed objects, are distorted because the length measure is not sensitive to corners and such junctions. In order to reconstruct such singularities we need to consider curvature depending energies. In [14] Coscia proposed a functional that includes the integral of square curvature $\int_C k^2 d\mathcal{H}^1$, where C is a family of curves.

More precisely, given three positive numbers α_L, α_K, α_P, we define for any family C of curves the functional

$$\mathcal{F}(C, P) = \sum_{\gamma \in C} \int_\gamma (\alpha_K k^2 + \alpha_L) d\mathcal{H}^1 + \alpha_P \# P,$$

where k is the curvature of the curve $\gamma \in C$, P is the set of the endpoints of all the curves in C with the exception of the regular closed ones, and $\#$ denotes the counting measure.

The functional considered by Coscia [14] is

$$G(u, C, P) = \mathcal{F}(C, P) + \int_{\Omega \setminus (C \cup P)} |\nabla u|^2 dx + \int_\Omega |u - g|^2 dx,$$

defined on all admissible families C of curves in Ω and all functions $u \in W^{1,2}(\Omega \setminus C)$. A family C of curves is admissible if it consists of a finite number of curves of class $W^{2,2}$ which do not cross each other or themselves, except possibly with the same tangent vector. Then, nontangent crossing points are penalized by forcing them to be considered as the endpoints of some curves. Tangent crossings must be allowed for semicontinuity reasons.

It should be noted that the functional G involves the recursive application of a one-dimensional version of the Mumford-Shah functional along the curves of the family C.

Existence of minimizers of G has been proved by Coscia in [14]. However, the numerical minimization of the functional G is a difficult task, which reflects the difficulties of the challenging problem of recovering geometrical properties of visible surfaces from two-dimensional image functions.

Braides and March [12] have proposed the approximation of G by means of a family of functionals G_ε which are numerically more tractable, and they have proved the Γ-convergence of the approximating functionals to G.

In the following we assume $\alpha_L = \alpha_K = \alpha_P = 1$.

The approximation result is divided in two parts: (i) a first approximization is performed by means of a new type of energies where points and curves are substituted by sets; (ii) the functionals defined on sets are then approximated by functionals defined on smooth functions.

The first step is the construction of a variational approximation of the term $\#P$ that counts the number of points of P by means of a functional whose minimizers are disks of small radius ε. Moreover, this functional is chosen in such a way that it can be approximated by an energy defined on functions by means of the coarea formula [11]. The functional is

$$\mathcal{E}_\varepsilon^{(1)}(D) = \frac{1}{4\pi} \int_{\partial D} (1/\varepsilon + \varepsilon k^2) d\mathcal{H}^1,$$

where D is a smooth set, k is the curvature of ∂D, and $\frac{1}{4\pi}$ is a normalization factor that derives from the fact that minimizers of $\mathcal{E}_\varepsilon^{(1)}$ are given by balls of radius ε.

The next step is the construction of another energy defined on sets that approximates the functional $\int_C (1+k^2) d\mathcal{H}^1$ (C is a finite union of $W^{2,2}$ curves with endpoints contained in P). The family C of curves is approximated, away from D, by smooth sets A which collapse, as ε tends to zero, on the curves of the family C. The energy is defined on sets A and D as

$$\mathcal{E}_\varepsilon^{(2)}(A, D) = \frac{1}{2} \int_{(\partial A)\setminus D} (1 + k^2) d\mathcal{H}^1.$$

Since in this functional there is not dependence on ε, then the constraint $meas(A) \leq a_\varepsilon = o(1)$ as $\varepsilon \to 0$ is imposed in order to approximate the curves in C by the set A. This means that the set A shrinks to the curves in C as ε tends to zero. The factor $1/2$ is due to the fact that each curve of C is the limit of two arcs of ∂A. Then the overall approximating functional is

$$\mathcal{E}_\varepsilon(u, A, D) = \mathcal{E}_\varepsilon^{(1)}(D) + \mathcal{E}_\varepsilon^{(2)}(A, D) + \int_{\Omega\setminus(A\cup D)} |\nabla u|^2 dx + \int_\Omega |u - g|^2 dx,$$

defined on smooth sets A, D compactly contained in Ω. Note that $A \cup D$ contains the singularities of u.

To obtain an energy defined on functions, Braides and March [12] used a gradient theory approach. Particularly, using the Modica-Mortola [25] approximation, the perimeter measures $\mathcal{H}^1(\partial A)$ and $\mathcal{H}^1(\partial D)$ are approximated by the measures $\mathcal{H}^1_\varepsilon(s, \nabla s)dx$ and $\mathcal{H}^1_\varepsilon(w, \nabla w)dx$, where

$$\mathcal{H}^1_\varepsilon(s, \nabla s)dx = \varepsilon|\nabla s|^2 + \frac{s^2(1-s)^2}{\varepsilon},$$

$$\mathcal{H}^1_\varepsilon(w, \nabla w)dx = \varepsilon|\nabla w|^2 + \frac{w^2(1-w)^2}{\varepsilon},$$

where s and w are functions approximating $1-\chi_A$ and $1-\chi_D$, respectively. Then the sets A and D will be replaced by the functions s and w in the new energy functional defined on functions. Furthermore, we need to define the curvature of the level sets of s and w:

$$k(\nabla s) = \begin{cases} \operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right) & \text{if } \nabla s \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$k(\nabla w) = \begin{cases} \operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) & \text{if } \nabla w \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the term $\mathcal{E}^{(1)}_\varepsilon(D)$ is approximated by

$$G^{(1)}_\varepsilon(w) = \int_\Omega (1/\beta_\varepsilon + \beta_\varepsilon k^2(\nabla w))\mathcal{H}^1_\varepsilon(w, \nabla w)dx,$$

with $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$, the term $\mathcal{E}^{(2)}_\varepsilon(A, D)$ is approximated by

$$G^{(2)}_\varepsilon(s, w) = \int_\Omega w^2(1 + k^2(\nabla s))\mathcal{H}^1_\varepsilon(s, \nabla s)dx,$$

and the constraint $meas(A) \leq a_\varepsilon$ is approximated by

$$I_\varepsilon(s, w) = 1/\mu_\varepsilon \int_\Omega ((1-s)^2 + (1-w)^2)dx,$$

where $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. Note that I_ε forces s and w to be equal to 1 almost everywhere in the limit as $\varepsilon \to 0$. Then the functional proposed by Braides and March [12] is

$$\begin{aligned} G_\varepsilon(u, s, w) \quad = \quad & \frac{1}{4\pi b_0} G^{(1)}_\varepsilon(w) + \frac{1}{2b_0} G^{(2)}_\varepsilon(s, w) + \int_\Omega s^2|\nabla u|^2 dx + \\ & + \int_\Omega |u - g|^2 dx + I_\varepsilon(s, w), \end{aligned}$$

where b_0 is a normalization constant. The functionals G_ε are defined on the space

$$W(\Omega) = \left\{(u, s, w) : u \in W^{1,2}(\Omega); 1 - s, 1 - w \in \mathcal{C}_0^\infty(\Omega; [0, 1])\right\}.$$

For an appropriate choice of the infinitesimal coefficients β_ε and μ_ε, in [12], Theorem 3.9, it has been proved that the functionals G_ε Γ-converge to G as $\varepsilon \to 0^+$ with respect to a suitable convergence of triplets of functions (u, s, w) to triplets (u, C, P). According to such a convergence of functions, level sets D_ε of w_ε collapse on points of P, and level sets A_ε of s_ε collapse on curves of C as ε tends to zero (as in the first step of the approximation process).

Though the approximating functionals G_ε are numerically more convenient than the original functional G, we observe that the presence of terms of the type

$$\int_{\Omega\setminus\{|\nabla s|=0\}} \left(\operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right)\right)^2 dx, \qquad \int_{\Omega\setminus\{|\nabla w|=0\}} \left(\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right)\right)^2 dx\,, \tag{2.1}$$

still makes the minimization of G_ε a difficult numerical problem. Hence we replace such terms with functionals more convenient for numerical computations. For this purpose we adapt to our problem a method proposed by Ballester, Bertalmio, Caselles, Sapiro and Verdera in [6].

Let $\theta : \Omega \to \mathbb{R}^2$ and $\omega : \Omega \to \mathbb{R}^2$ be vector fields such that $|\theta(x)| \leq 1$ and $|\omega(x)| \leq 1$ for any $x \in \Omega$. The vector fields θ and ω should be related to the functions s and w, respectively, by trying to impose that

$$\langle\theta, \nabla s\rangle = |\nabla s|, \qquad \langle\omega, \nabla w\rangle = |\nabla w|, \tag{2.2}$$

where $\langle , \rangle$ is the scalar product of $\mathbb{R}^2$, i.e., we should impose that θ and ω are related to the vector fields of directions of the gradient of s and w, respectively. Ideally, the quantities $|\theta(x)|$ and $|\omega(x)|$ should be equal either to 1 or 0, for any $x \in \Omega$, so that

$$\theta(x) = \frac{\nabla s(x)}{|\nabla s(x)|}, \qquad \omega(x) = \frac{\nabla w(x)}{|\nabla w(x)|},$$

at any point x where $\nabla s(x) \neq 0$ and $\nabla w(x) \neq 0$, otherwise $|\theta(x)| = 0$ and $|\omega(x)| = 0$. The conditions $|\theta(x)| \leq 1$ and $|\omega(x)| \leq 1$ should be interpreted as a relaxation of this.

Collecting all the observations above, we propose to replace the functionals (2.1) with the following functionals:

$$\begin{aligned} &\int_\Omega (\mathrm{div}(\theta))^2\,dx + \eta \int_\Omega \left(|\nabla s| - \langle \theta, \nabla s\rangle\right) dx, \\ &\int_\Omega (\mathrm{div}(\omega))^2\,dx + \eta \int_\Omega \left(|\nabla w| - \langle \omega, \nabla w\rangle\right) dx, \end{aligned} \tag{2.3}$$

where η is a positive weight. Moreover, the vector fields θ and ω are submitted to the constraints $|\theta(x)| \leq 1$ and $|\omega(x)| \leq 1$ for any $x \in \Omega$.

The conditions (2.2) are now incorporated in the functionals (2.3) via the penalization terms

$$\eta \int_\Omega \left(|\nabla s| - \langle \theta, \nabla s\rangle\right) dx, \qquad \eta \int_\Omega \left(|\nabla w| - \langle \omega, \nabla w\rangle\right) dx. \tag{2.4}$$

The overall energy G_ε is then approximated by the following functional

$$\begin{aligned} &M_{\varepsilon,\xi}(u, s, w, \theta, \omega) = \int_\Omega (s^2 + \zeta_\varepsilon)|\nabla u|^2 dx \\ &+\frac{1}{2b_0} \int_\Omega (w^2 + k_\varepsilon) \left[\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon} s^2(1-s)^2 + \lambda_\varepsilon\right] \left(1 + (\mathrm{div}\,\theta)^2\right) dx \\ &+\frac{1}{4\pi b_0} \int_\Omega \left[\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon} w^2(1-w)^2 + \lambda_\varepsilon\right] \left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon(\mathrm{div}\,\omega)^2\right) dx \\ &+\eta \int_\Omega \left(|\nabla s| - \langle \theta, \nabla s\rangle\right) dx + \eta \int_\Omega \left(|\nabla w| - \langle \omega, \nabla w\rangle\right) dx \\ &+ \int_\Omega |u-g|^2 dx + \frac{1}{\mu_\varepsilon} \int_\Omega (1-s)^2 dx + \frac{1}{\mu_\varepsilon} \int_\Omega (1-w)^2 dx \\ &+\xi \int_\Omega (\Delta s)^2 dx + \xi \int_\Omega (\Delta w)^2 dx, \end{aligned}$$

where $\xi > 0$ is a constant. The positive coefficients ζ_ε, k_ε and λ_ε are introduced to make the functional $M_{\varepsilon,\xi}$ coercive. We need to make precise the admissible class of functions where the modified functionals $M_{\varepsilon,\rho}$ have to be minimized. The functionals G_ε are defined on classes of smooth functions s and w. The suitable function space for $M_{\varepsilon,\rho}$ will be introduced in Section 2.2.

In Section 2.5 we prove existence of minimizers of $M_{\varepsilon,\xi}$ for fixed ε and ξ in a suitable function space where $u \in W^{1,2}(\Omega)$, $s, w \in W^{2,2}(\Omega)$ and $\theta, \omega \in W^{1,2}(\mathrm{div}, \Omega)$. Then in Sections 2.6.1 and 2.6.2 we study the behaviour of minimizers when ξ tends to zero and η tends to infinity. Particularly, in Section 2.6.2 we let η tend to infinity (with fixed ε and ξ) and we prove that sequences of minimizers of $M_{\varepsilon,\xi}$ admit subsequences converging as $\eta \to +\infty$ to a minimizer of a version of the functional G_ε, which is modified by means of the perturbation

$$\xi \int_\Omega (\Delta s)^2 dx + \xi \int_\Omega (\Delta w)^2 dx,$$

and the coefficients ξ_ε, k_ε and λ_ε. Denoting by $G_{\varepsilon,\xi}$ the resulting modified functionals, in Section 2.7 we prove the Γ-convergence of $G_{\varepsilon,\xi}$ to the original functional G depending on curves and points.

In Section 8 we compute the system of Euler equations of $M_{\varepsilon,\xi}$ and we propose an iterative numerical method for their solution. The numerical scheme is based on alternate minimizations using the conjugate gradient method. The results of some computer experiments are then shown and discussed.

2.2 Preliminary definitions

We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the usual euclidean norm and scalar product in $\mathbb{R}^2$. We denote by $\text{meas}(B)$ the Lebesgue measure of the set $B \subseteq \mathbb{R}^2$, by $\mathcal{H}^1$ the one-dimensional Hausdorff measure, and by $\#$ the counting measure. We will use standard notation for the Lebesgue and Sobolev spaces L^p and $W^{k,p}$.

We say that a set A is of class $\mathcal{C}^\infty$ if A is open, and its restriction to some neighbourhood of any $x \in \partial A$ is the subgraph of a function of class $\mathcal{C}^\infty$ with respect to a suitable orthogonal coordinate system. If $\Omega \subseteq \mathbb{R}^2$ is a bounded open set then we write $A \in \mathcal{C}^\infty_c(\Omega)$ if A is of class $\mathcal{C}^\infty$ and $A \subset\subset \Omega$.

The *Hausdorff distance* between two closed sets C and K is defined as $\mathrm{d}_{\mathcal{H}}(C, K) = \inf\{r > 0 : C \subset (K)_r, K \subset (C)_r\}$, where $(A)_r = \{x \in \mathbb{R}^2 : \text{dist}(x, A) < r\}$, for a generic set $A \subseteq \mathbb{R}^2$.

2.2.1 Admissible families of curves

A *curve* is a function $\gamma : [a, b] \to \mathbb{R}^2$ in $W^{2,2}(a, b)$ such that $|\dot{\gamma}| \neq 0$ in $[a, b]$. The points $\gamma(a)$ and $\gamma(b)$ are the *endpoints* of γ, the set $[\gamma] = \{\gamma(t) : t \in [a, b]\}$ is the *trace* of γ. A curve will be identified with its representative in $C^1([a, b])$. A curve is *simple* if $\gamma(t_1) = \gamma(t_2)$ only if $t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$. A *regular closed curve* is a curve on some interval $[a, b]$ that may be extended to a $(b - a)$-periodic $W^{2,2}_{\text{loc}}$ function on $\mathbb{R}$ (i.e., its endpoints join smoothly).

Let γ be a curve defined on $[a,b]$; we define the *unit tangent vector* at the point $t \in [a,b]$ as $\tau(t) = \dot\gamma(t)/|\dot\gamma(t)|$, and the *curvature* $\kappa(t)$ by $\kappa(t) = |\dot\tau(t)|/|\dot\gamma(t)|$. Note that the functionals

$$L(\gamma) = \int_a^b |\dot\gamma(t)|dt, \qquad K(\gamma) = \int_a^b \kappa^2(t)|\dot\gamma(t)|dt$$

are independent of the particular parametrization chosen. $L(\gamma)$ is the *length* of γ and K is the *integral of the square of the curvature* along γ.

Let $C = \{\gamma^i\}_i$ be a family of curves. We denote by $[C]$ the *trace* of C, defined as the union of all the traces of the curves in C. We denote by $P(C)$ the set of the endpoints of all the curves in C with the exception of those regular and closed. We define the functional

$$\mathcal{A}(C) = \sum_{\gamma\in C}\big(K(\gamma) + L(\gamma)\big) + \#P(C).$$

Following [14] we say that the family C of curves is *admissible* if the following conditions are satisfied:

(i) $\mathcal{A}(C) < +\infty$;

(ii) $\dot\gamma^i(t_1)$ and $\dot\gamma^j(t_2)$ are parallel whenever $\gamma^i(t_1) = \gamma^j(t_2)$ with t_1 and t_2 interior to the domains of γ^i and γ^j, respectively, and with possibly $i = j$.

We say that C is an *admissible family of curves in* Ω if in addition $[C] \subset \overline{\Omega}$.

We say that a family of curves C satisfies the *finiteness property* if C is finite and there exists a finite set of points F such that $[C]\setminus F$ can be written locally as the graph of a function of class $W^{2,2}$.

The following notion of equivalent families of curves is useful when dealing with different parameterizations of $[C]$.

Let C and C' be two admissible families of curves in Ω. We say that C and C' are *equivalent* if $[C] = [C']$, $P(C) = P(C')$ and

$$\sum_{\gamma\in C} L(\gamma) = \sum_{\gamma\in C'} L(\gamma), \qquad \sum_{\gamma\in C} K(\gamma) = \sum_{\gamma\in C'} K(\gamma).$$

2.2.2 Function spaces for the vector fields θ and ω

We introduce suitable function spaces that will be useful to find a domain for the functional $M_{\varepsilon,\xi}$.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with a Lipschitz boundary. We define the function space for the vector field θ. We set

$$W^{1,p}(\mathrm{div}, \Omega) = \left\{ \theta \in L^p(\Omega)^2 : \mathrm{div}(\theta) \in L^p(\Omega) \right\}, \quad 1 \leq p < \infty,$$

where the divergence operator is intended in the sense of distributions. The Trace Theorem [4] guarantees that the normal component $\langle \theta, \nu \rangle|_{\partial\Omega}$ is well defined for vector fields θ in $W^{1,p}(\mathrm{div}, \Omega)$, where ν is the unit normal to $\partial\Omega$.

Remark 4. If $p = 2$, $W^{1,2}(\mathrm{div}, \Omega)$ is a Hilbert space with respect to the scalar product

$$(\theta, \omega)_W = \int_\Omega \langle \theta, \omega \rangle \, dx + \int_\Omega \mathrm{div}\, \theta \, \mathrm{div}\, \omega \, dx \, .$$

Moreover, $W^{1,2}(\mathrm{div}, \Omega)$ is a reflexive Banach space with the norm $\|\theta\| = \int_\Omega |\theta|^2 dx + \int_\Omega (\mathrm{div}\, \theta)^2 dx$.

For reflexive Banach spaces the following theorem holds.

Theorem 7. *Let* $(B, \|\cdot\|_B)$ *be a reflexive Banach space and let* $\{v_h\}_h \subset B$ *be a sequence. If there exists* $K > 0$ *such that* $\|v_h\|_B \leq K$ *for any* h, *then there exists a subsequence* $\{v_{h_k}\}_k$ *weakly convergent in* B.

In our case, a sequence $\{\theta_h\} \subset W^{1,2}(\mathrm{div}, \Omega)$ weakly converges to $\theta \in W^{1,2}(\mathrm{div}, \Omega)$ if

$$\lim_{h \to \infty} \int_\Omega \left(\langle \theta_h, f \rangle + \mathrm{div}\, \theta_h \, \mathrm{div}\, f \right) dx = \int_\Omega \left(\langle \theta, f \rangle + \mathrm{div}\, \theta \, \mathrm{div}\, f \right) dx$$

for any $f \in W^{1,2}(\mathrm{div}, \Omega)$.

Analogously we set $\omega \in W^{1,2}(\mathrm{div}, \Omega)$.

We define [4]

$$X(\Omega)_p = \left\{ z \in L^\infty(\Omega; \mathbb{R}^2) : \, \mathrm{div}(z) \in L^p(\Omega) \right\}.$$

In [4] a weak trace on $\partial\Omega$ of $z \in X(\Omega)_p$ is defined. If $\nu(x)$ denotes the outer unit normal at $x \in \partial\Omega$, in [4] it is proved that there exists a linear operator $\gamma : X(\Omega)_p \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty, \qquad \gamma(z)(x) = \langle z(x), \nu(x) \rangle \; \forall x \in \partial\Omega \text{ if } z \in C^1(\overline{\Omega}, \mathbb{R}^2).$$

We shall denote $\gamma(z)(x)$ by $[z, \nu](x)$.

Let $p \geq 1$ and $q \geq 1$ be such that $(1/p) + (1/q) = 1$. If $z \in X(\Omega)_p$ and $v \in W^{1,2}(\Omega) \cap L^q(\Omega)$, the following Green's formula holds:

$$\int_\Omega v \, \mathrm{div}(z) dx + \int_\Omega \langle z, \nabla v \rangle dx = \int_{\partial\Omega} [z, \nu] v \, d\mathcal{H}^1. \tag{2.5}$$

2.3 The approximation framework

2.3.1 The curvature depending functional

For any admissible family C of curves in Ω, and any finite set of points $P \subset \overline{\Omega}$ such that $P(C) \subseteq P$, we define the functionals [12]:

$$\mathcal{F}(C,P) = \sum_{\gamma \in C} \big(\alpha_K K(\gamma) + \alpha_L L(\gamma)\big) + \alpha_P \# P,$$

$$\mathcal{F}_0(C,P) = \inf\{\mathcal{F}(\widehat{C},\widehat{P}) : [\widehat{C}] = [C],\ \widehat{P} \setminus [\widehat{C}] = P \setminus [C]\}.$$

The functional $\mathcal{F}_0$ allows us to deal with the trace of a family of curves independently of the parametrization of the curves themselves.

The definition of the functional is itself in terms of a minimization procedure. Note that the infimum in the definition of $\mathcal{F}_0(C,P)$ is a minimum; namely, there exist an admissible family C^* of curves in Ω and a finite set of points $P^* \subset \overline{\Omega}$ such that

$$[C^*] = [C], \quad P^* \setminus [C^*] = P \setminus [C], \qquad \mathcal{F}(C^*,P^*) = \mathcal{F}_0(C,P). \tag{2.6}$$

This can be easily proved by reasoning similarly as in the proof of [14] Theorem 4.2.

We denote by $X(\Omega)$ the family of all triplets (u,C,P) such that $P \subset \overline{\Omega}$ is a finite set of points, C is an admissible family of curves in Ω such that $P(C) \subseteq P$ and $u \in W^{1,2}(\Omega \setminus [C])$, and we introduce the functional $G : X(\Omega) \to [0,+\infty]$ defined by

$$G(u,C,P) = \int_{\Omega \setminus [C]} |\nabla u|^2 dx + \mathcal{F}_0(C,P) + \int_{\Omega} |u-g|^2 dx.$$

2.3.2 The approximating functionals G_ε

We define

$$W(\Omega) = \{(u,s,w) : u \in W^{1,2}(\Omega);\ 1-s, 1-w \in \mathcal{C}_0^\infty(\Omega;[0,1])\}.$$

If $1-s \in \mathcal{C}_0^\infty(\Omega;[0,1])$, using Sard's theorem (see e.g. [2]), for a.e. $\lambda \in (0,1)$ we have

$$\{s=\lambda\} = \partial\{s<\lambda\}, \quad \{s<\lambda\} \in \mathcal{C}_c^\infty(\Omega), \quad |\nabla s| \neq 0 \text{ on } \{s=\lambda\}.$$

Then we set

$$\kappa(\nabla s) = \operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right) \qquad \text{on } \{s=\lambda\} \quad \text{for a.e. } \lambda \in (0,1)\ ,$$

and, $V : \mathbb{R} \to [0, +\infty)$ being defined by $V(t) = t^2(1-t)^2$,

$$\mathcal{H}^1_\varepsilon(s, \nabla s) = \varepsilon |\nabla s|^2 + \frac{V(s)}{\varepsilon}. \tag{2.7}$$

The quantity $\kappa(\nabla s)$ is the curvature of the level set $\{s = \lambda\}$, and $\mathcal{H}^1_\varepsilon(s, \nabla s)$ is the *Modica-Mortola density* of elliptic functionals approximating the perimeter functional (see [25], [10]). If $1 - w \in \mathcal{C}^\infty_0(\Omega; [0, 1])$ the quantities $\kappa(\nabla w)$ and $\mathcal{H}^1_\varepsilon(w, \nabla w)$ are defined analogously.

Let now β_ε, μ_ε be positive infinitesimals as $\varepsilon \to 0^+$ such that

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon |\log \varepsilon|}{\beta_\varepsilon} = 0, \qquad \lim_{\varepsilon \to 0^+} \frac{\beta_\varepsilon}{\mu_\varepsilon} = 0. \tag{2.8}$$

For every $\varepsilon > 0$ we define [12]

$$G^{(1)}_\varepsilon(w) = \int_{\Omega \setminus \{|\nabla w| = 0\}} \Big(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(\nabla w) \Big) \mathcal{H}^1_\varepsilon(w, \nabla w) dx,$$

and

$$G^{(2)}_\varepsilon(s, w) = \int_{\Omega \setminus \{|\nabla s| = 0\}} w^2 (1 + \kappa^2(\nabla s)) \mathcal{H}^1_\varepsilon(s, \nabla s) dx.$$

We denote by $G_\varepsilon : W(\Omega) \to [0, +\infty]$ the functional defined by

$$\begin{aligned} G_\varepsilon(u, s, w) = & \int_\Omega s^2 |\nabla u|^2 dx + \frac{1}{4\pi b_0} G^{(1)}_\varepsilon(w) + \frac{1}{2b_0} G^{(2)}_\varepsilon(s, w) \\ & + \int_\Omega |u - g|^2 dx + \frac{1}{\mu_\varepsilon} \int_\Omega (1 - s)^2 dx + \frac{1}{\mu_\varepsilon} \int_\Omega (1 - w)^2 dx, \end{aligned} \tag{2.9}$$

where

$$b_0 = 2 \int_0^1 \sqrt{V(t)} dt.$$

2.3.3 The topology of Γ-convergence

We need the following notions of convergence for sequences of compact sets [12].

Definition 13. *We say that a sequence of compact sets $\{\mathcal{K}_h\}_h$* converges in the Hausdorff metric *to the compact set $\mathcal{K}$* up to the finite set of points P *if there exists a sequence of compact sets $\{\widehat{\mathcal{K}}_h\}_h$ such that $\widehat{\mathcal{K}}_h \subseteq \mathcal{K}_h$ for any h, $\{\widehat{\mathcal{K}}_h\}_h$ converges to $\mathcal{K}$ in the Hausdorff metric, and the maximum distance of $\mathcal{K}_h \setminus \widehat{\mathcal{K}}_h$ from the set P goes to zero.*

We set

$$Y(\Omega) = \{(u, A, D) : u \in W^{1,2}(\Omega);\ A, D \in \mathcal{C}_c^\infty(\Omega)\}.$$

We define the following convergence for sequences $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$.

Definition 14. *We say that a sequence* $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ converges weakly *to the triplet* $(u, C, P) \in X(\Omega)$, *if* $\text{meas}(A_h \cup D_h) \to 0$ *and the following properties hold:*

(i) $\{\partial D_h\}_h$ *converges in the Hausdorff metric to the set* P*;*

(ii) $\{\partial A_h\}_h$ *converges in the Hausdorff metric to* $[C]$ *up to the set* P*;*

(iii) $u_h \to u$ *in* $L^1(\Omega)$.

The above definition describes the concentration of the smooth sets D_h and A_h on sets of points and traces of curves, respectively.

We define the following convergence for sequences $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ that describes the concentration of the level sets of the smooth functions w_h and s_h on sets of points and traces of curves, respectively [12].

Definition 15. *We say that a sequence* $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ converges weakly *to the triplet* $(u, C, P) \in X(\Omega)$, *if, after setting*

$$\{x \in \Omega : s_h(x) < \lambda\} = A_h^\lambda, \qquad \{x \in \Omega : w_h(x) < \theta\} = D_h^\theta,$$

the following properties hold:

(i) for any $\theta, \lambda \in (0, 1)$ *there exist a finite set of points* $P^\theta \subset \overline{\Omega}$ *and an admissible family* C^λ *of curves in* Ω *such that the sequence* $\{(u_h, A_h^\lambda, D_h^\theta)\}_h$ *converges weakly to* (u, C^λ, P^θ)*;*

(ii) we have $[C] = \bigcap\{[C^\lambda] : 0 < \lambda < 1\}$ *and* $P = \bigcap\{P^\theta : 0 < \theta < 1\}$.

Now we define the Γ-convergence of the functionals G_ε to the functional G with respect to the convergence above.

Definition 16. *We say that* G_ε Γ-converge *to* G *as* $\varepsilon \to 0^+$ *if for every sequence* $\{\varepsilon_h\}_h$ *of positive numbers converging to zero and for every triplet* $(u, C, P) \in X(\Omega)$ *the following two conditions are fulfilled:*

(i) (liminf inequality) *for every sequence* $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ *converging weakly to* (u, C, P)*, we have*

$$\liminf_{h\to+\infty} G_{\varepsilon_h}(u_h, s_h, w_h) \geq G(u, C, P); \tag{2.10}$$

(ii) (limsup inequality) *there exists a sequence* $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ *converging weakly to* (u, C^*, P^*) *such that*

$$\limsup_{h\to+\infty} G_{\varepsilon_h}(u_h, s_h, w_h) \le G(u, C, P), \tag{2.11}$$

where C^* *and* P^* *are as in* (2.6).

In [12] the following theorem has been proved, which states that a sequence of triplets in $W(\Omega)$, asymptotically minimizing the functional G_ε, admits a subsequence converging weakly to a minimizer of G.

Theorem 8. *The functionals* G_ε Γ*-converge to* G *as* $\varepsilon \to 0^+$. *Moreover, if* $\{\varepsilon_h\}_h$ *is a sequence of positive numbers converging to zero, and* $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ *is a sequence such that*

$$\lim_{h\to+\infty}\Big(G_{\varepsilon_h}(u_h, s_h, w_h) - \inf_{W(\Omega)} G_{\varepsilon_h}\Big) = 0,$$

then there exist a subsequence $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ *and a minimizer* (u, C, P) *of* G *such that* $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ *converges weakly to* (u, C, P).

2.4 The modified approximating functionals $M_{\varepsilon,\xi}$

In this section we define the modified functionals $M_{\varepsilon,\xi}$. First we introduce the function space $V(\Omega)$ on which the modified approximating functionals are defined:

$$\begin{aligned} V(\Omega) \;=\; & \big\{(u, s, w, \theta, \omega) : u \in W^{1,2}(\Omega),\ 1-s, 1-w \in W^{2,2}_0(\Omega; [0,1]), \\ & \theta, \omega \in W^{1,2}(\mathrm{div}, \Omega),\ |\theta(x)| \le 1, |\omega(x)| \le 1 \ a.e.\ x \in \Omega, \\ & [\theta, \nu](x) = [\omega, \nu](x) = 0 \ x \in \partial\Omega\big\}, \end{aligned}$$

where $\nu(x)$ denotes the outer unit normal at $x \in \partial\Omega$.

In the following we denote by $\upsilon = (u, s, w, \theta, \omega)$ the elements of $V(\Omega)$. On the space $V(\Omega)$ we define the following convergence.

Definition 17. *We say that a sequence* $\{\upsilon_n\}_n = \{(u_n, s_n, w_n, \theta_n, \omega_n)\}_n \subset V(\Omega)$ *converges weakly to* $\upsilon = (u, s, w, \theta, \omega) \in V(\Omega)$ *if* $u_n \to u$ *weakly in* $W^{1,2}(\Omega)$, $s_n \to s$, $w_n \to w$ *weakly in* $W^{2,2}(\Omega)$, *and* $\theta_n \to \theta$, $\omega_n \to \omega$ *weakly in* $W^{1,2}(\mathrm{div}, \Omega)$.

Let $\varepsilon > 0$ be fixed, and let $\eta > 0$ and $\xi > 0$ be constants. For any $v = (u, s, w, \theta, \omega) \in V(\Omega)$ the functional $M_{\varepsilon,\xi}$ is defined by

$$
\begin{aligned}
&M_{\varepsilon,\xi}(u, s, w, \theta, \omega) = \int_\Omega (s^2 + \zeta_\varepsilon)|\nabla u|^2 dx \\
&+\frac{1}{2b_0}\int_\Omega (w^2 + k_\varepsilon)\left[\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2 + \lambda_\varepsilon\right]\left(1 + (\operatorname{div}\theta)^2\right) dx \\
&+\frac{1}{4\pi b_0}\int_\Omega \left[\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}w^2(1-w)^2 + \lambda_\varepsilon\right]\left(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon(\operatorname{div}\omega)^2\right) dx \\
&+\eta\int_\Omega (|\nabla s| - \langle\theta, \nabla s\rangle)\, dx + \eta\int_\Omega (|\nabla w| - \langle\omega, \nabla w\rangle)\, dx \\
&+\int_\Omega |u-g|^2 dx + \frac{1}{\mu_\varepsilon}\int_\Omega (1-s)^2 dx + \frac{1}{\mu_\varepsilon}\int_\Omega (1-w)^2 dx \\
&+\xi\int_\Omega (\Delta s)^2 dx + \xi\int_\Omega (\Delta w)^2 dx.
\end{aligned}
$$

The regularization by means of the terms involving $(\Delta s)^2$ and $(\Delta w)^2$ is introduced in order to achieve the lower semicontinuity of the functional $M_{\varepsilon,\xi}$. Moreover, $\xi_\varepsilon, k_\varepsilon, \lambda_\varepsilon \to 0^+$ as $\varepsilon \to 0$. Such infinitesimals are introduced to make the functional $M_{\varepsilon,\xi}$ coercive on the space $V(\Omega)$ for a fixed ε.

The functional $M_{\varepsilon,\xi}$ replaces G_ε for computational purposes.

2.5 Existence of minimizers of $M_{\varepsilon,\xi}$

In this section we prove the existence of minimizers of the functional $M_{\varepsilon,\xi}$ in the space $V(\Omega)$ when both ε and ξ are kept fixed. In this section we drop the dependence of $M_{\varepsilon,\xi}$ on ξ and we simply write M_ε. First, we recall two theorems that will be used in the sequel.

Theorem 9. (Egorov) *Let Ω be open with finite Lebesgue measure. Let $\{f_n\}_n$ be a sequence of measurable functions in Ω such that*

$$f_n(x) \to f(x) \text{ a.e.}, \qquad |f_n(x)| < \infty \text{ a.e.}$$

Then, for any $\delta > 0$ there exists a measurable set $A \subset \Omega$ such that $|\Omega \setminus A| < \delta$ and $f_n \to f$ uniformly on A. We say that $\{f_n\}_n$ almost uniformly converges to f.

Theorem 10. (Ioffe) *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\psi : \Omega \times \mathbb{R}^{m+k} \to [0, +\infty]$ be a normal function, i.e., $\mathcal{L}^n \times \mathcal{B}(\mathbb{R}^{m+k})$-measurable and lower semicontinuous with respect to variables in $\mathbb{R}^{m+k}$ for $\mathcal{L}^n$-a.e. $x \in \Omega$. We assume that $z \mapsto \psi(x, y, z)$ is convex in $\mathbb{R}^k$ for any $x \in \Omega$ and any $y \in \mathbb{R}^m$. Then*

$$\liminf_{n\to+\infty} \int_\Omega \psi(x, u_n, v_n)dx \geq \int_\Omega \psi(x, u, v)dx$$

whenever $\{u_n\}_n \subset [L^1(\Omega)]^m$ strongly converges to u and $\{v_n\}_n \subset [L^1(\Omega)]^k$ weakly converges to v.

Now we prove a compactness result for M_ε.

Theorem 11. (Compactness) *Let $\{\upsilon_n\}_n = \{(u_n, s_n, w_n, \theta_n, \omega_n)\}_n \subset V(\Omega)$ be a sequence such that $M_\varepsilon(\upsilon_n) \leq H$ for any $n \in \mathbb{N}$, where H is a positive constant independent of n.*

Then there exist a subsequence $\{\upsilon_{n_h}\}_h = \{(u_{n_h}, s_{n_h}, w_{n_h}, \theta_{n_h}, \omega_{n_h})\}_h$ and $\upsilon \in V(\Omega)$ such that υ_{n_h} converges weakly to υ as h tends to infinity.

Proof. Each term of $M_\varepsilon(\upsilon_n)$ is positive, since the integrands are square except the following terms

$$\eta \int_\Omega (|\nabla s_n| - \langle \theta_n, \nabla s_n\rangle)\, dx, \qquad \eta \int_\Omega (|\nabla w_n| - \langle \omega_n, \nabla w_n\rangle)\, dx,$$

which are also positive because $|\theta_n(x)| \leq 1$ and $|\omega_n(x)| \leq 1$ for a.e. $x \in \Omega$.

Then, since $M_\varepsilon(\upsilon_n) \leq H$, we have

$$\begin{aligned}\int_\Omega \left(|u_n|^2 + \xi_\varepsilon |\nabla u_n|^2\right) dx &\leq \int_\Omega \left(2|u_n - g|^2 + 2|g|^2 + \xi_\varepsilon|\nabla u_n|^2\right) dx \\ &\leq 2\left(H + \|g\|^2_{L^2(\Omega)}\right).\end{aligned}$$

Since ξ_ε is fixed it follows that the sequence $\{u_n\}_n$ is uniformly bounded in $W^{1,2}(\Omega)$, so that there exist a subsequence $\{u_{n_h}\}_h$ and a function $u \in W^{1,2}(\Omega)$ such that $u_{n_h} \to u$ weakly in $W^{1,2}(\Omega)$ as h tends to infinity.

Then we have

$$\frac{k_\varepsilon \lambda_\varepsilon}{2b_0} \int_\Omega (\operatorname{div} \theta_n)^2 dx \leq H, \qquad \frac{\lambda_\varepsilon \beta_\varepsilon}{4\pi b_0} \int_\Omega (\operatorname{div} \omega_n)^2 dx \leq H, \tag{2.12}$$

from which it follows that $\operatorname{div} \theta_n$ and $\operatorname{div} \omega_n$ are uniformly bounded in $L^2(\Omega)$. Since $|\theta_n(x)| \leq 1$ and $|\omega_n(x)| \leq 1$ for a.e. $x \in \Omega$, and $k_\varepsilon, \lambda_\varepsilon, \beta_\varepsilon$ are fixed, it follows that the sequences $\{\theta_n\}_n$ and $\{\omega_n\}_n$ are uniformly bounded in $W^{1,2}(\operatorname{div}, \Omega)$. Then, there exist subsequences, which we still denote by the same index n_h, $\{\theta_{n_h}\}_h$ and $\{\omega_{n_h}\}_h$, and vector fields $\theta, \omega \in W^{1,2}(\mathrm{div}, \Omega)$ such that $\theta_{n_h} \to \theta$ and $\omega_{n_h} \to \omega$ weakly in $W^{1,2}(\mathrm{div}, \Omega)$ as h tends to infinity.

Since $\{1 - s_n\}_n, \{1 - w_n\}_n \subset W_0^{2,2}(\Omega)$ are uniformly bounded in $L^\infty(\Omega)$ and

$$\int_\Omega (\Delta s_n)^2 dx \le \frac{H}{\xi}, \qquad \int_\Omega (\Delta w_n)^2 dx \le \frac{H}{\xi},$$

using Lemma 3.1 and Lemma 3.2 of Chapter 4 of [20], it follows that the sequences $\{s_n\}_n$ and $\{w_n\}_n$ are uniformly bounded in $W^{2,2}(\Omega)$. Then, there exist subsequences, which we still denote by the same index n_h, $\{s_{n_h}\}_h$ and $\{w_{n_h}\}_h$, and functions $1-s, 1-w \in W_0^{2,2}(\Omega)$ such that $s_{n_h} \to s$ and $w_{n_h} \to w$ weakly in $W^{2,2}(\Omega)$ as h tends to infinity.

Eventually, the subsequence $\{\upsilon_{n_h}\}_h = \{(u_{n_h}, s_{n_h}, w_{n_h}, \theta_{n_h}, \omega_{n_h})\}_h$ converges to $\upsilon = (u, s, w, \theta, \omega)$ weakly in $V(\Omega)$ and the theorem is proved. □

Then we prove a lower semicontinuity result for M_ε.

Theorem 12. (Lower semicontinuity) *For any $\upsilon = (u, s, w, \theta, \omega) \in V(\Omega)$ and any sequence $\{\upsilon_n\}_n = \{(u_n, s_n, w_n, \theta_n, \omega_n)\}_n$ converging weakly to υ in $V(\Omega)$ as n tends to infinity, the following inequality holds:*

$$\liminf_{n\to+\infty} M_\varepsilon(u_n, s_n, w_n, \theta_n, \omega_n) \ge M_\varepsilon(u, s, w, \theta, \omega).$$

Proof. Up to the extraction of a subsequence we may assume that

$$\liminf_{n\to+\infty} M_\varepsilon(\upsilon_n) = \lim_{n\to+\infty} M_\varepsilon(\upsilon_n) < +\infty, \tag{2.13}$$

otherwise the inequality is trivial.

We subdivide the integrals in M_ε in groups, in such a way that for each group we prove the lower semicontinuity of integral functionals of the same kind.

Step 1. First we consider the following terms:

$$\frac{1}{2b_0}\int_\Omega (w^2 + k_\varepsilon)\left[\frac{1}{\varepsilon}s^2(1-s)^2 + \lambda_\varepsilon\right] dx,$$

$$\frac{1}{4\pi b_0 \beta_\varepsilon}\int_\Omega \left[\frac{1}{\varepsilon}w^2(1-w)^2 + \lambda_\varepsilon\right] dx,$$

$$\frac{\varepsilon}{2b_0}\int_\Omega (w^2 + k_\varepsilon)|\nabla s|^2 dx,$$

$$\frac{\varepsilon}{4\pi b_0 \beta_\varepsilon}\int_\Omega |\nabla w|^2 dx,$$

$$\int_\Omega |u - g|^2 dx, \quad \frac{1}{\mu_\varepsilon}\int_\Omega (1-s)^2 dx, \quad \frac{1}{\mu_\varepsilon}\int_\Omega (1-w)^2 dx.$$

Since all the integrands are non-negative measurable functions, the lower semicontinuity of these integral functionals follows from Fatou's Lemma provided that the sequences $\{u_n\}_n$, $\{s_n\}_n$, $\{w_n\}_n$, $\{\nabla s_n\}_n$ and $\{\nabla w_n\}_n$ converge a.e. in Ω.

Using (2.13), the sequence $\{M_\varepsilon(v_n)\}_n$ is uniformly bounded with respect to n. Then using Theorem 11, up to the extraction of a subsequence, $u_n \to u$ weakly in $W^{1,2}(\Omega)$, and $s_n \to s$, $w_n \to w$ weakly in $W^{2,2}(\Omega)$. Then the sequences $\{u_n\}_n$, $\{s_n\}_n$ and $\{w_n\}_n$ converge in $L^2(\Omega)$ and, possibly extracting further subsequences, they converge almost everywhere. Moreover, the sequences $\{\nabla s_n\}_n$ and $\{\nabla w_n\}_n$ converge in $L^2(\Omega)$ and, possibly extracting further subsequences, they also converge almost everywhere.

Then the lower semicontinuity of the first group of integral functionals follows from Fatou's Lemma.

Step 2. Then we consider the following terms:

$$\frac{1}{2b_0}\int_\Omega (w^2+k_\varepsilon)\left[\varepsilon|\nabla s|^2+\frac{1}{\varepsilon}s^2(1-s)^2+\lambda_\varepsilon\right](\operatorname{div}\theta)^2dx,$$

$$\frac{\beta_\varepsilon}{4\pi b_0}\int_\Omega \left[\varepsilon|\nabla w|^2+\frac{1}{\varepsilon}w^2(1-w)^2+\lambda_\varepsilon\right](\operatorname{div}\omega)^2dx.$$

Using (2.13) and Theorem 11, up to the extraction of a subsequence, $\theta_n \to \theta$ and $\omega_n \to \omega$ weakly in $W^{1,2}(\operatorname{div},\Omega)$. Hence the sequences $\{\operatorname{div}\theta_n\}_n$ and $\{\operatorname{div}\omega_n\}_n$ converge weakly in $L^2(\Omega)$ to $\operatorname{div}\theta$ and $\operatorname{div}\omega$, respectively. In order to prove the lower semicontinuity of the integral functionals of the second group, we have to prove that

$$\liminf_{n\to+\infty}\int_\Omega f_n(\operatorname{div}\varphi_n)^2dx \geq \int_\Omega f(\operatorname{div}\varphi)^2dx, \tag{2.14}$$

where $\{f_n\}_n$ is a sequence of non-negative measurable functions converging a.e. to f, and $\{\operatorname{div}\varphi_n\}_n$ converges weakly in $L^2(\Omega)$ to $\operatorname{div}\varphi$.

Using Egorov Theorem 9, for any $\delta > 0$ there exists a subset $A_\delta \subset \Omega$ such that $|\Omega \setminus A_\delta| < \delta$ and $f_n \to f$ uniformly on A_δ. Then we get

$$\begin{aligned}
\liminf_{n\to+\infty} \int_\Omega f_n(\operatorname{div}\varphi_n)^2 dx &\geq \liminf_{n\to+\infty} \int_{A_\delta} f_n(\operatorname{div}\varphi_n)^2 dx \\
&= \liminf_{n\to+\infty} \int_{A_\delta} (f_n - f + f)(\operatorname{div}\varphi_n)^2 dx \\
&= \lim_{n\to+\infty} \int_{A_\delta} (f_n - f)(\operatorname{div}\varphi_n)^2 dx \\
&+ \liminf_{n\to+\infty} \int_{A_\delta} f(\operatorname{div}\varphi_n)^2 dx,
\end{aligned}$$

where, using (2.12), we have

$$\begin{aligned}
\int_{A_\delta} (f_n - f)(\operatorname{div}\varphi_n)^2 dx &\leq \sup_{x\in A_\delta} |f_n(x) - f(x)| \int_{A_\delta} (\operatorname{div}\varphi_n)^2 dx \\
&\leq C \sup_{x\in A_\delta} |f_n(x) - f(x)| \to 0,
\end{aligned}$$

where C is a positive constant independent of n. Then we obtain

$$\liminf_{n\to+\infty} \int_\Omega f_n(\operatorname{div}\varphi_n)^2 dx \geq \liminf_{n\to+\infty} \int_{A_\delta} f(\operatorname{div}\varphi_n)^2 dx. \tag{2.15}$$

Now we use Ioffe's Theorem 10, with $n = 2$, $m = 0$, $k = 1$ and the function $\psi : \Omega \times \mathbb{R} \to [0, +\infty]$ defined by $(x, z) \mapsto f(x)z^2$. Since the sequence $\{\operatorname{div}\varphi_n\}_n$ converges to $\operatorname{div}\varphi$ weakly in $L^2(\Omega)$, and then in $L^1(\Omega)$, Ioffe's theorem yields

$$\liminf_{n\to+\infty} \int_{A_\delta} f(\operatorname{div}\varphi_n)^2 dx \geq \int_{A_\delta} f(\operatorname{div}\varphi)^2 dx.$$

Using (2.15) we find

$$\liminf_{n\to+\infty} \int_\Omega f_n(\operatorname{div}\varphi_n)^2 dx \geq \int_{A_\delta} f(\operatorname{div}\varphi)^2 dx,$$

from which, by letting $\delta \to 0^+$, we obtain the inequality (2.14) and the lower semicontinuity of the second group of integral functionals.

Step 3. Then we consider the following term:

$$\int_\Omega (s^2 + \xi_\varepsilon)|\nabla u|^2 dx.$$

Using (2.13) and Theorem 11, up to the extraction of a subsequence, $u_n \to u$ weakly in $W^{1,2}(\Omega)$. Hence the sequence $\{\nabla u_n\}_n$ converges weakly in $L^2(\Omega)$ to ∇u. Since, up to the extraction of a subsequence, $\{s_n\}_n$ converges to s a.e., the lower semicontinuity of the integral functional under consideration follows from Egorov Theorem and Ioffe's Theorem by using the same method of proof of Step 2.

Step 4. We consider the following terms:

$$\eta \int_\Omega (|\nabla s| - \langle \theta, \nabla s \rangle)\, dx,$$

$$\eta \int_\Omega (|\nabla w| - \langle \omega, \nabla w \rangle)\, dx.$$

Using the Green's formula (2.5) and the boundary conditions $[\theta, \nu](x) = [\omega, \nu](x) = 0$ for $x \in \partial\Omega$, we have

$$-\int_\Omega \langle \theta_n, \nabla s_n \rangle dx = \int_\Omega s_n \mathrm{div}(\theta_n) dx,$$

$$-\int_\Omega \langle \omega_n, \nabla w_n \rangle dx = \int_\Omega w_n \mathrm{div}(\omega_n) dx.$$

Then we have

$$\begin{aligned}
-\lim_{n\to+\infty} \int_\Omega \langle \theta_n, \nabla s_n \rangle dx &= \lim_{n\to+\infty} \int_\Omega s_n \mathrm{div}(\theta_n) dx \\
&= \lim_{n\to+\infty} \int_\Omega (s_n - s + s) \mathrm{div}(\theta_n) dx \\
&= \lim_{n\to+\infty} \int_\Omega (s_n - s) \mathrm{div}(\theta_n) dx \\
&+ \lim_{n\to+\infty} \int_\Omega s\, \mathrm{div}(\theta_n) dx.
\end{aligned}$$

Using (2.13) and Theorem 11, up to the extraction of a subsequence, $s_n \to s$ weakly in $W^{2,2}(\Omega)$. Then the sequence $\{s_n\}_n$ converges to s in $L^2(\Omega)$. Therefore, using Hölder's inequality and (2.12), we get

$$\int_\Omega (s_n - s) \mathrm{div}(\theta_n) dx \le \|s_n - s\|_2 \, \|\mathrm{div}(\theta_n)\|_2 \le C \|s_n - s\|_2 \to 0,$$

where C is a positive constant independent of n. Moreover, since $\{\mathrm{div}\, \theta_n\}_n$ converges to $\mathrm{div}\, \theta$ weakly in $L^2(\Omega)$ and $s \in L^2(\Omega)$, we have

$$\lim_{n\to+\infty} \int_\Omega s\, \mathrm{div}(\theta_n) dx = \int_\Omega s\, \mathrm{div}(\theta) dx.$$

Then we obtain

$$-\lim_{n\to+\infty}\int_\Omega \langle\theta_n, \nabla s_n\rangle dx = \int_\Omega s\,\mathrm{div}(\theta)dx = -\int_\Omega \langle\theta, \nabla s\rangle dx. \tag{2.16}$$

Analogously, we obtain

$$-\lim_{n\to+\infty}\int_\Omega \langle\omega_n, \nabla w_n\rangle dx = \int_\Omega w\,\mathrm{div}(\omega)dx = -\int_\Omega \langle\omega, \nabla w\rangle dx. \tag{2.17}$$

Then, by the convergence of $|\nabla s_n|$ and $|\nabla w_n|$ a.e. in Ω, using Fatou's Lemma we have

$$\liminf_{n\to+\infty}\int_\Omega |\nabla s_n|dx \geq \int_\Omega |\nabla s|dx, \qquad \liminf_{n\to+\infty}\int_\Omega |\nabla w_n|dx \geq \int_\Omega |\nabla w|dx. \tag{2.18}$$

Collecting (2.16), (2.17) and (2.18), we obtain the lower semicontinuity of the fourth group of integral functionals.

Step 5. Eventually we consider the following terms:

$$\xi\int_\Omega (\Delta s)^2 dx,$$

$$\xi\int_\Omega (\Delta w)^2 dx.$$

Up to the extraction of a subsequence, $s_n \to s$ and $w_n \to w$ weakly in $W^{2,2}(\Omega)$, hence $\Delta s_n \to \Delta s$ and $\Delta w_n \to \Delta w$ weakly in $L^2(\Omega)$. The lower semicontinuity of the last group of integral functionals then follows from the weak lower semicontinuity of the L^2 norm. This completes the proof of the theorem. □

By using the compactness and lower semicontinuity theorems 11 and 12, we obtain an existence result of minimizers for the functional M_ε.

Theorem 13. *Let* $\varepsilon > 0$ *and* $\rho > 0$ *be fixed. Then the functional* M_ε *admits a minimizer in* $V(\Omega)$.

2.6 Limits whith varying parameters

2.6.1 Limit when the regularization parameter tends to zero

In this section we let the regularization parameter to vary, and we study the behaviour of minimizers of $M_{\varepsilon,\xi}$ when ξ tends to zero. Since we now consider the dependence on the regularization parameter, in the following we do not drop anymore the index ξ from the modified functionals. We prove the following compactness lemma.

Proposition 2. *Let* $\{v_\xi\}_\xi = \{(u_\xi, s_\xi, w_\xi, \theta_\xi, \omega_\xi)\}_\xi \subset V(\Omega)$ *be a sequence such that* v_ξ *minimizes* $M_{\varepsilon,\xi}$ *in* $V(\Omega)$ *for any* $\xi > 0$, *and* $\xi \to 0^+$.

Then there exist $v = (u, s, w, \theta, \omega)$ *such that* $u, s, w \in W^{1,2}(\Omega)$, $\theta, \omega \in W^{1,2}(\mathrm{div}, \Omega)$, *and a subsequence* $\{v_{\xi_h}\}_h = \{(u_{\xi_h}, s_{\xi_h}, w_{\xi_h}, \theta_{\xi_h}, \omega_{\xi_h})\}_h$ *such that* $u_{\xi_h} \to u$, $s_{\xi_h} \to s$, $w_{\xi_h} \to w$ *weakly in* $W^{1,2}(\Omega)$, *and* $\theta_{\xi_h} \to \theta$, $\omega_{\xi_h} \to \omega$ *weakly in* $W^{1,2}(\mathrm{div}, \Omega)$ *as* $h \to +\infty$.

Proof. Since for any $\xi > 0$ we have

$$M_{\varepsilon,\xi}(u_\xi, s_\xi, w_\xi, \theta_\xi, \omega_\xi) \le M_{\varepsilon,\xi}(0, 1, 1, 0, 0) = L,$$

where $L > 0$ is a constant independent of ξ, then $M_{\varepsilon,\xi}(v_\xi)$ is uniformly bounded with respect to ξ.

Arguing as in the proof of Theorem 11, there exist $u \in W^{1,2}(\Omega)$, $\theta, \omega \in W^{1,2}(\mathrm{div}, \Omega)$, and subsequences $\{u_{\xi_h}\}_h, \{\theta_{\xi_h}\}_h, \{\omega_{\xi_h}\}_h$ such that $u_{\xi_h} \to u$ weakly in $W^{1,2}(\Omega)$and $\theta_{\xi_h} \to \theta$, $\omega_{\xi_h} \to \omega$ weakly in $W^{1,2}(\mathrm{div}, \Omega)$ as $h \to +\infty$.

Since $\|s_\xi\|_\infty \le 1$, $\|w_\xi\|_\infty \le 1$ and

$$\int_\Omega |\nabla s_\xi|^2 dx \le \frac{2b_0}{\varepsilon k_\varepsilon} L, \qquad \int_\Omega |\nabla w_\xi|^2 dx \le \frac{4\pi b_0 \beta_\varepsilon}{\varepsilon} L,$$

for any ξ, it follows that the sequences $\{s_\xi\}_\xi$ and $\{w_\xi\}_\xi$ are uniformly bounded in $W^{1,2}(\Omega)$. Then, possibly extracting further subsequences $\{s_{\xi_h}\}_h$ and $\{w_{\xi_h}\}_h$, there exist functions $s, w \in W^{1,2}(\Omega)$ such that $s_{\xi_h} \to s$ and $w_{\xi_h} \to w$ weakly in $W^{1,2}(\Omega)$. This completes the proof of the proposition. □

2.6.2 Limit when the penalization parameter tends to infinity

In this section we let the penalization parameter η to vary, and we study the behaviour of minimizers of $M_{\varepsilon,\xi}$ when ε and ξ are kept fixed, while η tends to infinity. In the sequel of this section, for the sake of simplicity we denote the functional $M_{\varepsilon,\xi}$ by M_η. We set

$$\begin{aligned} Z(\Omega) \;=\; & \big\{(u, s, w, \theta, \omega) \in V(\Omega) : 1 - s, 1 - w \in W_0^{2,2}(\Omega; [0, 1]), \\ & \theta(x) = \frac{\nabla s(x)}{|\nabla s(x)|} \text{ if } \nabla s(x) \neq 0, \text{ otherwise } \theta(x) = 0, \\ & \omega(x) = \frac{\nabla w(x)}{|\nabla w(x)|} \text{ if } \nabla w(x) \neq 0, \text{ otherwise } \omega(x) = 0\big\}. \end{aligned}$$

For any $\upsilon = (u, s, w, \theta, \omega) \in V(\Omega)$ we set

$$\hat{M}(\upsilon) = M_\eta(\upsilon) - \eta \int_\Omega (|\nabla s| - \langle \theta, \nabla s \rangle) dx - \eta \int_\Omega (|\nabla w| - \langle \omega, \nabla w \rangle) dx, \quad (2.19)$$

and for any $(u, s, w, \theta, \omega) \in Z(\Omega)$ we define

$$M_\infty(u, s, w) = \hat{M}(u, s, w, \theta, \omega).$$

The functional M_∞ is a version of the functional G_ε modified by means of the perturbation

$$\xi \int_\Omega (\Delta s)^2 dx + \xi \int_\Omega (\Delta w)^2 dx,$$

and the coefficients ξ_ε, k_ε and λ_ε.

We prove the following convergence result.

Theorem 14. *Let $\{\upsilon_\eta\}_\eta = \{(u_\eta, s_\eta, w_\eta, \theta_\eta, \omega_\eta)\}_\eta \subset V(\Omega)$ be a sequence such that υ_η minimizes M_η in $V(\Omega)$ for any $\eta > 0$, and $\eta \to +\infty$.*

Then, possibly extracting a subsequence, υ_η converges weakly as $\eta \to +\infty$ to $\upsilon = (u, s, w, \theta, \omega) \in Z(\Omega)$, and such that (u, s, w) minimizes M_∞ in $Z(\Omega)$.

Proof. Arguing as in the proof of the compactness theorem 11, there exists $\upsilon = (u, s, w, \theta, \omega) \in V(\Omega)$ such that, possibly extracting a subsequence, υ_η converges weakly to υ as $\eta \to +\infty$.

For any $\eta > 0$ we have

$$M_\eta(\upsilon_\eta) \leq M_\eta(0, 1, 1, 0, 0) = L,$$

where L is a positive constant independent of η. Then it follows

$$\eta \int_\Omega (|\nabla s_\eta| - \langle \theta_\eta, \nabla s_\eta \rangle) dx \leq L,$$
$$\eta \int_\Omega (|\nabla w_\eta| - \langle \omega_\eta, \nabla w_\eta \rangle) dx \leq L,$$

for any η. Since $|\theta| \leq 1$ and $|\omega| \leq 1$ the above integrands are nonnegative, so that we have

$$\lim_{\eta \to +\infty} \int_\Omega (|\nabla s_\eta| - \langle \theta_\eta, \nabla s_\eta \rangle) dx = 0, \quad (2.20)$$

$$\lim_{\eta \to +\infty} \int_\Omega (|\nabla w_\eta| - \langle \omega_\eta, \nabla w_\eta \rangle) dx = 0. \quad (2.21)$$

Arguing as in the proof of the lower semicontinuity theorem 12, it follows

$$\begin{aligned}\lim_{\eta\to+\infty}\int_\Omega(|\nabla s_\eta|-\langle\theta_\eta,\nabla s_\eta\rangle)dx &\geq \liminf_{\eta\to+\infty}\int_\Omega(|\nabla s_\eta|-\langle\theta_\eta,\nabla s_\eta\rangle)dx\\ &\geq \int_\Omega(|\nabla s|-\langle\theta,\nabla s\rangle)dx\geq 0.\end{aligned}$$

Then, using (2.20), we have

$$\int_\Omega(|\nabla s|-\langle\theta,\nabla s\rangle)dx=0,$$

from which, since the integrand is nonnegative, we get

$$|\nabla s|-\langle\theta,\nabla s\rangle=0\quad\text{a.e. in }\Omega.$$

Since $|\theta(x)|\leq 1$ a.e. in Ω we have

$$\theta(x)=\frac{\nabla s(x)}{|\nabla s(x)|}\quad\text{a.e. in }\Omega.$$

Analogously we find

$$\omega(x)=\frac{\nabla w(x)}{|\nabla w(x)|}\quad\text{a.e. in }\Omega.$$

Hence it follows that $\upsilon=(u,s,w,\theta,\omega)\in Z(\Omega)$.

Now for any $\eta>0$ we have

$$\hat{M}(\upsilon)\leq M_\eta(\upsilon)\quad\forall\upsilon=(u,s,w,\theta,\omega)\in V(\Omega),\tag{2.22}$$

$$M_\infty(u,s,w)=M_\eta(u,s,w,\theta,\omega)\quad\forall(u,s,w,\theta,\omega)\in Z(\Omega).\tag{2.23}$$

Moreover, from the proof of the lower semicontinuity theorem 12, for any sequence $\{(u_h,s_h,w_h,\theta_h,\omega_h)\}_h\subset V(\Omega)$ converging weakly to $(u,s,w,\theta,\omega)\in Z(\Omega)$ it follows

$$\liminf_{h\to+\infty}\hat{M}(u_h,s_h,w_h,\theta_h,\omega_h)\geq M_\infty(u,s,w).\tag{2.24}$$

Let now $\upsilon^*=(u^*,s^*,w^*,\theta^*,\omega^*)\in Z(\Omega)$. Since $M_\eta(\upsilon^*)\geq M_\eta(\upsilon_\eta)$ for any η, using (2.22), (2.23) and (2.24) we have

$$\begin{aligned}M_\infty(u^*,s^*,w^*) &= M_\eta(\upsilon^*)\geq\limsup_{\eta\to+\infty}M_\eta(\upsilon_\eta)\geq\liminf_{\eta\to+\infty}M_\eta(\upsilon_\eta)\\ &\geq \liminf_{\eta\to+\infty}\hat{M}(\upsilon_\eta)\geq M_\infty(u,s,w),\end{aligned}$$

from which we get

$$M_\infty(u^*,s^*,w^*)\geq M_\infty(u,s,w).$$

Since $\upsilon^*\in Z(\Omega)$ is arbitrary, then (u,s,w) minimizes M_∞ in $Z(\Omega)$. □

2.7 Γ-convergence to the functional depending on curves and points

For any $(u, s, w, \theta, \omega) \in Z(\Omega)$ we set

$$G_{\varepsilon,\xi}(u, s, w) = M_\infty(u, s, w) = \hat{M}(u, s, w, \theta, \omega),$$

hence

$$G_{\varepsilon,\xi}(u, s, w) = G_\varepsilon(u, s, w) + \xi \int_\Omega (\Delta s)^2 dx + \xi \int_\Omega (\Delta w)^2 dx.$$

We prove the following Γ-convergence theorem.

Theorem 15. *Let Ω be a star-shaped bounded open set, and let $\zeta_\varepsilon = O(\varepsilon^{1+p})$ with $p > 1$, $k_\varepsilon = o(\varepsilon|\log \varepsilon|)$, and $\xi_\varepsilon = O(\varepsilon^{3+q})$, with $q > 1$.*

Then there exists $\wp > 0$ and a family of infinitesimals $\{\lambda_\varepsilon = o(\varepsilon^\wp)\}_\varepsilon$ such that the functionals $G_{\varepsilon,\xi}$ Γ-converge to G as $\varepsilon \to 0^+$.

Moreover, if $\{\varepsilon_h\}_h$ is a sequence of positive numbers converging to zero, and $\{(u_h, s_h, w_h)\}_h \subset Z(\Omega)$ is a sequence of minimizers of $G_{\varepsilon,\xi}$, then there exist a subsequence $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ and a minimizer (u, C, P) of G such that $\{(u_{h_k}, s_{h_k}, w_{h_k})\}_k$ converges weakly to (u, C, P).

Proof. Since $G_{\varepsilon,\xi}(u, s, w) \geq G_\varepsilon(u, s, w)$ for any $\{(u, s, w)\}_h \subset W(\Omega)$, the lower inequality of Γ-convergence follows immediately from the Γ-convergence of the functionals G_ε, which has been proved in [12]. Hence, only the upper inequality of Γ-convergence has to be proved.

Let $\{\varepsilon_h\}_h$ be a sequence of positive numbers converging to zero. We prove that, for every $(u, C, P) \in X(\Omega)$, there exists $\{(u_h, s_h, w_h)\}_h \subset W(\Omega)$ converging weakly to (u, C^*, P^*) such that

$$\limsup_{h \to +\infty} G_{\varepsilon_h,\xi_h}(u_h, s_h, w_h) \leq G(u, C, P),$$

where C^* and P^* are as in (2.6).

First we assume that C^* satisfies the finiteness property, $[C^*] \subset \Omega$, and $P^* \subset \Omega$. We set $k_{\varepsilon_h} = k_h$, $\lambda_{\varepsilon_h} = \lambda_h$, $\mu_{\varepsilon_h} = \mu_h$, $\zeta_{\varepsilon_h} = \zeta_h$, $\xi_{\varepsilon_h} = \xi_h$ and $\beta_{\varepsilon_h} = \beta_h$, with β_h and μ_h satisfying (2.8).

Step 1. Construction of the sequences of sets $\{A_h\}_h$ and $\{D_h\}_h$.

Arguing as in [12], proof of Theorem 6.3, there exists a sequence $C'_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ of families of simple curves of class $\mathcal{C}^\infty$ such that $\{\gamma_h^i\}_h$ converges strongly in $W^{2,2}$ to a curve γ^i for any $i = 1, \dots, m$, the family

$C' = \{\gamma^1, \dots, \gamma^m\}$ is admissible and satisfies the finiteness property, C' and C^* are equivalent and the following properties hold for any h:

$$P(C'_h) = P(C'), \qquad \big([\gamma^i_h] \cap [\gamma^j_h]\big) \setminus P(C') = \emptyset \quad \text{for all } i, j,\ i \neq j. \tag{2.25}$$

Let $C'_h = \{\gamma^1_h, \dots, \gamma^m_h\}$ with m independent of h. Since the curves γ^i_h are of class $\mathcal{C}^\infty$ and are converging strongly in $W^{2,2}$, for any $p \in P(C')$ and any curve $\gamma^i_h \in C'_h$ having p as an endpoint, the following properties hold for any h large enough:

(i) if γ^i_h is not a closed curve, then $[\gamma^i_h]$ intersects transversally $\partial B_r(p)$ in only one point for any $r \leq \beta_h$;

(ii) if γ^i_h is a closed curve, then $[\gamma^i_h]$ intersects transversally $\partial B_r(p)$ in only two points for any $r \leq \beta_h$.

Here $B_r(p)$ denotes the open ball with center p and radius r. Then we define

$$D_h = \bigcup \{B_{\beta_h}(p) :\ p \in P^*\}. \tag{2.26}$$

Moreover, for h large enough and any regular closed curve $\gamma^i_h \in C'_h$ we have $[\gamma^i_h] \cap D_h = \emptyset$.

For any set $A \subset \mathbb{R}^2$ let δ_A denote the signed distance function from ∂A negative inside A:

$$\delta_A(x) = \mathrm{dist}(x, A) - \mathrm{dist}(x, \mathbb{R}^2 \setminus A).$$

We set

$$D^0_h = \{x \in \Omega : \delta_{D_h}(x) < -2\varepsilon_h |\log \varepsilon_h|\}.$$

Since $\gamma^i_h \to \gamma^i$ strongly in $W^{2,2}$ for any $i = 1, \dots, m$, using (2.25) and properties (i) and (ii), we may find m sequences of sets $\{A^i_h\}_h \subset \mathcal{C}^\infty_c(\Omega)$ such that $\mathrm{meas}(A^i_h) \to 0$ for any i, and the following properties hold for any $i = 1, \dots, m$ and for any h:

$$\begin{aligned} &[\gamma^i_h] \setminus D_h \subset A^i_h, \qquad \overline{A}^i_h \cap \overline{A}^j_h = \emptyset \quad \text{for all } i \neq j, \\ &\partial A^i_h \setminus D^0_h \ = \ [\gamma^{i+}_h] \cup [\gamma^{i-}_h], \end{aligned} \tag{2.27}$$

where γ^{i+}_h and γ^{i-}_h are simple curves of class $\mathcal{C}^\infty$ such that $[\gamma^{i+}_h] \cap [\gamma^{i-}_h] = \emptyset$, and $\gamma^{i+}_h \to \gamma^i$ and $\gamma^{i-}_h \to \gamma^i$ strongly in $W^{2,2}$. Then we set

$$A_h = \bigcup_{i=1}^{m} A^i_h.$$

For h large enough we have $A_h, D_h \in \mathcal{C}_c^\infty(\Omega)$, and

$$\{x \in \Omega : \mathrm{dist}(x, A_h) < 2\varepsilon_h|\log\varepsilon_h|\} \subset\subset \Omega,$$
$$\{x \in \Omega : \mathrm{dist}(x, D_h) < 2\varepsilon_h|\log\varepsilon_h|\} \subset\subset \Omega.$$

Using (2.27), for any h we may write

$$\partial A_h \setminus D_h^0 = \bigcup_{i=1}^m [\gamma_h^{i+}] \cup [\gamma_h^{i-}]. \tag{2.28}$$

Moreover, the curves $\gamma_h^{i\pm}$ can be chosen in such a way that

$$\{x \in \Omega : \mathrm{dist}(x, [\gamma_h^{i\pm}]) \leq 3\varepsilon_h|\log\varepsilon_h|\} \cap \{x \in \Omega : \mathrm{dist}(x, [\gamma_h^{j\pm}]) \leq 3\varepsilon_h|\log\varepsilon_h|\} = \emptyset \tag{2.29}$$

for any $i \neq j$, $i, j \in \{1, \dots, m\}$, and for large enough h. Furthermore, we choose the sequence $\{A_h\}_h$ in such a way that

$$\frac{1}{\kappa(x)} = O(\varepsilon_h|\log\varepsilon_h|) \quad \text{for } x \in \partial A_h \cap D_h^0, \qquad \mathcal{H}^1(\partial A_h \cap D_h^0) = O(\varepsilon_h|\log\varepsilon_h|), \tag{2.30}$$

and

$$\lim_{h \to +\infty} \frac{\mathrm{meas}(A_h)}{\beta_h} = 0. \tag{2.31}$$

Step 2. Construction of an optimal profile function φ_h and of sequences $\{w_h\}_h$, $\{s_h\}_h$.

For any $h \in \mathbb{N}$ let $V_h(t) = V(t) + \varepsilon_h\lambda_h$, and let $\varphi_h^{(0)}$ be the solution of the ordinary differential equation

$$\frac{d\varphi_h^{(0)}}{dt} = \sqrt{V_h(\varphi_h^{(0)}(t))}, \qquad \varphi_h^{(0)}(0) = \frac{1}{2}. \tag{2.32}$$

The function $\varphi_h^{(0)}$ has the following properties: $\varphi_h^{(0)}$ is defined on an open bounded interval

$$(-a_h, a_h) \subset \mathbb{R}, \qquad a_h > 0, \quad \lim_{h \to +\infty} a_h = +\infty,$$

$\varphi_h^{(0)}$ is monotone increasing on $(-a_h, a_h)$,

$$\lim_{t \to \pm a_h} \varphi_h^{(0)}(t) = \pm\infty \quad \forall h \in \mathbb{N}, \qquad \lim_{h \to +\infty} \varphi_h^{(0)}(t) = \varphi(t) \quad \forall t \in \mathbb{R},$$

where

$$\frac{d\varphi}{dt} = \sqrt{V(\varphi(t))}, \qquad \varphi(t) = \frac{1}{2}\left(1 + \tanh\left(\frac{t}{2}\right)\right),$$

and $\varphi_h^{(0)}(-t) = 1 - \varphi_h^{(0)}(t)$. Furthermore, we choose $\wp$, and then the sequence $\{\lambda_h\}_h$, in such a way that the following properties hold:

$$2|\log\varepsilon_h| \in (0, a_h) \quad \forall h \in \mathbb{N}, \qquad \lim_{h\to+\infty} \varphi_h^{(0)}\Big(\frac{t}{\varepsilon_h}\Big) = \lim_{h\to+\infty} \varphi\Big(\frac{t}{\varepsilon_h}\Big) = 1 \ \forall t > 0, \tag{2.33}$$

there exist $C > 0$ independent of h and $h_0 \in \mathbb{N}$, such that for any $h > h_0$ we have

$$\begin{aligned} \Big|\varphi_h^{(0)}(t) - 1\Big| &\leq C\varepsilon_h \qquad \forall t \in (|\log\varepsilon_h|, 2|\log\varepsilon_h|), \\ \Big|\varphi_h^{(0)}(t)\Big| &\leq C\varepsilon_h \qquad \forall t \in (-2|\log\varepsilon_h|, -|\log\varepsilon_h|). \end{aligned} \tag{2.34}$$

For any $h \in \mathbb{N}$ let $\sigma_h : [0, +\infty) \to [0, 1]$ be a $\mathcal{C}^\infty$ function such that

$$\sigma_h = 1 \quad \text{on} \quad [0, |\log\varepsilon_h|], \qquad \sigma_h = 0 \quad \text{on} \quad [2|\log\varepsilon_h|, +\infty),$$

$$\sigma_h' < 0 \quad \text{in} \quad (|\log\varepsilon_h|, 2|\log\varepsilon_h|), \qquad \|\sigma_h'\|_{L^\infty([|\log\varepsilon_h|, 2|\log\varepsilon_h|])} = O(1/|\log\varepsilon_h|).$$

As in [12], we set

$$\varphi_h(t) = \begin{cases} \varphi_h^{(0)}\Big(\dfrac{t}{\varepsilon_h}\Big)\sigma_h\Big(\dfrac{t}{\varepsilon_h}\Big) + 1 - \sigma_h\Big(\dfrac{t}{\varepsilon_h}\Big) & \text{if } 0 \leq t \leq 2\varepsilon_h|\log\varepsilon_h|, \\ \varphi_h(t) = 1 & \text{if } t > 2\varepsilon_h|\log\varepsilon_h|, \\ 1 - \varphi_h(-t) & \text{if } t < 0. \end{cases}$$

We now construct the sequences of functions $\{s_h\}_h$ and $\{w_h\}_h$. We define

$$s_h(x) = \varphi_h(\delta_{A_h}(x)), \qquad w_h(x) = \varphi_h(\delta_{D_h}(x)) \qquad \text{for all } x \in \Omega. \tag{2.35}$$

Step 3. Estimate from above of $\limsup_{h\to+\infty} G^{(1)}_{\varepsilon_h,\xi_h}(w_h)$.

We set

$$\begin{aligned} D_h^1 &= \{x \in \Omega : |\delta_{D_h}(x)| < \varepsilon_h|\log\varepsilon_h|\}, \\ D_h^2 &= \{x \in \Omega : \varepsilon_h|\log\varepsilon_h| < |\delta_{D_h}(x)| < 2\varepsilon_h|\log\varepsilon_h|\}. \end{aligned}$$

Using the definition of $G^{(1)}_{\varepsilon,\xi}$ we have

$$\begin{aligned} G^{(1)}_{\varepsilon_h,\xi_h}(w_h) = & \int_{D_h^1} \Big(\frac{1}{\beta_h} + \beta_h\kappa^2(\nabla w_h)\Big)\left(\varepsilon_h|\nabla w_h|^2 + \frac{V(w_h)}{\varepsilon_h} + \lambda_h\right) dx \qquad (2.36) \\ & + \int_{D_h^2} \Big(\frac{1}{\beta_h} + \beta_h\kappa^2(\nabla w_h)\Big)\left(\varepsilon_h|\nabla w_h|^2 + \frac{V(w_h)}{\varepsilon_h} + \lambda_h\right) dx = I_h + II_h. \end{aligned}$$

Using (2.32), the equality $|\nabla w_h| = \left|\frac{d\varphi_h}{dt}(\delta_{D_h})\right|$, and the coarea formula we get for h large enough

$$\begin{aligned} I_h &= 2\int_{D_h^1} |\nabla w_h| \sqrt{V_h(w_h)}\Big(\frac{1}{\beta_h} + \beta_h \kappa^2(\nabla w_h)\Big)dx \\ &= 2\int_{-\varepsilon_h|\log\varepsilon_h|}^{\varepsilon_h|\log\varepsilon_h|} \frac{d\varphi_h}{dt}\sqrt{V_h(\varphi_h(t))}\int_{\{\delta_{D_h}=t\}}\Big(\frac{1}{\beta_h}+\beta_h\kappa^2\Big)d\mathcal{H}^1 dt. \end{aligned}$$

Then we get

$$I_h = 2\int_{-|\log\varepsilon_h|}^{|\log\varepsilon_h|} \frac{d\varphi_h^{(0)}}{d\theta}\sqrt{V_h(\varphi_h^{(0)}(\theta))}\int_{\{\delta_{D_h}=\varepsilon_h\theta\}}\Big(\frac{1}{\beta_h}+\beta_h\kappa^2\Big)d\mathcal{H}^1 d\theta.$$

Since for h large enough we have (see the proof of [9] Theorem 4.3)

$$\int_{\{\delta_{D_h}=\theta\}}\Big(\frac{1}{\beta_h}+\beta_h\kappa^2\Big)d\mathcal{H}^1 = \int_{\partial D_h}\Big(\frac{1}{\beta_h}+\beta_h\kappa^2\Big)d\mathcal{H}^1 + O(\varepsilon_h|\log\varepsilon_h|),$$

it follows

$$\begin{aligned} I_h &= 2\int_{\varphi_h^{(0)}(-|\log\varepsilon_h|)}^{\varphi_h^{(0)}(|\log\varepsilon_h|)} \sqrt{V_h(\tau)}d\tau \int_{\partial D_h}\Big(\frac{1}{\beta_h}+\beta_h\kappa^2\Big)d\mathcal{H}^1 \\ &+ O(\varepsilon_h|\log\varepsilon_h|)\int_{\varphi_h^{(0)}(-|\log\varepsilon_h|)}^{\varphi_h^{(0)}(|\log\varepsilon_h|)} \sqrt{V_h(\tau)}d\tau. \end{aligned}$$

Using (2.34) we have

$$\lim_{h\to+\infty}\varphi_h^{(0)}(-|\log\varepsilon_h|) = 0, \qquad \lim_{h\to+\infty}\varphi_h^{(0)}(|\log\varepsilon_h|) = 1,$$

from which it follows

$$\lim_{h\to+\infty}\int_{\varphi_h^{(0)}(-|\log\varepsilon_h|)}^{\varphi_h^{(0)}(|\log\varepsilon_h|)} \sqrt{V_h(\tau)}d\tau = \int_0^1 \sqrt{V(\tau)}d\tau. \tag{2.37}$$

Then we have

$$\lim_{h\to+\infty} I_h = 2\int_0^1\sqrt{V(\tau)}d\tau \lim_{h\to+\infty}\int_{\partial D_h}\Big(\frac{1}{\beta_h}+\beta_h\kappa^2\Big)d\mathcal{H}^1,$$

and, from the proof of Theorem 6.3 of [12], we have

$$\lim_{h\to+\infty}\int_{\partial D_h}\Big(\frac{1}{\beta_h}+\beta_h\kappa^2\Big)d\mathcal{H}^1 = 4\pi\#P^*,$$

from which we obtain

$$\lim_{h\to+\infty} I_h = 8\pi\#P^* \int_0^1 \sqrt{V(\tau)}d\tau. \tag{2.38}$$

We now consider the term II_h. Again using the coarea formula we get

$$\begin{aligned} II_h &= \int_{\varepsilon_h|\log\varepsilon_h|}^{2\varepsilon_h|\log\varepsilon_h|} \left[\varepsilon_h(d\varphi_h/dt)^2 + V_h(\varphi_h(t))/\varepsilon_h\right] \int_{\{\delta_{D_h}=t\}} \Big(\frac{1}{\beta_h} + \beta_h\kappa^2\Big) d\mathcal{H}^1 dt \\ &+ \int_{-2\varepsilon_h|\log\varepsilon_h|}^{-\varepsilon_h|\log\varepsilon_h|} \left[\varepsilon_h(d\varphi_h/dt)^2 + V_h(\varphi_h(t))/\varepsilon_h\right] \int_{\{\delta_{D_h}=t\}} \Big(\frac{1}{\beta_h} + \beta_h\kappa^2\Big) d\mathcal{H}^1 dt \\ &= O(1) \int_{\varepsilon_h|\log\varepsilon_h|}^{2\varepsilon_h|\log\varepsilon_h|} \left[\varepsilon_h(d\varphi_h/dt)^2 + V_h(\varphi_h(t))/\varepsilon_h\right] dt. \end{aligned}$$

Using the definition of φ_h, the properties of the function σ_h and (2.34), for $t \in (\varepsilon_h|\log\varepsilon_h|, 2\varepsilon_h|\log\varepsilon_h|)$ we have

$$\begin{aligned} V_h(\varphi_h(t)) &= \varphi_h^2(t)(1-\varphi_h(t))^2 + \varepsilon_h\lambda_h \\ &= \Big[1 + \sigma_h(t/\varepsilon_h)(\varphi_h^{(0)}(t/\varepsilon_h) - 1)\Big]^2 \sigma_h^2(t/\varepsilon_h)(\varphi_h^{(0)}(t/\varepsilon_h) - 1)^2 + \varepsilon_h\lambda_h \\ &\leq (1 + C\varepsilon_h)^2(C\varepsilon_h)^2 + \varepsilon_h\lambda_h = O(\varepsilon_h^2), \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} \frac{d\varphi_h}{dt} &= \frac{1}{\varepsilon_h}\frac{d\varphi_h^{(0)}}{dt}(t/\varepsilon_h)\sigma_h(t/\varepsilon_h) + \frac{1}{\varepsilon_h}\frac{d\sigma_h}{dt}(t/\varepsilon_h)(\varphi_h^{(0)}(t/\varepsilon_h) - 1) \\ &= \frac{1}{\varepsilon_h}\sigma_h(t/\varepsilon_h)\sqrt{(\varphi_h^{(0)}(t/\varepsilon_h))^2(1-\varphi_h^{(0)}(t/\varepsilon_h))^2 + \varepsilon_h\lambda_h} \\ &+ \frac{1}{\varepsilon_h}\frac{d\sigma_h}{dt}(t/\varepsilon_h)(\varphi_h^{(0)}(t/\varepsilon_h) - 1), \end{aligned}$$

from which we get

$$\left|\frac{d\varphi_h}{dt}\right| = \frac{1}{\varepsilon_h}\sigma_h(t/\varepsilon_h)O(\varepsilon_h) + \frac{1}{\varepsilon_h}O(\varepsilon_h/|\log\varepsilon_h|). \tag{2.40}$$

Using (2.39) and (2.40) we obtain

$$II_h = O(1)\int_{\varepsilon_h|\log\varepsilon_h|}^{2\varepsilon_h|\log\varepsilon_h|} \left[\varepsilon_h(d\varphi_h/dt)^2 + V_h(\varphi_h(t))/\varepsilon_h\right] dt = o(1).$$

Then, using (2.36) and (2.38) we have

$$\lim_{h\to+\infty} G^{(1)}_{\varepsilon_h,\xi_h}(w_h) = 4\pi b_0 \# P^*. \tag{2.41}$$

Step 4. Estimate from above of $\limsup_{h\to+\infty} G^{(2)}_{\varepsilon_h,\xi_h}(w_h)$.
We now set

$$\begin{aligned} A_h^1 &= \{x\in\Omega : |\delta_{A_h}(x)| < \varepsilon_h|\log\varepsilon_h|\}, \\ A_h^2 &= \{x\in\Omega : \varepsilon_h|\log\varepsilon_h| < |\delta_{A_h}(x)| < 2\varepsilon_h|\log\varepsilon_h|\}. \end{aligned}$$

Using the definition of $G^{(2)}_{\varepsilon,\xi}$ we have

$$\begin{aligned} G^{(2)}_{\varepsilon_h,\xi_h}(s_h,w_h) &= \int_{A_h^1}(w_h^2+k_h)\left(1+\kappa^2(\nabla s_h)\right)\left(\varepsilon_h|\nabla s_h|^2+\frac{V(s_h)}{\varepsilon_h}+\lambda_h\right)dx \\ &\quad + \int_{A_h^2}(w_h^2+k_h)\left(1+\kappa^2(\nabla s_h)\right)\left(\varepsilon_h|\nabla s_h|^2+\frac{V(s_h)}{\varepsilon_h}+\lambda_h\right)dx. \end{aligned}$$

We set

$$\begin{aligned} &\int_{A_h^1} w_h^2\left(1+\kappa^2(\nabla s_h)\right)\left(\varepsilon_h|\nabla s_h|^2+\frac{V(s_h)}{\varepsilon_h}+\lambda_h\right)dx \\ &+\int_{A_h^2} w_h^2\left(1+\kappa^2(\nabla s_h)\right)\left(\varepsilon_h|\nabla s_h|^2+\frac{V(s_h)}{\varepsilon_h}+\lambda_h\right)dx = \widetilde{I}_h + \widetilde{II}_h. \end{aligned}$$

Using (2.32), (2.34), the equality $|\nabla s_h| = \left|\frac{d\varphi_h}{dt}(\delta_{A_h})\right|$, the coarea formula, and arguing as in the estimate of I_h, we get for h large enough

$$\begin{aligned} \widetilde{I}_h &\le 2(1+C\varepsilon_h)^2\int_{A_h^1\setminus D_h^0}|\nabla s_h|\sqrt{V_h(s_h)}(1+\kappa^2(\nabla s_h))dx \\ &= 2(1+C\varepsilon_h)^2\int_{\varphi_h^{(0)}(-|\log\varepsilon_h|)}^{\varphi_h^{(0)}(|\log\varepsilon_h|)}\sqrt{V_h(\tau)}\int_{\{s_h=\tau\}\setminus D_h^0}(1+\kappa^2)\,d\mathcal{H}^1 d\tau. \end{aligned} \tag{2.42}$$

Then it follows that

$$\begin{aligned} \widetilde{I}_h &\le 2(1+C\varepsilon_h)^2\int_{\varphi_h^{(0)}(-|\log\varepsilon_h|)}^{\varphi_h^{(0)}(|\log\varepsilon_h|)}\sqrt{V_h(\tau)}\int_{\partial A_h\setminus D_h^0}(1+\kappa^2)\,d\mathcal{H}^1 d\tau \\ &+ O(\varepsilon_h|\log\varepsilon_h|)\int_{\varphi_h^{(0)}(-|\log\varepsilon_h|)}^{\varphi_h^{(0)}(|\log\varepsilon_h|)}\sqrt{V_h(\tau)}d\tau, \end{aligned}$$

from which, arguing as in the estimate of I_h, we obtain

$$\limsup_{h\to+\infty} \widetilde{I}_h \leq 2\int_0^1 \sqrt{V(\tau)}d\tau \lim_{h\to+\infty} \int_{\partial A_h \setminus D_h^0} \Big(1+\kappa^2\Big) d\mathcal{H}^1.$$

Using (2.28), from the proof of Theorem 6.3 of [12], we have

$$\begin{aligned}\limsup_{h\to+\infty} \widetilde{I}_h &\leq 2\int_0^1 \sqrt{V(\tau)}d\tau \lim_{h\to+\infty} \sum_{i=1}^m \big(K(\gamma_h^{i+}) + L(\gamma_h^{i+}) + K(\gamma_h^{i-}) + L(\gamma_h^{i-})\big) \\ &= 4\sum_{\gamma\in C^*} \big(K(\gamma)+L(\gamma)\big) \int_0^1 \sqrt{V(\tau)}d\tau. \qquad (2.43)\end{aligned}$$

Analogously we have that $\lim_h \widetilde{II}_h = 0$. We now set

$$\begin{aligned}&k_h \int_{A_h^1} \big(1+\kappa^2(\nabla s_h)\big) \left(\varepsilon_h |\nabla s_h|^2 + \frac{V(s_h)}{\varepsilon_h} + \lambda_h\right) dx \\ &+k_h \int_{A_h^2} \big(1+\kappa^2(\nabla s_h)\big) \left(\varepsilon_h |\nabla s_h|^2 + \frac{V(s_h)}{\varepsilon_h} + \lambda_h\right) dx = \widehat{I}_h + \widehat{II}_h.\end{aligned}$$

Using (2.42) we get

$$\widehat{I}_h = 2k_h \int_{\varphi_h^{(0)}(-|\log\varepsilon_h|)}^{\varphi_h^{(0)}(|\log\varepsilon_h|)} \sqrt{V_h(\tau)} \int_{\{s_h=\tau\}} (1+\kappa^2)\, d\mathcal{H}^1 d\tau.$$

Since the curves $\gamma_h^{i+}, \gamma_h^{i-}$ converge strongly to γ^i in $W^{2,2}$ for any $i = 1, \ldots, m$, using (2.29), (2.30) and taking into account that $k_h = o(\varepsilon_h |\log\varepsilon_h|)$ we find

$$k_h \int_{\{s_h=\tau\}} (1+\kappa^2)\, d\mathcal{H}^1 = o(1) \quad \forall \tau \in (\varphi_h^{(0)}(-|\log\varepsilon_h|), \varphi_h^{(0)}(|\log\varepsilon_h|)).$$

Using (2.37) it then follows

$$\lim_{h\to+\infty} \widehat{I}_h = 0.$$

Analogously we have that $\lim_h \widehat{II}_h = 0$.

Using (2.43) we have

$$\limsup_{h\to+\infty} G^{(2)}_{\varepsilon_h,\xi_h}(s_h, w_h) \leq 2b_0 \sum_{\gamma\in C^*} \big(K(\gamma)+L(\gamma)\big). \qquad (2.44)$$

Step 5. Estimates of $\lim_{h\to+\infty} \xi_h \int_\Omega (\Delta w_h)^2 dx$ *and* $\lim_{h\to+\infty} \xi_h \int_\Omega (\Delta s_h)^2 dx$.

Using the definition (2.35) of w_h, for any $x \in D_h^1 \cup D_h^2$ we have for h large enough

$$\begin{aligned} \Delta w_h(x) &= \frac{d^2\varphi_h}{dt^2}(\delta_{D_h}(x)) + \frac{d\varphi_h}{dt}(\delta_{D_h}(x))\Delta\delta_{D_h}(x) \\ &= \frac{d^2\varphi_h}{dt^2}(\delta_{D_h}(x)) + \kappa_h^t(x)\frac{d\varphi_h}{dt}(\delta_{D_h}(x)), \end{aligned} \tag{2.45}$$

where $\kappa_h^t(x)$ is the curvature of the level set $\{x : \delta_{D_h}(x) = t\}$. Using the definition of φ_h, for $t \in (0, 2\varepsilon_h|\log\varepsilon_h|)$ we have

$$\begin{aligned} \varepsilon_h^2\frac{d^2\varphi_h}{dt^2} &= \frac{d^2\varphi_h^{(0)}}{dt^2}(t/\varepsilon_h)\sigma_h(t/\varepsilon_h) + 2\frac{d\varphi_h^{(0)}}{dt}(t/\varepsilon_h)\frac{d\sigma_h}{dt}(t/\varepsilon_h) \\ &+ \frac{d^2\sigma_h}{dt^2}(t/\varepsilon_h)(\varphi_h^{(0)}(t/\varepsilon_h) - 1). \end{aligned} \tag{2.46}$$

Using (2.32) we find

$$\frac{d^2\varphi_h^{(0)}}{dt^2} = \frac{1}{2}\frac{dV_h}{dt}(\varphi_h^{(0)}) = \varphi_h^{(0)}(1-\varphi_h^{(0)})(1-2\varphi_h^{(0)}),$$

from which, using (2.34), we get

$$\|d^2\varphi_h^{(0)}/dt^2\|_{L^\infty([0,2|\log\varepsilon_h|])} = O(1).$$

Using (2.32), (2.34), (2.46) and the properties of the function σ_h we get

$$\|d^2\varphi_h/dt^2\|_{L^\infty([0,2\varepsilon_h|\log\varepsilon_h|])} = \frac{1}{\varepsilon_h^2}O(1). \tag{2.47}$$

Using the coarea formula and (2.45), since the level sets $\{x : \delta_{D_h}(x) = t\}$ are circles, we get

$$\xi_h \int_\Omega (\Delta w_h)^2 dx = \xi_h \int_0^{2\varepsilon_h|\log\varepsilon_h|} \left[d^2\varphi_h/dt^2 + \kappa_h^t d\varphi_h/dt\right]^2 \mathcal{H}^1(\{x : \delta_{D_h}(x) = t\})dt.$$

Since $1/\kappa_h^t = O(\beta_h)$ for any $t \in (0, 2\varepsilon_h|\log\varepsilon_h|)$, using (2.47) it follows

$$\begin{aligned} \xi_h \int_\Omega (\Delta w_h)^2 dx &= \frac{\xi_h}{\varepsilon_h^4}O(1)\int_0^{2\varepsilon_h|\log\varepsilon_h|} \mathcal{H}^1(\{x : \delta_{D_h}(x) = t\})dt \\ &= \frac{\xi_h}{\varepsilon_h^3}O(\beta_h|\log\varepsilon_h|). \end{aligned}$$

Since $\xi_h = O(\varepsilon_h^{3+q})$ with $q > 1$, it follows

$$\lim_{h\to+\infty} \xi_h \int_\Omega (\Delta w_h)^2 dx = 0. \tag{2.48}$$

Using the definition (2.35) of s_h, for any $x \in A_h^1 \cup A_h^2$ we have for h large enough

$$\Delta s_h(x) = \frac{d^2\varphi_h}{dt^2}(\delta_{D_h}(x)) + \kappa_h^t(x)\frac{d\varphi_h}{dt}(\delta_{D_h}(x)), \tag{2.49}$$

where $\kappa_h^t(x)$ now denotes the curvature of the level set $\{x : \delta_{A_h}(x) = t\}$. We have

$$\begin{aligned} \xi_h \int_\Omega (\Delta s_h)^2 dx &= \xi_h \int_{A_h^1 \cup A_h^2} \left[\frac{d^2\varphi_h}{dt^2}(\delta_{A_h}(x)) + \kappa_h^t(x)\frac{d\varphi_h}{dt}(\delta_{A_h}(x))\right]^2 dx \\ &\leq 2\xi_h \int_{A_h^1 \cup A_h^2} \left[\frac{d^2\varphi_h}{dt^2}(\delta_{A_h}(x))\right]^2 dx + 2\xi_h \int_{A_h^1 \cup A_h^2} \left[\kappa_h^t(x)\frac{d\varphi_h}{dt}(\delta_{A_h}(x))\right]^2 dx, \end{aligned}$$

from which, using the coarea formula it follows

$$\begin{aligned} \xi_h \int_\Omega (\Delta s_h)^2 dx &\leq 2\xi_h \int_0^{2\varepsilon_h|\log\varepsilon_h|} \left[d^2\varphi_h/dt^2\right]^2 \mathcal{H}^1(\{x : \delta_{A_h}(x) = t\})dt \\ &+ 2\xi_h \int_0^{2\varepsilon_h|\log\varepsilon_h|} [d\varphi_h/dt]^2 \int_{\{\delta_{A_h}=t\}} \kappa^2 d\mathcal{H}^1\, dt. \end{aligned}$$

Using (2.30) and (2.47) we get

$$\xi_h \int_\Omega (\Delta s_h)^2 dx \leq \xi_h \left(\frac{1}{\varepsilon_h^4} O(\varepsilon_h|\log\varepsilon_h|) + \frac{1}{\varepsilon_h^2} O(1)\right),$$

from which, since $\xi_h = O(\varepsilon_h^{3+q})$ with $q > 1$, it follows

$$\lim_{h\to+\infty} \xi_h \int_\Omega (\Delta s_h)^2 dx = 0. \tag{2.50}$$

Step 6. Estimate from above of $\limsup_{h\to+\infty} \int_\Omega (s_h^2 + \zeta_h)\,|\nabla u_h|^2 dx$.
We set

$$\mathcal{K}_h = \textstyle\bigcup_{i=1}^m [\gamma_h^i] \setminus D_h.$$

The sequence of compact sets $\{\mathcal{K}_h\}_h$ converges in the Hausdorff metric to the set $[C'] = [C^*]$; moreover the number of connected components of $\mathcal{K}_h$ is m for all h and $\sup_{h\in\mathbb{N}} \mathcal{H}^1(\mathcal{K}_h) < +\infty$. Then, since $u \in W^{1,2}(\Omega \setminus [C^*])$, using a result by Chambolle and Doveri ([13], Proposition 1 of Appendix), there exists a sequence $\{\widehat{u}_h\}_h$ with $\widehat{u}_h \in W^{1,2}(\Omega \setminus \mathcal{K}_h)$ for any h, such that $\widehat{u}_h \to u$ strongly in $L^2(\Omega)$ and $\nabla\widehat{u}_h \to \nabla u$ strongly in $L^2(\Omega;\mathbb{R}^2)$.

Let $\{\rho_h\}_h$ be a sequence of positive numbers converging to zero such that $\rho_h = o(\varepsilon_h)$ and

$$\{x \in \Omega : \mathrm{dist}(x, \mathcal{K}_h) < \rho_h\} \subset A_h \quad \text{for any } h.$$

Let then $\{g_h\}_h \subset \mathcal{C}^\infty(\mathbb{R}^2)$ be a sequence of functions such that

$$0 \leq g_h \leq 1, \qquad |\nabla g_h| \leq \frac{C}{\rho_h}, \tag{2.51}$$

with C positive constant independent of h, and

$$g_h(x) = \begin{cases} 1 & \text{on } \{x \in \Omega : \operatorname{dist}(x, \mathcal{K}_h) < \rho_h/2\} \\ 0 & \text{on } \{x \in \Omega : \operatorname{dist}(x, \mathcal{K}_h) \geq \rho_h\} \end{cases} \tag{2.52}$$

for all h. Then we define $u_h = (1 - g_h)\widehat{u}_h$ for any h so that $u_h \in W^{1,2}(\Omega)$.

Since $A_h, D_h \in \mathcal{C}_c^\infty(\Omega)$, by construction we have $\{(u_h, A_h, D_h)\}_h \subset Y(\Omega)$ and $\{(u_h, A_h, D_h)\}_h$ converges weakly to (u, C^*, P^*). Moreover we have $(u_h, s_h, w_h) \in W(\Omega)$ for h large enough and, using (2.33), one can check that $\{(u_h, s_h, w_h)\}_h$ converges weakly to (u, C^*, P^*).

Since $\widehat{u}_h \to u$ in $L^2(\Omega)$ and $\operatorname{meas}(A_h) \to 0$, using the definition of the function g_h, we have $\|u_h - \widehat{u}_h\|_{L^2} \to 0$ and

$$\lim_{h \to +\infty} \int_\Omega |u_h - g|^2 dx = \int_\Omega |u - g|^2 dx. \tag{2.53}$$

We take ρ_h in (2.52) such that $\rho_h = o(\varepsilon_h |\log \varepsilon_h|)$: then we have $u_h(x) = \widehat{u}_h(x)$ if $x \notin A_h^0$, with

$$A_h^0 = \{x \in \Omega : \delta_{A_h}(x) < -2\varepsilon_h |\log \varepsilon_h|\}.$$

Since $\nabla \widehat{u}_h \to \nabla u$ in $L^2(\Omega; \mathbb{R}^2)$, it follows

$$\limsup_{h \to +\infty} \int_\Omega s_h^2 |\nabla u_h|^2 dx \leq \lim_{h \to +\infty} \int_{\Omega \setminus A_h^0} |\nabla \widehat{u}_h|^2 dx = \int_{\Omega \setminus [C^*]} |\nabla u|^2 dx. \tag{2.54}$$

Now for any set $A \subset \mathbb{R}^2$ and any $r > 0$ we set

$$(A)_r = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, A) < r\}.$$

Then we have

$$\zeta_h \int_\Omega |\nabla u_h|^2 dx \leq \zeta_h \int_{\Omega \setminus (\mathcal{K}_h)_{\rho_h/2}} |\nabla u_h|^2 dx \leq 2\zeta_h \int_{\Omega \setminus (\mathcal{K}_h)_{\rho_h/2}} \left(|\nabla \widehat{u}_h|^2 + |\widehat{u}_h \nabla g_h|^2\right) dx.$$

Since $\zeta_h = O(\varepsilon_h^{1+p})$ with $p > 1$, using (2.51), by choosing an infinitesimal $\{\rho_h\}_h$ intermediate between $\varepsilon_h |\log \varepsilon_h|$ and ζ_h (for instance $\rho_h = \varepsilon_h^{1+p'}$ with $0 < p' < p$), we find

$$\lim_{h \to +\infty} \zeta_h \int_\Omega |\nabla u_h|^2 dx = 0. \tag{2.55}$$

Using (2.54) and (2.55), we get

$$\limsup_{h\to+\infty}\int_\Omega \left(s_h^2+\zeta_h\right)|\nabla u_h|^2dx \le \int_{\Omega\setminus[C^*]}|\nabla u|^2dx. \tag{2.56}$$

Step 7. Conclusion of the proof of Γ*-convergence.*

Finally, using (2.8), (2.26) and (2.31) we get

$$\limsup_{h\to+\infty}\frac{1}{\mu_h}\int_\Omega(1-w_h)^2dx \le \lim_{h\to+\infty}\frac{\mathrm{meas}((D_h)_{2\varepsilon_h|\log\varepsilon_h|})}{\mu_h}=0, \tag{2.57}$$

and

$$\limsup_{h\to+\infty}\frac{1}{\mu_h}\int_\Omega(1-s_h)^2dx \le \lim_{h\to+\infty}\frac{\mathrm{meas}((A_h)_{2\varepsilon_h|\log\varepsilon_h|})}{\mu_h}=0. \tag{2.58}$$

The upper inequality of Γ-convergence then follows collecting (2.41), (2.44), (2.48), (2.50), (2.53), (2.56), (2.57) and (2.58).

Eventually, the assumptions that C^* satisfies the finiteness property, $[C^*]\subset\Omega$, and $P^*\subset\Omega$ are removed as in Theorem 6.3 of [12].

Step 8. Convergence of minimizers.

Let $\{(u_h,s_h,w_h)\}_h\subset Z(\Omega)$ be a sequence of minimizers of $G_{\varepsilon,\xi}$, which exist by theorem 14. Since

$$G_\varepsilon(u_h,s_h,w_h)\le G_{\varepsilon,\xi}(u_h,s_h,w_h)$$

for any h, the equi-coerciveness of functionals $G_{\varepsilon,\xi}$ follows immediately from Theorem 5.1 of [12]. Then there exist a subsequence $\{(u_{h_k},s_{h_k},w_{h_k})\}_k$ and a triplet$(u,C,P)\in X(\Omega)$ such that $\{(u_{h_k},s_{h_k},w_{h_k})\}_k$ converges weakly to (u,C,P).

Eventually, using the variational properties of Γ-convergence (see [10] Section 1.5) and the equi-coerciveness of $G_{\varepsilon,\xi}$, it follows that (u,C,P) minimizes G. □

2.8 Euler equations

In the previous section we have proved the existence of minimizers of $M_{\varepsilon,\xi}$. In this section, for the purpose of numerical computations, we derive formally the Euler equations of the functional $M_{\varepsilon,\xi}$. We set

$$M_{\varepsilon,\xi}=M_{\varepsilon,0}+\xi\int_\Omega(\Delta s)^2dx+\xi\int_\Omega(\Delta w)^2dx.$$

First we examine $M_{\varepsilon,0}$ and we compute formally the Euler equations for such a functional, then we add the contribution of the terms involving the squared Laplacian of s and w.

Let us consider an integral functional of the type

$$I(v_1,\dots,v_m)=\int_\Omega f(x,v_1(x),\dots,v_m(x),\nabla v_1(x),\dots,\nabla v_m(x))dx,$$

where $\Omega\subset\mathbb{R}^n$ is an open set with Lipschitz boundary, $v_i:\Omega\to\mathbb{R}$ for $i=1,\dots,m$, and $f:\Omega\times\mathbb{R}^m\times\mathbb{R}^{nm}\to\mathbb{R}$. Minimizers of functional I, if they exist, satisfy the necessary condition that the Gâteaux derivative of I vanishes at minimizers. By computing the Gâteaux derivative of I, we suppose that the functions $(v_1,\dots,v_m)$ formally satisfy the Euler equations

$$\nabla_{v_i}I(v_1,\dots,v_m):=\frac{\partial f}{\partial v_i}-\operatorname{div}\left[\frac{\partial f}{\partial(\nabla v_i)}\right]=0\qquad\text{in }\Omega,\quad i=1,\dots,m,$$

supplemented with the boundary conditions

$$\langle\frac{\partial f}{\partial(\nabla v_i)},\nu_n\rangle=0\qquad\text{on }\partial\Omega,\quad i=1,\dots,m,$$

where ν_n is the unit outer normal of $\partial\Omega$. In our case $n=2$, $m=7$, $I=M_{\varepsilon,0}$, $(v_1,...,v_7)=(u,s,w,\theta_1,\theta_2,\omega_1,\omega_2)$. In the application to image segmentation the various terms in the energy functional have different weights as follows:

$$\begin{aligned}
&M_{\varepsilon,0}(u,s,w,\theta,\omega)=\int_\Omega(s^2+\xi_\varepsilon)|\nabla u|^2dx\\
&+\int_\Omega(w^2+k_\varepsilon)\left[\varepsilon|\nabla s|^2+\frac{1}{\varepsilon}s^2(1-s)^2\right]\left(\alpha+\tau(\operatorname{div}\theta)^2\right)dx\\
&+\alpha_1\int_\Omega\left[\varepsilon|\nabla w|^2+\frac{1}{\varepsilon}w^2(1-w)^2\right]\left(\frac{1}{\beta_\varepsilon}+\beta_\varepsilon(\operatorname{div}\omega)^2\right)dx\\
&+\eta\int_\Omega\left(|\nabla s|-\langle\theta,\nabla s\rangle\right)dx+\eta_1\int_\Omega\left(|\nabla w|-\langle\omega,\nabla w\rangle\right)dx\\
&+\mu\int_\Omega|u-g|^2dx+\frac{\alpha}{\mu_\varepsilon}\int_\Omega(1-s)^2dx+\frac{\alpha_1}{\mu_\varepsilon}\int_\Omega(1-w)^2dx,
\end{aligned}$$

where $\alpha,\tau,\alpha_1,\eta,\eta_1,\mu$ are positive weights. Here we set $\lambda_\varepsilon=0$.

Then we compute formally the Euler equations for an integral functional of the type

$$M_{\varepsilon,0}(u,s,w,\theta,\omega)=\int_\Omega f(u,s,w,\theta,\omega,\nabla u,\nabla s,\nabla w,\operatorname{div}\theta,\operatorname{div}\omega)dx.$$

Equation for u. Since we have

$$\frac{\partial f}{\partial u} = 2\mu(u-g),$$

$$\frac{\partial f}{\partial(\nabla u)} = 2(s^2+\xi_\varepsilon)\nabla u,$$

then the Euler equation and the boundary conditions are

$$\mu(u-g) = \operatorname{div}((s^2+\xi_\varepsilon)\nabla u) \qquad \text{in } \Omega,$$

$$\langle \nabla u, \nu_n\rangle = 0 \qquad \text{on } \partial\Omega.$$

If Ω is a rectangle the boundary conditions require $\frac{\partial u}{\partial x} = 0$ on vertical sides and $\frac{\partial u}{\partial y} = 0$ on horizontal sides. Note that the equation is linear in u.

Equation for s. First we compute the terms $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial(\nabla s)}$:

$$\frac{\partial f}{\partial s} = 2s|\nabla u|^2 + \frac{2}{\varepsilon}(w^2+k_\varepsilon)(2s^3-3s^2+s)\left[\alpha+\tau\,(\operatorname{div}\theta)^2\right] - 2\,\frac{\alpha}{\mu_\varepsilon}(1-s)\,,$$

$$\frac{\partial f}{\partial(\nabla s)} = 2\,\varepsilon\,(w^2+k_\varepsilon)\,\left[\alpha+\tau\,(\operatorname{div}\theta)^2\right]\nabla s + \eta\frac{\nabla s}{|\nabla s|} - \eta\,\theta\,.$$

Then the Euler equation for s is

$$\begin{aligned}
\nabla_s M_{\varepsilon,0} &= 2\left[|\nabla u|^2 + \frac{1}{\varepsilon}(w^2+k_\varepsilon)[\alpha+\tau\,(\operatorname{div}\theta)^2] + \frac{\alpha}{\mu_\varepsilon}\right] s \\
&- 2\,\alpha\,\varepsilon\operatorname{div}\big((w^2+k_\varepsilon)\nabla s\big) \\
&- 2\,\tau\,\varepsilon\operatorname{div}\big((w^2+k_\varepsilon)\,(\operatorname{div}\theta)^2\nabla s\big) - \eta\operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right) \\
&- \frac{2\alpha}{\mu_\varepsilon} - \frac{2}{\varepsilon}(w^2+k_\varepsilon)\,[\alpha+\tau\,(\operatorname{div}\theta)^2](-2\,s^3+3s^2) + \eta\operatorname{div}\theta = 0\,.
\end{aligned}$$

Note that the terms with $(-2s^3+3s^2)$ and $\frac{1}{|\nabla s|}$ are the only terms of the equation which are not linear in s. Since we use the conjugate gradient method, such terms will be replaced by $(-2s_0^3+3s_0^2)$ and $\frac{1}{|\nabla s_0|}$, where s_0 denotes the function s at the previous iteration.

The boundary conditions are

$$\langle 2\,\varepsilon\,(w^2+k_\varepsilon)\,(\alpha+\tau\,(\operatorname{div}\theta)^2)\nabla s + \eta\frac{\nabla s}{|\nabla s|} - \eta\,\theta, \nu_n\rangle = 0.$$

If Ω is a rectangle, on vertical sides we have

$$2\,\varepsilon\,(w^2+k_\varepsilon)\,(\alpha+\tau\,(\operatorname{div}\theta)^2)\frac{\partial s}{\partial x} + \eta\frac{1}{|\nabla s|}\frac{\partial s}{\partial x} - \eta\,\theta_1 = 0,$$

which is satisfied if $\frac{\partial s}{\partial x} = 0$ and $\theta_1 = 0$. Analogously, for the horizontal sides we have

$$2\,\varepsilon\,(w^2 + k_\varepsilon)\,(\alpha + \tau\,(\operatorname{div}\theta)^2)\frac{\partial s}{\partial y} + \eta\frac{1}{|\nabla s|}\frac{\partial s}{\partial y} - \eta\,\theta_2 = 0,$$

which is satisfied if $\frac{\partial s}{\partial y} = 0$ and $\theta_2 = 0$.

Equation for w. This equation is similar to the previous one. We compute the following derivatives:

$$\begin{aligned}\frac{\partial f}{\partial w} &= 2\,(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}\,s^2(1-s)^2)[\alpha + \tau\,(\operatorname{div}\theta)^2]\,w - 2\,\frac{\alpha_1}{\mu_\varepsilon}(1-w) \\ &+ 2\,\frac{\alpha_1}{\varepsilon}\,(2\,w^3 - 3w^2 + w)[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2]\,,\end{aligned}$$

$$\frac{\partial f}{\partial(\nabla w)} = 2\,\varepsilon\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2]\nabla w + \eta_1\frac{\nabla w}{|\nabla w|} - \eta_1\,\omega\,.$$

Then the Euler equation for w is

$$\begin{aligned}\nabla_w M_{\varepsilon,0} &= 2\,(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}\,s^2(1-s)^2)[\alpha + \tau\,(\operatorname{div}\theta)^2]\,w \\ &+ 2\,\frac{\alpha_1}{\mu_\varepsilon}\,w + 2\,\frac{\alpha_1}{\varepsilon}\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2]\,w \\ &- 2\,\varepsilon\left[\frac{1}{\beta_\varepsilon}\Delta w + \beta_\varepsilon\operatorname{div}(\,(\operatorname{div}\omega)^2\nabla w)\right] - \eta_1\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) \\ &- \frac{2\alpha_1}{\mu_\varepsilon} - 2\frac{\alpha_1}{\varepsilon}\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2](-2\,w^3 + 3w^2) + \eta_1\operatorname{div}\omega = 0\,.\end{aligned}$$

Note again that the terms with $(-2w^3 + 3w^2)$ and $\frac{1}{|\nabla w|}$ are the only terms of the equation which are not linear in w. Since we use the conjugate gradient method, such terms will be replace by $(-2w_0^3 + 3w_0^2)$ and $\frac{1}{|\nabla w_0|}$ respectively, where w_0 denoted the function w at the previous iteration.

The boundary conditions are

$$\langle 2\,\alpha_1\,\varepsilon\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2]\nabla w + \eta_1\frac{\nabla w}{|\nabla w|} - \eta_1\,\omega, \nu_n\rangle = 0.$$

If Ω is a rectangle, on vertical sides we have

$$2\,\alpha_1\,\varepsilon\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2]\frac{\partial w}{\partial x} + \eta_1\frac{1}{|\nabla w|}\frac{\partial w}{\partial x} - \eta_1\,\omega_1 = 0,$$

which is satisfied if $\frac{\partial w}{\partial x} = 0$ and $\omega_1 = 0$. Analogously, for the horizontal sides we have

$$2\,\alpha_1\,\varepsilon\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2]\frac{\partial w}{\partial y} + \eta_1\frac{1}{|\nabla w|}\frac{\partial w}{\partial y} - \eta_1\,\omega_2 = 0,$$

which is satisfied if $\frac{\partial w}{\partial y} = 0$ and $\omega_2 = 0$.

Equations for $\theta = (\theta_1, \theta_2)$. Since the Euler equation for θ_i, $i = 1, 2$, is

$$\nabla_{\theta_i} M_{\varepsilon,0} = \frac{\partial f}{\partial \theta_i} - \frac{\partial f}{\partial \nabla\theta_i} = 0,$$

we have

$$\frac{\partial f}{\partial \theta_1} = -\eta\frac{\partial}{\partial \theta_1}\langle\theta, \nabla s\rangle\,.$$

Since

$$\frac{\partial}{\partial \theta_1}\langle\theta, \nabla s\rangle = \frac{\partial}{\partial \theta_1}\,[\theta_1\frac{\partial s}{\partial x} + \theta_2\frac{\partial s}{\partial y}]\; = \frac{\partial s}{\partial x},$$

we obtain

$$\frac{\partial f}{\partial \theta_1} = -\eta\frac{\partial s}{\partial x}\,. \tag{2.59}$$

Analogously, for $i = 2$ we obtain

$$\frac{\partial f}{\partial \theta_2} = -\eta\frac{\partial s}{\partial y}. \tag{2.60}$$

Now we compute

$$\frac{\partial f}{\partial(\nabla\theta_1)} = \tau(w^2 + k_\varepsilon)(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2)\frac{\partial}{\partial(\nabla\theta_1)}(\operatorname{div}\theta)^2,$$

from which we get

$$\frac{\partial}{\partial(\nabla\theta_1)}(\operatorname{div}\theta)^2 = 2\operatorname{div}\theta\frac{\partial}{\partial(\nabla\theta_1)}(\operatorname{div}\theta) = 2\operatorname{div}\theta\,(1,0).$$

Then we have

$$\operatorname{div}\left(\frac{\partial f}{\partial(\nabla\theta_1)}\right) = \frac{\partial}{\partial x}[2\tau(w^2 + k_\varepsilon)(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2)\operatorname{div}\theta]. \tag{2.61}$$

Analogously, for θ_2 we have

$$\operatorname{div}\left(\frac{\partial f}{\partial(\nabla\theta_2)}\right) = \frac{\partial}{\partial y}[2\tau(w^2 + k_\varepsilon)(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2)\operatorname{div}\theta]. \tag{2.62}$$

For reasons of numerical stability, before discretizing the Euler equations, it is convenient to compute explicitly the derivatives appearing in the expressions (2.61) and (2.62). Hence, denoting $R = \varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2$, we find

$$\begin{aligned}
&\frac{\partial}{\partial x}[2\tau(w^2+k_\varepsilon)R\mathrm{div}\theta] = 2\tau\frac{\partial w^2}{\partial x}R\mathrm{div}\theta \\
+&\ \ 2\tau(w^2+k_\varepsilon)\frac{\partial R}{\partial x}\mathrm{div}\theta + 2\tau(w^2+k_\varepsilon)R\frac{\partial}{\partial x}(\mathrm{div}\theta) \\
=&\ \ 2\tau\mathrm{div}\theta\left[2w\frac{\partial w}{\partial x}R + (w^2+k_\varepsilon)\left(2\varepsilon\langle\nabla s, \frac{\partial\nabla s}{\partial x}\rangle + \frac{1}{\varepsilon}(4s^3-6s^2+2s)\frac{\partial s}{\partial x}\right)\right] \\
+&\ \ 2\tau(w^2+k_\varepsilon)R\left[\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_2}{\partial x\partial y}\right],
\end{aligned}$$

where

$$\frac{\partial\nabla s}{\partial x} = (\frac{\partial^2 s}{\partial x^2}, \frac{\partial^2 s}{\partial x\partial y}).$$

The equation for θ_2 is computed by using the same method. Hence, using (2.59) and (2.60), the Euler equations for θ are

$$\begin{aligned}
\nabla_{\theta_1}M_{\varepsilon,0} &= -\eta\frac{\partial s}{\partial x} - 2\tau\mathrm{div}\theta\left[2w\frac{\partial w}{\partial x}R+\right. \\
&+ \left.(w^2+k_\varepsilon)\left(2\varepsilon(\frac{\partial s}{\partial x}\frac{\partial^2 s}{\partial x^2} + \frac{\partial s}{\partial y}\frac{\partial^2 s}{\partial x\partial y}) + \frac{1}{\varepsilon}(4s^3-6s^2+2s)\frac{\partial s}{\partial x}\right)\right] \\
&- 2\tau(w^2+k_\varepsilon)R\left[\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_2}{\partial x\partial y}\right] = 0, \\
\nabla_{\theta_2}M_{\varepsilon,0} &= -\eta\frac{\partial s}{\partial y} - 2\tau\mathrm{div}\theta\left[2w\frac{\partial w}{\partial y}R+\right. \\
&+ \left.(w^2+k_\varepsilon)\left(2\varepsilon(\frac{\partial s}{\partial x}\frac{\partial^2 s}{\partial y\partial x} + \frac{\partial s}{\partial y}\frac{\partial^2 s}{\partial y^2}) + \frac{1}{\varepsilon}(4s^3-6s^2+2s)\frac{\partial s}{\partial y}\right)\right] \\
&- 2\tau(w^2+k_\varepsilon)R\left[\frac{\partial^2\theta_1}{\partial y\partial x} + \frac{\partial^2\theta_2}{\partial y^2}\right] = 0\,.
\end{aligned}$$

Note that the equations are linear in θ_1, θ_2 respectively.

The boundary conditions for θ_1 and θ_2 are given by

$$\langle 2\tau(w^2+k_\varepsilon)(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2)\mathrm{div}\,\theta\cdot(1,0), \nu_n\rangle = 0,$$

$$\langle 2\tau(w^2+k_\varepsilon)(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2)\mathrm{div}\,\theta\cdot(0,1), \nu_n\rangle = 0,$$

which imply

$$(w^2 + k_\varepsilon)(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2)\mathrm{div}\,\theta = 0 \qquad \text{on } \partial\Omega.$$

This boundary condition is satisfied if $\mathrm{div}\,\theta = 0$ on $\partial\Omega$. If Ω is a rectangle and $\theta_1 = 0$ on vertical sides (see the boundary conditions on s), the condition $\mathrm{div}\,\theta = 0$ requires $\frac{\partial\theta_2}{\partial y} = 0$ on vertical sides. On horizontal sides, if $\theta_2 = 0$, this boundary condition becomes $\frac{\partial\theta_1}{\partial x} = 0$.

Equations for $\omega = (\omega_1, \omega_2)$. The computations are similar to the previous ones, so that the equations are

$$\begin{aligned}
\nabla_{\omega_1} M_{\varepsilon,0} &= -\eta_1\frac{\partial w}{\partial x} - 2\alpha_1\beta_\varepsilon\frac{\partial}{\partial x}[\varepsilon|\nabla w|^2\mathrm{div}\,\omega] - \\
&\quad -2\alpha_1\beta_\varepsilon\frac{\partial}{\partial x}[\frac{1}{\varepsilon}w^2(1-w)^2\mathrm{div}\,\omega] = 0, \\
\nabla_{\omega_2} M_{\varepsilon,0} &= -\eta_1\frac{\partial w}{\partial y} - 2\alpha_1\beta_\varepsilon\frac{\partial}{\partial y}[\varepsilon|\nabla w|^2\mathrm{div}\,\omega] - \\
&\quad -2\alpha_1\beta_\varepsilon\frac{\partial}{\partial y}[\frac{1}{\varepsilon}w^2(1-w)^2\mathrm{div}\,\omega] = 0.
\end{aligned}$$

Computing the derivatives we find

$$\begin{aligned}
\nabla_{\omega_1} M_{\varepsilon,0} &= -\eta_1\frac{\partial w}{\partial x} - 2\alpha_1\beta_\varepsilon\mathrm{div}\,\omega\cdot \\
&\quad \left[2\varepsilon(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}) + \frac{1}{\varepsilon}(4w^3 - 6w^2 + 2w)\frac{\partial w}{\partial x}\right] \\
&\quad -2\alpha_1\beta_\varepsilon R_1\left[\frac{\partial^2\omega_1}{\partial x^2} + \frac{\partial^2\omega_2}{\partial x\partial y}\right] = 0\,, \\
\nabla_{\omega_2} M_{\varepsilon,0} &= -\eta_1\frac{\partial w}{\partial y} - 2\alpha_1\beta_\varepsilon\mathrm{div}\,\omega\cdot \\
&\quad \left[2\varepsilon(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y\partial x} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}) + \frac{1}{\varepsilon}(4w^3 - 6w^2 + 2w)\frac{\partial w}{\partial y}\right] \\
&\quad -2\alpha_1\beta_\varepsilon R_1\left[\frac{\partial^2\omega_1}{\partial y\partial x} + \frac{\partial^2\omega_2}{\partial y^2}\right] = 0\,,
\end{aligned}$$

where $R_1 = \varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}w^2(1-w)^2$. The boundary conditions are

$$\langle(\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}w^2(1-w)^2)\mathrm{div}\,\omega\cdot(1,0), \nu_n\rangle = 0,$$

$$\langle(\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}w^2(1-w)^2)\mathrm{div}\,\omega\cdot(0,1), \nu_n\rangle = 0,$$

which imply

$$(\varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}w^2(1-w)^2)\operatorname{div}\omega = 0.$$

This boundary condition is satisfied if $\operatorname{div}\omega = 0$ on $\partial\Omega$. If Ω is a rectangle the condition $\operatorname{div}\omega = 0$ requires $\frac{\partial\omega_2}{\partial y} = 0$ on vertical sides and $\frac{\partial\omega_1}{\partial x} = 0$ on horizontal sides (see the boundary conditions on w).

Note again that the equations are linear in ω_1 and ω_2 respectively.

The nonlinear system of Euler equations will be solved numerically by means of an iterative method, and by replacing in the nonlinear terms the functions s and w with s_0 and w_0 computed at the previous iteration. Then all the equations become linear and the conjugate gradient method will be used for each equation.

Further observations are required. Recalling the definition of θ and ω, the constraints $|\theta| \leq 1$ and $|\omega| \leq 1$ will be imposed by means of projection after each iteration. Moreover, since the terms $\operatorname{div}(\frac{\nabla s}{|\nabla s|})$ and $\operatorname{div}(\frac{\nabla w}{|\nabla w|})$ are not defined in regions where either $|\nabla s| = 0$ or $|\nabla w| = 0$, such terms will be replaced with

$$\operatorname{div}\left(\frac{\nabla s}{\sqrt{|\nabla s|^2 + \varepsilon_1}}\right), \quad \operatorname{div}\left(\frac{\nabla w}{\sqrt{|\nabla w|^2 + \varepsilon_1}}\right),$$

where $\varepsilon_1 > 0$ is a small positive constant. In the next section we give further details on the numerical implementation.

Eventually, the terms involving the squared Laplacian of s and w yield the following contribution to the Euler equations:

$$\nabla_s M_{\varepsilon,\xi} = \nabla_s M_{\varepsilon,0} + 2\xi\Delta^2 s, \qquad \nabla_w M_{\varepsilon,\xi} = \nabla_w M_{\varepsilon,0} + 2\xi\Delta^2 w,$$

where Δ^2 is the bilaplacian operator. A numerical experiment, executed using a small value of ξ, has exhibited the same result obtained with the algorithm discussed in the next section, where we set $\xi = 0$. Hence we do not discuss these terms further.

2.9 A numerical method

In this section we discretize the Euler equations of the functional, and we discuss some numerical experiments. As it was discussed in the previous section, we use an iterative scheme such that at each iteration we have to solve linear equations. Then we use the conjugate gradient method to solve such linear equations. The numerical method has the following structure:

```
Initialization of variables

for i=1:P
solve $\nabla_u M_{\varepsilon,0} = 0$ with respect to $u$ by conjugate gradient method
solve $\nabla_s M_{\varepsilon,0} = 0$ with respect to $s$ by conjugate gradient method
solve $\nabla_\theta M_{\varepsilon,0} = 0$ with respect to $\theta$ by conjugate gradient method
solve $\nabla_w M_{\varepsilon,0} = 0$ with respect to $w$ by conjugate gradient method
solve $\nabla_\omega M_{\varepsilon,0} = 0$ with respect to $\omega$ by conjugate gradient method
end
```

where P is the number of cycles. Furthermore, in each cycle we impose the constraint $|\theta| = 1$. The starting point of the conjugate gradient method at the first cycle is given by the initialization of variables. In subsequent cycles it is given by the values computed in the previous cycle.

Moreover, we choose the weights recalling that:

i) μ is the weight of the term

$$\int_\Omega |u-g|^2 dx\,,$$

that imposes u to approximate g.

ii) η, η_1 are the weights of the terms

$$\int_\Omega (|\nabla s| - \langle \theta, \nabla s\rangle)\, dx\,, \qquad \int_\Omega (|\nabla w| - \langle \omega, \nabla w\rangle)\, dx$$

respectively, that impose θ, ω to approximate $\frac{\nabla s}{|\nabla s|}$, $\frac{\nabla w}{|\nabla w|}$ respectively.

iii) α is the weight of the length integral

$$\int_\Omega (w^2 + k_\varepsilon)\left[\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2\right]$$

and of $\frac{1}{\mu_\varepsilon}\int_\Omega (1-s)^2 dx$ that imposes s is close to one out from the edge of u.

iv) τ is the weight of the length integral multiplied by the curvature term of s

$$\int_\Omega (w^2 + k_\varepsilon)\left[\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2\right](\operatorname{div}\theta)^2)dx$$

v) the behavior of α_1 is similar to α and τ.

2.9.1 The conditions on $\partial\Omega$

We give the discrete form of the boundary conditions derived in the previous section. In the sequel the open set Ω will be a square. Therefore, we fix a grid $N \times N$ with N integer. Since $u, s, w : \Omega \to \mathbb{R}$ and $\theta, \omega : \Omega \to \mathbb{R}^2$, the variables are matrices with dimension $N \times N$ for u, s, w and $2N \times N$ for θ, ω.

In order to express the boundary conditions by means of finite differences we augment the number of rows and columns of the matrices. Then u becomes a $(N+2)\times(N+2)$ matrix, s and w become $(N+4)\times(N+4)$ matrices. The variables θ and ω become matrices $(2N+4)\times(N+2)$.

Taking into account the observations of previous section in the case Ω is a rectangle:

- we ask that $\frac{\partial u}{\partial x} = 0$ on vertical sides and $\frac{\partial u}{\partial y} = 0$ on horizontal sides, therefore using forward finite differences, we obtain

$$u(0,j) = u(1,j) \quad u(N+1,j) = u(N,j)$$

$$u(i,0) = u(i,1) \quad u(i,N+1) = u(i,N) \quad \forall i,j = 1,...,N;$$

- we ask that $\frac{\partial s}{\partial x} = 0$ on vertical sides and $\frac{\partial s}{\partial y} = 0$ on horizontal sides, therefore when we use forward finite differences, we obtain

$$s(0,j) = s(1,j) \quad s(N+1,j) = s(N,j)$$

$$s(i,0) = s(i,1) \quad s(i,N+1) = s(i,N) \quad \forall i,j = 1,...,N,$$

while when we use central finite differences, we obtain

$$s(0,j) = s(2,j) \quad s(N+1,j) = s(N-1,j)$$

$$s(i,0) = s(i,2) \quad s(i,N+1) = s(i,N-1) \quad \forall i,j = 1,...,N$$

$$s(-1,j) = s(1,j) \quad s(N+2,j) = s(N,j) \quad s(i,-1) = s(i,1)$$

$$s(i,N+2) = s(i,N) \quad \forall i,j = 1,...,N;$$

- we ask that $\theta_1 = \theta_2 = 0$ on $\partial\Omega$, then we have simply

$$\theta(1,j) = \theta(N,j) = \theta(i,1) = \theta(i,N) = 0$$

$$\theta(1,j+N) = \theta(N,j+N) = \theta(i,N+1) = \theta(i,2N) = 0$$

$$\forall i,j = 1,...,N;$$

- we ask that $\frac{\partial w}{\partial x} = 0$ on vertical sides and $\frac{\partial w}{\partial y} = 0$ on horizontal sides, therefore when we use forward finite differences, we obtain

$$w(0,j) = w(1,j) \quad w(N+1,j) = w(N,j)$$

$$w(i,0) = w(i,1) \quad w(i,N+1) = w(i,N) \quad \forall i,j = 1,...,N,$$

  while when we use central finite differences, we obtain

$$w(0,j) = w(2,j) \quad w(N+1,j) = w(N-1,j)$$

$$w(i,0) = w(i,2) \quad w(i,N+1) = w(i,N-1) \quad \forall i,j = 1,...,N$$

$$w(-1,j) = w(1,j) \quad w(N+2,j) = w(N,j) \quad w(i,-1) = w(i,1)$$

$$w(i,N+2) = w(i,N) \quad \forall i,j = 1,...,N;$$

- we ask that $\omega_1 = \omega_2 = 0$ on $\partial\Omega$, then we have simply

$$\omega(1,j) = \omega(N,j) = \omega(i,1) = \omega(i,N) = 0$$

$$\omega(1,j+N) = \omega(N,j+N) = \omega(i,N+1) = \omega(i,2N) = 0$$

$$\forall i,j = 1,...,N.$$

2.9.2 Discretization of the equations

Now we discretize the Euler equations. For terms of the form $\mathrm{div}(A(i,j)\nabla F(i,j))$, forward finite differences are used for the gradient $\nabla F(i,j)$, while backward finite differences are used for $\mathrm{div}(\cdot)$. Denoting $C = A\nabla F$, we have

$$C^1(i,j) = A(i,j)\frac{F(i+1,j) - F(i,j)}{p},$$

$$C^2(i,j) = A(i,j)\frac{F(i,j+1) - F(i,j)}{p},$$

where p is the mesh size. Then we get

$$\begin{aligned}
\operatorname{div}(A(i,j)\nabla F(i,j)) &= \frac{C^1(i,j)-C^1(i-1,j)}{p}+\frac{C^2(i,j)-C^2(i,j-1)}{p}\\
&= A(i,j)\frac{F(i+1,j)-F(i,j)}{p^2}\\
&\quad -A(i-1,j)\frac{F(i,j)-F(i-1,j)}{p^2}+\\
&\quad +A(i,j)\frac{F(i,j+1)-F(i,j)}{p^2}\\
&\quad -A(i,j-1)\frac{F(i,j)-F(i,j-1)}{p^2}=\\
&= \frac{1}{p^2}[A(i,j)F(i+1,j)+A(i,j)F(i,j+1)+\\
&\quad +A(i-1,j)F(i-1,j)+A(i,j-1)F(i,j-1)-\\
&\quad -(2A(i,j)+A(i-1,j)+A(i,j-1))F(i,j)].
\end{aligned}$$

Now we compute the discrete Euler equations.

Equation for u. The equation is

$$\mu(u-g)=\operatorname{div}\left((s^2+\xi_\varepsilon)\nabla u\right)\quad \text{in } \Omega.$$

Setting $f=u$ and $A(i,j)=s(i,j)^2+\xi_\varepsilon$, the discretized equation is

$$\begin{aligned}
&[s(i,j)^2u(i+1,j)+s(i-1,j)^2u(i-1,j)+\\
&s(i,j-1)^2u(i,j-1)+s(i,j)^2u(i,j+1)]-\\
-[(2s(i,j)^2+s(i,j-1)^2+s(i-1,j)^2)+p^2\mu] &= -\mu p^2 g(i,j)\,.
\end{aligned}$$

Equation for s. The equation is

$$\begin{aligned}
&2\left[|\nabla u|^2+\frac{1}{\varepsilon}(w^2+k_\varepsilon)[\alpha+\tau(\operatorname{div}\theta)^2]+\frac{\alpha}{\mu_\varepsilon}\right]s\\
-&\ 2\,\alpha\,\varepsilon\operatorname{div}\big((w^2+k_\varepsilon)\nabla s\big)\\
-&\ 2\,\tau\,\varepsilon\operatorname{div}\big((w^2+k_\varepsilon)(\operatorname{div}\theta)^2\nabla s\big)-\eta\operatorname{div}\left(\frac{\nabla s}{|\nabla s|}\right)\\
=&\ \frac{2\alpha}{\mu_\varepsilon}+\frac{2}{\varepsilon}(w^2+k_\varepsilon)[\alpha+\tau(\operatorname{div}\theta)^2](-2s^3+3s^2)-\eta\operatorname{div}\theta\,.
\end{aligned}$$

We denote by

$$V(i,j) = \frac{1}{p^2}((u(i+1,j) - u(i,j))^2 + (u(i,j+1) - u(i,j))^2)$$

$$\forall i,j = 2, ...N-1,$$

with $V(i,j) = 0$ if $i = 1, N$ or $j = 1, N$,

the discretization by forward finite differences of $|\nabla u|^2$. The discretization of $\operatorname{div}\theta$ is

$$K(i,j) = \frac{1}{p}(\theta(i+1,j) - \theta(i,j) + \theta(i,j+1+N) - \theta(i,j+N))$$

$$\forall i,j = 2, ...N-1,$$

with $K(i,j) = 0$ if $i = 1, N$ or $j = 1, N$.

We remind that, in the iterative method, the function s is replaced in the nonlinear terms with the function s_0 computed in the previous cycle. Moreover, the term $\frac{1}{\nabla s}$ is replaced by $\frac{1}{\sqrt{\varepsilon_1 + |\nabla s_0|^2}}$, and the discretization of such a term is given by

$$B(i,j) = \frac{1}{\sqrt{\varepsilon_1 + \frac{1}{p^2}((s_0(i+1,j) - s_0(i,j))^2 + (s_0(i,j+1) - s_0(i,j))^2)}}$$

$\forall i,j = 2, ...N-1$, with $B(i,j) = \frac{1}{\sqrt{\varepsilon_1}}$ if $i = 1, N$ or $j = 1, N$.

We apply the discretization method for terms of type $\operatorname{div}(A\nabla F)$ to cases:
(i) $F = s\,,\ A = w(i,j)^2 + k_\varepsilon$;
(ii) $F = s\,,\ A = (w(i,j)^2 + k_\varepsilon)K(i,j)^2$;
(iii) $F = s\,,\ A(i,j) = B(i,j)$.
Moreover, we define the coefficients

$$E(i,j) = w(i,j)^2 + k_\varepsilon\,,$$

$$D(i,j) = E(i,j)K(i,j)^2\,,$$

$$H = \frac{1}{p^2}(\alpha E + \tau D + \frac{\eta}{2\varepsilon}B)\,.$$

Using forward finite differences for the other terms, we obtain the discretized equation

$$(H(i,j)s(i+1,j) + H(i-1,j)s(i-1,j) + H(i,j)s(i,j+1) +$$

$$H(i,j-1)s(i,j-1)) - \{\frac{1}{\varepsilon}V(i,j) + \frac{\alpha}{\varepsilon^2}E(i,j) + \frac{\alpha}{\mu_\varepsilon \varepsilon} + \frac{\tau}{\varepsilon^2}D(i,j) + \frac{4\alpha}{p^2} +$$
$$\frac{\tau}{p^2}(2D(i,j) + D(i-1,j) + D(i,j-1)) + \frac{\eta}{2\varepsilon p^2}(2B(i,j) + B(i-1,j) +$$
$$B(i,j-1))\}s(i,j) = \frac{\eta}{2\varepsilon}K(i,j) + \frac{1}{\varepsilon^2}(2s_0(i,j)^3 - 3s_0(i,j)^2)(\alpha E(i,j) +$$
$$\tau D(i,j)) - \frac{\alpha}{\mu_\varepsilon \varepsilon}\,.$$

Equations for θ. We set

$$R = \varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}s^2(1-s)^2,$$

$$T = (w^2+k_\varepsilon)[2\varepsilon \frac{\partial s}{\partial x}\frac{\partial^2 s}{\partial x^2} + \frac{\partial s}{\partial y}\frac{\partial^2 s}{\partial x \partial y}) + \frac{1}{\varepsilon}(w^2+k_\varepsilon)(4s^3-6s^2+2s)\frac{\partial s}{\partial x} + 2Rw\frac{\partial w}{\partial x}],$$

$$T^* = (w^2+k_\varepsilon)[2\varepsilon \frac{\partial s}{\partial x}\frac{\partial^2 s}{\partial y \partial x} + \frac{\partial s}{\partial y}\frac{\partial^2 s}{\partial y^2}) + \frac{1}{\varepsilon}(w^2+k_\varepsilon)(4s^3-6s^2+2s)\frac{\partial s}{\partial y} + 2Rw\frac{\partial w}{\partial y}].$$

The Euler equations may be written as

$$\begin{cases} 2\tau(R\frac{\partial^2\theta_1}{\partial x^2} + T\frac{\partial\theta_1}{\partial x} + R\frac{\partial^2\theta_2}{\partial x\partial y} + T\frac{\partial\theta_2}{\partial y}) = -\eta\frac{\partial s}{\partial x}, \\ \\ 2\tau(R\frac{\partial^2\theta_1}{\partial y\partial x} + T^*\frac{\partial\theta_1}{\partial x} + R\frac{\partial^2\theta_2}{\partial y^2} + T^*\frac{\partial\theta_2}{\partial y}) = -\eta\frac{\partial s}{\partial y}\,. \end{cases}$$

To discretize the equations we use the following finite differences:

- replace $\frac{\partial^2\theta_1}{\partial x^2}$ with $(\theta(i+1,j) + \theta(i-1,j) - 2\theta(i,j))\frac{1}{p^2}$
- replace $\frac{\partial\theta_1}{\partial x}$ with $(\theta(i+1,j) - \theta(i-1,j)))\frac{1}{2p}$
- replace $\frac{\partial^2\theta_2}{\partial x\partial y}$ with $[\theta(i+1,j+1+N) - \theta(i+1,j-1+N) - \theta(i-1,j+1+N) + \theta(i-1,j-1+N))]\frac{1}{4p^2}$
- replace $\frac{\partial\theta_2}{\partial y}$ with $(\theta(i,j+1+N) - \theta(i,j-1+N))\frac{1}{2p}$
- replace $\frac{\partial^2\theta_1}{\partial y\partial x}$ with $[\theta(i+1,j+1) - \theta(i-1,j+1) - \theta(i+1,j-1) + \theta(i-1,j-1))]\frac{1}{4p^2}$
- replace $\frac{\partial^2\theta_2}{\partial y^2}$ with $(\theta(i,j+1+N) + \theta(i,j-1+N) - 2\theta(i,j+N))\frac{1}{p^2}$

and we use the same finite differences for s and w. Then the discretized equations are

- $2\tau[(\frac{1}{p^2}Rd(i,j)+\frac{1}{2p}Td(i,j))\theta(i+1,j)+(\frac{1}{p^2}Rd(i,j)-\frac{1}{2p}Td(i,j))\cdot$

  $\cdot\theta(i-1,j)-\frac{2}{p^2}Rd(i,j)\theta(i,j)+\frac{1}{4p^2}Rd(i,j)(\theta(i+1,j+1+N)-$

  $-\theta(i+1,j-1+N)-\theta(i-1,j+1+N)+\theta(i-1,j-1+N))+$

  $+\frac{1}{2p}Td(i,j)(\theta(i,j+1+N)-\theta(i,j-1+N))]=-\frac{\eta}{2p}(s(i+1,j)-s(i-1,j))$

- $2\tau[\frac{1}{4p^2}Rd(i,j)(\theta(i+1,j+1)-\theta(i-1,j+1)-\theta(i+1,j-1)+\theta(i-1,j-1))+$

  $\frac{1}{2p}Td^*(i,j)(\theta(i+1,j)-\theta(i-1,j))-\frac{2}{p^2}Rd(i,j)\theta(i,j+N)+(\frac{1}{p^2}Rd(i,j)+$

  $+\frac{1}{2p}Td^*(i,j))\theta(i,j+1+N)+(\frac{1}{p^2}Rd(i,j)-\frac{1}{2p}Td^*(i,j))\theta(i,j-1+N)]=$

  $=-\frac{\eta}{2p}(s(i,j+1)-s(i,j-1))\,,$

where

$$Rd(i,j)=\frac{\varepsilon}{4p^2}((s(i+1,j)-s(i-1,j))^2+(s(i,j+1)-s(i,j-1))^2)+$$

$$+\frac{1}{\varepsilon}(s(i,j)^2(1-s(i,j)^2)),$$

$$Td(i,j)=2(w(i,j)^2+k_\varepsilon)[\varepsilon\frac{1}{2p^3}(s(i+1,j)-s(i-1,j))(s(i+1,j)+$$

$$+s(i-1,j)-2s(i,j))+\frac{1}{2p\varepsilon}s(i,j)(2s(i,j)^2-3s(i,j)+1)(s(i+1,j)-$$

$$-s(i-1,j))]+\frac{1}{p}Rdw(i,j)(w(i+1,j)-w(i-1,j)),$$

$$Td^*(i,j)=2(w(i,j)^2+k_\varepsilon)[\varepsilon\frac{1}{8p^3}(s(i+1,j)-s(i-1,j))(s(i+1,j+1)-$$

$$-s(i-1,j+1)-s(i+1,j-1)+s(i-1,j-1))+\frac{1}{2p\varepsilon}s(i,j)(2s(i,j)^2-$$

$$-3s(i,j)+1)(s(i,j+1)-s(i,j-1))]+\frac{1}{p}Rdw(i,j)(w(i,j+1)-w(i,j-1)),$$

are the discretizations of the functions R, T, T^*.

Equation for w. The equation is

$$
\begin{aligned}
& 2\,(\varepsilon|\nabla s|^2 + \frac{1}{\varepsilon}\,s^2(1-s)^2)[\alpha + \tau\,(\operatorname{div}\theta)^2]\,w \\
+ \quad & 2\,\frac{\alpha_1}{\mu_\varepsilon}\,w + 2\,\frac{\alpha_1}{\varepsilon}\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2]\,w \\
- \quad & 2\,\varepsilon\left[\frac{1}{\beta_\varepsilon}\Delta w + \beta_\varepsilon \operatorname{div}(\,(\operatorname{div}\omega)^2\nabla w)\right] - \eta_1 \operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) \\
= \quad & \frac{2\alpha_1}{\mu_\varepsilon} + 2\frac{\alpha_1}{\varepsilon}\,[\frac{1}{\beta_\varepsilon} + \beta_\varepsilon\,(\operatorname{div}\omega)^2](-2\,w^3 + 3w^2) - \eta_1\,\operatorname{div}\omega\,.
\end{aligned}
$$

We denote the discretization of $\operatorname{div}\omega$ by

$$K_1(i,j) = \frac{1}{p}(\omega(i+1,j) - \omega(i,j) + \omega(i,j+1+N) - \omega(i,j+N))$$

$$\forall i,j = 2,...N-1,$$

with $K_1(i,j) = 0$ if $i = 1, N$ or $j = 1, N$.

In the iterative method, the function w is replaced in the nonlinear terms with the function w_0 computed in the previous cycle. Moreover, the term $\frac{1}{\nabla w}$ is replaced by $\frac{1}{\sqrt{\varepsilon_1 + |\nabla w_0|^2}}$, and the discretization of such a term is given by

$$B_1(i,j) = \frac{1}{\sqrt{\varepsilon_1 + \frac{1}{p^2}((w_0(i+1,j) - w_0(i,j))^2 + (w_0(i,j+1) - w_0(i,j))^2)}}$$

$\forall i,j = 2,...N-1$, with $B_1(i,j) = \frac{1}{\sqrt{\varepsilon_1}}$ if $i = 1, N$ or $j = 1, N$..

We apply the discretization method for terms of type $\operatorname{div}(A\nabla F)$ to cases:
(i) $F = w\,,\ A(i,j) = \frac{1}{\beta_\varepsilon}$;
(ii) $F = w\,,\ A(i,j) = \beta_\varepsilon K_1(i,j)^2$;
(iii) $F = w\,,\ A(i,j) = B(i,j)$.
Moreover, we define the coefficients

$$D_1(i,j) = K_1(i,j)^2\,,$$

$$G(i,j) = Rd(i,j)(\alpha + \tau(K(i,j))^2) + \frac{1}{\varepsilon}(\frac{\alpha_1}{\beta_\varepsilon} + \alpha_1\beta_\varepsilon(K_1(i,j))^2) + \frac{2\alpha_1}{\mu_\varepsilon}\,,$$

$$H_1 = \frac{1}{p^2}(\frac{2\varepsilon\alpha_1}{\beta_\varepsilon} I_N + 2\alpha_1\beta_\varepsilon D_1 + \eta_1 B_1)\,,$$

where I_N denotes the identity matrix. Hence the discretized equation is

$$\begin{aligned}(H_1(i,j)w(i+1,j) + H_1(i-1,j)w(i-1,j) + H_1(i,j)w(i,j+1)+\\ +H_1(i,j-1)w(i,j-1)) - \{\frac{8\varepsilon\alpha_1}{p^2\beta_\varepsilon} + 2G(i,j) + \frac{2\varepsilon\alpha_1\beta_\varepsilon}{p^2}(2D_1(i,j)+\\ +D_1(i-1,j) + D_1(i,j-1)) + \frac{\eta_1}{2\varepsilon p^2}(2B_1(i,j) + B_1(i-1,j)+\\ +B_1(i,j-1))\}w(i,j) =\\ = \eta_1 K(i,j) + \frac{2}{\varepsilon}(2w_0(i,j)^3 - 3w_0(i,j)^2)(\frac{\alpha_1}{\beta_\varepsilon} + \alpha_1\beta_\varepsilon D_1(i,j)) - \frac{2\alpha_1}{\mu_\varepsilon}.\end{aligned}$$

Equations for ω. By using the same discretization method applied to the equations for θ, we define the following functions:

$$R_1 = \varepsilon|\nabla w|^2 + \frac{1}{\varepsilon}w^2(1-w)^2,$$

$$T_1 = 2\varepsilon(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}) + \frac{1}{\varepsilon}(4w^3 - 6w^2 + 2w)\frac{\partial w}{\partial x},$$

$$T_1^* = 2\varepsilon(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y\partial x} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}) + \frac{1}{\varepsilon}(4w^3 - 6w^2 + 2w)\frac{\partial w}{\partial y}.$$

Then the Euler equations may be written as

$$\begin{cases} 2\alpha_1\beta_\varepsilon(R_1\frac{\partial^2\omega_1}{\partial x^2} + T_1\frac{\partial\omega_1}{\partial x} + R_1\frac{\partial^2\omega_2}{\partial x\partial y} + T_1\frac{\partial\omega_2}{\partial y}) = -\eta_1\frac{\partial w}{\partial x},\\ \\ 2\alpha_1\beta_\varepsilon(R_1\frac{\partial^2\omega_1}{\partial y\partial x} + T_1^*\frac{\partial\omega_1}{\partial x} + R_1\frac{\partial^2\omega_2}{\partial y^2} + T_1^*\frac{\partial\omega_2}{\partial y}) = -\eta_1\frac{\partial w}{\partial y}.\end{cases}$$

To discretize the equations we use the same finite differences chosen for the equations for θ. Hence the discretized equations are

- $2\alpha_1\beta_\varepsilon[(\frac{1}{p^2}Rd_1(i,j) + \frac{1}{2p}Td_1(i,j))\omega(i+1,j) + (\frac{1}{p^2}Rd_1(i,j) - \frac{1}{2p}Td_1(i,j))\cdot$

  $\cdot\omega(i-1,j) - \frac{2}{p^2}Rd_1(i,j)\omega(i,j) + \frac{1}{4p^2}Rd_1(i,j)(\omega(i+1,j+1+N)-$

  $-\omega(i+1,j-1+N) - \omega(i-1,j+1+N) + \omega(i-1,j-1+N))+$

  $+\frac{1}{2p}Td_1(i,j)(\omega(i,j+1+N) - \omega(i,j-1+N))] =$

$= -\frac{\eta_1}{2p}(w(i+1,j) - w(i-1,j)),$

- $2\alpha_1\beta_\varepsilon[\frac{1}{4p^2}Rd_1(i,j)(\omega(i+1,j+1) - \omega(i-1,j+1) - \omega(i+1,j-1)+$

  $+\omega(i-1,j-1)) + \frac{1}{2p}Td_1^*(i,j)(\omega(i+1,j) - \omega(i-1,j)) - \frac{2}{p^2}Rd_1(i,j)\cdot$

  $\omega(i,j+N) + (\frac{1}{p^2}Rd_1(i,j) + \frac{1}{2p}Td_1^*(i,j))\omega(i,j+1+N)+$

  $+(\frac{1}{p^2}Rd_1(i,j) - \frac{1}{2p}Td_1^*(i,j))\omega(i,j-1+N)] =$

  $= -\frac{\eta_1}{2p}(w(i,j+1) - w(i,j-1))\,,$

where

$$Rd_1(i,j) = \frac{\varepsilon}{4p^2}((w(i+1,j) - w(i-1,j)^2 + (w(i,j+1) - w(i,j-1)^2)+$$

$$+\frac{1}{\varepsilon}(w(i,j)^2(1 - w(i,j)^2)),$$

$$Td(i,j) = 2[\varepsilon\frac{1}{2p^3}(w(i+1,j) - w(i-1,j))(w(i+1,j) + w(i-1,j)-$$

$$-2w(i,j)) + \frac{1}{2p\varepsilon}w(i,j)(2w(i,j)^2 - 3w(i,j) + 1)(w(i+1,j) - w(i-1,j))],$$

$$Td^*(i,j) = 2[\varepsilon\frac{1}{8p^3}(w(i+1,j) - w(i-1,j))(w(i+1,j+1) - w(i-1,j+1)-$$

$$-w(i+1,j-1) + w(i-1,j-1)) + \frac{1}{2p\varepsilon}w(i,j)(2w(i,j)^2 - 3w(i,j) + 1)\cdot$$

$$\cdot(w(i,j+1) - w(i,j-1))],$$

are the discretizations of the functions R_1, T_1, T_1^*, respectively.

We consider the unknowns s, w, u and θ, ω as column matrices $N^2 \times 1$ and $2N^2 \times 1$, respectively. Then the discretized equations yield linear systems $Ax = b$, with A square matrix of dimensions $N^2 \times N^2$ and $2N^2 \times 2N^2$, respectively. Then we solve the linear systems by using the conjugate gradient method. We use the Matlab 7 function *bicg*, which implements the conjugate gradient method for square, not symmetric matrices A.

To conclude this subsection we discuss the choice of initial values of the iterative method. We compute u, s as solutions of the system of Euler equations by imposing w, θ, ω fixed and equal to $w(i,j) = 1$, $\theta(i,j) = \theta(i,j+N) =$

0, $\omega(i,j) = \omega(i,j+N) = 0$ $\forall i,j = 1,...,N$. For u and s we consider the following initial values: $u(i,j) = g(i,j)$ and $s(i,j) = 1$ $\forall i,j = 1,...,N$. Then we use the conjugate gradient method, for several cycles, for the equations of u and s with the other variables kept constant.

The functions u and s computed in such a way are used as initial values for the solution of the system of equations in the next step, and to compute the initial values of the other variables according to the following scheme:

- if Grads is the gradient of s, i.e.,

  ```
  Grads=sparse(2*N^2,1);
  r=1;
  for i=1:N^2-1
  l=i/N;
  if l==r
  Grads(i+N^2)=(s(i+N)-s(i))/p;
  r=r+1;
  elseif i>N^2-N
  Grads(i)=(s(i+1)-s(i))/p;
  else
  Grads(i)=(s(i+1)-s(i))/p;
  Grads(i+N^2)=(s(i+N)-s(i))/p;
  end
  end
  ```

  and H is the modulus of ∇s, then θ has the initial value given by

  ```
  Invs0=sparse(N,N);
  for i=1:N
  for j=1:N
  Invs0(i,j)=1/sqrt(e1+H(i,j));
  end
  end
  teta=sparse(2*N^2,1);
  for i=1:N^2
  if H(i)<threshold_θ
  ```

```
teta(i)=0;
teta(i+N^2)=0;
else
teta(i)=Grads(i)*Invs0(i);
teta(i+N^2)=Grads(i+N^2)*Invs0(i);
end
end
```

- w has the initial value given by

```
Dct=sparse(N,2*N);
for i=1:N^2
Dct(i)=Invs0(i)*Grads(i);
Dct(i+N^2)=Invs0(i)*Grads(i+N^2);
end Dctr=divergenza(Dct,N);
for i=1:N^2
Dctr(i)=Dctr(i)^2;
end
w=sparse(ones(N^2,1));
for i=1:N^2
if Dctr(i)>threshold_w
w(i)=s(i);
end
end
```

- if Gradw is the gradient of w, defined in the same way as Grads, and $H1$ is the modulus of ∇w, then ω has the initial value given by

```
Invw0=sparse(N,N);
for i=1:N
for j=1:N
Invw0(i,j)=1/sqrt(e1+H1(i,j));
end
end
```

```
omega=sparse(2*N^2,1);
for i=1:N^2
if H1(i)<threshold_ω
omega(i)=0;
omega(i+N^2)=0;
else
omega(i)=Gradw(i)*Invw0(i);
omega(i+N^2)=Gradw(i+N^2)*Invw0(i);
end
end
```

In the above algorithm we use the parameters threshold_w,threshold_θ, and threshold_ω to control some variables. The parameters threshold_θ and threshold_ω are used in regions where $|\nabla s|$ and $|\nabla w|$ are close to zero, i.e., where s and w are nearly constant. In such regions the direction of the vectors ∇s and ∇w are not controlled by the functional so that they may have large changes. Therefore the vector fields θ and ω are not close to $(0, 0)$ in these regions as they should be. Then we impose that the vector fields θ and ω are equal to $(0, 0)$ by using thresholds on $|\nabla s|$ and $|\nabla w|$.

The parameter threshold_w is used to detect corner points of the level sets of the function s. At such points the curvature $\text{div}(\nabla s/|\nabla s|)^2$ is large, so that they can be detected by using a threshold on curvature. Hence we impose $w = s$ at such points, where s is close to zero, and $w = 1$ at other points.

The conjugate gradient method uses the above initialization of variables as initial data in the first cycle. In subsequent cycles the values computed in the previous cycle are used as initial data.

2.10 Numerical experiments

The numerical method has been experimented on three images 256 × 256, with the CPU AMD Athlon XP 2400 and 1 GByte of RAM. The first image contains a rectangle and a triangle, the second contains a grayscale within a rectangle, while the third one is a real image showing a cherry.. A grey level representation is used for the functions s and w: white corresponds to the value 1 and black corresponds to the value zero. The corner points of the two geometric shapes are detected by the function w.

Figure 2.1: the function g, $N = 256$.

The first image. Figure 2.10 shows the datum g. Figure 2.10 shows the functions u, s and w computed by the program.

In this case the following values of the parameters have been used: $\alpha = 10^{-1}$ for initialization and $\alpha = 10^{-2}$ in subsequent cycles; $\tau = 10^{-2}$, $\varepsilon = 10^{-2}$, and the other parameters are $\mu = 10^5$, $\alpha_1 = 10$, $\beta_\varepsilon = 10^{-3}$, $\mu_\varepsilon = \varepsilon \cdot 10^{-2}$, $threshold_w = 10^3$, $threshold_\omega = 10$, $threshold_\theta = 100$. The number of cycles is 3 and they are executed in about eight hours.

Figure 2.10 shows the value of the computed energy density of the functional $M_{\varepsilon,0}$ at each point of the square Ω.

Note that the algorithm detects the corner points along the boundaries, since at such points the value of the energy density exhibits peaks.

Now, in order to control the accuracy of the solutions obtained, we compute the (normalized) residue. In fact, once we have computed a solution x of a linear system $Ax = b$ using an iterative method, we can compute the normalized residue $\mathbf{r} = norm(b - Ax)$ ($\mathbf{n} = norm(b - Ax)/norm(b)$) with respect to a fixed norm. The Table 2.1 lists the values of normalized residues of linear systems of u, s, θ, w, ω after the last iteration in the norms:

$$\|x\|_1 = \sum_{i=1}^{M} |x_i|,$$

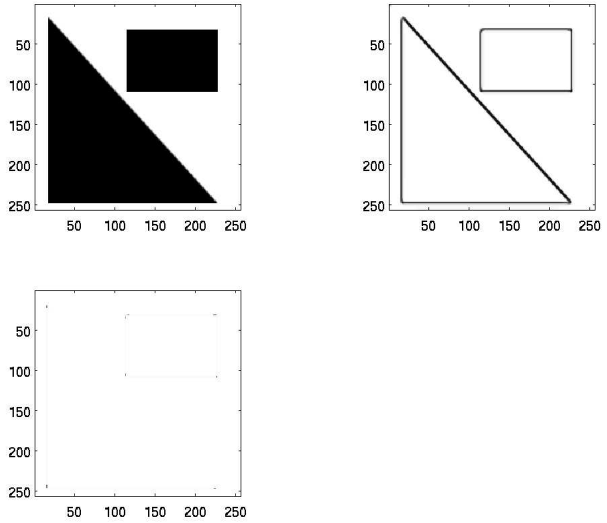

Figure 2.2: the computed functions u, s, w with $N = 256$.

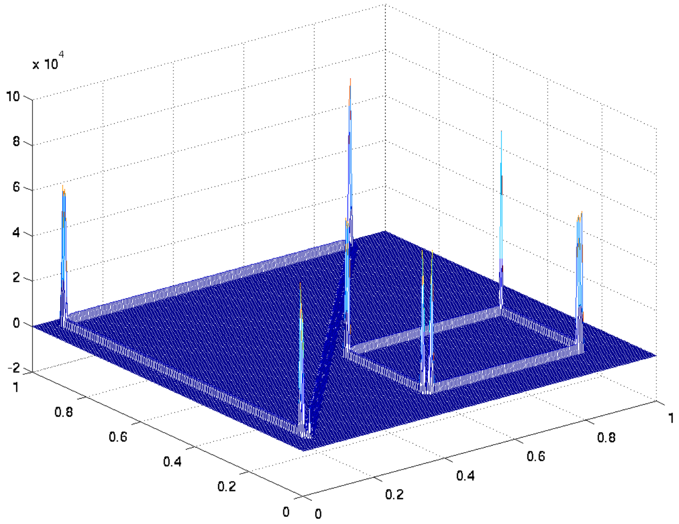

Figure 2.3: the energy density, $N = 256$.

$$\|x\|_2 = |x| = \left(\sum_{i=1}^{M} x_i^2 \right)^{\frac{1}{2}} ,$$

$$\|x\|_{inf} = \max_i x_i,$$

$\forall x \in \mathbb{R}^M$, where $M = N$ for u, s, w and $M = 2N$ for θ, ω. We denote such residues, respectively, by $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_{inf}$.

variable	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_{inf}$
u	1.1110e-07	3.1756e-07	6.2583e-06
s	2.8902e-05	9.8419e-06	2.2215e-06
θ	0.0881	0.0751	0.2354
w	2.3700e-06	5.5032e-06	1.8558e-04
ω	0.0596	0.0477	0.1116

Table 2.1: the residue values.

The second image. The following values of the parameters have been used: $\alpha = 0.5$, $\tau = 0.4$ for initialization and $\alpha = 1.4 \cdot 10^{-2}$, $\tau = 1.2 \cdot 10^{-2}$ in subsequent cycles;$\varepsilon = 1.01 \cdot 10^{-3}$, $\mu = 2 \cdot 10^5$, $\alpha_1 = 2$, $\beta_\varepsilon = 10^{-3}$, $\mu_\varepsilon = \varepsilon \cdot 10^{-2}$, $threshold_w = 5 \cdot 10^2$, $threshold_\omega = 20$, $threshold_\theta = 400$. The number of cycles is 3 and they are executed in about eight hours.

Figure 2.4 shows the datum g and Figure 2.5 shows the functions u, s, w computed by the program. Note that s is white within the rectangle, being the grayscale without jumps.

Figure 2.6 shows the value of the computed energy density of the functional $M_{\varepsilon,0}$ at each point of the square Ω. We see that the algorithm detects the corner points of the rectangle, since at such points the value of the energy density exhibits peaks.

variable	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_{inf}$
u	4.3826e-07	5.0744e-06	2.2767e-04
s	5.1184e-05	9.9203e-06	4.3015e-06
θ	0.0412	0.0353	0.1160
w	4.7578e-07	2.5174e-06	1.3896e-04
ω	0.0519	0.0453	0.1

Table 2.2: the residue values.

Finally, Table 2.2 list the values of normalized residues of linear systems for u, s, θ, w, ω after the last iteration in the norms chosen above.

Figure 2.4: the function g.

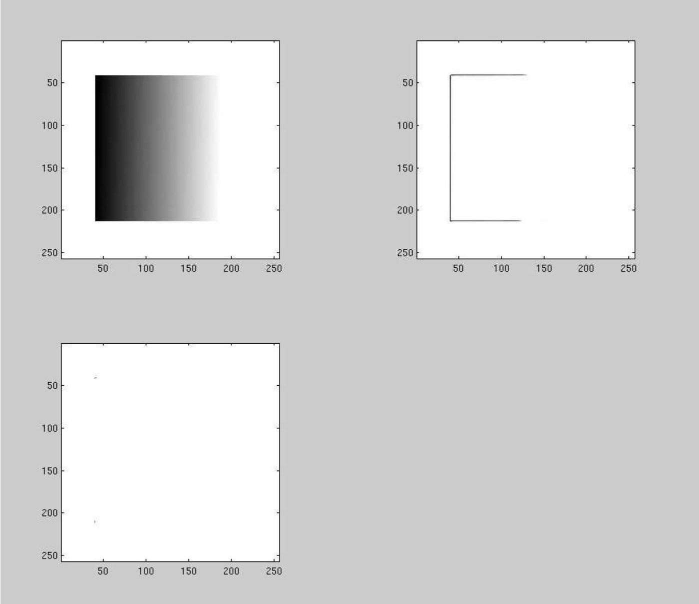

Figure 2.5: the computed functions u, s, w.

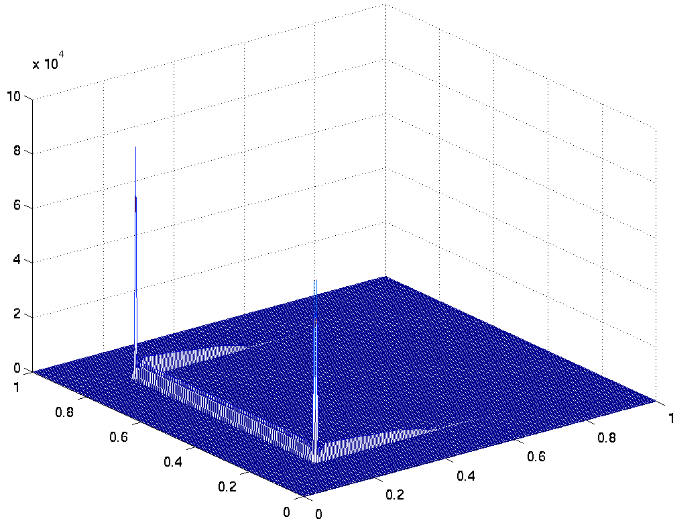

Figure 2.6: the energy density.

The third image. The following values of the parameters have been used: $\alpha = 2 \cdot 10^{-2}$, $\tau = 10^{-2}$ for initialization and $\alpha = 4 \cdot 10^{-3}$, $\tau = 2 \cdot 10^{-3}$ in subsequent cycles; $\varepsilon = 1.01 \cdot 10^{-3}$, $\mu = 2 \cdot 10^{5}$, $\alpha_1 = 10^{-1}$, $\beta_\varepsilon = 10^{-3}$, $\mu_\varepsilon = \varepsilon \cdot 10^{-2}$, $threshold_w = 6.2 \cdot 10^{5}$, $threshold_\omega = 80$, $threshold_\theta = 400$. The number of cycles is 3 and they are executed in about nine hours.

Figure 2.7 shows the datum g and Figure 2.8 shows the functions u, s, w computed by the program.

Figure 2.9 shows the value of the computed energy density of the functional $M_{\varepsilon,0}$ at each point of the square Ω. We see that the algorithm detects the corner points of the cherry, since at such points the value of the energy density exhibits peaks, however two peaks appear along the curvilinear boundary. In order to improve this experiment further work is required.

The table 2.3 list the values of normalized residues of linear systems for u, s, θ, w, ω after the last iteration in the same norms chosen for other experiments.

Figure 2.7: the function g.

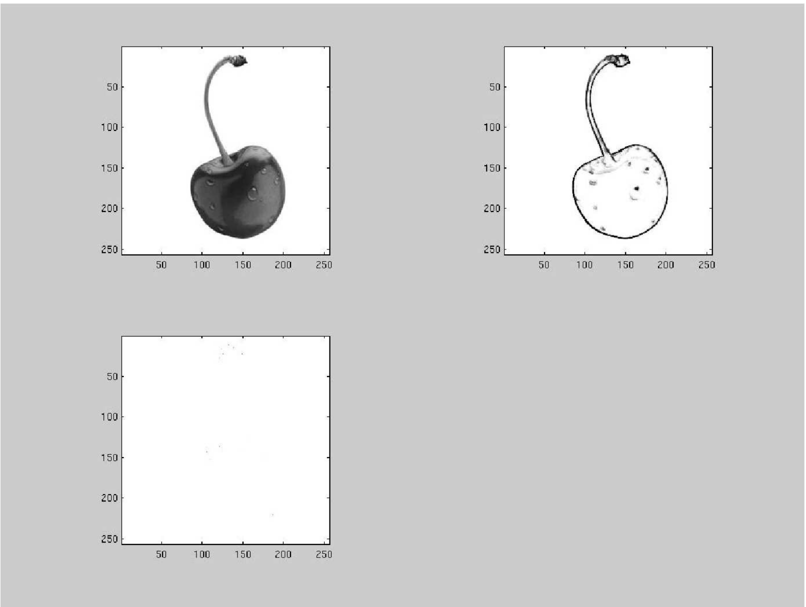

Figure 2.8: the computed functions u, s, w.

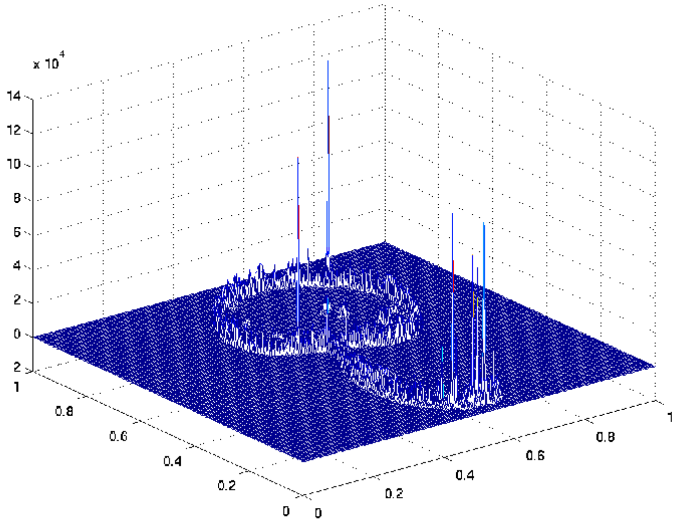

Figure 2.9: the energy density.

variable	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_{inf}$
u	1.8329e-06	7.6153e-06	2.9550e-04
s	2.0505e-05	9.1350e-06	5.1213e-06
θ	0.0955	0.0958	0.0995
w	2.1964e-06	4.5558e-06	1.4199e-04
ω	0.0761	0.0819	0.1305

Table 2.3: the residue values.

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