A thermodynamically-consistent theory of the ferro/paramagnetic transition

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Abstract: We propose a continuum theory describing the evolution of magnetization and temperature in a rigid magnetic body. The theory is based on a microforce balance, an energy balance, and an entropy imbalance. We motivate the choice of a class of constitutive equations, consistent with the entropy imbalance, that appear appropriate to describe the phase transition taking place in a ferromagnet at the Curie point. By combining these constitutive equations with the balance laws, we formulate an initial-boundary value problem for the magnetization and temperature fields, and we prove existence of weak solutions.

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Contents

1 Introduction

Micromagnetics, a variational theory whose name was coined by W.F. Brown [6], describes the equilibria of saturated ferromagnets, as determined by the competition among the magnetostatic, exchange and anisotropy energies (see, in particular, [10, 11, 20, 21, 29] and, for a survey, [22]); among other things, the theory predicts the formation of domain structures and the occurrence of hysteretic phenomena. The object of *dynamic micromagnetics*, a continuum theory developed in [3, 12], is to study the space-time evolution of the magnetization vector when the applied field changes, with or without concurrent mechanical deformations; in particular, the theory provides a framework within which phenomenological equations like Landau-Lifshitz' [23] and Gilbert's [16] are given a precise position.

In micromagnetics, dynamic or not, mechanical and thermal effects are generally left out of the picture: the former explicitly, by restricting attention to ferromagnetic bodies at mechanical rest [3]; the latter implicitly, by imposing the saturation constraint, that is to say, by imposing that the magnetization vector field is unimodular everywhere and at all times, as is the case when the temperature is well below the Curie point. Given that the Curie temperature of most ferromagnetic materials is of the order of 10^3K (see e.g. [2, Appendix D]), dynamic micromagnetics is bound to describe evolution processes taking place at room temperature. The Landau-Lifshitz and Gilbert equations are widely accepted mathematical models for such processes; a number of analytical investigations has been devoted to existence [1, 3, 17, 32], regularity [7, 8, 17, 26], and qualitative behavior [18, 19, 33] of solutions to that equation.

There are, however, evolutionary processes of technological importance, such as H(eat)A(ssisted)M(agnetic)R(ecording) processes, where the temperature field of the magnetic medium is temporarily raised in a small portion of the body so as to reduce its local magnetic coercivity [25, 31]: a description of such processes calls for a nonisothermal extension of dynamic micromagnetics. Such an extension has been pursued by Garanin *et al.* [15], who used techniques from statistical mechanics, and by Suhl [31] (see p. 11-15), who gave an argument to motivate a system of evolution equations for magnetization and temperature, with magnetization as the heat source driving temperature changes. To our knowledge, however, except for the attempt in [30], no continuum theory effectively coupling an evolution equation for the magnetization with the heat equation has been proposed so far. The theory we put forward in this paper under the name of *dynamic thermomicromagnetics* is meant to fill this gap. Its derivation is outlined in § 2, where we arrive at a strong formulation of a large class of initial/boundary value problems; in § 3, a weak notion of solution is introduced and a global-in-time existence theorem is stated, whose proof is given in § 3.2.

1.1 The model

Two distinctive features of our model are that (i) *the saturation constraint is dropped*, and that (ii) *the free energy includes a Ginzburg-Landau part* inducing saturation at the phase transition temperature (a G-L free energy is incorporated also in the model by Suhl [31]). Having dropped saturation, we are led to require that, in addition to the parallel component [12, 3], (iii) *the component of the magnetic-microforce balance perpendicular to the magnetization vector vanish*. Next, (iv) we postulate an *energy balance* that takes into account both thermal conduction and the working of magnetic microforces. Finally, (v) we supplement the balance laws by an *entropy imbalance principle*, that filters a fairly general class of constitutive equations, and helps giving them a convenient partial representation; our choice of a subclass amenable to mathematical analysis is carefully motivated in the last part of §2.

1.2 The mathematical problem

An informal strong formulation helps us delineate the features of the initial/boundary value problem we study.

Let Ω be a bounded domain of $\mathbb{R}^3$ with a reasonably smooth boundary $\partial\Omega$, occupied by the magnetic material. Given: an initial state $(\mathbf{m}_0, \theta_0) : \Omega \to \mathbb{R}^3 \times \mathbb{R}^+$, a finite time interval $(0, T)$, a magnetic field $\mathbf{h} : (0, T) \times \Omega \to \mathbb{R}^3$, and a *thermostat*, that is, an external temperature field $\theta_e : (0, T) \times \partial\Omega \to \mathbb{R}^+$; we seek a (*magnetization, temperature*) *process* $(\mathbf{m}, \theta) : [0, T) \times \Omega \to \mathbb{R}^3 \times \mathbb{R}^+$ satisfying the evolution equations

$$\alpha\,\dot{\mathbf{m}} - \tau\,\Delta\dot{\mathbf{m}} - \frac{1}{g(|\mathbf{m}|)}\mathbf{m} \times \dot{\mathbf{m}} - \mu\,\Delta\mathbf{m} + \varphi_0'(\mathbf{m}) = -\theta\varphi_1'(\mathbf{m}) + \mathbf{h}, \tag{1.1a}$$

$$c(\theta)\dot{\theta} - \kappa\,\Delta\theta = \alpha|\dot{\mathbf{m}}|^2 + \tau|\nabla\dot{\mathbf{m}}|^2 + \theta\varphi_1'(\mathbf{m}) \cdot \dot{\mathbf{m}} \tag{1.1b}$$

on $(0, T) \times \Omega$, with initial conditions

$$\mathbf{m}(0, x) = \mathbf{m}_0(x), \qquad \theta(0, x) = \theta_0(x) \quad \text{in } \Omega, \tag{1.2}$$

and boundary conditions

$$\partial_{\mathbf{n}}\mathbf{m} = \mathbf{0}, \qquad \kappa\,\partial_{\mathbf{n}}\theta + b(\theta - \theta_{\mathrm{e}}) = 0 \quad \text{on } (0,T)\times\partial\Omega. \tag{1.3}$$

Here a superposed dot denotes time differentiation, an apostrophe differentiation with respect to the corresponding argument (in the present case, $\mathbf{m}$), and $\partial_{\mathbf{n}}$ differentiation in the outward normal direction $\mathbf{n}$ to the boundary. The given field $b : \partial\Omega \to \mathbb{R}^+$ regulates contact heat condution at the boundary. The physical meaning and the mathematical qualifications of the constants α, τ, μ, and κ, as well as of the functions $c, g : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi_0, \varphi_1 : \mathbb{R}^3 \to \mathbb{R}$ will be stated later; at this stage, it suffices to point out that the thermomagnetic interactions we envisage are encapsulated in the right sides of the field equations (1.1a) and (1.1b).

The main mathematical difficulties with this problem are due to the L^1-structure of the right-hand side of (1.1b), which may even be negative. Our existence result relies on a Galerkin approximation scheme, combined with a regularization and a sequence of judiciously ordered limit passages. We first prove non-negativity of temperature, then we establish certain physically relevant a-priori estimates for $\nabla\mathbf{m}$, $\dot{\mathbf{m}}$, $\tau\nabla\dot{\mathbf{m}}$, and θ. Moreover, using a sophisticated technique developed in [4, 5] combined with a careful interpolation of the 'adiabatic' term $\theta\varphi_1'(\mathbf{m})\cdot\dot{\mathbf{m}}$, we estimate also the temperature gradient. The other delicate point in the limit passage is the 'identification' of the nonlinear term $\alpha|\dot{\mathbf{m}}|^2 + \tau|\nabla\dot{\mathbf{m}}|^2$, for which we rely on lower semicontinuity arguments accompanied by energetic estimates.

Our existence theorem holds also in the case when $\tau = 0$; interestingly, our study of the asymptotical behaviour of solutions for $\tau \to 0$ demonstrates that they tend to the solution we know to exist in that case.

2 Dynamic thermomicromagnetics: Modelization

2.1 Balance equations

Dynamic micromagnetics [3, 12, 28] regards ferromagnets as continuous material bodies endowed with two interacting physical structures: the one *mechanical*, identified with the atom lattice, the other *magnetic*, ascribed to the electron population; accordingly, balance laws are posited both for the composite continuum and for the magnetic structure alone. In this paper, while we neglect mechanical deformations for simplicity, we introduce an additional *thermal* structure, accounting for atom fluctuations about their lattice sites. Precisely, we consider a rigid material body resulting from the composition of two interacting structures, the one mechanical-magnetic, the other thermal.

For our composite continuum, the dynamical descriptors are: the *microstress* tensor $\mathbf{C}$; the *internal microforce* vector $\mathbf{k}$, accounting for structure interactions; and the *external microforce* vector $\mathbf{b}$. We postulate that these dynamical descriptors are consistent with a balance statement which – on denoting volume and surface measures by $\mathrm{d}x$ and $\mathrm{d}S$, respectively $-$ reads:

$$\int_{\partial\mathcal{P}} \mathbf{C}\mathbf{n}\,\mathrm{d}S + \int_{\mathcal{P}} (\mathbf{k}+\mathbf{b})\,\mathrm{d}x = \mathbf{0},$$

for every part $\mathcal{P}$ with smooth boundary $\partial\mathcal{P}$ of the region Ω of a three-dimensional Euclidean point space occupied by the body in question. We also postulate the following partwise energy balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \varepsilon\,\mathrm{d}x = \int_{\partial\mathcal{P}} (-\mathbf{q} + \mathbf{C}^{\top}\dot{\mathbf{m}})\cdot\mathbf{n}\,\mathrm{d}S + \int_{\mathcal{P}} \mathbf{b}\cdot\dot{\mathbf{m}}\,\mathrm{d}x, \quad \text{for every } \mathcal{P}\subset\Omega, \tag{2.1}$$

where ε is the volume density of *internal energy* and $\mathbf{q}$ is the *heat flux*, and where the additional energy flux $\mathbf{C}^{\top}\dot{\mathbf{m}}\cdot\mathbf{n}$ and the energy bulk-supply $\mathbf{b}\cdot\dot{\mathbf{m}}$ are due to the working of, respectively, contact and distance microforces; for simplicity, we do not include a bulk heat supply in our theory.

By a standard argument, these two partwise balances yield, respectively, the local *microforce balance*

$$\operatorname{div}\mathbf{C} + \mathbf{k} + \mathbf{b} = \mathbf{0} \quad \text{in } \Omega \tag{2.2}$$

and the local *energy balance*:

$$\dot{\varepsilon} = -\operatorname{div}\mathbf{q} - \mathbf{k}\cdot\dot{\mathbf{m}} + \mathbf{C}\cdot\nabla\dot{\mathbf{m}} \quad \text{in } \Omega; \tag{2.3}$$

equations (2.2) and (2.3) are the pointwise balances upon which our theory is built.

Remark 2.1 In this paper we measure magnetization density and magnetic fields in M_0 units, with M_0 the saturation magnetization at zero temperature: e.g., for $\mathbf{M}$ the *magnetization density*, we set

$$\mathbf{m} := M_0^{-1}\mathbf{M}.$$

Likewise, for Θ_c the Curie temperature, we measure the *absolute temperature* Θ in Θ_c units, and set

$$\theta := \Theta_c^{-1}\Theta.$$

2.2 Magnetic inertia

As a first step toward transforming the balance equation (2.2) into an evolution equation, we split $\mathbf{b}$ into *noninertial* and *inertial* parts:

$$\mathbf{b} = \mathbf{b}^{\mathrm{ni}} + \mathbf{b}^{\mathrm{in}}.$$

In standard continuum theories of ferromagnetic materials, the noninertial part is composed of the *applied field* $\mathbf{h}$, regarded as a control variable, and the self-induced field $\mathbf{h}_s = -\nabla u$, with u the solution of the quasistatic Maxwell equations. Without any substantial loss of generality (see Remark 4.3), we leave the self-induced magnetic field out of our present picture, and set

$$\mathbf{b}^{\mathrm{ni}} \equiv \mathbf{h}.$$

As to the inertial part, we take

$$\mathbf{b}^{\text{in}} = \frac{1}{g(|\mathbf{m}|)}\mathbf{m} \times \dot{\mathbf{m}}, \tag{2.4}$$

with g a continuous function such that

$$g(1) = \gamma < 0. \tag{2.5}$$

Here γ is proportional to the *gyromagnetic ratio*, a fundamental physical constant [2]; condition (2.5) ensures that the standard model of [3] for dynamic micromagnetics is recovered when the saturation constraint is enforced. Note that, since we scale magnetic fields by M_0, our scaled gyromagnetic ratio $g(|\mathbf{m}|)$ has dimension $(\text{time})^{-1}$. With (2.4), equation (2.2) takes the form:

$$-\frac{1}{g(|\mathbf{m}|)}\mathbf{m} \times \dot{\mathbf{m}} = \operatorname{div} \mathbf{C} + \mathbf{k} + \mathbf{h}\,. \tag{2.6}$$

Remark 2.2 The constitutive nature of any choice of distance forces of inertial nature is discussed in [27]. Generally, such forces expend power during an evolutionary process of the system they act upon. This is not the case for magnetic inertia forces, which are inherently powerless, because $\mathbf{b}^{\text{in}} \cdot \dot{\mathbf{m}} \equiv 0$.

Remark 2.3 For a scaled gyromagnetic function of the form

$$g(m) = \gamma m^2, \quad \gamma < 0, \tag{2.7}$$

the inertial distance microforce becomes

$$\mathbf{b}^{\text{in}} = \gamma^{-1}\boldsymbol{m} \times \dot{\boldsymbol{m}}, \quad \boldsymbol{m} := |\mathbf{m}|^{-1}\mathbf{m},$$

making a comparison of (2.6) with the Landau-Lifschitz-Bloch equation described in [14] possible (see §2.5). However, relation (2.7) is inconsistent with our assumption (3.1b), and thus unfit for our present proof of existence of weak solutions.

2.3 Entropy imbalance and general constitutive choices

The partwise *entropy imbalance* we postulate for the composite continuum we envisage is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \eta \,\mathrm{d}x \geq -\int_{\partial\mathcal{P}} \theta^{-1}\mathbf{q} \cdot \mathbf{n} \,\mathrm{d}S, \tag{2.8}$$

where η is the volume density of *entropy* and $\theta > 0$ is presumed strictly positive. Note that, at variance with the energy flux in (2.1), that consists of contributions from both the thermal and the magnetic structure, the latter structure does not contribute to the entropy flux, which is here taken proportional to the heat flux through the inverse of the absolute temperature, just as is standard in thermomechanics (see

Remark 2.4 here below). We also introduce the volume density of Helmholtz *free energy*:

$$\psi := \varepsilon - \eta\theta \,. \tag{2.9}$$

With this definition and (2.3), the local form of (2.8) reads:

$$\dot{\psi} \leq -\eta\dot{\theta} - \theta^{-1}\mathbf{q}\cdot\nabla\theta - \mathbf{k}\cdot\dot{\mathbf{m}} + \mathbf{C}\cdot\nabla\dot{\mathbf{m}}. \tag{2.10}$$

A first important use of this *dissipation inequality* is to suggest what fields should be the object of constitutive prescriptions, and in terms of what state variables, in order to close a theory based on the balance equations (2.2) and (2.3): a glance to (2.10) suffices to see that the relevant fields are:

$$\psi, \eta, \mathbf{q}, \mathbf{k}, \text{ and } \mathbf{C};$$

and that a reasonably inclusive list of state variables is:

$$\Lambda := (\theta, \nabla\theta, \mathbf{m}, \nabla\mathbf{m}, \dot{\mathbf{m}}, \nabla\dot{\mathbf{m}}).$$

Secondly, as proposed by Coleman and Noll in their classic paper [9], the constitutive prescriptions should be consistent with the dissipation inequality at any given state for whatever local continuation of any conceivable process attaining that state. Enforcing this consistency requirement yields partial representation results that apply to all admissible constitutive equations.

In our case, as is easy to see, the constitutive equations for free energy and entropy must have the forms:

$$\psi = \widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m}), \quad \eta = -\partial_\theta\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m}); \tag{2.11}$$

in addition, inequality (2.10) reduces provisionally to

$$0 \leq -\theta^{-1}\widetilde{\mathbf{q}}(\Lambda)\cdot\nabla\theta - \big(\widetilde{\mathbf{k}}(\Lambda) + \partial_{\mathbf{m}}\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m})\big)\cdot\dot{\mathbf{m}} + \big(\widetilde{\mathbf{C}}(\Lambda) - \partial_{\nabla\mathbf{m}}\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m})\big)\cdot\nabla\dot{\mathbf{m}}. \tag{2.12}$$

Now, let

$$\Lambda_0 = (\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \mathbf{0}),$$

and let

$$\mathbf{k}^{\mathrm{eq}} := \widetilde{\mathbf{k}}(\Lambda_0), \quad \mathbf{C}^{\mathrm{eq}} := \widetilde{\mathbf{C}}(\Lambda_0)$$

be the so-called *equilibrium parts* of the fields $\mathbf{k}$ and $\mathbf{C}$, with

$$\mathbf{k}^{\mathrm{neq}} = \widetilde{\mathbf{k}}^{\mathrm{neq}}(\Lambda) := \widetilde{\mathbf{k}}(\Lambda) - \widetilde{\mathbf{k}}(\Lambda_0), \quad \mathbf{C}^{\mathrm{neq}} = \widetilde{\mathbf{C}}^{\mathrm{neq}}(\Lambda) := \widetilde{\mathbf{C}}(\Lambda) - \widetilde{\mathbf{C}}(\Lambda_0),$$

their *nonequilibrium parts*. By definition,

$$\widetilde{\mathbf{k}}^{\mathrm{neq}}(\Lambda_0) = \mathbf{0}, \quad \widetilde{\mathbf{C}}^{\mathrm{neq}}(\Lambda_0) = \mathbf{0};$$

moreover, the reduced dissipation inequality (2.12) implies that

$$\widetilde{\mathbf{q}}(\Lambda_0) = \mathbf{0}.^1 \tag{2.13}$$

An application of the Coleman-Noll procedure to the reduced dissipation inequality (2.12) yields:

$$\begin{aligned} \mathbf{k}^{\mathrm{eq}} &= -\partial_{\mathbf{m}}\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m}), \\ \mathbf{C}^{\mathrm{eq}} &= \partial_{\nabla\mathbf{m}}\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m}), \end{aligned} \tag{2.14}$$

and further reduces (2.12) to the form:

$$0 \leq -\theta^{-1}\widetilde{\mathbf{q}}(\Lambda)\cdot\nabla\theta - \widetilde{\mathbf{k}}^{\mathrm{neq}}(\Lambda)\cdot\dot{\mathbf{m}} + \widetilde{\mathbf{C}}^{\mathrm{neq}}(\Lambda)\cdot\nabla\dot{\mathbf{m}}, \tag{2.15}$$

whence

$$\begin{aligned} &0 \leq -\widetilde{\mathbf{q}}(\theta, \nabla\theta, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \mathbf{0})\cdot\nabla\theta, \\ &0 \leq -\widetilde{\mathbf{k}}^{\mathrm{neq}}(\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \dot{\mathbf{m}}, \mathbf{0})\cdot\dot{\mathbf{m}}, \\ &0 \leq \widetilde{\mathbf{C}}^{\mathrm{neq}}(\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \nabla\dot{\mathbf{m}})\cdot\nabla\dot{\mathbf{m}}\,. \end{aligned} \tag{2.16}$$

Finally, repeated applications of a standard algebraic lemma (for a proof, see [3, Appendix B]) yield general thermodynamically admissible representations for the constitutive mappings delivering the nonequilibrium fields $\mathbf{q}, \mathbf{k}^{\mathrm{neq}}$, and $\mathbf{C}^{\mathrm{neq}}$. These representations are:

$$\begin{aligned} \widetilde{\mathbf{q}}(\theta, \nabla\theta, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \mathbf{0}) &= \widetilde{\mathbf{Q}}(\theta, \nabla\theta, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \mathbf{0})\nabla\theta, \\ \widetilde{\mathbf{k}}^{\mathrm{neq}}(\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \dot{\mathbf{m}}, \mathbf{0}) &= \widetilde{\mathbf{K}}(\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \dot{\mathbf{m}}, \mathbf{0})\dot{\mathbf{m}}, \\ \widetilde{\mathbf{C}}^{\mathrm{neq}}(\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \nabla\dot{\mathbf{m}}) &= \widetilde{\mathbb{C}}(\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \nabla\dot{\mathbf{m}})\nabla\dot{\mathbf{m}}, \end{aligned} \tag{2.17}$$

consistent with (2.16).

2.4 Basic integral estimates

We collect here certain formal estimates following from the balance and imbalance principles and the general constitutive equations we have considered so far; more special constitutive choices and estimates will come later.

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[1] Here is an argument leading to (2.13): fix $\mathbf{m}$ and $\nabla\mathbf{m}$, let

$$\mathfrak{h}(\mathbf{g}) := \theta^{-1}\widetilde{\mathbf{q}}(\theta, \mathbf{g}, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \mathbf{0})\cdot\mathbf{g}\,,$$

and note that the reduced dissipation inequality (2.12) implies that $\mathfrak{h}$ has nonnegative values; then, since $\mathfrak{h}(\mathbf{0}) = 0$ and $\widetilde{\mathbf{q}}$ is assumed smooth, it must be that

$$\partial_{\mathbf{g}}\mathfrak{h}(\mathbf{0}) = \theta^{-1}\widetilde{\mathbf{q}}(\theta, \mathbf{0}, \mathbf{m}, \nabla\mathbf{m}, \mathbf{0}, \mathbf{0}) = \mathbf{0}.$$

ENERGY. We begin by manipulating the local form of the energy balance. From (2.9) and the second of (2.11) we have that

$$\varepsilon = \psi - \theta\partial_\theta\psi,$$

whence, with the use also of (2.14),

$$\dot{\varepsilon} = \partial_{\mathbf{m}}\psi \cdot \dot{\mathbf{m}} + \partial_{\nabla\mathbf{m}}\psi \cdot \nabla\dot{\mathbf{m}} - \theta(\partial_\theta\psi)^{\boldsymbol{\cdot}} = -\mathbf{k}^{\mathrm{eq}} \cdot \dot{\mathbf{m}} + \mathbf{C}^{\mathrm{eq}} \cdot \nabla\dot{\mathbf{m}} - \theta(\partial_\theta\psi)^{\boldsymbol{\cdot}}\,.$$

Thus, given that

$$\mathbf{k} = \mathbf{k}^{\mathrm{eq}} + \mathbf{k}^{\mathrm{neq}} \quad \text{and} \quad \mathbf{C} = \mathbf{C}^{\mathrm{eq}} + \mathbf{C}^{\mathrm{neq}}, \tag{2.18}$$

the local energy balance (2.3) can be written as follows:

$$-\,\theta(\partial_\theta\psi)^{\boldsymbol{\cdot}} + \operatorname{div}\mathbf{q} = -\mathbf{k}^{\mathrm{neq}} \cdot \dot{\mathbf{m}} + \mathbf{C}^{\mathrm{neq}} \cdot \nabla\dot{\mathbf{m}}\,. \tag{2.19}$$

Next, consistent with the second of the boundary conditions (1.3), we assume that the outward heat flow at the boundary is proportional to the difference between θ and the thermostat temperature $\theta_{\mathrm{e}} \geq 0$:

$$\mathbf{q} \cdot \mathbf{n} = b(\theta - \theta_{\mathrm{e}}) \quad \text{on } \partial\Omega, \tag{2.20}$$

with $b = b(x) > 0$ and $\theta_{\mathrm{e}} = \theta_{\mathrm{e}}(t, x)$. Then, for all test functions w, we have the following general precursor of (3.7) below:

$$\begin{aligned}\int_\Omega \big(-\theta(\partial_\theta\psi)^{\boldsymbol{\cdot}} w + \mathbf{q} \cdot \nabla w\big)\,\mathrm{d}x + \int_{\partial\Omega} b\,\theta w\,\mathrm{d}S \\ = \int_\Omega \big(-(\mathbf{k}^{\mathrm{neq}} \cdot \dot{\mathbf{m}} - \mathbf{C}^{\mathrm{neq}} \cdot \nabla\dot{\mathbf{m}})w\big)\,\mathrm{d}x + \int_{\partial\Omega} b\,\theta_{\mathrm{e}} w\,\mathrm{d}S\,.\end{aligned}$$

Remark 2.4 With (2.20), the entropy imbalance (2.8) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta\,\mathrm{d}x \geq -\int_{\partial\Omega} b\Big(1 - \frac{\theta_{\mathrm{e}}}{\theta}\Big)\,\mathrm{d}S\,. \tag{2.21}$$

A feature (perhaps a physical limitation) of our model that this inequality makes evident is that, as a direct consequence of our assumption for the entropy flux, *entropy growth is bounded below only by the heat intake through the body's boundary.*

LYAPOUNOV STRUCTURE. From the microforce balance (2.6) we have that

$$\begin{aligned}0 &= \dot{\mathbf{m}} \cdot \operatorname{div}\mathbf{C} + \mathbf{k} \cdot \dot{\mathbf{m}} + \mathbf{h} \cdot \dot{\mathbf{m}} \\ &= \operatorname{div}(\mathbf{C}^\top\dot{\mathbf{m}}) - (-\mathbf{k}^{\mathrm{eq}} \cdot \dot{\mathbf{m}} + \mathbf{C}^{\mathrm{eq}} \cdot \nabla\dot{\mathbf{m}} - \mathbf{h} \cdot \dot{\mathbf{m}}) - (-\mathbf{k}^{\mathrm{neq}} \cdot \dot{\mathbf{m}} + \mathbf{C}^{\mathrm{neq}} \cdot \nabla\dot{\mathbf{m}}) \\ &= \operatorname{div}(\mathbf{C}^\top\dot{\mathbf{m}}) - (\partial_{\mathbf{m}}\psi \cdot \dot{\mathbf{m}} + \partial_{\nabla\mathbf{m}}\psi \cdot \nabla\dot{\mathbf{m}} - \mathbf{h} \cdot \dot{\mathbf{m}}) - (-\mathbf{k}^{\mathrm{neq}} \cdot \dot{\mathbf{m}} + \mathbf{C}^{\mathrm{neq}} \cdot \nabla\dot{\mathbf{m}})\,.\end{aligned}$$

Consider the following functional:

$$\Psi(\theta, \mathbf{m}) := \int_\Omega \big(\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m}) + \psi^{\mathrm{ext}}(\mathbf{m})\big)\,\mathrm{d}x,$$

where

$$\psi^{\text{ext}}(\mathbf{m}) := -\mathbf{h}\cdot\mathbf{m}\,.$$

Let the field $\mathbf{h}$ be differentiable with respect to time and, consistent with the first of the boundary conditions (1.3), let*the microscopic tractions be null at the boundary*:

$$\mathbf{C}\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{2.22}$$

Then,

$$\dot{\Psi} + \int_\Omega \big(-\dot{\theta}\,\partial_\theta\psi + \theta^{-1}\mathbf{q}\cdot\nabla\theta\big)\,\mathrm{d}x + \int_\Omega \dot{\mathbf{h}}\cdot\mathbf{m}\,\mathrm{d}x$$
$$= -\int_\Omega \big(-\theta^{-1}\widetilde{\mathbf{q}}(\Lambda)\cdot\nabla\theta - \widetilde{\mathbf{k}}^{\text{neq}}(\Lambda)\cdot\dot{\mathbf{m}} + \widetilde{\mathbf{C}}^{\text{neq}}(\Lambda)\cdot\nabla\dot{\mathbf{m}}\big)\,\mathrm{d}x\,.$$

Thus, in view of (2.15),

$$\dot{\Psi} + \int_\Omega \big(-\dot{\theta}\,\partial_\theta\psi + \theta^{-1}\mathbf{q}\cdot\nabla\theta\big)\,\mathrm{d}x + \int_\Omega \dot{\mathbf{h}}\cdot\mathbf{m}\,\mathrm{d}x \le 0,$$

which is the sort of basic *Lyapunov structure* our evolution problem has.

2.5 Specialization of constitutive choices

So far, initial and boundary conditions apart, our mathematical model for dynamic thermomicromagnetics consists of:

- the evolution equations (2.6) and (2.3) for the magnetization and the internal energy;

- the constitutive equation for the free energy, the first of equations (2.11), which determines the constitutive equations for the entropy, the equilibrium internal microforce $\mathbf{k}^{\text{eq}}$ and the equilibrium microstress $\mathbf{C}^{\text{eq}}$ via the second of (2.11) and (2.14);

- the constitutive relations (2.16) and (2.17) for the heat flux $\mathbf{q}$, the nonequilibrium internal microforce $\mathbf{k}^{\text{neq}}$ and the nonequilibrium microstress $\mathbf{C}^{\text{neq}}$.

This model is far too general to be amenable to mathematical analysis. To make it tractable, a simplification of the constitutive assumptions is in order, hopefully not as drastic as to trivialize or exclude any of the effects the theory aims to encompass.

As a part of such a delicate task, in the following we take each of the tensor-valued mappings $\widetilde{\mathbf{Q}}, \widetilde{\mathbf{K}}$ and $\widetilde{\mathbb{C}}$ in (2.17) to be a constant multiple of the appropriate identity mapping:

$$\mathbf{q} = -\kappa\nabla\theta, \qquad \kappa > 0; \tag{2.23a}$$
$$\mathbf{k}^{\text{neq}} = -\alpha\,\dot{\mathbf{m}}, \qquad \alpha > 0; \tag{2.23b}$$
$$\mathbf{C}^{\text{neq}} = \tau\nabla\dot{\mathbf{m}}, \qquad \tau \ge 0. \tag{2.23c}$$

The choice of a simple but significant form of the free-energy mapping requires a thorough discussion.

We restrict attention to free energies having the following form:

$$\widetilde{\psi}(\theta,\mathbf{m},\nabla\mathbf{m})=\varphi(\theta)+\varphi_0(\mathbf{m})+\theta\,\varphi_1(\mathbf{m})+\frac{1}{2}\,\mu\,|\nabla\mathbf{m}|^2,\quad \mu>0. \tag{2.24}$$

Consequently, the other two state functions, entropy and internal energy, specialize as follows:

$$\begin{aligned}\tilde{\eta}(\theta,\mathbf{m}) &= -\big(\varphi'(\theta)+\varphi_1(\mathbf{m})\big),\\ \tilde{\varepsilon}(\theta,\mathbf{m},\nabla\mathbf{m}) &= \varphi(\theta)-\theta\,\varphi'(\theta)+\varphi_0(\mathbf{m})+\tfrac{1}{2}\,\mu\,|\nabla\mathbf{m}|^2.\end{aligned} \tag{2.25}$$

It is important to note that *thermomagnetic coupling is carried* by one contribution to the free energy, namely, *by the value of the mapping* $(\mathbf{m},\theta)\rightarrow\theta\varphi_1(\mathbf{m})$; and that, since entropy depends on both temperature and magnetization, constitutive choices entailing order/disorder phase transitions when temperature becomes larger and/or magnetization smaller are conceivable for the mappings φ and φ_1 (that our model supports such phase transitions will be demonstrated in the first remark of our last section).

We shall now show that, although special, our present choice of a free energy is general enough to incorporate in our model all of the individual material structures relevant to a description of the ferro/paramagnetic transition as an evolutionary phenomenon taking place in a ferromagnetic heat conductor, namely: (i) the purely thermal structure leading to the heat equation; (ii) the algebraic order-parameter structure of the ferromagnetic transition; (iii) the mechanical-magnetic structure leading to the Gilbert equation for the isothermal evolution of the magnetization vector in a saturated ferromagnet at rest.

The Heat Equation. As is well known, the theory of heat conduction in a rigid material body with null volume heat supply is based on the instance of (2.3) that obtains for $\dot{\mathbf{m}}\equiv 0$. On taking $\widetilde{\psi}(\theta,\mathbf{m},\nabla\mathbf{m})=\varphi(\theta)$, and adopting (2.23a) for the heat flux $\mathbf{q}$, that instance of the energy balance reduces to the nonlinear *heat equation*: $c(\theta)\dot{\theta}=\kappa\Delta\theta$, where

$$c(\theta):=-\theta\varphi''(\theta) \tag{2.26}$$

is the *heat capacity*.

The para/ferromagnetic transition. Figure 1 illustrates the typical behavior of magnetization as a function of temperature, in the absence of an applied field. On denoting by

$$m:=|\mathbf{m}|, \tag{2.27}$$

the order parameter, the abrupt transition between ferromagnetic and paramagnetic behavior taking place when temperature falls below the critical temperature is customarily described as a bifurcation with stability exchange at the Curie point $\theta_c \equiv 1$. The simplest *coarse-grain* free energy that captures such a phenomenology has the following form, borrowed from Landau (*cf.* [24] or also e.g. [13, §10.6]):

$$\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m}) = \psi_L(\theta, \mathbf{m}) := a_0(\theta - 1)m^2 + b_0 m^4, \qquad a_0, b_0 > 0; \tag{2.28}$$

in fact, the minimum of $\psi_L(\theta, \cdot)$ is attained on the orbit of $\mathbf{m}$ satisfying (2.27) for $m = \sqrt{\frac{a_0}{2b_0}(1-\theta)}$ if $0 \leq \theta \leq 1$, and at $\mathbf{m} = \mathbf{0}$ if $\theta \geq 1$ (see Figure 1).

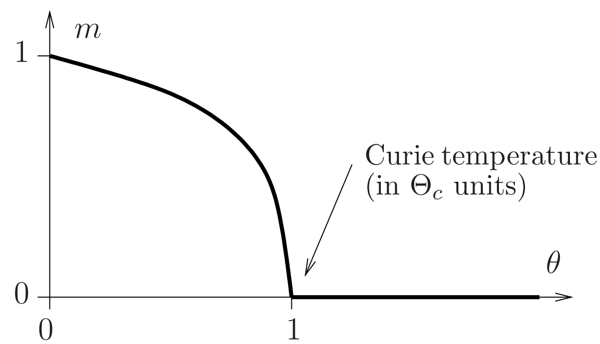

Fig. 1: Typical dependence of magnetization (in M_0 units) vs. *absolute temperature (in Θ_c units) under zero applied field* $\mathbf{h}$, *cf. e.g. [2].*

The Gilbert Equation. For a saturated ferromagnet, the magnetization field is unimodular: $|\mathbf{m}| = 1$; consequently, when the external microforce is split into its inertial and noninertial parts in the manner of §2.2, the microforce equation (2.6) reduces to

$$\gamma^{-1}\dot{\mathbf{m}} = \mathbf{m} \times (\mathrm{div}\mathbf{C} + \mathbf{k} + \mathbf{h}); \tag{2.29}$$

the constitutive choices for the free energy, the internal microforce and the microstress must be consistent with the dissipation inequality:

$$\dot{\psi} \leq -\mathbf{k} \cdot \dot{\mathbf{m}} + \mathbf{C} \cdot \nabla\dot{\mathbf{m}},$$

that is to say, the appropriate version of the general inequality (2.10). Let

$$\widetilde{\psi}(\theta, \mathbf{m}, \nabla\mathbf{m}) = \psi_{\mathrm{mm}}(\mathbf{m}, \nabla\mathbf{m}) := \psi_{\mathrm{a}}(\mathbf{m}) + \frac{1}{2}\,\mu\,|\nabla\mathbf{m}|^2,$$

with $\mu > 0$ the *exchange constant* and with $\psi_{\mathrm{a}} : \mathbb{R}^3 \to \mathbb{R}$ the *anisotropy energy*. Then, in view of (2.14), the equilibrium parts of internal microforce and microstress become, respectively,

$$\mathbf{k}^{\mathrm{eq}} = -\psi_{\mathrm{a}}'(\mathbf{m}) \quad \text{and} \quad \mathbf{C}^{\mathrm{eq}} = \mu\nabla\mathbf{m}.$$

On taking the nonequilibrium parts of the interaction microforce and microstress to be just as in (2.23), the evolution equation (2.29) reduces to:

$$\gamma^{-1}\dot{\mathbf{m}} + \alpha\mathbf{m} \times \dot{\mathbf{m}} - \tau\mathbf{m} \times \Delta\dot{\mathbf{m}} = \mathbf{m} \times (\mu\,\Delta\mathbf{m} - \psi_{\mathrm{a}}'(\mathbf{m}) + \mathbf{h})\,,$$

an equation proposed in [3] as a regularization of the *Gilbert equation*:

$$\gamma^{-1}\dot{\mathbf{m}} + \alpha \mathbf{m} \times \dot{\mathbf{m}} = \mathbf{m} \times (\mu\, \Delta \mathbf{m} - \psi_{\mathrm{a}}'(\mathbf{m}) + \mathbf{h}).$$

In conclusion, the phenomenology we wish to describe is captured provided in (2.24) we pick:

$$\varphi_0(\mathbf{m}) = -a_0|\mathbf{m}|^2 + b_0|\mathbf{m}|^4 + \psi_a(\mathbf{m}) \quad \text{and} \quad \varphi_1(\mathbf{m}) = a_0|\mathbf{m}|^2. \tag{2.30}$$

Final Remarks. With the choice (2.24), the microforce balance (2.6), combined with (2.18) and (2.14), becomes

$$-\frac{\mathbf{m}}{g(|\mathbf{m}|)} \times \dot{\mathbf{m}} = \operatorname{div}(\mu \nabla \mathbf{m} + \mathbf{C}^{\mathrm{neq}}) - \varphi_0'(\mathbf{m}) - \theta \varphi_1'(\mathbf{m}) + \mathbf{k}^{\mathrm{neq}} + \mathbf{h},$$

while the energy balance (2.19), with the use of (2.26), becomes

$$c(\theta)\dot{\theta} - \theta\, \varphi_1'(\mathbf{m}) \cdot \dot{\mathbf{m}} + \operatorname{div}\mathbf{q} = -\mathbf{k}^{\mathrm{neq}} \cdot \dot{\mathbf{m}} + \mathbf{C}^{\mathrm{neq}} \cdot \nabla \dot{\mathbf{m}}.$$

The field equations (1.1) follow by choosing $\mathbf{q}$, $\mathbf{C}^{\mathrm{neq}}$, and $\mathbf{k}^{\mathrm{neq}}$, as in (2.23); the boundary conditions (1.3) are obtained from (2.22) and (2.20) in a similar fashion. This concludes the derivation of the initial-boundary-value problem (1.1)–(1.3).

3 Dynamic thermomicromagnetics: Analysis

3.1 Weak formulation and analytical results

Henceforth we let $I := (0, T)$, $\Sigma := I \times \partial\Omega$, and $Q := I \times \Omega$. We use the standard notation $C^\infty(\cdot)$ for the space of smooth (vector- or tensor-valued) functions, $L^p(\cdot)$ for p-power Lebesgue integrable functions, $W^{k,p}(\cdot)$ for the Sobolev spaces of functions whose k-th derivatives are in $L^p(\cdot)$, and $W^{k,p}(\cdot)^*$ for its dual space. Moreover, for X a Banach space, we denote by $L^p(I; X)$ the L^p-Bochner space of X-valued functions, by $W^{k,p}(I; X)$ the corresponding Sobolev-Bochner space, and by $\mathcal{M}(\bar{I}; X)$ the space of X-valued measures on $\bar{I} = [0, T]$.

In addition to the modeling assumptions listed in the previous section, the mathematical analysis of the problem formulated in § 1.2 requires some additional qualification on the constitutive equations and on the other data. For the reader's convenience, we gather these qualifications here below.

Constitutive Assumptions.

$$\alpha,\mu>0,\ \kappa>0,\ \tau\geq 0; \tag{3.1a}$$

$$g:\mathbb{R}^+\to\mathbb{R}^+ \text{ continuous, } g(m)\geq g_0+g_1 m,\ g_0>0,\ g_1\begin{cases} =0 & \text{if } \tau>0,\\ >0 & \text{if } \tau=0;\end{cases} \tag{3.1b}$$

$$\varphi_0,\varphi_1:\mathbb{R}\to\mathbb{R} \text{ are continuously differentiable;} \tag{3.1c}$$

$$c:\mathbb{R}^+\to\mathbb{R}\ \text{ is continuously differentiable; moreover,} \tag{3.1d}$$

$$\exists\ c_{\max}\geq c_{\min}>0,\ \omega\in[1,3):\quad c_{\min}(1+\theta^{\omega-1})\leq c(\theta)\leq c_{\max}(1+\theta^{\omega-1}); \tag{3.1e}$$

$$\exists\ \zeta>0,\quad C_{\max}\in\mathbb{R}:\quad c'(\theta)\leq C_{\max}\frac{c(\theta)^2}{(1{+}\theta)^{1+\zeta}}; \tag{3.1f}$$

$$\exists\ q\geq 1,\quad C_{\min}>0:\quad C_{\min}|\mathbf{m}|^q\leq\varphi_0(\mathbf{m}); \tag{3.1g}$$

$$\exists\ q_0<\widehat{q}:=\max(6,q),\ C_{\max}{\in}\mathbb{R}:\ |\varphi_0'(\mathbf{m})|\leq\begin{cases} C_{\max}\big(1{+}|\mathbf{m}|^{q_0}\big) & \text{if } \tau>0,\\ C_{\max}\big(1{+}|\mathbf{m}|^{q_0/2}\big) & \text{if } \tau=0;\end{cases} \tag{3.1h}$$

$$\exists\ q_1<\widehat{q}\,\min\Big(\frac{\omega}{\omega{+}1},\frac{1}{\nu'}-\frac{\omega{+}3}{6\omega}\Big),\ C_{\max}{\in}\mathbb{R}:\quad \big|\varphi_1'(\mathbf{m})\big|<C_{\max}\big(1{+}|\mathbf{m}|^{q_1}\big). \tag{3.1i}$$

Assumptions on the Data.

$$\Omega\subset\mathbb{R}^3 \text{ is a bounded Lipschitz domain;} \tag{3.2a}$$

$$\mathbf{h}\in W^{1,1}(I;L^{\nu'}(\Omega;\mathbb{R}^3)); \tag{3.2b}$$

$$\mathbf{m}_0\in W^{1,2}(\Omega;\mathbb{R}^3),\qquad \theta_0\in L^\omega(\Omega),\qquad \theta_0\geq 0; \tag{3.2c}$$

$$b\in L^\infty(\partial\Omega),\qquad b\geq 0,\qquad \theta_{\rm e}\in L^1(\Sigma),\qquad \theta_{\rm e}\geq 0. \tag{3.2d}$$

Here, the exponent ν' is defined as follows:

$$\nu':=\frac{\nu}{\nu{-}1},\quad \text{where}\quad \nu=\begin{cases} 6 & \text{if } \tau>0\\ 2 & \text{if } \tau=0\end{cases}. \tag{3.3}$$

Note that $c(\theta)=\text{const.}$ satisfies (3.1d,e, f) for $\omega=1$; and that, for any $\omega\geq 1$, an example of heat-capacity function c satisfying (3.1d,e,f) is $c(\theta):=c_{\min}(1+\theta)^{\omega-1}$, provided $\zeta\leq\omega-1$. We anticipate that the rather unconventional condition (3.1f) will be expedient later to establish (3.57), while the restriction on $q_1/\widehat{q}$ in (3.1i) will bear on (3.19) and (3.55).

Definition 3.1 ***(Very weak solutions)*** *Let assumptions* (3.1) *and* (3.2) *hold. Moreover, let*

$$\mathbf{m}\in W^{1,2}(I;L^2(\Omega;\mathbb{R}^3))\cap L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{3.4a}$$

$$\theta\in L^r(I;W^{1,r}(\Omega))\cap L^\infty(I;L^\omega(\Omega))\quad \textit{for any } r:\ 1\leq r<\frac{3+2\omega}{3+\omega}, \tag{3.4b}$$

and let

$$\tau\mathbf{m}\in W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^3)). \tag{3.5}$$

We say that the pair $(\mathbf{m}, \theta)$ *is a very weak solution to the initial-boundary-value problem (1.1)–(1.3) if the following holds true:*
(i) $\mathbf{m}$ *satisfies the initial conditions (1.2);*
(ii) for all $\mathbf{z} \in C^1(\Omega; \mathbb{R}^3)$ *and for a.a.* $t \in I$,

$$\int_\Omega \Big(\alpha\dot{\mathbf{m}} - \frac{\mathbf{m}}{g(|\mathbf{m}|)} \times \dot{\mathbf{m}} + \varphi_0'(\mathbf{m})\Big) \cdot \mathbf{z} + \nabla\Big(\mu\mathbf{m} + \tau\dot{\mathbf{m}}\Big) : \nabla\mathbf{z}\, \mathrm{d}x$$
$$= \int_\Omega \Big(- \theta\varphi_1'(\mathbf{m}) + \mathbf{h}\Big) \cdot \mathbf{z}\, \mathrm{d}x\,; \quad (3.6)$$

(iii) for all $z \in C^1(Q)$ *such that* $z(T, \cdot) = 0$,

$$\int_Q \Big(- \widehat{c}(\theta)\dot{z} + \kappa\nabla\theta \cdot \nabla z\Big)\, \mathrm{d}x\, \mathrm{d}t + \int_\Sigma b\,\theta\, z \mathrm{d}S\ \mathrm{d}t$$
$$= \int_Q \Big(\alpha\big|\dot{\mathbf{m}}\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}\big|^2 + \theta\varphi_1'(\mathbf{m}){\cdot}\dot{\mathbf{m}}\Big) z\ \mathrm{d}x\, \mathrm{d}t + \int_\Sigma b\, \theta_{\mathrm{e}} z\ \mathrm{d}S\, \mathrm{d}t + \int_\Omega \widehat{c}(\theta_0(x)) z(0, x)\ \mathrm{d}x\,, \quad (3.7)$$

where $\widehat{c}$ *denotes a primitive of function* c.

Our main analytical result is the following

Theorem 3.2 *(**Existence of very weak solutions**) Assume that* (3.1) *and* (3.2) *hold with* ν *as in* (3.3). *Then, the initial-boundary value problem (1.1)–(1.3) has a solution* $(\mathbf{m}, \theta)$ *in the sense of Definition 3.1. Moreover,*

$$\dot{\theta} \in \mathcal{M}(I; W^{3,2}(\Omega)^*), \quad (3.8a)$$
$$(\widehat{c}(\theta))^{\boldsymbol{\cdot}} \in L^1(I; W^{3,2}(\Omega)^*). \quad (3.8b)$$

In addition, we study the asymptotical behaviour of very weak solutions in the limit $\tau \to 0$, that is to say, when the higher-order dissipation parameter τ vanishes (see [3] for a similar study in the isothermal case). Here is the result we are able to prove.

Theorem 3.3 *(**Asymptotics for** $\tau \to 0$) Let* $(\mathbf{m}_\tau, \theta_\tau)$ *denote a very weak solution for* $\nu = 2$ *to the initial-boundary-value problem (1.1)–(1.3).Then, there exists a sequence* $\{\tau_j\}_{j\in\mathbb{N}}$ *such that* $\tau_j \to 0$ *for* $j \to \infty$ *and*

$$\mathbf{m}_{\tau_j} \to \mathbf{m} \qquad \textit{weakly* in } W^{1,2}(I; L^2(\Omega; \mathbb{R}^3)) \cap L^\infty(I; W^{1,2}(\Omega; \mathbb{R}^3)), \quad (3.9a)$$
$$\theta_{\tau_j} \to \theta \qquad \textit{weakly* in } L^r(I; W^{1,r}(\Omega)) \cap L^\infty(I; L^\omega(\Omega)), \quad (3.9b)$$

with r *from (3.4b) as* $j \to \infty$. *Moreover, the limit* $(\mathbf{m}, \theta)$ *is a solution of (1.1)–(1.3) in the sense of Definition 3.1, with* $\tau = 0$.

Remark 3.4 Formally, the identity (3.6) follows from (1.1a) after multiplication of both sides by the test function $\mathbf{z}$ and integration by parts in space with the use of the boundary conditions (1.3). Likewise, condition (3.7) is obtained by multiplying

(1.1b) by the test function w and integrating by parts with respect to space and time with the use of (1.3) and the initial conditions (1.2). The qualifier "very weak" in definition 3.1 is motivated by the fact that, for the heat equation, an additional time integration is in order, because, as suggested by (3.8), one cannot expect the time derivative of θ to belong to any standard function space.

3.2 Proofs

We will prove Theorem 3.2 by careful successive passages to the limit in a suitably regularized Galerkin approximation and by adapting to our coupled system, with the use of function-space interpolation theory, certain sophisticated *a priori* estimates for the temperature gradient in the L^1-type theory for the heat equation. The above-mentioned τ-regularization ensures existence of the Galerkin solution; needless to say, the regularizing term is eventually set to zero by a passage to the limit.

Given a parameter $k \in \mathbb{N}$, we take an increasing sequence of finite-dimensional subspaces $V_k \subset W^{1,\infty}(\Omega)$ approximating the whole $W^{1,2}(\Omega)$ with respect to the strong topology, i.e.

$$V_k \subset V_{k+1} \qquad \text{and} \qquad \mathrm{cl}\Big(\bigcup_{k\in\mathbb{N}} V_k\Big) = W^{1,2}(\Omega) =: V, \tag{3.10}$$

where "cl" denotes closure in the $W^{1,2}$-norm. We also approximate the initial data $\mathbf{m}_0$ and θ_0 and the boundary datum $\theta_{\rm e}$ by choosing appropriate sequences $\mathbf{m}_{0,k}$, $\theta_{0,k}$, and $\theta_{{\rm e},k}$ such that

$$\mathbf{m}_{0,k} \in V_k^3 \qquad \text{and} \qquad \lim_{k\to\infty} \mathbf{m}_{0,k} = \mathbf{m}_0 \qquad \text{in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{3.11a}$$

$$\theta_{0,k} \in V_k \qquad \text{and} \qquad \lim_{k\to\infty} \theta_{0,k} = \theta_0 \qquad \text{in } L^{\omega}(\Omega), \tag{3.11b}$$

$$\theta_{{\rm e},k} \in W^{1,1}(I;L^\infty(\Gamma)) \qquad \text{and} \qquad \lim_{k\to\infty} \theta_{{\rm e},k} = \theta_{\rm e} \qquad \text{in } L^1(\Sigma). \tag{3.11c}$$

Observe that the heat capacity $c(\theta_{kln})$ is defined only for $\theta_{kln} \geq 0$. However, non-negativity of θ_{kln} (a finite linear combination of basis functions) is a property that, in general, cannot be expected. Therefore we introduce an extension of the function $\theta \mapsto c(\theta)$ to the range of negative temperatures by setting:

$$\bar{c}(\theta) = c(|\theta|).$$

For $n \in \mathbb{N}$, let $1/n > 0$ be a regularization parameter. The first step towards the construction of a solution consists in choosing, for each triplet (k,l,n), with $l \geq k$, a pair of functions $(\mathbf{m}_{kln}, \theta_{kln})$ with

$$\mathbf{m}_{kln} \in W^{1,p}(I;V_k^3),$$
$$\theta_{kln} \in L^\infty(I;V_l) \ \cap \ W^{1,1}(I;V_l^*),$$

satisfying the initial conditions

$$\mathbf{m}_{kln}(0,\cdot) = \mathbf{m}_{0,k}, \qquad \theta_{kln}(0,\cdot) = \theta_{0,l}, \tag{3.13}$$

and such that, for a.e. $t \in [0,T]$, for all $z \in V_k^3$ and $w \in V_l$,

$$\int_\Omega \left(\alpha \dot{\mathbf{m}}_{kln} - \frac{\mathbf{m}_{kln}}{g(|\mathbf{m}_{kln}|)} \times \dot{\mathbf{m}}_{kln} + \varphi_0'(\mathbf{m}_{kln}) + \frac{1}{n} |\dot{\mathbf{m}}_{kln}|^{p-2} \dot{\mathbf{m}}_{kln} \right) \cdot \mathbf{z} \,\mathrm{d}x \\ + \int_\Omega \left(\mu \nabla \mathbf{m}_{kln} + \tau \nabla \dot{\mathbf{m}} \right) \cdot \nabla \mathbf{z} \,\mathrm{d}x = \int_\Omega \left(-\theta_{kln} \varphi_1'(\mathbf{m}_{kln}) + \mathbf{h} \right) \cdot \mathbf{z} \,\mathrm{d}x, \quad (3.14)$$

and

$$\int_\Omega \kappa \nabla \theta_{kln} \cdot \nabla w + \bar{c}(\theta_{kln}) \dot{\theta}_{kln} w \,\mathrm{d}x + \int_{\partial\Omega} b\, \theta_{kln}\, w \mathrm{d}S \\ = \int_\Omega \left(\alpha \big|\dot{\mathbf{m}}_{kln}\big|^2 + \tau \big|\nabla \dot{\mathbf{m}}_{kln}\big|^2 + \theta_{kln} \varphi_1'(\mathbf{m}_{kln}) \cdot \dot{\mathbf{m}}_{kln} \right) w \,\mathrm{d}x + \int_{\partial\Omega} b\, \theta_{\mathrm{e},k}\, w \,\mathrm{d}S. \quad (3.15)$$

In other words, we seek a weak solution $(\mathbf{m}_{kln}, \theta_{kln})$ of

$$\alpha \dot{\mathbf{m}}_{kln} - \tau \Delta \dot{\mathbf{m}}_{kln} - \frac{\mathbf{m}_{kln}}{g(|\mathbf{m}_{kln}|)} \times \dot{\mathbf{m}}_{kln} + \frac{1}{n} |\dot{\mathbf{m}}_{kln}|^{p-2} \dot{\mathbf{m}}_{kln} - \Delta \mathbf{m}_{kln} + \varphi_0'(\mathbf{m}_{kln}) \\ = -\theta_{kln} \varphi_1'(\mathbf{m}_{kln}) + \mathbf{h} + \mathbf{g}_{kln}^{(1)}, \quad (3.16a)$$

$$\bar{c}(\theta_{kln}) \dot{\theta}_{kln} - \mathrm{div}\big(\kappa \nabla \theta_{kln}\big) = \alpha \big|\dot{\mathbf{m}}_{kln}\big|^2 + \tau \big|\nabla \dot{\mathbf{m}}_{kln}\big|^2 + \theta_{kln} \varphi_1'(\mathbf{m}_{kln}) \cdot \dot{\mathbf{m}}_{kln} + g_{kln}^{(1)} \quad (3.16b)$$

with boundary conditions

$$\partial_{\mathbf{n}} \mathbf{m}_{kln} = \mathbf{g}_{kln}^{(2)}, \qquad \kappa \partial_{\mathbf{n}} \theta_{kln} + b\big(\theta_{kln} - \theta_{\mathrm{e},k}\big) = g_{kln}^{(2)}, \quad (3.17)$$

and initial conditions (3.13). Here the residua $\mathbf{g}_{kln}^{(1)}$, $\mathbf{g}_{kln}^{(2)}$, $g_{kln}^{(1)}$, $g_{kln}^{(2)}$ satisfy, for a.a. $t \in I$,

$$(\mathbf{g}_{kln}^{(1)}, \mathbf{g}_{kln}^{(2)})(t) := \left(\mathbf{z} \mapsto \int_\Omega \mathbf{g}_{kln}^{(1)}(t) \cdot \mathbf{z} \,\mathrm{d}x + \int_{\partial\Omega} \mathbf{g}_{kln}^{(2)}(t) \cdot \mathbf{z} \,\mathrm{d}S \right) \in (V_k^3)^\perp \quad (3.18)$$

and similarly $(g_{kln}^{(1)}, g_{kln}^{(2)})(t) := (w \mapsto \int_\Omega g_{kln}^{(1)}(t)\, w \,\mathrm{d}x + \int_{\partial\Omega} g_{kln}^{(2)}(t)\, w \,\mathrm{d}S) \in V_k^\perp$.

The p-regularizing term on the left-hand side of (3.16a) compensates for the quadratic growth of the terms containing time derivative of $\mathbf{m}$ on the right-hand side of (3.16b) and allows us to prove global-in-time existence of solutions to (3.16a)–(3.16b), as we do in the next lemma.

Lemma 3.5 ***(Existence of Galerkin solution, a-priori estimates)*** *Let (3.1), (3.2), and (3.11) hold. Let the exponent p be chosen large enough to satisfy*

$$p > \frac{2(1+\omega)}{\omega}, \qquad \frac{1}{1+\omega} + \frac{q_1}{\widehat{q}} + \frac{1}{p} < 1. \quad (3.19)$$

Then, problem (3.16a)–(3.16b) *with initial conditions* (3.13) *and boundary conditions* (3.17) *admits a global-in-time solution* $(\mathbf{m}_{kln}, \theta_{kln})$. *Moreover, there exist*

$C_{i,kn} < +\infty$, $i = 1, \ldots, 4$ *(which depend on k and ε, but not on l) such that*

$$\big\|\mathbf{m}_{kln}\big\|_{W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^3))\cap L^\infty(I;L^{\hat q}(\Omega;\mathbb{R}^3))} \leq C_{1,kn}, \tag{3.20a}$$

$$\big\|\dot{\mathbf{m}}_{kln}\big\|_{L^p(Q;\mathbb{R}^3)} \leq C_{2,kn}, \tag{3.20b}$$

$$\big\|\theta_{kln}\big\|_{L^\infty(I;L^{1+\omega}(\Omega))\,\cap\, L^2(I;W^{1,2}(\Omega))} \leq C_{3,kn}, \tag{3.20c}$$

$$\big\|\dot{\theta}_{kln}\big\|_{L^2(Q)} \leq C_{4,kn}\,. \tag{3.20d}$$

Proof. Local existence of the Galerkin approximation follows from a standard application of ODE theory. Global-in-time existence follows by successive prolongation, using the bounds (3.20), which we now prove.

To prove (3.20a-c), let us introduce the *auxiliary potential* $\mathfrak{C} : \mathbb{R} \to \mathbb{R}$ defined by $\mathfrak{C}(\theta) := \int_0^\theta \vartheta\bar{c}(\vartheta)\,\mathrm{d}\vartheta$. Note that $\theta\bar{c}(\theta)\dot{\theta} = (\mathfrak{C}(\theta))^{\boldsymbol{\cdot}}$ and that (3.1e) ensures

$$c_{\min}\Big(\frac{1}{2}\theta^2 + \frac{1}{1+\omega}|\theta|^{1+\omega}\Big) \leq \mathfrak{C}(\theta) \leq c_{\max}\Big(\frac{1}{2}\theta^2 + \frac{1}{1+\omega}|\theta|^{1+\omega}\Big). \tag{3.21}$$

Note also that p satisfying (3.19) can be chosen by virtue of (3.1i). We test (3.14) by $\dot{\mathbf{m}}_{kln}$ and (3.15) by θ_{kln}. Summing the resulting equations we get the following identity:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\big|\nabla\mathbf{m}_{kln}\big|^2 + \varphi_0(\mathbf{m}_{kln}) + \mathfrak{C}(\theta_{kln})\,\mathrm{d}x \\
&\quad + \int_\Omega \alpha\big|\dot{\mathbf{m}}_{kln}\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_{kln}\big|^2 + \frac{1}{n}\big|\dot{\mathbf{m}}_{kln}\big|^p + \kappa|\nabla\theta_{kln}|^2\,\mathrm{d}x + \int_{\partial\Omega} b\,\theta_{kln}^2\,\mathrm{d}S \\
&= \int_\Omega \Big(\alpha\big|\dot{\mathbf{m}}_{kln}\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_{kln}\big|^2\Big)\,\theta_{kln} \\
&\quad + \theta_{kln}^2\varphi_1'(\mathbf{m}_{kln})\cdot\dot{\mathbf{m}}_{kln} - \theta_{kln}\varphi_1'(\mathbf{m}_{kln})\cdot\dot{\mathbf{m}}_{kln} + \mathbf{h}\cdot\dot{\mathbf{m}}_{kln}\,\mathrm{d}x + \int_{\partial\Omega} b\,\theta_{\mathrm{e},k}\theta_{kln}\,\mathrm{d}S.
\end{aligned} \tag{3.22}$$

Then, we use the first of (3.19) along with Hölder's and Young's inequalities to obtain the estimate

$$\int_\Omega \alpha\big|\dot{\mathbf{m}}_{kln}\big|^2\theta_{kln}\,\mathrm{d}x \leq \delta\big\|\dot{\mathbf{m}}_{kln}\big\|^p_{L^p(\Omega;\mathbb{R}^3)} + C_{\alpha,\delta,\omega,p}\|\theta_{kln}\|^{1+\omega}_{L^{1+\omega}(\Omega)}.$$

Next, we note that for each k fixed there exists a constant $C_{k,p}$ such that

$$\big\|w\big\|_{W^{1,2}(\Omega)} \leq C_{k,p}\big\|w\big\|_{L^p(\Omega)} \qquad \forall w \in V_k.$$

From this fact, we can derive the estimate

$$\int_\Omega \tau\big|\nabla\dot{\mathbf{m}}_{kln}\big|^2\theta_{kln} \leq \delta\big\|\dot{\mathbf{m}}_{kln}\big\|^p_{L^p(\Omega;\mathbb{R}^3)} + C_{k,\tau,\delta,\omega,p}\big\|\theta_{kln}\big\|^{1+\omega}_{L^{1+\omega}(\Omega)}.$$

Next, we use the second condition in (3.19) to obtain

$$\int_\Omega \theta_{kln}^2 \varphi_1'(\mathbf{m}_{kln})\cdot\dot{\mathbf{m}}_{kln}\,\mathrm{d}x \\ \le C_{\tau,\delta,\omega}(1+\|\theta_{kln}\|_{L^{1+\omega}(\Omega)}^{1+\omega}) + \|\mathbf{m}_{kln}\|_{L^{\widehat{q}}(\Omega;\mathbb{R}^3)}^{\widehat{q}} + \delta\big\|\dot{\mathbf{m}}_{kln}\big\|_{L^p(\Omega;\mathbb{R}^3)}^p.$$

As to the remaining term $\int_\Omega \theta_{kln}\varphi_1'(\mathbf{m}_{kln})\cdot\dot{\mathbf{m}}_{kln}\,\mathrm{d}x$, the same type of estimate (although with different constants) holds. Finally,

$$\int_\Omega \mathbf{h}\cdot\dot{\mathbf{m}}_{kln}\,\mathrm{d}x \le C_{5,\delta}\|\mathbf{h}\|_{L^{\nu'}(\Omega)}^{\nu'} + \delta\big\|\dot{\mathbf{m}}_{kln}\big\|_{L^\nu(\Omega;\mathbb{R}^3)}^\nu. \tag{3.23}$$

With the above estimates, choosing $\delta = 1/(4n)$, it follows from (3.22) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac12|\nabla\mathbf{m}_{kln}|^2 + \varphi_0(\mathbf{m}_{kln}) + \mathfrak{C}(\theta_{kln})\,\mathrm{d}x + \int_\Gamma b\theta_{kln}^2\,\mathrm{d}S\\
&\le C_{6,kn} + C_{7,kn}\int_\Omega |\mathbf{m}_{kln}|^{\widehat{q}} + \theta_{kln}^2 + |\theta_{kln}|^{1+\omega}\,\mathrm{d}x + \int_\Gamma b\theta_{\mathrm{e},k}\theta_{kln}\,\mathrm{d}S\\
&\le C_{6,kn} + C_{8,kn}\int_\Omega \frac12|\nabla\mathbf{m}_{kln}|^2 + \varphi_0(\mathbf{m}_{kln}) + \mathfrak{C}(\theta_{kln})\,\mathrm{d}x + \int_\Gamma b\theta_{\mathrm{e},k}(1+\theta_{kln}^2)\,\mathrm{d}S.
\end{aligned}$$

In the last inequality we use the embedding $W^{1,2}(\Omega)\subset L^6(\Omega)$ (in the case $\tau>0$) along with the coercivity of φ_0 and $\mathfrak{C}$ (*cf.* (3.1g) and (3.21)). The bounds (3.20a-c) follow from Gronwall's lemma.

The bound (3.20d) is obtained in a similar fashion by testing (3.14) by $\dot{\mathbf{m}}_{kln}$ and (3.15) by $\dot{\theta}_{kln}$. Summing the resulting equations, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac12|\nabla\mathbf{m}_{kln}|^2 + \varphi_0(\mathbf{m}_{kln}) + \frac{\kappa}{2}|\nabla\theta_{kln}|^2\,\mathrm{d}x + \int_\Gamma \frac{b}{2}\theta_{kln}^2\,\mathrm{d}S\Big)\\
&\qquad\qquad + \int_\Omega \frac1n\big|\dot{\mathbf{m}}_{kln}\big|^p + c_{\min}\big|\dot{\theta}_{kln}\big|^2\,\mathrm{d}x\\
&\le \int_\Omega \alpha\big|\dot{\mathbf{m}}_{kln}\big|^2\dot{\theta}_{kln} + \tau\big|\nabla\dot{\mathbf{m}}_{kln}\big|^2\dot{\theta}_{kln} + \theta_{kln}\varphi_1'(\mathbf{m}_{kln})\cdot\dot{\mathbf{m}}_{kln}\dot{\theta}_{kln}\\
&\qquad\qquad + \theta_{kln}\varphi_1'(\mathbf{m}_{kln})\cdot\dot{\mathbf{m}}_{kln} + \mathbf{h}\cdot\dot{\mathbf{m}}_{kln}\,\mathrm{d}x + \int_\Gamma b\theta_{\mathrm{e},k}\dot{\theta}_{kln}\,\mathrm{d}S.
\end{aligned}\tag{3.24}$$

Then we use Hölder's and Young's inequalities to deduce the estimate

$$\int_\Omega \alpha\big|\dot{\mathbf{m}}_{kln}\big|^2\dot{\theta}_{kln} \le C_{\alpha,\delta} + \delta\big\|\dot{\mathbf{m}}_{kln}\big\|_{L^p(\Omega;\mathbb{R}^3)}^p + \delta\big\|\dot{\theta}_{kln}\big\|_{L^2(\Omega)}^2,$$

where we have used the first condition in (3.19), which along with (3.1e) guarantees that $p>4$. We also note that

$$\int_\Omega \tau\big|\nabla\dot{\mathbf{m}}_{kln}\big|^2\dot{\theta}_{kln} \le C_{\alpha,\delta,k} + \delta\big\|\dot{\mathbf{m}}_{kln}\big\|_{L^p(\Omega;\mathbb{R}^3)}^p + \delta\big\|\dot{\theta}_{kln}\big\|_{L^2(\Omega)}^2.$$

Furthermore, by the first condition in (3.19) there exists $s > 1$ such that $1/p+1/2+1/s = 1$. Then,

$$\int_\Omega \theta_{kln}\varphi_1'(\mathbf{m}_{kln})\dot{\mathbf{m}}_{kln}\dot{\theta}_{kln} \\ \leq \delta\big\|\dot{\mathbf{m}}_{kln}\big\|^p_{L^p(\Omega;\mathbb{R}^3)} + \frac{c_{\min}}{2}\big\|\dot{\theta}_{kln}\big\|^2_{L^2(\Omega)} + C_{\delta,p,c_{\min}}\big\|\varphi_1'(\mathbf{m}_{kln})\theta_{kln}\big\|^s_{L^s(\Omega)}. \quad (3.25)$$

Let C_k be a constant such that $\|w\|_{L^\infty(\Omega)} \leq C_k\|w\|_{L^{\hat{q}}(\Omega)}$ for every $w \in V_k$. Then $\|\varphi_1'(\mathbf{m}_{kln})\|^s_{L^\infty(\Omega;\mathbb{R}^3)} \leq 2^{s-1}C^s_{\max}(1 + C_k^{sq_1}\|\mathbf{m}_{kln}\|^{sq_1}_{L^{\hat{q}}(\Omega;\mathbb{R}^3)}) \leq C_{5,k\varepsilon}$, where the last inequality comes from the bound (3.20a) and the embedding $W^{1,2}(\Omega) \subset L^{\hat{q}}(\Omega)$. Therefore, since $s < \omega + 1$ (this is guaranteed by the first in (3.19)), then

$$\big\|\varphi_1'(\mathbf{m}_{kln})\theta_{kln}\big\|^s_{L^s(\Omega)} \leq C_{5,kj}C_{\omega,p}\big\|\theta_{kln}\big\|^{\omega+1}_{L^{\omega+1}(\Omega)} \leq C_{5,kj}C_{\omega,p}C_{3,j}.$$

We omit the estimate of the term $\theta_{kln}\varphi_1'(\mathbf{m}_{kln})\cdot\dot{\mathbf{m}}_{kln}$, which is even simpler. The last term in (3.24) is to be treated by integration by-part in time and using the trace theorem. The bound (3.20d) is now obtained by arguing as before: we integrate (3.24) over $[0, t]$, perform the above-mentioned by-part integration, we take $\delta = 1/(4n)$, we absorb in the left-hand side of (3.25) the terms on the right-hand side containing $\dot{\mathbf{m}}_{kln}$, and we use the integral version of Gronwall's lemma. $\square$

Remark 3.6 To get the estimate on $\dot{\theta}_{kln}$ without imposing any unnecessary conditions on the exponents ω, q and q_1, we have used the fact that for k fixed all $L^p(\Omega)$ norms with $1 \leq p \leq \infty$ are equivalent when restricted on V_k.

Proposition 3.7 ***(Limit passage** $l \to \infty$**)*** *For each $k, n \in \mathbb{N}$, it is possible to extract from $\{(\mathbf{m}_{kln}, \theta_{kln})\}_{l\in\mathbb{N}}$ a subsequence such that, as $l \to \infty$:*

$$\mathbf{m}_{kln} \to \mathbf{m}_{kn} \qquad \text{weakly in } W^{1,2}(I; W^{1,2}(\Omega;\mathbb{R}^3)), \quad (3.26a)$$
$$\theta_{kln} \to \theta_{kn} \qquad \text{weakly* in } W^{1,2}(I; L^2(\Omega)) \cap L^\infty(I; L^{1+\omega}(\Omega)). \quad (3.26b)$$

Moreover, the limit $(\mathbf{m}_{kn}, \theta_{kn})$ is a weak solution of

$$\alpha\dot{\mathbf{m}}_{kn} - \tau\Delta\dot{\mathbf{m}}_{kn} - \frac{\mathbf{m}_{kn}}{g(|\mathbf{m}_{kn}|)}\times\dot{\mathbf{m}}_{kn} + \frac{1}{n}\big|\dot{\mathbf{m}}_{kn}\big|^{p-2}\dot{\mathbf{m}}_{kn} - \mu\Delta\mathbf{m}_{kn} + \varphi_0'(\mathbf{m}_{kn}) \\ = -\theta_{kln}\varphi_1'(\mathbf{m}_{kn}) + \mathbf{h} + \mathbf{g}^{(1)}_{kn}, \quad (3.27a)$$

$$\bar{c}(\theta_{kn})\frac{\partial\theta_{kn}}{\partial t} - \mathrm{div}\big(\kappa\nabla\theta_{kn}\big) = \alpha\big|\dot{\mathbf{m}}_{kn}\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_{kn}\big|^2 + \theta_{kn}\varphi_1(\mathbf{m}_{kn})\cdot\dot{\mathbf{m}}_{kn}, \quad (3.27b)$$

with initial conditions

$$\mathbf{m}_{kn}(0,\cdot) = \mathbf{m}_{0,k}, \qquad \theta_{kn}(0,\cdot) = \theta_0, \quad (3.28)$$

and boundary conditions

$$\partial_{\mathbf{n}}\mathbf{m}_{kn} = \mathbf{g}^{(2)}_{kn}, \qquad \kappa\partial_{\mathbf{n}}\theta_{kn} + b(\theta_{kn} - \theta_{\mathrm{e},k}) = 0, \quad (3.29)$$

where $(\mathbf{g}^{(1)}_{kn}, \mathbf{g}^{(2)}_{kn})(t) \in (V_k^3)^\perp$ for a.a. $t \in I$ as in (3.18).

Proof. The existence of the limit for some subsequence follows from the estimates of Lemma 3.5 using weak and weak* compactness.

The functions $\{\mathbf{m}_{kln}(t)\}_{l\in\mathbb{N}}$ belong to the same finite-dimensional space V_k^3 for a.a. $t \in I$, hence the first equation of (3.26) actually implies that

$$\mathbf{m}_{kln} \to \mathbf{m}_{kn} \text{ strongly in } L^\infty(Q;\mathbb{R}^3). \tag{3.30}$$

Hence by the smoothness of φ_0 and φ_1 we have

$$\left.\begin{array}{l} \varphi_0'(\mathbf{m}_{kln}) \to \varphi_0'(\mathbf{m}_{kn}) \\ \varphi_1'(\mathbf{m}_{kln}) \to \varphi_1'(\mathbf{m}_{kn}) \end{array}\right\} \text{ strongly in } L^\infty(Q;\mathbb{R}^3). \tag{3.31}$$

By (3.20c,d) and by Aubin-Lions' theorem and a standard interpolation, we have that

$$\theta_{kln} \to \theta_{kn} \text{ strongly in } L^{(8+2\omega)/3-\epsilon}(Q), \quad \epsilon > 0. \tag{3.32}$$

In view of (3.31) and (3.32), in order to show that (3.27a) holds, the only nontrivial part is the passage to the limit in the nonlinear term $|\dot{\mathbf{m}}_{kln}|^{p-2}\dot{\mathbf{m}}_{kln}$. We achieve this by showing that $\dot{\mathbf{m}}_{kln}$ converges strongly to $\dot{\mathbf{m}}_{kn}$ in $L^p(Q;\mathbb{R}^3)$. Indeed, we shall prove a stronger assertion: there exists a (small) constant $c_p > 0$ such that

$$\alpha\big\|\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn}\big\|^2_{L^2(Q;\mathbb{R}^3)} + \tau\big\|\nabla\dot{\mathbf{m}}_{kln} - \nabla\dot{\mathbf{m}}_{kn}\big\|^2_{L^2(Q;\mathbb{R}^3)} \\ + \frac{c_p}{n}\big\|\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn}\big\|^p_{L^p(Q;\mathbb{R}^3)} =: I_{kln} \to 0. \tag{3.33}$$

To prove (3.33), observe that by the first equation of (3.19), we have $p \geq 2$, hence the map $\mathbf{m} \mapsto |\mathbf{m}|^{p-2}\mathbf{m}$ is uniformly monotone. Thus, taking the difference between (3.16a) and (3.27a), testing it by $\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn}$ (note that this is a legal test since the space V_k^3 is the same for all l), and integrating the resulting equation with respect to time, it is easy to obtain the following inequality:

$$\begin{aligned} I_{kln} \leq & \int_Q \Big| (\theta_{kln}\varphi_1'(\mathbf{m}_{kln}) - \theta_{kn}\varphi_1'(\mathbf{m}_{kn})) \cdot (\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn})\Big| \,\mathrm{d}x\,\mathrm{d}t \\ & + \int_Q \Big|\Big(\frac{\mathbf{m}_{kln}}{g(|\mathbf{m}_{kln}|)}\times\dot{\mathbf{m}}_{kln} - \frac{\mathbf{m}_{kn}}{g(|\mathbf{m}_{kn}|)}\times\dot{\mathbf{m}}_{kn}\Big) \cdot (\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn})\Big| \,\mathrm{d}x\,\mathrm{d}t \\ =:\ & I_{1,kln} + I_{2,kln}. \end{aligned}$$

The first integral is easily estimated:

$$I_{1,kln} \leq \delta\big\|\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn}\big\|^p_{L^p(Q;\mathbb{R}^3)} + C_\delta\big\|\theta_{kln}\varphi_1'(\mathbf{m}_{kln}) - \theta_{kn}\varphi_1'(\mathbf{m}_{kn})\big\|^{p/(p-1)}_{L^{p/(p-1)}(Q;\mathbb{R}^3)}. \tag{3.34}$$

The last term goes to zero because of (3.31) and (3.32) and because certainly $p > (8{+}2\omega)/(5{+}2\omega)$ (this follows from (3.19) and (3.1e)). Using (3.30), the remaining integral is estimated as follows:

$$\begin{aligned} I_{2,kln} = & \int_Q \bigg(\Big(\frac{\mathbf{m}_{kln}}{g(|\mathbf{m}_{kln}|)} - \frac{\mathbf{m}_{kn}}{g(|\mathbf{m}_{kn}|)}\Big) \times \dot{\mathbf{m}}_{kn}\bigg) \cdot \big(\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn}\big) \,\mathrm{d}x\,\mathrm{d}t \\ \leq\ & \delta\big\|\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn}\big\|^p_{L^p(Q;\mathbb{R}^3)} \\ & + C_{p,\delta}\Big\|\frac{\mathbf{m}_{kln}}{g(|\mathbf{m}_{kln}|)} - \frac{\mathbf{m}_{kn}}{g(|\mathbf{m}_{kn}|)}\Big\|^{p/(p-1)}_{L^\infty(Q;\mathbb{R}^3)}\big\|\dot{\mathbf{m}}_{kn}\big\|^{p/(p-1)}_{L^2(Q;\mathbb{R}^3)}. \end{aligned} \tag{3.35}$$

Now, we choose $\delta < c_p/(2n)$ so that the terms $\delta\|\dot{\mathbf{m}}_{kln} - \dot{\mathbf{m}}_{kn}\|^p_{L^p(Q;\mathbb{R}^3)}$ from (3.34) and (3.35) can be absorbed in the left-hand side of (3.33). Thus the convergence (3.33) is proved. To conclude the proof, we note that (3.33) gives strong convergence of the quadratic terms on the right-hand side of (3.16b). Passing to the limit in the remaining terms of the heat equation is standard.

□

Proposition 3.8 ***(Estimates for*** $\mathbf{m}_{kn}$ ***and*** θ_{kn}***)*** *Let the assumptions of Lemma 3.5 hold. Then,* $\mathbf{m}_{kn}$ *and* θ_{kn} *satisfy also the following inequalities:*

$$\big\|\mathbf{m}_{kn}\big\|_{W^{1,2}(I;L^2(\Omega;\mathbb{R}^3))\,\cap\,L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3)\cap L^{\widehat{q}}(\Omega;\mathbb{R}^3))} \leq C_5, \tag{3.36a}$$

$$\big\|\sqrt{\tau}\dot{\mathbf{m}}_{kn}\big\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))} \leq C_6, \tag{3.36b}$$

$$\big\|\theta_{kn}\big\|_{L^\infty(I;L^\omega(\Omega))} \leq C_7, \tag{3.36c}$$

$$\big\|\dot{\mathbf{m}}_{kn}\big\|_{L^p(Q;\mathbb{R}^3)} \leq C_8\sqrt[p]{n} \tag{3.36d}$$

$$\big\|\nabla\theta_{kn}\big\|_{L^r(Q;\mathbb{R}^3)} \leq C_{9,r}, \qquad 1 \leq r < \frac{3+2\omega}{3+\omega}, \tag{3.36e}$$

$$\big\|\dot{\theta}_{kn}\big\|_{L^1(I;W^{3,2}(\Omega)^*)} \leq C_{10}, \tag{3.36f}$$

where $C_{9,r}$ *depends on* r*. Moreover,* $\theta_{kn} \geq 0$*.*

Proof. Testing the heat equation (3.27b) by the negative part of θ_{kn}, we get $\theta_{kn} \geq 0$, hence $\bar{c}(\theta_{kn}) = c(\theta_{kn})$. This allows us to use the coercivity of $\widehat{c}$ to obtain estimates that do not depend on k and n. We test (3.27a) by $\dot{\mathbf{m}}_{kn}$ and we add (3.27b) tested by 1. Integrating over a time interval $[0,t]$, we obtain the energy estimate:

$$\begin{aligned}
&\int_\Omega \varphi_0(\mathbf{m}_{kn}(t,\cdot)) + \frac{\mu}{2}|\nabla\mathbf{m}_{kn}(t,\cdot)|^2 + \widehat{c}(\theta_{kn}(t,\cdot))\,\mathrm{d}x + \frac{1}{n}\int_0^t\int_\Omega |\dot{\mathbf{m}}_{kn}|^p \\
&\quad= \int_\Omega \varphi_0(\mathbf{m}_{0,k}) + \frac{\mu}{2}|\nabla\mathbf{m}_{0,k}|^2 + \widehat{c}(\theta_{0,k})\,\mathrm{d}x + \int_0^t\int_\Omega \mathbf{h}\cdot\dot{\mathbf{m}}_{kn}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_\Omega \varphi_0(\mathbf{m}_{0,k}) + \frac{\mu}{2}|\nabla\mathbf{m}_{0,k}|^2 + \widehat{c}(\theta_{0,k}) - \mathbf{h}(0,\cdot)\cdot\mathbf{m}_{0,k}\,\mathrm{d}x \\
&\quad\quad+ \int_\Omega \mathbf{h}(t,\cdot)\cdot\mathbf{m}_{kn}(t,\cdot)\,\mathrm{d}x - \int_0^t\int_\Omega \dot{\mathbf{h}}\cdot\mathbf{m}_{kn}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.37}$$

The term on the last line can be estimated as $\int_\Omega \mathbf{h}(t,\cdot)\cdot\mathbf{m}_{kn}(t,\cdot)\,\mathrm{d}x \leq \frac{1}{4\delta}\|\mathbf{h}(t,\cdot)\|^{\nu'}_{L^{\nu'}(\Omega;\mathbb{R}^3)} + \delta\|\mathbf{m}_{kn}(t,\cdot)\|^{\nu}_{L^\nu(\Omega;\mathbb{R}^3)}$. Also, we have

$$\begin{aligned}
&\Big|\int_0^t\int_\Omega \dot{\mathbf{h}}\cdot\mathbf{m}_{kn}\,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq N\int_0^t \big\|\dot{\mathbf{h}}(t,\cdot)\big\|_{L^{\nu'}(\Omega;\mathbb{R}^3)}\Big(1 + \big\|\mathbf{m}_{kn}(t,\cdot)\big\|^{\widehat{q}}_{L^{\widehat{q}}(\Omega;\mathbb{R}^3)} + \big\|\mathbf{m}(t,\cdot)\big\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\Big)\,\mathrm{d}t
\end{aligned} \tag{3.38}$$

where N comes from Sobolev's embedding. Using the coercivity (3.1g) of φ_0 and Gronwall's lemma, from the above estimates we conclude that (3.36b,c,d) hold, and also that

$$\|\mathbf{m}_{kn}\|_{L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3))} \le C_5.$$

We now prove (3.36e) and (3.36a). Note that the right-hand side of the heat equation is $L^1(Q)$-integrable

$$R_{kn} := \alpha\big|\dot{\mathbf{m}}_{kn}\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_{kn}\big|^2 + \theta_{kn}\varphi_1'(\mathbf{m}_{kn})\cdot\dot{\mathbf{m}}_{kn} \in L^1(Q),$$

although we have not shown yet that $\|R_{kn}\|_{L^1(Q)}$ is uniformly bounded with respect to k and n. Following [4, 5] we introduce a "cutoff function" $\phi_j : \mathbb{R}^+ \to [0,1]$ defined by

$$\phi_j(\theta) := \begin{cases} 0 & \text{if } \theta \le j, \\ \theta - j & \text{if } j \le \theta \le j+1, \\ 1 & \text{if } \theta \ge j+1, \end{cases} \tag{3.39}$$

and we observe that $\phi_j(\theta_{kn})$ is a legal test for (3.27b). Using this test, integrating with respect to x and t, and using Green's formula, we obtain

$$\begin{aligned}
&\kappa\int_{B_j}|\nabla\theta_{kn}|^2\,\mathrm{d}x\,\mathrm{d}t = \kappa\int_Q \phi_j'(\theta_{kn})|\nabla\theta_{kn}|^2\,\mathrm{d}x\,\mathrm{d}t = \kappa\int_Q \nabla\theta_{kn}\cdot\nabla\phi_j(\theta_{kn})\,\mathrm{d}x\,\mathrm{d}t \\
&\le \kappa\int_Q \nabla\theta_{kn}\cdot\nabla\phi_j(\theta_{kn})\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \widehat{c\phi}_j(\theta_{kn}(T,\cdot))\,\mathrm{d}x + \int_\Sigma b\theta_{kn}\phi_j(\theta_{kn})\,\mathrm{d}S\,\mathrm{d}t \\
&= \int_\Omega \widehat{c\phi}_j(\theta_0)\,\mathrm{d}x + \int_Q R_{kn}\phi_j(\theta_{kn})\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma b\theta_{\mathrm{e},k}\phi_j(\theta_{kn})\,\mathrm{d}S\,\mathrm{d}t \\
&\le c_{\max}\Big(\big\|\theta_0\big\|_{L^1(\Omega)} + \frac{1}{\omega}\big\|\theta_0\big\|_{L^\omega(\Omega)}^\omega\Big) + \big\|R_{kn}\big\|_{L^1(Q)} + \big\|b\big\|_{L^\infty(\Sigma)}\big\|\theta_{\mathrm{e},k}\big\|_{L^1(\Sigma)},
\end{aligned} \tag{3.40}$$

where $B_j := \{(t,x) \in Q :\ j \le \theta_{kn}(t,x) \le j+1\}$ and $\widehat{c\phi}_j(\theta) = \int_0^\theta c(s)\phi_j(s)ds$. The last inequality is a consequence of (3.1e), which implies that $\widehat{c\phi}_j(\theta) \le c_{\max}(\theta + \frac{1}{\omega}\theta^\omega)$.

For every $\zeta > 0$,

$$\begin{aligned}
\int_Q \frac{|\nabla\theta_{kn}|^2}{(1+\theta_{kn})^{1+\zeta}}\,\mathrm{d}x\,\mathrm{d}t &= \sum_{j=0}^\infty \int_{B_j} \frac{|\nabla\theta_{kn}|^2}{(1+\theta_{kn})^{1+\zeta}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \sum_{j=0}^\infty \frac{1}{(1+j)^{1+\zeta}}\int_{B_n}|\nabla\theta_{kn}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{1}{\kappa}\sum_{j=0}^\infty \frac{1}{(1+j)^{1+\zeta}}\Big(c_{\max}\big\|\theta_0\big\|_{L^1(\Omega)} + \frac{c_{\max}}{\omega}\big\|\theta_0\big\|_{L^\omega(\Omega)}^\omega \\
&\qquad + \big\|R_{kn}\big\|_{L^1(Q)} + \big\|b\big\|_{L^\infty(\Sigma)}\big\|\theta_{\mathrm{e},k}\big\|_{L^1(\Sigma)}\Big) \\
&\le \bar{C}_1 + \bar{C}_2\big\|R_{kn}\big\|_{L^1(\Omega)} =: C_{kn},
\end{aligned} \tag{3.41}$$

where $\bar{C}_1$ and $\bar{C}_2$ are positive constants. Let further $r < 2$. By Hölder inequality and (3.41),

$$\begin{aligned}
\int_Q \left|\nabla\theta_{kn}\right|^r \mathrm{d}x\,\mathrm{d}t &= \int_Q \frac{|\nabla\theta_{kn}|^r}{(1+\theta_{kn})^{(1+\zeta)r/2}}(1+\theta_{kn})^{(1+\zeta)r/2}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \Big(\int_Q \frac{|\nabla\theta_{kn}|^2}{(1+\theta_{kn})^{1+\zeta}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{r/2}\Big(\int_Q \big(1+\theta_{kn}\big)^{(1+\zeta)r/(2-r)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(2-r)/2} \\
&\leq \Big(\bar{C}_1+\bar{C}_2\big\|R_{kn}\big\|_{L^1(\Omega)}\Big)^{r/2}\,\Big(\int_0^T\big\|1+\theta_{kn}(t,\cdot)\big\|_{L^{(1+\zeta)r/(2-r)}(\Omega)}^{(1+\zeta)r/(2-r)}\,\mathrm{d}t\Big)^{(2-r)/2}. \qquad (3.42)
\end{aligned}$$

We observe that, if we adopt the norm $(\|\cdot\|_{L^\omega(\Omega)}+\|\nabla\cdot\|_{L^r(\Omega;\mathbb{R}^3)})$ for $W^{1,r}(\Omega)$, then, for every $\lambda\in(0,1)$ such that

$$\frac{2-r}{(1+\zeta)r} \geq \lambda\Big(\frac{1}{r}-\frac{1}{3}\Big)+\frac{1-\lambda}{\omega}, \qquad (3.43)$$

we can apply the standard Gagliardo-Nirenberg's inequality to get the following estimate:

$$\begin{aligned}
&\big\|1{+}\theta_{kn}(t,\cdot)\big\|_{L^{(1+\zeta)r/(2-r)}(\Omega)} \\
&\qquad\leq C_{\mathrm{GN}}\Big(\big\|1{+}\theta_{kn}(t,\cdot)\big\|_{L^\omega(\Omega)}+\big\|\nabla\theta_{kn}(t,\cdot)\big\|_{L^r(\Omega;\mathbb{R}^3)}\Big)^{\lambda}\big\|1{+}\theta_{kn}(t,\cdot)\big\|_{L^\omega(\Omega)}^{1-\lambda} \\
&\qquad\leq C_{\mathrm{GN}}\Big(|\Omega|^{1/\omega}{+}C_7\Big)^{1-\lambda}\Big(|\Omega|^{1/\omega}{+}C_7{+}\big\|\nabla\theta_{kn}(t,\cdot)\big\|_{L^r(\Omega;\mathbb{R}^3)}\Big)^{\lambda}. \qquad (3.44)
\end{aligned}$$

We also observe that, if (3.44) holds with

$$\lambda = \frac{2-r}{1+\zeta}\,, \qquad (3.45)$$

then it can be substituted into (3.42) to obtain

$$\begin{aligned}
&\Big(\int_0^T \|1+\theta_{kn}(t,\cdot)\|_{L^{(1+\zeta)r/(2-r)}(\Omega)}^{(1+\zeta)r/(2-r)}\,\mathrm{d}t\Big)^{(2-r)/2} \\
&\quad\leq \Big(\int_0^T C_{\mathrm{GN}}^{(1+\zeta)r/(2-r)}\big(|\Omega|^{1/\omega}{+}C_7\big)^{(1-\lambda)(1+\zeta)r/(2-r)} \\
&\qquad\qquad \Big(|\Omega|^{1/\omega}{+}C_7{+}\|\nabla\theta_{kn}(t,\cdot)\|_{L^r(\Omega;\mathbb{R}^3)}\Big)^{\lambda(1+\zeta)r/(2-r)}\,\mathrm{d}t\Big)^{(2-r)/2} \\
&\quad\leq \Big(\int_0^T 3^{r-1}C_{\mathrm{GN}}^{(1+\zeta)r/(2-r)}\big(|\Omega|^{1/\omega}{+}C_7\big)^{(1-\lambda)(1+\zeta)r/(2-r)} \\
&\qquad\qquad \Big(|\Omega|^{r/\omega}{+}C_7^r{+}\|\nabla\theta_{kn}(t,\cdot)\|_{L^r(\Omega;\mathbb{R}^3)}^r\Big)\,\mathrm{d}t\Big)^{(2-r)/2} \\
&\quad=: \bar{C}_3+\bar{C}_4\Big(\int_Q|\nabla\theta_{kn}|^r\,\mathrm{d}x\,\mathrm{d}t\Big)^{(2-r)/2} \qquad (3.46)
\end{aligned}$$

for suitable constants $\bar{C}_3$ and $\bar{C}_4$.

With some algebra, it is easy to check that, for λ as in (3.45), condition (3.43) reads

$$r \le \frac{2\omega + 3 - 3\zeta}{\omega + 3}.$$

Hence if $r \le (2\omega + 3)/(\omega + 3)$ we can join (3.42) with (3.46) to obtain the estimate

$$\big\|\nabla\theta_{kn}\big\|^r_{L^r(Q;\mathbb{R}^3)} \le \bar{C}_5 + \bar{C}_6\big\|R_{kn}\big\|_{L^1(\Omega)}.$$

In principle, the above estimate may not be uniform with respect to n and k. Anyway, it provides L^r-integrability of the temperature gradient. This suffices for us to go further. We test (3.27b) by $\dot{\mathbf{m}}_{kn}$, we integrate with respect to time and we use the above inequality multiplied by $1/(2\bar{C}_6)$ to get

$$\begin{aligned}\frac{1}{2}\tau\big\|\nabla\dot{\mathbf{m}}_{kn}\big\|^2_{L^2(Q;\mathbb{R}^{3\times3})} &+ \frac{1}{2\bar{C}_6}\big\|\nabla\theta_{kn}\big\|^r_{L^r(Q;\mathbb{R}^3)} \\ &\le \frac{\bar{C}_5}{2\bar{C}_6} + \frac{3}{2}\left|\int_Q \theta_{kn}\varphi_1'(\mathbf{m}_{kn})\cdot\dot{\mathbf{m}}_{kn}\,\mathrm{d}x\,\mathrm{d}t\right|. \end{aligned} \tag{3.47}$$

To estimate the right-hand side of (3.47), we use (3.1h) and we apply Hölder's and Young's inequalities to obtain:

$$\begin{aligned}&\left|\int_Q \theta_{kn}\varphi_1'(\mathbf{m}_{kn})\cdot\dot{\mathbf{m}}_{kn}\;\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad\le \int_0^T \big\|\theta_{kn}(t,\cdot)\big\|_{L^\sigma(\Omega)}\big\|\varphi_1'(\mathbf{m}_{kn}(t,\cdot))\big\|_{L^s(\Omega;\mathbb{R}^3)}\big\|\dot{\mathbf{m}}_{kn}(t,\cdot)\big\|_{L^\nu(\Omega;\mathbb{R}^3)}\,\mathrm{d}t, \end{aligned} \tag{3.48}$$

where

$$s := \frac{\widehat{q}}{q_1}, \qquad \sigma := \frac{\widehat{q}\nu'}{\widehat{q} - \nu' q_1}.$$

By the second estimate in (3.36a) we have the following bound, which is uniform with respect to k and n:

$$\begin{aligned}\|\varphi_1'(\mathbf{m}_{kn})\|_{L^s(\Omega;\mathbb{R}^3)} &\le C_{\max}\|1 + |\mathbf{m}_{kn}|^{q_1}\|_{L^s(\Omega)} \\ &\le C_{\max}C_{q_1,s}(1 + \|\mathbf{m}_{kn}\|^{q_1}_{L^{q_1 s}(\Omega)}) \\ &\le C_{\max}C_{q_1,s,\widehat{q}}(1 + \|\mathbf{m}_{kn}\|^{q_1}_{L^{\widehat{q}}(\Omega)}) \le \bar{C}_7. \end{aligned} \tag{3.49}$$

Note that if $\xi \in (0,1)$ satisfies

$$\frac{1}{\nu'} - \frac{q_1}{\widehat{q}} = \frac{1}{\sigma} \ge \xi\Big(\frac{1}{r} - \frac{1}{3}\Big) + \frac{1-\xi}{\omega}, \tag{3.50}$$

then by the Gagliardo-Nirenberg inequality with some $C_{\mathrm{GN}} \in \mathbb{R}$ and the norm $\|\cdot\|_{L^\omega(\Omega)} + \|\nabla\cdot\|_{L^r(\Omega;\mathbb{R}^3)}$ on $W^{1,r}(\Omega)$, and by the second estimate in (3.36a) , we have

$$\begin{aligned}\big\|\theta\big\|_{L^\sigma(\Omega)} &\le C_{\mathrm{GN}}\big\|\theta\big\|^{1-\xi}_{L^\omega(\Omega)}\Big(\big\|\theta\big\|_{L^\omega(\Omega)} + \big\|\nabla\theta\big\|_{L^r(\Omega;\mathbb{R}^3)}\Big)^\xi \\ &\le C_{\mathrm{GN}}C_6^{1-\xi}\Big(C_6 + \big\|\nabla\theta\big\|_{L^r(\Omega;\mathbb{R}^3)}\Big)^\xi, \end{aligned} \tag{3.51}$$

and therefore:

$$\left|\int_Q \theta_{kn}\varphi_1'(\mathbf{m}_{kn})\cdot\dot{\mathbf{m}}_{kn}\,\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\le C_{\rm GN}C_6^{1-\xi}\bar{C}_7\int_0^T \big(C_6+\|\nabla\theta_{kn}(t,\cdot)\|_{L^r(\Omega;\mathbb{R}^3)}\big)^{\xi}\big\|\dot{\mathbf{m}}_{kn}(t,\cdot)\big\|_{L^\nu(\Omega;\mathbb{R}^3)}\,\mathrm{d}t\,. \tag{3.52}$$

If, in addition to (3.50), ξ satisfies

$$\xi<\frac{r}{2}, \tag{3.53}$$

then another application of Hölder's and Young's inequalities in (3.52) gives the following bound:

$$\left|\int_Q \theta_{kn}\varphi_1'(\mathbf{m}_{kn})\cdot\dot{\mathbf{m}}_{kn}\,\,\mathrm{d}x\,\mathrm{d}t\right| \;\le\; C_\delta \;+\; \delta\big\|\nabla\theta\big\|_{L^r(Q;\mathbb{R}^3)}^r \;+\; \delta\big\|\dot{\mathbf{m}}_{kn}\big\|_{L^\nu(Q;\mathbb{R}^3)}^2. \tag{3.54}$$

As the right-hand side of (3.50) is decreasing with respect to ξ, we take ξ as large as possible, compatibly with (3.53). Substituting (3.53) into (3.50) and realizing that $r<(3{+}2\omega)/(3{+}\omega)$, we obtain

$$\frac{1}{\nu'}-\frac{q_1}{\widehat{q}}>\frac{\omega+3}{6\omega}. \tag{3.55}$$

This is guaranteed by (3.1i). Thus the bound (3.54) holds, and it can be easily combined with (3.47) to obtain (3.36a) and (3.36e) (by taking δ small enough, and by absorption on the left-hand side).

Also (3.36f) follows from (3.20a-c). This is a bit technical, however. Due to (3.20c), we have $\dot{\theta}_{kn}\in L^2(Q)$ for each particular k, n (although the whole sequence is not bounded in this space). Therefore, the following computation, applied to (3.27b) divided by $c(\theta_{kn})$ with usage of the latter boundary condition in (3.29) is legal:

$$\begin{aligned}\int_Q \dot{\theta}_{kn}z\,\,\mathrm{d}x\,\mathrm{d}t &= \int_Q \frac{R_{kn}+\kappa\Delta\theta_{kn}}{c(\theta_{kn})}z\,\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q \frac{R_{kn}z}{c(\theta_{kn})}-\kappa\nabla\theta_{kn}\cdot\nabla\frac{z}{c(\theta_{kn})}\,\,\mathrm{d}x\,\mathrm{d}t+\int_\Sigma \frac{b(\theta_{kn}{-}\theta_{{\rm e},k})z}{c(\theta_{kn})}\,\,\mathrm{d}S\,\mathrm{d}t\\ &= \int_Q \frac{R_{kn}z}{c(\theta_{kn})}-\frac{\kappa\nabla\theta_{kn}{\cdot}\nabla z}{c(\theta_{kn})}+\frac{c'(\theta_{kn})\kappa\nabla\theta_{kn}{\cdot}\nabla\theta_{kn}z}{c(\theta_{kn})^2}\,\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\qquad+\int_\Sigma \frac{b(\theta_{kn}{-}\theta_{{\rm e},k})z}{c(\theta_{kn})}\,\,\mathrm{d}S\,\mathrm{d}t\end{aligned} \tag{3.56}$$

for any $z \in L^\infty(I;W^{3,2}(\Omega))$. Thus,

$$
\begin{aligned}
\Big\|\frac{\partial\theta_{kn}}{\partial t}\Big\|_{L^1(I;W^{3,2}(\Omega)^*)} &= \sup_{z\in L^\infty(I;W^{3,2}(\Omega))}\int_Q \dot\theta_{kn} z \,\mathrm{d}x\,\mathrm{d}t \\
&= \sup_{z\in L^\infty(I;W^{3,2}(\Omega))}\int_Q \frac{R_{kn}\, z}{c(\theta_{kn})} - \frac{\kappa\nabla\theta_{kn}\cdot\nabla z}{c(\theta_{kn})} \\
&\qquad + \frac{c'(\theta_{kn})\kappa\nabla\theta_{kn}\cdot\nabla\theta_{kn} z}{c(\theta_{kn})^2}\,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \frac{b(\theta_{kn}-\theta_{\mathrm{e},k})z}{c(\theta_{kn})}\,\mathrm{d}S\,\mathrm{d}t.
\end{aligned}
$$

Now we can estimate it by using ∇z bounded in $L^\infty(Q;\mathbb{R}^3)$ and the already proved estimates (3.20a-c) as well as the assumption that $1/c(\cdot)$ is bounded, cf. (3.1f). In particular, we estimate

$$
\begin{aligned}
\int_Q \frac{c'(\theta_{kn})\kappa\nabla\theta_{kn}\cdot\nabla\theta_{kn} z}{c(\theta_{kn})^2}\,\mathrm{d}x\,\mathrm{d}t &\le \int_Q C_{\max}\kappa\frac{|\nabla\theta_{kn}|^2}{(1+\theta_{kn})^{1+\zeta}}|z|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C_{\max}\kappa C_{kn}\big\|z\big\|_{L^\infty(Q)},
\end{aligned}
\tag{3.57}
$$

with $C_{\max}$ from (3.1f) and C_{kn} from (3.41). We use $\zeta > 0$ small enough to make (3.1f) effective. It is important that, although the regularity estimates that allowed for (3.56) were not uniform, the last estimate is again uniform with respect to n and k. $\square$

Lemma 3.9 *(**Limit passage** $n \to \infty$) Under the assumptions of Proposition 3.8, there exists a subsequence of $\{(\mathbf{m}_{kn}, \theta_{kn})\}_{n\in\mathbb{N}}$ such that, as $n \to \infty$,*

$$
\mathbf{m}_{kn} \to \mathbf{m}_k \quad \begin{cases} \textit{weakly in } W^{1,2}(I;W^{1,2}(\Omega;\mathbb{R}^3)) & \textit{if } \tau > 0, \\ \textit{weakly* in } W^{1,2}(I;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3)) & \textit{if } \tau = 0, \end{cases}
\tag{3.58a}
$$

$$
\theta_{kn} \to \theta_k \qquad \textit{weakly* in } W^{1,2}(I;L^2(\Omega)) \cap L^\infty(I;L^\omega(\Omega)).
\tag{3.58b}
$$

Moreover, $(\mathbf{m}_k, \theta_k)$ is a very weak solution of the system:

$$
\alpha\dot{\mathbf{m}}_k - \tau\Delta\dot{\mathbf{m}}_k - \frac{\mathbf{m}_k}{g(|\mathbf{m}_k|)}\times\dot{\mathbf{m}}_k = \Delta\mathbf{m}_k - \varphi_0'(\mathbf{m}_k) - \theta_k\varphi_1'(\mathbf{m}_k) - \mathbf{h} + \mathbf{g}_k, \tag{3.59a}
$$

$$
c(\theta_k)\dot\theta_k - \mathrm{div}\big(\kappa\nabla\theta_k\big) = \alpha\big|\dot{\mathbf{m}}_k\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_k\big|^2 + \theta_k\varphi_1'(\mathbf{m}_k)\cdot\dot{\mathbf{m}}_k, \tag{3.59b}
$$

with initial conditions

$$
\mathbf{m}_k(0,\cdot) = \mathbf{m}_{k,0}, \qquad \theta_k(0,\cdot) = \theta_0, \tag{3.60}
$$

and with boundary conditions

$$
\partial_{\mathbf{n}}\mathbf{m}_k = \mathbf{g}_k^{(2)}, \qquad \kappa\partial_{\mathbf{n}}\theta_k + b(\theta_k - \theta_{\mathrm{e},k}) = 0, \tag{3.61}
$$

where the residua $\mathbf{g}_k^{(1)}$, $\mathbf{g}_k^{(2)}$, *satisfy* $(\mathbf{g}_k^{(1)}, \mathbf{g}_k^{(2)})(t) \in (V_k^3)^\perp$ *as in (3.18). Moreover, for each* k *the functions* $\mathbf{m}_k$ *and* θ_k *inherit from* $\mathbf{m}_{kn}$ *and* θ_{kn} *the following bounds*

$$\|\mathbf{m}_k\|_{W^{1,2}(I;L^2(\Omega;\mathbb{R}^3))\cap L^\infty(I;W^{1,2}(\Omega;\mathbb{R}^3)\cap L^{\widehat{q}}(\Omega;\mathbb{R}^3))} \le C_5, \tag{3.62a}$$

$$\|\sqrt{\tau}\dot{\mathbf{m}}_k\|_{L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))} \le C_6, \tag{3.62b}$$

$$\|\theta_k\|_{L^\infty(I;L^\omega(\Omega))} \le C_7, \tag{3.62c}$$

$$\|\nabla\theta_k\|_{L^r(Q;\mathbb{R}^3)} \le C_{9,r}, \qquad 1 \le r < \frac{3+2\omega}{3+\omega}, \tag{3.62d}$$

$$\|\dot{\theta}_k\|_{L^1(I;W^{3,2}(\Omega)^*)} \le C_{10}. \tag{3.62e}$$

Furthermore, $\theta_k \ge 0$.

Sketch of the proof. The passage to the limit from (3.27) to (3.59) is the same as that of Proposition 3.7, since k is still constant. The only difference is that the estimate (3.36d) is now used to show that the regularizing term in the magnetic part vanishes in the limit. Indeed, we have

$$\left|\int_Q \frac{1}{n}|\dot{\mathbf{m}}_{kn}|^{p-2}\dot{\mathbf{m}}_{kn}\cdot\mathbf{z}\,\mathrm{d}x\,\mathrm{d}t\right| \le \frac{1}{n}\|\dot{\mathbf{m}}_{kn}\|_{L^p(Q;\mathbb{R}^3)}^{p-1}\|\mathbf{z}\|_{L^p(Q;\mathbb{R}^3)} \le C_8\frac{1}{\sqrt[p]{n}}\|\mathbf{z}\|_{L^p(Q;\mathbb{R}^3)} \to 0 \tag{3.63}$$

for any $\mathbf{z} \in L^p(Q;\mathbb{R}^3)$. As a consequence, the estimates are uniform with respect to n. □

The next proposition concludes the proof of Theorem 3.2.

Proposition 3.10 ***(Limit passage*** $k \to \infty$*) Under the assumptions of Proposition 3.8, there exists a subsequence of* $\{(\mathbf{m}_k, \theta_k)\}_{k\in\mathbb{N}}$ *which converges weakly* in the topologies specified in Lemma 3.9. Its limit* $(\mathbf{m}, \theta)$ *is a very weak solution of (1.1) with initial conditions (1.2) and boundary conditions (1.3), in the sense of Definition 3.1. Moreover,* θ *satisfies (3.8).*

Proof. By (3.62a) we have that $\mathbf{m}_k \to \mathbf{m}$ weakly* in the topologies specified there. Hence by the Aubin-Lions theorem we can extract a subsequence from $\{(\mathbf{m}_k, \theta_k)\}_{k\in\mathbb{N}}$, which we still label by k, such that $\mathbf{m}_k \to \mathbf{m}$ strongly in $L^\eta(I;L^{6-1/\eta}(\Omega;\mathbb{R}^3))$ for every $1 \le \eta < +\infty$. If $\widehat{q} > 6$, this result can be improved by an interpolation between $L^\eta(I;L^{6-1/\eta}(\Omega;\mathbb{R}^3))$ and $L^\infty(I;L^{\hat{q}}(\Omega;\mathbb{R}^3))$, which gives that $\mathbf{m}_k \to \mathbf{m}$ strongly in $L^\eta(I;L^{\widehat{q}-1/\eta}(\Omega;\mathbb{R}^3))$ for every $1 \le \eta < +\infty$. It follows from (3.62c) that we can further extract a subsequence from $\{(\mathbf{m}_k, \theta_k)\}_{k\in\mathbb{N}}$ (again without changing labels) such that $\theta_k \to \theta$ weakly* in $L^\infty(I;L^\omega(\Omega))$. Thus, by (3.62d), θ_k is bounded in $L^r(I;W^{1,r}(\Omega))$. This along with (3.62e) and Aubin-Lions' theorem gives that $\theta_k \to \theta$ strongly $L^\eta(I;L^{\omega-1/\eta}(\Omega))$. Also, using the Gagliardo-Nirenberg inequality to interpolate between $W^{1,r}(\Omega)$ and $L^\omega(\Omega)$, we easily get strong convergence of θ_k in $L^{1+2\omega/3}(Q)$.

To prove that $(\mathbf{m},\theta)$ satisfies (3.6), it is sufficient to prove convergence of the nonlinear terms in (3.59a). First, from the growth condition (3.1h) it follows that $\varphi_0'(\mathbf{m}_k)\to\varphi_0'(\mathbf{m})$ strongly in $L^\eta(I;L^{1+1/\eta}(\Omega))$ for all $1\le\eta<+\infty$. Second, by virtue of (3.1i), we obtain that $\theta_k\varphi_1'(\mathbf{m}_k)\to\theta\varphi_1'(\mathbf{m})$ strongly in $L^2(I;L^{\nu'}(\Omega;\mathbb{R}^3))$. Finally, we have also $\frac{\mathbf{m}_k}{g(|\mathbf{m}_k|)}\to\frac{\mathbf{m}}{g(|\mathbf{m}|)}$ strongly in $L^\eta(L^{\widehat{q}-1/\eta}(Q;\mathbb{R}^3))$ due to (3.1b).

It remains for us to show that $(\mathbf{m},\theta)$ satisfies (3.6). To pass to the limit in (3.59b), we test it by w and integrate over Q. Then, we integrate by parts in time on the left-hand side to get:

$$\int_Q\Big(-\widehat{c}(\theta_k)\dot{w}+\kappa\nabla\theta_k\cdot\nabla w\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\widehat{c}(\theta_0(x))w(0,x)\;\mathrm{d}x+\int_\Sigma b\,\theta_k\,w\mathrm{d}S\,\mathrm{d}t$$
$$=\int_Q\Big(\alpha\big|\dot{\mathbf{m}}_k\big|^2+\tau\big|\nabla\dot{\mathbf{m}}_k\big|^2+\theta_k\varphi_1'(\mathbf{m}_k)\cdot\dot{\mathbf{m}}_k\Big)w\,\mathrm{d}x\,\mathrm{d}t+\int_\Sigma b\,\theta_{\mathrm{e},k}\,w\mathrm{d}S\mathrm{d}t. \tag{3.64}$$

As to the left-hand side of (3.64), the passage to the limit is carried out using the above-proven strong convergence of θ_k in $L^{1+2\omega/3}(Q)$. Here we need $\omega<3$, as assumed in (3.1e), to guarantee convergence of $\widehat{c}(\theta_k)$ in $L^1(Q)$. Next, we study the convergence of the right-hand side of (3.64). We claim that:

$$\lim_{k\to\infty}\Big(\alpha\big|\dot{\mathbf{m}}_k\big|^2+\tau\big|\nabla\dot{\mathbf{m}}_k\big|^2\Big)=\alpha\big|\dot{\mathbf{m}}\big|^2+\tau\big|\nabla\dot{\mathbf{m}}\big|^2\qquad\text{in }L^1(Q). \tag{3.65}$$

To prove (3.65), we test (3.6) by $\dot{\mathbf{m}}$, which is in $L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))$ if $\tau>0$ and thus is a legal test function. This gives

$$\int_\Omega\frac{1}{2}|\nabla\mathbf{m}(T)|^2+\varphi_0(\mathbf{m}(T))\,\mathrm{d}x+\int_Q\alpha\big|\dot{\mathbf{m}}\big|^2+\tau\big|\nabla\dot{\mathbf{m}}\big|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_\Omega\frac{1}{2}|\nabla\mathbf{m}_0|^2+\varphi_0(\mathbf{m}_0)\,\mathrm{d}x+\int_Q(\mathbf{h}-\theta\varphi_1'(\mathbf{m}))\cdot\dot{\mathbf{m}}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.66}$$

In case $\tau=0$, we need to use a subtler argument. By (3.1b), we have $\frac{\mathbf{m}}{g(\mathbf{m})}\in L^\infty(Q;\mathbb{R}^3)$ hence $\frac{\mathbf{m}}{g(\mathbf{m})}\times\dot{\mathbf{m}}\in L^2(Q;\mathbb{R}^3)$, by (3.1i) we have $\varphi_0'(\mathbf{m})\in L^2(Q;\mathbb{R}^3)$, by (3.2b) also $\mathbf{h}\in L^2(Q;\mathbb{R}^3)$, and $\theta\varphi_1'(\mathbf{m})\in L^2(Q;\mathbb{R}^3)$ has already been used (and proved). Since it has also been proved that $\dot{\mathbf{m}}\in L^2(Q;\mathbb{R}^3)$, it follows from (1.1a) that, in addition, $\Delta\mathbf{m}\in L^2(Q;\mathbb{R}^3)$. Moreover, $\mathbf{m}:I\to W^{1,2}(\Omega;\mathbb{R}^3)$ is actually a weakly continuous function (alhough not necessarily strongly continuous). Then, the formula

$$\int_Q\Delta\mathbf{m}\cdot\dot{\mathbf{m}}\;\mathrm{d}x\,\mathrm{d}t=\frac{1}{2}\int_\Omega|\nabla\mathbf{m}_0|^2-|\nabla\mathbf{m}(T)|^2\;\mathrm{d}x \tag{3.67}$$

needed to show the energy equality (3.66) holds. The integration-by-parts formula (3.67) can be proved by mollifing $\mathbf{m}$ with respect to the spatial variables (not with respect to both space and time). On denoting by M_ε the mollification operator (a linear mapping) and on setting $\mathbf{m}_\varepsilon:=M_\varepsilon\mathbf{m}$, we have $\dot{\mathbf{m}}_\varepsilon=(M_\varepsilon\mathbf{m})^{\bullet}=M_\varepsilon\dot{\mathbf{m}}\in L^2(I;C^1(\Omega;\mathbb{R}^3))$. By standard calculus, we obtain that (3.67) holds for the mollified function $\mathbf{m}_\varepsilon$, namely: $\int_Q \Delta\mathbf{m}_\varepsilon \cdot \dot{\mathbf{m}}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_\Omega |\nabla\mathbf{m}_0|^2 - |\nabla\mathbf{m}_\varepsilon(T)|^2 \,\mathrm{d}x$. We then obtain (3.67) by letting $\varepsilon \to 0$, using the fact that $\Delta\mathbf{m}_\varepsilon \to \Delta\mathbf{m}$ in $L^2(Q;\mathbb{R}^3)$, $\dot{\mathbf{m}}_\varepsilon \to \dot{\mathbf{m}}$ in $L^2(Q;\mathbb{R}^3)$, and $\nabla\mathbf{m}_\varepsilon(T) \to \nabla\mathbf{m}(T)$ in $L^2(\Omega;\mathbb{R}^3)$.

Thus, in both cases ($\tau = 0$ and $\tau > 0$), we obtain:

$$
\begin{aligned}
\int_Q \alpha\big|\dot{\mathbf{m}}\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}\big|^2 \,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{k\to\infty} \int_Q \alpha\big|\dot{\mathbf{m}}_k\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_k\big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\le \limsup_{k\to\infty} \int_Q \alpha\big|\dot{\mathbf{m}}_k\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_k\big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \limsup_{k\to\infty} \int_\Omega \frac{1}{2}|\nabla\mathbf{m}_{0,k}|^2 + \varphi_0(\mathbf{m}_{0,k}) - \frac{1}{2}|\nabla\mathbf{m}_k(T)|^2 \\
&\quad - \varphi_0(\mathbf{m}_k(T))\,\mathrm{d}x + \int_Q (h - \theta_k\varphi_1'(\mathbf{m}_k)) \cdot \dot{\mathbf{m}}_k \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_\Omega \frac{1}{2}|\nabla\mathbf{m}_0|^2 + \varphi_0(\mathbf{m}_0) - \frac{1}{2}|\nabla\mathbf{m}(T)|^2 - \varphi_0(\mathbf{m}(T))\,\mathrm{d}x \\
&\quad + \int_Q (\mathbf{h} - \theta\varphi_1'(\mathbf{m})) \cdot \dot{\mathbf{m}} \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.68)
\end{aligned}
$$

Here we have used successively the weak-lower semicontinuity of the L^2-norms, the convergence $\mathbf{m}_k \to \mathbf{m}$ weakly* in the topology specified in (3.58a), equation (3.59a) tested by $\dot{\mathbf{m}}_k$, the convergence $\mathbf{m}_k(T) \to \mathbf{m}(T)$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $\theta_k\varphi_1'(\mathbf{m}_k) \cdot \dot{\mathbf{m}}_k \to \theta\varphi_1'(\mathbf{m}) \cdot \dot{\mathbf{m}}$ weakly in $L^1(Q)$. Combining (3.66) and (3.68), we get

$$
\lim_{k\to\infty} \int_Q \alpha\big|\dot{\mathbf{m}}_k\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}_k\big|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_Q \alpha\big|\dot{\mathbf{m}}\big|^2 + \tau\big|\nabla\dot{\mathbf{m}}\big|^2 \,\mathrm{d}x\,\mathrm{d}t. \qquad (3.69)
$$

Taking into account the fact that $\dot{\mathbf{m}}_k \to \dot{\mathbf{m}}$ weakly in $L^2(I;W^{1,2}(\Omega;\mathbb{R}^3))$ (if $\tau > 0$) or in $L^2(Q;\mathbb{R}^3)$ (if $\tau = 0$), equation (3.69) alone suffices to conclude that $\dot{\mathbf{m}}_k \to \dot{\mathbf{m}}$ strongly in the mentioned spaces, whence (3.65). Now we can pass to the limit on the right-hand side of (3.59b), using (3.65) for the dissipative terms while the weak convergence of $\theta_k\varphi_1'(\mathbf{m}_k) \cdot \dot{\mathbf{m}}_k$ in $L^1(Q)$ has been already proved (and used for passing to the limit in the magnetic part, which we then employed to deduce (3.66)–(3.68)); now we proved that the adiabatic term converges even strongly. Eventually, (3.8a) follows from (3.62e) while (3.8b) follows from (3.62) through the equation $(\widehat{c}(\theta))^{\boldsymbol{\cdot}} = \kappa\Delta\theta + \alpha|\dot{\mathbf{m}}|^2 + \tau|\nabla\dot{\mathbf{m}}|^2 + \theta\varphi_1'(m) \cdot \dot{\mathbf{m}}$ itself. $\square$

Sketch of the proof of Theorem 3.3. Note that the estimates (3.62) (with τ insted of k) are uniform with respect to τ except (3.62b), which however says that $\|\nabla\dot{\mathbf{m}}_{\tau_j}\|^2_{L^2(Q;\mathbb{R}^{3\times3})} \le C_6/\sqrt{\tau}$. The same compactness arguments used before to handle the case $\tau = 0$ apply, and the only point is to show that $\tau_j\|\nabla\dot{\mathbf{m}}_{\tau_j}\|^2_{L^2(Q;\mathbb{R}^{3\times3})}$ is not only bounded, but even converges to 0 as $j \to \infty$. We proceed as in the previous proof and then we observe

$$\Big|\int_Q \tau_j \nabla\dot{\mathbf{m}}_{\tau_j} : \nabla\mathbf{z}\,\mathrm{d}x\,\mathrm{d}t\Big| \le \tau_j \big\|\nabla\dot{\mathbf{m}}_{\tau_j}\big\|_{L^2(Q;\mathbb{R}^{3\times3})}\big\|\nabla\mathbf{z}\big\|_{L^2(Q;\mathbb{R}^{3\times3})}$$
$$\le C_6\sqrt{\tau_j}\big\|\nabla\mathbf{z}\big\|_{L^2(Q;\mathbb{R}^{3\times3})} \to 0 \tag{3.70}$$

for any $\mathbf{z} \in L^2(I;W^{1,2}(\Omega;\mathbb{R}^3)$ (observe that (3.63) is based on a similar argument). Thus we obtain (3.6) with $\tau = 0$, and testing it by $\mathbf{z} := \dot{\mathbf{m}}$, we prove that the limit $(\mathbf{m},\theta)$ satisfies the energy equality:

$$\int_\Omega \frac{1}{2}\big|\nabla\mathbf{m}(T)\big|^2 + \varphi_0(\mathbf{m}(T))\,\mathrm{d}x + \int_Q \alpha\big|\dot{\mathbf{m}}\big|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_\Omega \frac{1}{2}\big|\nabla\mathbf{m}_0\big|^2 + \varphi_0(\mathbf{m}_0)\,\mathrm{d}x + \int_Q (\mathbf{h} - \theta\varphi_1'(\mathbf{m}))\cdot\dot{\mathbf{m}}\,\mathrm{d}x\,\mathrm{d}t; \tag{3.71}$$

here the argument (3.67) had to be used again. Then, using the analogous arguments employed in (3.68), we obtain the following chain of inequalities:

$$\begin{aligned}\int_Q \alpha\big|\dot{\mathbf{m}}\big|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{j\to\infty}\int_Q \alpha\big|\dot{\mathbf{m}}_{\tau_j}\big|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le \liminf_{j\to\infty}\int_Q \alpha\big|\dot{\mathbf{m}}_{\tau_j}\big|^2 + \tau_j\big|\nabla\dot{\mathbf{m}}_{\tau_j}\big|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le \limsup_{j\to\infty}\int_Q \alpha\big|\dot{\mathbf{m}}_{\tau_j}\big|^2 + \tau_j\big|\nabla\dot{\mathbf{m}}_{\tau_j}\big|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_\Omega \frac{1}{2}|\nabla\mathbf{m}_0|^2 + \varphi_0(\mathbf{m}_0) - \frac{1}{2}|\nabla\mathbf{m}(T)|^2 - \varphi_0(\mathbf{m}(T))\,\mathrm{d}x\\
&\quad + \int_Q (\mathbf{h} - \theta\varphi_1'(\mathbf{m}))\cdot\dot{\mathbf{m}}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.72}$$

Combination of (3.71) with (3.72) yields:

$$\lim_{j\to\infty}\int_Q \alpha\big|\dot{\mathbf{m}}_{\tau_j}\big|^2 + \tau_j\big|\nabla\dot{\mathbf{m}}_{\tau_j}\big|^2\,\mathrm{d}x\,\mathrm{d}t = \int_Q \alpha\big|\dot{\mathbf{m}}\big|^2\,\mathrm{d}x \tag{3.73}$$

from which and from the weak convergence of $\dot{\mathbf{m}}_{\tau_j} \to \dot{\mathbf{m}}$ in $L^2(Q;\mathbb{R}^3)$ we have that both $\dot{\mathbf{m}}_{\tau_j} \to \dot{\mathbf{m}}$ strongly in $L^2(Q;\mathbb{R}^3)$ and $\tau_j\big|\nabla\dot{\mathbf{m}}_{\tau_j}\big|^2 \to 0$ in $L^1(Q)$. The limit passage in the heat equation is then straightforward. □

4 Concluding remarks

Remark 4.1 (*Ferro/paramagnetic transition processes*) One might wonder legitimately whether our model actually supports existence of thermomagnetic processes during which the ferro/paramagnetic transition occurs. To demonstrate that this is indeed the case, we take function φ_1 to be nonnegative-valued (as exemplified by the second of (2.30)), and we consider a thermomagnetic process $(\mathbf{m}, \theta)$ satisfying (1.1)–(1.3), with

$$\operatorname*{ess\,sup}_{x\in\Omega} \theta_0(x) < 1,$$

so that the magnetic body under study is initially all in a ferromagnetic phase, and with

$$\operatorname*{ess\,inf}_{(t,x)\in\Sigma} \theta_{\mathrm{e}}(t,x) =: \bar{\theta}_{\mathrm{e}} > 1.$$

With a view towards arriving at a contradiction, assume that for every choice of $T > 0$ the process $(\mathbf{m}, \theta)$ would be such that $\theta < \bar{\theta}_{\mathrm{e}}$ a.e. in Q. To begin with, from the entropy imbalance (2.21) we deduce that

$$C_0 + C_1 T \leq \int_\Omega \eta(T,x)\,\mathrm{d}x, \tag{4.1}$$

where

$$C_0 = \int_{\mathcal{P}} \eta(0,x)\,\mathrm{d}x, \qquad C_1 = -\int_{\partial\Omega} b\left(1 - \frac{\theta_{\mathrm{e}}}{\bar{\theta}_{\mathrm{e}}}\right)\mathrm{d}S > 0.$$

Moreover, by combining (2.26) and (3.1e), we have that

$$c(\theta) = -\theta\varphi''(\theta) \geq c_{\min}(1 + \theta^{\omega-1}),$$

whence

$$-\varphi'(1) + c_{\min}\left(\log\theta + \frac{1}{\omega-1}(\theta^{\omega-1} - 1)\right) \geq -\varphi'(\theta).$$

With this, recalling (2.25) and using the assumptions that $\varphi_1(\mathbf{m}) \geq 0$ and that $\theta < \bar{\theta}_{\mathrm{e}}$ a.e. in Q, we get:

$$\eta < -\varphi'(1) + c_{\min}\Big(\log\bar{\theta}_{\mathrm{e}} + \frac{1}{\omega-1}(\bar{\theta}_{\mathrm{e}}^{\omega-1} - 1)\Big). \tag{4.2}$$

For $T \to \infty$, combination of (4.1) and (4.2) induces a contradiction. We conclude that at least some part of our magnet must undergo a ferro-to-paramagnetic transition during a process of the type here considered.

Remark 4.2 (*Anisotropy energies*) We have not paused to detail what types of anisotropy energy ψ_a pertain to the magnetic materials whose temperature-driven transitions fall within the reach of our present theorems. In fact, since the first of relations (2.30) kind of 'buries' ψ_a into φ_0, the constitutive assumption (3.1c) is satisfied as long as we take a continuously differentiable ψ_a.

Remark 4.3 (*Self-induced demagnetizing field*) In order to account for the self-induced magnetic field, the initial-boundary-value problem we studied in this paper must be modified as follows. For one, the system (1.1) must be augmented by the so-called *magnetostatic equation*:

$$\operatorname{div}(\nabla u - \chi_\Omega \mathbf{m}) = 0 \tag{4.3}$$

a quasistatic limit of Maxwell's equations; here χ_Ω denotes the indicator function of Ω, and solutions are sought in $H^1(\Omega)$ in the sense of distributions. For two, the additional term ∇u (the demagnetizing field) must be added to the right-hand side of (1.1.a); the free energy becomes:

$$\Phi(\theta, \mathbf{m}, \nabla\mathbf{m}) := \int_\Omega \widetilde{\varphi}(\theta, \mathbf{m}, \nabla\mathbf{m})\,\mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2\,\mathrm{d}x.$$

The term in question has been omitted only for simplicity; it could be included in our analytical treatment, at the expense of performing some changes in the proofs. What would be mostly affected are the estimates that are obtained by testing (1.1.a) by $\dot{\mathbf{m}}$ (for example, the estimate (3.22) in the proof of Lemma 3.5). Indeed, when testing by $\dot{\mathbf{m}}$, an additional term $\int_\Omega \nabla u\cdot\dot{\mathbf{m}}\,\mathrm{d}x$ would appear on the right-hand side of each such estimate. This term can be handled by integration by parts in time, using (4.3) differentiated with respect to time, and then using the integral version of Gronwall's inequality. Proceeding formally, one finds:

$$\begin{aligned}
\int_0^t\int_\Omega \nabla u\cdot\dot{\mathbf{m}}\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \nabla u(t,\cdot)\cdot\mathbf{m}(t,\cdot) - \nabla u(0,\cdot)\cdot\mathbf{m}(0,\cdot)\,\mathrm{d}x - \int_0^t\int_\Omega \nabla\dot{u}\cdot\mathbf{m}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega \nabla u(t,\cdot)\cdot\mathbf{m}(t,\cdot) - \nabla u(0,\cdot)\cdot\mathbf{m}(0,\cdot)\,\mathrm{d}x - \int_0^t\int_{\mathbb{R}^3} \nabla\dot{u}\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega \nabla u(t,\cdot)\cdot\mathbf{m}(t,\cdot) - \frac{1}{2}|\nabla u(t,\cdot)|^2 - \nabla u(0,\cdot)\cdot\mathbf{m}(0,\cdot) + \frac{1}{2}|\nabla u(0,\cdot)|^2\,\mathrm{d}x,
\end{aligned}$$

an expression that can further treated by Hölder inequality and incorporated into the integral Gronwall inequality.

Remark 4.4 (*Comparison with the Landau-Lifschitz-Bloch Equation*) In our notation and units, the Landau-Lifschitz-Bloch equation proposed in [14] reads:

$$\dot{\mathbf{m}} = \gamma\mathbf{m}\times\mathbf{h}_{\text{eff}} + \frac{\hat{\ell}_1(\theta)}{|\mathbf{m}|^2}(\mathbf{m}\cdot\mathbf{h}_{\text{eff}})\mathbf{m} - \frac{\hat{\ell}_2(\theta)}{|\mathbf{m}|^2}\mathbf{m}\times(\mathbf{m}\times\mathbf{h}), \tag{4.4}$$

where $\hat{\ell}_i(\theta) = \gamma\hat{m}_{\text{e}}(\theta)\alpha_i$ with $\alpha_i > 0$ for $i = 1, 2$. Here the function $\hat{m}_{\text{e}} : \mathbb{R}^+ \to (0, 1)$ specifies the dependence of the spontaneous magnetization on temperature, while $\mathbf{h}_{\text{eff}}$ is an "effective field" obtained by taking the negative variation of a free energy whose density has an expression that does not fit within our framework (see (4.18) and (4.19) of [14]).

To compare (4.4) with our model, set $\mathbf{h}^{\text{eff}} = \mathrm{div}\mathbf{C}^{\text{eq}} + \mathbf{k}^{\text{eq}} + \mathbf{h}$; assume that g has the form (2.7); and take

$$\mathbf{C}^{\text{neq}} = \mathbf{0}, \quad -\mathbf{k}^{\text{neq}} = \alpha_\parallel|\mathbf{m}|^{-2}(\mathbf{m}\otimes\mathbf{m})\dot{\mathbf{m}} + \alpha_\perp|\mathbf{m}|^{-2}(\mathbf{I}-\mathbf{m}\otimes\mathbf{m})\dot{\mathbf{m}}, \tag{4.5}$$

where

$$\alpha_\parallel \equiv \hat{\alpha}_\parallel(\mathbf{m}, \theta) = \gamma^{-1}\alpha_1^{-1}\hat{m}_{\text{e}}^{-1}(\theta), \quad \alpha_\perp \equiv \hat{\alpha}_\perp(\mathbf{m}, \theta) = \gamma^{-1}\alpha_2|\mathbf{m}|^{-2}\hat{m}_{\text{e}}(\theta). \tag{4.6}$$

Starting from (2.6), with some algebra one obtains:

$$\alpha_\perp \mathbf{m}\times\dot{\mathbf{m}} + \gamma^{-1}\dot{\mathbf{m}} = \mathbf{m}\times\mathbf{h}_{\rm eff} + \gamma^{-1}\alpha_\parallel^{-1}|\mathbf{m}|^{-2}(\mathbf{m}\cdot\mathbf{h}_{\rm eff})\mathbf{m}. \tag{4.7}$$

In the above equatoin, the last term on the right-hand side accounts for longitudinal relaxation of $\mathbf{m}$. With some additional effort, (4.7) can be given the form:

$$\dot{\mathbf{m}} = \hat{\beta}(\theta,\mathbf{m})\gamma\mathbf{m}\times\mathbf{h} + \frac{\hat{\ell}_1(\theta)}{|\mathbf{m}|^2}(\mathbf{m}\cdot\mathbf{h}_{\rm eff})\mathbf{m} - \eta\frac{\hat{\ell}_2(\theta)}{|\mathbf{m}|^2}\mathbf{m}\times(\mathbf{m}\times\mathbf{h}_{\rm eff}), \tag{4.8}$$

where

$$\hat{\beta}(\theta,\mathbf{m}) = \left(1+\left(\alpha_2\frac{\hat{m}_{\rm e}(\theta)}{|\mathbf{m}|}\right)^2\right)^{-1}.$$

Then (4.4) is recovered in the limit:

$$\alpha_2\frac{\hat{m}_{\rm e}(\theta)}{|\mathbf{m}|} \to 0. \tag{4.9}$$

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