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"ASSET PRICE ANOMALIES UNDER BOUNDED RATIONALITY"

EMILIO BARUCCI, ROBERTO MONTE AND ROBERTO RENÒ

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Asset Price Anomalies under Bounded Rationality*

Emilio Barucci

Dipartimento di Statistica e Matematica Applicata all'Economia

Università di Pisa.

Via Cosimo Ridolfi, 10 - 56124 Pisa, ITALY

e-mail: ebarucci@ec.unipi.it

Roberto Monte

Dipartimento di Studi Economici, Finanziari e Metodi Quantitativi

Università di Roma "Tor Vergata".

Via Columbia, 2 - 00133 Roma, ITALY

e-mail: monte@sefemeq.uniroma2.it

Roberto Renò

Dipartimento di Economia Politica

Università di Siena

Piazza S.Francesco 7 - 53100 Siena, ITALY

e-mail: reno@unisi.it

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Abstract

We analyze the classical asset pricing model assuming non fully rational agents. Agents forecast future prices cum dividend through an adaptive learning rule. This assumption provides an explanation of some anomalies encountered in the empirical analysis of asset prices under full rationality: returns are serially correlated (positively over a short horizon and negatively over a longer horizon) and the dividend yield predicts future returns (positive correlation). Considering the continuous time limit process, the same regularities are established analytically for price increments.

Keywords: Asset Prices, Returns correlation, Bounded Rationality, Dividends, Diffusion Processes.

Classification: (JEL 1995) C61, C62, D83, D84, E32

1 Introduction

The classical asset pricing theory, based on the absence of arbitrage opportunities in the market, provides us with some testable implications. Assuming a stationary investment opportunity set, or risk neutrality and constant tastes, excess returns should not be predictable and no sign of autocorrelation should be observed (the so-called random walk hypothesis). Many studies have tested these implications empirically. An evaluation of the literature allows us to establish that returns are predictable. They are characterized by mean reversion over long horizons and trend over short horizons; excess returns over a period larger than one year are negatively autocorrelated, see Fama and French (1988), while returns over a horizon up to one year are positively autocorrelated, see Lo and MacKinlay (1988). Moreover, excess returns can be predicted through the lagged dividend yield, see Shiller (1984), and asset price volatility is not explained through classical pricing models (excess volatility), see Shiller (1989).

Two well established schools of thought can be identified on these "anomalies": the classical asset pricing school, and the so called behavioral finance school. Those belonging to the first school explain these phenomena by relaxing the stationarity assumption for dividends and returns. In this context, equity premia are not constant over time and expected returns are autocorrelated, see Fama and French (1988). Partisans of the behavioral school argue that the assumption of time varying expected returns is inadequate to explain the regularities and invoke the presence of some elements of "irrationality" in the market, see Cutler, Poterba and Summers (1991), Lakoniskok, Schleifer and Vishny (1994). On this debate see also Fama (1991), Fama (1998).

This paper aims to contribute to this debate by analyzing asset prices under bounded rationality. We assess whether asset price anomalies can be explained by assuming the presence of non "fully rational" agents in the market. Consider the classical no arbitrage equation, under rational expectations, agents compute the expectation as the conditional expectation with respect to the probability measure of the model and the information available in the market. Under bounded rationality, the expected price cum dividend is computed through a rule of thumb. In what follows, we assume that agents update their expectations through a first order autoregressive learning mechanism (adaptive expectations): today's expectation for tomorrow's price is a convex combination of yesterday's expectation for today's price and yesterday's price, see Hommes (1994), Barucci (2000). The learning rule analyzed below can also be interpreted as an extrapolating technical analysis trading strategy: agents compute their expectation through an extrapolating rule.

We compare the asset prices bounded rationality evolution to the one obtained under rational expectations. The literature on asset price anomalies is large. In what follows, we concentrate on the following puzzle: predictability of asset returns through past returns and past dividend yields. Similar analyses have been developed in Barsky and De Long (1993), Buckley and Tonks (1989), Timmermann (1993), Timmermann (1994), Timmermann (1996), Sogner and Mitlohner (2002). The main novelty in our analysis is in the learning rule. The above papers assume that agents believe in a misspecified model for the dividend price process, which is correctly specified only under rational expectations. Model parameters are estimated through a learning mechanism such as recursive ordinary least squares. Unless the parameters of the model are time varying, see Barsky and De Long (1993), Buckley and Tonks (1989), agents learn the rational expectations parameters of the model in the long run, and therefore the classical dynamics is observed, see Timmermann (1993), Timmermann (1996), Sogner and Mitlohner (2002). Given the information set, the learning rule employed in those papers is optimal from an econometric point of view. In our setting, agents are not so clever; they use a simpler learning rule. They predict prices and dividends directly, instead of estimating their law of motion. The learning mechanism considered below is sounder from a behavioral perspective. The dynamics under bounded rationality does not converge toward the one obtained under rational expectations.

In Timmermann (1993), Timmermann (1996), it is shown that asset prices under bounded rationality are characterized by a volatility higher than that obtained under rational expectations. Moreover, asset returns can be predicted through the lagged dividend yield. However, returns show only weak evidence of serial correlation. In our analysis, we show that, under bounded rationality, returns can be predicted through lagged dividend yields and that they are serially correlated. In particular, the correlation is positive when returns are computed over a small time window and negative when the window is long. Therefore, the bounded rationality learning mechanism generates positive-negative serial correlation in returns, as shown in the empirical literature on asset prices. It is difficult to build a model that generates positive-negative serial correlation of the returns depending on the horizon. In particular, positive serial correlation over short horizons is not obtained with classical asset price models, and almost all the behavioral finance models either predict negative or positive serial correlation, see Fama (1998). We stress that we do not intend to calibrate the model to reproduce some features of financial time series, but only to show the effect of the bonded rationality assumption.

The memory of the learning mechanism plays a crucial role. In the analysis of deterministic models (cobweb and overlapping generations models), it is shown that memory stabilizes the economy, see Barucci (2000), Hommes (1994). A similar effect is shown in our setting: a longer memory induces a smaller degree of dependence when the horizon of the return is long (the mean reversion effect is weaker), while over a short horizon the serial correlation of returns and their correlation with the dividend yield is always significant and positive. The interesting point is that long run dependencies in financial time series can be explained by short memory in the learning mechanism of the agents. The rationale for this type of effect is simple: short memory in the expectation information mechanism induces both overreaction and delayed overreaction to dividend news, inducing positive [resp. negative] serial correlation on returns over a short [resp. long] horizon. This result confirms the other ones obtained in the financial markets literature with non fully rational agents. In Brock and Hommes (1998), considering a simple asset pricing problem, an evolutionary approach has been proposed to motivate the adoption of a rule of thumb by the agents in order to address a forecasting problem. The agents have to choose the forecasting model. There are heterogeneous agents in the economy: fundamentalists and agents who forecast future price by looking at lagged deviations of the price from the fundamental price (trend followers, contrarian, technical analysts). Agents switch to more successful forecasting models according to their past performances. The evolution of this type of economy is characterized by highly irregular, chaotic asset price fluctuations when the switching intensity is high (agents assign a strong weight to recent observations). This is not the case when the intensity rate is small (which corresponds to a long memory).

Following Follmer and Schweizer (1993), we analyze the continuous time limit of the discrete time model. The goal is to provide a micro-foundation for the continuous time stochastic processes for asset prices used in the mathematical finance literature. For this type of process we are able to show analytically the regularities for price increments mentioned above. Price increments are positively [resp. negatively] correlated when the time interval is short [resp. long].

The paper is organized as follows. Section 2 presents the asset price model under bounded rationality. Section 3 analyzes the predictability of returns through simulations. Section 4 analyzes the continuous time limit of the discrete-time process.

2 Asset Prices under Bounded Rationality

Let us consider a discrete time economy with two assets, a risk free asset and a risky asset. The risk free rate, which is assumed to be constant over time, is r. The risky asset delivers dividends D_t, which are modeled through a trend stationary AR(1) process

$$D_{t+1} = \theta + \beta(t+1) + \gamma D_t + \sigma Z_{t+1}, \tag{1}$$

where $\theta, \beta, \gamma, \sigma$ are parameters ($\gamma < 1$, $\sigma > 0$), and $(Z_t)_{t\geq 1}$ is a sequence of i.i.d. Normal random variables with $\mathbf{E}[Z_t] = 0$ and $\mathbf{D}^2[Z_t] = 1$ for $t \geq 0$. This process has been proposed in many papers to describe the evolution of dividends and has been analyzed under bounded rationality in Timmermann (1996).

In the market there are rational traders aiming to exploit arbitrage opportunities. They are risk neutral and therefore the no arbitrage condition entails that today's price be equal to the discounted expectation of tomorrow's price plus the dividend. In a rational expectations environment, the expectation is given by the conditional expectation, and therefore

$$S_t = \frac{1}{1+r} E[S_{t+1} + D_{t+1} | \mathcal{F}_t]. \tag{2}$$

Under full rationality, there is a unique rational expectations solution for (1)-(2), see Timmermann (1996):

$$S_t = \frac{\gamma}{1+r-\gamma} D_t + \frac{1+r}{r(1+r-\gamma)}(\theta + \frac{1+r}{r}\beta + \beta t). \tag{3}$$

In what follows, agents are not assumed fully rational. Let $X_t = S_t + D_t$ the price cum dividend at time t. Agents compute the expected price cum dividend according to an adaptive learning mechanism as a smoothed average of observed prices, see Barucci (2000). Let $\hat{X}_t$ denote the expectation at time t of the price cum dividend at time $t+1$. According to the adaptive learning scheme, we have

$$\hat{X}_t = \hat{X}_{t-1} + \alpha_t (X_{t-1} - \hat{X}_{t-1}), \tag{4}$$

where α_t is the learning coefficient ($0 \leq \alpha_t \leq 1$). In what follows, α_t is a constant (α). By (4), $\hat{X}_t$ is a geometric average of X_s ($1 \leq s \leq t-1$) with coefficient $1-\alpha$. The parameter α describes the memory of the learning mechanism: α near zero means that agents have a long memory. In this case, remote and recent observations are weighted almost in the same way $(1/t)$. When α is near 1 agents have a short memory. Recent observations have a weight larger than remote ones. For $\alpha = 1$, $\hat{X}_t = X_{t-1}$. By decreasing α, we have that agents have more memory.

The no arbitrage condition (2) gives us

$$S_t = \frac{1}{1+r}\hat{X}_t. \tag{5}$$

Hence, the evolution of the asset price is described by (1),(4),(5).

3 Simulations

We consider the bounded rationality model described above and we seek to determine how the bounded rationality hypothesis may explain some empirical results that are not justified inside the classical asset pricing theory with rational expectations. Simulations are needed since no closed form expression is available for serial correlation of logarithmic returns and for the correlation between returns and dividend yield.

We simulate the model by using the GNU Fortran Monte Carlo generator, with parameters $\theta = 0.47$, $\beta = 0.022$, $\gamma = 0.9$, and $\sigma^2 = 0.25$, as estimated in Timmermann (1996) on the Standard&Poor 500 time series for the period $1873 - 1992$. To have a stationary time series, we set $\beta = 0$. [1] We also set $r = 0.05$.

3.1 Predicting Excess Returns by Using Past Returns

The ability of a given time series to predict future returns can be evaluated through their correlation. Here we consider the correlation between the log return in the time interval

[1] We also performed computations with $\beta = 0.022$. Results described below are robust to the introduction of a trend component.

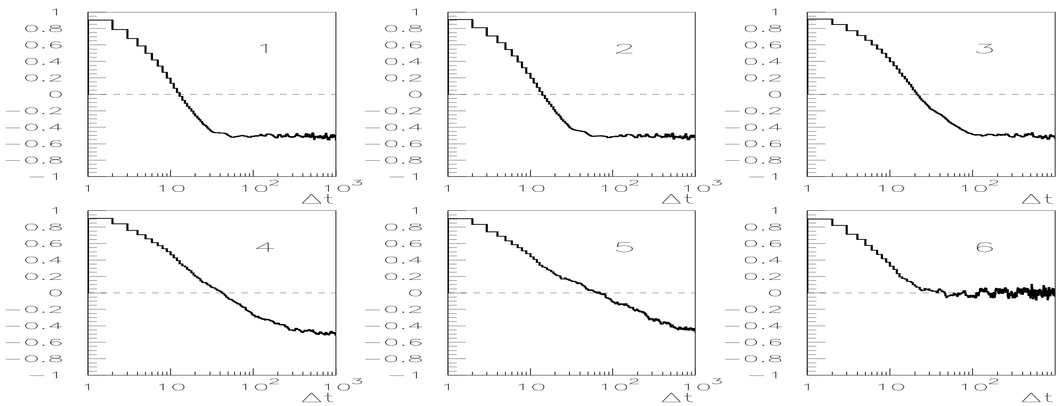

Figure 1: Correlation (6) as a function of Δt for six different values of $\alpha:$ (1) $\alpha = 1$ (perfect memory); (2) $\alpha = 0.8$; (3) $\alpha = 0.3$; (4) $\alpha = 0.1$; (5) $\alpha = 0.05$; (6) $\alpha = 0$ (no memory).

$(t - \Delta t, t)$ and that in the interval $(t, t + \Delta t)$

$$\rho_{yy} = corr\left[\log\left(\frac{S_{t+\Delta t} + D_{t+\Delta t}}{S_t}\right), \log\left(\frac{S_t + D_t}{S_{t-\Delta t}}\right)\right]. \tag{6}$$

If such a correlation is significantly different from zero, then future returns can be predicted from past returns. According to the random walk hypothesis, the correlation in (6) should not differ significantly from zero.

The correlation is computed by generating 1000 numerical simulations of our model. For a given pair $(t, \Delta t)$, denoted by $Y(t, \Delta t) = \log\left(\frac{S_{t+\Delta t}+D_{t+\Delta t}}{S_t}\right)$, the correlation in (6) is computed as

$$\hat{\rho}_{yy} = \frac{\sum_{i=1}^{1000}\left(Y_i(t, \Delta t) - \bar{Y}\right) \cdot \left(Y_i(t - \Delta t, t) - \bar{Y}\right)}{\sum_{i=1}^{1000}\left(Y_i(t, \Delta t) - \bar{Y}\right)^2}, \tag{7}$$

where the suffix i denotes the $i-$th simulation and $\bar{Y} = \frac{1}{1000}\sum_{i=1}^{1000} Y_i(t, \Delta t)$. We check that such a correlation is independent of t, i.e. calculating $\hat{\rho}_{yy}$ for different values of t with given Δt and α and evaluating the RMS of the distribution, which turns out to be negligible. For a fixed t, we can express the results as a function of Δt and of the learning rate α. The resulting correlation is shown in Figure 1. The model reproduces, with differing degree depending on the learning rate α, the anomalies encountered in testing the classical asset price model: positive correlation over short horizons and negative correlation over a long horizon. Furthermore, while positive correlation over a short horizon seems not to be affected by the memory of the agents, long memory weakens the serial correlation computed over a long horizon. When $\alpha \longrightarrow 0$, the long run negative correlation vanishes. For $\alpha = 0$, the price is constant, and therefore the evolution of returns is given only by the dividend process, which is mean reverting, yielding a null long run correlation over a long horizon.

The statistical error of the estimator is of the order of $\frac{1}{\sqrt{1000}} \simeq 3\%$, which provides us with a lower bound on the error of the estimate. However, the discrepancy from zero is so sharp that it is hard to think of a non significant correlation. To check this point, we compute the Ljung-Box statistic on the time series generated by simulation:

$$Q(m) = T(T+2)\sum_{k=1}^{m} \frac{1}{T-k}\rho_k^2, \tag{8}$$

where T is the number of data points used in the estimate and ρ_k is the k-th lag autocorrelation coefficient of log returns. If the null hypothesis of zero serial correlation holds, then Q is asymptotically distributed as a χ^2 variable. In the case of an $ARMA(p,q)$ model, it is a χ^2 with $m-p-q$ degrees of freedom.

We apply this test to our simulations for $m = 1, \ldots, 50$, and different learning coefficients α. In all cases, the test rejects the null hypothesis of zero correlation overwhelmingly, thus confirming previous results. Table 1 shows the percentage of significantly correlated time series in the generated sample for different values of Δt and α, where significance is again evaluated with the Ljung-Box statistic. Note that the findings shown in Figure 1 are confirmed. The memory effect is observed: bounded rationality produces positive serial correlation for returns over a short period for every level of memory and negative serial correlation over a long period only if agents have short memory.

Table 1: Percentage of significantly serially correlated time series in the generated simulation sample, evaluated with the 5% percentile of the Ljung-Box statistics (3.84146).

$\alpha \downarrow$ $\quad \Delta t \rightarrow$	10	50	100	200
0	100	51.1	52.4	52.8
0.05	100	76.6	79.7	91.4
0.1	100	71.6	92.0	97.9
0.3	100	99.0	99.8	100
0.8	100	100	100	100
1	100	100	100	100

3.2 Predicting Excess Returns by Using the Lagged Dividend Yield

Many authors (Fama and French (1988), Shiller (1984)) argue that the dividend to price ratio is a good predictor of future excess returns. Using again 1000 simulations, as explained in the previous subsection, we compute the correlation

$$\rho_{dy} = corr\left[\frac{D_t}{S_t}, \log\left(\frac{S_{t+\Delta t} + D_{t+\Delta t}}{S_t}\right)\right]. \tag{9}$$

We find again that the correlation is independent from t; so we fix a t and let Δt and α vary. The results are shown in Figure 2. Our model exhibits a positive correlation, as observed in the empirical literature, over a short intermediate horizon and null correlation for a long horizon.

To evaluate the results, we run a least squares regression of excess returns on the lagged dividend yield. We test the significance of the yield coefficient obtained in such a regression by making use of the $t-$ test at the 5% confidence level. Results are reported in Table 2 together with the average R^2 of the simulation, which displays a significant yield coefficient. The results confirm Figure 2. Memory plays a role similar to that observed before: while memory does not affect the presence of correlation between the dividend yield and returns

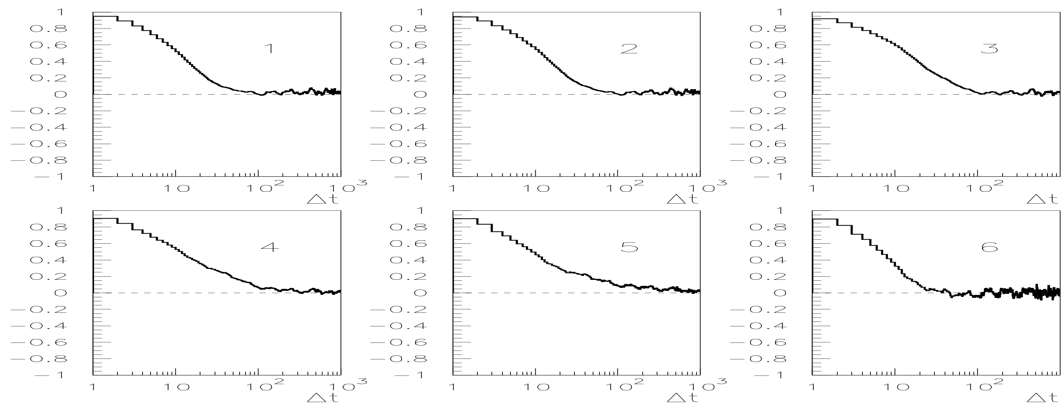

Figure 2: Correlation (9) as a function of Δt for six different values of α : (1) $\alpha = 1$ (perfect memory); (2) $\alpha = 0.8$; (3) $\alpha = 0.3$; (4) $\alpha = 0.1$; (5) $\alpha = 0.05$; (6) $\alpha = 0$ (no memory).

computed over a short horizon, the correlation becomes weaker as memory and the returns horizon become longer.

4 Continuous Time Limit

In this section, following Follmer and Schweizer (1993), and Barucci, Giuli and Monte (2000), we examine the continuous time limit of the discrete time model introduced in Section 2 as the time interval goes to zero. For the reader's convenience, we summarize the system of discrete equations involved:

$$X_k = S_k + D_k, \tag{10}$$

$$\hat{X}_k = \hat{S}_k + \hat{D}_k, \tag{11}$$

$$S_k = e^{-\rho}\hat{X}_k, \tag{12}$$

$$\hat{X}_k = \hat{X}_{k-1} + \alpha(X_{k-1} - \hat{X}_{k-1}), \tag{13}$$

$$D_k = \theta + e^{-\eta}D_{k-1} + \sigma Z_k, \tag{14}$$

Table 2: Percentage of time series with significant yield coefficient in the regression of excess returns against lagged dividend yield. In parenthesis we report the average R^2 multiplied by 100 of the regressions with a significant yield.

$\alpha \downarrow$ $\Delta t \rightarrow$	10	50	100	200
0	100 (11.6)	35.4 (0.6)	35.6 (0.5)	37.6 (0.5)
0.05	100 (24.0)	99.6 (5.4)	97.5 (3.0)	85.3 (1.6)
0.1	100 (31.6)	99.9 (6.8)	98.2 (2.8)	81.3 (1.1)
0.3	100 (41.3)	99.9 (5.1)	92.7 (1.2)	59.9 (0.5)
0.8	100 (40.2)	99.8 (1.7)	69.4 (0.4)	56.7 (0.4)
1	100 (38.4)	97.5 (1.3)	67.9 (0.4)	61.5 (0.4)

$$k = 1, 2, \ldots.$$

Here we have written k for the discrete time index to emphasize the difference from the continuous one, which will be denoted by t in the sequel, and we have introduced the instantaneous interest rate and the instantaneous autoregressive parameter

$$\rho \equiv \ln(1+r) \qquad \text{and} \qquad \eta \equiv -\ln(\gamma).$$

Combining (10) with (12), substituting into (13), and setting $Y_k \equiv \hat{X}_{k+1}$, we reduce the above system to the following one, which describes the dynamics of the relevant variables in innovation form:

$$\begin{aligned} Y_k &= Y_{k-1} - \alpha\left(1 - e^{-\rho}\right) Y_{k-1} + \alpha\left(\theta + e^{-\eta} D_{k-1} + \sigma Z_k\right), \\ D_k &= \theta + e^{-\eta} D_{k-1} + \sigma Z_k. \end{aligned} \tag{15}$$

Since $S_0 \equiv S$, $D_0 \equiv D$, and $\hat{X}_0 \equiv \hat{X}$ are assumed to be constant, and the random variables of the sequence $(Z_k)_{k \geq 1}$ are assumed to be independent, it is well known that the solution $(Y_k, D_k)_{k \geq 0}$ to (15) is a Markov chain with respect to the filtration $(\mathcal{F}_k)_{k \geq 0}$ generated by the sequence $(Z_k)_{k \geq 1}$ itself. Then, by means of a standard stepwise time rescaling, it is not difficult to show that it is possible to obtain the weak convergence of the solutions of the rescaled systems to the solution of a system of diffusive stochastic differential equations, see Barucci, Giuli and Monte (2000). To this end, first, we rewrite (15) in the equivalent difference form

$$\begin{aligned} Y_k - Y_{k-1} &= \alpha\theta - \alpha\left(1 - e^{-\rho}\right) Y_{k-1} + \alpha e^{-\eta} D_{k-1} + \alpha\sigma Z_k, \\ D_k - D_{k-1} &= \theta - \left(1 - e^{-\eta}\right) D_{k-1} + \sigma Z_k. \end{aligned} \tag{16}$$

Then, for each $n \geq 1$, we consider the partition of the interval $[k-1, k[$ by means of the n points $k - 1 \equiv t_{n(k-1)} < t_{n(k-1)+1} < \ldots < t_{nk-1} < t_{nk} \equiv k$, where $t_j - t_{j-1} \equiv \Delta t = 1/n$, for every $j \geq 1$, and we rescale the system proportionally by writing

$$\begin{aligned} Y_{t_j}^{(n)} - Y_{t_{j-1}}^{(n)} &= \alpha\theta\Delta t - \alpha\rho Y_{t_{j-1}}^{(n)}\Delta t + \alpha D_{t_{j-1}}^{(n)}\Delta t + \alpha\sigma Z_{t_j}^{(n)}, \\ D_{t_j}^{(n)} - D_{t_{j-1}}^{(n)} &= \theta\Delta t - \eta D_{t_{j-1}}^{(n)}\Delta t + \sigma Z_{t_j}^{(n)}, \end{aligned} \tag{17}$$

where $(Z_{t_j}^{(n)})_{j\geq 1}$ is a sequence of independent, normally distributed random variables with mean 0 and variance Δt, and we are taking into account that, for the role of the parameters ρ and η, the terms $\left(1 - e^{-\rho}\right) Y_{k-1}$, $e^{-\eta} D_{k-1}$, and $\left(1 - e^{-\eta}\right) D_{k-1}$ have to be rescaled to $\rho Y_{t_{j-1}}^{(n)}\Delta t$, $D_{t_{j-1}}^{(n)}\Delta t$, and $\eta D_{t_{j-1}}^{(n)}\Delta t$, respectively. Note that, similarly to (15), the solution $(Y_{t_j}^{(n)}, D_{t_j}^{(n)})_{j\geq 0}$ of (17) is a Markov chain with respect to the filtration $(\mathcal{F}_{t_j}^{(n)})_{j\geq 0}$ generated by the sequence $(Z_{t_j}^{(n)})_{j\geq 1}$. Now, defining the sequence $(W_{t_j}^{(n)})_{j\geq 0}$ by

$$W_{t_j}^{(n)} \stackrel{def}{=} \begin{cases} \sum_{i=1}^{j} Z_{t_i}^{(n)} & \text{if } j \geq 1 \\ 0 & \text{if } j = 0 \end{cases},$$

and writing

$$Y_t^{(n)} \stackrel{def}{=} Y_{t_j}^{(n)}, \quad D_t^{(n)} \stackrel{def}{=} D_{t_j}^{(n)}, \quad W_t^{(n)} \stackrel{def}{=} W_{t_j}^{(n)}, \quad \text{for } t_j \leq t < t_{j+1},$$

we introduce the processes $(Y_t^{(n)})_{t\geq 0} \equiv Y^{(n)}$, $(D_t^{(n)})_{t\geq 0} \equiv D^{(n)}$, and $(W_t^{(n)})_{t\geq 0} \equiv W^{(n)}$ having right continuous paths with finite left-hand limits (RCLL paths), which arise as step right-continuation of the point paths of the processes $(Y_{t_j}^{(n)})_{j\geq 0}$, $(D_{t_j}^{(n)})_{j\geq 0}$, and $(W_{t_j}^{(n)})_{j\geq 0}$, respectively. It is well known that, for each $n \geq 0$, we may consider all processes introduced above as single random variables taking values in the Polish space $D([0,+\infty[\,;\mathbb{R})$ of all RCLL paths endowed with the Skorohod distance. Therefore, $(Y^{(n)})_{n\geq 1}$, $\left(D^{(n)}\right)_{n\geq 1}$, and $\left(W^{(n)}\right)_{n\geq 1}$ turn out to be sequences of $D([0,+\infty[\,;\mathbb{R})$-valued random variables. In this setting, it is well known that the sequence $\left(W^{(n)}\right)_{n\geq 1}$ converges weakly to the Wiener process $W \equiv (W_t)_{t\geq 0}$ starting at 0, which may be clearly also considered as a single random variable taking values in $D([0,+\infty[\,;\mathbb{R})$. In addition, we claim that, as n goes to infinity, the sequence $(Y^{(n)}, D_t^{(n)})_{n\geq 0}$ converges to the solution of a system of stochastic differential equations. More precisely our result is the following:

Proposition 1 *As n goes to infinity, the sequence of couples of $D([0,+\infty[\,;\mathbb{R})$-valued random variables $(Y^{(n)}, D^{(n)})_{n\geq 0}$ converges weakly to the solution of the system of stochastic differential equations*

$$\begin{cases} dY_t = \alpha\theta\, dt - \alpha\rho Y_t\, dt + \alpha D_t\, dt + \alpha\sigma\, dW_t, \\ dD_t = \theta\, dt - \eta D_t\, dt + \sigma\, dW_t, \end{cases} \tag{18}$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process.

The proof of the above Proposition is analogous to that of Barucci, Giuli and Monte (2000), Proposition 1, and, for brevity, we omit it here. However, we note that (18) can be integrated by means of a standard procedure and the solution $(Y_t, D_t)_{t\geq 0}$ is given by

$$D_t = De^{-\eta t} + \frac{\theta}{\eta}\left(1 - e^{-\eta t}\right) + \sigma \int_0^t e^{-\eta(t-s)}\, dW_s. \tag{19}$$

$$Y_t = \phi D_t + (Y - \phi D)\, e^{-\alpha\rho t} + \frac{\alpha - \phi}{\alpha\rho}\theta\left(1 - e^{-\alpha\rho t}\right) + (\alpha - \phi)\,\sigma \int_0^t e^{-\alpha\rho(t-s)}\, dW_s, \tag{20}$$

where

$$\phi \equiv \frac{\alpha}{\alpha\rho - \eta}. \tag{21}$$

From (20) it follows

Proposition 2 *For all $t, \Delta t \geq 0$ we have*

$$
\begin{aligned}
& Cov\left(Y_t - Y_{t-\Delta t}, Y_{t+\Delta t} - Y_t\right) \\
= \; & -\frac{1}{2}\frac{(\alpha-\phi)^2}{\alpha\rho}\sigma^2\left(1+e^{-\alpha\rho(2t-\Delta t)}\right)\left(1-e^{-\alpha\rho\Delta t}\right)^2 \\
& +\frac{\alpha\left(1-\alpha\rho+\eta\right)\phi}{\alpha^2\rho^2-\eta^2}\sigma^2\left(\left(1-e^{-\eta\Delta t}\right)^2+e^{-(\alpha\rho+\eta)t}\left(e^{\alpha\rho\Delta t}-1\right)\left(1-e^{-\eta\Delta t}\right)\right) \\
& +\frac{\alpha\left(1-\alpha\rho+\eta\right)\phi}{\alpha^2\rho^2-\eta^2}\sigma^2\left(\left(1-e^{-\alpha\rho\Delta t}\right)^2+e^{-(\alpha\rho+\eta)t}\left(e^{(1-\gamma)\Delta t}-1\right)\left(1-e^{-\alpha\rho\Delta t}\right)\right) \\
& -\frac{1}{2}\frac{\phi^2}{\eta}\sigma^2\left(1+e^{-\eta(2t-\Delta t)}\right)\left(1-e^{-\eta\Delta t}\right)^2. \qquad (22)
\end{aligned}
$$

Notice that,

$$\frac{\phi}{\alpha^2\rho^2-\eta^2} = \frac{\alpha}{\left(\alpha\rho+\eta\right)\left(\alpha\rho-\eta\right)^2}, \qquad (23)$$

which is positive independently on the sign of ϕ. Moreover, on account of (21) and (23), we can write

$$
\begin{aligned}
& Cov\left(Y_t - Y_{t-\Delta t}, Y_{t+\Delta t} - Y_t\right) \\
= \; & -\frac{1}{2}\frac{\left(1-\alpha\rho+\eta\right)^2}{\alpha\rho}\psi^2\left(1+e^{-\alpha\rho(2t-\Delta t)}\right)\left(1-e^{-\alpha\rho\Delta t}\right)^2 \\
& +\frac{1-\alpha\rho+\eta}{\alpha\rho+\eta}\psi^2\left(\left(1-e^{-\eta\Delta t}\right)^2+e^{-(\alpha\rho+\eta)t}\left(e^{\alpha\rho\Delta t}-1\right)\left(1-e^{-\eta\Delta t}\right)\right) \\
& +\frac{1-\alpha\rho+\eta}{\alpha\rho+\eta}\psi^2\left(\left(1-e^{-\alpha\rho\Delta t}\right)^2+e^{-(\alpha\rho+\eta)t}\left(e^{\eta\Delta t}-1\right)\left(1-e^{-\alpha\rho\Delta t}\right)\right) \\
& -\frac{1}{2}\frac{1}{\eta}\psi^2\left(1+e^{-(\eta)(2t-\Delta t)}\right)\left(1-e^{-(\eta)\Delta t}\right)^2, \qquad (24)
\end{aligned}
$$

where

$$\psi^2 \equiv \frac{\alpha^2\sigma^2}{\left(\alpha\rho-\eta\right)^2}. \qquad (25)$$

Notice also that we have clearly

$$Cov\left(\hat{X}_t - \hat{X}_{t-\Delta t}, \hat{X}_{t+\Delta t} - \hat{X}_t\right) = Cov\left(Y_t - Y_{t-\Delta t}, Y_{t+\Delta t} - Y_t\right).$$

In addition, from (12), by time rescaling it follows

$$S_t - S_{t-\Delta t} = e^{-\rho\Delta t}\left(\hat{X}_t - \hat{X}_{t-\Delta t}\right).$$

Therefore, Equation (24) provides us with all relevant information about the price process covariance.

Now, when Δt is small, ($\Delta t \ll 1/\alpha\rho \wedge 1/\eta$ from a computational point of view), then it is easily seen that (24) gives

$$\begin{aligned}
& Cov\left(Y_t - Y_{t-\Delta t}, Y_{t+\Delta t} - Y_t\right) \\
= & -\frac{1}{2}\alpha\rho\left(1-\alpha\rho+\eta\right)^2\psi^2\left(1+e^{-2\alpha\rho t}\right)\left(\Delta t\right)^2 \\
& +\frac{1-\alpha\rho+\eta}{\alpha\rho+\eta}\psi^2\left(\alpha^2\rho^2+2\alpha\rho\eta e^{-(\alpha\rho+\eta)t}+\eta^2\right)\left(\Delta t\right)^2 \\
& -\frac{1}{2}\eta\psi^2\left(1+e^{-2\eta t}\right)\left(\Delta t\right)^2 \\
& +o\left(\left(\Delta t\right)^2\right),
\end{aligned} \tag{26}$$

and it is not difficult to show that the right-hand side in (26) is positive for every $t > 0$.

Indeed, setting

$$\begin{aligned}
f\left(t\right) \equiv & -\frac{1}{2}\alpha\rho\left(1-\alpha\rho+\eta\right)^2\left(1+e^{-2\alpha\rho t}\right) \\
& +\frac{1-\alpha\rho+\eta}{\alpha\rho+\eta}\left(\alpha^2\rho^2+2\alpha\rho\eta e^{-(\alpha\rho+\eta)t}+\eta^2\right) \\
& -\frac{1}{2}\eta\left(1+e^{-2\eta t}\right),
\end{aligned}$$

we have

$$f\left(0\right) = \left(1-\alpha\rho\right)\left(\eta-\alpha\rho\right)^2,$$

and

$$f'\left(t\right) = \left(\alpha\rho\left(1-\alpha\rho+\eta\right)e^{-\alpha\rho t} - \eta e^{-\eta t}\right)^2.$$

Hence, we can conclude that, for small values of Δt the price-process increments are positively correlated at any time instant t, provided a plausible value of ρ. On the other hand, when Δt is large ($\Delta t \gg 1/\alpha\rho \vee 1/\left(1-\gamma\right)$), then t is also necessarily large ($t \geq \Delta t$) and, from (24), we obtain

$$Cov\left(Y_t - Y_{t-\Delta t}, Y_{t+\Delta t} - Y_t\right) \simeq -\frac{1}{2}\left(\frac{\left(1-\alpha\rho+\eta\right)^2}{\alpha\rho} - 4\frac{1-\alpha\rho+\eta}{\alpha\rho+\eta} + \frac{1}{\eta}\right)\psi^2 \tag{27}$$

In this case, a straightforward computation allows to reduce the right-hand side in (27) to

$$-\frac{1}{2}\frac{(\eta-\alpha\rho)^2\,(1+2\eta+\eta\,(1+\alpha\rho))}{\alpha\rho\,(\alpha\rho+\eta)\,\eta}\psi^2,$$

and the latter clearly shows that for large values of Δt the price-process increments are negatively correlated.

Continuous time limit analysis is useful to compare the covariance of the price increments by computing it in a closed form, as Δt changes. A task which is very difficult in a discrete time.

5 Conclusions

By assuming that agents exploit arbitrage opportunities in a financial market and form their expectations according to an adaptive learning rule, we show that the price dynamics reproduces some of the anomalies encountered in the empirical literature of financial markets: asset price returns are serially correlated and in particular they are positively correlated over a short horizon and negatively correlated over a long horizon. Results are illustrative of the effect of bounded rationality in financial markets, we do not attempt to calibrate the model in order to reproduce stylized facts. The above feature is not reproduced in previous studies on asset prices under bounded rationality. Moreover, returns can be predicted through past dividend yields. We have established that short memory in the learning process induces long-range dependencies in the time series (mean reversion). The result confirms Brock and Hommes (1998), where it is shown that if agents switch among forecasting rules according to their past performance with a strong switching intensity (which corresponds to a short memory), then an irregular dynamics arises in financial markets. The interpretation is that a short memory induces both overreaction and delayed overreaction.

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