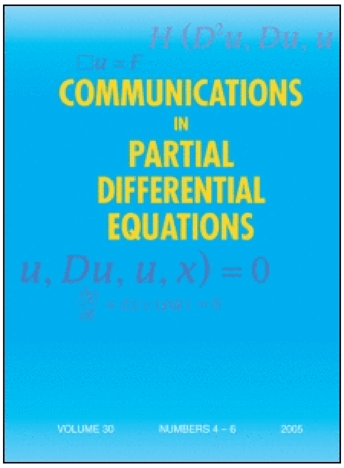
This article was downloaded by: *[Sinestrari, Carlo]*
On: *28 December 2009*
Access details: *Access Details: [subscription number 913232801]*
Publisher *Taylor & Francis*
Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



Communications in Partial Differential Equations
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597240

On a Differential Model for Growing Sandpiles with Non-Regular Sources

Piermarco Cannarsa [a]; Pierre Cardaliaguet [b]; Carlo Sinestrari [a]
[a] Dipartimento di Matematica, Università di Roma Tor Vergata, Rome, Italy [b] Laboratoire de Mathématique (UMR CNRS 6205), Université de Brest, Brest, France

To cite this Article Cannarsa, Piermarco, Cardaliaguet, Pierre and Sinestrari, Carlo(2009) 'On a Differential Model for Growing Sandpiles with Non-Regular Sources', Communications in Partial Differential Equations, 34: 7, 656 — 675
To link to this Article: DOI: 10.1080/03605300902909966
URL: http://dx.doi.org/10.1080/03605300902909966

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: http://www.informaworld.com/terms-and-conditions-of-access.pdf

This article may be used for research, teaching and private study purposes. Any substantial or systematic reproduction, re-distribution, re-selling, loan or sub-licensing, systematic supply or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.

Communications in Partial Differential Equations, 34: 656–675, 2009
Copyright © Taylor & Francis Group, LLC
ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300902909966

Taylor & Francis
Taylor & Francis Group

On a Differential Model for Growing Sandpiles with Non-Regular Sources

PIERMARCO CANNARSA[1], PIERRE CARDALIAGUET[2], AND CARLO SINESTRARI[1]

[1]Dipartimento di Matematica, Università di Roma Tor Vergata, Rome, Italy
[2]Laboratoire de Mathématique (UMR CNRS 6205), Université de Brest, Brest, France

We consider a variational model that describes the growth of a sandpile on a bounded table under the action of a vertical source. The possible equilibria of such a model solves a boundary value problem for a system of nonlinear partial differential equations that we analyze when the source term is merely integrable. In addition, we study the asymptotic behavior of the dynamical problem showing that the solution converges asymptotically to an equilibrium that we characterize explicitly.

Keywords Asymptotic profile; Distance function; Granular matter; Uniqueness of solutions.

Mathematics Subject Classification Primary 35C15, 35F30; Secondary 47J20, 35Q99.

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

1. Introduction

Several differential systems have been proposed for describing the growth of a sandpile on a bounded table under the action of a vertical source. Here we investigate the one proposed by L. Prigozhin in the seminal paper [17]. This problem is strongly related to the fast/slow diffusion model studied by many authors (see, e.g., [1, 13, 14]) as well as to the so–called BCRE models (named after the authors of [3], see also, e.g., [4, 16]).

In the model we consider, the table $\Omega \subset \mathbb{R}^n$ is a given bounded connected domain with smooth boundary. The source $f \geq 0$ is an integrable function in $\overline{\Omega}$. The height of the sand, denoted by u, satisfies the following parabolic problem:

$$u_t = \operatorname{div}(vDu) + f \quad \text{in } \mathbb{R}^+ \times \Omega$$

Received February 1, 2008; Accepted March 14, 2009
Address correspondence to Carlo Sinestrari, Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, Rome 00133, Italy; E-mail: sinestra@mat.uniroma2.it

$$|Du| \leq 1, \quad |Du| < 1 \Rightarrow v = 0 \quad \text{in } \mathbb{R}^+ \times \Omega \tag{1}$$
$$u = 0 \quad \text{on } \partial\Omega, \quad u(0, \cdot) = u_0 \quad \text{in } \Omega,$$

where $v(t, x) \geq 0$ is an auxiliary function, to be determined, and u_0 is the initial configuration, such that $\|Du_0\|_\infty \leq 1$, $u_0 = 0$ on $\partial\Omega$.

In [17], it is proved—under very general assumptions—that system (1) has a unique weak solution (u, v). Moreover, the first component u of the solution is characterized by the variational inequality

$$\begin{cases} f - u_t \in \partial I(u) & \text{in } L^2(\Omega) \\ u(0, \cdot) = u_0. \end{cases} \tag{2}$$

Here, ∂I denotes the subdifferential of the convex function $I : L^2(\Omega) \to [0, \infty]$ defined by

$$I(u) = \begin{cases} 0 & \text{if } u \in \mathbb{K}_0 \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathbb{K}_0 := \{u \in W^{1,\infty}(\Omega) : \|Du\|_\infty \leq 1, u|_{\partial\Omega} = 0\}.$$

Once well-posedness is established, the next natural question is whether the solution $u(t, \cdot)$ converges as $t \to \infty$. At least formally, one would expect the asymptotic limit to be an equilibrium configuration of the dynamical system and, therefore, to satisfy

$$-\text{div}(vDu) = f, \quad (1 - |Du|)v = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{3}$$

This system has also been found by Hadeler and Kuttler [16] in order to describe the equilibria of the aforementioned BCRE model. In that same paper, the authors gave the explicit solution for this equilibrium for $n = 1$. Later on, system (3) was analyzed in [5] for $n = 2$ and then in [6] for arbitrary space dimension, obtaining the existence, partial uniqueness and representation formula for the solution when the source, f, is continuous. In this case, the solution (u, v) turns out to be continuous in Ω, with u equal to the distance from $\partial\Omega$ on the support of v. In particular, the continuity of v is a special—to some extent, surprising—property of the solution that cannot be expected if f is discontinuous.

On the other hand, from both the theoretical and the applied points of view it is interesting to study problem (3) for an integrable source term. This is one of the aims of this paper: we will show that the theory of [5, 6] (existence, partial uniqueness, representation) can be extended to $f \in L^1(\Omega)$. For this, several new ideas will be necessary.

For instance, the representation formula (19) for the solution of (3) involves integrals of f along line segments: when f is just measurable, it should be checked that such a formula remains meaningful, and we do this in Section 4.1. As for uniqueness, in the continuous case an important step of the proof was to show that v vanishes on the cut locus $\overline{\Sigma}$ of Ω, which is a set of measure zero. Clearly, the sense

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

of such a property should be made precise when v is just integrable. In fact, it is even false if f is unbounded (see Example 4.7). Therefore, we have to develop a new strategy to prove uniqueness: the new proof we give in Section 4.2 turns out to be both simpler and more powerful than the one given in [5, 6]. Like in the continuous case, full uniqueness holds just for the v component of the solution. Indeed, one cannot expect uniqueness for u: the structure of (3) only allows to determine u on the support of v (where it coincides with the distance from $\partial\Omega$). Let us mention that a representation formula for the solutions has been recently given for similar problems with anisotropic metrics [9] or with different boundary conditions [8]; it can be expected that our approach for uniqueness can be applied to these cases as well.

So, returning to the original problem of studying the asymptotic limit of the solution of (2), the above discussion explains why the stationary problem (3) does not suffice to uniquely determine such a function on the whole domain Ω. For this purpose, we will study problem (2) directly, showing that the solution $u(t, \cdot)$ converges, as $t \to \infty$, to a limit that we characterize in Section 3.1. Such a limit depends on Ω, the support of f, and the initial condition u_0. Moreover, if f is bounded away from zero in a neighborhood of the cut locus, then the equilibrium is attained in finite time, as we prove in Section 3.2.

Finally, setting up the theory in $L^1(\Omega)$—or, more generally, in $L^p(\Omega)$ for $1 \leq p \leq \infty$—one can derive Lipschitz estimates in $L^p(\Omega)$ for the v-component of the solution. Such estimates, that are reminiscent of the L^p-regularity results obtained in [11] for the Monge–Kantorovich problem, are derived in Section 4.3 of this paper.

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

2. Notation and Preliminaries

Let $n \geq 2$ be an integer. We denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the Euclidean scalar product and norm in $\mathbb{R}^n$ respectively. For any $x_0 \in \mathbb{R}^n$ and $r > 0$ we set

$$B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}.$$

For a given function $g \in L^1(A)$, where A is an open subset of $\mathbb{R}^n$, we call the *support* of g the set of all $x \in A$ such that $\int_{B_\varepsilon(x)\cap A} |g(y)|dy > 0$ for all $\varepsilon > 0$ (or, equivalently, such that $\{y \in B_\varepsilon(x) \cap A : g(y) \neq 0\}$ has positive measure for all $\varepsilon > 0$). It is easy to see that the support of g is a closed set (in the relative topology of A) and that it coincides with the usual notion of support if g is continuous. Clearly, if $\phi \in C(A)$ is nonnegative and $g \in L^1(A)$ is such that $\int_A \phi(x)|g(x)|dx = 0$, then $\phi \equiv 0$ on $\mathrm{spt}(g)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with $\mathscr{C}^2$ boundary $\partial\Omega$. We briefly recall some properties of the distance function in Ω; some more details can be found e.g., in [5, 7]. In what follows we denote by $d : \overline{\Omega} \to \mathbb{R}$ the distance function from the boundary of Ω and by Σ the singular set of d, that is, the set of points $x \in \Omega$ at which d is not differentiable. Since d is Lipschitz continuous, Σ has Lebesgue measure zero. Introducing the projection $\Pi(x)$ of x onto $\partial\Omega$ in the usual way, Σ is also the set of all points x at which $\Pi(x)$ is not a singleton.

For any $x \in \partial\Omega$ and $i = 1, \dots, n-1$, the number $\kappa_i(x)$ denotes the $i-th$ principal curvature of $\partial\Omega$ at the point x, corresponding to a principal direction $e_i(x)$ orthogonal to $Dd(x)$, with the sign convention $\kappa_i \geq 0$ if the normal section of Ω

along the direction e_i is convex. Also, we will label in the same way the extension of κ_i to $\overline{\Omega}\backslash\Sigma$ given by

$$\kappa_i(x) = \kappa_i(\Pi(x)) \quad \forall x \in \overline{\Omega}\backslash\Sigma. \tag{4}$$

Notice that the regularity of Ω guarantees that the principal curvatures κ_i are continuous functions on $\partial\Omega$. For any $x \in \overline{\Omega}$ and any $y \in \Pi(x)$ we recall that

$$\kappa_i(y)d(x) \leq 1 \quad \forall i = 1, \ldots, n-1. \tag{5}$$

If, in addition, $x \in \overline{\Omega}\backslash\overline{\Sigma}$, then

$$\kappa_i(x)d(x) < 1 \quad \text{and} \quad D^2d(x) = -\sum_{i=1}^{n-1} \frac{\kappa_i(x)}{1-\kappa_i(x)d(x)} e_i(x) \otimes e_i(x),$$

where $e_i(x)$ is the unit eigenvector corresponding to $\frac{\kappa_i(x)}{1-\kappa_i(x)d(x)}$ and $\otimes$ stands for tensor product (see, e.g., [15]).

Remark 2.1. The set Γ of points $x \in \Omega\backslash\Sigma$ such that the equality sign holds in (5) for some index i is called the set of regular *focal* (or *conjugate*) points. It represents the "boundary" of the singular set Σ in the sense that $\overline{\Sigma} \subset \Omega$ and $\overline{\Sigma} = \Sigma \cup \Gamma$. The set $\overline{\Sigma}$ is called the *cut locus* (or the *ridge*) of Ω. We recall that under our assumptions, $\overline{\Sigma}$ is a set of zero Lebesgue measure.

Let us introduce the function

$$\tau(x) = \begin{cases} \min\{t \geq 0 : x + tDd(x) \in \overline{\Sigma}\} & \forall x \in \overline{\Omega}\backslash\overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}. \end{cases} \tag{6}$$

Since the map $x \mapsto x + \tau(x)Dd(x)$ is a natural retraction of $\overline{\Omega}$ onto $\overline{\Sigma}$, we will refer to $\tau(\cdot)$ as the *maximal retraction length* of Ω onto $\overline{\Sigma}$ or *normal distance to* $\overline{\Sigma}$. It is easy to see that

$$d(x + tDd(x)) = d(x) + t, \quad \forall t \in [-d(x), \tau(x)]. \tag{7}$$

It can be proved that τ is continuous in $\overline{\Omega}$ (see [5, Lemma 2.14]). The function τ actually enjoys finer regularity properties, which will not be needed in this paper.

To a closed set $C \subset \Omega$, let us associate the map u_C defined as follows:

$$u_C(x) = \max_{y \in C}[d(y) - |y - x|]_+, \quad x \in \Omega. \tag{8}$$

If C is empty then we set $u_C \equiv 0$.

Proposition 2.2. *The function u_C satisfies the following properties*:

(i) *u_C is the smallest nonnegative function on Ω such that $u_C \equiv d$ on C and* $\mathrm{Lip}(u_C) \leq 1$.
(ii) *$u_C \leq d$ in Ω; in addition $u_C \equiv d$ in Ω if and only if $\overline{\Sigma} \subset C$.*

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

Proof. Property (i) is a straightforward consequence of the definition. Since d is nonnegative with Lipschitz constant one, we also deduce from (i) that $u_C \leq d$. To prove the equivalence in (ii), suppose first that $\overline{\Sigma} \subset C$. Then (i) implies that $u_C = d$ on $\overline{\Sigma}$. If we take any $x \in \Omega$, we have that $x + \tau(x)Dd(x) \in \overline{\Sigma}$, and therefore

$$u_C(x) \geq u_C(x + \tau(x)Dd(x)) - |\tau(x)Dd(x)| = d(x + \tau(x)Dd(x)) - \tau(x) = d(x),$$

where we have also used (7). Thus we have that $u_C \equiv d$ everywhere in Ω. To prove the converse implication, we argue by contradiction and suppose that $u_C \equiv d$ but $\overline{\Sigma} \not\subset C$. Since C is a closed set, we can find $x_0 \in \Sigma$ such that $x_0 \notin C$. For a smooth set Ω as in our hypotheses, the singular set Σ lies at positive distance from $\partial\Omega$; since we are assuming that $u_C \equiv d$, we deduce that $u_C(x_0) = d(x_0) > 0$. By definition of u_C, there exists $y_0 \in C$ such that $u_C(x_0) = [d(y_0) - |y_0 - x_0|]_+$. Since $u_C(x_0) > 0$, the argument of the positive part has to be strictly positive and we deduce that $u_C(x_0) = d(y_0) - |y_0 - x_0|$. Let us now take any $z_0 \in \Pi(x_0)$. Then $d(y_0) \leq |y_0 - z_0|$ and

$$d(x_0) = u_C(x_0) = d(y_0) - |y_0 - x_0| \leq |y_0 - z_0| - |y_0 - x_0| \leq |x_0 - z_0| = d(x_0).$$

So equality holds everywhere in the above inequalities. In particular, $d(y_0) = |y_0 - z_0|$ and x_0 belongs to the interior of the segment $[z_0, y_0]$. This is impossible, since it is well known (see e.g., [7, Cor. 3.4.5(iii)]) that all points of the segment joining a point of Ω to one of its projections on $\partial\Omega$ do not belong to Σ, except possibly for the initial endpoint. $\square$

3. Asymptotic Behavior

In this section we investigate the variational inequality

$$\begin{cases} f - u_t \in \partial I(u) & \text{in } L^2(\Omega) \\ u(\cdot, 0) = u_0 \end{cases} \tag{9}$$

where $f \in L^2(\Omega)$, $I(u)$ is defined by

$$I(u) = \begin{cases} 0 & \text{if } u \in \mathbb{K}_0 \\ +\infty & \text{otherwise,} \end{cases}$$

and where

$$\mathbb{K}_0 := \{u \in W^{1,\infty}(\Omega) : \|Du\|_\infty \leq 1, u|_{\partial\Omega} = 0\}.$$

The initial position u_0 is also assumed to belong to $\mathbb{K}_0$.

Equation (9) has been interpreted by several authors [1, 13, 14, 17] as a natural model for growing sandpiles. We are interested in the behavior as $t \to +\infty$ of the solution of (9). We recall that u is a solution of (9) if, for any $T > 0$, $u \in H^1((0, T), L^2(\Omega))$, $u(t, \cdot) \in \mathbb{K}_0$ for any $t \geq 0$ and $f - u_t(t, \cdot) \in \partial I(u(t, \cdot))$ a.e., where $\partial I(u(t, \cdot))$ denotes the subdifferential (in the sense of convex analysis) of the convex map I at $u(t, \cdot)$. Note that this inclusion is equivalent to

$$\langle u_t(t, \cdot) - f, \phi - u(t, \cdot)\rangle_{L^2(\Omega)} \geq 0 \quad \forall \phi \in \mathbb{K}_0, \text{ for almost all } t \geq 0$$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

where $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ stands for the scalar product in $L^2(\Omega)$. It is well-known that (9) has a unique solution, see, e.g., [2].

The following comparison principle for solutions can be found in [17].

Lemma 3.1. *Suppose that $f^1 \geq f^2$ and that $u_0^1 \geq u_0^2$. Then $u^1 \geq u^2$, where u^i, $i = 1, 2$, is the solution of the variational problem* (9) *with $f = f^i$ and $u(0, \cdot) = u_0^i$.*

In particular, a solution u of (9) is non–decreasing in time (compare u with the constant solution $u^2 \equiv u_0$ given for $f^2 \equiv 0$). We give the proof of the above lemma for the reader's convenience.

Proof. Let us set

$$u^+(t, x) = \max\{u^1(t, x), u^2(t, x)\} \quad \text{and} \quad u^-(t, x) = \min\{u^1(t, x), u^2(t, x)\}.$$

Then $u^\pm$ are continuous functions with $u_t^\pm \in L^2((0, T) \times \Omega)$ and $u^\pm \in \mathbb{K}_0$. Using u^+ as a test function in the variational problem for u^1, we obtain

$$\langle f^1 - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \leq 0 \quad \text{a.e. } t \in [0, T].$$

Since $f^1 \geq f^2$ and $u^+ \geq u^1$, then also

$$\langle f^2 - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \leq 0 \quad \text{a.e. } t \in [0, T].$$

Analogously, by looking at the variational problem for u^2, we get

$$\langle f^2 - u_t^2, u^- - u^2 \rangle_{L^2(\Omega)} \leq 0.$$

Now, let us denote by $\mathbf{1}_A$ the characteristic function of a set A, that is $\mathbf{1}_A(x) = 1$ for $x \in A$ and $\mathbf{1}_A(x) = 0$ otherwise. Thus, since $\{(t, x) : u^1(t, x) < u^2(t, x)\}$ is an open set (u^i are continuous functions), we have

$$u^- - u^2 = (u^1 - u^2)\mathbf{1}_{\{u^1 < u^2\}} = (u^1 - u^+)\mathbf{1}_{\{u^1 < u^2\}} = u^1 - u^+,$$

while $u_t^2 \mathbf{1}_{\{u^1 < u^2\}} = u_t^+ \mathbf{1}_{\{u^1 < u^2\}}$. Therefore,

$$\langle f^2 - u_t^2, u^- - u^2 \rangle_{L^2(\Omega)} = \langle f^2 - u_t^+, u^1 - u^+ \rangle_{L^2(\Omega)} \leq 0 \quad \text{a.e. } t \in [0, T],$$

and then

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\left|u^+ - u^1\right|^2_{L^2(\Omega)} &= \langle u_t^+ - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \\ &= \langle u_t^+ - f^2, u^+ - u^1 \rangle_{L^2(\Omega)} + \langle f^2 - u_t^1, u^+ - u^1 \rangle_{L^2(\Omega)} \leq 0. \end{aligned}$$

Since $u^+ = u^1$ at time $t = 0$ and the functions u^i, $u^\pm$ are continuous, we conclude that $u^+ \equiv u^1$. $\square$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

3.1. *Identification of the Asymptotic Limit*

Let u be a solution of (9) defined on $[0, +\infty)$. Since u is nondecreasing and bounded from above by d—as are all elements of $\mathbb{K}_0$—the limit

$$u_\infty(x) = \lim_{t\to+\infty} u(t, x)$$

exists and satisfies

$$u_0(x) \le u_\infty(x) \le d(x) \quad \forall x \in \Omega.$$

Moreover $u_\infty \in \mathbb{K}_0$ because $u(\cdot, t) \in \mathbb{K}_0$ for any t.

Theorem 3.2. *We have*

$$u_\infty(x) = \max\{u_0(x), u_f(x)\} \quad \forall x \in \Omega, \tag{10}$$

where u_f is the map defined by

$$u_f(x) = \max_{y\in spt(f)} [d(y) - |y - x|]_+ \quad x \in \Omega. \tag{11}$$

Proof. Let us introduce the function

$$\psi(t) = \int_\Omega u(t, x)dx \quad \forall t \ge 0.$$

Since $u_t \in L^2([0, T] \times \Omega)$ for any $T > 0$, ψ is absolutely continuous. The map $t \mapsto u(t, x)$ being nondecreasing for any x, we have that $u_t \ge 0$ a.e. and

$$\psi'(t) = \int_\Omega u_t(t, x)dx \ge 0 \quad \text{for almost all } t \ge 0.$$

Since $\psi(t) \to \int_\Omega u_\infty$ as $t \to +\infty$, there is a sequence $t_k \to +\infty$ such that $\psi'(t_k) \to 0$ and for which $u_t(t_k, \cdot)$ exists and satisfies

$$\langle f - u_t(t_k, \cdot), \phi - u(t_k, \cdot)\rangle \le 0 \quad \forall \phi \in \mathbb{K}_0.$$

Note that $\psi'(t_k) \to 0$ implies that $u_t(t_k, \cdot) \to 0$ in $L^1(\Omega)$. Passing to the limit in the above equation gives

$$\langle f, \phi - u_\infty\rangle_{L^2(\Omega)} \le 0 \quad \forall \phi \in \mathbb{K}_0. \tag{12}$$

In particular, plugging $\phi = d$ in the above inequality entails

$$\int_\Omega f(d - u_\infty) \le 0.$$

Since $f \ge 0$ and $u_\infty \le d$, we conclude that $u_\infty = d$ on spt(f).

To complete the proof of the theorem, we first observe that $\bar{u} := \max\{u_0, u_f\}$ is a stationary solution of (9) because $\bar{u} = d \ge \phi$ on spt(f) for any $\phi \in \mathbb{K}_0$ and $f \ge 0$. Since $u_0 \le \bar{u}$, we get, by comparison, that $u(t, x) \le \bar{u}$ for any $t \ge 0$. Hence $u_\infty \le \bar{u}$.

Conversely, we already know that $u_0 \le u_\infty$. Since $u_\infty \in \mathbb{K}_0$ and $u_\infty = d$ on $\mathrm{spt}(f)$, we obtain $u_\infty \ge u_f$ because u_f is the smallest function in $\mathbb{K}_0$ which coincides with d on $\mathrm{spt}(f)$ (Proposition 2.2). Thus, $u_\infty \ge \bar{u}$. □

3.2. *Convergence in Finite Time*

In this subsection we assume that f is positive in a neighborhood of the ridge, that is,

$$\exists r > 0 \quad \text{such that } f \ge r \quad \text{a.e. in } B_r(x) \text{ for any } x \in \Sigma. \tag{13}$$

Such an assumption implies, in particular, that $\overline{\Sigma} \subset \mathrm{spt}(f)$. Therefore, by Proposition 2.2 and Theorem 3.2, the asymptotic limit u_∞ of the solution to (9) is given by the distance function d. Our next result shows that, in this case, convergence takes place in finite time.

Theorem 3.3. *Under assumption* (13) *there is a time T such that, for any initial position $u_0 \in \mathbb{K}_0$, the solution $u(\cdot, t)$ of* (9) *becomes stationary after T, that is,*

$$u(t, \cdot) = d \quad \forall t \ge T. \tag{14}$$

Proof. Let $R = \max_\Omega d$ and let $r > 0$ be given by assumption (13). Let us set $T = R^{n+1}/((n+1)r^{n+1} + 1$. We will show that (14) holds for such a choice of T. Fix $\bar{x} \in \Sigma$ and define, for all $x \in \Omega$ and $t \ge 0$,

$$f^1(t, x) = \begin{cases} (r - |x - \bar{x}|)_+ & \text{if } t \in [0, 1) \\ r & \text{if } t \ge 1 \text{ and } |x - \bar{x}| \le r \\ 0 & \text{otherwise.} \end{cases}$$

Let u^1 be the solution of (9) with initial condition $u_0^1 :\equiv 0$ and source f^1. One readily checks that

$$u^1(t, x) = t\,(r - |x - \bar{x}|)_+ \quad \forall t \in [0, 1], \; \forall x \in \Omega.$$

Let α be given by

$$\alpha(t) = \left(r^{n+1} + (n+1)r^{n+1}(t-1)\right)^{\frac{1}{n+1}} \quad \text{for } t \ge 1.$$

Observe that $\alpha'(t) = r^{n+1}\alpha(t)^{-n}$. We claim that

$$u^1(t, x) = (\alpha(t) - |x - \bar{x}|)_+ \quad \text{if } t \in [1, \bar{t}],$$

where $\bar{t} = (d^{n+1}(\bar{x}) - r^{n+1})/((n+1)r^{n+1}) + 1$. To prove this, let us denote by u^2 the right-hand side of the equality. Then $u^2(\cdot, t) \in \mathbb{K}_0$ for any $t \in [1, \bar{t}]$, $u^2(1, \cdot) = u^1(1, \cdot)$ and $u_t^2 \in L^2((1, \bar{t}) \times \Omega)$. Let us now check that u^2 satisfies the variational inequality

$$\langle u_t^2 - f^1, \phi - u^2 \rangle_{L^2(\Omega)} \ge 0 \quad \forall \phi \in \mathbb{K}_0.$$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

For any $\phi \in \mathbb{K}_0$ we have (in polar coordinates)

$$\begin{aligned}&\langle u_t^2 - f^1, \phi - u^2\rangle_{L^2(\Omega)}\\&\quad = \int_{S^{n-1}} d\mathscr{H}^{n-1}(\omega)\int_0^r (\alpha'(t) - r)\big(\phi(t,\bar{x}+\rho\omega) - u^2(t,\bar{x}+\rho\omega)\big)\rho^{n-1}d\rho\\&\qquad + \int_{S^{n-1}} d\mathscr{H}^{n-1}(\omega)\int_r^{\alpha(t)} \alpha'(t)\big(\phi(t,\bar{x}+\rho\omega) - u^2(t,\bar{x}+\rho\omega)\big)\rho^{n-1}d\rho.\end{aligned}$$

From the definition of u^2 and the fact that $\mathrm{Lip}(\phi) \le 1$ we deduce that the map

$$\rho \mapsto \phi(t,\bar{x}+\rho\omega) - u^2(t,\bar{x}+\rho\omega)$$

is nondecreasing on $[0,\alpha(t)]$. Therefore, since $0 < \alpha'(t) \le r$, we have

$$(\alpha'(t) - r)(\phi(t,\bar{x}+\rho\omega) - u^2(t,\bar{x}+\rho\omega)) \ge (\alpha'(t) - r)(\phi(t,\bar{x}+r\omega) - u^2(t,\bar{x}+r\omega))$$

for all $\rho \in [0,r]$. Similarly we have, for all $\rho \in [r,\alpha(t)]$,

$$\alpha'(t)(\phi(t,\bar{x}+\rho\omega) - u^2(t,\bar{x}+\rho\omega)) \ge \alpha'(t)(\phi(t,\bar{x}+r\omega) - u^2(t,\bar{x}+r\omega)).$$

From these two inequalities we obtain that

$$\begin{aligned}&\langle u_t^2 - f^1, \phi - u^2\rangle_{L^2(\Omega)}\\&\quad \ge \int_{S^{n-1}} d\mathscr{H}^{n-1}(\omega)(\phi(t,\bar{x}+r\omega) - u^2(t,\bar{x}+r\omega))\int_0^{\alpha(t)} \rho^{n-1}(\alpha'(t) - r\mathbf{1}_{[0,r]})d\rho = 0\end{aligned}$$

since, by the definition of α, $\int_0^{\alpha(t)} \rho^{n-1}(\alpha'(t) - r\mathbf{1}_{[0,r]}(\rho))d\rho = 0$. This shows that u^2 is a solution and therefore $u^1 = u^2$ on $[0,\bar{t}]$.

By assumption (13), we have that $f^1 \le f$. Therefore, since $u^1(0,\cdot) = 0 \le u_0$, a comparison argument shows that $u^1(t,\cdot) \le u(t,\cdot)$ for any $t \in [0,\bar{t}]$. Thus, since $\bar{t} \le T$,

$$u(T,x) \ge u(\bar{t},x) \ge u^1(\bar{t},x) = (d(\bar{x}) - |x-\bar{x}|)_+ \quad \forall \bar{x} \in \Sigma. \tag{15}$$

This implies that $u(T,x) \ge u_f(x)$, where $u_f(x)$ is defined in (11). So, in view of Proposition 2.2(ii) and assumption (13), to obtain the conclusion it suffices to note that $u_f(x)$ coincides with d. □

4. Analysis of the Stationary Problem

In this section we analyze the system of partial differential equations

$$\begin{cases} -\mathrm{div}(vDu) = f & \text{in } \Omega\\ v \ge 0, \quad |Du| \le 1 & \text{in } \Omega\\ |Du| - 1 = 0 & \text{in } \{v > 0\},\end{cases} \tag{16}$$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

complemented with the conditions

$$\begin{cases} u \geq 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Such a system describes the stationary states of problem (1). The solution of system (16)–(17) is intended in the following sense:

Definition 4.1. A pair of functions $(u, v) \in W_0^{1,\infty}(\Omega) \times L^1(\Omega)$ is a solution of (16)–(17) if

1. $u, v \geq 0$ and $|Du(x)| \leq 1$ almost everywhere in Ω;
2. for every test function $\phi \in \mathscr{C}_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)dx; \tag{18}$$

3. $\int_\Omega v(x)(|Du(x)|^2 - 1)dx = 0$.

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

4.1. *Existence*

In this subsection we prove that the pair (d, v_f), where d is the distance function from $\partial\Omega$ and

$$v_f(x) = \int_0^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt \quad \text{almost every } x \in \Omega \tag{19}$$

is a solution of system (16)–(17). In spite of the terms of the form $(1 - d(x)\kappa_i(x))^{-1}$, the product appearing inside the integral is a uniformly bounded function; in fact, it is easy to check (see [6, Proposition 3.2]) that

$$0 < \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} \leq 1 + \|[\kappa_i]_-\|_\infty \|\tau\|_\infty, \quad 0 < t < \tau(x). \tag{20}$$

However, when $f \in L^1(\Omega)$ it is not obvious, at first sight, that the integral in (19) is finite for a.e. x; thus, our first step will be to show that v_f is a well-defined function in $L^1(\Omega)$.

Given $y \in \partial\Omega$, we denote by $\nu(y)$ the interior normal to Ω at y. Then $Dd(y + t\nu(y)) = \nu(y)$ for all $t \in [0, \tau(y))$. Let $\mathscr{O}$ be the subset of $\partial\Omega \times \mathbb{R}_+$ defined by

$$\mathscr{O} = \{(y, t) \in \partial\Omega \times \mathbb{R}_+ \mid 0 < t < \tau(x)\}.$$

Then the mapping $X : \mathscr{O} \to \Omega \backslash \overline{\Sigma}$ defined by

$$\forall (y, t) \in \mathscr{O}, \quad X(y, t) = y + t\nu(y)$$

is one-to-one and $\mathscr{C}^1$ on its domain. Moreover, the volume element changes according to

$$dx = \prod_{i=1}^{n-1}(1 - t\kappa_i(y))dt\, d\mathscr{H}^{n-1}(y).$$

Since $|\overline{\Sigma}| = 0$, we deduce the following formula, valid for any $h \in L^1(\Omega)$,

$$\int_\Omega h(x)dx = \int_{\partial\Omega} \int_0^{\tau(y)} h(y + sDd(y)) \prod_{i=1}^{n-1} (1 - s\kappa_i(y))ds\, d\mathscr{H}^{n-1}(y). \tag{21}$$

Lemma 4.2. *For $\mathscr{H}^{n-1}$-a.e. $y \in \partial\Omega$ the function $s \to f(y + sv(y)) \prod_{i=1}^{n-1}(1 - s\kappa_i(y))$ is in $L^1([0, \tau(y)])$.*

Proof. It is an immediate consequence of formula (21) above. □

In particular, we deduce that for a.e. $y \in \partial\Omega$ one of the two following properties holds: either (i) the map $t \to f(y + tv(y))$ is in $L^1([0, \tau(y)])$, or (ii) $\kappa_i(y)\tau(y) = 1$ for some i, i.e., the normal ray starting at y ends at a focal point. Simple examples show that the set of the points $y \in \partial\Omega$ which satisfy (ii) but not (i) can have positive $\mathscr{H}^{n-1}$-measure (see Example 4.7 later).

Lemma 4.3. *Let $g : \Omega \to \mathbb{R}$ be such that the map $x \to d(x)g(x)$ belongs to $L^1(\Omega)$. Then*

$$\int_\Omega \int_0^{\tau(x)} g(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt\, dx = \int_\Omega d(x)g(x)dx.$$

Proof. It suffices to consider the case where $g \in L^\infty(\Omega)$, since the general case follows by approximation. Let us consider the function

$$h(x) = \int_0^{\tau(x)} g(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt,$$

which is in $L^\infty(\Omega)$ since we are assuming that $g \in L^\infty(\Omega)$. We first observe that, given any $y \in \partial\Omega$ and $s \in [0, \tau(y))$, we have

$$h(y + sv(y)) = \int_0^{\tau(y)-s} g(y + (t + s)v(y)) \prod_{i=1}^{n-1} \frac{1 - (s + t)\kappa_i(y)}{1 - s\kappa_i(y)} dt$$

because $d(y + sv(y)) = s$, $\tau(y + sv(y)) = \tau(y) - s$, $Dd(y + sv(y)) = v(y)$ and $\kappa_i(y + sv(y)) = \kappa_i(y)$, for $s \in [0, \tau(y))$. Thus (21) implies

$$\begin{aligned}
\int_\Omega h(x)dx &= \int_{\partial\Omega} \int_0^{\tau(y)} h(y + sv(y)) \prod_{i=1}^{n-1} (1 - s\kappa_i(y))ds\, d\mathscr{H}^{n-1}(y) \\
&= \int_{\partial\Omega} \int_0^{\tau(y)} \int_0^{\tau(y)-s} g(y + (t + s)v(y)) \prod_{i=1}^{n-1} (1 - (s + t)\kappa_i(y))dt\, ds\, d\mathscr{H}^{n-1}(y) \\
&= \int_{\partial\Omega} \int_0^{\tau(y)} \int_s^{\tau(y)} g(y + tv(y)) \prod_{i=1}^{n-1} (1 - t\kappa_i(y))dt\, ds\, d\mathscr{H}^{n-1}(y) \\
&= \int_{\partial\Omega} \int_0^{\tau(y)} \int_0^{t} g(y + tv(y)) \prod_{i=1}^{n-1} (1 - t\kappa_i(y))ds\, dt\, d\mathscr{H}^{n-1}(y)
\end{aligned}$$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

$$= \int_{\partial\Omega} \int_0^{\tau(y)} tg(y + tv(y)) \prod_{i=1}^{n-1} (1 - t\kappa_i(y)) dt \, d\mathscr{H}^{n-1}(y)$$

$$= \int_\Omega d(x) g(x) dx$$

where we have again used (21) in the last equality. This proves our lemma. □

Corollary 4.4. *The function*

$$v_f(x) := \int_0^{\tau(x)} f(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt$$

is well-defined for almost every $x \in \Omega$ and is in $L^1(\Omega)$.

Now we can prove that the pair (d, v_f) is a solution of our system.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathscr{C}^2$ and f be in $L^1(\Omega)$ and nonnegative. Then, the pair (d, v_f) defined above satisfies* (16)–(17) *in the sense of Definition* 4.1.

Proof. Let $\{f_k\}$ be a sequence of continuous functions such that $f_k \to f$ in $L^1(\Omega)$ as $k \to \infty$ and set

$$v_{f_k}(x) = \begin{cases} \int_0^{\tau(x)} f_k(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt & \forall x \in \Omega \backslash \overline{\Sigma} \\ 0 & \forall x \in \overline{\Sigma}. \end{cases}$$

By [6, Theorem 3.1] the pair (d, v_{f_k}) satisfies, for every test function $\phi \in \mathscr{C}_c^\infty(\Omega)$,

$$\int_\Omega v_{f_k}(x) \langle Dd(x), D\phi(x) \rangle dx = \int_\Omega f_k(x) \phi(x) dx. \tag{22}$$

In addition, we have, setting $g_k = |f - f_k|$ and applying Lemma 4.3,

$$\|v_f - v_{f_k}\|_1 \le \int_\Omega \int_0^{\tau(x)} g_k(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x) + t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dx$$

$$= \int_\Omega d(x) g_k(x) dx \le \operatorname{diam}(\Omega) \|f - f_k\|_1.$$

This shows that $v_{f_k} \to v_f$ in $L^1(\Omega)$. Passing to the limit in (22), we obtain that v_f satisfies point 2 of Definition 4.1. Points 1 and 3 follow immediately from well known properties of the distance function. □

Proposition 4.6. *For $\mathscr{H}^{n-1}$-a.e. $y \in \partial\Omega$ we have*

$$\lim_{t \uparrow \tau(y)} v_f(y + tDd(y)) \prod_{i=1}^{n-1} (1 - t\kappa_i(y)) = 0.$$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

Proof. We have

$$v_f(y + tDd(y)) \prod_{i=1}^{n-1}(1 - t\kappa_i(y)) = \int_t^{\tau(y)} f(y + s\nu(y)) \prod_{i=1}^{n-1}(1 - s\kappa_i(y))ds,$$

and we know from Lemma 4.2 that the function inside the integral is in $L^1([0, \tau(y)])$ for a.e. $y \in \partial\Omega$. Therefore the integral tends to zero as the interval of integration shrinks to a point. □

If $f \in L^\infty$ then it is easy to see, directly from the definition, that $v_f(y + tDd(y)) \to 0$ if $t \to \tau(y)$ for a.e. $y \in \partial\Omega$. If f is unbounded, this is no longer true in general, as the following example shows.

Example 4.7. Let $\Omega = B_1 \subset \mathbb{R}^2$ and let $f(x) = 1/|x|$. Then it is easily checked that $d(x) = 1 - |x|$, $\Sigma = \{0\}$, $k(x) \equiv 1$, $\tau(x) = |x|$ and $v_f(x) \equiv 1$.

We conclude by proving a property of the function u_f defined in (11). By construction, $u_f \equiv d$ on $\mathrm{spt}(f)$; the next result shows that the same holds on $\mathrm{spt}(v_f)$, which is in general a larger set. The set where d and u_f coincide is important in the analysis of the uniqueness of the stationary system, as we shall see in the next subsection.

Lemma 4.8. *We have $d(x) = u_f(x)$ for every $x \in \mathrm{spt}(v_f)$.*

Proof. Let us first show that, for any $x \in \mathrm{spt}(v_f)$, there exists $t \in [0, \tau(x)]$ such that $x + tDd(x) \in \mathrm{spt}(f)$. To prove this, let us first consider the case where $x \notin \overline{\Sigma}$. We argue by contradiction and suppose that $x + tDd(x) \notin \mathrm{spt}(f)$ for all $t \in [0, \tau(x)]$. Then there exists a neighborhood of the segment joining x to $x + \tau(x)Dd(x)$ where $f \equiv 0$ a.e.. Using the definition of v_f and the continuity of τ and of Dd, this easily implies that $v_f \equiv 0$ a.e. in a neighborhood of x. Thus, x cannot belong to $\mathrm{spt}(v_f)$. If $x \in \overline{\Sigma}$, we can prove that $x \in \mathrm{spt}(f)$ by a similar argument.

Let us now take any $x \in \mathrm{spt}(v_f)$, and choose t as above. Using the properties that $\|Du_f\|_\infty \leq 1$ and $u_f \equiv d$ on $\mathrm{spt}(f)$ (see Proposition 2.2), we have that

$$u_f(x) \geq u_f(x + tDd(x)) - |tDd(x)| = d(x + tDd(x)) - t = d(x) + t - t = d(x),$$

where we have also used (7). On the other hand, $u_f \leq d$ everywhere, again by Proposition 2.2, and this proves the assertion. □

Remark 4.9. Simple examples show that the set where d and u_f coincide is in general even larger than $\mathrm{spt}(v_f)$. Take for instance $\Omega = B_1(0) \subset \mathbb{R}^2$ and choose f to be a nonnegative function such that $\mathrm{spt}(f) = \{(x, y) \in \Omega : y \geq 0\}$, e.g., $f(x, y) = [y]_+$. Then it is easily seen that $\mathrm{spt}(v_f) = \mathrm{spt}(f)$; on the other hand, since $\mathrm{spt}(f)$ contains the origin, which is the unique point of $\overline{\Sigma}$ in this case, we have that $d \equiv u_f$ in Ω by Proposition 2.2. A detailed study of the set where $d \equiv u_f$, in the more general setting of anisotropic geometries, can be found in [10].

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

4.2. *Uniqueness*

In this subsection we give a complete characterization of the solutions of the stationary system, which is summarized in the next statement.

Theorem 4.10. *A pair of functions* $(u, v) \in W_0^{1,\infty} \times L^1(\Omega)$ *is a solution of* (16)–(17) *in the sense of Definition* 4.1 *if and only if*

(i) $v = v_f$ *a.e. in* Ω;
(ii) $\|Du\|_\infty \leq 1$ *and* $u_f \leq u \leq d$ *in* Ω, *where* u_f *is given by* (11).

In addition, the solution of system (16)–(17) *is unique if and only if* $\overline{\Sigma} \subset \operatorname{spt} f$.

Thus, the v-component of the solution must coincide with v_f, while the u-component is unique only if $\overline{\Sigma} \subset \operatorname{spt} f$; if this does not happen, then u is uniquely determined only on the subset of Ω where $u_f = d$. We split the proof Theorem 4.10 in a sequence of intermediate results, which we state and prove separately.

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

Lemma 4.11. *Let* (u, v) *be a solution of system* (16)–(17). *Then* $u \equiv d$ *in* spt(f) *and* $v(x)Du(x) = v(x)Dd(x)$ *almost everywhere in* Ω.

Proof. It is well known that $d \geq \phi$ for all $\phi \in \mathbb{K}_0$; in particular, we have that $d \geq u$. By Definition 4.1 we have, for every $\phi \in C_c^\infty(\Omega)$,

$$\int_\Omega v(x)\langle Du(x), D\phi(x)\rangle dx = \int_\Omega f(x)\phi(x)dx.$$

By approximation, the same property holds for $\phi \in W_0^{1,\infty}(\Omega)$, including the case $\phi = u - d$. Hence,

$$\int_\Omega v(x)\langle Du(x), Du(x) - Dd(x)\rangle dx = \int_\Omega f(x)(u(x) - d(x))dx \leq 0.$$

On the other hand,

$$\begin{aligned}
&\int_\Omega v(x)\langle Du(x), Du(x) - Dd(x)\rangle dx \\
&\quad = \int_\Omega \frac{v(x)}{2}(|Du(x) - Dd(x)|^2 + |Du(x)|^2 - |Dd(x)|^2)dx \\
&\quad = \int_\Omega \frac{v(x)}{2}(|Du(x) - Dd(x)|^2)dx + \int_\Omega \frac{v(x)}{2}(|Du(x)|^2 - 1)dx \\
&\quad = \int_\Omega \frac{v(x)}{2}(|Du(x) - Dd(x)|^2)dx \geq 0,
\end{aligned}$$

where we have used property 3 of Definition 4.1. We conclude that

$$\int_\Omega \frac{v(x)}{2}(|Du(x) - Dd(x)|^2)dx = \int_\Omega f(x)(u(x) - d(x))dx = 0.$$

It follows that both integrands are zero almost everywhere. Since u, d are continuous and $d - u \geq 0$, the vanishing of the first integral implies that $vDu = vDd$

almost everywhere, while the vanishing of the second one is equivalent to $u \equiv d$ in spt(f). □

Remark 4.12. The previous lemma shows that, if (u, v) is a solution of system (16)–(17), then (d, v) is a solution of the same system. In fact, by the lemma, the pair (d, v) satisfies point 2 of Definition 4.1, while points 1 and 3 are immediate consequences of the properties of d.

Observe that, if d, v were smooth functions, then we could integrate equation (16) and apply the divergence theorem to obtain

$$\int_\Omega f(x)dx = -\int_\Omega \operatorname{div}(v(x)Dd(x))dx = \int_{\partial\Omega} v(y)\langle Dd(y), Dd(y)\rangle d\mathcal{H}^{n-1}(y)$$
$$= \int_{\partial\Omega} v(y)d\mathcal{H}^{n-1}(y),$$

since Dd coincides with the inner normal on $\partial\Omega$. The next proposition contains a weak formulation of the above equality.

Lemma 4.13. *Let us set $\Omega_\varepsilon = \{x \in \Omega : d(x) \leq \varepsilon\}$. Then,*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} v(x)dx = \int_\Omega f(x)dx.$$

Proof. For any $\varepsilon > 0$, let us set $\phi_\varepsilon(x) = \min\{1, \varepsilon^{-1}d(x)\}$. Then

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} v(x)dx = \lim_{\varepsilon\to 0^+}\int_\Omega v(x)\langle D\phi_\varepsilon(x), Dd(x)\rangle dx$$
$$= \lim_{\varepsilon\to 0^+}\int_\Omega f(x)\phi_\varepsilon(x)dx = \int_\Omega f(x)$$

as required. □

We are now ready to prove the uniqueness of the v-component of our system.

Lemma 4.14. *If* (u, v) *is a solution of system* (16)–(17), *then* $v = v_f$ *a.e..*

Proof. It is convenient to change coordinates. Let us consider a parametrization of a portion of boundary of Ω, given by $\Phi : A \to \partial\Omega$, with $A \subset \mathbb{R}^{n-1}$. Then the map $(z, t) \longrightarrow \Phi(z) + t\nu(\Phi(z))$ (where $\nu(y)$ is the inner normal) is a diffeomorphism for $(z, t) \in \widetilde{A}$, where

$$\widetilde{A} = \{(z, t) : z \in A, t \in (0, \tau(\Phi(z)))\}.$$

Given a function h defined on Ω, let us denote by $\tilde{h}$ the corresponding function on $\widetilde{A}$ defined by $\tilde{h}(z, t) = h(\Phi(z) + t\nu(\Phi(z)))$. If h is differentiable, then we have that

$$\frac{\partial\tilde{h}}{\partial t}(t, z) = \langle Dh(x), Dd(x)\rangle\big|_{x=\Phi(z)+t\nu(\Phi(z))}.$$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

In addition, the volume element changes according to $dx = \prod_{i=1}^{n-1}(1-\kappa_i(z)t)m(z)dz\,dt$, where $\kappa_i(z) = \kappa_i(\Phi(z))$ and $m(z) = J\Phi(z)$ is the jacobian of Φ defined as in [12, Section 3.2.2]. Since, by Remark 4.12, the pair (d, v) solves our system, $\tilde{v}$ satisfies

$$\int_{\widetilde{A}} \tilde{v}(z,t)\frac{\partial\psi}{\partial t}(z,t)\prod_{i=1}^{n-1}(1-\kappa_i(z)t)m(z)dz\,dt = \int_{\widetilde{A}} \tilde{f}(z,t)\psi(z,t)\prod_{i=1}^{n-1}(1-\kappa_i(z)t)m(z)dz\,dt.$$

for any $\psi \in W_0^{1,\infty}(\widetilde{A})$; indeed, any such ψ can be seen as $\psi = \tilde{\phi}$ for some $\phi \in W_0^{1,\infty}(\Omega)$. Since (d, v_f) is also a solution, the same relation is satisfied by the function $\tilde{v}_f$. Therefore, taking $w(z,t) = \tilde{v}(z,t) - \tilde{v}_f(z,t)$, we have

$$\int_{\widetilde{A}} w(z,t)\frac{\partial\psi}{\partial t}(z,t)\prod_{i=1}^{n-1}(1-t\kappa_i(z))m(z)dz\,dt = 0$$

for any $\psi \in W_0^{1,\infty}(\widetilde{A})$. From this it is easy to deduce that $w(z,t)\prod_{i=1}^{n-1}(1-t\kappa_i(z)) = \bar{w}(z)$ a.e. in $\widetilde{A}$ for a suitable function $\bar{w}$ of z only. Since the argument can be repeated on any part of $\partial\Omega$, we conclude that there exists a function $W \in L^1(\partial\Omega)$ such that

$$v(y+t\nu(y)) = v_f(y+t\nu(y)) + W(y)\prod_{i=1}^{n-1}(1-t\kappa_i(y))^{-1}, \quad y \in \partial\Omega,\ \ t \in [0,\tau(y)) \text{ a.e.}$$

We need to show that $W = 0$ a.e.. First we show it is nonnegative. In fact, we have

$$W(y) = [v(y+t\nu(y)) - v_f(y+t\nu(y))]\prod_{i=1}^{n-1}(1-t\kappa_i(y)), \quad y \in \partial\Omega,\ \ t \in [0,\tau(y)) \text{ a.e.}$$

Thus, letting $t \to \tau(y)$ and using Proposition 4.6, we obtain that

$$W(y) = \lim_{t\to\tau(y)} v(y+t\nu(y))\prod_{i=1}^{n-1}(1-t\kappa_i(y)),$$

which is nonnegative a.e. since both factors are nonnegative. Next we observe that, by Lemma 4.13,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[v(x) - v_f(x)]dx = 0.$$

On the other hand

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[v(x) - v_f(x)]dx \\
&\quad = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\partial\Omega}\int_0^\varepsilon [v(y+tDd(y)) - v_f(y+tDd(y))]\prod_{i=1}^{n-1}(1-t\kappa_i(y))dt\,d\mathcal{H}^{n-1}(y) \\
&\quad = \int_{\partial\Omega} W(y)d\mathcal{H}^{n-1}(y).
\end{aligned}$$

Since the integrand is nonnegative, we have a contradiction unless $W(y) = 0$ a.e. $\square$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

We now turn to the issue of the uniqueness of u.

Lemma 4.15. *If* (u, v) *is a solution of system* (16)–(17) *in the sense of Definition* 4.1, *then* $u_f \le u \le d$ *in* Ω, *where* u_f *is given by* (11). *In addition,* $u \equiv d$ *in* $\mathrm{spt}(v_f)$.

Proof. Suppose that the pair (u, v) is a solution. Then $v = v_f$ a.e. in Ω by Lemma 4.14. In addition, $u = d$ on $\mathrm{spt}(f)$ by Lemma 4.11. By definition, u is nonnegative, vanishes on $\partial\Omega$ and has Lipschitz constant at most one. Then $u \ge u_f$ by Proposition 2.2(i) and $u \le d$ by the maximality of d. This proves that $u_f \le u \le d$. The property that $u \equiv d$ in $\mathrm{spt}(v_f)$ then follows from Lemma 4.8. □

We can now conclude the proof of our main result.

Proof of Theorem 4.10. Suppose that the pair (u, v) is a solution. Then Lemmas 4.14 and 4.15 show that u, v satisfy properties (i)–(ii) of the statement. Conversely, suppose that u, v satisfy (i)–(ii). Then, by Lemma 4.8, we have that $u = d$ on $\mathrm{spt}(v) = \mathrm{spt}(v_f)$. Since, by Theorem 4.5, the pair (d, v_f) is a solution, we easily verify using the definition that (u, v) is also a solution.

Thus, if $u_f \equiv d$ everywhere in Ω, the solution to the system is unique. Otherwise, there are infinitely many choices for u; for example, setting $u_\lambda = \lambda u_f + (1-\lambda)d$, we have that the pair (u_λ, v_f) is a solution for any $\lambda \in [0, 1]$. Since, by Proposition 2.2(ii), the property that $u_f \equiv d$ is equivalent to $\overline{\Sigma} \subset \mathrm{spt}(f)$, this concludes our proof. □

Remark 4.16. Results related to those of Subsections 4.1 and 4.2 have been recently obtained by Crasta and Finzi Vita in [8]. The authors consider a stationary problem with an integrable source in the presence of walls on some parts of the boundary, obtaining existence of solutions as in Theorem 4.5. However, the uniqueness of v is left as an open problem in [8]. It is likely that the ideas of our paper can be applied to prove the uniqueness of v for the problem with walls as well.

4.3. *Regularity*

In this last part of our paper, we investigate the regularity properties of the mapping which associates to a function $f \in L^\infty(\Omega)$ the solution (u, v_f) of (16)–(17). Since we can always choose $u = d$, we only consider the second component $f \mapsto v_f$ of this mapping.

Proposition 4.17. *We have, for any* $p \in [1, +\infty]$,

$$\|v_{f_1} - v_{f_2}\|_p \le C_p(\Omega)\|f_1 - f_2\|_p \quad \forall f_1, f_2 \in L^\infty(\Omega), \;\; f_1, f_2 \ge 0.$$

where

$$C_p(\Omega) = \mathrm{diam}(\Omega)\,(1 + \|[\kappa]_-\|_\infty \mathrm{diam}(\Omega))^{(n-1)(1-\frac{1}{p})}$$

with $[\kappa]_- = \max_{1 \le i \le n-1} \max\{0, -\kappa_i\}$.

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

Remark 4.18.

1. If we choose $f_2 = 0$, then $v_{f_2} = 0$ and we have the following bounds on v_f:

$$\|v_f\|_p \leq C_p(\Omega)\|f\|_p \quad \forall f \in L^\infty(\Omega).$$

2. If $p = 1$ or if Ω is convex, then the constant $C_p(\Omega)$ only depends on p, n and the diameter of Ω.
3. The above estimates still hold if $p > 1$ and $\partial\Omega$ is of class $\mathscr{C}^{1,1}$.

Proof of Proposition 4.17. Let us compute $\|v_{f_1} - v_{f_2}\|_p^p$. We have

$$\begin{aligned}
\|v_{f_1} - v_{f_2}\|_p^p &= \int_\Omega \left| \int_0^{\tau(x)} (f_1 - f_2)(x + tDd(x)) \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt \right|^p dx \\
&\leq \int_\Omega (\tau(x))^{p-1} \int_0^{\tau(x)} |(f_1 - f_2)(x + tDd(x))|^p \left(\prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} \right)^p dt\, dx
\end{aligned}$$

thanks to Hölder's inequality. Taking $C = 1 + \|[\kappa]_-\|_\infty \|\tau\|_\infty$, we obtain, by (20),

$$\begin{aligned}
&\|v_{f_1} - v_{f_2}\|_p^p \\
&\quad \leq \int_\Omega (\tau(x))^{p-1} C^{(n-1)(p-1)} \int_0^{\tau(x)} |(f_1 - f_2)(x + tDd(x))|^p \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt\, dx \\
&\quad \leq C^{(n-1)(p-1)} (\|\tau\|_\infty)^{p-1} \int_\Omega \int_0^{\tau(x)} |(f_1 - f_2)(x + tDd(x))|^p \prod_{i=1}^{n-1} \frac{1 - (d(x)+t)\kappa_i(x)}{1 - d(x)\kappa_i(x)} dt\, dx.
\end{aligned}$$

Hence, by Lemma 4.3, we have

$$\begin{aligned}
\|v_{f_1} - v_{f_2}\|_p^p &\leq C^{(n-1)(p-1)} (\|\tau\|_\infty)^{p-1} \int_\Omega d(x) |(f_1 - f_2)(x)|^p dx \\
&\leq C^{(n-1)(p-1)} (\|\tau\|_\infty)^{p-1} \operatorname{diam}(\Omega) \int_\Omega |(f_1 - f_2)(x)|^p dx.
\end{aligned}$$

We can then complete the proof noting that $\tau(x) \leq \operatorname{diam}(\Omega)$ for any x. □

Let us underline that, in Proposition 4.17, $C_p(\Omega)$ strongly depends on the curvature of the set Ω. However, we can get rid of this dependence introducing a weight in the L^p norm.

Proposition 4.19. *For any* $p \in [1, +\infty]$, *we have*

$$\|d^{(n-1)(1-\frac{1}{p})}(v_{f_1} - v_{f_2})\|_p \leq C'_p(\Omega)\|f_1 - f_2\|_p \quad \forall f_1, f_2 \in L^\infty(\Omega),$$

where

$$C'_p(\Omega) = (\operatorname{diam}(\Omega))^{(np-n+1)/p}.$$

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

Proof. One can argue as in the proof of Proposition 4.17, replacing estimate (20) by the following one: for all $x \in \Omega \backslash \overline{\Sigma}$ and $t \in [0, \tau(x))$ we have

$$\frac{1-(d(x)+t)\kappa_i(x)}{1-d(x)\kappa_i(x)} \leq 1 + \frac{\tau(x)}{d(x)} \quad \forall 1 \leq i \leq n-1.$$

We omit the easy details. □

Remark 4.20.

1. In particular, the above proposition implies that the map $f \mapsto v_f$ can be defined on any bounded domain Ω and that, for any $p \in (1, +\infty]$, v_f belongs to $L^p_{loc}(\Omega)$ if f belongs to $L^p(\Omega)$.
2. In general, if Ω is not smooth, one cannot expect v_f to be in $L^p(\Omega)$ for f in $L^p(\Omega)$, unless Ω is convex or $p = 1$. For instance, in the case of $n = 2$, $\Omega = B_2(0)\backslash\{0\}$ and $f = 1$, we have that $v_f(x) = (1 - |x|^2)/(2|x|)$ in $B_1(0)\backslash\{0\}$, which is unbounded although $f \in L^\infty(\Omega)$. Notice, however that the map $x \mapsto d(x)v_f(x)$ is bounded.

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009

Acknowledgments

We thank Graziano Crasta for stimulating discussions and for bringing to our attention the behavior described in Remark 4.9. We also thank the anonymous referees for their useful comments. This work has been partially supported by the Italian PRIN 2005 and PRIN 2007 Programs "Metodi di viscosità, metrici e di teoria del controllo in equazioni alle derivate parziali nonlineari".

References

[1] Aronsson, G., Evans, L. C., Wu, Y. (1996). Fast/slow diffusion and growing sandpiles. *J. Differential Equations* 131:304–335.
[2] Barbu, V. (1993). *Analysis and Control of Nonlinear Infinite Dimensional Systems*. Mathematics in Sciences and Engineering. San Diego, CA: Academic Press Inc.
[3] Bouchaud, J. P., Cates, M. E., Ravi Prakash, J., Edwards, S. F. (1994). A model for the dynamics of sandpile surfaces. *J. Phys. I France* 4:1383–1410.
[4] Boutreux, T., de Gennes, P.-G. (1996). Surface flows of granular mixtures. I. General principles and minimal model. *J. Phys. I France* 6:1295–1304.
[5] Cannarsa, P., Cardaliaguet, P. (2004). Representation of equilibrium solutions to the table problem for growing sandpile. *J. Eur. Math. Soc.* 6:1–30.
[6] Cannarsa, P., Cardaliaguet, P., Crasta, G., Giorgieri, E. (2005). A boundary value problem for a PDE model in mass transfer theory: representation of solutions and applications. *Calc. Var.* 24:431–457.
[7] Cannarsa, P., Sinestrari, C. (2004). *Semiconcave Functions, Hamilton–Jacobi Equations, and Optimal Control*. Boston: Birkhäuser.
[8] Crasta, G., Finzi Vita, S. (2008). An existence result for the sandpile problem on at tables with walls. *Netw. Heterog. Media* 3:815–830.
[9] Crasta, G., Malusa, A. (2007). The distance function from the boundary in a Minkowski space. *Trans. Amer. Math. Soc.* 359:5725–5759.
[10] Crasta, G., Malusa, A. (2007). A sharp uniqueness result for a class of variational problems solved by a distance function. *J. Differential Equations* 243:427–447.
[11] De Pascale, L., Evans, L. C., Pratelli, A. (2004). Integral estimates for transport densities. *Bull. London Math. Soc.* 36:383–395.

[12] Evans, L. C., Gariepy, R. F. (1992). *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. Boca Raton, FL: CRC Press.
[13] Evans, L. C., Feldman, M., Gariepy, R. F. (1997). Fast/slow diffusion and collapsing sandpiles. *J. Differential Equations* 137:166–209.
[14] Feldman, M. (1999). Variational evolution problems and nonlocal geometric motion. *Arch. Ration. Mech. Anal*. 146:221–274.
[15] Gilbarg, D., Trudinger, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*. Grundlehren der Mathematischen Wissenschaften 224. Berlin: Springer-Verlag.
[16] Hadeler, K. P., Kuttler, C. (1999). Dynamical models for granular matter. *Granular Matter* 2:9–18.
[17] Prigozhin, L. (1996). Variational model of sandpile growth. *Euro. J. Appl. Math*. 7:225–235.

Downloaded By: [Sinestrari, Carlo] At: 19:32 28 December 2009