Current Trends in Potential Theory

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Conference Proceedings

Bucharest (Romania), September 2002 and 2003

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Theta
Bucharest 2005

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CIP Description of the National Library
Descrierea CIP a Bibliotecii Naţionale a României
CURRENT TRENDS IN POTENTIAL THEORY.
CONFERENCE (2002 ; Bucureşti)
Current trends in potential theory : conference
proceedings : Bucharest - Romania, September 2002-2003 /
ed. by: Dominique Bakry, Lucian Beznea, Gheorghe Bucur,
Michael Röckner. - Bucureşti : Theta, 2005
ISBN 973-85432-6-6

I. Bakry, Dominique (ed.)
II. Beznea, Lucian (ed.)
III. Bucur, Gheorghe (ed.)
IV. Röckner, Michael (ed.)

V. Current trends in potential theory. Conference (2003 ;
Bucureşti)
517(063)

ISBN 973-85432-6-6

Printed on acid free paper.

Printed and bound by ARTA GRAFICĂ, Bucharest, Romania

A POTENTIAL THEORETIC APPROACH TO TWISTING

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To the 70^{th} *anniversary of the birthday of Nicu Boboc*

ABSTRACT. We establish a new, potential theoretic approach to the study of twist points in the boundary of simply connected planar domains.

1. MOTIVATION AND OUTLINE OF OUR RESULTS

This paper is an overview of some results of ours. Proofs appear in [1]. We introduce a geometric, potential theoretic approach to the study of twist points in the boundary of planar domains. Our main motivation is the general principle that the use of potential theoretic methods will improve our understanding of the correspondence between geometric properties of the domain and analytic properties of the conformal map of the unit disc onto the domain, and hint at possible higher dimensional versions; cf. [3].

1.1. **Background.** The Riemann mapping theorem states, in essence, that all bounded, connected, simply connected open sets in the plane are analytically isomorphic to each other. Let $\mathbb{S}_b$ be the family of all such planar domains. If $D \in \mathbb{S}_b$ then let ∂D denote the boundary of D. Let $U \overset{\text{def}}{=} \{z \in \mathbb{R}^2 : |z| < 1\}$ be the unit disc of center 0. If $D \in \mathbb{S}_b$ and $x \in D$ then an analytic isomorphism

$$f : U \to D$$

such that $f(0) = x$ is called a *Riemann map of D with pole at x*. This map is determined modulo a rotation of U. Let $\mathscr{F}(f) \subset \partial U$ be the set of points $\theta \in \partial U$ where the *angular limit* of f at θ, denoted by $f_\flat(\theta)$, exists; see [25], p. 6. A theorem of Fatou [10] implies that the set $\mathscr{F}(f)$ has full Lebesgue measure. Denote by $\partial_\flat D$ the image of the map

$$f_\flat : \mathscr{F}(f) \to \partial D.$$

In general, $f_\flat$ is not onto, but its image is independent of f and dense in ∂D. Indeed, if $w \in \partial D$, the following conditions are equivalent:

(a) there is a point $\theta \in \partial U$ such that $f_\flat(\theta) = w$;
(b) there is a *Jordan half-open arc in D ending at w*.

Observe that condition (a) is expressed in terms of the Riemann map of the domain, while (b) is expressed entirely in terms of the geometry of the domain. Properties of the first kind are called *analytic* while those of the second are called *geometric*; cf. [2]. The

2000 *Mathematics Subject Classification.* Primary 31A15; Secondary 30C85.
Key words and phrases. Harmonic measure, twist points, McMillan Theorem.

study of the correspondence between analytic conditions or quantities and geometric ones is *one of the main aims of the theory of the boundary behavior of conformal maps* (we quote from [25] and refer to it for further background; see also [4]).

This correspondence may reveal itself in subtle guises. For example, the analytic Bloch function

$$\log f' : U \to \mathbb{C}$$

has the following remarkable property:

For Lebesgue a.e. $\theta \in \partial U$, the angular boundary behavior of $\log f'$ *at θ determines whether D is twisting or sectorially accessible at $f_b(\theta) \in \partial D$;*

see below for precise definitions of these terms. Recall that the statement holds for almost every point, but *not* at every point; see [25] and [21[.

Let S be the class of all analytic univalent functions $f : U \to \mathbb{C}$, normalized by $f(0) = 0$ and $f'(0) = 1$. If log is the branch equal to 0 at $\zeta = 0$ and $\zeta \in U$ then the quantity

$$\log \frac{\zeta f'(\zeta)}{f(\zeta)}$$

is an *analytic functional of $f \in S$* (see [25, p. 123]) about which a number of properties are known. We mention the following:

- Seidel [26]: A domain D is *starlike relative to $x = 0 \in D$* if and only if $\left|\arg\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\right| < \frac{\pi}{2}$ for all $\zeta \in U$ (cf. [25]);
- Grunsky [15]: the inequality

$$\left|\arg \frac{\zeta f'(\zeta)}{f(\zeta)}\right| \leqslant \log \frac{1+|\zeta|}{1-|\zeta|}$$

holds and is sharp for $f \in S$ and $\zeta \in U$; cf. [13, p. 117] and [24, p. 168].

1.2. **Outline of our results.** Let $\mathscr{H}(D)$ be the space of real valued functions defined on D and harmonic therein. For each $D \in \mathbb{S}_b$, we define and study a certain function

$$h_D : D \to \mathscr{H}(D),$$

that recaptures

$$\arg \frac{\zeta f'(\zeta)}{f(\zeta)}$$

directly in terms of geometric data of the domain $D \in \mathbb{S}_b$, where $D = f(U)$. Here are the most salient properties of the function h_D.

For fixed $x \in D$ and for a.e. $w \in \partial D$, relative to harmonic measure, the behavior of $h_D(x)(z)$ when $z \to w$ predicts whether D is twisting or sectorially accessible at w. For NTA domains we establish the previous assertion in a quantitative way. The definition of NTA domains is found in [17]. Thus, h_D plays the role of $\log f'$ and is instrumental for a potential theoretic approach to twisting.

The definition of h_D is not based on the Riemann map of D, it is purely geometric and potential theoretic.

For each $x \in D$, the function

$$h_D(x) : D \to \mathbb{R}$$

is a harmonic Bloch function on D.

The harmonicity of $h_D(x): D \to \mathbb{R}$ does not appear to be obvious from direct inspection of its definition; indeed, it will be seen to be the expression of a symmetry under reflections on lines — even for domains that are not symmetric.

If f is a Riemann map of D with pole at $x \in D$, then the composition

$$h_D(x) \circ f : U \to \mathbb{R}$$

is equal to the map $\zeta \mapsto \arg \frac{\zeta f'(\zeta)}{f(\zeta)}$. Thus, our function h_D yields a geometric, potential theoretic representation for the analytic quantity $\arg \frac{\zeta f'(\zeta)}{f(\zeta)}$.

2. PRELIMINARY RESULTS AND DEFINITIONS

Throughout this paper, $D \in \mathbb{S}_b$, f is the Riemann map of D with pole at $x \in D$, and $G_D : D \times D \to (-\infty, \infty[$ is the Green function of D.

If $w \in \mathbb{R}^2$ we denote $\mathrm{dist}(w) : \mathbb{R}^2 \to [0, \infty)$ the function $\mathrm{dist}(w)(z) \stackrel{\mathrm{def}}{=} |z - w|$ and by $\mathrm{dist}(\partial D)$ the function on $\mathbb{R}^2$ given by $\mathrm{dist}(\partial D) \stackrel{\mathrm{def}}{=} \min_{w \in \partial D} \mathrm{dist}(w)$. We write $\mathrm{dist}(\partial D, z)$ for $\mathrm{dist}(\partial D)(z)$. Let $B(w,r) \stackrel{\mathrm{def}}{=} \{z \in \mathbb{R}^2 : |z - w| < r\}$ and $\overline{B}(w,r) \stackrel{\mathrm{def}}{=} \{z \in \mathbb{R}^2 : |z - w| \leq r\}$. Following [9, Section 1.I.2[the unweighted average of a function u over $\partial B(w,r)$ is denoted by $L(u, w, r)$.

If $A, B \subset \mathbb{C}$, the set of continuous maps from A to B is denoted $C(A,B)$. We let $J(A,B)$ be the subset of $C(A,B)$ consisting of injective maps. If $g \in C(A, \mathbb{C})$ and $w \in \mathbb{C}$ then $g - w \in C(A, \mathbb{C})$ is the function $s \mapsto g(s) - w$. We use $\mathrm{Im}(z)$ to denote the imaginary part of $z \in \mathbb{C}$, $\mathrm{Re}(z)$ for its real part and $z^\star$ for its complex conjugate. Let $C(\partial D)$ be the Banach space of real valued continuous functions on ∂D, endowed with the uniform norm.

2.1. **Harmonic measure.**

If $\phi \in C(\partial D)$ then the unique harmonic function on D whose boundary values are equal to ϕ is denoted $\phi^\natural$ and called the *harmonic extension on D of ϕ*. The *harmonic extension operator*

$$\natural : C(\partial D) \to \mathscr{H}(D)$$

is the map $\phi \mapsto \phi^\natural$. If $z \in D$, the functional $\phi \mapsto \phi^\natural(z)$ is given by integration with respect to a Borel probability measure on ∂D, denoted by μ_D^z and called *harmonic measure for D with pole at z*. Thus,

$$\phi^\natural(z) = \int_{\partial D} \phi(w) \mu_D^z(\mathrm{d}w), \quad \forall z \in D, \forall \phi \in C(\partial D). \tag{2.1}$$

Harmonic measures with different poles are mutually absolutely continuous. Observe that if $B = B(x,r)$ and $u \in C(\partial D)$ then

$$L(u, x, r) = \int_{\partial B} u(z) \mu_B^x(\mathrm{d}z).$$

2.2. **NTA domains.** NTA domains in $\mathbb{R}^n$ were introduced in [17[, where the precise definition can be found. Let NTA_2 be the collection of all planar NTA domains. Then $\mathrm{NTA}_2 \subset \mathbb{S}_b$ with proper inclusion, since a planar NTA domain is a quasidisc; see [17] and [25[. The boundary of a planar NTA domain need not be differentiable; see, for example, the von Koch snowflake [18]. If $D \in \mathrm{NTA}_2$, $w \in \partial D$ and $\alpha > 0$ then $\Gamma_\alpha(w) \subset D$ is defined as

$$\Gamma_\alpha(w) = \{z \in D : |z - w| < (1+\alpha)\mathrm{dist}(\partial D, z)\}.$$

If D has smooth boundary then the approach regions $\Gamma_\alpha(w)$ are comparable to open triangles having vertex at w and contained in D. In general, $\Gamma_\alpha(w)$ may not contain any triangle having vertex at w.

Theorem 2.1. *If $D \in \mathrm{NTA}_2$ and $U \in \mathscr{H}(D)$ then $\partial D = N \cup P \cup L$ where:*

(i) *N has harmonic measure zero;*

(ii) *if $w \in P$ then U is unbounded above and below in $\Gamma_\alpha(w) \cap B(w,r)$ for each positive α, r;*

(iii) *if $w \in L$ then U has a finite limit along $\Gamma_\alpha(w)$ for each $\alpha > 0$.*

2.3. **Sectorial accessibility.** The set $\mathrm{Sect}(f)$, defined in [25, p. 144] is a subset of ∂U. We now define the corresponding subset of ∂D.

Definition 2.2. We define $\mathrm{Sect}(D) \subset \partial D$ as follows: a point $w \in \partial D$ belongs to $\mathrm{Sect}(D)$ if and only if D contains an open triangle having a vertex at w.

Remark. The correspondence between $\mathrm{Sect}(f)$ and $\mathrm{Sect}(D)$ is given by the following relation:

$$f_b(\mathrm{Sect}(f)) = \mathrm{Sect}(D). \tag{2.2}$$

2.4. **Curves.** If $B \subset \mathbb{C}$, the elements of $C([0,1[, B)$ are called *curves in B*. The points $c(0)$, $c(1)$ are called the *endpoints of* the curve c; $c(0)$ is the *initial point of* c, $c(1)$ its *endpoint* and c is said to be a curve *from* $c(0)$ *to* $c(1)$. The image of c is also denoted by c. If convenient, we may assume, after change of parameter, that the parameter space is equal to $[\alpha, \beta[$, where $-\infty < \alpha < \beta < \infty$; see [20[. A curve in B whose endpoints coincide is called a *closed curve in B*. A change of parameter shows that closed curves in B are elements of $C(\partial U, B)$. We denote by $\Sigma_B(y,z)$ the set of smooth curves in B from $y \in B$ to $z \in B$.

2.5. **Half-open arcs.** If $B \subset \mathbb{C}$, the elements of $C([0,1), B)$ are called *half-open arcs in B*. We say that the half-open arc c *ends at* $w \in \mathbb{C}$ or that c is a half-open arc *from* $c(0)$ *to* w if the limit $\lim_{s \to 1} c(s)$ exists and is equal to w. Elements of $J([0,1), B)$ are called *Jordan half-open arcs in B*. We denote $J([0,1), B)$ by $J(B)$.

2.6. **Accessibility.** If $\theta \in \partial U$ and $s \in [0,1)$ let $\rho_w(s) \stackrel{\mathrm{def}}{=} s\theta$. Then $\rho_\theta \in J(U)$ and ρ_θ ends at θ. If $c \in J(U)$ ends at some point of ∂U then it is not necessarily true that $f \circ c$ ends at some point of ∂D. However, if $c \in J(D)$ ends at some point of ∂D then $f^{-1} \circ c$ ends at some point of ∂U; see [20[. Indeed, the set $\partial_b D$, image of f_b, can be described geometrically.

Lemma 2.3. *If $w \in D$ then the following conditions are equivalent:*
(i) *there is $\theta \in \mathscr{F}(f)$ such that $f_b(\theta) = w$;*
(ii) *there is $\theta \in \partial U$ such that $f \circ \rho_\theta$ ends at w;*
(iii) *there is $\theta \in \partial U$ and $c \in J(U)$ ending at θ such that $f \circ c$ ends at w;*
(iv) *there is $c \in J(D)$ ending at w.*

The proof of the following result can be found in [20, Section III.6] and [9].

Lemma 2.4. *The set $\partial_b D$ is dense in ∂D and it has full harmonic measure.*

2.7. **Liftings.** If A is a topological space, $g \in C(A, \mathbb{C}\setminus\{0\})$, $\gamma \in C(A, \mathbb{C})$ and $e^\gamma = g$ then we say that γ is a *lifting of* g. Let $L(g)$ denote the set of all liftings of g. If A is connected and simply connected, then each $g \in C(A, \mathbb{C}\setminus\{0\})$ has a lifting and, if γ_1 and γ_2 are liftings of g, then there is a unique $n \in \mathbb{Z}$ such that $\gamma_1 = \gamma_2 + 2\pi i n$; see [14].

2.8. **Twisting half-open arcs.** Let c be a half-open arc in $\mathbb{C}\setminus\{0\}$ and assume that c has a lifting γ such that $\mathrm{Im}(\gamma)$ is unbounded above and below on $[0, 1)$. Then each lifting of c has the same property and we say that c is *twisting*. Let $\mathscr{T}$ be the set of twisting half-open arcs in $\mathbb{C}\setminus\{0\}$.

Proposition 2.5. *Let $c_1, c_2 \in J(\mathbb{C}\setminus\{0\})$ be Jordan half-open arcs having 1 as initial point and ending at the origin. Assume that $c_1(s_1) = c_2(s_2)$ if and only if $s_1 = s_2 = 0$. Then $c_1 \in \mathscr{T}$ if and only if $c_2 \in \mathscr{T}$.*

2.9. **Twist points of the Riemann map.** Let $\mathrm{Twist}(f) \subset \partial U$ be the set of points $\theta \in \mathscr{F}(f)$ such that $f \circ \rho_\theta - f_b(\theta) \in \mathscr{T}$.

Lemma 2.6. *If $\theta \in \mathscr{F}(f)$ then the following conditions are equivalent:*
(i) *$\theta \in \mathrm{Twist}(f)$;*
(ii) *$f \circ c - f_b(\theta) \in \mathscr{T}$ for each $c \in J(U)$ ending at θ;*
(iii) *for each $c \in J(U)$ ending at θ and each $g \in L(f - f_b(\theta))$, the map $\mathrm{Im}(g) \circ c : [0, 1) \to \mathbb{R}$ is unbounded above and below.*

2.10. **Twist points of the domain.** Let $\mathrm{twist}(D)$ be the set of points $w \in \partial_b D$ such that $c - w \in \mathscr{T}$ for each $c \in J(D)$ ending at w. The following result clarifies the relation between $\mathrm{Twist}(f)$ and $\mathrm{twist}(D)$. Since we could not find it in the literature, we record it for future reference.

Proposition 2.7. *If $w \in \partial_b D$ then the following conditions are equivalent:*
(1) *$w \in \mathrm{twist}(D)$;*
(2) *$\{\theta \in \mathscr{F}(f) : f_b(\theta) = w\} \subset \mathrm{Twist}(f)$;*
(3) *$\{\theta \in \mathscr{F}(f) : f_b(\theta) = w\} \cap \mathrm{Twist}(f) \neq \emptyset$.*

The following result follows from the Twist Point Theorem [21]; cf. [25].

Proposition 2.8. *If $D \in \mathbb{S}_b$ then $\mathrm{Sect}(D) \cup \mathrm{twist}(D)$ has full harmonic measure in ∂D.*

2.11. **The quasihyperbolic metric and harmonic Bloch functions.** The *quasihyperbolic distance*[1] $k_D(z, y)$ in D from z to y is defined as the minimum of the arc length

[1] Introduced in [12]; see also [11] and [25, p. 92].

integrals

$$\int_c \frac{1}{\operatorname{dist}(\partial D)}\,\mathrm{d}s,$$

evaluated along all rectifiable paths c from z to y contained in D. The quasihyperbolic distance is a geometric quantity; cf. [25[. Observe that if $z, y \in D$ and $|z-y| < \frac{1}{2}\operatorname{dist}(\partial D, z)$ then

$$\frac{1}{2}\frac{|z-y|}{\operatorname{dist}(\partial D, z)} < k_D(z,y) < 2\frac{|z-y|}{\operatorname{dist}(\partial D, z)}.$$

Indeed, $k_D(z,y) \geqslant \log\left(1+\frac{|z-y|}{\operatorname{dist}(z,\partial D)}\right)$, by [12, Lemma 2.1[. It follows that a function $U : D \to \mathbb{R}$ is Lipschitz relative to the metric (D, k_D), i.e.

$$\sup_{\substack{z,z'\in D\\ z\neq z'}} \frac{|U(z)-U(z')|}{k_D(z,z')} < \infty, \tag{2.3}$$

if and only if U satisfies

$$\sup_D \operatorname{dist}(\partial D)|\operatorname{grad} U| < \infty. \tag{2.4}$$

If $U \in \mathscr{H}(D)$ satisfies (2.3) then U is called a *harmonic Bloch function*; cf. [19].

2.12. **The hyperbolic metric and the Green function.** Let λ_U be the hyperbolic metric of the unit disc, with the normalization such that $\lambda_U(0,\zeta) = \frac{1}{2}\log\frac{1+|\zeta|}{1-|\zeta|}$; see [25, p. 6[. Other normalizations can be found in the literature. If $z, y \in D$ then $\lambda_D(z,y) \stackrel{\text{def}}{=} \lambda_U(f^{-1}(z), f^{-1}(y))$ is the *hyperbolic distance* in D between z and y. The Koebe distortion theorem implies that

$$\lambda_D(z,y) \leqslant k_D(z,y) \leqslant 4\lambda_D(z,y), \quad \forall\, z, y \in D. \tag{2.5}$$

See [25, p. 92[.

The hyperbolic geodesics in D emanating from x admit both an analytic description and a potential theoretic one. Given $z \in D$, we denote by $g = g_D(x,z)$ the hyperbolic geodesic in D from x to z. If $f(\zeta) = z$ then a parametric representation of $g_D(x,z)$ is given by $t \mapsto f(t\zeta)$, $0 \leqslant t \leqslant 1$. The arc $g_D(x,z)$ is precisely the integral curve from x to z of the gradient of the Green function $G_D(x,\cdot)$.

2.13. **The winding angle.** Fix an orientation in $\mathbb{R}^2$. Let $c \in J([0,1[,\mathbb{R}^2)$ and assume that c is smooth. Let $c_\star$ be the potential of the double layer of constant unit density over c. The function $c_\star$ is harmonic on the open set $\mathbb{R}^2\setminus c$, complement of the image of c, and it extends by continuity at the endpoints of c but not at the other points of c, since therein it is subject to a jump; see [6[. Recall that the value of $c_\star$ at w is given by the arc length integral

$$c_\star(w) \stackrel{\text{def}}{=} \int_c \frac{\partial}{\partial n}\log\frac{1}{\operatorname{dist}(w)}\,\mathrm{d}s, \tag{2.6}$$

where n is the positively oriented normal to c. We call $c_\star(w)$ the *winding angle of c as seen from w*, since it is the signed variation of the argument of $y - w$, when y goes from

x to z along c, as can be seen via the Green formula. Observe that $c_*(w) = (c-w)_*(0)$. The following estimate will be useful:

$$|c_*(w)| \leqslant \int_c \frac{1}{\operatorname{dist}(w)}\,ds. \tag{2.7}$$

Indeed,

$$\left|\frac{\partial}{\partial n}\log\frac{1}{\operatorname{dist}(w)}\right| \leqslant \left|\operatorname{grad}\log\frac{1}{\operatorname{dist}(w)}\right| = \frac{1}{\operatorname{dist}(w)}.$$

If $0 \notin c$ then $c_*(0)$ equals the integral along c of the closed differential form $\frac{x dy - y dx}{x^2+y^2}$ modulo a sign that depends on the orientation of the plane. Since a closed differential form can be integrated along any curve, without assuming any smoothness on the curve (see [5, p. 58]) then $c_*(w)$ can be defined for any curve c in $\mathbb{R}^2$ and

$$|c_*(w)| = \int_{c-w} \frac{x dy - y dx}{x^2+y^2}.$$

Similarly, since $\frac{d\zeta}{z}$ is a closed differential form in $\mathbb{C}\setminus\{0\}$, if c is curve in $\mathbb{R}^2$ then $c_*(w)$ is equal (modulo a sign) to the imaginary part of the complex line integral $\int_c \frac{d\zeta}{\zeta - w}$.

If $\gamma \in L(c-w)$ then $c_*(w) = \operatorname{Im}(\gamma(1)) - \operatorname{Im}(\gamma(0))$; see [22]. Indeed, if c is given by the parametric representation $\phi(t), 0 \leqslant t \leqslant 1$ and we let $\Phi(t) = \frac{\phi(t)-w}{|\phi(t)-w|}$, then $\Phi : [0,1[\to \partial U$. Now, let $\Phi^{\sim} : [0,1] \to \mathbb{R}$ be any lifting of Φ via the universal covering map $\mathbb{R} \to \partial U$, $r \mapsto e^{ir}$. Then $c_*(w) = \Phi^{\sim}(1) - \Phi^{\sim}(0)$ (the right-hand side being independent on the choice of the lifting).

2.14. **The relative winding angle.** Consider the continuous function

$$\langle\cdot,\cdot\rangle_D : D \times D \to C(\partial D)$$

defined by evaluating the winding angle along curves in D: If $y, z \in D$, $w \in \partial D$ and $c \in \Sigma_D(y,z)$, then

$$\langle y,z\rangle_D(w) \overset{\text{def}}{=} c_*(w).$$

Since D is simply connected and $c_*(w)$ is given by integrating along c a differential 1-form defined on the punctured plane $\mathbb{R}^2\setminus\{w\}$ and closed therein, Stokes' theorem implies that the function $\langle\cdot,\cdot\rangle_D$ is well defined. Moreover, $\langle y,z\rangle_D(w)$ is separately harmonic in $y, z \in D$ for fixed w.

Thus, $\langle y,z\rangle_D(w)$, called the *winding angle relative to D*, measures the signed variation of the argument of $y' - w$ as y' goes from y to z staying within D.

Observe that $\langle x,x\rangle_D = 0$, $\langle x,z\rangle_D = -\langle z,x\rangle_D$, and in fact

$$\langle x,y\rangle_D + \langle y,z\rangle_D = \langle x,z\rangle_D, \quad \forall x,y,z \in D. \tag{2.8}$$

Lemma 2.9. *If $w \in \partial_b D$ then the following conditions are equivalent:*

(i) $w \in \operatorname{twist}(D)$;

(ii) $\limsup_{s\to 1}\langle x,c(s)\rangle_D(w) = +\infty$ *and* $\liminf_{s\to 1}\langle x,c(s)\rangle_D(w) = -\infty$ *for each half-open arc in D ending at w.*

Remark. Property (ii) is independent of the choice of x.

2.15. The harmonic winding angle. If $y_1, y_2 \in D$ then the function $\langle y_1, y_2\rangle_D : \partial D \to \mathbb{R}$ has the harmonic extension $\langle y_1, y_2\rangle_D^\natural : D \to \mathbb{R}$. Following (2.1), $\langle y_1, y_2\rangle_D^\natural$ is given by

$$\langle y_1, y_2\rangle_D^\natural(z) = \int_{\partial D} \langle y_1, y_2\rangle_D(w)\mu_D^z(dw), \quad z \in D.$$

Moreover, $\langle y_1, y_2\rangle_D^\natural(z)$ is harmonic in each variable separately, and

$$\lim_{z \to w} \langle y_1, y_2\rangle_D^\natural(z) = \langle y_1, y_2\rangle_D(w) = c_*(w) \tag{2.9}$$

for each $w \in \partial D$ and $c \in \Sigma_D(y_1, y_2)$.

Lemma 2.10.

$$|\langle y_1, y_2\rangle_D^\natural(z)| \leq k_D(y_1, y_2), \quad \forall z \in D. \tag{2.10}$$

3. The function h_D

3.1. The original potential theoretic definition. We now define a function $D \to \mathcal{H}(D)$. Let $h_D : D \times D \to \mathbb{R}$ be the function

$$h_D(y, z) \stackrel{\text{def}}{=} \langle y, z\rangle_D^\natural(z) = \int_{\partial D} \langle y, z\rangle_D(w)\mu_D^z(dw), \quad y, z \in D.$$

Notation. Whenever convenient, we shall write $h_D(y)(z)$ for $h_D(y, z)$, so that $h_D(y) : D \to \mathbb{R}$ denotes $h_D(y, \cdot)$ as function of the *second* variable.

Remark. The functional h_D is covariant under translations, rotations and dilations of D. We shall see that it uniquely determines the domain D (apart from the scale) since it determines (the inverse of) its Riemann map.

Since $h_D(x, z)$, as function of x, is a superposition of functions harmonic in x, its harmonicity in x follows from standard arguments. Its harmonicity with respect to z does not seem to be equally immediate or obvious from the viewpoint of potential theory; we now show that it is related, in fact equivalent, to a certain invariance property under reflections on lines.

Proposition 3.1. *If $D \in \mathbb{S}_b$ then the following conditions are equivalent:*

(a) *for each $x \in D$ then function $h_D(x)$ is harmonic at each point of D;*

(b) *for each $x \in D$ then function $h_D(x)$ has vanishing mean-value over each sufficiently small ball centered at the point x;*

(c) *for each $x \in D$, if z' denotes $x + (z - x)^*$ and $D' \stackrel{\text{def}}{=} \{z' : z \in D\}$ then*

$$L(h_D(x), x, r) = L(h_{D'}(x), x, r)$$

for each $r > 0$ such that $\overline{B}(x, r) \subset D$.

Theorem 3.2. *If $D \in \mathbb{S}_b$ and $y \in D$ then $h_D(y)$ is harmonic on D.*

Theorem 3.3. *If $D \in \mathbb{S}_b$ and $y \in D$ then $h_D(y)$ is a harmonic Bloch function on D.*

Remark. A natural issue, related to Lemma 2.9, is to determine sufficient conditions that make the quantity

$$|h_D(x, z) - \langle x, z\rangle_D(w)|$$

bounded when $z \to w \in \partial_b D$. See Sections 3.5 and 4.

3.2. The twist function of a smoothly bounded domain. The function h_D also admits a second, less direct, potential theoretic description, based on an approximation argument. Let $\mathbb{S}_b^\infty \subset \mathbb{S}_b$ be the set of all domains in $\mathbb{S}_b$ with C^∞ boundary. If $D \in \mathbb{S}_b^\infty$ then there is $r_D > 0$ such that if $0 < r \leqslant r_D$ and $w \in \partial D$, then there is a point in D, denoted $n_D^r(w)$, at distance r from w, on the inner normal to ∂D at w.

Definition 3.4. If $D \in \mathbb{S}_b^\infty$ and $x \in D$ we define $t_D : D \to C(\partial D)$ for $0 < r \leqslant r_D$

$$t_D(x)(w) \stackrel{\text{def}}{=} (x, n_D^r(w))_D(w), \quad w \in \partial D. \tag{3.1}$$

The values of $t_D(x)$ are independent of r. The function t_D is called the *twist function of D*.

Remark. The function t_D measures the twisting of D around ∂D. Thus, $t_D(x)$ gauges the difference between, say, a disc and a domain shaped like a snake. Indeed, these domains have no twist points but the disc twists much less around its boundary than the others.

Notation. Whenever convenient, we shall write $t_D(x, w)$ for $t_D(x)(w)$.

Lemma 3.5. *If $D \in \mathbb{S}_b^\infty$ then the following conditions are equivalent:*

(a) $h_D(x) = t_D(x)^\natural$ *for each* $x \in D$;

(b) $t_D(x)^\natural(x) = 0$ *for each* $x \in D$.

Theorem 3.6. $h_D(x) = t_D(x)^\natural$ *for each* $D \in \mathbb{S}_b^\infty$ *and* $x \in D$.

Corollary 3.7. *If $D \in \mathbb{S}_b^\infty$ then for each $w \in \partial D$*

$$\lim_{z \to w} h_D(x, z) = t_D(x, w). \tag{3.2}$$

Remark. For domains whose boundary is not smooth, the function $t_D(x)$ cannot be directly defined as in (3.1) and, in particular, the function $h_D(x, \cdot)$ does not possess boundary values as in (3.2). Consider, for example, the von Koch snowflake [18[. In fact, almost every point, in the boundary of the von Koch snowflake, relative to harmonic measure, is a twist point [8, 25[.

Regular exhaustions. If $D \in \mathbb{S}_b$ then there are domains $D_n \in \mathbb{S}_b^\infty$, $n \geqslant 1$, such that $\overline{D_n} \subset D_{n+1}$ and $\bigcup_{n=1}^{\infty} D_n = D$. Any such sequence $\{D_n\}_n$ is called a *regular exhaustion of D*.

Theorem 3.8. *If $D \in \mathbb{S}_b$, $\{D_n\}_n$ is a regular exhaustion of D and $x \in D_1$ then the harmonic functions $t_{D_n}(x)^\natural$ converge, uniformly on compact subsets of D, to a function that is independent of the choice of the regular exhaustion of D; indeed, the limit function is precisely $h_D(x)$.*

Remark. Observe that Theorem 3.8 provides another, less direct but natural[2] description of h_D. We now show that if we choose a special exhaustion of D then there is no need to use a limiting argument.

[2] The statement of Theorem 3.8 is inspired by the construction of the generalized solution of Dirichlet's problem in arbitrary domains in $\mathbb{R}^n$, due to N. Wiener [28].

The Green exhaustion. We now choose the regular exhaustion given by the superlevel sets of the Green function of the domain. If $z \in D$ and $z \neq x$ then the subdomain $D_x^z \subset D$ defined by

$$D_x^z \stackrel{\text{def}}{=} \{\zeta \in D : G_D(x,\zeta) > G_D(x,z)\}$$

has smooth boundary and contains x. Moreover, $z \in \partial D_x^z$ and $\overline{D_x^z} \subset D$. In particular, the function $t_{D_x^z}$ is defined at the point (x,z). Recall that $h_D(x,x) \equiv 0$.

Theorem 3.9. *If $D \in \mathbb{S}_b$, $x \in D$, $z \in D$ and $z \neq x$, then*

$$h_D(x,z) = t_{D_x^z}(x,z), \tag{3.3}$$

thus, the restriction of $h_D(x)$ to the subdomain D_x^z is equal to $h_{D_x^z}(x)$.

3.3. **The h function and the hyperbolic geodesics.** The following corollaries of Theorem 3.9 show that h_D can be described in terms of the hyperbolic geodesics of the domain. Recall the definition of $g_D(x,z) \in \Sigma_D(x,z)$ given in Section 2. Recall also that, if $c \in \Sigma_D(x,z)$, then $c_\star$ is also defined at the endpoints x and z of c.

Corollary 3.10. *If $D \in \mathbb{S}_b$, $x, z \in D$ and $x \neq z$, then*

$$h_D(x,z) = g_D(x,z)_\star(z). \tag{3.4}$$

If c is a smooth curve and c' denotes the tangent vector to c then c' is, in itself, a curve in $\mathbb{R}^2$. In the following corollary, we consider the curve $t_D(x,z)$ defined as g' where $g = g_D(x,z)$ is the hyperbolic geodesic in D from x to z.

Corollary 3.11. *If $D \in \mathbb{S}_b$, $x, z \in D$, then*

$$h_D(x,z) - h_D(z,x) = t_D(x,z)_\star(0). \tag{3.5}$$

Thus, $h_D(x,z) - h_D(z,x)$ is the winding angle of $t_D(x,z)$ as seen from 0.

An application to starlike domains. A domain D is called *starlike with respect to $x \in D$* if

$$|\langle x,z\rangle_D(w)| < \pi$$

for each $w \in \partial D$ and $z \in D$. An equivalent definition is that the line segment from x to z is entirely contained in D for each $z \in D$. The following result recaptures the well known analytic characterization of starlike domains, due to W. Seidel; see Section 3.4 and, in particular, (3.9).

Proposition 3.12. *A domain $D \in \mathbb{S}_b$ is starlike with respect to $x \in D$ if and only if $|h_D(x,z)| < \frac{\pi}{2}$ for all $z \in D$.*

3.4. **An analytic description.** We now give a purely analytic description of h_D. On the polydisc $U^3 \stackrel{\text{def}}{=} U \times U \times U$ define the nonvanishing analytic function

$$f_\star : U^3 \to \mathbb{C} \setminus \{0\}$$

by

$$f_\star(\zeta,\xi,\eta) \stackrel{\text{def}}{=} \frac{(\infty, f(\eta), f(\xi), f(\zeta))}{(\infty,\eta,\xi,\zeta)}, \tag{3.6}$$

where $(z_1, z_2, z_3, z_4) \stackrel{\text{def}}{=} \frac{z_1-z_2}{z_1-z_4}\frac{z_3-z_4}{z_3-z_2}$ is the cross ratio of the points z_j in the Riemann sphere [16, p. 58]. In particular, $(\infty, a, c, b) = \frac{b-c}{a-c}$. Observe that $f_\star(0,0,0) = 1$. The functional $f \mapsto f_\star$ is invariant under translations, rotations and dilations. If $\zeta = \xi$, $\eta = \theta \in \partial U$, and $f_\flat$ exists at θ, then (3.6), defined by continuity, is precisely the Visser–Ostrowski quotient

$$f_\star(\zeta,\zeta,\theta) = \frac{f'(\zeta)(\zeta-\theta)}{f(\zeta)-f_\flat(\theta)};$$

see [25]. If $\zeta = \xi$ and $\eta = 0$ then (3.6) yields

$$f_\star(\zeta,\zeta,0) = \frac{\zeta f'(\zeta)}{f(\zeta)-f(0)},$$

the analytic quantity used by W. Seidel in order to characterize starlike domains[3]. Let arg be the branch equal to zero for $\zeta = 0$.

Theorem 3.13. *Under the hypotheses given above, the following identities hold for all $\zeta, \xi \in U$, with $x = f(0)$, $z = f(\zeta)$, $w = f_\flat(\theta)$ and $z' = f(\xi)$,*

$$\arg f_\star(\zeta,\theta,0) = \langle x,z\rangle_D(w) - \langle 0,\zeta\rangle_U(\theta), \tag{3.7}$$

$$\arg f_\star(\zeta,\xi,0) = \langle x,z\rangle_D^\natural(z') - \langle 0,\zeta\rangle_U^\natural(\xi), \tag{3.8}$$

$$\arg f_\star(\zeta,\zeta,0) = h_D(x,z). \tag{3.9}$$

Remark. Theorem 3.2 follows from (3.9). Our proof of Theorem 3.2 is purely potential theoretic and independent of (3.9).

Remark. From (3.4) and (3.9) we obtain

$$\arg f_\star(\zeta,\zeta,0) = g_D(x,z)_\star(z),$$

where $x = f(0)$ and $z = f(\zeta)$. The previous identity can also be proved directly, as in [27, p. 672] without employing our function h_D. See also [7].

3.5. **The h function and relative winding.** If f is the Riemann map of D and $f_\flat$ exists at θ, then the following identity is verified by computation, left to the reader.

$$\frac{f_\star(\zeta,\zeta,\theta) f_\star(0,0,\zeta) f_\star(\zeta,\theta,0)}{f_\star(\zeta,\zeta,0)} = f_\star(0,0,\theta), \quad \forall \zeta \in U. \tag{3.10}$$

Now, choose a value for $\arg f_\star(0,0,\theta)$, such that $|\arg f_\star(0,0,\theta)| < \pi$. This choice uniquely determines a branch of $\arg f_\star(\zeta,\zeta,\theta)$ with the given initial condition for $\zeta = 0$. Now, select the branch of $\arg f_\star(\zeta,\zeta,0)$, equal to 0 for $\zeta = 0$ and do the same for $\arg f_\star(\zeta,0,0)$ and $\arg f_\star(\zeta,\theta,0)$. The following result follows from (3.10) and Theorem 3.13.

Proposition 3.14. *If $f_\flat(\theta)$ exists then, under the choices described above, for all $\zeta \in U$*

$$h_D(x,z) - \langle x,z\rangle_D(w) = \arg f_\star(\zeta,\zeta,\theta) + h_D(z,x) + \varepsilon(z), \tag{3.11}$$

where $z = f(\zeta)$, $x = f(0)$ and $|\varepsilon(z)| < \frac{3\pi}{2}$.

Indeed, $\varepsilon(z) = -\langle 0,\zeta\rangle_U(\theta) - \arg f_\star(0,0,\theta)$. The quantity $h_D(z,x)$ does not always remain bounded when $z \to w$; the reader may verify that a negative example is given by the domain in [13, Figure 2, p. 36].

[3]See [26, p. 206] and [25, p. 66].

4. The function h_D and twisting

Theorem 4.1. *If $D \subset \mathbb{R}^2$ is NTA then for each $\alpha > 0$ and each $w \in \partial D$*

$$\sup_{z \in \Gamma_\alpha(w)} |h_D(x,z) - \langle x,z\rangle_D(w)| < \infty. \tag{4.1}$$

Remark. Observe that (4.1) is a general, intrinsic form of the estimate (1) given in [8] in the proof of Lemma 3 of that paper. Examples show that (4.1) may fail if D is not NTA at w; see [21].

Theorem 4.2. *If $D \in \mathbb{S}_b$ then for a.e. point $w \in \partial D$, relative to harmonic measure, the boundary behaviour of $h_D(x)$ at w predicts whether $w \in$ Sect(D) or $w \in$ twist(D).*

Acknowledgements

This research was supported in part by MIUR (Cofin. 2000).

The third-named author acknowledges hospitality from the Chalmers University of Technology, through the grant *Jubileumsfonden* from Göteborg University (1998–2002), from the University of Rome "Tor Vergata", through an INdAM grant (1997–1998), and from C.U.N.Y. (Nov–Dec 2002).

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