

Classical Invariant Theory for the Quantum Symplectic Group

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INTRODUCTION

In this paper we establish the quantum analogues of the first and second fundamental theorem for vector invariants for the symplectic group. Let us briefly recall this result. Consider a vector space V endowed with a non degenerate antisymmetric bilinear form \langle, \rangle . Let $\mathrm{Sp}(V)$ denote the group of isometries. The first and second fundamental theorems for vector invariants for $\mathrm{Sp}(V)$ tells us that any invariant polynomial function on

$$\underbrace{V \oplus V \oplus \cdots \oplus V}_{n\text{-times}}$$

is a polynomial in the functions $a_{i,j}$, $1 \leq i < j \leq n$, whose value on a n -tuple of vectors v_1, v_2, \dots, v_n is given by $\langle v_i, v_j \rangle$, while the ideal of relations is described as follows: if $n \leq \dim V$ the $a_{i,j}$ are algebraically independent. If $n > \dim V$, then it is generated by the order $\dim V + 1$ Pfaffians of the $n \times n$ antisymmetric matrix whose i, j -th entry equals $a_{i,j}$. Our scope in this paper is to give a q -analogue of this theorem. For this purpose we introduce in Section 1 a q -analogue of the ring of functions on a generic antisymmetric matrix, give various bases for this ring and study various quotients of it. Here the basic idea is to introduce a q -analogue of the notion of a Pfaffian.

In Section 2 we connect this idea with the invariant theory for the direct sum of m copies of the standard representation of the quantum enveloping algebra for type C_n . We define a q -analogue of the ring of functions on m copies of the standard representation of the quantum enveloping algebra for type C_n and prove our main theorem using the results of Section 1.

1. QUANTUM ANTISYMMETRIC MATRICES

Let k be a field of characteristic zero. Consider the field $K = k(q)$ of rational functions in the variable q . The algebra of q -polynomial functions

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on a *generic antisymmetric* $n \times n$ q -matrix A is the free K -algebra generated by variables $a_{i,j}$ for $1 \leq i < j \leq n$ modulo the ideal generated by the relations

$$\begin{aligned}
 a_{i,j}a_{i,h} &= qa_{i,h}a_{i,j} & \text{for } j < h \\
 a_{i,j}a_{i,t} &= qa_{i,t}a_{i,j} \\
 a_{i,j}a_{s,j} &= qa_{s,j}a_{i,j} & \text{for } i < s \\
 a_{i,j}a_{i+s,j-r} &= a_{i+s,j-r}a_{i,j} & \text{for } r, s > 0 \\
 a_{i,j}a_{i+s,j+r} &= a_{i+s,j+r}a_{i,j} + (q - q^{-1})a_{i+s,j}a_{i,j+r} \\
 & \text{for } r, s > 0 & \text{if } j > i + s \\
 a_{i,j}a_{i+s,j+r} &= a_{i+s,j+r}a_{i,j} - q^{-1}q_{i,i+s}a_{j,j+r} + qa_{j,j+r}a_{i,i+s} \\
 & \text{for } r, s > 0 & \text{if } j < i + s.
 \end{aligned} \tag{1.1}$$

We start by proving that this is actually a q -analogue of a polynomial ring. For this let us order lexicographically the set of pairs (i, j) with $i < j$.

PROPOSITION 1.1. *The monomials*

$$a_{i_1, j_1} a_{i_2, j_2} \cdots a_{i_m, j_m}$$

with $(i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_m, j_m)$, are a basis for A .

Proof. The fact that these monomials span A is clear from the defining relations. To show their linear independence let us define a representation of A . For this let us consider the polynomial ring $K[y_{i,j}]$ with $1 \leq i < j \leq n$ and define a A -module structure on $K[y_{i,j}]$ by defining the action of $a_{i,j}$ as follows. Take the basis of $K[y_{i,j}]$ formed by the monomials

$$M = y_{i_1, j_1} y_{i_2, j_2} \cdots y_{i_m, j_m}$$

$(i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_m, j_m)$. Set

$$a_{i,j} \circ 1 = y_{i,j}.$$

Suppose now that the action of the $a_{i,j}$'s has been defined on monomials of degree smaller than m . Set

$$a_{i,j} \circ M = y_{i,j} M$$

if $(i, j) \leq (i_1, j_1)$. Notice that this completely defines the action of $a_{1,1}$, so that we can suppose that by induction the action of $a_{i',j'}$ for $(i', j') < (i, j)$ has been defined. Suppose now that $(i, j) > (i_1, j_1)$. Then using the defining relations (1.1), we can write $a_{i,j}a_{i_1, j_1}$ as a linear combination of monomials $a_{s,t}a_{h,k}$ with $(s, t) < (i, j)$. Using this, the fact that $M = a_{i_1, j_1} \circ y_{i_2, j_2} \cdots y_{i_m, j_m}$

and the inductive hypotheses we immediately deduce that in order to obtain an action of A , the action of $a_{i,j}$ can be defined in exactly one way. It is then easy to see that this indeed defines an action of A on $K[y_{i,j}]$.

The linear independence of the monomials M then implies the claim. ■

Consider now the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ (see [2, 3]), with generators E_1, \dots, E_{n-1} , F_1, \dots, F_{n-1} , and $L_1^{\pm 1}, \dots, L_n^{\pm 1}$. We claim that we have a natural action of $U_q(\mathfrak{gl}_n)$ on A . To define it, we first define it on the vector space with basis the elements $a_{i,j}$ by setting

$$\begin{aligned}
 E_s \circ a_{i,j} &= 0 & \text{if } i, j \neq s+1 & \quad \text{or } i=j-1=s \\
 E_s \circ a_{s+1,j} &= a_{s,j} \\
 E_s \circ a_{j,s+1} &= a_{j,s} \\
 F_s \circ a_{i,j} &= 0 & \text{if } i, j \neq s & \quad \text{or } i=j-1=s \\
 F_s \circ a_{s,j} &= a_{s+1,j} \\
 F_s \circ a_{j,s} &= a_{j,s+1} \\
 L_s \circ a_{i,j} &= a_{i,j} & \text{if } i, j \neq s \\
 L_s \circ a_{s,j} &= qa_{s,j} \\
 L_s \circ a_{j,s} &= qa_{j,s}.
 \end{aligned} \tag{1.2}$$

It is easy to see that defines the irreducible representation corresponding to the fundamental weight ω_2 . The fact that $U_q(\mathfrak{gl}_n)$ is a Hopf algebra, immediately implies that we get a representation on the free algebra generated by the $a_{i,j}$'s. We have

PROPOSITION 1.2. *The action of $U_q(\mathfrak{gl}_n)$ defined on the vector space with basis the elements $a_{i,j}$, induces an action of $U_q(\mathfrak{gl}_n)$ on A . Furthermore, all the defining relations for A can be deduced from the relation*

$$a_{1,2}a_{1,3} = qa_{1,3}a_{1,2} \tag{1.3}$$

and the fact that $U_q(\mathfrak{gl}_n)$ acts on A .

Proof. We shall prove the second part. The proof of the first follows along the same lines. Let us proceed by induction on the lexicographic ordering of the fourtuple (i, j, h, k) to show that the compatibility with the action of $U_q(\mathfrak{gl}_n)$ can be used to deduce the corresponding relation for $a_{i,j}a_{h,k}$. We can clearly assume that $i \leq h$ and, if $i=h$, $j < k$. We have various cases.

If $i=h=1$ and $j=2$, consider the relation $a_{1,2}a_{1,k-1} = qa_{1,k-1}a_{1,2}$ which we suppose to hold by induction and apply F_{k-1} . We deduce that $a_{1,2}a_{1,k} = qa_{1,k}a_{1,2}$ as desired.

The treatment of all cases in which $i = h$ is completely analogous and we leave it to the reader.

Assume now that $i < h$. If $j = k$, then if $i > 1$, consider the relation $a_{i-1,j}a_{h,j} = qa_{h,j}a_{i-1,j}$. We deduce, applying F_i , the desired relation $a_{i,j}a_{h,j} = qa_{h,j}a_{i,j}$. By similar reasoning we reduce to the case $i = 1, h = 2$. In this case if $j > 3$ start with the relation $a_{1,j}a_{2,j-1} = a_{2,j-1}a_{1,j}$ and apply F_j to deduce the desired relation. We remain with the case $j = 3$. In this case we start with the relation $a_{1,2}a_{2,3} = qa_{2,3}a_{1,2}$ and apply F_2 , getting the desired relation.

Assume now $i < h$ and $j > k$. In this case if $i > 1$ we start from the relation $a_{i-1,j}a_{h,k} = a_{h,k}a_{i-1,j}$ and apply F_i to get the relation $a_{i,j}a_{h,k} = a_{h,k}a_{i,j}$, so we can assume $i = 1$. Proceeding in a completely analogous way, we reduce to the case in which $h = 2, k = 3, j = 4$. We start with the relation $a_{1,3}a_{2,3} = qa_{2,3}a_{1,3}$ and apply F_3 , we get $q^{-1}a_{1,4}a_{2,3} + a_{1,3}a_{2,4} = a_{2,4}a_{1,3} + qa_{2,3}a_{1,4}$. On the other hand, if we start with $a_{1,3}a_{1,4} = qa_{1,4}a_{1,3}$ and apply F_1 , we get $q^{-1}a_{2,3}a_{1,4} + a_{1,3}a_{2,4} = a_{2,4}a_{1,3} + qa_{1,4}a_{2,3}$. Subtracting and dividing by $q + q^{-1}$, we get the desired relation.

The remaining cases are completely analogous and we leave them to the reader. ■

In view of the above proposition we propose now to study the representation of $U_q(\mathfrak{gl}_n)$ on our algebra A and find its irreducible components. Notice that A is graded and that if we consider its homogeneous component of degree m , A_m , a basis for it is given by the monomials

$$a_{i_1,j_1}a_{i_2,j_2} \cdots a_{i_m,j_m} \quad (1.4)$$

with $(i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_m, j_m)$. Thus by the definition of the action of $U_q(\mathfrak{gl}_n)$ we deduce that the character of A_n equals the character of $S^n(\Lambda^2 k^n)$ considered as a representation of $\mathrm{Gl}(n, k)$. It follows from the well known decomposition of this representation, that each irreducible component in it has multiplicity one and the irreducible components appearing in it are exactly those with Young diagram having rows of even length and containing $2m$ boxes. Since one knows, [4, 5], that the irreducible representations of $U_q(\mathfrak{gl}_n)$ for which the spectrum of the L_s consists of powers of q have the same indexing and characters as in the classical case of $\mathrm{Gl}(n)$, we deduce

PROPOSITION 1.3. *If we consider A as a $U_q(\mathfrak{gl}_n)$ -module, then its homogeneous component of degree m , A_m decomposes as the direct sum*

$$A_m = \bigoplus_Y V_Y, \quad (1.5)$$

where Y runs through the Young diagrams with rows of even length such that $|Y| = 2m$, and V_Y denotes the irreducible $U_q(\mathfrak{gl}_n)$ -module corresponding to Y . ■

We want to exhibit now a highest weight vector $P_Y \in A_m$, for any Y with V_Y appearing in the irreducible decomposition of A_m . In order to do so, let us introduce certain elements of A . Given an ordered sequence $1 \leq i_1 < i_2 < \dots < i_{2h} \leq m$, we define the corresponding q -Pfaffian $[i_1, i_2, \dots, i_{2h}]$ inductively as follows. If $h = 1$, $[i_1, i_2] = a_{i_1, i_2}$. If $h > 1$

$$[i_1, i_2, \dots, i_{2h}] = \sum_{r=2}^{2h} (-q)^{r-2} a_{i_1, i_r} [i_2, \dots, i_r^\vee, \dots, i_{2h}]. \quad (1.6)$$

LEMMA 1.4. Consider the element $[i_1, i_2, \dots, i_{2h}] \in A$ for a given sequence $I = \{i_1 < i_2 < \dots < i_{2h}\}$. Then,

$$E_s[i_1, i_2, \dots, i_{2h}] = 0 \quad \text{if either } s+1 \notin I \quad \text{or} \quad \{s, s+1\} \subseteq I$$

If $s = i_t - 1$ and $s \notin I$, $E_s[i_1, i_2, \dots, i_{2h}] = [i_1, i_2, \dots, i_{t-1}, s, i_{t+1}, \dots, i_{2h}]$,

$$F_s[i_1, i_2, \dots, i_{2h}] = 0 \quad \text{if either } s \notin I \quad \text{or} \quad \{s, s+1\} \subseteq I$$

If $s = i_t$ and $s+1 \notin I$, $F_s[i_1, i_2, \dots, i_{2h}] = [i_1, i_2, \dots, i_{t-1}, s+1, i_{t+1}, \dots, i_{2h}]$.

Proof. We prove our statement for the E_i 's, the case of the F_i 's being completely analogous.

If $h = 1$ the statement of the lemma is just part of (1.2), so we proceed by induction on h . The fact that $E_s[i_1, i_2, \dots, i_{2h}] = 0$ if $s+1 \notin I$ is obvious, and so is the fact that $E_s[i_1, i_2, \dots, i_{2h}] = [i_1, i_2, \dots, i_{t-1}, s, i_{t+1}, \dots, i_{2h}]$ if $s = i_t - 1$, and $s \notin I$.

Assume now that $i_{t-1} = s$ and $i_t = s+1$. For the moment assume also $t > 2$. We get

$$\begin{aligned} E_s[i_1, i_2, \dots, i_{2h}] &= (-q)^{t-2} a_{i_1, s} [i_2, \dots, i_t^\vee, \dots, i_{2h}] \\ &\quad + (-q)^{t-3} q a_{i_1, s} [i_2, \dots, i_t^\vee, \dots, i_{2h}] = 0. \end{aligned}$$

On the other hand, if $i_1 = s$ and $i_2 = s+1$ then

$$\begin{aligned} [s, s+1, i_3, \dots, i_{2h}] &= a_{s, s+1} [i_3, \dots, i_{2h}] \\ &\quad + \sum_{3 \leq r < t \leq 2h} ((-q)^{r+t-5} a_{s, i_r} a_{s+1, i_t} + (-q)^{r+t-4} a_{s, i_t} a_{s+1, i_r}) \\ &\quad \times [i_3, \dots, i_r^\vee, \dots, i_t^\vee, \dots, i_{2h}]. \end{aligned}$$

From this formula, if we apply E_s and use (1.1) and (1.2) we immediately deduce that $E_s[s, s+1, i_3, \dots, i_{2h}] = 0$ as desired. ■

Let us now fix a Young diagram $Y = (2h_1 \geq 2h_2 \geq \dots \geq 2h_t)$ with $n \geq 2h_1$. Given a Young tableau of shape Y

$$T = \begin{array}{ccccccc} i_{11} & i_{12} & \cdots & \cdots & \cdots & & i_{1h_1} \\ i_{21} & i_{22} & \cdots & \cdots & & & i_{2h_2} \\ \vdots & \vdots & & & & & \\ & & & & & & \\ i_{t1} & i_{t2} & \cdots & & & & i_{th_t} \end{array}$$

with $i_{hk} < i_{hk+1}$ for all h, k and $1 \leq i_{hk} \leq n$ we set, by abuse of notation, $T = \prod_{r=1}^t [i_{r1}, i_{r2}, \dots, i_{rh_r}] \in A$ and say that T is standard if $i_{hk} \leq i_{h+1k}$ for all h, k (1 will be taken as the empty standard tableau). We shall call the tableau $K_Y = \prod_{r=1}^t [1, 2, \dots, h_r]$ the canonical tableau of shape Y . We have

THEOREM 1.5. (1) *The set of standard tableaux is a basis of A as a $k(q)$ vector space.*

(2) *The set of canonical tableaux of shape Y is a complete set of highest weight vectors for the action of $U_q(\mathfrak{gl}_n)$ on A (by this we mean that the K_Y, s are linearly independent and any highest weight vector in A is a multiple of K_Y for a suitable Y).*

Proof. To see the first part, we work on the algebra over $k[q, q^{-1}]$ \bar{A} , generated by the $a_{i,j}$ with relations (1.1) and remark a few facts. The natural map $\bar{A} \rightarrow A$ is an injection since clearly \bar{A} is spanned as a $k[q, q^{-1}]$ -module by the monomials (1.2) and these are linearly independent in A by Proposition 1.1. In particular q_1 is not a zero divisor in \bar{A} and the monomials (1.2) are a basis of \bar{A} over $k[q, q^{-1}]$. Second, our standard tableaux T lie in \bar{A} . Having made these remarks let us prove the linear independence of the standard tableaux. Suppose $\sum_{i=1}^k b_i T_i$ is a linear relation with $b_i \in k(q)$ and T_i standard. Removing the denominators we can assume that the b_i 's are in $k[q, q^{-1}]$ and are not all divisible by $q-1$. Now reduce mod $q-1$. By the above remarks, we have that $\bar{A}/(q-1)$ is the ring of polynomials with coefficients in k in the variables $a_{i,j}$, and one knows, [1], that in this ring the standard tableaux are linearly independent. We deduce that for all $i=1, \dots, k$, $b_i \equiv 0 \pmod{q-1}$ getting a contradiction. It remains to prove that the standard tableaux span A . This follows immediately from the fact that, for all m , A_m has the same dimension as the space of homogeneous forms in the $a_{i,j}$'s of degree m and that, on the other hand, the standard tableaux of degree m are a basis for this space.

The second part is a immediate consequence of the first and of Lemma 1.4. ■

We want now to consider some quotient algebras of A . Notice that, if we fix an even number $h \leq n$, the subspace of A with basis the q -Pfaffians $[i_1, \dots, i_h]$ is stable under the action of $U_q(\mathfrak{gl}_n)$ as follows immediately from Lemma 1.4. We deduce that, if we consider the two sided ideal $I_h \subset A$ generated by those elements, I_h is stable under $U_q(\mathfrak{gl}_n)$. We want to give a basis for I_h and A/I_h and describe their decomposition into irreducible modules.

THEOREM 1.6. (1) I_h has a basis consisting of the standard tableaux whose shape has first row of length at least h .

(2) A/I_h has a basis consisting of the standard tableaux whose shape has first row of length at most $h-1$.

(3) As a $U_q(\mathfrak{gl}_n)$ -module, I_h is the direct sum of the irreducible modules V_Y , with $Y = (2h_1 \geq 2h_2 \geq \dots \geq h_r)$ and $n \geq 2h_1 \geq h$.

(4) As a $U_q(\mathfrak{gl}_n)$ -module, A/I_h is the direct sum of the irreducible modules V_Y , with $Y = (2h_1 \geq 2h_2 \geq \dots \geq h_r)$ and $2h_1 < h$.

Proof. All the statements are immediate consequence, using Theorem 1.5, of the first. So, let us prove (1).

Set \mathcal{T}_h equal to the set of Young diagrams for $\mathrm{Gl}(n)$ whose first row has length at least h . Set J_h equal to the span in A of the standard tableaux whose shape lies in \mathcal{T}_h . It follows from the definitions that $J_h \subset I_h$. Also both I_h and J_h are graded, and we have that the dimension of the degree m component of J_h equals $\sum_{Y \in \mathcal{T}_h, |Y|=2m} \dim V_Y$. Thus in order to obtain our statement, it suffices to see that, if V_Y is contained in I_h , then $Y \in \mathcal{T}_h$. To see this, let us recall that by Pieri formula, if we take any Young diagram Y and consider $V_{\square\square} \otimes V_Y \cong V_Y \otimes V_{\square\square}$, then in its decomposition into irreducibles there appear only modules $V_{Y'}$ with $Y' \supset Y$. This and an easy induction, immediately imply our claim. ■

2. INVARIANT THEORY FOR THE QUANTUM SYMPLECTIC GROUP

Consider the free algebra $k(q) \langle x_{i,j} \rangle$ with $i = 1, \dots, 2n, j = 1, \dots, m$. Let J be the ideal generated by the relations

$$x_{h,i} x_{h,j} = q x_{h,j} x_{h,i} \quad \text{for } 1 \leq i < j \leq m$$

$$x_{h,i} x_{k,j} = x_{k,j} x_{h,i} + (q - q^{-1}) x_{h,j} x_{k,i}$$

$$\text{for } 1 \leq i < j \leq m, 1 \leq h < k \leq 2n, h+k \neq 2n$$

$$x_{h,j}x_{k,i} = x_{k,i}x_{h,j}$$

$$\text{for } 1 \leq i < j \leq m, 1 \leq h < k \leq 2n, h+k \neq 2n+1$$

$$x_{h,i}x_{k,i} = qx_{k,i}x_{h,i} \quad \text{for } 1 \leq h < k \leq 2n, h+k \neq 2n+1, \quad (2.1)$$

$$x_{h,i}x_{2n+1-h,i} = q^2x_{2n+1-h,i}x_{h,i} + (q - q^{-1}) \sum_{s=1}^{h-1} q^{s+1}x_{2n+1-h+s,i}x_{h-s,i} \quad (2.2)$$

and

$$x_{1,h}x_{2n,k} - q^2x_{2n,h}x_{1,k} = qx_{2n,k}x_{1,h} - q^{-1}x_{1,k}x_{2n,h}$$

$$\text{for } h < k$$

$$q^{-2}x_{n+1,h}x_{n,k} + x_{n,h}x_{n+1,k} = qx_{n,k}x_{n+1,h} + q^{-1}x_{n+1,k}x_{n,h}$$

$$\text{for } h < k$$

$$x_{2n-s+1,h}x_{s,k} - q^{-1}x_{2n-s+2,h}x_{s-1,k} =$$

$$q^{-1}x_{s,k}x_{2n-s+1,h} - x_{s-1,k}x_{2n-s+2,h} \quad \text{for } h < k, n \geq s \geq 2 \quad (2.3)$$

$$q^{-1}x_{s,h}x_{2n-s+1,k} - x_{s-1,h}x_{2n-s+2,k} =$$

$$x_{2n-s+1,k}x_{s,h} - q^{-1}x_{2n-s+2,k}x_{s-1,h}$$

$$+ (q - q^{-1})(q^{-1}x_{s,k}x_{2n-s+1,h} - x_{s-1,k}x_{2n-s+2,h})$$

$$\text{for } h < k, n \geq s \geq 2.$$

We set $B = k(q) \langle x_{i,j} \rangle / J$. Notice that since J is a homogeneous ideal, the algebra B is naturally graded.

Given sequences $I = \{i_1, \dots, i_t\}$ with $1 \leq i_s \leq 2n$ and $J = \{j_1, \dots, j_t\}$ with $1 \leq j_s \leq m$, we can consider the monomial

$$M_I^J = \prod_{h=1}^t x_{i_h, j_h}.$$

We have

PROPOSITION 2.1. *The monomials M_I^J with $(j_1, i_1) \geq (j_2, i_2) \cdots \geq (j_t, i_t)$ in the lexicographic ordering are a $k(q)$ basis of the algebra B .*

Proof. We need first to show that these monomials linearly span B . By an easy induction, it suffices to see that any degree two monomial $x_{i,h}x_{j,k}$ with $(k, j) < (h, i)$, can be expressed as a linear combination of degree two monomials $x_{i_r, h_r}x_{j_r, k_r}$ with $(h_r, i_r) \geq (k_r, j_r)$ and $(h_r, i_r) > (h, i)$. By (2.1) and (2.2) this is clear if $j \neq 2n - i + 1$ or if $h = k$. We need to consider the

monomials $x_{i,h}x_{2n-i+1,k}$ with $h < k$ and show that they can be expressed as linear combinations of the monomials $x_{j,k}x_{2n-j+1,h}$. For these, using the relations (2.3) one is clearly reduced to show that the $2n \times 2n$ matrix.

$$\mathcal{X} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -q^2 \\ 0 & 0 & \cdots & 1 & q^{-2} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & -q^{-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 0 & 1 & -q^{-1} & \cdots & 0 \\ 1 & -q^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \\ 0 & \cdots & 1 & -q^{-1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is invertible. An easy direct computation then shows that $\det \mathcal{X} = q^2 + q^{-2n}$, giving the claim.

We now need to show linear independence. To see this, consider the polynomial ring $k(q)[z_{i,j}]$, $i = 1, \dots, 2n$; $j = 1, \dots, m$. This ring has a basis given by the monomials in the $z_{i,j}$, N_I^J with $(j_1, i_1) \geq (j_2, i_2) \cdots \geq (j_t, i_t)$ in the lexicographic ordering. We now define operators $x_{i,h}$ sending polynomials of degree t to polynomials of degree $t+1$, on this ring as follows.

We define $x_{i,h} \circ 1 = z_{i,h}$. By induction on the degree, we can now assume that we have defined the operator $x_{i,h}$ on polynomials of degree less than t . We now define it on polynomials of degree t .

First we define $x_{2n,m}$ by setting for a monomial N_I^J as above $x_{2n,m} \circ N_I^J := z_{2n,m} N_I^J$.

Suppose now we have defined the operators $x_{s,t}$ for $(t, s) > (i, h)$. Let N_I^J be as above. Then we set $x_{i,h} \circ N_I^J := z_{i,h} N_I^J$ if $(h, i) \geq (j_1, i_1)$. If on the other hand, $(j_1, i_1) > (h, i)$ we first write $x_{i,h} x_{i_1, j_1} = \sum_r = 1^k x_{s_r, t_r} x_{d_r, h_r}$ with $(t_r, s_r) \geq (d_r, h_r)$ and $(t_r, s_r) > (h, i)$ and remark that the operator $\sum_{r=1}^k x_{s_r, t_r} x_{d_r, h_r}$ has been defined on polynomials of degree $t-1$. We now set $x_{i,h} \circ N_I^J := (\sum_{r=1}^k x_{s_r, t_r} x_{d_r, h_r}) \circ (\prod_{p=2}^t z_{i_p, j_p})$.

A straightforward but lengthy computation shows that this defines an action of B on $k(q)[z_{i,j}]$. In particular since by definition, given a monomial M_I^J as in the statement of the proposition we have $M_I^J 1 = N_I^J$, the linear independence of the N_I^J implies that of the M_I^J 's, as desired. ■

Remark. (1) If we consider the algebra \mathcal{B} generated by the $x_{i,j}$ over the ring $R = k[q, q^{-1}, q^2 + q^{-2n}]$ with relations (2.1), (2.2), (2.3), the proof given in the proposition implies that the monomials M_I^J are an R -basis for this algebra.

(2) The algebra $\mathcal{B}/(q-1)$ is the polynomial ring $k[x_{i,j}]$.

COROLLARY 2.2. B has no zero divisors.

Proof. Suppose $a, b \in B$ are two nonzero elements such that $ab = 0$. Then, by removing denominators, we can assume that $a, b \in \mathcal{B}$ and furthermore, since \mathcal{B} is a free R -module, that a and b are nonzero modulo $q - 1$. Then the fact that $ab = 0$ modulo $q - 1$ gives a contradiction. ■

Consider now the quantized enveloping algebra $U_q(\mathfrak{sp}(n))$ (see [2, 3]) with generators $e_1, \dots, e_n, f_1, \dots, f_n$ and $K_1^{\pm 1}, \dots, K_n^{\pm 1}$. We claim that we have a natural action of $U_q(\mathfrak{sp}(n)) \otimes U_q(\mathfrak{gl}(m))$ on B .

For this we first define an action on the degree one part B_1 of B which has as basis the elements $x_{i,j}$ as

$$\begin{aligned} K_i x_{j,h} &= q^{\delta_{i,j} - \delta_{i,j-1} + \delta_{i,2n-j} - \delta_{i,2n-j+1}} x_{j,h} & \text{for } 1 \leq i \leq n-1 \\ f_i x_{j,h} &= \delta_{i,j} x_{j+1,h} - \delta_{i,2n-j} x_{j+1,h} & \text{for } 1 \leq i \leq n-1 \\ e_i x_{j,h} &= \delta_{i,j-1} x_{j-1,h} - \delta_{i,2n-j+1} x_{j-1,h} & \text{for } 1 \leq i \leq n-1 \\ K_n x_{j,h} &= q^{2\delta_{n,j} - 2\delta_{2,h-1}} x_{j,h} \\ f_n x_{j,h} &= \delta_{n,j} x_{j+1,h} \\ e_n x_{j,h} &= \delta_{n,j-1} x_{j-1,h} \end{aligned}$$

and

$$\begin{aligned} L_i x_{j,h} &= q^{\delta_{i,h}} x_{j,h} \\ F_i x_{j,h} &= \delta_{i,h} x_{j,h+1} \\ E_i x_{j,h} &= \delta_{i,h-1} x_{j,h-1}. \end{aligned}$$

It is clear from the definitions that the two actions of $U_q(\mathfrak{sp}(n))$ and $U_q(\mathfrak{gl}(m))$ commute, so that we get an action of $U_q(\mathfrak{sp}(n)) \otimes U_q(\mathfrak{gl}(m))$ on B_1 . Now by a straightforward computation one verifies that if we extend this action to a Hopf algebra action on the free algebra $k(q) \langle x_{i,j} \rangle$, the ideal J is preserved so that we obtain an action on B . In particular, if we consider the ring C of invariants under the $U_q(\mathfrak{sp}(n))$ action, i.e., the subring $C = \{a \in B \mid xa = \varepsilon(x)a\}$, ε being the counit for $U_q(\mathfrak{sp}(n))$, the algebra $U_q(\mathfrak{gl}(m))$ acts on C . Our goal is to describe C by generators and relations.

Let us first exhibit some elements in C . For each $h = 1, \dots, m$ set v_h equal to the column vector such that ${}^t v_h = (x_{1,h}, \dots, x_{2n,h})$. Set

$$\langle v_h, v_k \rangle = \sum_{i=1}^n q^{i-1-n} x_{i,h} x_{2n-i+1,k} - \sum_{i=1}^n q^{n-i+1} x_{2n-i+1,h} x_{i,k}. \quad (2.4)$$

We have

LEMMA 2.3. $\langle v_h, v_k \rangle \in C$ for all $h, k = 1, \dots, m$. Furthermore,

$$\langle v_h, v_k \rangle = -q^{-1} \langle v_k, v_h \rangle,$$

if $h < k$. In particular $\langle v_h, v_k \rangle = 0$.

Proof. The fact that $\langle v_h, v_k \rangle \in C$ for all $h, k = 1, \dots, m$ is an easy verification that we leave to the reader.

To see the rest let us first show that $\langle v_h, v_h \rangle = 0$. We claim that for $s < n$,

$$\begin{aligned} & \sum_{i=1}^s q^{i-1-n} x_{i,h} x_{2n-i+1,h} - \sum_{i=1}^s q^{n-i+1} x_{2n-i+1,h} x_{i,h} \\ &= \sum_{i=1}^s (q^{2s-i-n+1} - q^{n-i+1}) x_{2n-i+1,h} x_{i,h}. \end{aligned}$$

This follows immediately from relations (2.2) for $s = 1$. Assume it for $s - 1$. Thus we have

$$\begin{aligned} & \sum_{i=1}^s q^{i-1-n} x_{i,h} x_{2n-i+1,h} - \sum_{i=1}^s q^{n-i+1} x_{2n-i+1,h} x_{i,h} \\ &= \sum_{i=1}^{s-1} (q^{2s-2-i-n+1} - q^{n-i+1}) x_{2n-i+1,h} x_{i,h} \\ &+ (q^{s-1-n} x_{s,h} x_{2n-s+1,h} - q^{n-s+1}) x_{2n-s+1,h} x_{s,h} \end{aligned}$$

Substituting relation (2.2) for $x_{s,h} x_{2n-s+1,k}$, the claim follows.

Using this relation we now have

$$\begin{aligned} \langle v_h, v_h \rangle &= \sum_{i=1}^{n-1} (q^{n-i-1} - q^{n-i+1}) x_{2n-i+1,h} x_{i,h} + q^{-1} x_{n,h} x_{n+1,h} \\ &- q x_{n+1,h} x_{n,h} = 0, \end{aligned}$$

again by (2.2), as desired.

Applying now the operator $F - h$, we deduce

$$0 = F_h(\langle v_h, v_h \rangle) = \langle v_h, v_{h+1} \rangle + q^{-1} \langle v_{h+1}, v_h \rangle.$$

Thus our claim follows for $h + 1$. Assume it by induction for $k - 1$. Applying F_{k-1} to the identity $\langle v_h, v_{k-1} \rangle = -q^{-1} \langle v_{k-1}, v_h \rangle$ everything follows. ■

Before proceeding, let us recall that as in the case of $U_q(\mathfrak{gl}(m))$ also for $U_q(\mathfrak{sp}(n)) \otimes U_q(\mathfrak{gl}(m))$ (see [4, 5]), if we fix two sequences of integers $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_n)$ with $\lambda_i \geq 0$ and $\mu_i - \mu_{i+1} \geq 0$, there is a unique irreducible $U_q(\mathfrak{sp}(n)) \otimes U_q(\mathfrak{gl}(m))$ -module $V_{\lambda, \mu}$ of highest weight (λ, μ) whose dimension and character equal the dimension and character of the corresponding classical $\mathrm{Sp}(n) \times \mathrm{Gl}(m)$ -module $\bar{V}_{\lambda, \mu}$ (recall that if $\mu_i \geq 0$ μ corresponds to the Young diagram having μ_m rows of length m and for each $1 \leq i \leq m-1$ $\mu_i - \mu_{i-1}$ rows of length i).

Now consider the ring $\mathcal{B}/(q-1) = \bar{\mathcal{B}}$. Proposition 2.1 clearly implies that if we consider $\bar{\mathcal{B}}$ as a $\mathrm{Sp}(n) \times \mathrm{Gl}(m)$ -module, then for each $h \geq 0$, the multiplicity $m_{\lambda, \mu}(h) = \dim \mathrm{Hom}_{U_q(\mathfrak{sp}(n)) \otimes U_q(\mathfrak{gl}(m))}(V_{\lambda, \mu}, B_h)$ of $V_{\lambda, \mu}$ in the degree h component B_h of $\bar{\mathcal{B}}$, equals the multiplicity $\bar{m}_{\lambda, \mu}(h) = \dim \mathrm{Hom}_{\mathrm{Sp}(n) \times \mathrm{Gl}(m)}(\bar{V}_{\lambda, \mu}, B_h)$ of $\bar{V}_{\lambda, \mu}$ in the degree h component \bar{B}_h of $\bar{\mathcal{B}}$. In particular we can apply this when $\lambda = 0$ and deduce that if we consider the ring of $\mathrm{Sp}(n)$ -invariant polynomials $\bar{C} \subset \bar{\mathcal{B}}$ as a $\mathrm{Gl}(m)$ -module and C as a $U_q(\mathfrak{gl}(m))$ -module, then for any Young diagram Y , the multiplicities of V_Y in C and of \bar{V}_Y in \bar{C} are the same.

Now recall that, [1], as a $\mathrm{Gl}(m)$ -module, $\bar{C} = \bigoplus_{Y \in \mathcal{T}_n} \bar{V}_Y$, where \mathcal{T}_n is the set of Young diagrams for $\mathrm{Gl}(m)$ with even rows of length at most $2n$. Thus we deduce

PROPOSITION 2.4. *As a $U_q(\mathfrak{gl}(m))$ -module*

$$C = \bigoplus_{Y \in \mathcal{T}_n} V_Y.$$

We are now in the position to prove our main result.

THEOREM 2.5. (1) *The ring C is generated by the elements $\langle v_h, v_k \rangle$ for $h < k$.*

(2) *Let A be the algebra of functions on a quantum antisymmetric matrix considered in Section 1. There is a $U_q(\mathfrak{gl}(m))$ -equivariant surjective homomorphism*

$$\phi: A \rightarrow C$$

defined by $\phi(a_{h,k}) = \langle v_h, v_k \rangle$. Furthermore ϕ is an isomorphism if $m \leq 2n+1$, while $\ker \phi = I_{2n+2}$ if $m > 2n+1$.

Proof. We begin by remarking that the linear map $\phi: A_1 \rightarrow C_2$ defined by $\phi(a_{h,k}) = \langle v_h, v_k \rangle$ is $U_q(\mathfrak{gl}(m))$ -equivariant. In order to extend it to an algebra homomorphism $\phi: A \rightarrow C$, we clearly have to verify that the

elements $\langle v_h, v_k \rangle$ satisfy relations (1.1). But by Proposition 1.2 it suffices to verify that

$$\langle v_1, v_2 \rangle \langle v_1, v_3 \rangle = q \langle v_1, v_3 \rangle \langle v_1, v_2 \rangle.$$

To see this, notice that the elements $\langle v_1, v_2 \rangle \langle v_1, v_3 \rangle$ and $\langle v_1, v_3 \rangle \langle v_1, v_2 \rangle$ have the same weight $(2, 1, 1, 0, \dots, 0)$ and by the above proposition the corresponding weight space has dimension one in C_4 . We deduce that $\langle v_1, v_2 \rangle \langle v_1, v_3 \rangle = a \langle v_1, v_3 \rangle \langle v_1, v_2 \rangle$ for some $a \in k(q)$. Applying E_2 to this equality, we get $q(\langle v_1, v_2 \rangle)^2 = a(\langle v_1, v_2 \rangle)^2$. Since by Corollary 2.2 $(\langle v_1, v_2 \rangle)^2$ is nonzero, we deduce $a = q$ as desired.

Having established the existence of the morphism ϕ , we remark that, since I_{2n+2} decomposes by Theorem 1.5 as the direct sum of irreducible representations V_Y whose Young diagram has first row of length greater than $2n$, again by the above Proposition, if $m > 2n$, $I_{2n+2} \subset \ker \phi$.

At this point in order to show our claims, we need to show that the homomorphism $\tilde{\phi}: A/I_{2n+2} \rightarrow C$ is an isomorphism. Since by Theorem 1.5 and the above Proposition A/I_{2n+2} and C have the same decomposition into irreducible $U_q(\mathfrak{gl}(m))$ -module, it clearly suffices to show that $\tilde{\phi}$ is injective.

By Theorem 1.5, we thus need to see that under the composed homomorphism $A \xrightarrow{\phi} C \xrightarrow{j} B$, j being the inclusion, the standard tableaux whose shape lies in \mathcal{T}_n , map to linearly independent vectors. Set $\psi = j \circ \phi$. Consider now the R -subalgebra $\mathcal{A} \subset A$ generated by the $a_{h,k}$'s. Clearly $\psi(\mathcal{A}) \subset \mathcal{B} \cdot$ and we get an induced homomorphism $\bar{\psi}(\bar{\mathcal{A}}) \rightarrow \bar{\mathcal{B}}$, defined by

$$\bar{\psi}(\bar{a}_{h,k}) = \sum_{i=1}^n \bar{x}_{i,h} \bar{x}_{2n+1-i,k} - \sum_{i=1}^n \bar{x}_{2n+1-i,h} \bar{x}_{i,k}.$$

We know [1], that $\bar{\psi}$ maps our set of standard tableaux to a linearly independent set. Using now the same argument as in Corollary 2.2, we deduce that under ψ the standard tableaux whose shape lies in \mathcal{T}_n map to linearly independent vectors and hence our theorem. ■

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