Università degli studi di Roma
Tor Vergata
facoltà di ingegneria

Dottorato di Ricerca in
Informatica e Ingegneria dell'Automazione
Ciclo XX

Control of Underactuated Mechanical Systems via Passivity-Based and Geometric Techniques

Candidato:
Giuseppe Viola

Relatore:
Prof. Antonio Tornambè

Co-relatori:
Prof. Romeo Ortega
Prof. Jessy W. Grizzle

Coordinatore:
Prof. Daniel P. Bovet

Control of Underactuated Mechanical Systems via Passivity-Based and Geometric Techniques

Candidato:	Relatore:
Giuseppe Viola	*Prof. Antonio Tornambè*
Co-relatori:	Coordinatore:
Prof. Romeo Ortega	*Prof. Daniel P. Bovet*
Prof. Jessy W. Grizzle	

For all those who encouraged me
to pursue my dream
even though I still don't know
what that dream looks like.

Acknowledgments

The research that has gone into this thesis has been thoroughly enjoyable. That enjoyment is largely a result of the interaction that I have had with my supervisors and colleagues.

I feel very privileged to have worked with all of my supervisors, who taught me a great deal about control science, by sharing with me the joy and the hardship of research. I must first express my gratitude towards my advisor, Professor Antonio Tornambè. His support and encouragement have led me through the most difficult moments, and his love for science has set an example that I will always have as reference. I would also like to thank Professors Romeo Ortega and Jessy W. Grizzle, who have kindly welcomed me during my stays in Paris and Ann Arbor, Michigan. I enjoyed their help and sharing of knowledge, which made me achieve important results for this thesis.

I am grateful to all of the faculty at Dipartimento di Informatica, Sistemi e Produzione, University of Tor Vergata: Prof. Alessandro Astolfi, Dr. Sergio Galeani, Prof. Osvaldo Maria Grasselli, Dr. Francesco Martinelli, Prof. Laura Menini, Prof. Salvatore Nicosia and Prof. Luca Zaccarian for their assistance and support. I especially thank Prof. Laura Menini, for her technical and human support, and Prof. Salvatore Nicosia, for his wise and caring advice.

I would like to thank my fellow PhD students at DISP: Claudio Argento, Daniele Carnevale, Fulvio Forni, Simona Onori, Luigi Pangione, Serena Pani, Fabio Piedimonte and Alessandro Potini; my time would have been far less enjoyable without all the fun that we had. A special thank goes to Alessandro

Potini, for all our talking about expectations and everything that is valuable in life.

I would also like to express a great deal of affection to my friends from Gaeta: Antonello, for inspiring me with his profound view of life, Marialucia, for encouraging me like who knows what you are going through only can, Annatereresa, for her shining smile, and Mariacristina, for her sweet unpredictability. Thank you all for passing over all the times I was not there.

Finally, I thank my family, for their love and comprehension. I will always endeavor to make it worthwhile.

Contents

List of Figures

Chapter 1

Introduction

Control of mechanical systems is currently among one of the most active fields of research, due to the diverse applications of mechanical systems in real life. However, the study of mechanical systems goes back to Euler and Lagrange in the 1700's, and it was not until the 1850's that mechanical control systems came to the picture in regulation of steam engines. During the past century, a series of scientific, industrial, and military applications motivated rigorous analysis and control design for mechanical systems. On the other hand, the theoretically challenging nature of analysis of the behavior of mechanical systems attracted many mathematicians to study their intrinsic properties, leading to the development of the articulated theories based on the differential geometric approach.

The last decades have shown an increasing interest in the control of underactuated mechanical systems. These systems are characterized by the fact of possessing more degrees of freedom than actuators, i.e., one or more degrees of freedom are unactuated. This class of mechanical systems are abundant in real life; examples of such systems include, but are not limited to, surface vessels, spacecraft, underwater vehicles, helicopters, road vehicles, mobile robots, space robots and underactuated manipulators. The underactuation property of underactuated systems may come from one of the following reasons:

- the dynamics of the system (e.g. aircrafts, spacecrafts, helicopters, underwater vehicles);
- by design for reduction of the cost or some practical purposes (e.g. satellites with two thrusters and flexible-link robots);
- actuator failure (e.g. in a surface vessel or aircraft);
- imposed artificially to create complex low-order nonlinear systems for the purpose of gaining insight in control of high-order underactuated systems (e.g. the Acrobot, the Pendubot, the Beam-and-Ball system, the Pendulum on a cart system, the Rotating Pendulum, the TORA system.

The underactuation properties generates interesting control problems which require fundamental nonlinear approaches. The linear approximation around equilibrium points may, in general, not be controllable and the feedback stabilization problem, in general, cannot be transformed into a linear control problem. Therefore, in many cases linear control methods cannot be used to solve the feedback stabilization problem, not even locally. As a result, the last years saw the birth of many new nonlinear control techniques, sometimes antipodal with respect to their inspiration, which have represented the basis for any theoretical and practical developments in this field.

Among these, it is worth citing two of the major research lines: the passivity-based and geometric-based ones. Analysis of mechanical systems from a truly differential geometric perspective has its foundations in the classic work of Abraham and Marsden [1]. Another classic text in geometric methods in mechanics is that of Arnold [8]. On the other hand, mathematical control theory is a younger subject; recent differential geometric treatments may be found in the books of Agrachev and Sachkov [6], Isidori [40], Nijmeijer and van der Schaft [60]. Such works have laid the ground for plenty of subsequent developments in the field of control of underactuated mechanical systems. An example is given by the book of Bullo and Lewis [16], which gives many theoretical tools based on differential and Riemannian geometry for modeling, analyzing and controlling mechanical systems, with an emphasis on motion planning for underactuated mechanical systems. Another work, which is directly related to the results given in the thesis, is the book by J. Grizzle and his collaborators [91], "Feedback control of dynamic bipedal robot locomotion". It extends the concepts of feedback linearization and zero dynamics to the special cases of underactuated mechanical systems with impulse effects exemplified by walking robots. Namely, the classical control scheme based on the method of asymptotically driving a set of outputs to zero, which leads to the definition of zero dynamics, is extended to include the impact events of the robot's foot touching the ground, leading to the concept of Hybrid Zero Dynamics (HZD). Since the first paper on the subject [36], which gave the first formally proved stability result of a walking gait for a robot with stiff legs and torso, a lot of research effort has been devoted to extending the class of walking robots and the achievable gaits. The results are striking: provably asymptotically stable walking and running gaits are achieved in [91] for a footless rigid robot with knees and torso, and research is under way to provide extensions to robots with series compliant actuators.

The other major cited line of research in the field of underactuated mechanical systems is represented by the work of R. Ortega and his collaborators, who gave birth, in their book [65] "Passivity based control of Euler-Lagrange systems: mechanical, electrical and electromechanical applications", to the passivity-based paradigm. It shifts the designer's attention from manipulating input and output signals to shaping the system energy, towards the goal of achieving stability and performance. The field of passivity has received many contributions, starting from the ground-breaking work of Takegaki and Arimoto [83], and Byrnes and Isidori [15]. The book [86] by van der Schaft introduced the concept of port-controlled Hamiltonian systems, a unified framework for modeling interconnected systems exchanging energy among each other; such a framework constituted the basis for the development of the theory of "interconnection and damping assignment passivity-based control" (IDA–PBC) [67], whose major application in the field of mechanical systems is indeed underactuation. A lot of research effort has been devoted to the case of underactuation degree one, which includes many benchmark applications such pendulum systems and vertical take-off landing (VTOL) aircrafts.

The present work configures as an attempt of generalizing some of the existing results on the control of linear and nonlinear underactuated mechanical systems by using techniques borrowed from both the approaches cited above. The work is divided in three chapters. Each of them focuses on a different control problem and uses different tools to provide solutions. In particular, the following goals are achieved:

- *Input-output decoupling for linear underactuated mechanical systems*: a class of linear mechanical systems is identified for which the problem of input-output decoupling with asymptotic stability can be achieved through interconnection with a controller that is required to be a mechanical system itself. Techniques such as passivity, positive realness, and polynomial matrix descriptions are used to address such a problem;

- *asymptotic stabilization of arbitrary equilibria in nonlinear mechanical systems with underactuation degree one*: some extensions are provided to the theory of IDA–PBC to enlarge the class of mechanical systems with underactuation degree one that can be stabilized. Such extensions have been achieved through explicit solutions and/or homogenization of the partial differential equations describing the matching conditions between the given and the desired dynamics;

- *exponential stabilization of periodic orbits in nonlinear underactuated mechanical systems with impulse effects*: the class of biped robots to which the HZD-based technique can be applied is extended to include compliant actuation, which raises the underactuation degree and renders it more difficult to deal with invariance of the zero dynamics manifold. Techniques borrowed from singular-perturbations analysis and Brouwer's fixed point theorem are used to achieve the goal.

Chapter 2

Input-output decoupling of linear mechanical systems

2.1 Introduction

Most of the recent work devoted to the control of mechanical systems has focused on the problem of guaranteeing classical requirements for a control system by means of special classes of controllers, or with some additional requirements for the closed-loop system, e.g., it can be required that it is an Hamiltonian system with a structure similar to the given one. This is also the main characteristic of this chapter, in which the classical problem of input-output decoupling is dealt with for a class of m-inputs m-outputs underactuated multi-body linear mechanical systems, with the requirement that the controller has to be another mechanical system to be physically connected to the given one (the terminal points of the controller are to be physically attached to the actuated bodies). A similar approach is quite classical in vibration control, where the possibility of reducing the vibrations of a mechanical structure by connecting to it either mechanical dampers or electric RLC circuits is called passive control. With respect to the standard way of designing a controller, constituted by a generic dynamical system taking as inputs the available outputs of the system (often the whole state), and giving as outputs the forces or torques to be applied to the actuated bodies, the approach taken here has many differences, that render it interesting. Some of such differences are actually restrictions, in fact the proposed controller has to be a very special dynamical system, with a strong structure: this limits severely the possible choices for the designer. On the other hand, as will be specified later, with the approach taken here it is possible to use non-causal controllers, which is quite unusual in control theory.

2.2 Preliminaries and problem formulation

Consider a linear mechanical system constituted by ideal point bodies, linear springs and linear dampers, moving on a line. Or, equivalently, a linear mechanical system constituted by a set of bodies rotating around the same axis, connected by torsional springs and dampers.

Let $q_i(t)$ be the position at time $t \in \mathbb{R}$ with respect to an inertial reference frame of the i-th body, $i = 1, 2, ..., n$, where n is the number of the bodies and let $q(t) := [q_1(t), \ldots, q_n(t)]^T$; let $M_i \in \mathbb{R}, M_i > 0$, be the mass of the i-th body, $i = 1, 2, ..., n$. When present, let $K_{i,j} \in \mathbb{R}$ ($F_{i,j} \in \mathbb{R}$) be the coefficient of elasticity (the damping factor) of the spring (the damper) possibly connecting the i-th body with the j-th one, $i = 1, 2, ..., n, j = i + 1, ..., n$; when present, let $K_{0,i} \in \mathbb{R}$ ($F_{0,i} \in \mathbb{R}$) be the coefficient of elasticity (the damping coefficient) of the spring (the damper) possibly connecting the i-th body ($i = 1, 2, ..., n$) with the ground, constituted by an infinitely massive body (numbered with the index 0). Without loss of generality, the length at rest of all the springs can be considered null.

Notation 1. $A > 0$ (respectively, $A \geq 0$) means that matrix A is real, symmetric and positive definite (respectively, semi-definite).

Let the system be described by the following kinetic and potential energies and by the following dissipation function, respectively: $\mathcal{K} = \frac{1}{2}\dot{q}^T D\dot{q} =$

$\frac{1}{2}\sum_{i=1}^{n} M_i\dot{q}_i^2$, $\mathcal{V} = \frac{1}{2}q^T H q = \frac{1}{2}\sum_{i=1}^{n} K_{0,i}q_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} K_{i,j}\left(q_i - q_j\right)^2$, $\mathcal{F} = \frac{1}{2}\dot{q}^T F \dot{q} = \frac{1}{2}\sum_{i=1}^{n} F_{0,i}\dot{q}_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} F_{i,j}\left(\dot{q}_i - \dot{q}_j\right)^2$, where D is the generalized inertia matrix which is diagonal and positive definite (since all the bodies have non-null mass), F is symmetric positive semidefinite, and H is symmetric positive semidefinite if all the springs have non-negative coefficients of elasticity.

Now, assume that m bodies (without loss of generality, the first m ones) are actuated by external forces $u_i(t), i = 1, ..., m$, and let $u(t) := [u_1(t), \ldots, u_m(t)]^T$ be the input of the system, whence the underactuation degree is equal to $n - m$. The relevant outputs of the system are both the positions $y_q(t) = q_1(t), \ldots, q_m(t)]^T$ of the first m bodies and their velocities $y_v(t) = \dot{q}_1(t), \ldots, \dot{q}_m(t)]^T$. The considered mechanical system is then described by the following equations:

$$D\ddot{q}(t) + F\dot{q}(t) + Hq(t) = Bu(t), \tag{2.1}$$

$$y_q(t) = B^T q(t), \tag{2.2}$$

$$y_v(t) = B^T \dot{q}(t), \tag{2.3}$$

where $B \in \mathbb{R}^{n\times m}, B = \left[\begin{array}{cc} I_m & 0 \end{array}\right]^T$.

Note that $\det\left(D\ s^2 + F\ s + H\right)$ is not the null function since D is nonsingular. By Laplace transformation, we have:

$$\begin{aligned} y_q(s) &= B^T\ (D\ s^2 + F\ s + H)^{-1} B\ u(s), \\ y_v(s) &= B^T\ (D\ s + F + H\ \frac{1}{s})^{-1} B\ u(s), \end{aligned}$$

where $y_q(s) = \mathcal{L}\left\{B^T\ q(t)\right\}$, $y_v(s) = \mathcal{L}\left\{B^T\ \dot{q}(t)\right\}$, $u(s) = \mathcal{L}\left\{u(t)\right\}$. In the following, the impedance matrix $Z(s) = B^T\ (D\ s + F + H\ \frac{1}{s})^{-1} B$ and the admittance matrix $Y(s) = Z^{-1}(s)$ will be used repeatedly.

It is well known that a square rational matrix function $Z(s)$ is positive real if $\mathrm{Re}(Z(s))$ is positive semidefinite for all s having $\mathrm{Re}(s) \geq 0$; $Z(s)$ is BIBO stable if each entry $Z_{i,j}(s) = \dfrac{N_{i,j}(s)}{F_{i,j}(s)}$, with $N_{i,j}(s)$ and $F_{i,j}(s)$ being coprime polynomials, is proper and its denominator $F_{i,j}(s)$ has all the roots with negative real part; the system (2.1), (2.2), (2.3) described by the impedance

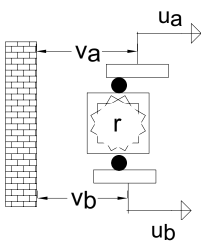

Figure 2.1: The pictorial representation of the speed reducer.

$Z(s)$ is asymptotically stable if all the roots of $\det(D\ s^2 + F\ s + H) = 0$ have negative real part. The following lemma gives sufficient conditions for the impedance matrix to be positive real.

Lemma 2.2.1. Since $\det(D\ s^2 + F\ s + H)$ is not the null function, if H is positive semidefinite, in addition to D and F which are positive semidefinite as well, then the square rational matrix function $B^T(D\ s + F + H\ \frac{1}{s})^{-1}B$ is positive real.

Proof. If $\det(D\ s^2 + F\ s + H)$ is not the null function, it is well known that $(D\ s + F + H\ \frac{1}{s})^{-1}$ is positive real if and only if $D\ s + F + H\ \frac{1}{s}$ is positive real. Setting $s = a + ib$, $a \geq 0$, we have

$$\mathrm{Re}(D\ s + F + H\ \frac{1}{s}) = aD + F + \frac{a}{a^2 + b^2}H,$$

which is a linear combination with non-negative coefficients of positive semidefinite matrices, and therefore it is positive semidefinite. Finally, if $(D\ s + F + H\ \frac{1}{s})^{-1}$ is positive semidefinite, then $B^T(D\ s + F + H\ \frac{1}{s})^{-1}B$ is positive semidefinite for any real B. □□□

Taking into account that, when it exists, the inverse of a positive real matrix is positive real, under the hypotheses of Lemma 2.2.1, both the impedance and the admittance of system (2.1) (2.2), (2.3), are positive real.

The proposed controller will not be a generic dynamical system taking as input $y_q(t)$ and/or $y_v(t)$ and giving as output $u(t)$, but, rather, the controller will be another mechanical system having m terminal points to be physically connected to the first m bodies of the system. The connection can be either a direct one (i.e., the terminal point is glued to the mass of the body) or through an (ideal) speed reducer (e.g., an ideal gear reduction unit). The speed reducer, represented schematically in Figure 2.1, is a two terminal points object, without mass, friction and elasticity, characterized by the transmission ratio r. Denoting by v_i and u_i, $i \in \{a, b\}$, respectively, the velocity and the force applied to the i-th terminal point of the speed reducer, the equations describing its behaviour are

$$v_b = r v_a, \quad (2.4)$$

$$u_b = \frac{1}{r} u_a. \quad (2.5)$$

By integrating equation (2.4), if q_i, $i \in \{a, b\}$, denotes the position of the i-th terminal point of the speed reducer, we have $q_b = r q_a + c$, with c being an arbitrary constant that in this chapter is taken equal to 0, without loss of generality. In the special case where $r = 1$, the speed reducer is equivalent to the direct connection, whereas when $r = -1$, it corresponds to inverting the velocity.

If $r_1, ..., r_m$ are the transmission ratios of the reducers used for the connection (possibly, equal to 1), the controller is described by:

$$D_c \ddot{q}_c(t) + F_c \dot{q}_c(t) + H_c q_c(t) = 0, \quad (2.6)$$

$$y_{c,q}(t) = B_c^T q_c(t), \quad (2.7)$$

with $q_c(t) \in \mathbb{R}^{n_c}, B_c \in \mathbb{R}^{n_c \times m}, B_c = \begin{bmatrix} R & 0 \end{bmatrix}^T$, $R = \text{diag}(r_1^{-1}, ..., r_m^{-1})$, D_c diagonal and positive semidefinite, F_c symmetric and positive semidefinite, H_c symmetric and $\det(D_c\ s^2 + F_c\ s + H_c)$ being not the null function. The overall system is then described by the following equations:

$$D\ddot{q}(t) + F\dot{q}(t) + Hq(t) = Bu(t) + B\lambda(t), \quad (2.8)$$

$$D_c \ddot{q}_c(t) + F_c \dot{q}_c(t) + H_c q_c(t) = -B_c \lambda(t), \quad (2.9)$$

$$y_q(t) = y_{c,q}(t), \quad (2.10)$$

where $\lambda(t)$ is the vector of the Lagrange multipliers that takes into account the equality constraint (2.10), which represents the forces exchanged between the system and the controller. Note that, by eliminating the Lagrange multipliers and using the equality constraint (2.10), the overall system can be rewritten in the form (2.1), i.e., as an unconstrained mechanical system having $n + n_c - m$ degrees of freedom. The input of the overall system (2.8)–(2.10) is still $u(t)$ and the relevant outputs are still $y_q(t)$ and $y_v(t)$. The control problem studied here is stated formally as follows.

Problem 1. *Find, if any, a controller of the form (2.6)–(2.7) such that the overall system (2.8)–(2.10) is asymptotically stable and input-output decoupled (the latter being equivalent to have a non-singular and diagonal impedance matrix).*

The overall system (2.8)–(2.10) will be called the (mechanical) parallel connection of the system and the controller, because if $Y(s)$ and $Y_c(s)$ are the admittances of the mechanical system and of the controller, respectively, then the admittance of the parallel connection is $Y_p(s) = Y(s) + Y_c(s)$. In addition, as for the impedance $Z_p(s)$ of the parallel connection, it can be easily seen that $Z_p(s) = Z(s)\left(I + Z_c^{-1}(s)Z(s)\right)^{-1}$ (where $Z_c^{-1}(s) = Y_c(s)$), i.e. the parallel connection can be seen as a feedback system from the output $y_v(t)$. Note that $Z_c^{-1}(s)$ is not necessarily proper (hence the proposed connection results equivalent to use a standard non-causal controller) and, moreover, that we are interested in a controller whose inverse be the impedance of a mechanical system, whence the classical tools for designing a controller that guarantees input-output decoupling with stability cannot be used. A crucial property of the parallel connection of two mechanical systems is that if two systems having positive real impedance matrices $Z_1(s)$ and $Z_2(s)$ are connected in parallel, the impedance matrix of the parallel connection is still positive real (this is easily seen by taking into account that both the sum of two positive real matrices and the inverse of a positive real matrix — when it exists — are positive real). However, special care is to be used when the property of interest is the

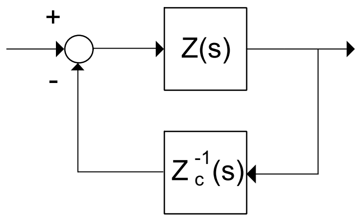

Figure 2.2: The mechanical parallel connection seen as a feedback system.

asymptotic stability of the system, which is stronger than the real positivity.

The following simple example shows that the parallel connection of two asymptotically stable mechanical systems needs not be asymptotically stable.

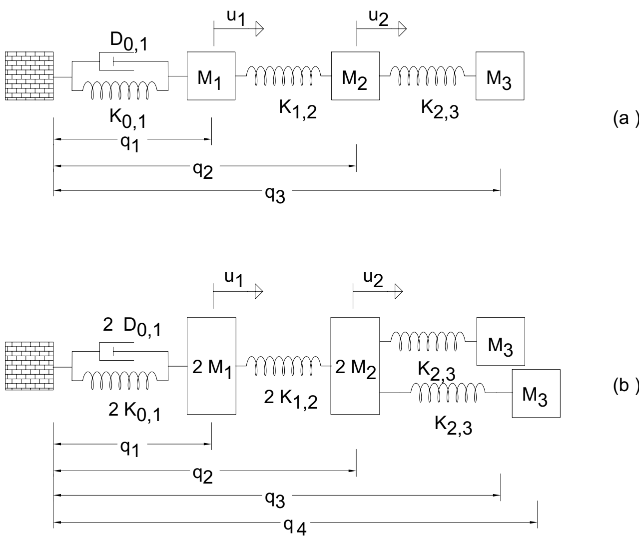

Figure 2.3: The mechanical systems considered in Example 1.

Example 1. Consider the mechanical system depicted in Figure 2.3-(a), which is constituted by three bodies moving on an horizontal line and connected by the springs having positive coefficients of elasticity $K_{0,1}, K_{1,2}$ and $K_{2,3}$, and one damper having damping coefficient $F_{0,1} > 0$ as shown in Figure 2.3-(a).

With the proposed notations, the system can be rewritten in the form (2.1)–(2.3), with

$$D = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}, F = \begin{bmatrix} F_{0,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} K_{0,1} + K_{1,2} & -K_{1,2} & 0 \\ -K_{1,2} & K_{1,2} + K_{2,3} & -K_{2,3} \\ 0 & -K_{2,3} & K_{2,3} \end{bmatrix},$$

from which it is easy to compute

$$\det \left(D\ s^2 + F\ s + H\right) = p_1(s) + sF_{0,1}p_2(s),$$

where $p_1(s)$ and $p_1(s)$ are even polynomials. Now, it is easy to recast the ratio $\dfrac{p_1(s)}{sp_2(s)}$ as follows:

$$\frac{p_1(s)}{sp_2(s)} = h_1 s + \cfrac{1}{h_2 s + \cfrac{1}{h_3 s + \cfrac{1}{h_4 s + \cfrac{1}{h_5 s + \cfrac{1}{h_6 s}}}}},$$

where $h_1 = M_1$, $h_2 = 1/(K_{1,2} + K_{0,1})$, $h_3 = M_2(K_{1,2} + K_{0,1})^2/K_{1,2}^2$, $h_4 = K_{1,2}^2/((K_{0,1}K_{2,3} + K_{1,2}K_{2,3} + K_{0,1}K_{1,2})(K_{1,2} + K_{0,1}))$, $h_5 = M_3(K_{0,1}K_{2,3} + K_{1,2}K_{2,3} + K_{0,1}K_{1,2})^2/(K_{1,2}^2 K_{2,3}^2)$, $h_6 = K_{1,2}K_{2,3}/(K_{0,1}(K_{0,1}K_{2,3} + K_{1,2}K_{2,3} + K_{0,1}K_{1,2}))$. Since such coefficients are positive and the polynomials $p_1(s)$ and $sp_2(s)$ are coprime for positive values of masses and coefficients of elasticity, the ratio $\dfrac{p_1(s)}{sp_2(s)}$ is positive real, whence (by well known results) the polynomial $p_1(s) + sF_{0,1}p_2(s)$ has all the roots with negative real part for all positive $F_{0,1}$.

Taking two identical systems as the one in Figure 2.3-(a) and connecting them in mechanical parallel, the mechanical system depicted in Figure 2.3-(b) is obtained. Its description as unconstrained mechanical system has $q =$

$[q_1, q_2, q_3, q_4]^T$ as position vector, and it can be rewritten in the form (2.1)–(2.3), with

$$D = \begin{bmatrix} 2M_1 & 0 & 0 & 0 \\ 0 & 2M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_3 \end{bmatrix}, F = \begin{bmatrix} 2F_{0,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 2K_{0,1} + 2K_{1,2} & -2K_{1,2} & 0 & 0 \\ -2K_{1,2} & 2K_{1,2} + 2K_{2,3} & -K_{2,3} & -K_{2,3} \\ 0 & -K_{2,3} & K_{2,3} & 0 \\ 0 & -K_{2,3} & 0 & K_{2,3} \end{bmatrix},$$

from which it is easy to compute

$$\det\left(D\ s^2 + F\ s + H\right) = 2\left(M_3 s^2 + K_{2,3}\right)\hat{p}(s),$$

where $\hat{p}(s)$ is a polynomial having all the roots with negative real part. The first factor of this polynomial has $\pm j\sqrt{\frac{K_{2,3}}{M_3}}$ as roots for any value of $F_{0,1}$, which shows how the parallel connection of two asymptotically stable mechanical systems may not be asymptotically stable. □

We recall the well known fact (see [43]) that, for mechanical systems of the form (2.1)–(2.3), the stabilizability from the input $u(t)$ and the detectability from the output $y_v(t)$ can be tested, respectively, by means of the following two necessary and sufficient conditions:

$$\text{rank}\left(\begin{bmatrix} D\ s^2 + F\ s + H & B \end{bmatrix}\right) = n, \forall s \in \mathbb{C}, \text{Re}(s) \geq 0, \tag{2.11}$$

$$\text{rank}\left(\begin{bmatrix} D\ s^2 + F\ s + H \\ s\ B^T \end{bmatrix}\right) = n, \forall s \in \mathbb{C}, \text{Re}(s) \geq 0. \tag{2.12}$$

Remark 2.2.2. If $\det(H) \neq 0$, then the stabilizability and detectability conditions (2.11), (2.12) are equivalent. As a matter of fact, if $\det(H) \neq 0$, then $\text{rank}\left(\left[D\ s^2 + F\ s + H\right]\right)$ is maximum for $s = 0$, and therefore we can assume without loss of generality that $s \neq 0$; under such an assumption,

$$\operatorname{rank}\left(\begin{bmatrix} D\ s^2 + F\ s + H \\ s\ B^T \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} D\ s^2 + F\ s + H \\ B^T \end{bmatrix}\right)$$
$$= \operatorname{rank}\left(\begin{bmatrix} D\ s^2 + F\ s + H & B \end{bmatrix}\right).$$

Remark 2.2.3. The structural properties of stabilizability and detectability can be lost by the mechanical parallel connection even if the original mechanical system and the controller are stabilizable and detectable.

The goal of this chapter is to find a controller having admittance matrix $Y_c(s)$ such that the overall system is input-output decoupled and asymptotically stable. The next three lemmas recall important facts that will be useful in the proof of the main result. The first one is concerned with the possibility of stabilizing a mechanical system — having positive real impedance $Z(s)$ — by connecting the m actuated bodies with the ground by means of m identical dampers having damping coefficient equal to $F > 0$. Such a connection can be seen as the parallel connection of the given mechanical system and of the controller with singular D_c constituted by just the m dampers, having admittance matrix $Y_c(s) = [\text{diag}\{F, ..., F\}] = Z_c^{-1}(s)$.

Lemma 2.2.4. If $D > 0$ and $Z(s)$ is positive real, then $Z_p(s) = Z(s)(I + D\ Z(s))^{-1}$ is BIBO stable. If the stabilizability and detectability conditions (2.11) and (2.12) hold, then the parallel connection having $Z_p(s)$ as impedance matrix is asymptotically stable.

Proof. Since $Z(s)(I+D\ Z(s))^{-1} = \frac{1}{F}\left(I - (I + D\ Z(s))^{-1}\right)$, the poles of $Z_p(s)$ are the zeros of $I + D\ Z(s)$. By contradiction, assume the existence of $\hat{s}$, $\text{Re}(\hat{s}) \geq 0$, (possibly, the point at infinity) such that $I + Z(\hat{s})D$ is singular; then, there exists $v \neq 0$ such that

$$v^T\ \ (I + Z(\hat{s})D)\ \ v = 0; \tag{2.13}$$

this implies $D = -\dfrac{v^T\ v}{v^T\ Z(\hat{s})\ v}$, which contradicts the assumption that D is positive, as $v^T\ v > 0$ and $v^T\ Z(\hat{s})\ v > 0$ for any $v \neq 0$; as a matter of fact, note that if $v^T\ Z(\hat{s})\ v = 0$, then by (2.13), $v^T\ v = 0$ in contradiction with the assumption that $v \neq 0$. Finally, it is standard that the BIBO stability implies asymptotic stability under stabilizability and detectability conditions. □□□

The second intermediate result (Lemmas 2.2.5 and 2.2.6) is concerned with the possibility of rendering positive real the impedance matrix of a mechanical system by connecting the m actuated bodies with the ground by means of m identical springs having a positive and sufficiently high coefficient of elasticity K. Such a connection can be seen as the parallel connection of the given mechanical system and of the controller (with a singular D_c) constituted by just the m springs, having admittance matrix $Y_c(s) = \frac{K}{s} I = Z_c^{-1}(s)$. Moreover, the description of the parallel connection in the form (2.1)–(2.3) has the same D and F matrices of the given mechanical system, whereas for its matrix H_p we have:

$$H_p = H + \operatorname{diag}\left(\underbrace{K, ..., K}_{m \text{ times}}, \underbrace{0, ..., 0}_{n-m \text{ times}}\right). \tag{2.14}$$

Theorem 1. *If $D > 0$ and $F \geq 0$, then all the roots of* $\det(Ds^2 + Fs + H)$ *have non-positive real part if and only if $H \geq 0$.*

Proof. Let λ be a non-real root of $\det(Ds^2 + Fs + H)$. Since D, F and H are real,

$$\begin{aligned} (D\lambda^2 + F\lambda + H)v &= 0, \\ (D\,(\lambda^*)^2 + F\lambda^* + H)v^* &= 0. \end{aligned}$$

By multiplying the first for $(v^*)^T$ and the second for v^T, we obtain:

$$\begin{aligned} (v^*)^T\,(D\lambda^2 + F\lambda + H)v &= 0, \\ v^T(D\,(\lambda^*)^2 + F\lambda^* + H)v^* &= 0. \end{aligned}$$

Since D, F and H are symmetric, we have that $(v^*)^T\,Bv = v^T Bv^*$, $(v^*)^T\,Dv = v^T Dv^*$ and $(v^*)^T\,Hv = v^T Hv^*$, thus obtaining:

$$\begin{aligned} v^T Bv^*\lambda^2 + v^T Dv^*\lambda + v^T Hv^* &= 0, \\ v^T Bv^*\,(\lambda^*)^2 + v^T Dv^*\lambda^* + v^T Hv^* &= 0. \end{aligned}$$

By subtracting the second equation from the first one, we obtain

$$\begin{aligned} v^T B v^* \left(\lambda^2 - (\lambda^*)^2\right) + v^T D v^* (\lambda - \lambda^*) &= 0, \\ &\downarrow \\ v^T B v^* (\lambda - \lambda^*) (\lambda + \lambda^*) + v^T D v^* (\lambda - \lambda^*) &= 0. \end{aligned}$$

Since λ is not real, then $(\lambda - \lambda^*) \neq 0$, whence

$$\begin{aligned} v^T B v^* (\lambda + \lambda^*) + v^T D v^* &= 0 \\ (\lambda + \lambda^*) &= -\frac{v^T D v^*}{v^T B v^*} \\ &= 2\mathrm{Re}(\lambda) \end{aligned}$$

from which we observe that if λ is a non-real root of $\det(Ds^2 + Fs + H)$, then its real part is non-positive (independently of H).

Now, assume that λ is a non-negative real root of $\det(Ds^2+Fs+H)$, then

$$\begin{aligned} v^T (D\lambda^2 + F\lambda + H) v &= 0, \\ v^T B v \lambda^2 + v^T D v \lambda + v^T H v &= 0, \\ v^T H v &= -\left(v^T B v \lambda^2 + v^T D v \lambda\right); \end{aligned}$$

since H is symmetric, then H must have at least one non-positive eigenvalue.

Assume that H has at least one negative eigenvalue (it is not positive semidefinite). Let (λ, v) be a pair (eigenvalue, eigenvector):

$$(D\lambda^2 + F\lambda + H)v = 0.$$

Let $\lambda = \gamma + a$, with a being real and non-negative; then,

$$\begin{aligned} \left(D\gamma^2 + (F + 2aD)\gamma + H + aF + a^2 D\right) v &= 0, \\ &\downarrow \\ (\hat{D}(a)\gamma^2 + \hat{F}(a)\gamma + \hat{H}(a))v &= 0, \end{aligned}$$

i.e. (γ, v) is a pair (eigenvalue, eigenvector) relative to the triple $\hat{D}(a), \hat{F}(a)$ and $\hat{H}(a)$, where $\hat{D}(a) = D, \hat{F}(a) = F + 2aD$ and $\hat{H}(a) = H + aF + a^2 D$.

Since D is positive definite, there exist $\bar{a} > 0$ such that $\hat{D}(\bar{a}), \hat{F}(\bar{a})$ and $\hat{H}(\bar{a})$ are positive definite. Since, by assumption, $\hat{H}(0) = H$ has at least one negative eigenvalue, and $\hat{H}(\bar{a})$ has all positive eigenvalues since the eigenvalues of $\hat{H}(a)$ are continuous functions of a, there exists $a^* \in (0, \bar{a})$ such that $\hat{H}(a^*)$ has at least one eigenvalue equal to zero, i.e. $\det(\hat{H}(a^*)) = 0$; this means that $\det(\hat{D}(a^*)\gamma^2 + \hat{F}(a^*)\gamma + \hat{H}(a^*))$ has at least one root equal to 0 (it easy to see that $\left[\det(\hat{D}(a^*)\gamma^2 + \hat{F}(a^*)\gamma + \hat{H}(a^*))\right]_{\gamma=0} = \det(\hat{H}(a^*)) = 0$), which means that $\det(D\lambda^2 + F\lambda + H)$ has at least one eigenvalue equal to $\lambda = a^* > 0$. □□□

The following lemma gives a necessary and sufficient condition for the impedance matrix of the mechanical system to be positive real.

Lemma 2.2.5. If $D > 0$, $F \geq 0$ and the stabilizability and detectability conditions (2.11) and (2.12) hold, then $B^T(D\ s + F + H\ \frac{1}{s})^{-1}B$ is positive real if and only if $H \geq 0$.

Proof. Assume that $H \geq 0$. Since $\det(D\ s^2 + F\ s + H)$ is not the null function (remember that $D > 0$), it is well known that $(D\ s + F + H\ \frac{1}{s})^{-1}$ is positive real if and only if $D\ s + F + H\ \frac{1}{s}$ is positive real. Setting $s = a + ib$, $a \geq 0$, we have

$$\mathrm{Re}(D\ s + F + H\ \frac{1}{s}) = aD + F + \frac{a}{a^2 + b^2}H,$$

which is a linear combination with non-negative coefficients of positive semidefinite matrices, and therefore it is positive semidefinite. Finally, if $(D\ s + F + H\ \frac{1}{s})^{-1}$ is positive semidefinite, then $B^T(D\ s + F + H\ \frac{1}{s})^{-1}B$ is positive semidefinite for any real B.

Assume that $B^T(D\ s + F + H\ \frac{1}{s})^{-1}B$ is positive real. By absurd, assume there exists at least one negative eigenvalue of matrix H. By Theorem 1, there exists at least one positive root of $\det(D\ s^2 + D\ s + H)$, which (by the stabilizability and observability conditions) is also a pole of $B^T(D\ s + F + H\ \frac{1}{s})^{-1}B$, which is a contradiction since it cannot be positive real. □□□

As for the possibility of rendering matrix H_p positive definite through an appropriate choice of K, a necessary and sufficient condition is given by the following lemma (whose proof can be found in [60]).

Lemma 2.2.6. The matrix H_p in (2.14) can be rendered positive definite with a suitable choice of K if and only if the matrix $H_{mm} \in \mathbb{R}^{(n-m)\times(n-m)}$ obtained by removing the first m rows and columns of H is positive definite. Moreover, if $H_{mm} > 0$, then there exists $\bar{K} \geq 0$ such that $H_p > 0$ for all $K > \bar{K}$.

2.3 The $(p, q)-$block decoupling problem

In this section, the $(p, q)-$block decoupling problem is dealt with; the results obtained here will be used in the rest of the chapter to design a controller solving Problem 1.

Let us formally define a block decoupled matrix as follows:

Definition 2.3.1. An $m \times m$ admittance matrix is said to be (p, q)-block decoupled if it is of the form blockdiag(Y_1, Y_2), where Y_1 and Y_2 are, respectively, $p \times p$ and $q \times q$ matrices and $p + q = m$.

Now, in order to design a controller solving Problem 1, the (p, q)-block decoupling problem is dealt with first. Consider the pictorial representation of the given mechanical system as a non-directed graph having $n + 1$ vertices, one for each body-mass and one for the ground, and one edge for each spring and damper. The following assumption can be made without loss of generality.

Assumption 1. *The graph associated with the given mechanical system is connected.*

Denote by $\bar{M}_1$ the system constituted by the first p actuated bodies and by the springs and dampers connecting such bodies with each other, and by

$\bar{M}_2$ the system constituted by the latter $q = m - p$ actuated bodies and by the springs and dampers connecting such bodies with each other.

Denote by $\mathcal{X}_1$ the set of the vertices (masses) that are connected by a path of the graph with the system $\bar{M}_1$ (that is, with a body in $\bar{M}_1$), after removing the vertices corresponding to the bodies in $\bar{M}_2$ and the ground, and all the edges connecting such vertices. Symmetrically, define $\mathcal{X}_2$ by removing $\bar{M}_1$, the ground and the relevant edges. Let $S_{12} = \mathcal{X}_1 \cap \mathcal{X}_2$, $S_1 = \mathcal{X}_1 \setminus \{\mathcal{X}_1 \cap S_{12}\}$ and $S_2 = \mathcal{X}_2 \setminus \{\mathcal{X}_2 \cap S_{12}\}$. Let n_1, n_2 and n_3 be the cardinalities of S_1, S_2 and S_{12}, respectively.

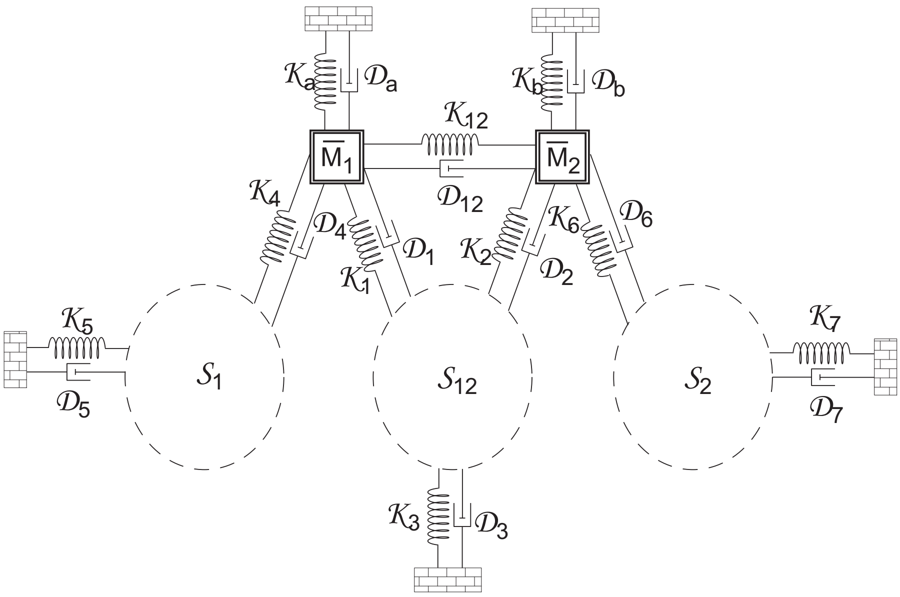

Figure 2.4: Decomposition of the given system. For space reasons springs and dampers are depicted in different directions, but the reader should imagine all the motions as horizontal.

In this way, the n degrees of freedom of the given system can be partitioned into 5 sets, represented pictorially in Figure 2.4, with $n = n_1 + n_2 + n_3 + m$. In Figure 2.4, the spring labeled by $\mathcal{K}_1$ represents a set of springs with possibly different coefficients of elasticity, each one connecting a different mass of the set S_{12} with $\bar{M}_1$ ($\mathcal{K}_1$ can be understood as the vector of such coefficients of elasticity); the same happens for the springs labeled by $\mathcal{K}_2, ..., \mathcal{K}_7, \mathcal{K}_a, \mathcal{K}_D$ and the dampers labeled by $\mathcal{F}_1, ..., \mathcal{F}_7, \mathcal{F}_a, \mathcal{F}_D$. Furthermore, not all such springs and dampers need to be actually present, since the case when a spring is missing can be considered by letting its coefficient of elasticity be equal to zero, and similarly for the dampers. However, in order to be consistent with the definition of S_1, S_2 and S_{12}, for each $i \in \{1, 2, 4, 6\}$ either $\mathcal{F}_i$ or $\mathcal{K}_i \neq 0$.

The controller proposed to solve the (p, q)-block decoupling problem is a n_c-degrees of freedom mechanical system, with $n_c = n_3 + m$, constituted by a copy of the masses in $\bar{M}_1$ and $\bar{M}_2$, whose coordinates will be denoted by $q_{c,1}$ and $q_{c,2}$, respectively, and all the masses contained in the set S_{12}, with (i) a copy of all the springs and dampers that in the given system connect such masses with each other and with the ground, (ii) m additional dampers having damping coefficient $D > 0$ connecting the bodies with coordinates $q_{c,1}$ and $q_{c,2}$ with the ground and (iii) m additional springs with sufficiently high coefficient of elasticity K connecting the same m bodies with the ground. Such a coefficient of elasticity is to be chosen (as it will be clear in the proof) to guarantee the asymptotic stability of the overall system. Moreover, q speed reducers characterized by $r = -1$ are to be used to connect the bodies having coordinates $q_{c,2}$ with the corresponding bodies in $\bar{M}_2$, whereas the bodies having coordinates $q_{c,1}$ are to be glued with the corresponding ones in $\bar{M}_1$. In this way, the matrix R used in the description of the controller is $R = \text{blockdiag}(I_p, -I_q)$. In order to prove the effectiveness of the proposed controller, let Σ_1 denote the $p \times p$ MIMO mechanical system obtained from the given one by fixing to the ground the masses in $\bar{M}_2$, and removing the masses in S_2 and all the springs and dampers directly connected with the removed masses so to obtain a system with $n_1 + n_3 + p$ degrees of freedom, whose inputs and outputs are denoted by u_1 and y_1, respectively. Symmetrically, define Σ_2 (a $q \times q$ MIMO system) by fixing the masses in $\bar{M}_1$ and removing all the masses in S_1, with the relevant springs and dampers, so to obtain a system with $n_2 + n_3 + q$ degrees of freedom, whose inputs and outputs are denoted by u_2 and y_2, respectively.

The following result gives sufficient conditions for solving the (p,q)-block decoupling problem.

Theorem 2. *Under Assumption 1, if (i) the matrix H_{mm} defined as in Lemma 2.2.6 is positive definite, (ii) Σ_1 and Σ_2 are reachable, then there exists $\bar{K} \geq 0$ such that for each $K > \bar{K}$ the mechanical parallel connection of the given system with the proposed controller is asymptotically stable and (p,q)-block decoupled.*

Proof. Consider the admittance matrix

$$Y(s) = \begin{bmatrix} Y_{11}(s) & \cdots & Y_{1m}(s) \\ \vdots & \ddots & \vdots \\ Y_{m1}(s) & \cdots & Y_{mm}(s) \end{bmatrix} =: \begin{bmatrix} Y_{11}(s) & Y_{12}(s) \\ Y_{12}^T(s) & Y_{22}(s) \end{bmatrix}$$

of the given system and the admittance matrix

$$Y_c(s) = \begin{bmatrix} Y_{c,11}(s) & \cdots & Y_{c,1m}(s) \\ \vdots & \ddots & \vdots \\ Y_{c,m1}(s) & \cdots & Y_{c,mm}(s) \end{bmatrix} =: \begin{bmatrix} Y_{c,11}(s) & Y_{c,12}(s) \\ Y_{c,12}^T(s) & Y_{c,22}(s) \end{bmatrix}$$

of the controller, where $Y_{11}(s)$ and $Y_{c,11}(s)$ are $p \times p$ matrices and $Y_{22}(s)$ and $Y_{c,22}(s)$ are $q \times q$ matrices. Considering the meaning of the first row block of $Y(s)$, it follows that $Y_{12}(s)$ is the transfer matrix from the velocity $v_2(s)$ (the vector of the velocities of the bodies in $\bar{M}_2$), to the force $u_1(s)$ (the vector of the forces acting on the bodies in $\bar{M}_1$), when the p bodies in $\bar{M}_1$ are rigidly fixed to the ground. Hence, looking at the pictorial representation in Figure 2.4, it is clear that $Y_{12}(s)$ is due only to the springs and dampers possibly connecting the masses in $\bar{M}_1$ with those in $\bar{M}_2$ and to the masses belonging to S_{12}, their interconnections, and the springs and dampers connecting them with the masses in $\bar{M}_1$ and in $\bar{M}_2$ and with the ground. Such a subsystem is exactly replicated in the controller, and, due to the speed reducers with

$r = -1$ used to connect the q masses belonging to the subsystem $\bar{M}_2$ of the controller, it follows that

$$Y_{c,12}(s) = -Y_{12}(s).$$

This shows that the mechanical parallel connection of the system and the controller has a (p, q)-block decoupled admittance matrix, since $Y_p(s) = Y(s) + Y_c(s)$. In order to show that the overall system is asymptotically stable, consider the following slight modification of the overall system that is totally equivalent to the proposed one. Rather than connecting the m bodies having coordinates $q_{c,1}$ and $q_{c,2}$ with the ground by means of m springs having coefficient of elasticity K, it is possibile to use $2m$ springs having coefficient of elasticity $\frac{K}{2}$, m of them connecting the bodies with coordinates $q_{c,1}$ and $q_{c,2}$ with the ground, and the other m connecting the bodies with coordinates q_1 and q_2 with the ground. It can be seen that the proposed modification amounts to have a modified system with admittance matrix

$$\bar{Y}(s) = Y(s) + \begin{bmatrix} \frac{K}{2} I_p & 0 \\ 0 & \frac{K}{2} I_q \end{bmatrix} = \begin{bmatrix} \bar{Y}_{11}(s) & Y_{12}(s) \\ Y_{12}^T(s) & \bar{Y}_{22}(s) \end{bmatrix}$$

and a modified controller with admittance matrix

$$\bar{Y}_c(s) = Y_c(s) - \begin{bmatrix} \frac{K}{2} I_p & 0 \\ 0 & \frac{K}{2} I_q \end{bmatrix} = \begin{bmatrix} \bar{Y}_{c,11}(s) & Y_{c,12}(s) \\ Y_{c,12}^T(s) & \bar{Y}_{c,22}(s) \end{bmatrix}$$

By computing the potential energy of the overall system, it is clear that this modification, which will be helpful in the rest of the proof, is irrelevant; hence, the asymptotic stability of $Y_p(s)$ will be proven for the modified system. Now, assume that the overall system is written in the form (2.1), with matrices D_p, F_p and H_p replacing matrices D, F and H, and with the overall state vector given by $q_p = \begin{bmatrix} q_1^T & q_2^T & q_3^T & q_{3c}^T & q_4^T & q_5^T \end{bmatrix}^T \in \mathbb{R}^{n+n_3}$, $q_1 \in \mathbb{R}^p$, $q_2 \in \mathbb{R}^q$, $q_3 \in \mathbb{R}^{n_3}$, $q_{3c} \in \mathbb{R}^{n_3}$, $q_4 \in \mathbb{R}^{n_1}$, $q_5 \in \mathbb{R}^{n_2}$, with q_1, q_2, q_3, q_4 and q_5 being the coordinates of the bodies in $\bar{M}_1$, $\bar{M}_2$, S_{12}, S_1, and S_2, respectively, and q_{3c} the coordinates of the bodies that constitute the controller, apart from the m

having coordinates $q_{c,1}$ and $q_{c,2}$ (such coordinates disappear from the overall description since $q_{c,1} = q_1$ and $q_{c,2} = -q_2$). Matrix H_p has the following form:

$$H_p = \begin{bmatrix} 2H_1 + KI_p & 0 & H_{13} & H_{13} & H_{14} & 0 \\ 0 & 2H_2 + KI_q & H_{23} & -H_{23} & 0 & H_{25} \\ H_{13}^T & H_{23}^T & H_3 & 0 & 0 & 0 \\ H_{13}^T & -H_{23}^T & 0 & H_3 & 0 & 0 \\ H_{14}^T & 0 & 0 & 0 & H_4 & 0 \\ 0 & H_{25}^T & 0 & 0 & 0 & H_5 \end{bmatrix}, \tag{2.15}$$

where $H_1 \in \mathbb{R}^{p\times p}$, $H_2 \in \mathbb{R}^{q\times q}$, $H_3 \in \mathbb{R}^{n_3\times n_3}$, $H_4 \in \mathbb{R}^{n_1\times n_1}$, $H_5 \in \mathbb{R}^{n_2\times n_2}$. It is clear that if the matrix H_{mm} of the given system,

$$H_{mm} = \text{blockdiag}(H_3, H_4, H_5),$$

is positive definite as guaranteed by condition (i), then the corresponding matrix for the parallel connection, given by

$$H_{p,mm} = \text{blockdiag}(H_3, H_3, H_4, H_5),$$

is definite positive too; hence, by Lemma 2.2.6, a coefficient $K = K_0$ can be chosen sufficiently high so to guarantee that $H_p > 0$. Thus, with a choice of a coefficient K greater than or equal to K_0, the asymptotic stability of the parallel connection can be proven through Lemmas 2.2.4 and 2.2.5, by showing that the overall system is stabilizable and detectable. In the following, it will be shown that, under condition (ii), the overall system is actually reachable and observable. First, note that the system $\bar{\Sigma}_1$, obtained by adding to Σ_1 p springs with coefficient of elasticity $\frac{K}{2}$ that connect the masses belonging to $\bar{M}_1$ with the ground is reachable if and only if Σ_1 is reachable. This can be easily proved as follows.

Denote by D_{Σ_1}, F_{Σ_1}, H_{Σ_1} and B_{Σ_1} the matrices used in the description of Σ_1 and by $D_{\bar{\Sigma}_1}$, $F_{\bar{\Sigma}_1}$, $H_{\bar{\Sigma}_1}$ and $B_{\bar{\Sigma}_1}$ those used in the description of $\bar{\Sigma}_1$. It follows that $D_{\Sigma_1} = D_{\bar{\Sigma}_1}$, $F_{\Sigma_1} = F_{\bar{\Sigma}_1}$, $H_{\Sigma_1} = H_{\bar{\Sigma}_1} + \frac{K}{2}I_p$ and $B_{\Sigma_1} = B_{\bar{\Sigma}_1} = \begin{bmatrix} I_p & 0 \end{bmatrix}$.

The following holds:

$$\begin{aligned}
&\operatorname{rank}\left(\left[\begin{array}{cc} D_{\bar\Sigma_1}\ s^2+F_{\bar\Sigma_1}\ s+H_{\bar\Sigma_1} & B_{\bar\Sigma_1}\end{array}\right]\right)\\
=\ &\operatorname{rank}\left(\begin{array}{c}\left[\begin{array}{cc} D_{\Sigma_1}\ s^2+F_{\Sigma_1}\ s+H_{\Sigma_1} & B_{\Sigma_1}\end{array}\right]\cdot\\ \cdot\left[\begin{array}{cc} I & 0\\ \frac{K}{2}B_{\Sigma_1}^T & I_p\end{array}\right]\end{array}\right)\\
=\ &\operatorname{rank}\left(\left[\begin{array}{cc} D_{\Sigma_1}\ s^2+F_{\Sigma_1}\ s+H_{\Sigma_1} & B_{\Sigma_1}\end{array}\right]\right)
\end{aligned}$$

Note also that, in view of Remark 2.2.2, system $\bar\Sigma_1$ is also observable if its matrix $H_{\bar\Sigma_1}$ is non-singular. By using the same symbols as above, the matrix $H_{\bar\Sigma_1}$ is given by

$$H_{\bar\Sigma_1}=\left[\begin{array}{ccc} H_1+\frac{K}{2}I_p & H_{13} & H_{14}\\ H_{13}^T & H_3 & 0\\ H_{14}^T & 0 & H_4\end{array}\right],$$

which, by Shur complements, is positive definite if the following relation is satisfied:

$$H_1+\frac{K}{2}I_p-H_{13}H_3^{-1}H_{13}^T-H_{14}H_4^{-1}H_{14}^T>0 \tag{2.16}$$

(note that the term on the left hand side is a $p\times p$ matrix).

By using Shur complements on H_p, which is positive definite if $K=K_0$, one obtains

$$2H_1+K_0I_p-H_{13}H_3^{-1}H_{13}^T-H_{14}H_4^{-1}H_{14}^T>0, \tag{2.17}$$

$$2H_2+K_0I_q-H_{23}H_3^{-1}H_{23}^T-H_{25}H_5^{-1}H_{25}^T>0. \tag{2.18}$$

It can be shown that, by choosing a value for the coefficient K according to the following inequality,

$$K\geq 2K_0+2\max\{\lambda_{\max}(H_1),\lambda_{\max}(H_2)\},$$

the matrices $H_{\bar\Sigma_1}$ and $H_{\bar\Sigma_2}$ can be made positive definite (and, consequently, non-singular).

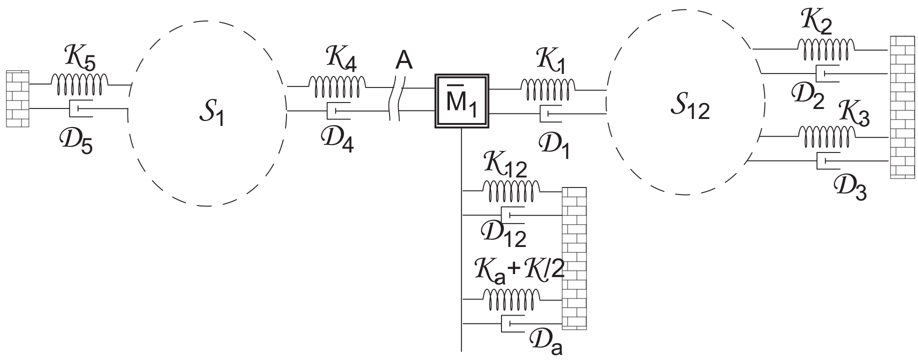

Figure 2.5: The subsystem $\bar{\Sigma}_1$ considered in the proof of Theorem 2.

Now, consider the pictorial representation of $\bar{\Sigma}_1$ reported in Figure 2.5. It is clear that the admittance of $\bar{\Sigma}_1$, which coincides with $\bar{Y}_{11}(s)$, can be decomposed as $\bar{Y}_{11}(s) = \bar{Y}_{11,R}(s) + \bar{Y}_{11,L}(s)$, where $\bar{Y}_{11,R}(s)$ corresponds to the forces that are due to the inertia of the masses belonging to $\bar{M}_1$and all the masses in S_{12} (the part of the system on the right of point A in Figure 2.5), whereas $\bar{Y}_{11,L}(s)$ corresponds to the forces due to the masses in S_1 (the part of the system on the left of point A in Figure 2.5). The term $\bar{Y}_{11,R}(s)$, which is the term duplicated by the proposed (modified) controller, is a non-proper rational matrix, since it contains the inertia of the masses belonging to $\bar{M}_1$, whereas $\bar{Y}_{11,L}(s)$ is a strictly proper rational matrix; the proofs of these facts can be made easily through the electric circuit analog to the mechanical system (see [48]) or by simple algebraic manipulations in the Laplace domain. Obviously, a wholly similar decomposition $\bar{Y}_{22}(s) = \bar{Y}_{22,R}(s) + \bar{Y}_{22,L}(s)$ holds for the admittance of $\bar{\Sigma}_2$, where $\bar{Y}_{22,R}(s)$ corresponds to S_{12} e $\bar{Y}_{22,L}(s)$ to S_2. The admittance matrix of the parallel connection is given by:

$$\begin{aligned}\bar{Y}_p(s) &= \begin{bmatrix} 2\bar{Y}_{11,R}(s) + \bar{Y}_{11,L}(s) & 0 \\ 0 & 2\bar{Y}_{22,R}(s) + \bar{Y}_{22,L}(s) \end{bmatrix} \\ &=: \begin{bmatrix} Y_{p,1}(s) & 0 \\ 0 & Y_{p,2}(s) \end{bmatrix}\end{aligned}$$

In view of the block diagonal structure of $\bar{Y}_p(s)$, it can be shown that, under the hypothesis that the systems represented by the admittance matrices $Y_{p,1}(s)$ and $Y_{p,2}(s)$ are irreducible (i.e. reachable and observable, see [43]), the overall system is irreducible. To prove this, let $Y_{p,1}(s)$ be represented by the following polynomial matrix description (PMD):

$$\begin{bmatrix} P_1(s) & Q_1(s) \\ -R_1(s) & W_1(s) \end{bmatrix} \begin{bmatrix} \xi_1(s) \\ -v_1(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -u_1(s) \end{bmatrix}$$

where $\xi_1(s)$ is a pseudostate for $Y_{p,1}(s)$, $v_1(s)$ is the vector of velocities of the masses belonging to $\bar{M}_1$ and $u_1(s)$ the vector of the forces acting on them. Since $Y_{p,1}(s)$ is irreducible, the matrices $P_1(s)$ and $Q_1(s)$ are left coprime and the matrices $R_1(s)$ and $P_1(s)$ are right coprime. A similar representation can be found for $Y_{p,2}(s)$. The PMD of the overall system is the following:

$$\begin{bmatrix} P_1(s) & 0 & Q_1(s) & 0 \\ 0 & P_2(s) & 0 & Q_2(s) \\ -R_1(s) & 0 & W_1(s) & 0 \\ 0 & -R_2(s) & 0 & W_2(s) \end{bmatrix} \begin{bmatrix} \xi_1(s) \\ \xi_2(s) \\ -v_1(s) \\ -v_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -u_1(s) \\ -u_2(s) \end{bmatrix}$$

The matrix

$$\begin{bmatrix} P_1(s) & 0 & Q_1(s) & 0 \\ 0 & P_2(s) & 0 & Q_2(s) \end{bmatrix}$$

has full row rank for all $s \in \mathbb{C}$, since its rank is equal to the sum of ranks of the two block rows, which are full by hypothesis. Consequently, the overall system is reachable. In a similar way, it can be shown that the overall system is also observable, thus proving that the overall system is irreducible.

It remains to show that the systems represented by $Y_{p,1}(s) = 2\bar{Y}_{11,R}(s) + \bar{Y}_{11,L}(s)$ and $\bar{Y}_{22}(s) = 2\bar{Y}_{22,R}(s) + \bar{Y}_{22,L}(s)$ are irreducible, under the hypothesis that $\bar{\Sigma}_1$ and $\bar{\Sigma}_2$ are irreducible. The proof will be outlined only for $Y_{p,1}(s)$, because the other case is analogous.

Since the admittance matrix of $\bar{\Sigma}_1$ is $\bar{Y}_{11,R}(s) + \bar{Y}_{11,L}(s)$, it can be represented as the parallel connection of the two PMDs that represent $\bar{Y}_{11,R}(s)$ and

$\bar{Y}_{11,L}(s)$. These PMDs are, respectively,

$$\begin{bmatrix} P_{1,L}(s) & Q_{1,L}(s) \\ -R_{1,L}(s) & 0 \end{bmatrix} \begin{bmatrix} \xi_{1,L}(s) \\ -v_1(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -u_{1,L}(s) \end{bmatrix}$$

$$\begin{bmatrix} P_{1,R}(s) & Q_{1,R}(s) \\ -R_{1,R}(s) & W_{1,R}(s) \end{bmatrix} \begin{bmatrix} \xi_{1,R}(s) \\ -v_1(s) \end{bmatrix} = \begin{bmatrix} 0 \\ -u_{1,R}(s) \end{bmatrix}$$

where $\xi_{1,L}(s)$ and $\xi_{1,R}(s)$ are the pseudostates of $\bar{Y}_{11,R}(s)$ and $\bar{Y}_{11,L}(s)$ and $u_{1,L}(s)$ and $u_{1,R}(s)$ are the vectors of forces acting on the masses belonging to $\bar{M}_1$. The element $(2, 2)$ of the first PMD is 0 because $\bar{Y}_{11,L}(s)$ is a strictly proper rational matrix. The PMD of the parallel connection (i.e. of $\bar{\Sigma}_1$) is:

$$\begin{bmatrix} P_{1,L}(s) & 0 & Q_{1,L}(s) \\ 0 & P_{1,R}(s) & Q_{1,R}(s) \\ -R_{1,L}(s) & -R_{1,R}(s) & W_{1,R}(s) \end{bmatrix} \begin{bmatrix} \xi_{1,L}(s) \\ \xi_{1,R}(s) \\ -v_1(s) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -(u_{1,L}(s) + u_{1,R}(s)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -u_1(s) \end{bmatrix}$$

where $u_1(s) = u_{1,L}(s) + u_{1,R}(s)$ is the total force applied on the masses belonging to $\bar{M}_1$. Since $\bar{\Sigma}_1$ is irreducible, the matrix

$$\begin{bmatrix} P_{1,L}(s) & 0 & Q_{1,L}(s) \\ 0 & P_{1,R}(s) & Q_{1,R}(s) \end{bmatrix} \tag{2.19}$$

has full row rank for all $s \in \mathbb{C}$.

Now, a PMD for the system $Y_{p,1}(s)$ is given by:

$$\begin{bmatrix} P_{1,L}(s) & 0 & Q_{1,L}(s) \\ 0 & 2P_{1,R}(s) & Q_{1,R}(s) \\ -R_{1,L}(s) & -R_{1,R}(s) & 2W_{1,R}(s) \end{bmatrix} \begin{bmatrix} \xi_{1,L}(s) \\ \xi_{1,R}(s) \\ -v_1(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -u_1(s) \end{bmatrix}$$

and the matrix

$$\begin{bmatrix} P_{1,L}(s) & 0 & Q_{1,L}(s) \\ 0 & 2P_{1,R}(s) & Q_{1,R}(s) \end{bmatrix} = \begin{bmatrix} P_{1,L}(s) & 0 & Q_{1,L}(s) \\ 0 & P_{1,R}(s) & Q_{1,R}(s) \end{bmatrix} \cdot \begin{bmatrix} I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & I \end{bmatrix}$$

has the same rank of the matrix in (2.19), for all $s \in \mathbb{C}$. Thus, the system represented by $Y_{p,1}(s)$ is reachable. A similar argument can be used to prove the observability of such a system, thus showing its irreducibility. Analogously, it can be shown that also the system represented by $Y_{p,2}(s)$ is irreducible, thus completing the proof of the asymptotic stability. □□□

2.4 Main result

In order to obtain a full input-output decoupled system, the block decoupling operation described in Section 3.4.3 has to be iterated on the remaining admittance submatrices. The latter theorem gives only sufficient conditions to perform the first block decoupling operation and gives no information about what could happen if such an operation is iterated on the remaining submatrices. Therefore, in the following it will be stated that, if at each block decoupling step the obtained system satisfies some sufficient conditions, then it will be possible to iterate the process in order to obtain a full input-output decoupled system.

Remark 2.4.1. After an admittance matrix has been made (p, q)-block decoupled, the $p \times p$ admittance submatrix on the main diagonal can be seen as the admittance matrix of the system in which the last q inputs have been zeroed, (that is, the last q actuated bodies have been fixed to the ground) and the masses connected by a path only to the last q fixed bodies have been removed. Obviously, a symmetrical result holds for the $q \times q$ admittance submatrix.

In the following, an algorithm will be presented which, under some conditions, can be used to design a controller that gives an overall parallel connection which input-output decoupled and asymptotically stable. After, its effectiveness will be shown.

Algorithm 1. 1. Let $i = 1$ and $\mathfrak{I}_i = \{1, \ldots, m\}$. Let Sys_i be the given system. Let $m_i = m$.

2. If $i = m$ then STOP (the system is input-output block decoupled). Otherwise, choose an actuated body M_{j_i}, where $j_i \in \mathfrak{I}_i$.

3. Let $\bar{M}_2$ be M_{j_i}. Let $\bar{M}_1$ be constituted by the remaining actuated bodies of Sys_i. In this way, $p = 1$ and $q = m_i - i$.

4. If the system Sys_i satisfies the hypotheses of Theorem 2 (with m_i replacing m), then perform the $(1, m_i - i)$-block decoupling operation on the system Sys_i and go to step 6). Otherwise go to step 5).

5. Choose another actuated body for M_{j_i}, with $j_i \in \mathfrak{I}_i$, and go to step 3). If there are no more different bodies that can be chosen, then FAIL.

6. Consider the system obtained from the parallel connection of the system Sys_i and the controller designed at step 4) by fixing to the ground the chosen M_{j_i} and removing all the masses connected by a path only to M_{j_i} (and the relevant springs and dampers). Let Sys_{i+1} be such system and $\mathfrak{I}_{i+1} = \mathfrak{I}_i \setminus \{j_i\}$. Let $m_{i+1} = m_i - 1$. Let $i \leftarrow i + 1$. Go to step 2.

The strategy at the basis of the proposed procedure consists of isolating, one by one, all the actuated bodies, and trying to find a way that makes possible to apply Theorem 2 at each step, thus obtaining, as the overall controller, the mechanical parallel connection of $m - 1$ subcontrollers, designed as described in Theorem 2. The following proposition, [54], states the effectiveness of the proposed algorithm.

Theorem 3. *If Algorithm 1 terminates with STOP, the resulting overall system is input-output decoupled and asymptotically stable.*

The proposition will be proved by induction on the number of already decoupled input-output pairs.

Basis. To show: After choosing M_1, if the given system satisfies the conditions of Theorem 2, the parallel connection of the given system and the first designed controller is $(1, m - 1)$-block decoupled and asymptotically stable.

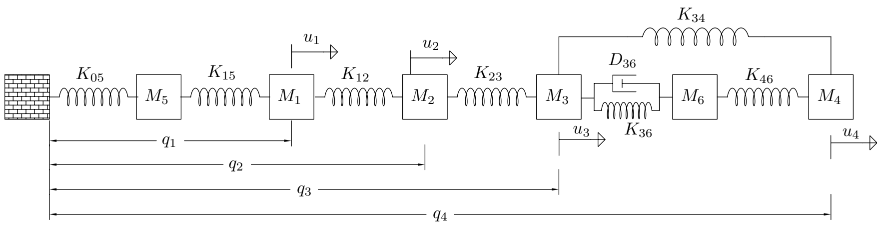

Figure 2.6: The mechanical system considered in Example 2.

Proof.

The proof of the Basis clause is a trivial application of Theorem 2.

Step. Assume: after having performed $i < m-1$ block decoupling operations through the parallel connection of i subcontrollers, the obtained system is asymptotically stable and is represented by the admittance matrix

$$\bar{Y}_p^i(s) = \begin{bmatrix} Y_{p,1}(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & Y_{p,i}(s) & 0 \\ 0 & \cdots & 0 & \bar{Y}_{p,i+1}^i(s) \end{bmatrix},$$

where $\bar{Y}_{p,i+1}^i(s)$ is a $(m-i) \times (m-i)$ matrix (the superscript i denotes the step number in the decoupling process).

To show: if it is possible, as described in Algorithm 1, to choose a new actuated body in a way that Theorem 2 can be applied, then the parallel connection of the system described in the Assume clause and the $(i+1)$-th designed subcontroller is asymptotically stable and is represented by the admittance matrix

$$\bar{Y}_p^{i+1}(s) = \begin{bmatrix} Y_{p,1}(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & Y_{p,i+1}(s) & 0 \\ 0 & \cdots & 0 & \bar{Y}_{p,i+2}^{i+1}(s) \end{bmatrix},$$

where $\bar{Y}_{p,i+2}^{i+1}(s)$ is a $(m-i-1) \times (m-i-1)$ matrix.

Consider the system obtained by fixing to the ground the i actuated bodies corresponding to the already decoupled input-output pairs and removing all

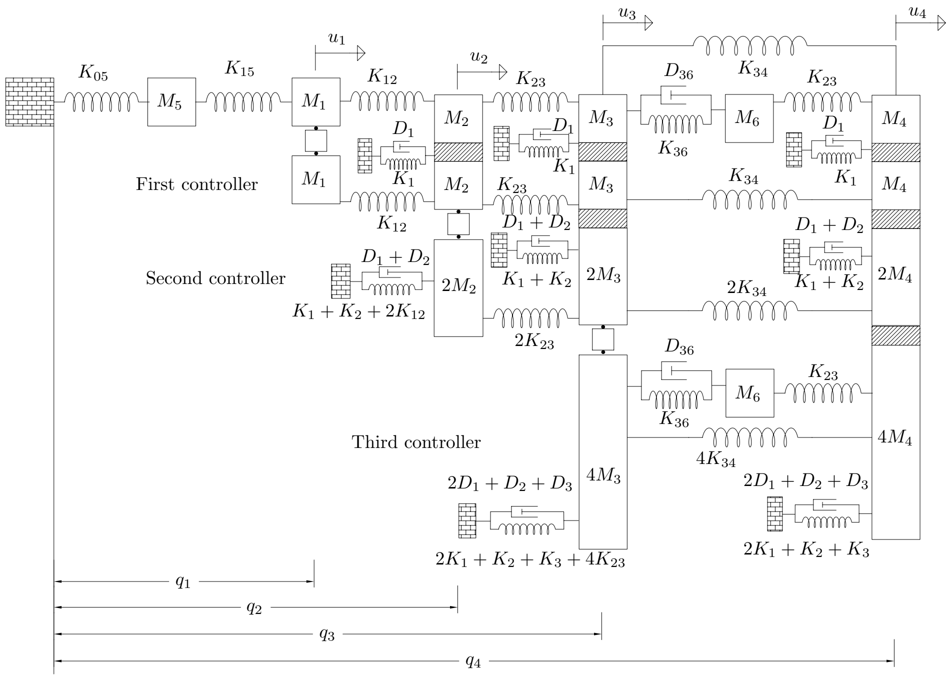

Figure 2.7: The overall mechanical system considered in Example 2.

the masses connected by a path only to such bodies, with the relevant springs and dampers. Such a system is represented by the admittance matrix $\bar{Y}^{i}_{p,i+1}(s)$. Since it is possible to choose an actuated body among the remaining $m-i$ in a way that it is possible to perform the block decoupling operation, the parallel connection of this system and the $(i+1)$-th controller will be asymptotically stable and its $(m-i) \times (m-i)$ admittance matrix will be

$$\bar{Y}^{i+1}_{p,i+1}(s) = \begin{bmatrix} Y_{p,i+1}(s) & 0 \\ 0 & \bar{Y}^{i+1}_{p,i+2}(s) \end{bmatrix}.$$

Therefore, the overall system will be represented by the following admittance matrix (where the dependence on s is omitted):

$$\bar{Y}^{i+1}_{p} = \begin{bmatrix} Y_{p,1} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & Y_{p,i} & 0 & \vdots \\ 0 & \cdots & 0 & Y_{p,i+1} & 0 \\ 0 & \cdots & \cdots & 0 & \bar{Y}^{i+1}_{p,i+2} \end{bmatrix}.$$

The asymptotic stability of such system follows easily and it is also clear that the latter admittance matrix is non-singular, thus completing the proof of the correctness of the proposed algorithm. □□□

Example 2. Consider the mechanical system depicted in Figure 2.6, where $n = 6$, $M_i > 0$, $F_{36} > 0$, and $K_{i,j} > 0$. At step 1, j_1 is chosen to be 1. Therefore, the relevant sets for the design of the first controller are $S_{12} = \{\}$, $S_1 = \{M_5\}$, $S_2 = \{M_6\}$. The first controller is characterized by $n_{c,1} = 4$ and by the values K_1 and F_1 for the spring and damper. At step 2, $j_2 = 2$, and the sets for the design of the second controller are $S_{12} = \{\}$, $S_1 = \{\}$, $S_2 = \{M_6\}$.The second controller is characterized by $n_{c,2} = 3$ and by K_2 and F_2. At step 3, $j_3 = 3$, and the sets are $S_{12} = \{M_6\}$, $S_1 = \{\}$, $S_2 = \{\}$. The third controller has $n_{c,3} = 3$ and K_3 and F_3 as spring and damper. The overall system, i.e., the mechanical parallel connection of the given system and the

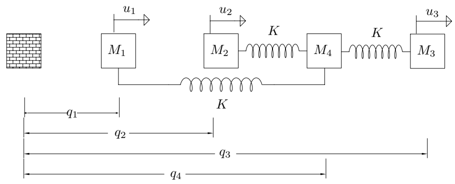

Figure 2.8: The system considered in Example 3.

three designed controllers is depicted in Figure 2.7. In the figure, the little boxes filled with oblique segments represent the glue that joins two masses together.

2.5 Further Considerations

The proposed algorithm gives sufficient conditions in order to obtain a full input-output decoupled system. But, as illustrated below, there exist many systems that, even not satisfying the conditions of the algorithm, can be decoupled using a different approach, as can be seen from the following example.

Example 3. Consider the system represented in Figure 2.8. The peculiarity of this system is represented by mass M_4 (which is not actuated), linked through springs to the three actuated bodies. It is easy to understand that there is no way to apply the algorithm. Indeed, after the first step, it is not possbile to choose a new actuated mass in such a way that the corresponding Σ_1 and Σ_2 subsystems are reachable. This is due to the fact that, in either way the new actuated body is chosen, there is always a Σ subsystem composed by two exact copies of masses and springs, that turns out to be unreachable. However, there are at least two different ways of decoupling such system. In the following, one of these strategies will be illustrated. Consider the mechanical parallel connection reported in Figure 2.9. The equations of the system are:

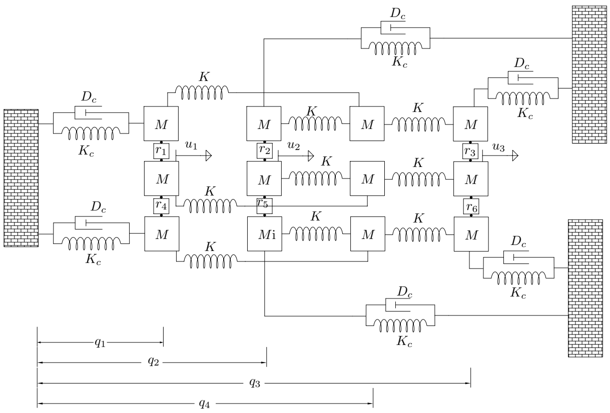

Figure 2.9: The parallel connection of the given system and the controller.

$$
\begin{aligned}
M\ddot{q}_1 + Kq_1 - Kq4 &= u_1 + \lambda_1 + \lambda_4 \\
M\ddot{q}_2 + Kq_2 - Kq4 &= u_2 + \lambda_2 + \lambda_5 \\
M\ddot{q}_3 + Kq_3 - Kq4 &= u_3 + \lambda_3 + \lambda_6 \\
M\ddot{q}_4 - Kq_1 - Kq_2 - Kq_3 + 3Kq_4 &= 0 \\
M\ddot{q}_{c,1} + F_c\dot{q}_{c,1} + (K + K_c)q_{c,1} - Kq_{c,4} &= -\frac{1}{r_1}\lambda_1 \\
M\ddot{q}_{c,2} + F_c\dot{q}_{c,2} + (K + K_c)q_{c,2} - Kq_{c,4} &= -\frac{1}{r_2}\lambda_2 \\
M\ddot{q}_{c,3} + F_c\dot{q}_{c,3} + (K + K_c)q_{c,3} - Kq_{c,4} &= -\frac{1}{r_3}\lambda_3 \\
M\ddot{q}_{c,4} - Kq_{c,1} - Kq_{c,2} - Kq_{c,3} + 3Kq_{c,4} &= 0 \\
M\ddot{q}_{F,1} + F_c\dot{q}_{F,1} + (K + K_c)q_{F,1} - Kq_{F,4} &= -\frac{1}{r_4}\lambda_4 \\
M\ddot{q}_{F,2} + F_c\dot{q}_{F,2} + (K + K_c)q_{F,2} - Kq_{F,4} &= -\frac{1}{r_5}\lambda_5 \\
M\ddot{q}_{F,3} + F_c\dot{q}_{F,3} + (K + K_c)q_{F,3} - Kq_{F,4} &= -\frac{1}{r_6}\lambda_6 \\
M\ddot{q}_{F,4} - Kq_{F,1} - Kq_{F,2} - Kq_{F,3} + 3Kq_{F,4} &= 0 \\
q_{c,1} &= r_1q_1 \\
q_{c,2} &= r_2q_2 \\
q_{c,3} &= r_3q_3 \\
q_{F,1} &= r_4q_1 \\
q_{F,2} &= r_5q_2 \\
q_{F,3} &= r_6q_3
\end{aligned}
$$

By eliminating the Lagrange multipliers, the closed loop system equations can be obtained:

$$\begin{aligned}
M(1+r_1^2+r_4^2)\ddot{q}_1 + F_c(r_1^2+r_4^2)\dot{q}_1 + (K+(r_1^2+r_4^2)(K+K_c))q_1 & \\
-Kq_4 - Kr_1q_{c,4} - Kr_4q_{F,4} &= u_1 \\
M(1+r_2^2+r_5^2)\ddot{q}_2 + F_c(r_2^2+r_5^2)\dot{q}_2 + (K+(r_2^2+r_5^2)(K+K_c))q_2 & \\
-Kq_4 - Kr_2q_{c,4} - Kr_5q_{F,4} &= u_2 \\
M(1+r_3^2+r_6^2)\ddot{q}_3 + F_c(r_3^2+r_6^2)\dot{q}_3 + (K+(r_3^2+r_6^2)(K+K_c))q_3 & \\
-Kq_4 - Kr_3q_{c,4} - Kr_6q_{F,4} &= u_3 \\
M\ddot{q}_4 - Kq_1 - Kq_2 - Kq_3 + 3Kq_4 &= 0 \\
M\ddot{q}_{c,4} - Kr_1q_1 - Kr_2q_2 - Kr_3q_3 + 3Kq_{c,4} &= 0 \\
M\ddot{q}_{F,4} - Kr_4q_1 - Kr_5q_2 - Kr_6q_3 + 3Kq_{F,4} &= 0
\end{aligned}$$

By letting $q(t) = [q_1(t), q_2(t), q_3(t), q_4(t), q_{c,4}(t), q_{F,4}(t)]^T$, the system can be described by means of Equations (2.1), (2.2), (2.3), with the following matrices:

$$D = \begin{bmatrix} M(1+r_1^2+r_4^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & M(1+r_2^2+r_5^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & (1+r_3^2+r_6^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix}$$

$$F = \begin{bmatrix} F_c(r_1^2+r_4^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & F_c(r_2^2+r_5^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & F_c(r_3^2+r_6^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} K_a & 0 & 0 & -K & -Kr_1 & -Kr_4 \\ 0 & K_b & 0 & -K & -Kr_2 & -Kr_5 \\ 0 & 0 & K_c & -K & -Kr_3 & -Kr_6 \\ -K & -K & -K & 3K & 0 & 0 \\ -Kr_1 & -Kr_2 & -Kr_3 & 0 & 3K & 0 \\ -Kr_4 & -Kr_5 & -Kr_6 & 0 & 0 & 3K \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $K_a = K + (r_1^2 + r_4^2)(K + K_c)$, $K_b = K + (r_2^2 + r_5^2)(K + K_c)$, $K_c = K + (r_3^2 + r_6^2)(K + K_c)$. In the following, for simplicity, it is assumed that $M = 1$ and $K = 1$. It turns out that the closed loop system, i.e. the mechanical parallel connection of the system and the controller, is reachable. This can be verified by computing the Smith form of the polynomial matrix $\begin{bmatrix} D\ s^2 + F\ s + H & B \end{bmatrix}$. Since it is $\begin{bmatrix} I & 0 \end{bmatrix}$, the system is reachable.

Using Equations (2.8), (2.9), (2.10), the admittance matrix of the mechanical parallel connection can be easily calculated. Assuming $M = 1$ and $K = 1$, $Y(s)$ turns out to be:

$$Y(s) = \begin{bmatrix} y_{11}(s) & -\frac{(r_2r_1+1+r_5r_4)}{(s^2+3)s} & -\frac{(r_3r_1+1+r_6r_4)}{(s^2+3)s} \\ -\frac{(r_2r_1+1+r_5r_4)}{(s^2+3)s} & y_{22}(s) & -\frac{(1+r_3r_2+r_6r_5)}{(s^2+3)s} \\ -\frac{(r_3r_1+1+r_6r_4)}{(s^2+3)s} & -\frac{(1+r_3r_2+r_6r_5)}{(s^2+3)s} & y_{33}(s) \end{bmatrix}$$

By imposing that the elements outside the main diagonal must be zero for all s, an algebraic system of three nonlinear equations in the unknowns $r_1, \ldots r_6$ is obtained.

A solution to this system is the following one:

$$\begin{aligned} r_1 &= -\frac{r_3(r_5 - r_6)}{r_6 + r_5 r_6^2 + r_5 r_3^2} \\ r_2 &= -\frac{1 + r_6 r_5}{r_3} \\ r_3 &= free \\ r_4 &= -\frac{1 + r_6 r_5 + r_3^2}{r_6 + r_5 r_6^2 + r_5 r_3^2} \\ r_5 &= free \\ r_6 &= free \end{aligned}$$

The values of the free unknowns have to be chosen on a way that all the speed reducer ratios are different from zero.

By choosing $r_1 = -\frac{2}{7}$, $r_2 = -\frac{3}{2}$, $r_3 = 1$, $r_4 = -\frac{10}{7}$, $r_5 = 1$, $r_1 = \frac{1}{2}$ the overall system turns out to be input-output decoupled. Finally, it has to be verified if the overall system is asymptotically stable. It can be seen that, using the same values as above for the masses and springs, the characteristic polynomial of the closed loop system, i.e. $\det\big(D\ s^2 + F\ s + H\big)$, has all its roots in the open left half plane. Hence, the given system, represented in Figure 2.8, can be made input-output decoupled and asimptotically stable through interconnection with another mechanical system.

2.6 Conclusions and future work

In this chapter the problem of input-output decoupling has been dealt with for linear mechanical systems under the requirement that the controller is another mechanical system to be physically connected to the given one. The problem has been solved for m-inputs m-outputs systems, under some conditions on the structural properties of the system. Further work will be devoted to enlarge the class of m-inputs m-outputs systems for which a solution can be found.

Chapter 3

IDA–PBC of nonlinear mechanical systems

3.1 Introduction

Control design problems have been traditionally approached adopting a signal-processing viewpoint. That is, the plant to be controlled and the controller are viewed as signal-processing devices that transform certain input signals into outputs. An alternative to this perspective, which is particularly suited for physical applications, is to view dynamical systems as energy-transformation devices. The action of a controller may also be understood in energy terms as another dynamical system (typically implemented in a computer) interconnected with the process to modify its behavior. The control problem can then be recast as finding a dynamical system and an interconnection pattern such that the overall energy function takes the desired form. There are at least two important advantages of adopting an *energy-shaping* perspective to control: 1) Shaping the energy permits to deal with not just stabilization, but also performance objectives — the latter being, of course, the main concern in applications; 2) Practitioners are familiar with energy concepts, hence it can serve as a common language to facilitate the communication with control theorists, incorporating prior knowledge and providing physical interpretations to the control action. The idea of energy-shaping has its roots in the ground-breaking work [83] in robot manipulator control, where robust controllers are derived with simple potential energy shaping. The principle was later formalized in [66] by using the fundamental notion of passivity. In such a paper the term passivity-based control (PBC) was coined to define a controller design methodology whose aim is to render the closed-loop system passive with some desired storage function. During the early years of story of PBC, a lot of research effort has been devoted to the application of PBC to electrical and electromechanical systems, which requires shaping of the total energy, instead of just the potential component. In carrying out this extension two approaches were pursued: in the first one, closer to classical Lyapunov-based design, the storage function to be assigned at closed-loop is selected first, and then the controller that ensures this objective is designed. Extensive applications of this line of research may be found in [65]. A drawback of this approach was that the closed-loop storage functions (typically taken as quadratic in errors) are not energy functions in any meaningful physical sense. In order to overcome this problem, a new passivity-based control strategy called interconnection and damping assignment (IDA) was introduced, in which the closed-loop storage function is indeed an energy function, obtained as a result of the designer's choice of desired subsystems interconnection and damping. In [52] this approach is proposed for stability analysis, and the extension for controller design is reported in [69] and [70]. Since then many successful applications have been given of IDA–PBC, including mass-balance systems [62], electrical motors [75], power systems [29, 42, 30, 31], magnetic levitation systems [79, 72], and especially underactuated mechanical systems. As a matter of fact, also for such a class of systems stabilization cannot be achieved just by potential energy shaping. There are currently two main approaches to the problem: the method of controlled Lagrangians [14] and IDA–PBC. In both cases stabilization (of a desired equilibrium) is achieved by identifying the class of systems—Lagrangian for the first method and Hamiltonian for IDA–PBC—that can possibly be obtained via feedback. The conditions under which such a feedback law exists are called *matching conditions*, and consist of a set of nonlinear partial differential equations (PDEs). In case these PDEs can be solved the original control system and the target dynamic system are said to *match*. A lot of research effort has been devoted to the solution of the matching equations. In [14] the authors give a series of conditions on the system and the assignable inertia matrices such that the PDEs can be solved. Also, techniques to solve the PDEs have been reported in [10, 13] and some geometric aspects of the equations are investigated in [47]. The case of systems with underactuation degree one has been studied in detail in [11] and [2]. In the latter it is proved that, if the inertia matrix and the force induced by the potential energy (on the unactuated coordinate) are independent of the unactuated coordinate, then the PDEs can be *explicitly solved*. This chapter aims at providing some extensions of the control strategy given in [2], that take into account certain classes of systems for which such a strategy fails to work. First, a brief overview will be given on the theory of passivity and dissipativity, the framework of port-controlled Hamiltonian systems, and finally the basic theory of IDA–PBC. In the second part of the chapter two extensions will be presented [49, 87], together with their motivating physical examples.

3.2 Background material

3.2.1 Passivity and dissipativity

The concept of passivity is quite general, in the sense that it can be defined for any kind of systems described by an input-output relation, and not necessarily for a dynamic system. Dissipativity, on the other hand, can be regarded to some extent as a particular kind of passivity property that may arise in dynamic systems, as it will be clear in the following discussion.

Consider a system described by a generic map from a certain input space to an output space. In order to define passivity, some concepts regarding those two spaces must be introduced. Consider a linear input space U (with dimension m), and let the output space Y be the dual space $U^\star$, that is, the set of linear functions on U). Denote the duality product between U and $U^\star$ by $< y|u >$, for $y \in U^\star, u \in U$ (that is, $< y|u >$ is the linear function $y : U \to I\!R$ evaluated at u). Furthermore, take any linear space of functions $u : I\!R^+ \to U$, denoted by $L(U)$, and any linear space of functions $y : I\!R^+ \to Y$, denoted by $L(U^\star)$. Define a duality pairing between $L(U)$ and $L(U^\star)$ by defining for $u \in L(U)$ and $y \in L(U^\star)$

$$< y|u >_T := \int_0^T < y(t)|u(t) > dt \tag{3.1}$$

assuming that the integral on the right hand side exists. In applications, the duality product $< y|u >$ will usually be the instantaneous power (electrical power if the components of u, y are voltages and currents, or mechanical power if they are generalized forces and velocities, respectively). Thus, $< y|u >_T$ denotes the externally supplied energy during the time interval $[0, T]$.

Definition 3.2.1. Let $G : L(U) \to L(U^\star)$. Then G is passive if there exists a positive constant β such that

$$< G(u)|u >_T \geq -\beta \qquad \forall u \in L(U), \forall T \geq 0. \tag{3.2}$$

Equivalently, (3.2) can be rewritten as

$$- < G(u)|u >_T \leq \beta \qquad \forall u \in L(U), \forall T \geq 0, \tag{3.3}$$

with the interpretation that the maximum amount of energy extractable from the system is bounded by a finite constant β.

Next comes the concept of dissipativity, which is defined for dynamic systems described by the following equations:

$$\Sigma : \begin{cases} \dot{x} = f(x, u), & u \in U \\ y = h(x, u), & y \in Y, \end{cases} \tag{3.4}$$

where $x \in X$. It is possible to associate with such a description an input-output map G_{x_0} defined by substituting in $h(x, u)$ the input function u and the solution x of $\dot{x} = f(x, u)$ starting from x_0.

Define the supply rate on the space $U \times Y$ as

$$s := U \times Y \to I\!R. \tag{3.5}$$

Definition 3.2.2. A system Σ is said to be dissipative with respect to the supply rate s if there exists a function $S : X \to I\!R^+$, called the storage function, such that for all $x_0 \in X$, all $t_1 \geq t_0$, and all input functions u

$$S(x(t_1)) \geq S(x(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t))dt, \tag{3.6}$$

where $x(t_0) = x_0$, and $x(t_1)$ is the state of Σ at time t_1 resulting from initial condition x_0 and input function u.

Inequality (3.6) is called the dissipation inequality. It expresses the fact that the "stored energy" $S(x(t_1))$ of Σ at any future time t_1 is at most equal to the sum of the stored energy $S(x(t_0))$ at present time t_0 and the total externally supplied energy $\int_{t_0}^{t_1} s(u(t), y(t))dt$ during the time interval $[t_0, t_1]$. Hence, there can be no internal "creation of energy", only internal dissipation is possible. If (3.6) holds with the equal sign, the system is said to be lossless.

One important choice of supply rate is $s(u, y) =< y|u >, u \in U, y \in U^\star$. Suppose Σ is dissipative with respect to this supply rate. Then for some function $S \geq 0$

$$\int_0^T < y(t)|u(t) > dt \geq S(x(T)) - S(x(0)) \geq -S(x(0))$$

for all $x(0) = x_0$, and all $T \geq 0$ and all input functions u. This means precisely that the input-output map G_{x_0} of Σ is passive for any $x_0 \in X$, and S can be given the interpretation of stored energy.

3.2.2 Stability of dissipative systems

When the storage function S is continuously differentiable, it is possible to give an interpretation of dissipativity in terms of stability in the sense of Lyapunov. As a matter of fact, inequality (3.6) can be rewritten in the following differential form:

$$S_x(x)f(x,u) \leq s(u,h(x,u)), \quad \forall x,u, \tag{3.7}$$

with $S_x(x)$ denoting the row vector of partial derivatives of $S(x)$ with respect to x. The following result [86] holds.

Lemma 3.2.3. Let $S : X \to I\!R^+$ be a C^1 storage function for Σ. Assume that the supply rate s satisfies

$$s(0,y) \leq 0, \forall y. \tag{3.8}$$

Suppose that $x^\star \in X$ is a strict local minimum for S. Then $x^\star$ is a stable equilibrium of the unforced system $\dot{x} = f(x,0)$ with Lyapunov function $V(x) = S(x) - S(x^\star) \geq 0$ for x around $x^\star$. Furthermore, suppose that no solution of $\dot{x} = f(x,0)$ other than $x(t) = x^\star$ remains in $\{x \in X : s(0,h(x,0))\}$ for all t. Then $x^\star$ is an asymptotically stable equilibrium, which is globally asymptotically stable if $V \geq 0$ is proper.

This result will be used extensively in what follows to prove effectiveness of the IDA–PBC control strategy.

3.2.3 Port-controlled Hamiltonian systems

In this section an important class of passive state space systems is introduced, namely, the port-controlled hamiltonian systems. A standard method for deriving the equations of motion for mechanical systems is via the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau \tag{3.9}$$

where $q = (q_1, \ldots, q_n)^T$ are generalized configuration coordinates for the systems with n degrees of freedom, the Lagrangian L equals the difference $K - V$ between kinetic energy K and potential energy V, and $\tau = (\tau_1, \ldots, \tau_n)^T$ is the vector of generalized forces acting on the system. In standard mechanical systems the kinetic energy K is often of the form

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} \tag{3.10}$$

where the $n \times n$ ineria matrix $D(q)$ is symmetric and positive definite for all q. In this case the vector of generalized momenta $p = (p_1, \ldots, p_n)^T$, defined as $p = \frac{\partial L}{\partial \dot{q}}$ is simply given by

$$p = D(q)\dot{q}, \tag{3.11}$$

and by defining the state vector $(q_1, \ldots, q_n, p_1, \ldots, p_n)^T$ the n second-order equations (3.9) transform into $2n$ first-order equations

$$\begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q,p) \quad (= D^{-1}(q)p) \\ \dot{p} & = & -\frac{\partial H}{\partial q}(q,p) + \tau \end{array} \tag{3.12}$$

where $H(q,p) = \frac{1}{2}p^T D^{-1}(q)p + V(q)$ is the total energy of the system. The equations (3.12) are called the Hamiltonian equations of motion, and H is called the Hamiltonian function. The following energy balance immediately follows from (3.12):

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial q}(q,p)\dot{q} + \frac{\partial^T H}{\partial p}(q,p)\dot{p} = \frac{\partial^T H}{\partial p}(q,p)\tau = \dot{q}\tau, \tag{3.13}$$

expressing that a Hamiltonian system, with output $y = \dot{q} = \frac{\partial H}{\partial p}(q,p)$ is dissipative (actually, lossless).

A generalization of the Hamiltonian equations is represented by the so called port-controlled Hamiltonian systems, which are described in local coordinates as

$$\begin{array}{rcl} \dot{x} & = & J(x)\frac{\partial H}{\partial x}(x) + g(x)u \quad x \in X, u \in I\!R^m \\ y & = & g^T(x)\frac{\partial H}{\partial x}(x), \quad y \in I\!R^m. \end{array} \tag{3.14}$$

where $J(x)$ is a skew-symmetric matrix (also called the interconnection matrix). As it can be easily seen, a port-controlled Hamiltonian systems is still lossless. A truly dissipative port-controlled Hamiltonian systems can be obtained if the above equations are modified as follows, to take into account the existence of dissipative forces such as friction:

$$\begin{array}{rcl} \dot{x} & = & [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u \\ y & = & g^T(x)\frac{\partial H}{\partial x}(x), \end{array} \tag{3.15}$$

with $R(x)$ being a positive semidefinite matrix.

3.2.4 IDA–PBC of port-controlled Hamiltonian systems

Motivated by lemma 3.2.3, PBC poses the stabilization problem in terms of the so called passivation objective , that is, select a control action $u = \beta(x)+v$ so that the closed loop dynamics satisfy the new energy-balancing equation

$$H_d(x(t)) - H_d(x(0)) = \int_0^t v^t(s)y'(s)ds - d_d(x(t)) \tag{3.16}$$

where $H_d(x)$ is the new desired total energy function, which has a minimum at $x^\star$, y' (which may be equal to y) is the new passive output, and the natural dissipation term has been replaced by some desired function $d_d(x)$ to increase the convergence rate. In IDA–PBC, the passivation objective is achieved by aiming at the closed-loop dynamics

$$\begin{array}{rcl} \dot{x} & = & [J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x}(x) + g(x)v \\ y & = & g^T(x)\frac{\partial H_d}{\partial x}(x), \end{array} \tag{3.17}$$

where $J_d(x) = -J_d(x)^T$ and $R_d(x) = R_d(x)^T \geq 0$ are some desired interconnection and damping matrices, respectively. It is clear that the solutions of (3.17) satisfy (3.16) with

$$d_d(x(t)) = \int_0^t \left(\frac{\partial H_d}{\partial x}(x(s))\right)^T R_d(x(s))\frac{\partial H_d}{\partial x}(x(s))ds.$$

The following result shows how the passivation objective reduces to solving a set of PDEs [64]. It is given in terms of a general nonlinear control-affine system to show point out its generality.

Proposition 1. *Consider the system*

$$\dot{x} = f(x) + g(x)u. \tag{3.18}$$

Assume that there exist matrices $g^{\perp}(x), J_d(x) = -J_d(x)^T, R_d(x) = R_d(x)^T \geq 0$ *and a function* $H_d(x) : I\!R^n \to I\!R$ *that verify the PDE*

$$g^{\perp}(x)f(x) = g^{\perp}(x)[J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x}, \tag{3.19}$$

where $g^{\perp}(x)$ *is a full-rank left annihilator of* $g(x)$ *(i.e.* $g^{\perp}(x)g(x) = 0$*), and* $H_d(x)$ *is such that*

$$x_\star = \arg\min H_d(x), \tag{3.20}$$

with $x_\star \in I\!R^n$ *the equilibrium to be stabilized. Then, the closed-loop system (3.18) with* $u = \beta(x)$*, where*

$$\beta(x) = [g^T(x)g(x)]^{-1}g^T(x)\{[J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x} - f(x)\}, \tag{3.21}$$

takes the port-controlled Hamiltonian form

$$\dot{x} = [J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x} \tag{3.22}$$

with $x_\star$ *a (locally) stable equilibrium. It will be asymptotically stable if, in addition,* $x_\star$ *is an isolated minimum of* $H_d(x)$ *and the largest invariant set under the closed loop dynamics (3.22) contained in*

$$\left\{ x \in I\!R^n : \left(\frac{\partial H_d}{\partial x}(x)\right)^T R_d(x)\frac{\partial H_d}{\partial x}(x) = 0 \right\}$$

equals $\{x_\star\}$*. Moreover, an estimate of its domain of attraction is given by the largest bounded level set* $\{x \in I\!R^n : H_d(x) \geq c\}$*.*

Proof. By Setting up the right hand side of (3.18), with $u = \beta(x)$, equal to the right hand side of (3.22) the following matching equation is obtained:

$$f(x) + g(x)\beta(x) = [J_d(x) - R_d(x)]\frac{\partial H_d}{\partial x}.$$

Multiplying on the left by $g^{\perp}(x)$ yields the PDE (3.19). The expression of the control law is obtained by multiplying on the left by the pseudo-inverse of $g(x)$. Stability of $x_\star$ is established by noting that, along the trajectories of (3.22), it holds that

$$\dot{H}_d = -\left(\frac{\partial H_d}{\partial x}(x)\right)^T R_d(x)\frac{\partial H_d}{\partial x}(x) \geq 0.$$

Hence, $H_d(x)$ qualifies as a Lyapunov function. Asymptotical stability follows immediately by invoking La Salle's invariance principle. Finally, in order to ensure that the solutions remain bounded, the estimate of the domain of attraction can be given as the largest bounded level set of $H_d(x)$. □□□

As clear from the proposition, the key step in the design procedure is solving equation (3.19). It is stressed that $J_d(x)$ and $R_d(x)$ are free, up to the constraints of skew-symmetry and positive semidefiniteness, respectively, and $H_d(x)$ may be totally or partially fixed, provided that (3.20) is ensured. As a consequence, there exist at least three ways to proceed:

- (Non-parameterized IDA) In one extreme case, the desired interconnection $J_d(x)$ and damping $R_d(x)$ matrices, as well as $g^{\perp}(x)$. This yields a PDE whose solutions define the admissible energy functions $H_d(x)$ for the given interconnection and damping matrices. Among the family of solutions the one that satisfies (3.20) is selected.

- (Algebraic IDA) The desired energy function is fixed, and thus (3.19) becomes an algebraic equation in $J_d(x), R_d(x)$ and $g^{\perp}(x)$.

- (Parameterized IDA) For some physical systems it is desirable to restrict the desired energy function to a certain class, for instance, for mechanical systems, the sum of a potential energy term that depends only on the generalized positions, and the kinetic energy that is quadratic in the generalized momenta. Fixing the structure of the energy function yields a new PDE for its unknown terms and, at the same time, imposes some constraints on the interconnection and damping matrices.

Sometimes it is natural to split the control action into energy shaping and damping injection terms as $u = u_{es} + u_{di}$. This fixes the matrix $R_d(x) = g(x)K_v g^T(x)$ with $K_v > 0$. Then, the PDE becomes

$$g^{\perp}(x)J_d(x)\frac{\partial H_d}{\partial x}(x) = g^{\perp}(x)f(x) \tag{3.23}$$

and

$$u_{es}(x) = [g^T(x)g(x)]^{-1}g^T(x)[J_d(x)\frac{\partial H_d}{\partial x}(x) - f(x)] \tag{3.24}$$

$$u_{di}(x) = -K_v g^T(x)\frac{\partial H_d}{\partial x}(x). \tag{3.25}$$

This partition proceeds from the premise that damping is introduced to enforce asymptotic stability to an otherwise stable system.

3.2.5 IDA-PBC of underactuated mechanical systems

The parameterized IDA variation of the design technique applied to mechanical systems simplifies as follows. Consider a mechanical system described by using the Hamiltonian formalism:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u \tag{3.26}$$

with total energy $H(q,p) = \frac{1}{2}p^T D^{-1}(q)p + V(q)$, where $q \in I\!R^n, p \in I\!R^n$ are the generalized position and momenta, respectively, $D(q) = D^T(q) > 0$ is the inertia matrix, $V(q)$ is the potential energy and rank $(B) = m < n$, meaning that the system is underactuated. Fix the desired energy function to be

$$H_d(q,p) = \frac{1}{2}p^T D_d^{-1}(q)p + V_d(q),$$

where $D_d(q)$ and $V_d(q)$ represent the closed-loop inertia matrix and potential energy function, respectively, and it is required that $D_d(q) = D_d^T(q) > 0$ and $q_\star = \arg\ \min V(q)$. Fixing the desired energy function also fixes the desired interconnection matrix as

$$J_d(q,p) = \begin{bmatrix} 0 & D^{-1}(q)D_d(q) \\ -D_d(q)D^{-1}(q) & J_2(q,p) \end{bmatrix} = -J_d^T(q,p),$$

where the skew-symmetric matrix $J_2(q,p)$ is a free parameter. On the other hand, when the control action is split into energy shaping and damping injection, the damping matrix results of the form

$$R_d(q) = \begin{bmatrix} 0 & 0 \\ 0 & B(q)K_v B^T(q) \end{bmatrix} \geq 0,$$

where $K_v > 0$.

As shown in [2], the PDEs of IDA–PBC can be naturally separated into the terms that depend on p and terms which are independent of p, i.e., those corresponding to the kinetic and potential energy, respectively. This leads to

$$\begin{aligned} B^{\perp}(q)\Big[\nabla_q(p^T D^{-1}(q)p) - D_d(q)D^{-1}(q)\nabla_q(p^T D_d^{-1}(q)p) & \\ +2J_2(q,p)D_d^{-1}(q)p\Big] &= 0, \quad (3.27) \\ B^{\perp}(q)\left[\nabla_q V - D_d(q)D^{-1}(q)\nabla_q V_d\right] &= 0, \quad (3.28) \end{aligned}$$

where $B^{\perp}(q)B(q) = 0$. As for the control law, (3.21) yields (the arguments are omitted for brevity)

$$u = (B^T B)^{-1} B^T (\nabla_q H - D_d D^{-1} \nabla_q H_d + J_2 D_d^{-1} p) \qquad (3.29)$$

The first equation is a nonlinear and nonhomogeneous PDE which is in general quite difficult to solve, whereas the second one is linear. A lot of research work has been done in order to provide conditions under which these two PDEs could be solved, including what is reported in this thesis. The first major result has been given in [2] and it deals with systems characterized by an underactuation degree equal to one.

Consider the class of mechanical systems described by equations of the following form:

$$\begin{aligned} \dot{q} &= D^{-1}(q_r)p \\ \dot{p} &= s(q_r) + B(q_r)u, \end{aligned} \qquad (3.30)$$

where q_r, with r an integer taking values in the set $\{1,\ldots,n\}$, is a distinguished element of $q \in I\!R^n$, $p \in I\!R^n$, and $u \in I\!R^{n-1}$. Structures of this form may result from the reduction of a certain class of mechanical systems via partial feedback linearization. Two key properties of (3.30) are exploited in [2]:

i) Since $D(q_r)$ and $D_d(q_r)$ depend only on the coordinate q_r the kinetic energy PDE (3.27) becomes an ODE;

ii) by restricting $D_d(q_r)$ to a particular structure, the latter can be explicitly solved with a suitable choice of $J_2(q_r)$ and $B^{\perp}$.

The main result of this section is the following:

Proposition 2. *Consider the system (3.30). Fix*

$$D_d(q_r) = \int_{q_r^\star}^{q_r} B(\mu)\Psi(\mu)B^T(\mu)d\mu + D_d^0,$$

and assume that there exist $\Psi(q_r) = \Psi^T(q_r) \in I\!R^{(n-1)\times(n-1)}$ *and* $D_d^0 = (D_d^0)^T > 0 \in I\!R^{n\times n}$ *such that for some left annihilator* $\tilde{B}^{\perp}(q_r)$ *of* $B(q_r)$

$$\left|\tilde{B}^{\perp}(q_r^\star)D_d(q_r^\star)D^{-1}(q_r^\star)e_r\right| \geq \epsilon > 0.$$

Then, there exists a matrix $J_2(q_r)$ *and a function* $\eta(q_r)$ *such that the kinetic energy PDE (3.27) with* $B^{\perp}(q_r) = \eta(q_r)B^{\perp}(q_r^\star)$ *is solved. Furthermore, the solution of the potential energy PDE (3.28)is given by*

$$V_d(q) = -\frac{1}{\rho}\int_0^{q_r} B^{\perp}(\mu)s(\mu)d\mu + \Phi(z(q)), \tag{3.31}$$

where $z(q)$ *is a* $n-1$ *dimensional vector whose components are of the form*

$$z_i(q) = -\frac{1}{\rho}\int_0^{q_r} B^{\perp}(\mu)D_d(\mu)D^{-1}(\mu)e_i d\mu, \qquad i = 1,\ldots,n, i \neq r,$$

where $\Phi(z)$ *is an arbitrary differentiable function, and* ρ *is a constant.*

The above result holds also when the inertia matrix and the force induced by the potential energy on the unactuated coordinate do not depend on the unactuated coordinate. In such a case, the kinetic energy PDE (3.27) becomes homogeneous. The third part of the chapter is devoted to removing such an assumption, thus enlarging the class of systems to which IDA–PBC can be applied. The next part, instead, deals with solving the potential energy PDE for a special class of underactuated systems, exemplified by the acrobot.

3.3 The Acrobot

3.3.1 Problem Formulation

In this section, a first generalization of the results reported above is presented. Namely, the hypothesis that the force induced by the potential energy on the unactuated coordinate do not depend on the unactuated coordinate itself. Hence, the following assumptions are kept:

A1) The system has underactuation degree one.

A2) The inertia matrix D depends only on the actuated coordinates.

Further, motivated by the interesting Acrobot example ([82, 50]), the following assumption is assumed to hold:

A3) The system has two degrees–of–freedom and, without loss of generality, take $B = [0 \;\; 1]^T$.

Assumption A3) is critical for the subsequent developments. Assumptions A1) and A2) ensure that the term $B^{\perp}\nabla_q(p^T D^{-1} p)$ in the PDE (3.28) is zero. In this case (3.28) can be solved with a constant D_d, taking $J_2 = 0$. Thus, it is possible to concentrate only on potential energy shaping and the PDE to be solved takes the form

$$\gamma^T(q_2)\nabla V_d = \nabla_1 V, \tag{3.32}$$

where it has been used used $B^{\perp} = [1\;0]$, defined

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} := D^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \tag{3.33}$$

and taken the constant matrix D_d as

$$D_d = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}. \tag{3.34}$$

The problem addressed in this chapter is the derivation of conditions on D and V such that an explicit solution to (3.32) can be obtained. For simplicity,

$q^\star$ will be taken equal to 0. Note that, under Assumptions A1) and A3), the origin is an assignable equilibrium for (3.26) only if

$$\nabla_1 V(0) = 0, \tag{3.35}$$

that will be assumed in the sequel.

3.3.2 Preliminary Lemma

The following lemma, of interest on its own, lies at the core of the subsequent developments. It identifies a class of PDEs of the form (3.32), whose solution can be transformed into the solution of ordinary differential equations (ODEs).

Lemma 3.3.1. Consider the PDE in the unknown function $V_d : \mathbb{R}^2 \to \mathbb{R}$,

$$a(q_2)\nabla_1 V_d + \nabla_2 V_d = b(q) \tag{3.36}$$

where $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^2 \to \mathbb{R}$. Assume b can be factored as

$$b(q) = c(q_2) + B^T(q_2)N(q_1) \tag{3.37}$$

for some $c : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}^p$ and $N : \mathbb{R} \to \mathbb{R}^p$ solution of the ODEs

$$N' = AN, \tag{3.38}$$

for some constant matrix $A \in \mathbb{R}^{p\times p}$ and initial conditions $N(0) \in \mathbb{R}^p$.[1]

Then, for all differentiable functions $\Phi : \mathbb{R} \to \mathbb{R}$, a solution of (3.36) is given by

$$V_d(q) = F(q_2) + H^T(q_2)N(q_1) + \Phi(q_1 - z(q_2)), \tag{3.39}$$

with $F : \mathbb{R} \to \mathbb{R}$, $H : \mathbb{R} \to \mathbb{R}^p$ and $z : \mathbb{R} \to \mathbb{R}$ solutions of the ODEs

$$\begin{aligned} F' &= c \\ H' &= -aA^T H + B \\ z' &= a. \end{aligned} \tag{3.40}$$

[1]This assumption implies that the dependence of b on q_1 consists of sine, cosine, exponential and polynomial functions generated as solutions of the linear ODEs (3.38).

Proof. The proof is obtained by direct substitution of (3.39) into (3.36). Indeed, differentiating (3.39)

$$\nabla V_d = \left[\begin{array}{c} H^T N' + \Phi' \\ F' + N^T H' - a\Phi' \end{array} \right] = \left[\begin{array}{c} H^T A N + \Phi' \\ c + B^T N - a\nabla_1 V_d \end{array} \right]$$

that, taking into account (3.37), clearly verifies (3.36). □□□

From (3.39), (3.40) it is clear that an explicit solution to the PDE (3.36) is available provided that the following integrals can be computed:

$$\int c(\xi) d\xi, \quad \int a(\xi) d\xi, \quad \int \Psi(\xi, q_2) B(\xi) d\xi,$$

where Ψ is the "state transition matrix" of the H equation in (3.40)

$$\Psi(\xi, q_2) = e^{-[\int_\xi^{q_2} a(s)ds]A^T}.$$

The evaluation of Ψ can sometimes be simplified introducing a change of coordinates as follows. Assume the function z is invertible, that is, there exists $\hat{q}_2 : \mathbb{R} \to \mathbb{R}$ such that $z(\hat{q}_2(s)) = s$ for all $s \in \mathbb{R}$. For all functions $x : \mathbb{R} \to \mathbb{R}$ introduce the following notation for the function compositions

$$\tilde{x}(z) := x(\hat{q}_2(z)).$$

A simple application of the chain rule, i.e. $\frac{dH}{dq_2} = \frac{d\tilde{H}}{dz}\frac{dz}{dq_2}$, transforms (3.40) into

$$\begin{aligned} \tilde{F}'(z) &= \frac{\tilde{c}(z)}{\tilde{a}(z)} \\ \tilde{H}'(z) &= -A^T \tilde{H}(z) + \frac{\tilde{B}(z)}{\tilde{a}(z)}, \end{aligned} \tag{3.41}$$

where the function a has been removed from the $\tilde{H}$ equation, yielding a linear forced ODE, whose state transition matrix can be trivially calculated.

From the function $\tilde{H}$ one can obtain the desired H noting that

$$\tilde{H}(z(s)) = H((\hat{q}_2(z(s)) = H(s), \ \forall s \in \mathbb{R}.$$

3.3.3 Solving the Potential Energy PDE

In this section Lemma 3.3.1 and the change of coordinates procedure described above are used to give conditions on D and V such that an explicit solution to (3.32) can be obtained.

The following assumptions are assumed to hold:

A4) $\nabla_1 V$ can be factored as

$$\nabla_1 V(q) = c_0(q_2) + B_0^T(q_2) N(q_1) \tag{3.42}$$

for some $c_0 : \mathbb{R} \to \mathbb{R}$, $B_0 : \mathbb{R} \to \mathbb{R}^p$ and $N : \mathbb{R} \to \mathbb{R}^p$ solution of the ODE

$$N' = AN \tag{3.43}$$

for some constant matrix $A \in \mathbb{R}^{p \times p}$ and $N(0) \in \mathbb{R}^p$.

A5) $k_1 > 0$, $k_2 \in \mathbb{R}$, elements of D_d, are such that

$$e_i^T D^{-1}(0) \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq 0, \ i = 1, 2$$

with $e_i \in \mathbb{R}^2$ the vectors of the orthonormal Euclidean basis.

A6) The integral

$$z(q_2) := \int_0^{q_2} \frac{\gamma_1(\sigma)}{\gamma_2(\sigma)} \, d\sigma, \tag{3.44}$$

where γ is defined in (3.33), is computable and $z(q_2)$ is a diffeomorphism with an inverse map $\hat{q}_2 : \mathbb{R} \to \mathbb{R}$, i.e., $z(\hat{q}_2(s)) = s, \forall\, s \in \mathbb{R}$. Furthermore, the following functions can be computed:

$$\begin{aligned} \tilde{F}(z) &:= \int_0^z \frac{c_0}{\gamma_1}(\hat{q}_2(\sigma)) \, d\sigma \\ \tilde{H}^0(z) &:= \int_0^z e^{(A^T \sigma)} \frac{B_0}{\gamma_1}(\hat{q}_2(\sigma)) \, d\sigma. \end{aligned} \tag{3.45}$$

Proposition 3.3.2. Assume $D(q_2)$ and $V(q)$ satisfy conditions A4)–A6). Then the function

$$V_d(q) = \tilde{F}(z(q_2)) + N^T(q_1)\tilde{H}(z(q_2)) + \Phi(q_1 - z(q_2)),$$

where

$$\tilde{H}(z) = e^{-A^T z}\left[\tilde{H}(0) + \tilde{H}^0(z)\right]. \tag{3.46}$$

solves the potential energy PDE (3.32) for all differentiable functions $\Phi : \mathbb{R} \to \mathbb{R}$ and all vectors $\tilde{H}(0) \in \mathbb{R}^2$.

Proof. The proof is a direct application of Lemma 3.3.1 and the change of coordinates described in the previous section. Indeed, under Assumption A5),[2] (3.32) and (3.42) can be written in the forms (3.36) and (3.37), respectively, with

$$a := \frac{\gamma_1}{\gamma_2}, \quad b := \frac{\nabla_1 V}{\gamma_2}, \quad c := \frac{c_0}{\gamma_2}, \quad B := \frac{B_0}{\gamma_2}.$$

The ODEs (3.40), in the new coordinate z, take the form (3.41) with

$$\frac{\tilde{c}}{\tilde{a}}(z) = \frac{c_0}{\gamma_1}(\hat{q}_2(z)), \quad \frac{\tilde{B}}{\tilde{a}}(z) = \frac{B_0}{\gamma_1}(\hat{q}_2(z)). \tag{3.47}$$

The solution of these ODEs, given in the proposition, completes the proof. □□□

3.3.4 Main Stabilization Result

In the previous section a parametrization of the assignable potential energy functions is proposed in terms of the first column of D_d (the reals $k_1 > 0$, $k_2 \in \mathbb{R}$), the vector $\tilde{H}(0) \in \mathbb{R}^2$ and the (differentiable) function Φ. Here some additional constraints will be imposed on these parameters to ensure stability of the closed–loop that, besides positivity of D_d (that is satisfied with a suitable choice of k_3), requires assignment of the desired minimum to V_d. Towards this end, the following assumptions are made:

[2]Because of continuity $|\gamma(q)| \neq 0$ for all q in a neighborhood of zero.

A7) There exists $k_1 > 0$, $k_2 \in \mathbb{R}$ such that

$$\rho := \begin{bmatrix} k_1 & k_2 \end{bmatrix} D^{-1}(0) \begin{bmatrix} \nabla_{11} V(0) \\ \nabla_{12} V(0) \end{bmatrix} > 0.$$

A8) $\tilde{H}(0)$ and Φ are such that

$$\begin{aligned} \tilde{H}^T(0) A N(0) + \Phi^{'}(0) &= 0 && (3.48) \\ \tilde{H}^T(0) A^2 N(0) + \Phi^{''}(0) &> \frac{[\nabla_{11} V(0)]^2}{\rho} && (3.49) \end{aligned}$$

Assumption A8) is stated in this form for ease of exposition, but there always exist a vector $\tilde{H}(0)$ such that it is satisfied with a quadratic function $\Phi(s) = \frac{k_p}{2} s^2,\ k_p > 0$. Indeed, in this case (3.48) holds taking $\tilde{H}(0)$ orthogonal to $AN(0)$. Furthermore, (3.49) will be satisfied taking k_p sufficiently large. Assumption A7), on the other hand, imposes a real constraint on the admissible D and V.

The main result can now be stated, which essentially boils down to proving that, if A7) and A8) hold, V_d has an isolated local minimum at zero.

Proposition 3.3.3. Consider the system (3.26) satisfying Assumptions A1)–A6). Then the IDA–PBC law (3.29) takes the form

$$u = \frac{1}{2} \nabla_2 \left(p^T D^{-1} p \right) + \nabla_2 V - \begin{bmatrix} k_2 & k_3 \end{bmatrix} D^{-1} \nabla V_d$$

and it yields the closed-loop dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^{-1} D_d \\ -D_d D^{-1} & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}, \qquad (3.50)$$

where D_d is the constant matrix (3.34) and $V_d(q)$ as in Proposition 3.3.2.

Furthermore, if $k_3 > \frac{k_2^2}{k_1}$ and Assumptions A7) and A8) hold, the origin is a stable equilibrium of (3.50) with Lyapunov function H_d.

The control expression follows from (3.29) taking into account Assumptions A1)–A3), and the fact that $J_2 = 0$ and D_d is constant.

Proposition 3.3.2 establishes that, under Assumptions A4)–A6), the proposed V_d solves the potential energy PDE (3.28). Now, the condition on k_3 ensures $D_d > 0$, hence H_d will be a Lyapunov function for the system if V_d has an isolated local minimum at zero. To prove this fact it is convenient to study the function in the coordinates (q_1, z), hence, define

$$\begin{aligned} \tilde{V}_d(q_1, z) &:= \tilde{F}(z) + N^T(q_1)\tilde{H}(z) + \Phi(q_1 - z) \\ \tilde{V}(q_1, z) &:= V(q_1, \hat{q}_2(z)). \end{aligned}$$

The gradient of $\tilde{V}_d$ is

$$\nabla\tilde{V}_d = \begin{bmatrix} \tilde{H}^T AN + \Phi' \\ \tilde{F}' + N^T\tilde{H}' - \Phi' \end{bmatrix} = \begin{bmatrix} \tilde{H}^T AN + \Phi' \\ \frac{\tilde{c}}{\tilde{a}} + N^T\frac{\tilde{B}}{\tilde{a}} - \nabla_1\tilde{V}_d \end{bmatrix},$$

where the second identity is obtained using (3.41). Now, from (3.42) and (3.47) it follows that

$$\frac{\tilde{c}}{\tilde{a}}(z) + N^T(q_1)\frac{\tilde{B}}{\tilde{a}}(z) = \frac{1}{\tilde{\gamma}_1(z)}\nabla_1\tilde{V}(q_1, z)$$

which, in view of (3.35) and the fact that $\hat{q}_2(0) = 0$, is zero at $(q_1, z) = (0, 0)$. Consequently, $\nabla\tilde{V}_d(0) = 0$ if and only if (3.48) holds.

Some simple calculation show that the Hessian of $\tilde{V}_d$ is given by

$$\nabla^2\tilde{V}_d(q_1, z) = \begin{bmatrix} \kappa_1(q_1, z) & \kappa_2(q_1, z) - \kappa_1(q_1, z) \\ \kappa_2(q_1, z) - \kappa_1(q_1, z) & \kappa_1(q_1, z) - \kappa_2(q_1, z) + \kappa_3(q_1, z) \end{bmatrix},$$

where the real-valued functions $\kappa_i, i = 1, \ldots, 3$ are defined as

$$\begin{aligned} \kappa_1 &= \tilde{H}^T A^2 N + \Phi'' \\ \kappa_2 &= N^T A^T \frac{\tilde{B}}{\tilde{a}} = \frac{1}{\tilde{\gamma}_1}\nabla_{11}\tilde{V} \\ \kappa_3 &= \nabla_2\left(\frac{1}{\tilde{\gamma}_1}\nabla_1\tilde{V}\right) = \frac{1}{\tilde{\gamma}_1^2}\left(\tilde{\gamma}_2\nabla_{12}V - \frac{\tilde{\gamma}_2}{\tilde{\gamma}_1}\gamma_1'\nabla_1 V\right). \end{aligned}$$

The Hessian is positive-definite if $(\kappa_2+\kappa_3) > 0$ and $\kappa_1 > \frac{\kappa_2^2}{\kappa_2+\kappa_3}$. By Evaluating it at the origin, and taking into account that $\nabla_1 V(0) = 0$, the following holds:

have

$$\begin{aligned}\kappa_1(0) &= \tilde{H}^T(0)A^2N(0) + \Phi''(0)\\ \frac{\kappa_2^2(0)}{\kappa_2(0)+\kappa_3(0)} &= \frac{[\nabla_{11}V(0)]^2}{\rho},\end{aligned}$$

with ρ as defined in Assumption A7), completing the proof.

□□□

3.3.5 The Acrobot Example

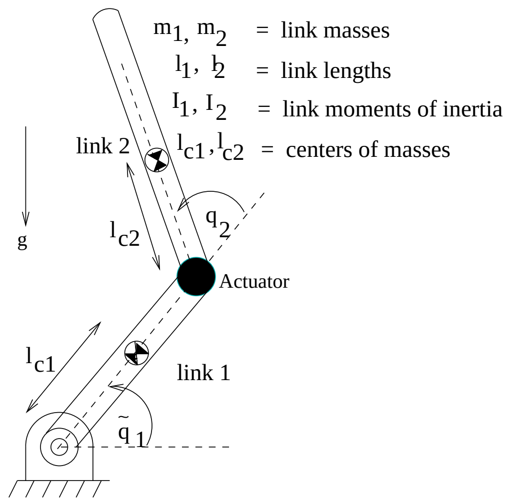

Figure 3.1: The Acrobot, with $q_1 = \tilde{q}_1 - \pi/2$.

In this section it is proved that the methodology described above applies to the interesting Acrobot system described in [82]. The equations of motion of the Acrobot (schematic shown in Figure 3.1) are given by (3.26) with $n = 2$, $m = 1$,

$$D(q_2) = \begin{bmatrix} c_1 + c_2 + 2c_3\cos q_2 & c_2 + c_3\cos q_2 \\ c_2 + c_3\cos q_2 & c_2 \end{bmatrix},$$

$$V(q) = g[c_4\cos q_1 + c_5\cos(q_1+q_2)],\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with g the gravitational constant and the parameters

$$c_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1, \qquad c_2 = m_2 l_{c2}^2 + I_2, \qquad c_3 = m_2 l_1 l_{c2},$$
$$c_4 = m_1 l_{c1} + m_2 l_1, \qquad c_5 = m_2 l_{c2}$$

verifying $c_1 c_2 > c_3^2$.

The control objective is to stabilize the upward equilibrium position, i.e., $q^\star = (0, 0)$.

Verifying the Assumptions

The Acrobot clearly satisfies Assumptions A1)–A3). Assumption A4) also holds with

$$c_0(q_2) = 0,\ B_0(q_2) = -\begin{bmatrix} c_4 g + c_5 g \cos q_2 \\ c_5 g \sin q_2 \end{bmatrix},$$

$$N(q_1) = \begin{bmatrix} \sin q_1 \\ \cos q_1 \end{bmatrix},\ A := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ N(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To verify Assumptions A5) and A6), the vector γ given in (3.33) must be computed. By using (3.34), it follows that

$$\gamma(q_2) = \frac{1}{\delta(q_2)} \begin{bmatrix} (k_1 - k_2) c_2 - k_2 c_3 \cos q_2 \\ k_2(c_1 + c_2) - k_1 c_2 + c_3(2k_2 - k_1) \cos q_2 \end{bmatrix} \tag{3.51}$$

where $\delta(q_2) := c_1 c_2 - c_3^2 \cos^2(q_2) > 0$. The ratio $\frac{\gamma_1}{\gamma_2}$ can be expressed as

$$\frac{\gamma_1}{\gamma_2} = \frac{a_3 + a_4 \cos q_2}{a_1 + a_2 \cos q_2},$$

where

$$a_1 := k_2(c_1 + c_2) - k_1 c_2,\ a_2 := (2k_2 - k_1) c_3,\ a_3 := (k_1 - k_2) c_2,\ a_4 := -k_2 c_3.$$

Even though the function $\frac{\gamma_1}{\gamma_2}$ above can be explicitly integrated, in order to simplify the derivations it is possible to exploit the fact that the ratio takes a constant value $\mu := \frac{a_4}{a_2}$ when $a_3 a_2 = a_1 a_4$.[3] The latter fixes the following relationship between the free parameters k_1 and k_2

$$k_1 = (1 \pm \sqrt{\frac{c_1}{c_2}})k_2, \tag{3.52}$$

where either the positive or the negative sign cab ne selected, and the ratio takes the values

$$\frac{\gamma_1}{\gamma_2} = \mu := \frac{1}{\pm\sqrt{\frac{c_1}{c_2}} - 1}.$$

From (3.52) it follows that, since $k_1 > 0$, the only choice when $c_1 = c_2$ is the positive sign but, unfortunately, μ is not defined in this case, hence the following is needed:

A6') $c_1 \neq c_2$.

Under Assumption A6') (3.44) becomes $z(q_2) = \mu q_2$, which is obviously a diffeomorphism. Interestingly, for the choice of parameters (3.52), Assumption A5) is satisfied for all q_2. Indeed, replacing (3.52) in (3.51) and doing some simple calculations yields

$$\gamma(q_2) = \frac{k_2(\pm\sqrt{c_1 c_2} - c_3 \cos q_2)}{\delta(q_2)} \left[\frac{1}{\pm\sqrt{\frac{c_1}{c_2}} - 1} \right]$$

which, in view of Assumption A6'), $c_1 c_2 > c_3^2$ and $\delta > 0$, is nonzero for all q_2.

Some lengthy but straightforward calculations yield that the remaining integrability conditions of Assumption A6) are easily verifiable with $\tilde{F} = 0$

[3] This fact is easily seen noting that

$$\frac{\gamma_1}{\gamma_2} = \frac{a_4}{a_2} + \frac{a_3 a_2 - a_1 a_4}{a_2} \frac{1}{a_1 + a_2 \cos q_2}.$$

and $\tilde{H}$ as in (3.46) with

$$e^{-A^T z} = \begin{bmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{bmatrix}$$

$$\tilde{H}^0(z) = \begin{bmatrix} -b_1 \sin z - b_2 \sin(\pm\sqrt{\frac{c_1}{c_2}} z) \\ b_1 \cos z + b_2 \cos(\pm\sqrt{\frac{c_1}{c_2}} z) + \end{bmatrix} + \begin{bmatrix} -b_3 \sin((\pm 2\sqrt{\frac{c_1}{c_2}} - 1)z) + b_4 \sin((\pm\sqrt{\frac{c_1}{c_2}} - 2)z) \\ b_3 \cos((\pm 2\sqrt{\frac{c_1}{c_2}} - 1)z) + b_4 \cos((\pm\sqrt{\frac{c_1}{c_2}} - 2)z) \end{bmatrix}$$

and the constants $b_i, i = 1 \dots 4$ defined as

$$\begin{aligned} b_1 &:= \frac{1}{k_2} g \left(\pm\sqrt{c_1 c_2} c_4 + \frac{c_3 c_5}{2} \right) \\ b_2 &:= \frac{1}{k_2} \frac{g\mu}{\mu + 1} \left(\frac{1}{2} c_3 c_4 \pm \sqrt{c_1 c_2} c_5 \right) \\ b_3 &:= \frac{1}{k_2} \frac{g c_3 c_5 \mu}{2\mu + 4} \\ b_4 &:= \frac{1}{k_2} \frac{g c_3 c_4}{2} \frac{\mu}{\mu - 1}, \end{aligned} \tag{3.53}$$

where the factor $\frac{1}{k_2}$ is pulled out to underscore its role as a "scaling gain" for V_d.

In the coordinates (q_1, q_2), the potential energy V_d takes the simple form

$$\begin{aligned} V_d(q_1, q_2) = {} & N^T(q_1) e^{A(\mu q_2)} \tilde{H}(0) + b_1 \cos q_1 + b_2 \cos(q_1 + q_2) \\ & + b_3 \cos(q_1 + 2q_2) + b_4 \cos(q_1 - q_2) + \Phi(q_1 - \mu q_2). \end{aligned} \tag{3.54}$$

Once the existence of a solution for the potential energy PDE has been established by following Proposition 3.3.2, it remains to verify the stability conditions of Proposition 3.3.3. For, (3.52) yields

$$\rho = \frac{-g k_2}{c_1 c_2 - c_3^2} \left[(-c_3 \pm \sqrt{c_1 c_2}) c_4 - c_5 \left(-c_1 \pm c_3 \sqrt{\frac{c_1}{c_2}} \right) \right], \tag{3.55}$$

hence, Assumption A7) is equivalent to

A7') The constants k_2 and c_i are such that

$$k_2 \left[(-c_3 \pm \sqrt{c_1 c_2}) c_4 - c_5 \left(-c_1 \pm c_3 \sqrt{\frac{c_1}{c_2}} \right) \right] < 0$$

Assumption A6') is generically satisfied and the only "real" assumption is A7'). However, note that—except for the case when $c_1 = c_2$—the sign of k_2 is not fixed, hence it is also always possible to fulfill this condition.

Finally, Assumption A8) can be shown to hold if and only if the following condition is satisfied:

A8') The elements of the vector $\tilde{H}(0)$ and the function Φ satisfy

$$\begin{aligned} \tilde{H}_1(0) &= -\Phi'(0) \\ \tilde{H}_2(0) &< \Phi''(0) - \frac{g^2(c_4+c_5)^2}{\rho}. \end{aligned}$$

Asymptotic Stability

The main stabilization result for the Acrobot can be now stated. In order to achieve asymptotic stability, a damping injection term has been injected [68].

Proposition 3.3.4. Consider the Acrobot system satisfying Assumptions A6') and A7'). Fix

$$k_1 = (1-\sqrt{\frac{c_1}{c_2}})k_2, \quad k_3 > \frac{k_2}{1-\sqrt{\frac{c_1}{c_2}}}.$$

Let $\Phi(q_1 - \mu q_2) = \frac{k_p}{2}(q_1 - \mu q_2)^2$ with

$$\mu = \frac{-1}{\sqrt{\frac{c_1}{c_2}}+1}, \quad k_p > \frac{g^2(c_4+c_5)^2}{\rho} + \tilde{H}_2(0),$$

with ρ given by (3.55) (with negative sign), $\tilde{H}_2(0)$ arbitrary and $\tilde{H}_1(0) = 0$. Then the (globally defined) IDA–PBC law

$$u = -\frac{1}{2}\nabla_{q_2}\left(p^T D^{-1} p\right) - \left[\begin{array}{cc} k_2 & k_3 \end{array}\right] D^{-1}\nabla V_d + \nabla_{q_2} V + \frac{k_v}{k_1 k_3 - k_2^2}(k_2 p_1 - k_1 p_2)$$

where $k_v > 0$ is the damping injection gain and

$$\begin{aligned} \nabla_1 V_d &= -\tilde{H}_2(0)\sin(q_1-\mu q_2) - b_1 \sin q_1 - b_2 \sin(q_1+q_2) - b_3 \sin(q_1+2q_2) \\ &\quad -b_4 \sin(q_1-q_2) + k_p(q_1-\mu q_2) \\ \nabla_2 V_d &= \tilde{H}_2(0)\mu \sin(q_1-\mu q_2) - b_2 \sin(q_1+q_2) - 2b_3 \sin(q_1+2q_2) \\ &\quad +b_4 \sin(q_1-q_2) - k_p\mu(q_1-\mu q_2), \end{aligned}$$

achieves (local) asymptotic stabilization of the trivial equilibrium with Lyapunov function H_d.

Proof. It is easy to verify that, with the given selection of Φ and $\tilde{H}(0)$, Assumption A8') is satisfied. Henceforth, stability follows from Proposition 3.3.3 and it only remains to prove the asymptotic part.

The derivative of H_d along the trajectories of the closed-loop system is given by

$$\dot{H}_d = -k_v|B^T D_d^{-1} p|^2 \lambda_\eta 0,$$

with $|\cdot|$ the Euclidean norm. Define the residual set as

$$\Pi := \{(q,p) \in \Omega_c \mid B^T D_d^{-1} p = 0\}.$$

Let $(q,p) \in \Pi$. It follows that

$$0 = B^T D_d^{-1} p = B^T D_d^{-1} D\dot{q} \Leftrightarrow \dot{q}_1 = \frac{\gamma_1}{\gamma_2}\dot{q}_2 = \mu\dot{q}_2. \tag{3.56}$$

Integrating (3.56) yields $q_1(t) = \mu q_2(t) + C$, where C is the integration constant.

Differentiating the left hand side of (3.56) yields

$$0 = B^T D_d^{-1}\dot{p} = -B^T D^{-1}\nabla V_d \tag{3.57}$$

where, to obtain the second identity, the fact that the closed–loop dynamics verify

$$\dot{p} = -D_d D^{-1}\nabla_q V_d(q) - k_v B B^T D_d^{-1} p$$

has been used. Now, since $B^T D^{-1}\nabla V_d$ is an analytic function of q_2, equation (3.57) (whose right hand side is not identically zero) has only isolated solutions. Thus, the only possible trajectories lying in Π are of the form $(q_1(t), q_2(t), p_1(t), p_2(t)) = (\bar{q}_2 + C, \bar{q}_2, 0, 0)$, with $\bar{q}_2$ constant.

Since the origin is an *isolated* and stable equilibrium point, it is possible to conclude that trajectories starting sufficiently near to the origin will eventually converge to it, thus proving local asymptotical stability.

□□□

Simulations

The Acrobot parameters—resulting from $m_1 = 4, m_2 = 1, I_1 = 0.3333, I_2 = 1.3333, l_1 = 1, l_2 = 4, l_{c1} = 0.5000, l_{c2} = 2$—are displayed in Table 3.1

c_1	c_2	c_3	c_4	c_5
2.3333	5.3333	2	3	2

Table 3.1: Acrobot parameters

In (3.52) the negative sign is selected, $k_2 = 1$ and $k_3 = 5.9073$ yielding

$$D_d = \begin{bmatrix} 0.3386 & 1 \\ 1 & 5.9073 \end{bmatrix} > 0, \quad \mu = -0.6019.$$

The level sets of V_d, for $k_p = 280, \tilde{H}_2(0) = 10$, are shown in Fig. 3.2. It can be observed from the figure that there are closed sets containing points in the lower half plane of the configuration space, that is, $|q_1| > \pi/2$. This ensures that, starting with zero velocities, the IDA–PBC will *swing–up* the Acrobot from the lower half plane and at the same time asymptotically stabilize its upward equilibrium. This is the first *continuous* controller that achieves these two objectives for the Acrobot. It should, however, be pointed out that the ability to enlarge the domain of attraction is severely stymied as the actual shape of V_d is essentially determined by the robot parameters. Indeed, from (3.54) it is clear that the effect of the design parameters at disposal $(\tilde{H}_2(0), k_2, k_p)$[4] "disappears" along the line $q_1 = \mu q_2$. On the other hand, V_d may have other extrema at some points arbitrarily close to this line, thus generating spurious equilibria for the closed–loop system.

The time-response of the system starting from the fully stretched downward position, i.e., $(q(0), p(0)) = (-\pi, 0, 0, 0)$ with $k_v = 12$ is depicted in Fig. 3.3. Although this initial condition does not belong to the estimated domain of attraction $\Omega_{\bar{c}}$, the simulation shows that the controller effectively swings–up

[4]Assuming a quadratic Φ.

the acrobot from the downward position. In order to justify theoretically this simulated evidence, current research is under way to improve the estimate of the basin of attraction.

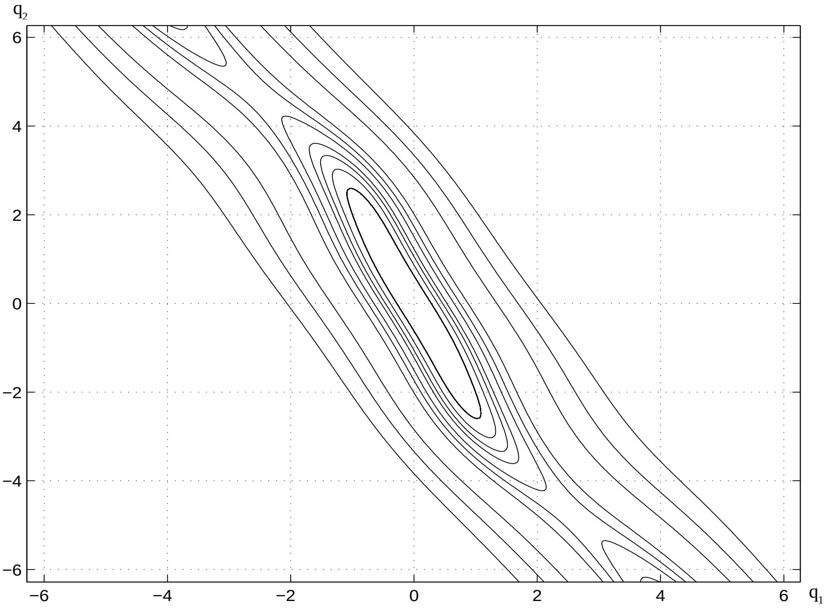

Figure 3.2: Level sets of V_d.

3.4 Simplifying the matching equations via change of coordinates

In this section another extension to the IDA–PBC methodology is presented. Such an extension is aimed at providing constructive solutions to the PDE associated to the kinetic energy, which is nonlinear and inhomogeneous and whose solution, that defines the desired inertia matrix, must be positive definite. The possibility of eliminating (or simplifying) the forcing term in this PDE is studied here, and the main contribution is the proof that it is possible to achieve this objective by *reparametrizing the target dynamics* and introducing a *change of coordinates* in the original system. The class of coordinate

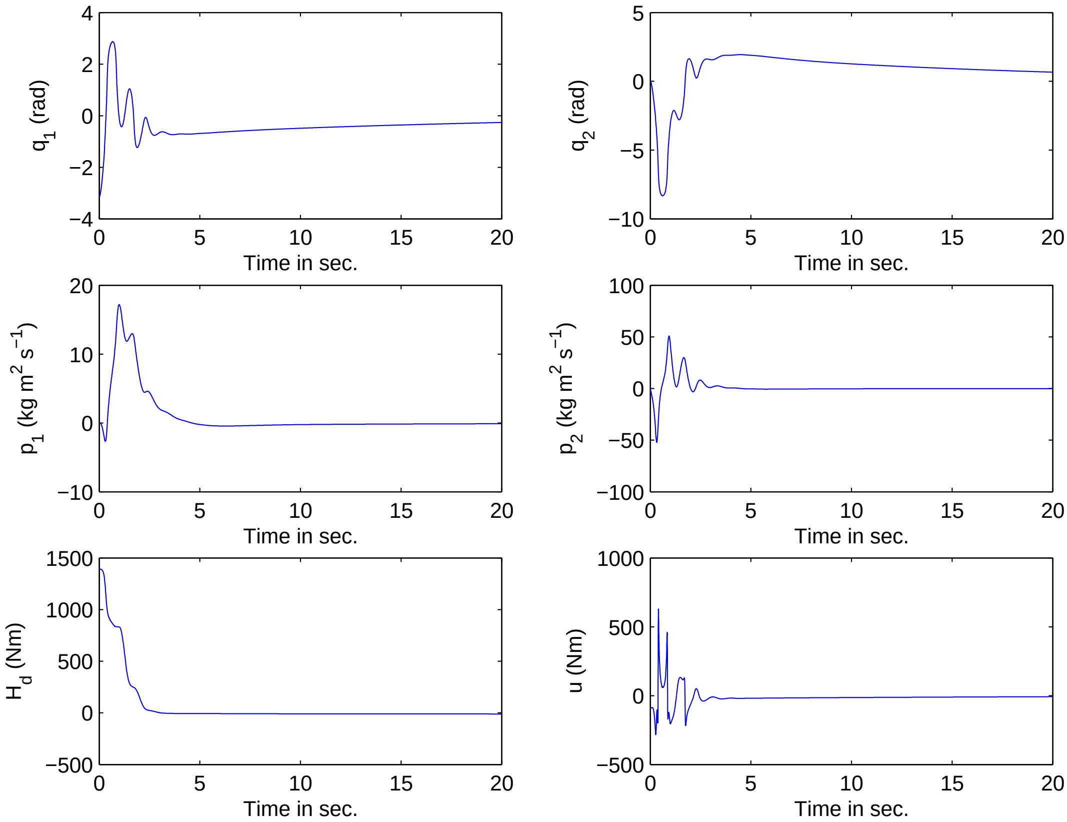

Figure 3.3: Transient responses.

changes that yields an homogenous PDE is a solution of another PDE—similar to the kinetic energy PDE—but this time without the requirement of positive definiteness. Furthermore, it is shown that, in the particular case of transformation to the Lagrangian coordinates, the possibility of simplifying the PDEs is determined by the interaction between the Coriolis forces and the actuation structure. The result is illustrated with the examples of the pendulum on a cart and Furuta's pendulum.

3.4.1 Problem formulation

The presence of the forcing term introduces a *quadratic term* in D_d in the KE PDE that renders very difficult its solution—even with the help of the free skew–symmetric matrix J_2. To reveal this deleterious effect it is convenient to eliminate the dependence on p and re-write the PDE in the equivalent form, see [68] and [2] for further details,

$$\begin{aligned}&\sum_{i=1}^{n}\left[(B_k^{\perp}D_dD^{-1}e_i)\frac{\partial D_d}{\partial q_i}-(B_k^{\perp}e_i)D_d\frac{\partial D^{-1}}{\partial q_i}D_d\right]\\&=-[\mathcal{J}(q)\mathcal{A}_k^T(q)+\mathcal{A}_k(q)\mathcal{J}^T(q)],\end{aligned}\tag{3.58}$$

for $k=1,\dots,n-m$, where $e_i\in\mathbb{R}^n$ is the i–th vector of the n–dimensional Euclidean basis,

$$\mathcal{J}(q)\triangleq\left[\alpha_1(q)\,\vdots\,\alpha_2(q)\,\vdots\,\cdots\,\vdots\,\alpha_{n_o}(q)\right]\in\mathbb{R}^{n\times n_o},$$

$$\alpha_i\in\mathbb{R}^n,\ i=1,\dots,n_o:=\frac{n}{2}(n-1)$$

is a *free* matrix and the row vectors $B_k^{\perp}\in\mathbb{R}^{1\times n}$ have been defined,

$$B^{\perp}:=\begin{bmatrix}B_1^{\perp}\\ \vdots\\ B_{n-m}^{\perp}\end{bmatrix},\mathcal{A}_k:=-\left[W_1\left(B_k^{\perp}\right)^T,W_2\left(B_k^{\perp}\right)^T,\dots,W_{n_o}\left(B_k^{\perp}\right)^T\right]\in\mathbb{R}^{n\times n_o},$$

with the $W_i\in\mathbb{R}^{n\times n}$ skew–symmetric matrices with elements 1s and 0s. For instance, for $n=3$, it holds that

$$W_1:=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},\ W_2:=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\ W_3:=\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}.\tag{3.59}$$

Remark 3.4.1. It is worth noting that the target dynamics Σ_d is *parameterized* by the triple $\{D_d,V_d,J_2\}$. Re–parameterization of the target dynamics is a key step introduced here.

Remark 3.4.2. As shown in [2], with the definitions above, the free matrix J_2 can be written as

$$J_2 = \sum_{i=1}^{n_o} p^T D_d^{-1} \alpha_i W_i, \tag{3.60}$$

and the terms $B_k^{\perp} J_2$, that appear in (3.27), become

$$B_k^{\perp} J_2 = p^T D_d^{-1} \mathcal{J} \mathcal{A}_k^T.$$

Equation (3.58) is obtained factoring (3.27) in the form $p^T D_d^{-1}[\ \cdot\ ]D_d^{-1}p$, taking the symmetric part of the matrices $\mathcal{J}\mathcal{A}_k^T$ and setting the expression in brackets, which is independent of p, equal to zero.

Remark 3.4.3. In [2] it is proved that, if $n - m = 1$ and the inertia matrix and the force induced by the potential energy (on the unactuated coordinate) are independent of the unactuated coordinate, then the PDEs can be *explicitly solved*. The first assumption on the inertia matrix implies, precisely, that the forcing term $B^{\perp}\nabla_q(p^T D^{-1} p) = 0$, which is essential for the construction of the solutions (see Proposition 5).

3.4.2 Generating an Homogeneous Kinetic Energy PDE

The strategy here to eliminate the forcing term in the KE PDE consists of two steps. First, the system (3.26) is expressed in the new coordinates $(q, \tilde{p})$, with $p = T(q)\tilde{p}$, where $T \in \mathbb{R}^{n\times n}$ is *full rank*, yielding:

$$\tilde{\Sigma} : \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & T^{-T} \\ -T^{-1} & Q_{22} \end{bmatrix} \begin{bmatrix} \nabla_q \tilde{H} \\ \nabla_{\tilde{p}} \tilde{H} \end{bmatrix} + \begin{bmatrix} 0 \\ T^{-1}B \end{bmatrix} u, \tag{3.61}$$

where $Q_{22} = -T^{-1}\left[S(q,\tilde{p}) - S^T(q,\tilde{p})\right] T^{-T}$, $\tilde{H}(q,\tilde{p}) = \frac{1}{2}\tilde{p}^T T^T(q) D^{-1}(q) T(q)\tilde{p}$ $+V(q)$, and

$$S(q,\tilde{p}) = \nabla_q (T(q)\tilde{p}). \tag{3.62}$$

It is worth noting that these equations can be seen as a particular form of the Boltzmann-Hamel equations (see [93]), in which $\tilde{p}$ is the vector of quasi-velocities, related to the velocity vector $\dot{q}$ by means of the relation $\tilde{p} = T^{-1}(q)D(q)\dot{q}$.

The second step is to define a *new target dynamics*, in the coordinates $(q, \tilde{p})$, as

$$\tilde{\Sigma}_d : \begin{bmatrix} \dot{q} \\ \dot{\tilde{p}} \end{bmatrix} = \begin{bmatrix} 0 & D^{-1}(q)T(q)\tilde{D}_d(q) \\ -\tilde{D}_d(q)T^T(q)D^{-1}(q) & \tilde{J}_2(q,\tilde{p}) \end{bmatrix} \begin{bmatrix} \nabla_q \tilde{H}_d \\ \nabla_{\tilde{p}} \tilde{H}_d \end{bmatrix}, \tag{3.63}$$

where

$$\tilde{H}_d(q,\tilde{p}) = \frac{1}{2}\tilde{p}^T \tilde{D}_d^{-1}(q)\tilde{p} + \tilde{V}_d(q), \tag{3.64}$$

$\tilde{D}_d \in \mathbb{R}^{n\times n}$ and $\tilde{J}_2 = -\tilde{J}_2^T$ is free. The proposed target dynamics are clearly "compatible" with the new system representation—in the sense that the first n equations are already "matched". In Section 3.4.3 the connection is established between (3.63) and the target dynamic system (3.17) expressed in the new coordinates—see also Remark 3.4.7.

To state the main result the following assumption is needed.

A) The full rank matrix T is such that, for $k = 1, \dots, n-m$,

$$\sum_{i=1}^{n} \left[T^T D^{-1} e_i B_k^{\perp} \frac{\partial T}{\partial q_i} + \frac{\partial T^T}{\partial q_i}(e_i B_k^{\perp})^T D^{-1} T + B_k^{\perp} e_i T^T \frac{\partial D^{-1}}{\partial q_i} T \right] = 0. \tag{3.65}$$

Proposition 3. *Consider the system (3.26) and the partial change of coordinates $p = T(q)\tilde{p}$ where T satisfies Assumption A. For all matrices $\tilde{D}_d(q) = \tilde{D}_d^T(q) \in \mathbb{R}^{n\times n}$ and functions $\tilde{V}_d(q)$ that satisfy the PDEs*

$$B^{\perp} T \left[\tilde{D}_d T^T D^{-1} \nabla_q (\tilde{p}^T \tilde{D}_d^{-1} \tilde{p}) - 2\tilde{J}_2 \tilde{D}_d^{-1} \tilde{p} \right] = 0 \tag{3.66}$$

$$B^{\perp} T \tilde{D}_d T^T D^{-1} \nabla \tilde{V}_d = B^{\perp} \nabla V, \tag{3.67}$$

for some $\tilde{J}_2(q,\tilde{p}) = -\tilde{J}_2^T(q,\tilde{p}) \in \mathbb{R}^{n\times n}$, the system (3.26) in closed–loop with the IDA–PBC $u = \hat{\tilde{u}}(q,\tilde{p})$, where

$$\hat{\tilde{u}}(q,\tilde{p}) = (B^T B)^{-1} B^T \left(\nabla_q H + \dot{T}\tilde{p} - T\tilde{D}_d T^T D^{-1} \nabla_q \tilde{H}_d + T\tilde{J}_2 \tilde{D}_d^{-1} \tilde{p} \right) \tag{3.68}$$

takes, in the coordinates $(q, \tilde{p})$, the Hamiltonian form (3.17).

Proof. Differentiating $p = T(q)\tilde{p}$ yields

$$T\dot{\tilde{p}} = \dot{p} - \dot{T}\tilde{p} = -\nabla_q(\frac{1}{2}p^T D^{-1} p) - \nabla V + Gu - \dot{T}\tilde{p}, \tag{3.69}$$

where (3.26) it is used to get the last identity. On the other hand, from (3.17), it follows that

$$T\dot{\tilde{p}} = -T\tilde{D}_d T^T D^{-1} \left[\nabla_q(\frac{1}{2}\tilde{p}^T \tilde{D}_d^{-1}\tilde{p}) + \nabla \tilde{V}_d\right] + T\tilde{J}_2\tilde{D}_d^{-1}\tilde{p}. \tag{3.70}$$

Multiplying the right hand sides of (3.69) and (3.70) by the $n \times n$ *full–rank* matrix $\begin{bmatrix} B^\perp \\ B^T \end{bmatrix}$ and setting them equal yields (3.67), (3.68) and

$$\begin{aligned} &B^\perp T \left[\tilde{D}_d T^T D^{-1}\nabla_q(\frac{1}{2}\tilde{p}^T \tilde{D}_d^{-1}\tilde{p}) - \tilde{J}_2\tilde{D}_d^{-1}\tilde{p}\right] \\ &= B^\perp \left[\nabla_q(\frac{1}{2}p^T D^{-1}p) + \sum_{i=1}^n e_i^T D^{-1} p \frac{\partial T}{\partial q_i} T^{-1} p\right], \end{aligned} \tag{3.71}$$

where the identity $\dot{q}_i = e_i^T D^{-1} p$ has been used. In view of (3.66), the left hand side of this equation is equal to zero. Proceeding row by row, the right hand side term can be factored as

$$p^T T^{-T} \sum_{i=1}^n \left(T^T \frac{1}{2} B_k^\perp e_i \frac{\partial D^{-1}}{\partial q_i} T + T^T D^{-1} e_i B_k^\perp \frac{\partial T}{\partial q_i}\right) T^{-1} p, \quad k = 1, \ldots, n-m.$$

From (3.65) of Assumption A, and the fact that the skew–symmetric part of the term in parenthesis does not contribute to the quadratic form, it follows that this term is equal to zero, which completes the proof. □□□

Proposition 4. *Assumption A holds with $T = D$, that is, transforming to the Lagrangian coordinates where $\tilde{p} = \dot{q}$, if and only if*

$$B^\perp(q)C(q, \dot{q})\dot{q} = 0, \tag{3.72}$$

where $C \in \mathbb{R}^{n \times n}$ is the matrix of Coriolis and centrifugal forces of the mechanical system (3.26).

Proof. From the derivations in the proof of Proposition 3, in particular from (3.69), it follows that the forcing term in the KE PDE disappears if and only if

$$B^{\perp}[\dot{T}\tilde{p} + \nabla_q(\frac{1}{2}p^T D^{-1}p)] = 0.$$

Now, recall that (e.g., equation (3.19) in [45])

$$\nabla_q(\frac{1}{2}p^T D^{-1}p) = -\nabla_q(\frac{1}{2}\dot{q}^T D\dot{q}) = [C(q,\dot{q}) - \dot{D}(q)]\dot{q}, \tag{3.73}$$

Also, with $T = D$ it follows that $\tilde{p} = \dot{q}$ and (3.65) is equivalent to (3.72). □□□

Now a slightly modified version of a result reported in [2] will be stated, which gives a constructive solution of the homogeneous KE PDE (3.66) for systems with underactuation degree one.

Proposition 5. *Consider equation (3.66). Suppose that $n - m = 1$ and the matrices B and T are function of a single element of q, say q_r, $r \in \{1, \ldots, n\}$. Then, for all desired locally positive definite inertia matrices of the form*

$$\tilde{D}_d(q_r) = \int_{q_r^\star}^{q_r} T^{-1}(\mu)B(\mu)\Psi(\mu)B^T(\mu)T^{-T}(\mu)d\mu + \tilde{D}_d^0 \tag{3.74}$$

where the matrix function $\Psi = \Psi^T \in \mathbb{R}^{(n-1)\times(n-1)}$ and the constant matrix $\tilde{D}_d^0 = (\tilde{D}_d^0)^T > 0 \in \mathbb{R}^{n\times n}$, may be arbitrarily chosen, there exists a matrix $\tilde{J}_2$ such that the KE PDE (3.66) holds in a neighborhood of $q_r^\star$.

Proof. Using the same notation as in [2], define the following matrices and vectors:

$$B_A^{\perp} = B^{\perp}T, \quad \mathcal{A}_A = \left[W_1(B_A^{\perp})^T, \cdots, W_{n_0}(B_A^{\perp})^T\right], \quad \gamma_A = D^{-1}T^T\tilde{D}_d(B_A^{\perp})^T, \tag{3.75}$$

where the W_i matrices are defined in [2]. Using the parametrization introduced in (3.60) for the matrix $\tilde{J}_2$, it is easy to see that the PDE (3.66) becomes:

$$\tilde{p}^T\tilde{D}_d^{-1}\left[\gamma_{A_r}(q)\frac{d\tilde{D}_d}{dq_r} + 2\mathcal{J}(q)\mathcal{A}_A(q)^T\right]\tilde{D}_d^{-1}\tilde{p} = 0, \tag{3.76}$$

where γ_{A_r} is the r–th element of γ_A. As a consequence of (3.75) and the definition of $\tilde{D}_d$ in (3.74), it follows that

$$B_A^\perp \mathcal{A}_A = 0, \quad B_A^\perp \frac{d\tilde{D}_d}{dq_i} = B^\perp T T^{-1} B \Psi B^\perp T^{-T} = 0.$$

By Lemma 1 and 2 of [2], it follows that there exists $\mathcal{J}$ such that the term in brackets in (3.76) is equal to zero, thus completing the proof. □□□

Remark 3.4.4. Comparing (3.27) with (3.66) the absence of the forcing term in the latter can be easily noticed. Therefore, the PDE that needs to be solved is (in principle) simpler. As it will be shown in Proposition 6, this simplification has been achieved without modifying the potential energy PDE (3.28), but it is subject to the condition of finding a matrix T satisfying Assumption A. In any case, the change of coordinates adds a new degree of freedom—the second term in the right hand side of the KE PDE (3.71)—that, as will be shown in the Furuta pendulum example below, can be used to solve the PDE.

Remark 3.4.5. It is interesting to compare the original KE PDE (3.58) and the additional PDE that needs to be solved (3.65). At first glance, it may be argued that (3.65) is as complicated as, if not more complicated than, (3.58). Notice, however, that the terms $e_i B_k^\perp$ in (3.65) are matrices with only one non–zero row, while $B_k^\perp D_d D^{-1} e_i$ in (3.58) is a scalar that "mixes" all the terms in D_d. Also, it is stressed that, contrary to D_d that must be symmetric and positive definite, the *only condition* on T is invertibility. For instance, in the example of the pendulum on a cart system, a solution to (3.65) is trivially obtained while no obvious solution for (3.58) is available.

Remark 3.4.6. Condition (3.72) imposes that the Coriolis and centrifugal forces that are not in the image of B, hence are not affected by the control action, should be zero. Hence, the part of the open loop dynamics complementary to the inputs is, in some sense, linear. This requirement translates, in its turn, into a condition on the inertia matrix D. Indeed, it is shown in

[45] that, for all vectors $x, y \in \mathbb{R}^n$,

$$C(q,x)y = \begin{bmatrix} x^T C^1(q) y \\ x^T C^2(q) y \\ \vdots \\ x^T C^n(q) y \end{bmatrix},$$

where the (ij) elements of the $n \times n$ symmetric matrices $C^k,\ k = 1, ..., n$, are defined by the Christoffel symbols of the second kind as

$$C^k_{ij} = \frac{1}{2}\left[\frac{\partial D_{kj}}{\partial q_i} + \frac{\partial D_{ki}}{\partial q_j} - \frac{\partial D_{ij}}{\partial q_k}\right] = C^k_{ji}. \tag{3.77}$$

It is clear that the k-th element of the Coriolis and centrifugal forces vector will be zero if and only if $C^k_{ij} = 0,\ i, j = 1, .., n$. It should be mentioned that (3.72) is not coordinate–invariant, leaving open the possibility of considering changes of coordinates $\tilde{q} = \phi(q)$ for which the condition will hold. An extreme case is a system with zero Riemmanian curvature, which has a constant inertia matrix in some suitably defined coordinates [16].

Remark 3.4.7. Clearly, changing coordinates of the system *and* the target dynamics does not affect the matching conditions and, consequently, the PDEs will be equivalent. The subtlety here is that, as indicated in Remark 3.4.1, the target dynamic systems Σ_d and $\tilde{\Sigma}_d$ are parameterized by the triples $\{D_d, V_d, J_2\}$ and $\{\tilde{D}_d, \tilde{V}_d, \tilde{J}_2\}$, respectively. Along the same lines, feedback actions of the form $u = \alpha(q,p) + \beta(q,p)v$, with $\beta \in \mathbb{R}^{m\times m}$ full rank, will not affect the PDEs—that "live in Ker B". Indeed, for a system of the form $\dot{x} = f(x) + g(x)u$ and target dynamics $\dot{x} = F(x)\nabla H_d$ the matching equations are $g^\perp f = g^\perp F \nabla H_d$, with $g^\perp g = 0$, independently of the feedback action.

3.4.3 Solving the Original PDEs

Proposition 3 establishes that, solving the new PDEs (3.66), (3.67), the system Σ (3.26) in closed–loop with the IDA–PBC (3.68) takes, in the coordinates $(q, \tilde{p})$, the Hamiltonian form $\tilde{\Sigma}_d$ (3.64). Three natural questions arise:

- What are the dynamics of the closed–loop system in the original coordinates (q,p)?
- What is the relationship between the solutions of the new matching problem $\{\tilde{D}_d, \tilde{V}_d, \tilde{J}_2\}$ and the solutions of the original matching problem $\{D_d, V_d, J_2\}$?
- What is the relationship between the original matching controller $\hat{u}(q,p)$ and the new one $\hat{\tilde{u}}(q,\tilde{p})$?

The answers to these questions are given in the proposition below. The rationale of the proposition is best explained referring to Fig. 1. The connections between the nodes Σ, $\tilde{\Sigma}$ and $\tilde{\Sigma}_d$ are given by Proposition 3. It remains to establish the connection with the original target dynamics node Σ_d. Towards this end, Σ_d in (3.17) will be written in the new coordinates and then it will be proved existence of a *bijective mapping* $\Psi : \{\tilde{D}_d, \tilde{V}_d, \tilde{J}_2\} \rightarrow \{D_d, V_d, J_2\}$, that makes the transformed system *equal* to $\tilde{\Sigma}_d$—that is, with the same structure matrix and the same Hamiltonian function.[5] This proves that Σ and Σ_d match and, consequently, the corresponding parameters $\{D_d, V_d, J_2\}$ solve the PDEs and define the control $\hat{u}(q,p)$.

Instrumental for the proof of the proposition is the following simple fact, whose proof is established via direct calculations.

Fact 1. *Consider the system* $\dot{x} = F(x)\nabla H_d$*, where* $x \in \mathbb{R}^n$*,* $F \in \mathbb{R}^{n\times n}$ *and* $H_d : \mathbb{R}^n \rightarrow \mathbb{R}$*, and the vector function* $x = \phi(z)$*, where* $\phi : U \rightarrow \mathbb{R}^n$ *with* $\det \frac{\partial \phi}{\partial z} \neq 0 \ \ \forall z \in U$*. Then,* $\dot{z} = \bar{F}(z)\nabla\tilde{H}_d$*, where*

$$\bar{F}(z) := D^{-1}(z)\tilde{F}(z)D(z), \quad \tilde{H}_d(z) := H_d(\phi(z)), \quad \tilde{F}(z) := F(\phi(z)), \quad (3.78)$$

with $D(z) := \frac{\partial \phi}{\partial z} \in \mathbb{R}^{n\times n}$*.*

[5]The mapping Ψ can also be derived computing the Poisson brackets of the coordinates $(q,\tilde{p})$, as done in [13] to establish the equivalence between controlled Lagrangians and IDA–PBC.

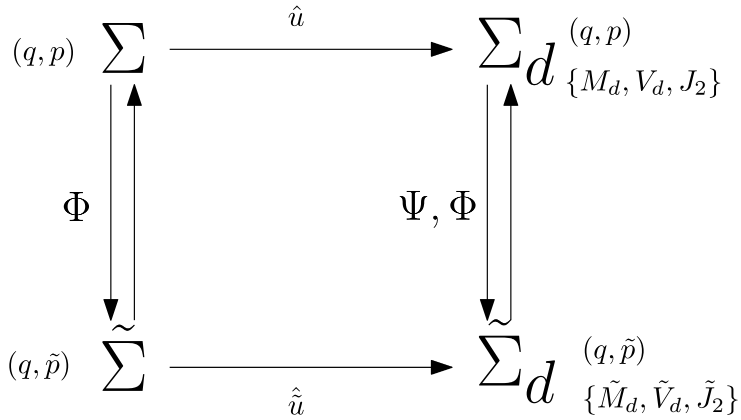

Figure 3.4: Diagram description of the systems transformations.

Proposition 6. *The triple* $\{\tilde{D}_d, \tilde{V}_d, \tilde{J}_2\}$ *solves the new matching equations (3.66),(3.67) if and only if the triple* $\{D_d, V_d, J_2\}$ *solves the original matching equations (3.27),(3.28), where*[6]

$$
\begin{aligned}
D_d &= T\tilde{D}_d T^T \\
V_d &= \tilde{V}_d \\
J_2(q,p) &= T\tilde{J}_2(q, T^{-1}p)T^T + S(q, T^{-1}p)D^{-1}T\tilde{D}_d T^T \\
&\quad -T\tilde{D}_d T^T D^{-1} S^T(q, T^{-1}p)
\end{aligned}
\tag{3.79}
$$

and $S(q,\tilde{p})$ *is given in (3.62). Furthermore, the control that matches* Σ *to* Σ_d *is obtained as* $\hat{u}(q,p) = \hat{\tilde{u}}(q, T^{-1}(q)p)$*, with* $\hat{\tilde{u}}$ *defined in (3.68).*

Proof. To simplify the expressions, denote $x = \mathrm{col}(q,p)$, $z = \mathrm{col}(q,\tilde{p})$,

$$
x = \phi(z) = \begin{bmatrix} q \\ T\tilde{p} \end{bmatrix}, \quad F(x) = \begin{bmatrix} 0 & D^{-1}(q)D_d(q) \\ -D_d(q)D^{-1}(q) & J_2(q,p) \end{bmatrix},
$$

and compute

$$
D(z) = \frac{\partial \phi}{\partial z} = \begin{bmatrix} I & 0 \\ S(q,\tilde{p}) & T(q) \end{bmatrix}, \quad D^{-1}(z) = \begin{bmatrix} I & 0 \\ -T^{-1}(q)S(q,\tilde{p}) & T^{-1}(q) \end{bmatrix}.
$$

[6]For simplicity, the argument q in the functions that depend only on q have been omitted.

A simple calculation yields

$$\bar{F}(z) = D^{-1}(z)\tilde{F}(z)D(z) = \begin{bmatrix} 0 & D^{-1}\tilde{D}_d T^{-T} \\ -T^{-1}D_d D^{-1} & \bar{F}_{22}(z) \end{bmatrix}$$

where

$$\bar{F}_{22}(z) := -T^{-1}S(q,\tilde{p})D^{-1}D_d T^{-T} + T^{-1}D_d D^{-1}S^T(q,\tilde{p})T^{-T} + T^{-1}J_2(q,T(q)\tilde{p})T^{-T}.$$

Replacing (3.79) above yields

$$\bar{F}(z) = \begin{bmatrix} 0 & D^{-1}T\tilde{D}_d \\ -\tilde{D}_d T^T D^{-1} & \tilde{J}_2(q,\tilde{p}) \end{bmatrix},$$

which is the structure matrix given in (3.17). Finally, calculating H_d in the coordinates $(q,\tilde{p})$ yields the expression $\frac{1}{2}\tilde{p}^T T^T D_d^{-1} T\tilde{p} + V_d$, which, upon replacement of (3.79), coincides with the $\tilde{H}_d(q,\tilde{p})$ defined in (3.64). □□□

3.4.4 Examples

Pendulum on a Cart

The dynamic equations of the pendulum on a cart are given by (3.26) with $n = 2$, $m = 1$, and

$$D(q_1) = \begin{bmatrix} 1 & b\cos q_1 \\ b\cos q_1 & c \end{bmatrix}, \; V(q_1) = a\cos q_1, \; B = e_2,$$

$$a = \frac{g}{\ell}, \; b = \frac{1}{\ell}, \; c = \frac{D+m}{m\ell^2}$$

where q_1 denotes the pendulum angle with the upright vertical, q_2 the cart position, m and ℓ are, respectively, the mass and the length of the pendulum, D is the mass of the cart and g is the gravity acceleration. The equilibrium to be stabilized is the upward position of the pendulum with the cart placed in *any desired location*, which corresponds to $q_{1\star} = 0$ and an arbitrary $q_{2\star}$.

Noting that $B^{\perp} = e_1^T$, the KE PDE (3.58) takes the form

$$\sum_{i=1}^{2} (e_1^T D_d D^{-1} e_i) \frac{\partial D_d}{\partial q_i} - D_d \frac{\partial D^{-1}}{\partial q_1} D_d = - \begin{bmatrix} 0 & \alpha_1(q) \\ \alpha_1(q) & 2\alpha_2(q) \end{bmatrix}, \tag{3.80}$$

with α_i being free functions. Using these functions it is possible to "solve" two of the three equations above, so it remains only one PDE to be solved. To simplify the expression of this equation, D_d is made function only of q_1, thus leading to the ODE

$$\begin{aligned} &(cm_{11} - bm_{12}\cos q_1)\frac{dm_{11}}{dq_1} \\ &= \frac{2b\sin q_1}{c - b^2\cos^2 q_1}[b\cos q_1(cm_{11}^2 + m_{12}^2) - (c + b^2\cos^2 q_1)m_{11}m_{12}], \end{aligned}$$

where $m_{ij}(q_1)$ is the ij–element of the matrix D_d. Even using m_{12} as a degree of freedom finding a solution to this ODE is a daunting task.

In order to simplify the PDE, the technique proposed above will be applied. By computing the Coriolis and centrifugal forces matrix using (3.77), it is easy to see that the condition $B^{\perp}C\dot{q} = 0$ is satisfied. Therefore, by taking $T = D$, the PDE (3.66) becomes:

$$B^{\perp} D \left[\tilde{D}_d \nabla_q (\tilde{p}^T \tilde{D}_d^{-1} \tilde{p}) - 2\tilde{J}_2 \tilde{D}_d^{-1} \tilde{p} \right] = 0. \tag{3.81}$$

It is easy to see that the hypotheses of Proposition 5 are satisfied with $r = 1$. By selecting

$$\Psi(\mu) = \frac{-k\sin\mu}{m_3 - b^2\cos^2\mu}, \quad \tilde{D}_d^0 = \begin{bmatrix} \frac{kb^2}{3}\cos^3 q_{1\star} & -\frac{kb}{2}\cos^2 q_{1\star} \\ -\frac{kb}{2}\cos^2 q_{1\star} & k\cos q_{1\star} + m_{22}^0 \end{bmatrix}$$

with $k > 0$ and $m_{22}^0 \geq 0$ free parameters, it follows that a solution is provided by

$$\tilde{D}_d = \begin{bmatrix} \frac{kb^2}{3}\cos^3 q_1 & -\frac{kb}{2}\cos^2 q_1 \\ -\frac{kb}{2}\cos^2 q_1 & k\cos q_1 + m_{22}^0 \end{bmatrix}, \quad \tilde{J}_2 = \tilde{p}^T \tilde{D}_d^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Such a solution is positive definite and bounded for all $q_1 \in (\frac{-\pi}{2}, \frac{\pi}{2})$.

Regarding the potential energy, it can be seen that a solution of the PDE (3.67) is given by

$$\tilde{V}_d = \frac{3a}{kb^2 \cos^2 q_1} + \frac{P}{2}\left[q_2 - q_{2\star} + \frac{3}{b}\ln(\sec q_1 + \tan q_1) + \frac{6m_{22}^0}{kb}\tan q_1\right]^2,$$

with $P > 0$ arbitrary and $q_{2\star}$ the cart position to be stabilized. The expression of the control law, the asymptotic analysis of the closed–loop system and some representative simulations may be found in [2].

Furuta's Pendulum

The dynamic equations of Furuta's pendulum are given by (3.26) with $n = 2$, $m = 1$, and

$$D(q_1) = \begin{bmatrix} 1 & b\cos q_1 \\ b\cos q_1 & c + \sin^2 q_1 \end{bmatrix}, V(q_1) = a\cos q_1, B = e_2,$$

$$a = \frac{mg\ell}{I_p}, b = \frac{mr\ell}{I_a}, c = \frac{I_a + mr^2}{I_p}$$

where q_1 denotes the pendulum angle with the upright vertical, q_2 the angular position of the arm, m and ℓ are, respectively, the mass and the length of the pendulum, r is the length of the arm, I_p and I_a are the inertia of the pendulum and the arm respectively, and g is the gravity acceleration. See [3, 33] for details. The equilibrium to be stabilized is the upward position of the pendulum with the arm placed at *any desired angle*, i.e., $q_{1\star} = 0$ and an arbitrary $q_{2\star}$.

Since $B^\perp = e_1^T$ the KE PDE (3.58) takes the form (3.80) where, again, only one PDE needs to be solved—which is, unfortunately, very complex. Furthermore, it is straightforward to see that the condition $B^\perp C\dot{q} = 0$ is *not satisfied*. Even though with $T = D$ the forcing term cannot be canceled, the KE PDE is considerably simplified with this selection of T. Indeed, the forcing term in the original coordinates (q,p) is

$$
\begin{aligned}
&B^{\perp}\nabla_q(p^TD^{-1}p)\\
&= -\frac{b\sin q_1}{(\det D)^2}\, p^T \begin{bmatrix} 2(c+1)b\cos q_1 & -((b^2+1)\cos^2 q_1 + c + 1) \\ -((b^2+1)\cos^2 q_1 + c+1) & \frac{2(b^2+1)}{b}\cos q_1 \end{bmatrix} p,
\end{aligned}
$$

where $\det D = c + \sin^2 q_1 - b^2\cos^2 q_1$. Taking $T = D$ yields the following forcing term (right hand side of (3.71)) in the new coordinates $(q,\tilde{p})$:

$$
B^{\perp}\left[\nabla_q(p^TD^{-1}p) + e_1^TD^{-1}p\frac{dD}{dq_1}D^{-1}p\right]_{p=D\tilde{p}} = -\frac{1}{2}\sin(2q_1)\ \tilde{p}^T\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}\tilde{p}.
$$

The dramatic simplification of the forcing term can hardly be overstated.

In the new coordinates $(q,\tilde{p})$ and choosing, as before, the matrix $\tilde{D}_d$ function only of the q_1–coordinate, the KE PDE (after factorization of $\tilde{p}$) becomes

$$
\begin{aligned}
\gamma_{A_1}(q_1)\frac{d\tilde{D}_d}{dq_1} &= \sin(2q_1)\begin{bmatrix} m_{12}^2 & m_{12}m_{22} \\ m_{12}m_{22} & m_{22}^2\end{bmatrix}\\
&\quad - \begin{bmatrix} \alpha_1 2b\cos q_1 & \alpha_2 b\cos q_1 - \alpha_1 \\ \alpha_2 b\cos q_1 - \alpha_1 & -2\alpha_2\end{bmatrix}, \qquad (3.82)
\end{aligned}
$$

where γ_{A_1}—the first element of the vector γ_A defined in (3.75)—is $\gamma_{A_1} = m_{11} + m_{12}b\cos q_1$. Before proceeding to solve this equation, it is worth recalling that the only motivation to modify the inertia matrix is to enforce a coupling term, captured by the vector γ_A, in the potential energy PDE (3.67), see Remark 1 of [2]. This equation takes the form $\gamma_{A_1}\nabla_{q_1}\tilde{V}_d + \gamma_{A_2}\nabla_{q_2}\tilde{V}_d = -a\sin q_1$, and its solution is obtained invoking Proposition 4 of [2] as

$$
\tilde{V}_d(q) = -a\int_0^{q_1}\frac{\sin\mu}{\gamma_{A_1}(\mu)}d\mu + \Phi\left(q_2 - \int_0^{q_1}\frac{\gamma_{A_2}(\mu)}{\gamma_{A_1}(\mu)}d\mu\right),
$$

with Φ an arbitrary differentiable function. It is clear from this expression that, if $\gamma_{A_1} = -k_1 < 0$, then the minimum condition for $\tilde{V}_d$ can be assigned with a suitable selection of Φ. Motivated by this argument, it is convenient to fix $m_{11} = k_1$ and $m_{12} = \frac{-2k_1}{b\cos q_1}$, and then solving for $\alpha_1,\ \alpha_2$ the $(1,1)$ and

(1, 2) equations of (3.82). This yields

$$\alpha_1 = \frac{4k_1^2}{b^3 \cos^2 q_1} \sin q_1, \ \alpha_2 = 2m_{22} - \frac{k_1}{\cos^2 q_1}.$$

Plugging back these expressions into the (2, 2) equation of (3.82) yields the ODE

$$k_1 \frac{dm_{22}}{dq_1} = -\sin(2q_1) m_{22}^2 - 4m_{22} + \frac{2k_1}{\cos^2 q_1}.$$

Solving this equation and putting all the pieces together yields

$$\tilde{D}_d = \begin{bmatrix} k_1 & -\frac{2k_1}{b\cos q_1} \\ -\frac{2k_1}{b\cos q_1} & \frac{2k_1}{b^2\cos^2 q_1} + \frac{k_1}{k_2^2 + \sin^2 q_1} \end{bmatrix},$$

where $k_2 > 0$ is a free parameter stemming from the initial conditions of m_{22}. This matrix is positive definite for all $q_1 \in \left(-\arccos\sqrt{\frac{1+k_2^2}{1+b^2}}, \arccos\sqrt{\frac{1+k_2^2}{1+b^2}} \right)$— which suggests to choose k_2 sufficiently small, with the largest admissible angle $\pm \arccos\sqrt{\frac{1}{1+b^2}}$.

Selecting a quadratic function for Φ, the potential energy becomes

$$\tilde{V}_d = -\frac{a}{k_1}\cos q_1 + \frac{P}{2}\left[q_2 - q_{2\star} + \frac{b}{k_2}\arctan\left(\frac{\sin q_1}{k_2}\right)\right]^2, \quad P > 0$$

that clearly has an isolated minimum at the desired equilibrium; hence, the IDA–PBC controller ensures its stability. A detailed presentation of this result, including the proof of asymptotic stability and *experimental* validation, is reported in [4]. It is also shown there that for all solutions of $m_{11} + m_{12} b\cos q_1 = -k_1$, the maximum interval of positive definiteness of $\tilde{D}_d$ is the one given above.

A more general class of mechanical systems

The pendulum on a cart and Furuta's pendulum are two of the most common benchmarks to which the theory developed in this section can be applied. However, it is possible to find a more general class of systems for which Assumption A holds. Consider a generic mechanical system characterized by the following inertia and input matrices:

$$M(q_1) = \begin{bmatrix} m_1 f^2(q_1) & m_2 f(q_1) \\ m_2 f(q_1) & m_3 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $f(q_1) \neq 0 \;\; \forall \, q_1$, $m_1 m_3 - m_2^2 > 0$, and $m_1 > 0$. It is straightforward to see that assumption A) holds with

$$T(q_1) = \begin{bmatrix} f(q_1) & 0 \\ 0 & 1 \end{bmatrix},$$

thus establishing a whole class of mechanical systems, with arbitrary potential energy function and with the given inertia matrix, for which it is possible to design an IDA–PBC control law. As an example, consider the two-DOF system depicted in Figure 3.5, where $h(q_1) = \int_0^{q_1} f(\theta)d\theta$, Q, R, S are double prismatic joints. The joint Q is constrained to on the curve represented by h, and R and S are connected through a rigid rod. Π is the horizontal plane, and it is parallel to the plane (r, q_2); the frame (p_1, p_2) is rotated by an angle α with respect to the projection (r', q_2') of (r, q_2) on Π.

It is easy to see that the coordinates q_1, q_2 can be taken as generalized coordinates, and the kinetic energy is given by

$$\frac{1}{2} \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} M_1 f^2(q_1) & M_2 f(q_1) \\ M_2 f(q_1) & M_3 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}.$$

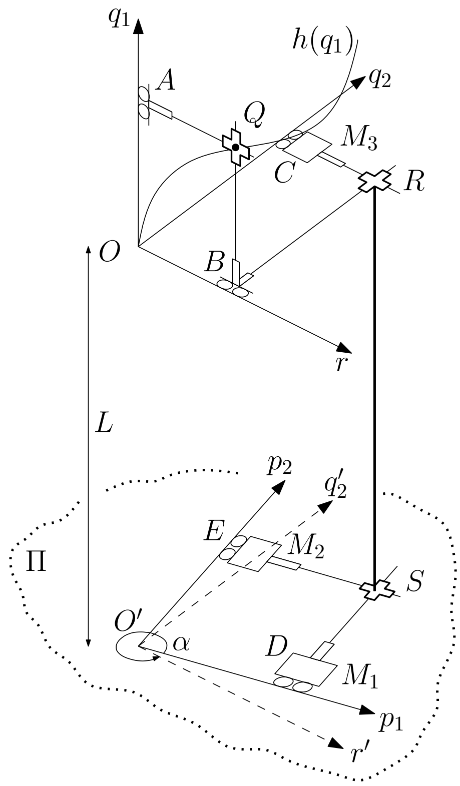

Figure 3.5: Schematic of the system considered in the example.

Chapter 4

Bipedal dynamic robot locomotion

4.1 Introduction

Mechanical biped locomotion has been studied for well over 30 years. It has been motivated by its potential use as a means of locomotion in rough terrain or environments with discontinuous supports, as well as its potential benefits to prostheses development and testing. A quick search of the literature, see for example [12], reveals over a hundred walking mechanisms built by public research laboratories, universities and major companies. Nevertheless, conceptual control breakthroughs have not kept pace with the technological developments. A canonical control problem in bipedal robots is how to design a controller that generates closed-loop motions, such as walking or running, that are periodic and stable, that is, stable limit cycles. Most of the state of the art is characterized by a heavy reliance on heuristics or on principles such as the zero moment point (ZMP) criterion [37, 88, 89] that do not ensure stability. As a result, only slow motions may be achieved. Truly dynamic motions, such as balancing, running or fast walking, are excluded with these approaches [34]. The most common approach to control is through the tracking of precomputed reference trajectories. The trajectories may be determined via analogy, either with biological systems [90], [7], or with simpler, passive, mechanical-biped systems [53, 84, 85]. They can be generated by an oscillator, such as van der Pol's oscillator [44], or computed through optimization of various cost criteria, such as minimum expended control energy over a walking cycle [17, 18, 22, 80]. Within the context of tracking, many different control methods have been explored, including continuous-time methods based on PID controllers, [25, 26, 73] computed torque and sliding mode control [56, 19, 78, 51], or essentially discrete-time methods, based on impulse control. Other control methods have been investigated that do not rely on precomputed reference trajectories for the angular positions. These include controlling energy, angular momentum, and others [77, 81, 46, 38, 39, 35, 76, 24].

A major breakthrough in this field is represented by the work of Jessy W. Grizzle and his collaborators, who in the past eight years have provided sound theoretical means and important technological insights for addressing the problem. In their recent book on the subject [91], the authors provide effective yet intuitive control schemes to create stable dynamic motions in biped robots with or without feet, including running. The book summarizes and extends the result reported in [36, 92, 57], most of which have received awards for their enlightening importance. The results reported in the book have been successfully tested on a robot prototype called RABBIT (see Figure 4.2). RABBIT is a rigid 5-link mechanism constituted by two legs with knees and a torso and having point feet. This latter choice is motivated, as thoroughly discussed in [91], by the necessity of considering underactuation in designing a controller. In legged robots, underactuation arises during the swing phase of the step, when the stance leg is pinned to the ground and no control effort can be exerted by the point foot. One may argue that in humans the ankle does provide such an effort, but the main point here is that even in humans there exists a phase of the motion, namely the pivoting at the toe, during which the system as a whole is truly underactuated. As al-

Figure 4.1: The five-link biped robot RABBIT.

ready presented in Chapter 3, underactuation greatly complicates the task of designing a controller and analyzing the stability of orbits of the closed-loop system. In the last years a lot of research effort has been devoted in designing controllers for RABBIT-like robots in which tuned springs are introduced in the rigid mechanism. Such a design choice has on one hand two benefits: within the strides of walking and running, springs can store and release some of the energy that would otherwise be lost as actuators do negative work [7]; and at foot touchdown events, springs isolate reflected motor inertias from the energy-dissipating effects of rigid collisions. On the other hand, delivering torque through compliant elements poses several challenges for control design, since there is an obvious increase in the degrees of freedom of the robot model, and hence, the degree of underactuation. In [58, 59] a solution to this problem is provided by using a dynamic extension of the controller designed in [92]. In this chapter a different approach to the problem will be described first, which is based on a singular perturbation-like analysis. Such an approach is suitable for a fairly general class of nonlinear systems with impulse effects. Then, for the particular case of biped robots, it will be shown that this general technique reduces to straightforward solution of the problem of compliant actuation, which does not require a dynamic extension as in [58].

In the next section, the notions of systems with impulse effects and a powerful technique for controlling such a class of systems, namely the hybrid zero dynamics, will be introduced. Next, the approach based on singular perturbations will be presented, and finally its application to bipedal robots with compliant actuation.

4.2 Background material

Consider a control system

$$\dot{x} = f(x) + g(x)u, \tag{4.1}$$

where $x \in \mathcal{X} \subset I\!R^n, u \in I\!R^m$. An impact (or switching) surface is a co-dimension one surface $S \subset \mathcal{X}$ at which the solutions of (4.1) are subject to an instantaneous transition. This transition can be modeled as a re-initialization of the differential equation, regulated by a map $\Delta : S \to \mathcal{X}$, which determines the new initial condition for (4.1) as a function of the point at which the solution impacts S. The resulting system may be written as

$$\Sigma : \begin{cases} \dot{x} &= f(x) + g(x)u \quad x \notin S \\ x^+ &= \Delta(x^-) \quad x^- \in S. \end{cases} . \tag{4.2}$$

In simple terms, a solution of (4.2) is determined by the differential equation (4.1), until its state "impacts" the surface S, when an instantaneous re-initialization occurs. Right after the impact, the state evolves again according to (4.1), starting from the new initial condition provided by Δ.

4.2.1 Periodic orbits

Consider the system (4.2) with $u = 0$, namely:

$$\Sigma : \begin{cases} \dot{x} &= f(x) \quad x \notin S \\ x^+ &= \Delta(x^-) \quad x^- \in S. \end{cases} . \tag{4.3}$$

A solution $\varphi(t)$ of (4.2) is *periodic* if there exists a finite $T > 0$ such that $\varphi(t+T) = \varphi(t)\ \forall t \in [0, +\infty)$. The set $\mathcal{O} = \{\varphi(t), t \geq 0\}$ with $\varphi(t)$ periodic is called a *periodic orbit* of (4.2). In biped robotics, the main control objective is creating periodic orbits in the closed-loop system that involve impacts. Such impacts correspond to the robot's swing leg touching the ground, and the periodic orbit to a walking gait. An important characteristic of this type of orbits is that they are not closed ([36, 57]). In the following, its closure will be denoted $\bar{\mathcal{O}}$. A periodic orbit $\mathcal{O}$ is *transversal* to S if its closure intersect S in exactly one point, and at the intersection $\mathcal{O}$ is not tangent to S (more formally, for $x^\star = \bar{\mathcal{O}} \cap S, L_f H(x^\star) := \frac{\partial H}{\partial x}(x^\star) f(x^\star) \neq 0$).

Analogously to equilibrium points, it is possible to define stability of orbits in the sense of Lyapunov. For, consider any norm $\|\cdot\|$ on $\mathcal{X}$, and define the distance between a point x and a set M as $\text{dist}(x, M) = \inf_{y \in M} \|x - y\|$. A periodic orbit $\mathcal{O}$ is stable in the sense of Lyapunov if for every $\epsilon > 0$ there exists an open neighborhood $\mathcal{V}$ of $\mathcal{O}$ such that for every $p \in \mathcal{V}$, there exists a solution $\varphi : [0, \infty) \to \mathcal{X}$ of (4.2) satisfying $\varphi(0) = p$ and $\text{dist}(\varphi(t), \mathcal{O}) < \epsilon$ for all $t \geq 0$. $\mathcal{O}$ is attractive if there exists an open neighborhood $\mathcal{V}$ of $\mathcal{O}$ such that, for every $p \in \mathcal{V}$, there exists a solution $\varphi : [0, \infty) \to \mathcal{X}$ of (4.2) satisfying $\varphi(0) = p$ and $\lim_{t \to \infty} \text{dist}(\varphi(t), \mathcal{O}) = 0$. $\mathcal{O}$ is asymptotically stable in the sense of Lyapunov if it is both stable and attractive. Moreover, a periodic orbit $\mathcal{O}$ is *exponentially stable* if there exists $\delta > 0, N > 0$ and $\gamma > 0$ such that

$$\forall t \geq 0, \ \text{dist}(\varphi(t, x), \mathcal{O}) \geq N e^{-\gamma t} \text{dist}(x, \mathcal{O}) \tag{4.4}$$

whenever $\text{dist}(x_0, \mathcal{O}) < \delta$.

4.2.2 Poincaré return map

The method of Poincaré sections is a primary tool for testing stability of periodic orbit in nonlinear systems. In the case of systems with impulse effects, the natural choice for the Poincaré section is represented by the impact surface S (see Figure).

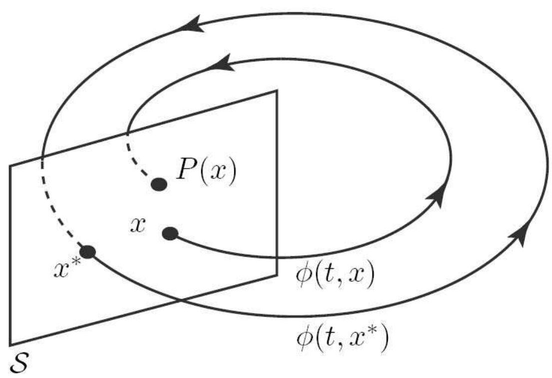

Figure 4.2: Geometric interpretation of the Poincaré map.

In order to calculate the Poincaré map, define the *time-to-impact* function as

$$T_I(x) = \begin{cases} \inf\{t \geq 0 : \varphi(t,x) \in S\}, & \text{if such } t \text{ exists,} \\ \infty, & \text{otherwise;} \end{cases} \tag{4.5}$$

The Poincaré map P associated with Σ can be written as

$$P : S \to S, \quad x \mapsto \varphi(T_I \Delta(x), \Delta(x)). \tag{4.6}$$

The method of Poincaré sections, in its version for testing exponential stability, can be stated as follows:

Theorem 4 (Method of Poincaré sections [36, 57])**.** *Under hypotheses H1), if system (4.2) has a transversal periodic orbit, then the following are equivalent:*

i) $x^\star$ is an exponentially stable fixed point of P:

ii) $\mathcal{O}$ is an exponentially stable periodic orbit.

An important property of P entailed by hypotheses H1) is its differentiability on a neighborhood of $x^\star$. Indeed, the differentiability of T_I is proven in [74] at each point of $\tilde{S} := \{x \in S : T_I(x) < \infty \text{ and } L_f H(P(x)) \neq 0\}$ and Δ and f are differentiable by hypothesis. Hence, exponential stability can be simply tested by computing the eigenvalues of the linearization of P at its fixed point.

4.2.3 The hybrid zero dynamics

In this section the control technique based on the concept of hybrid zero dynamics (HZD) will be briefly introduced. More details can be found in [92] and [91]. Intuitively, it is based on the method of computed torque or inverse dynamics. It consists of defining a set of outputs, equal in number to the inputs, and then designing a feedback controller that asymptotically drives the outputs to zero. The task that the robot is to achieve is encoded into the set of outputs in such a way that the nulling of the outputs is (asymptotically) equivalent to achieving the task, whether the task be asymptotic convergence to an equilibrium point, a surface, or a time trajectory. For a system modeled by ordinary differential equations (in particular, no impact dynamics), the maximal internal dynamics of the system that are compatible with the output being identically zero are called the zero dynamics [40, 41, 60]. Hence, the method of computed torque, which is asymptotically driving a set of outputs to zero, is indirectly designing a set of zero dynamics for the robot. Since in general the dimension of the zero dynamics is considerably less than the dimension of the model itself, the task to be achieved by the robot has been implicitly encoded into a lower dimensional system. One of the main points of applying such a technique to a system with impulse effects is that this process can be exploited in the design of feedback controllers even in the presence of impacts. The basic idea is that an output should give rise to a zero dynamics for the continuous portion of the model (4.2) and the resulting zero dynamics manifold should be invariant under the impact map. This is formalized as follows. Associate a C^∞ output $y = h(x)$ to (4.2) and assume that h has vector relative degree $(k, \ldots, k)$ with respect to the continuous portion of the hybrid model (that is, h_i has relative degree k_i with respect to (4.1) and the decoupling matrix is square and invertible [40]) and that there exists $x_0 \in \mathcal{X}$ such that $h(x_0) = 0$. Let Z be the zero dynamics manifold for the continuous portion of the dynamics and let $u^\star$ be the feedback (unique on Z) such that for all $x \in Z$, $f^\star(x) := f(x) + g(x)u^\star(x) \in T_x Z$. If in addition, $S \cap Z$ is a C^∞ manifold of dimension one less than Z and

$$\Delta(S \cap Z) \subset Z \tag{4.7}$$

then Z is a hybrid zero dynamics manifold for S and the restriction dynamics

$$\Sigma|_Z : \left\{ \begin{array}{rcll} \dot{z} & = & f^\star|_Z(z) & z \notin S \cap Z \\ z^+ & = & \Delta|_{S\cap Z}(z^-) & z^- \in S \cap Z. \end{array} \right. . \tag{4.8}$$

is called the hybrid zero dynamics of S, where $f^\star|_Z$ and Δ are the restrictions of $f^\star$ and Δ to Z and $S \cap Z$, respectively. Note that (4.7) is equivalent to

$$\forall x \in \Delta(S \cap Z) \text{and } \ \forall 0 \leq i \leq k-1, L_f^i h(x) = 0.$$

Suppose that an output $h(x)$ has already been designed so that there exists an asymptotically stable periodic orbit in the hybrid zero dynamics. The following theorem [57] gives sufficient conditions for stabilization of such a periodic orbit in the full model.

Theorem 5. *Consider a C^∞ system with impulse effects (4.2) with hybrid zero dynamics (4.8). Suppose that (4.8) contains a periodic orbit $\mathcal{O}$ that is exponentially stable and transversal to S. If in addition there exists a vector of functions $\phi : \mathcal{X} \to I\!R^{n-mk}$ such that $L_g\phi \equiv 0$ and*

$$\Phi(x) = \Big(h(x); L_f h(x); \cdots ; L_f^{k-1} h(x)\Big)$$

is a diffeomorphism on $\mathcal{X}$, then the orbit $\mathcal{O}$ is exponentially stabilizable. For any choice of matrices $K_0, K_1, \cdots, K_{k-1}$ satisfying that $s^k I + K_{k-1}s^{k-1} + \cdots + K0$ is Hurwitz, for $\epsilon > 0$ sufficiently small, the feedback

$$u(x) = -\Big(L_g L_f^{k-1} h(x)\Big)^{-1}\left(L_f^k h(x) + \sum_{i=0}^{k-1} \frac{K_i}{\epsilon^{k-i}} L_f^i h(x)\right)$$

applied to (4.2) renders $\mathcal{O}$ exponentially stable in the (full-dimensional) closed-loop system.

One critical aspect of applying Theorem 5 is the selection of an output $h(x)$ that leads to a hybrid zero dynamics. In [58, 59] a solution is provided based on a dynamic extension of (4.2). The following section addresses the same problem by using a different approach, namely assuming partial invariance of the zero dynamics manifold under impacts, and proving existence and exponential stability of orbits outside the zero dynamics manifold.

4.3 Partial impact invariance

Consider the system with impulse effects (4.2) and suppose that the following conditions are satisfied:

H1.1) $\mathcal{X} \in \mathbb{R}^n$ is open and connected, H1.2) $f : \mathcal{X} \to \mathbb{R}^n$ is $\mathcal{C}^1$,

H1.3) $g : \mathcal{X} \to \mathbb{R}^{n\times m}$ is $\mathcal{C}^1$ and $G := \text{span}(g_1, \ldots, g_m)$ is involutive,

H1.4) The system $\dot{x} = f(x) + g(x)u$ has a vector relative degree[1] $\{4, \ldots, 4\}$ with respect to the output $y = h(x),\ y \in \mathbb{R}^m$,

H1.5) $H : \mathcal{X} \to \mathbb{R}$ is $\mathcal{C}^1$,

H1.6) $S := \{x \in \mathcal{X} : H(x) = 0\}$ is non-empty and $\forall x \in S, \frac{\partial H}{\partial x} \neq 0$ (that is, S is $\mathcal{C}^1$ and has co-dimension one),

H1.7) $\Delta : S \to \mathcal{X}$ is $\mathcal{C}^1$, and

H1.8) $\Delta(S) \cap S = \emptyset$.

Then, there exist coordinates[2] $x := (\eta, z)$ with[3] $\eta = (\eta_1, \ldots, \eta_4) \in \mathcal{V} := \mathcal{V}_1 \times \cdots \times \mathcal{V}_4, \eta_i \in \mathcal{V}_i \subset \mathbb{R}^m, z \in \mathcal{W} \subset \mathbb{R}^{n-2m}$, and a state feedback $u = -(L_g L_f^3 h(x))^{-1}[L_f^4 h(x) + \sum_{i=0}^{3} K_i\ L_f^i h(x)]$ such that the continuous part of the closed loop system can be written as

$$\begin{aligned} \dot{\eta} &= A\eta \\ \dot{z} &= q(\eta, z), \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -K_0 & -K_1 & -K_2 & -K_3 \end{bmatrix}. \tag{4.9}$$

[1] the case of uniform vector relative degree $(k, \cdots, k)$ is easily obtained by a straightforward generalization.

[2] Without loss of generality, it will be assumed that the system is already written in these coordinates.

[3] $\mathcal{W}$ can be interpreted as the image through a homeomorphism ψ of Z, where (Z, ψ) is a coordinate chart on Z.

As in [57], a "high gain" version of the previously proposed controller will be used here, that is

$$u = u_\epsilon = -(L_g L_f^3 h(x))^{-1}[L_f^4 h(x) + \sum_{i=0}^{3} \frac{K_i}{\epsilon^{4-i}} L_f^i h(x)] \tag{4.10}$$

The resulting closed loop system, denoted by Σ_ϵ, can be written as follows:

$$\Sigma_\epsilon : \begin{cases} \begin{bmatrix} \dot{\eta} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A(\epsilon)\eta \\ q(\eta, z) \end{bmatrix} & x \notin S \\ x^+ = \Delta(x^-) & x^- \in S, \end{cases} \tag{4.11}$$

where

$$A(\epsilon) = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -\frac{K_0}{\epsilon^4} & -\frac{K_1}{\epsilon^3} & -\frac{K_2}{\epsilon^2} & -\frac{K_1}{\epsilon} \end{bmatrix}.$$

Denote with $\phi^\epsilon(t, x) := (\phi^\epsilon_\eta(t, \eta), \phi^\epsilon_z(t, \eta, z))$ the solution of the continuous part of Σ_ϵ starting from (η, z). Moreover, define the time-to-impact function T_I^ϵ as

$$T_I^\epsilon(x) = \begin{cases} \inf\{t \geq 0 : \phi^\epsilon(t, x) \in S\}, & \text{if such } t \text{ exists} \\ \infty. & \text{otherwise} \end{cases} \tag{4.12}$$

Then, the Poincaré map associated with Σ_ϵ is given by

$$P_\epsilon : S \to S, \quad x \mapsto \phi^\epsilon(T_I^\epsilon(\Delta(x)), \Delta(x)). \tag{4.13}$$

In [92, 57] the hybrid invariance of the zero dynamics manifold made possible to define the *restriction dynamics* (4.8), which is not possible here. However, inspired by [36], where it is shown that under certain hypotheses the zero dynamics manifold becomes invariant under high-gain control, the concept of *limit restriction dynamics* will be defined here, which can be regarded as the "slow part" in the framework of singular perturbation analysis. For, define the projection and extension operators π and σ as follows:

$$\pi : \mathcal{X} \to \mathcal{W}, \quad (\eta, z) \mapsto z.$$

$$\sigma : \mathcal{W} \to \mathcal{X}, \quad z \mapsto (0, z).$$

Then the limit restriction dynamics Σ_0 of Σ_ϵ is defined as:

$$\Sigma_0 : \begin{cases} \dot{z} = q(0,z) & \sigma(z) \notin S \\ z^+ = \Delta_0(z^-) & \sigma(z^-) \in S, \end{cases} \tag{4.14}$$

where $\Delta_0(z) = \pi \circ \Delta \circ \sigma(z)$. In other words, the limit restriction dynamics are given by the zero dynamics $q(0,z)$ for the continuous part, and by the projection on the zero dynamics manifold of the impact map Δ.

Call $\phi_z^0(t,z)$ the solution the continuous part of Σ_0 starting from z and define $\phi^0(t,z) := \sigma(\phi_z^0(t,z))$. Analogously to what has been done for Σ_ϵ, define the time–to–impact function as

$$T_I^0(z) = \begin{cases} \inf\{t \geq 0 : \phi^0(t,z) \in S\}, & \text{if such } t \text{ exists} \\ \infty, & \text{otherwise.} \end{cases} \tag{4.15}$$

Then, the Poincaré map associated with Σ_0 is:

$$P_0 : \pi(S \cap Z) \to \pi(S \cap Z), \quad z \mapsto \phi_z^0(T_I^0(\Delta_0(z)), \Delta_0(z)). \tag{4.16}$$

In order to state the main result, suppose the following hypotheses are met:

H2.1) K_0 through K_3 are such that A in (4.9) is Hurwitz;

H2.2) Partitioning z as $z = (z_1, z_2)$, $q(\eta, z)$ can be written as

$$q(\eta, z) = \begin{bmatrix} \alpha(\eta_1, z_1) z_2 + \beta(\eta_1, z_1)\eta_2 \\ \gamma(\eta_1, z_1) \end{bmatrix}, \tag{4.17}$$

where $\alpha(\eta_1, z_1), \beta(\eta_1, z_1), \gamma(\eta_1, z_1)$ are locally Lipschitz and uniformly bounded for all η_1, z_1;

H2.3) For $Z = \{(\eta, z) \in \mathcal{X} : \eta = 0\}$, $S \cap Z$ is a $(k-1)$–dimensional $\mathcal{C}^1$ embedded submanifold of Z, H2.4) Σ_0 has an exponentially stable periodic orbit $\mathcal{O}_0$ which is transversal to $S \cap Z$,

H2.5) $x^\star = (0, z^\star) := \bar{\mathcal{O}}_0 \cap S$ is such that the image $\Delta(S \cap Z)$ of $S \cap Z$ through Δ is tangent to Z at $\Delta(x^\star)$.

H2.6) h is such that $h \circ \Delta\big|_{S \cap Z} \equiv 0$.

The structure entailed by hypothesis H2.2) is characteristic of mechanical systems. In the following, whenever convenient, $\mathcal{W}$ will be partitioned accordingly as $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2$. Hypotheses H2.5) and H2.6) together are less strict than hypothesis H2.2) of [57], that is, $\Delta(S \cap Z) \subset Z$, which is equivalent to tangency of $\Delta(S \cap Z)$ to Z for all $x \in S \cap Z$. Hypothesis H2.4) can be given a useful interpretation in terms of properties of the Poincaré map P_0, as shown by the following lemma.

Lemma 4.3.1. Let $F : \mathbb{R}^p \to \mathbb{R}^p$ be a continuously differentiable map possessing an exponentially stable fixed point at the origin. Then, there exists a norm on $\mathbb{R}^p$ in which F is a contraction locally around the origin.

Proof. As a direct consequence of the hypotheses, $F(\xi)$ can be written as

$$F(\xi) = A\xi + G(\xi)$$

where A is Schur and G has the property that $\frac{\partial G}{\partial \xi}(0) = 0$.

The property of A being Schur implies the existence of positive definite matrices P and Q such that $A^T PA - P = -Q$. This is equivalent to say that all of the eigenvalues of the matrix $A^T PA - P$ are strictly less than zero. By continuity of eigenvalues on the entries of a matrix, it follows that there exists $\rho < 1$, sufficiently close to one, such that $A^T PA - \rho P =: -Q'$ is still negative definite.

The fact that $\frac{\partial G}{\partial \xi}(0) = 0$ implies that

$$\forall \gamma > 0 \; \exists r : \forall \, \|\xi\|_2 < r, \; \left\| \frac{\partial G}{\partial \xi}(\xi) \right\|_2 < \gamma.$$

By [61, Lemma 3.1], it follows that

$$\forall \gamma > 0 \ \exists r : \forall \|\xi\|_2 < r, \ \|G(\xi_1) - G(\xi_2)\|_2 < \gamma \|\xi_1 - \xi_2\|_2 .$$

Define $\|\xi\|_P := \left(\xi^T P \xi\right)^{1/2}$. It is easy to see that this is indeed a norm on $\mathbb{R}^p$. What has to be shown is that, for the value of $\rho < 1$ found above, it holds that $\|F(\xi_1) - F(\xi_2)\|_P^2 < \rho \|\xi_1 - \xi_2\|_P^2$, for ξ_1, ξ_2 in some neighborhood of the origin.

The following holds:

$$\begin{aligned}
&\|F(\xi_1) - F(\xi_2)\|_P^2 - \rho \|\xi_1 - \xi_2\|_P^2 \\
&= [F(\xi_1) - F(\xi_2)]^T P [F(\xi_1) - F(\xi_2)] - \rho(\xi_1 - \xi_2)^T P (\xi_1 - \xi_2) \\
&= [A(\xi_1 - \xi_2) + G(\xi_1) - G(\xi_2)]^T P [A(\xi_1 - \xi_2) + G(\xi_1) - G(\xi_2)] \\
&\quad -\rho(\xi_1 - \xi_2)^T P (\xi_1 - \xi_2) \\
&= -(\xi_1 - \xi_2)^T Q' (\xi_1 - \xi_2) + 2[G(\xi_1) - G(\xi_2)]^T P A (\xi_1 - \xi_2) \\
&\quad +[G(\xi_1) - G(\xi_2)]^T P [G(\xi_1) - G(\xi_2)] \\
&\leq -\lambda_{min}(Q') \|\xi_1 - \xi_2\|_2^2 + 2 \|G(\xi_1) - G(\xi_2)\|_2 \|P\|_2 \|A\|_2 \|\xi_1 - \xi_2\|_2 \\
&\quad +\lambda_{max}(P) \|G(\xi_1) - G(\xi_2)\|_2^2 \\
&\leq -(\lambda_{min}(Q') - 2\gamma \|P\|_2 \|A\|_2 + \lambda_{max}(P)\gamma^2) \|\xi_1 - \xi_2\|_2^2 \\
&=: -p(\gamma) \|\xi_1 - \xi_2\|_2^2 .
\end{aligned}$$

Choose $\gamma > 0$ such that $p(\gamma) > 0$ (this is possible because $p(0) > 0$). It follows that there exists $r > 0$ such that

$$\|F(\xi_1) - F(\xi_2)\|_P^2 < \rho \|\xi_1 - \xi_2\|_P^2 ,$$

for all ξ_1, ξ_2 less than r in norm. □□□

As a consequence, hypothesis H2.4) implies the existence of a norm $\|\cdot\|_{\mathcal{W}}$ on $\mathcal{W}$ such that P_0 is a contraction on some neighborhood of $z^\star$. More precisely, denoting $B_\delta(z^\star) = \{z : H(\sigma(z)) = 0, \|z - z^\star\|_{\mathcal{W}} \leq \delta\}$, it holds that

$$\exists \rho_0 < 1, \delta_0 > 0 : \forall z_a, z_b \in B_{\delta_0}(z^\star), \|P_0(z_a) - P_0(z_b)\|_{\mathcal{W}} \leq \rho_0 \|z_a - z_b\|_{\mathcal{W}} . \tag{4.18}$$

Clearly, in view of the partitioning introduced in hypothesis H2.2), it is legitimate to calculate the $\mathcal{W}$-norm of components of z, that is, interpret $\|z_2\|_{\mathcal{W}}$ as $\|(0, z_2)\|_{\mathcal{W}}$. This will be used in the subsequent developments.

It follows naturally to adopt a norm $\|\cdot\|_{\mathcal{X}}$ for $\mathcal{X} = \mathcal{V} \times \mathcal{W}$ given by the sum of a norm on $\mathcal{V}$, say, the 2-norm, and the norm $\|\cdot\|_{\mathcal{W}}$ on $\mathcal{W}$, that is:

$$\|(\eta, z)\|_{\mathcal{X}} = \|\eta\|_2 + \|z\|_{\mathcal{W}} \, .$$

Moreover, let $F_\alpha : \mathcal{W}_2 \to \mathcal{W}_1$ and $F_\beta : \mathcal{V}_2 \to \mathcal{W}$ be linear applications. Define their induced matrix norms as:

$$\|F_\alpha\|_\alpha := \max_{\|z_2\|_{\mathcal{W}}=1} \|F_\alpha \, z_2\|_{\mathcal{W}} \, ,$$

$$\|F_\beta\|_\beta := \max_{\|\eta_2\|_2=1} \|F_\beta \, \eta_2\|_{\mathcal{W}} \, ,$$

Finally, hypothesis H2.1) implies that there exist positive constants k_1, k_2 such that $\left\|e^{At}\right\|_2 \leq k_1 e^{-k_2 t}$. A similar bound can also be given for $\left\|e^{A(\epsilon)t}\right\|_2$. As a matter of fact, there exists a matrix $\Pi(\epsilon) = \text{blockdiag}(\epsilon^3 I, \epsilon^2 I, \epsilon I, I)$ such that $A(\epsilon) = \Pi(\epsilon)\frac{A}{\epsilon}\Pi^{-1}(\epsilon)$. Thus,

$$\left\|e^{A(\epsilon)t}\right\|_2 = \left\|\Pi(\epsilon) e^{A\frac{t}{\epsilon}} \Pi^{-1}(\epsilon)\right\|_2 \leq \max\{1, \epsilon^3\} \max\{1, \frac{1}{\epsilon^3}\} k_1 e^{-k_2 \frac{t}{\epsilon}} .$$

Here follows the main result of this note.

Theorem 6. *Suppose that Hypotheses H1) and H2) are met. Then, there exists $\bar{\epsilon} > 0$ such that, for each $\epsilon < \bar{\epsilon}$, P_ϵ has an exponentially stable fixed point.*

The main difference between this result and the one given in [57] is the fact that $\mathcal{O}_0$ is *not* the orbit that will be asymptotically stabilized. Instead, the trajectories of the closed loop system will converge to another exponentially stable orbit, say $\mathcal{O}_\epsilon$, corresponding to the fixed point of P_ϵ and lying *outside* the zero dynamics manifold. What happens is that $\mathcal{O}_\epsilon$ will "tend" to $\mathcal{O}_0$ in the high gain limit.

The main idea behind the proof of existence of a such a fixed point is based on Brouwer's theorem, which is reported here for convenience:

Theorem (Brouwer)**.** Let B be a set homeomorphic to a nonempty compact convex subset of $\mathbb{R}^p$. Any continuous function $P : B \to B$ has a fixed point.

Therefore, it suffices to prove the existence of a closed subset of S homeomorphic to a nonempty compact convex set that is mapped into itself by P_ϵ, for ϵ sufficiently small. This will be achieved by showing that some notion of uniform convergence of P_ϵ to P_0 can be given, which in turn requires proving uniform convergence of ϕ^ϵ to ϕ^0 and of $T_I^\epsilon \circ \Delta$ to $T_I^0 \circ \Delta_0$. Then, the property of P_0 begin a contraction will do the rest.

4.3.1 Uniform convergence of ϕ^ϵ to ϕ^0

The aim of this section is to analyze how the solutions of the continuous part of Σ_ϵ and Σ_0 behave in the high gain limit, that is, as ϵ tends to zero. In [36] a control law for a three-link biped robot has been designed such that the zero dynamics manifold is not invariant under impacts, but it is also shown how such invariance can be recovered by using high gain control.

Here, the analysis reported in [36] will be restated in a more general setting, allowing for nonfinite-time stabilizing feedback and applicable to the class of systems characterized by hypothesis H2.2).

The first step towards this goal is to study the Lipschitz properties of (4.17), and, in particular, uniformity of the Lipschitz constant with respect to ϵ. In order to prove this, denote with $\phi_\eta(t,\eta)$ and $\phi^\epsilon_\eta(t,\eta)$ the solution of $\dot\eta = A\eta$ and $\dot\eta = A(\epsilon)\eta$ respectively, starting from the initial condition η. A simple change of time scale shows that, for all $i = 1,\dots,4$,

$$\phi^\epsilon_{\eta_i}(t,\eta) = \frac{1}{\epsilon^{i-1}}\phi_{\eta_i}(t/\epsilon, \eta_1, \epsilon\eta_2, \epsilon^2\eta_3, \epsilon^3\eta_4). \tag{4.19}$$

As a consequence, for all $i = 1,\dots,4$,

$$\left\|\phi^\epsilon_{\eta_i}(t,\eta)\right\|_2 \le \frac{k_1}{\epsilon^{i-1}}\left\|\eta\right\|_\epsilon e^{-\frac{k_2}{\epsilon}t}, \tag{4.20}$$

where $\left\|\eta\right\|_\epsilon = \left\|\left(\eta_1, \epsilon\eta_2, \epsilon^2\eta_3, \epsilon^3\eta_4\right)\right\|_2$.

Moreover, it can be shown that the solution $\phi_z^\epsilon(t,x)$ of the continuous part of (4.11) is bounded by a function of (t,ϵ) that is continuous on $[0,+\infty)\times[0,+\infty)$. As a matter of fact,

$$\begin{aligned}
\|\phi_z^\epsilon(t)\|_{\mathcal{W}} &\leq \|z\|_{\mathcal{W}} + \int_0^t \left\| \begin{bmatrix} \alpha(\phi_{\eta_1}^\epsilon(\tau),\phi_{z_1}^\epsilon(\tau))\phi_{z_2}^\epsilon(\tau) \\ \gamma(\phi_{\eta_1}^\epsilon(\tau),\phi_{z_1}^\epsilon(\tau)) \end{bmatrix} \right\|_{\mathcal{W}} d\tau \\
&\quad + \int_0^\infty \left\|\beta(\phi_{\eta_1}^\epsilon(\tau),\phi_{z_1}^\epsilon(\tau))\right\|_\beta \left\|\phi_{\eta_2}^\epsilon(\tau)\right\|_2 d\tau \\
&\leq \|z\|_{\mathcal{W}} + \int_0^t \Big(\left\|\alpha(\phi_{\eta_1}^\epsilon(\tau),\phi_{z_1}^\epsilon(\tau))\right\|_\alpha \left\|\phi_{z_2}^\epsilon(\tau)\right\|_{\mathcal{W}} + \left\|\gamma(\phi_{\eta_1}^\epsilon(\tau),\phi_{z_1}^\epsilon(\tau))\right\|_{\mathcal{W}} \Big) d\tau \\
&\quad + \int_0^\infty \left\|\beta(\phi_{\eta_1}^\epsilon(\tau),\phi_{z_1}^\epsilon(\tau))\right\|_\beta \frac{k_1}{\epsilon} \|\eta\|_\epsilon \, e^{-\frac{k_2}{\epsilon}t} d\tau \\
&\leq \|z\|_{\mathcal{W}} + M_\beta \frac{k_1}{k_2} \|\eta\|_\epsilon + M_\gamma t + \int_0^t M_\alpha \|\phi_z^\epsilon(\tau)\|_{\mathcal{W}} \, d\tau;
\end{aligned}$$

by using the Gronwall-Bellman lemma, it follows that

$$\|\phi_z^\epsilon(t)\|_{\mathcal{W}} \leq \left(\|z\|_{\mathcal{W}} + M_\beta \frac{k_1}{k_2} \|\eta\|_\epsilon + \frac{M_\gamma}{M_\alpha} \right) e^{M_\alpha t}, \tag{4.21}$$

thus showing that $\phi_z^\epsilon(t)$ stays bounded for finite values of t as ϵ tends to zero. Therefore, whenever the vector x of initial conditions of $\phi^\epsilon(t)$ belongs to a given compact set, so will the vector function $\begin{bmatrix} \alpha(\phi_{\eta_1}^\epsilon(t),\phi_{z_1}^\epsilon(t))\phi_{z_2}^\epsilon(t) \\ \gamma(\phi_{\eta_1}^\epsilon(t),\phi_{z_1}^\epsilon(t)) \end{bmatrix}$ for finite values of t and for $\epsilon \to 0$, as, by hypothesis H2.2), α,β,γ are uniformly bounded for all η_1, z_1. It follows that, in such a case, the above mentioned vector function is Lipschitz with a constant L that is independent on ϵ, as ϵ tends to zero.

In the following, whenever needed, the vector x will be partitioned as $x = (\eta_1, \eta_{2:4}, z)$.

Lemma 4.3.2. Take any $t > 0$ and any $\bar{x} = (0, \bar{\eta}_{2:4}, \bar{z})$. Then,

$$\lim_{\substack{\epsilon\to 0 \\ x\to\bar{x}}} \phi^\epsilon(t,x) = \phi^0(t,\bar{z}). \tag{4.22}$$

Proof. From the definition of $\|\cdot\|_{\mathcal{X}}$, it follows that

$$\left\|\phi^\epsilon(t,x) - \phi^0(t,\bar{z})\right\|_{\mathcal{X}} = \left\|\phi_\eta^\epsilon(t,\eta)\right\|_2 + \left\|\phi_z^\epsilon(t,x) - \phi_z^0(t,\bar{z})\right\|_{\mathcal{W}}.$$

The first term on the right hand side is such that

$$\left\|\phi^\epsilon_\eta(t,\eta)\right\|_2 \le \left\|e^{A(\epsilon)t}\right\|_2 \|\eta\|_2 \to 0$$

as $\epsilon \to 0$. As for the second,

$$\begin{aligned}
&\left\|\phi^\epsilon_z(t) - \phi^0_z(t)\right\|_{\mathcal{W}} \le \\
&\le \|z-\bar{z}\|_{\mathcal{W}} + \int_0^t \left(\left\| \begin{bmatrix} \alpha(\phi^\epsilon_{\eta_1}(\tau), \phi^\epsilon_{z_1}(\tau))\phi^\epsilon_{z_2}(\tau) - \alpha(0,\phi^0_{z_1}(\tau))\phi^0_{z_2}(\tau) \\ \gamma(\phi^\epsilon_{\eta_1}(\tau), \phi^\epsilon_{z_1}(\tau)) - \gamma(0,\phi^0_{z_1}(\tau)) \end{bmatrix} \right\|_{\mathcal{W}} + \right.\\
&\qquad\qquad \left. + \left\|\beta(\phi^\epsilon_{\eta_1}(\tau), \phi^\epsilon_{z_1}(\tau))\right\|_\beta \left\|\phi^\epsilon_{\eta_2}(\tau)\right\|_2 \right) d\tau \\
&\le \|z-\bar{z}\|_{\mathcal{W}} + \int_0^t \left(L\left(\left\|\phi^\epsilon_{\eta_1}(\tau)\right\|_2 + \left\|\phi^\epsilon_z(t) - \phi^0_z(t)\right\|_{\mathcal{W}} \right) + M_\beta \left\|\phi^\epsilon_{\eta_2}(\tau)\right\|_2 \right) d\tau \\
&\le \|z-\bar{z}\|_{\mathcal{W}} + \int_0^\infty \left(L\left\|\phi^\epsilon_{\eta_1}(\tau)\right\|_2 + M_\beta \left\|\phi^\epsilon_{\eta_2}(\tau)\right\|_2 \right) d\tau + \int_0^t L\left\|\phi^\epsilon_z(t) - \phi^0_z(t)\right\|_{\mathcal{W}} d\tau,
\end{aligned}$$

where the Lipschitz constant L is calculated on a suitably large compact set. Applying Gronwall-Bellman's lemma to the above inequality yields:

$$\left\|\phi^\epsilon_z(t,x) - \phi^0_z(t,\bar{z})\right\|_{\mathcal{W}} \le \left(\|z-\bar{z}\|_{\mathcal{W}} + (\epsilon L + M_\beta)\frac{k_1}{k_2}\|\eta\|_\epsilon \right) e^{Lt},$$

which tends to zero as $\epsilon \to 0, x \to \bar{x}$. □□□

With this lemma at hand, it is possible to prove uniform convergence of ϕ_ϵ to ϕ_0.

Proposition 7. *Let C and D be compact subsets of $\mathcal{W}$and $\mathcal{V}_2 \times \mathcal{V}_3 \times \mathcal{V}_4$, respectively. Then, $\phi^\epsilon(t,\eta_1,\bar{\eta}_{2:4},\bar{z})$ converges to $\phi^0(t,\bar{z})$ as $\epsilon \to 0, \eta_1 \to 0$ uniformly with respect to $\bar{z} \in C, \bar{\eta}_{2:4} \in D$ and $t \in [a,b]$, with $b \ge a > 0$.*

Proof. Take $z = \bar{z}$, any $\eta_{2:4} = \bar{\eta}_{2:4}$ in the vector of initial conditions for ϕ^ϵ in (4.22) and any $t \in [a,b]$. Then, $\left\|\phi^\epsilon(t,\eta_1,\bar{\eta}_{2:4},\bar{z}) - \phi^0(t,\bar{z})\right\|_{\mathcal{W}}$ can be bounded as follows:

$$\begin{aligned}
&\left\|\phi^\epsilon(t,\eta_1,\bar{\eta}_{2:4},\bar{z}) - \phi^0(t,\bar{z})\right\|_{\mathcal{W}} \le \left\|e^{A(\epsilon)t}\right\|_2 \|\eta\|_2 + (\epsilon L + M_\beta)\frac{k_1}{k_2}\|\eta\|_\epsilon e^{Lt} \\
&\le \frac{k_1}{\epsilon^3}(\|\eta_1\|_2 + d)e^{-\frac{k_2}{\epsilon}a} + (\epsilon L + M_\beta)\frac{k_1}{k_2}(\|\eta_1\|_2 + d\sum_{i=1}^4 \epsilon^i)e^{Lb},
\end{aligned}$$

where $d = \max_{\eta_{2:4}\in D} \|\eta_{2:4}\|_2$. The proof follows by noting that the above expression does not contain $\bar{z}, \bar{\eta}_{2:4}$ or t and tends to zero as $\epsilon \to 0, \eta_1 \to 0$. □□□

4.3.2 Uniform convergence of $T_I^\epsilon \circ \Delta$ to $T_I^0 \circ \Delta$

The second step towards proving uniform convergence of P_ϵ to P_0 consists of showing convergence of $T_I^\epsilon \circ \Delta$ to $T_I^0 \circ \Delta$. For, two preliminary lemmas are needed.

Lemma 4.3.3. Let D be a compact subset of $\mathcal{V}_2 \times \mathcal{V}_3 \times \mathcal{V}_4$ and C be a compact subset of $\mathcal{W}$ such that every trajectory ϕ^0 starting from it is transversal to S. Take any $\bar{z} \in C, \bar{\eta}_{2:4} \in D$ and let $\bar{x} = (0, \bar{\eta}_{2:4}, \bar{z})$. Then,

$$\lim_{\substack{\epsilon\to 0\\ x\to\bar{x}}} T_I^\epsilon(x) = T_I^0(\bar{z}). \tag{4.23}$$

Proof. By contradiction, suppose first that there exists $\bar{x}$ for which the limit in (4.23) does not exist. Then, there exist two pairs of sequences $(\{\epsilon_n^1\}, \{x_n^1\})$ and $(\{\epsilon_n^2\}, \{x_n^2\})$ such that $\epsilon_n^i \to 0, x_n^i \to \bar{x}$, and $T_I^{\epsilon_n^i}(x_n^i) \to T_i$, with $T_1 > T_2$.

Claim 1. $\lim_{n\to+\infty} \phi^{\epsilon_n^i}(T_i, x_n^i) \in S, \quad i = 1, 2.$

Proof of Claim 1.

Take any $i = 1, 2$. By Lemma 4.3.2, $\lim_{n\to+\infty} \phi^{\epsilon_n^i}(T_i, x_n^i) = \phi^0(T_i, \bar{z})$. By contradiction, suppose that $\phi^0(T_i, \bar{z}) \notin S$. Let $d_i = \mathrm{dist}(\phi^0(T_i, \bar{z}), S) > 0$. By Lemma 4.3.2, there exist $\bar{\epsilon}, \lambda > 0$ such that, for all $\epsilon < \bar{\epsilon}$ and x such that $\|x - \bar{x}\|_{\mathcal{X}} < \lambda$, it holds that $\phi^\epsilon(T_i, x) \in B_{d_i/2}(\phi^0(T_i, \bar{z}))$ (the open ball of radius $d_i/2$ centered in $\phi^0(T_i, \bar{z})$). Consequently, there exists $\delta > 0$ such that $\phi^\epsilon(t, x) \in B_{d_i/2}(\phi^0(T_i, \bar{z}))$ for all $\epsilon < \bar{\epsilon}, \|x - \bar{x}\|_{\mathcal{X}} < \lambda$ and for all $t \in [T_i - \delta, T_i + \delta]$. Since $\lim_{n\to\infty} T_I^{\epsilon_n^1}(x_n^1) = T_1$, there exists $N > 0$ such that, for all $n \geq N$, $T_I^{\epsilon_n^1}(x_n^1) \in [T_1 - \delta, T_1 + \delta]$. Therefore, taking n such that $n \geq N, \epsilon_n^i < \bar{\epsilon}, \left\|x_n^i - \bar{x}\right\|_{\mathcal{X}} < \lambda$ ensures that $\phi^{\epsilon_n^i}(T_I^{\epsilon_n^i}(x_n^i), x_n^i) \in B_{d_i/2}(\phi^0(T_i, \bar{z}))$. This implies that $\mathrm{dist}(\phi^{\epsilon_n^i}(T_I^{\epsilon_n^i}(x_n^i), x_n^i), S) > 0$, which contradicts the fact that $\phi^{\epsilon_n^i}(T_I^{\epsilon_n^i}(x_n^i), x_n^i) \in S$ by definition of the time–to–impact function. □□□

Thus, $\phi^0(T_i, \bar{z}) \in S,\ i = 1, 2$. It follows from the definition of T_I^0 that $T_1 > T_2 \geq T_I^0(\bar{z})$.

Claim 2. $T_1 \leq T_I^0(\bar{z})$.

Proof of Claim 2.

By contradiction, suppose $T_1 > T_I^0(\bar{z})$. Since ϕ^0 is transversal to S for any initial condition in C, there exist T_a and T_b arbitrarily close to $T_I^0(\bar{z})$ such that $T_a < T_I^0(\bar{z}) < T_b < T_1$ and, without loss of generality, $H(\phi^0(T_a, \bar{z})) > 0, H(\phi^0(T_b, \bar{z})) < 0$. Now, choose two neighborhoods of $\phi^0(T_a, \bar{z})$ and $\phi^0(T_b, \bar{z})$ small enough that their intersection with S is empty. By Lemma 4.3.2, there exists $N > 0$ such that $\phi^{\epsilon_N^1}(T_a, x_N^1)$ and $\phi^{\epsilon_N^1}(T_b, x_N^1)$ belong to these two neighborhoods, respectively. If necessary, choose N large enough that $T_I^{\epsilon_N^1}(x_N^1) > T_b$ (this is possible because T_1 is strictly greater than T_b). It follows that $H(\phi^{\epsilon_N^1}(T_a, x_N^1)) > 0, H(\phi^{\epsilon_N^1}(T_b, x_N^1)) < 0$. Continuity of $H \circ \phi^\epsilon$ as a function of t implies that there exists $\bar{T} \in (T_a, T_b)$ such that $\phi^{\epsilon_N^1}(\bar{T}, x_N^1) \in S$. This is a contradiction, due to the fact that $T_I^{\epsilon_N^1}(x_N^1)$ is the smallest impact time of $\phi^{\epsilon_N^1}(t, x_N^1)$, and $\bar{T} < T_I^{\epsilon_N^1}(x_N^1)$. □□□

Combining the two claims above shows that $T_I^0(\bar{z}) \geq T_1 > T_2 \geq T_I^0(\bar{z})$, which contradicts the fact that $T_1 > T_2$. Consequently, the limit in (4.23) does exist for all $\bar{z} \in C, \bar{\eta}_{2:4} \in D$. Now, suppose that there exists $\bar{z}$ and $\bar{\eta}_{2:4}$ such that the limit in (4.23) is different from $T_I^0(\bar{z})$. An argument identical to the one just presented shows that this is a contradiction. □□□

Lemma 4.3.4. Let D be a compact subset of $\mathcal{V}_2 \times \mathcal{V}_3 \times \mathcal{V}_4$ and C be a compact subset of $\mathcal{W}$ such that every trajectory ϕ^0 starting from it is transversal to S. Then, $T_I^\epsilon(\eta_1, \bar{\eta}_{2:4}, \bar{z})$ converges to $T_I^0(\bar{z})$ as $\epsilon \to 0, \eta_1 \to 0$ uniformly with respect to $\bar{z} \in C$ and $\bar{\eta}_{2:4} \in D$.

Proof. Since uniform convergence is equivalent to convergence in the Lagrangian metric for function spaces, the statement to be proved can be rewritten as

$$\lim_{\substack{\epsilon\to 0\\ \eta_1\to 0}} \max_{\substack{z\in C\\ \eta_{2:4}\in D}} \left|T_I^\epsilon(\eta_1,\eta_{2:4},z) - T_I^0(z)\right| = 0. \tag{4.24}$$

By contradiction, suppose first that the limit in (4.24) does not exist. Then, there exist two pairs of sequences $(\{\epsilon_n^1\},\{\eta_n^1\})$ and $(\{\epsilon_n^2\},\{\eta_n^2\})$ such that $\epsilon_n^i \to 0, \eta_n^i \to 0$ and

$$\lim_{n\to\infty} \max_{\substack{z\in C\\ \eta_{2:4}\in D}} \left|T_I^{\epsilon_n^i}(\eta_n^i,\eta_{2:4},z) - T_I^0(z)\right| = l_i,$$

with $l_1 \neq l_2$. Since T_I^ϵ is a continuous function of its arguments, by the Weierstrass theorem it follows that there exist two pairs of sequences $(\{z_n^1\},\{\eta_{2:4,n}^1\})$, $(\{z_n^2\},\{\eta_{2:4,n}^2\}) \subset C \times D$ (which can be considered, without loss of generality, convergent to $(\bar\eta_{2:4}^1,\bar z^1)$ and $(\bar\eta_{2:4}^2,\bar z^2)$, respectively), for which, letting $x_n^i = (\eta_n^i,\eta_{2:4,n}^i,z_n^i)$, it holds that

$$\lim_{n\to\infty} \left|T_I^{\epsilon_n^i}(x_n^i) - T_I^0(z_n^i)\right| = l_i. \tag{4.25}$$

Now, take any $\nu > 0$. Continuity of T_I^0 implies that, for $i = 1, 2$,

$$\exists N_a^i : \forall n > N_a^i, \left|T_I^0(z_n^i) - T_I^0(\bar z^i)\right| < \nu/2,$$

and, by Lemma 4.3.3,

$$\exists N_b^i : \forall n > N_b^i, \left|T_I^{\epsilon_n^i}(x_n^i) - T_I^0(\bar z^i)\right| < \nu/2.$$

Let $N_i = \max\{N_a^i, N_b^i\}$. It follows that for all $\nu > 0$ there exists N_i such that

$$\left|T_I^{\epsilon_n^i}(x_n^i) - T_I^0(z_n^i)\right| \leq \left|T_I^{\epsilon_n^i}(x_n^i) - T_I^0(\bar z^i)\right| + \left|T_I^0(z_n^i) - T_I^0(\bar z^i)\right| < \nu,$$

which shows that $l_1 = l_2 = 0$, which is a contradiction. Therefore, the limit in (4.24) must exist. Suppose now that it is different from zero. Reasoning as above, it can be shown that this leads to a contradiction, thus concluding the proof. □□□

Proposition 8. *$T_I^\epsilon \circ \Delta(x)$ converges to $T_I^0 \circ \Delta_0(z)$ as $\epsilon \to 0, \eta \to 0$ uniformly with respect to $z \in B_{\delta_0}(z^\star)$.*

Proof. Let $\Delta(x) = (\Delta_\eta(x), \Delta_z(x))$. It follows from the definition of Δ_0 that $\Delta_0(z) = \Delta_z(0, z)$. The following has to be shown:

$$\forall \nu > 0\ \exists \bar{\epsilon}, \lambda > 0 : \forall \epsilon < \bar{\epsilon}, \|\eta\|_2 < \lambda, z \in B_{\delta_0}(z^\star), \left|T_I^\epsilon(\Delta(x)) - T_I^0(\Delta_z(0, z))\right| < \nu.$$

By lemma 4.3.4,

$$\forall \nu > 0\ \exists \bar{\epsilon}, \mu > 0 : \forall \epsilon < \bar{\epsilon}, \|\eta_1\|_2 < \mu, z \in C, \eta_{2:4} \in D, \left|T_I^\epsilon(x) - T_I^0(z)\right| < \nu, \tag{4.26}$$

where D is any compact set in $\mathcal{V}_2 \times \mathcal{V}_3 \times \mathcal{V}_4$ and C is any compact set in $\mathcal{W}$ with the property that every trajectory ϕ^0 starting from it is transversal to S. Continuity of Δ implies uniform continuity on compact sets, so $\lim_{\eta\to 0} \Delta_{\eta_1}(\eta, z) = \Delta_{\eta_1}(0, z) = 0, \forall z \in B_{\delta_0}(z^\star)$, that is:

$$\forall \mu > 0\ \exists \lambda > 0 : \forall \|\eta\|_2 < \lambda, z \in B_{\delta_0}(z^\star), \|\Delta_{\eta_1}(x)\|_2 < \mu. \tag{4.27}$$

Now, fix any $\nu > 0$. By (4.26), there exist $\bar{\epsilon}, \mu > 0$ such that, whenever $\|\Delta_{\eta_1}(x)_2\| < \mu$ and for all $\Delta_{\eta_{2:4}}(x) \in D, \Delta_z(x) \in C$, it holds that $\left|T_I^\epsilon(\Delta(x)) - T_I^0(\Delta_z(0, z))\right| < \nu$. By (4.27), in correspondence of μ there exists λ such that, for all $\|\eta\|_2 < \lambda$ and $z \in B_{\delta_0}(z^\star)$, $\|\Delta_{\eta_1}(x)\|_2 < \mu$. Choose C and D so that

$$C \times D \supset \{(\eta_{2:4}^\Delta, z^\Delta) : (\eta_1^\Delta, \eta_{2:4}^\Delta, z^\Delta) = \Delta(x) \text{ and } x \text{ is s.t. } \|\eta\|_2 \leq \lambda, z \in B_{\delta_0}(z^\star)\}.$$

Clearly, if λ is small enough, C can be chosen in order to meet the property mentioned above. Indeed, for $\lambda = 0$, C can be chosen as $C = \Delta_0(B_{\delta_0}(z^\star))$, which does meet such a property because of (4.18). Thus, if necessary, restrict the value of λ given by (4.27). Then, the proof follows easily. □□□

4.3.3 Uniform convergence of P_ϵ to P_0

In this section all of the previous results will be used to prove uniform convergence of P_ϵ to P_0. What will be actually proved is convergence of P_ϵ to $\sigma \circ P_0$, as P_0 is defined on a subset of $\mathcal{W}$, rather than $\mathcal{X}$.

Proposition 9. *$P_\epsilon(x)$ converges to $\sigma \circ P_0(z)$ as $\epsilon \to 0, \eta \to 0$ uniformly with respect to $z \in B_{\delta_0}(z^\star)$.*

Proof. The following holds:

$$\begin{aligned}\|P_\epsilon(x) - \sigma \circ P_0(z)\|_{\mathcal{W}} &= \left\| e^{A(\epsilon)\, T_I^\epsilon \circ \Delta(x)} \Delta_\eta(x) \right\|_2 \\ &+ \left\| \phi_z^\epsilon(T_I^\epsilon \circ \Delta(x), \Delta(x)) - \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_0(z)) \right\|_{\mathcal{W}}. \end{aligned} \tag{4.28}$$

It will be shown that each term on the right hand side of (4.28) tends to zero as $\epsilon \to 0, \eta \to 0$ uniformly with respect to $z \in B_{\delta_0}(z^\star)$.

Let R be the maximum value of $\|\Delta(x)\|_{\mathcal{X}}$ on a suitably large compact subset of $\mathcal{X}$ containing all x involved in this proof (this is clearly possible because all of the considered values of z lie in $B_{\delta_0}(z^\star)$, and $\eta \to 0$). Then, the first term of the right hand side of (4.28) can be bounded as

$$\left\| e^{A(\epsilon)\, T_I^\epsilon \circ \Delta(x)} \Delta_\eta(x) \right\|_2 \le \left\| e^{A(\epsilon)\, T_I^\epsilon \circ \Delta(x)} \right\|_2 \|\Delta_\eta(x)\|_2 \le \frac{k_1}{\epsilon^3} R\, e^{-\frac{k_2}{\epsilon} T_I^\epsilon \circ \Delta(x)}. \tag{4.29}$$

By hypotheses H1.8) and H2.5), there exists $\bar{T}_0 > 0$ such that $T_I^0 \circ \Delta_0(z) > \bar{T}_0, \forall z \in B_{\delta_0}(z^\star)$. By Proposition 8, it follows that for ϵ and η sufficiently small, $T_I^\epsilon \circ \Delta(x)$ will be greater than, say, $\bar{T}_0/2$, for all $z \in B_{\delta_0}(z^\star)$. Thus, as $\epsilon \to 0, \eta \to 0$, it holds that

$$\left\| e^{A(\epsilon)\, T_I^\epsilon \circ \Delta(x)} \Delta_\eta(x) \right\|_2 \le \frac{k_1}{\epsilon^3} R\, e^{-\frac{k_2 \bar{T}_0}{2\epsilon}} \to 0$$

uniformly with respect to $z \in B_{\delta_0}(z^\star)$.

The second term of the right hand side of (4.28) reads

$$\begin{aligned} &\left\| \phi_z^\epsilon(T_I^\epsilon \circ \Delta(x), \Delta(x)) - \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_0(z)) \right\|_{\mathcal{W}} \le \\ &\quad \left\| \phi_z^\epsilon(T_I^0 \circ \Delta_0(z), \Delta(x)) - \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_z(x)) \right\|_{\mathcal{W}} \\ &+ \left\| \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_0(z)) - \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_z(x)) \right\|_{\mathcal{W}} \\ &+ \left\| \phi_z^\epsilon(T_I^\epsilon \circ \Delta(x), \Delta(x)) - \phi_z^\epsilon(T_I^0 \circ \Delta_0(z), \Delta(x)) \right\|_{\mathcal{W}}. \end{aligned} \tag{4.30}$$

By hypothesis H2.6) and uniform continuity of Δ on compact sets, it holds that $\lim_{\eta \to 0} \Delta_{\eta_1}(\eta, z) = \Delta_{\eta_1}(0, z) = 0, \forall z \in B_{\delta_0}(z^\star)$. This fact and Lemma

4.3.2 imply that the first term on the right hand side of (4.30) tends to zero uniformly with respect to $z \in B_{\delta_0}(z^\star)$, as ϵ and η tend to zero.

Since $\lim_{\eta\to 0} \Delta_z(\eta, z) = \Delta_z(0, z) = \Delta_0(z)$, the second term on the right hand side of (4.30) is such that

$$\lim_{\eta\to 0} \max_{z\in B_{\delta_0}(z^\star)} \left\| \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_0(z)) - \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_z(x)) \right\|_{\mathcal{W}} =$$

$$= \max_{z\in B_{\delta_0}(z^\star)} \lim_{\eta\to 0} \left\| \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_0(z)) - \phi_z^0(T_I^0 \circ \Delta_0(z), \Delta_z(x)) \right\|_{\mathcal{W}} = 0,$$

where the first equality holds because of continuity of the max function. This shows that this term tends to zero as $\eta \to 0$, uniformly with respect to $z \in B_{\delta_0}(z^\star)$.

The third term reads

$$\begin{aligned}
&\left\| \phi_z^\epsilon(T_I^\epsilon \circ \Delta(x), \Delta(x)) - \phi_z^\epsilon(T_I^0 \circ \Delta_0(z), \Delta(x)) \right\|_{\mathcal{W}} \le \\
&\le \left| \int_{T_I^0\circ\Delta_0(z)}^{T_I^\epsilon\circ\Delta(x)} \left\| \begin{bmatrix} \alpha(\phi_{\eta 1}^\epsilon(\tau), \phi_{z1}^\epsilon(\tau))\phi_{z2}^\epsilon(\tau) + \beta(\phi_{\eta 1}^\epsilon(\tau), \phi_{z1}^\epsilon(\tau))\phi_{\eta 2}^\epsilon(\tau) \\ \gamma(\phi_{\eta 1}^\epsilon(\tau), \phi_{z1}^\epsilon(\tau)) \end{bmatrix} \right\|_{\mathcal{W}} d\tau \right| \\
&\le \left| \int_{T_I^0\circ\Delta_0(z)}^{T_I^\epsilon\circ\Delta(x)} \Bigg(\left\| \alpha(\phi_{\eta 1}^\epsilon(\tau), \phi_{z1}^\epsilon(\tau)) \right\|_\alpha \left\| \phi_{z2}^\epsilon(\tau) \right\|_{\mathcal{W}} + \left\| \beta(\phi_{\eta 1}^\epsilon(\tau), \phi_{z1}^\epsilon(\tau)) \right\|_\beta \left\| \phi_{\eta 2}^\epsilon(\tau) \right\|_2 \right. \\
&\qquad\qquad \left. + \left\| \gamma(\phi_{\eta 1}^\epsilon(\tau), \phi_{z1}^\epsilon(\tau)) \right\|_{\mathcal{W}} \Bigg) d\tau \right| \\
&\le \left| \int_{T_I^0\circ\Delta_0(z)}^{T_I^\epsilon\circ\Delta(x)} \Bigg(M_\alpha \left(\|\Delta_z(x)\|_{\mathcal{W}} + M_\beta \frac{k_1}{k_2} \|\Delta_\eta(x)\|_\epsilon + \frac{M_\gamma}{M_\alpha} \right) e^{M_\alpha \tau} \right. \\
&\qquad\qquad \left. + M_\beta \frac{k_1}{\epsilon} \|\Delta_\eta(x)\|_\epsilon e^{-\frac{k_2}{\epsilon}\tau} + M_\gamma \Bigg) d\tau \right|,
\end{aligned}$$

where the last bound has been obtained by using (4.19) and (4.21) . By following a reasoning similar to the one detailed above, it can be shown that there exist $0 < T_1 < T_2$ such that, for ϵ and η sufficiently small, $[T_I^0 \circ \Delta_0(z), T_I^\epsilon \circ$

$\Delta(x)] \subset [T_1, T_2], \forall z \in B_{\delta_0}(z^\star)$. Hence,

$$\begin{aligned}
&\left\|\phi_z^\epsilon(T_I^\epsilon \circ \Delta(x), \Delta(x)) - \phi_z^\epsilon(T_I^0 \circ \Delta_0(z), \Delta(x))\right\|_{\mathcal{W}} \le \\
&\le \left| \int_{T_I^0 \circ \Delta_0(z)}^{T_I^\epsilon \circ \Delta(x)} \left(M_\alpha \left(R + M_\beta R \frac{k_1}{k_2} p(\epsilon) + \frac{M_\gamma}{M_\alpha} \right) e^{M_\alpha T_2} \right. \right. \\
&\qquad \left. \left. + M_\beta R \frac{k_1}{\epsilon} p(\epsilon) e^{-\frac{k_2}{\epsilon} T_1} + M_\gamma \right) d\tau \right| \\
&=: \left| \int_{T_I^0 \circ \Delta_0(z)}^{T_I^\epsilon \circ \Delta(x)} M(\epsilon) d\tau \right| \\
&= M(\epsilon) \left| T_I^0 \circ \Delta_0(z) - T_I^\epsilon \circ \Delta(x) \right|,
\end{aligned}$$

where $p(\epsilon) = \sum_{i=0}^{3} \epsilon^i$. It is easy to see that $M(\epsilon)$ tends to a constant when $\epsilon \to 0$, and, by Proposition 8, $\left|T_I^0 \circ \Delta_0(z) - T_I^\epsilon \circ \Delta(x)\right|$ tends to zero uniformly with respect to $z \in B_{\delta_0}(z^\star)$, as $\epsilon \to 0, \eta \to 0$. □□□

4.3.4 Fixed points of P_ϵ

Proof of Theorem 1. The proof is divided in two steps. First, a set $U(x^\star)$ will be found such that $\mathcal{D}P_\epsilon$ is Schur on $U(x^\star)$ for small ϵ. Then, it will be proved that $U(x^\star)$ is mapped by P_ϵ into itself.

Hypotheses H2.4) and H2.5) imply that $\mathcal{D}P_\epsilon(x^\star)$ is Schur. This can be proved by closely following the proof of Theorem 2 in [57], and substituting the hypothesis $\Delta(S \cap Z) \subset Z$ with hypothesis H2.5). The resulting $\mathcal{D}P_\epsilon(x^\star)$ is a matrix of the form $\begin{bmatrix} M_{11}^\epsilon & 0 \\ M_{21}^\epsilon & M_{22} \end{bmatrix}$, where $M_{11}^\epsilon \to 0$ as $\epsilon \to 0$ and M_{22} is Schur, because it is the Jacobian of P_0 evaluated at $z^\star$. By continuity of eigenvalues on the entries of a matrix, it follows that there exists $\bar{\epsilon}_1 > 0$ such that, for all $\epsilon < \bar{\epsilon}_1$, the eigenvalues of $\mathcal{D}P_\epsilon(x^\star)$ have magnitude less than one. By continuous differentiability of P_ϵ, there exists a set $U_1(x^\star)$ such that $\mathcal{D}P_\epsilon(x)$ is Schur for all $x \in U_1(x^\star)$ and for all $\epsilon < \bar{\epsilon}_1$. Without loss of generality, $U_1(x^\star)$ can be taken as $U_1(x^\star) = \{(\eta, z) \in S : \|\eta\|_2 \le \lambda_1, \|z - z^\star\|_{\mathcal{W}} \le \delta_1\}$, for some $\lambda_1, \delta_1 > 0$.

Let $\delta = \min\{\delta_0, \delta_1\}$ and $U(x^\star) = \{(\eta, z) \in S : \|\eta\|_2 \le \lambda, \|z - z^\star\|_{\mathcal{W}} \le$

$\delta\}$. It will be shown that there exists $0 < \lambda \leq \lambda_1$ and $0 < \bar{\epsilon} \leq \bar{\epsilon}_1$ such that $P_\epsilon(U(x^\star)) \subset U(x^\star)$ for all $\epsilon < \bar{\epsilon}$. Take any $x \in U(x^\star)$ and denote $P_\epsilon(x) = (P_\epsilon^\eta(x), P_\epsilon^z(x))$. Then, $P_\epsilon(x) \in U(x^\star)$ if and only if $\|P_\epsilon^\eta(x)\|_2 \leq \lambda$ and $\|P_\epsilon^z(x) - z^\star\|_{\mathcal{W}} \leq \delta$. It follows from (4.18) that

$$\begin{aligned} \|P_\epsilon^z(x) - z^\star\|_{\mathcal{W}} &\leq \|P_\epsilon^z(x) - P_0^z(z)\|_{\mathcal{W}} + \|P_0^z(z) - z^\star\|_{\mathcal{W}} \\ &\leq \|P_\epsilon(x) - \sigma \circ P_0(z)\|_{\mathcal{X}} + \rho_0\delta. \end{aligned}$$

By Proposition 9, there exist $\bar{\epsilon}_2, \lambda_2 > 0$ such that, for all $\epsilon < \bar{\epsilon}_2, \|\eta\|_2 < \lambda_2$, $\|z - z^\star\|_{\mathcal{W}} \leq \delta$, it holds that $\|P_\epsilon(x) - \sigma \circ P_0(z)\|_{\mathcal{X}} < (1 - \rho_0)\delta$, thereby implying that

$$\exists \bar{\epsilon}_2, \lambda_2 > 0 : \forall \epsilon < \bar{\epsilon}_2, \|\eta\|_2 < \lambda_2, \|z - z^\star\|_{\mathcal{W}} \leq \delta, \|P_\epsilon^z(x) - z^\star\|_{\mathcal{W}} \leq \delta.$$

Now, as in the proof of Proposition 9, it can be shown that there exist $\bar{T} > 0$ and $\bar{\epsilon}_3, \lambda_3 > 0$ such that, for all $\epsilon < \bar{\epsilon}_3, \|\eta\|_2 < \lambda_3, \|z - z^\star\|_{\mathcal{W}} \leq \delta$, it holds that $T_I^\epsilon \circ \Delta(x) \geq \bar{T}$, and thus

$$\|P_\epsilon^\eta(x)\|_2 \leq \frac{k_1}{\epsilon^3} R\, e^{-\frac{k_2}{\epsilon}\bar{T}}.$$

Now, let $\lambda = \min\{\lambda_i, i = 1, 2, 3\}$ and $\bar{\epsilon}_4$ be such that $\frac{k_1}{\epsilon^4} R\, e^{-\frac{k_2}{\epsilon}\bar{T}} \leq \lambda\ \forall \epsilon < \bar{\epsilon}_4$. Letting $\bar{\epsilon} = \min\{1, \bar{\epsilon}_i, i = 1, \ldots, 4\}$ concludes the proof that $P_\epsilon(U(x^\star)) \subset U(x^\star)$ for all $\epsilon < \bar{\epsilon}$.

Now, in order to apply Brouwer's theorem, it remains to show that $U(x^\star)$ is homeomorphic to a nonempty compact convex set. By hypothesis H1.6), it follows that S is a smooth manifold, and thus, by [40, Sec. A.4], it is also an embedded manifold. Denote by $\text{int}(U(x^\star))$ the interior of $U(x^\star)$. Since $\text{int}(U(x^\star)) \subset S$, $\text{int}(U(x^\star))$ is an embedded manifold as well. Then, by the last theorem reported in [40, Sec. A.4], there exists a cubic coordinate chart $(\text{int}(U(x^\star)), \psi)$. It follows that $\psi(U(x^\star))$ is a closed cube, thus proving the existence of a homeomorphism between $U(x^\star)$ and a nonempty compact convex set. At this point, Brouwer's theorem can be applied to P_ϵ, concluding the existence of a fixed point $x_\epsilon^\star \in U(x^\star)$ of P_ϵ, for all $\epsilon < \bar{\epsilon}$. Since $\mathcal{D}P_\epsilon(x_\epsilon^\star)$ is Schur, exponential stability of such a fixed point can also be concluded. □□□

4.4 Total impact invariance

In the case of bipedal robots with compliant actuation, the above discussion greatly simplifies. Namely, it can be shown that even for a mechanical system with relative degree 4, tangency of $\Delta(S \cap Z)$ to Z at a point implies that $\Delta(S \cap Z) \subset Z$. For, consider a bipedal robot with series compliant actuators (see Figure 4.3).

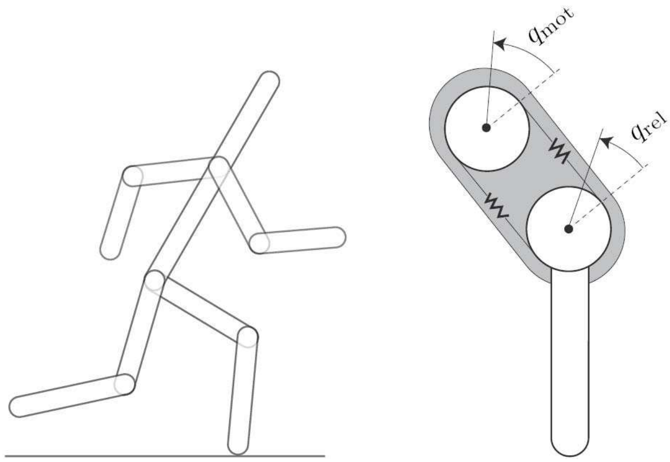

Figure 4.3: Left: A representative example of the class of N-link biped robot considered. Right: A schematic of a rotational joint with series compliant actuation.

Let $q := (q_a, \theta)$ denote the vector of its generalized coordinates, where q_a is a set of relative joint coordinates and θ is an absolute angular coordinate, and let q_m be the vector of motor angles. The stance phase dynamics are given by

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = BK(q_m - q_a) \tag{4.31}$$

$$J\ddot{q}_m + K(q_m - q_a) = u, \tag{4.32}$$

where J is a diagonal matrix of rotor inertias and K a diagonal matrix of spring constants. Let $\xi = (q, q_m, \dot{q}, \dot{q}_m)$ represent the state vector. Then, the model can be written as

$$\begin{cases} \dot{\xi} &= \begin{bmatrix} \dot{q} \\ \dot{q}_m \\ -D^{-1}(q)\left[C(q,\dot{q})\dot{q} + G(q) - BK(q_m - q_a)\right] \\ -J^{-1}K(q_m - q_a) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ J^{-1} \end{bmatrix} u, \qquad \xi \notin \mathcal{S} \\ \xi^+ &= \bar{\Delta}(\xi^-), \qquad \xi^- \in \mathcal{S}, \end{cases} \tag{4.33}$$

where $\mathcal{S} = \{\xi : \bar{H}(q) = 0\}$, with $H(q)$ the height of the swing leg end above the ground. Let the impact map $\bar{\Delta}$ be denoted as follows:

$$(q^+, \dot{q}^+, q_m^+, \dot{q}_m^+) = (\Delta_q\, q^-, \Delta_{\dot{q}}(q^-)\dot{q}^-, Rq_m^-, R\dot{q}_m^-), \tag{4.34}$$

where R is a relabeling matrix. Consider an output $y = h(q)$, with $h(q) = q_a - h_d(\theta)$. The system has relative degree 4 with respect to y, so it is possible to perform the following change of coordinates $x = (\eta, z) = \Phi(\xi)$:

$$\begin{aligned}
\eta_1 &= h(q) = q_a - h_d(\theta) \\
\eta_2 &= L_f h(q,\dot{q}) = \frac{\partial h}{\partial q}\dot{q} = \dot{q}_a - \frac{\partial h_d}{\partial \theta}\dot{\theta} \\
\eta_3 &= L_f^2 h(q,\dot{q},q_m) = \frac{\partial h}{\partial q}D^{-1}(q)BKq_m \\
&\quad -\frac{\partial h}{\partial q}D^{-1}(q)(C(q,\dot{q})\dot{q} + G(q) - BKq_a) - \frac{\partial^2 h_d}{\partial \theta^2}\dot{\theta}^2 \\
&=: \frac{\partial h}{\partial q}D^{-1}(q)BKq_m - \frac{\partial^2 h_d}{\partial \theta^2}\dot{\theta}^2 + \bar{E}_1(q,\dot{q}) \\
\eta_4 &= L_f^3 h(q,\dot{q},q_m,\dot{q}_m) = \frac{\partial L_f^2 h}{\partial q}\dot{q} \\
&\quad -\frac{\partial L_f^2 h}{\partial \dot{q}}D^{-1}(q)\left[C(q,\dot{q})\dot{q} + G(q) - BK(q_m - q_a)\right] + \frac{\partial h}{\partial q}D^{-1}(q)BK\dot{q}_m \\
&=: \frac{\partial h}{\partial q}D^{-1}(q)BK\dot{q}_m - \frac{\partial^3 h_d}{\partial \theta^3}\dot{\theta}^3 + \bar{E}_2(q,\dot{q},q_m) \\
z_1 &= \theta \\
z_2 &= \sigma := d_{NN}(q_a)R(q_a)\dot{q}_a + d_{NN}(q_a)\dot{\theta},
\end{aligned} \tag{4.35}$$

where $R(q_a) := \left[\frac{d_{N1}(q_a)}{d_{NN}(q_a)} \cdots \frac{d_{N,N-1}(q_a)}{d_{NN}(q_a)}\right]$. Note also that $\bar{E}_1(q,\dot{q})$ is quadratic in $\dot{q}$ and $\bar{E}_2(q, \dot{q}, q_m)$ is cubic in $\dot{q}$ and linear in q_m. The inverse of Φ is given by:

$$
\begin{aligned}
q_a &= \eta_1 + h_d(z_1) \\
\theta &= z_1 \\
\dot{q}_a &= \left(I + \frac{\partial h_d}{\partial z_1}(z_1) R(\eta_1 + h_d(z_1))\right)^{-1} \left[\eta_2 + \frac{z_2}{d_{NN}(\eta_1 + h_d(z_1))} \frac{\partial h_d}{\partial z_1}(z_1)\right] \\
\dot{\theta} &= \left(1 + R(\eta_1 + h_d(z_1)) \frac{\partial h_d}{\partial z_1}(z_1)\right)^{-1} \\
&\quad \cdot \left[\frac{z_2}{d_{NN}(\eta_1 + h_d(z_1))} - R(\eta_1 + h_d(z_1))\eta_2\right] \\
q_m &= \left[L_g L_f^3 h(\eta_1 + h_d(z_1)) J\right]^{-1} \\
&\quad \cdot \left(\eta_3 + \frac{\partial^2 h_d}{\partial z_1^2}(z_1) \dot{\theta}^2(\eta_1, \eta_2, z_1, z_2) - E_1(\eta_1, \eta_2, z_1, z_2)\right) \\
\dot{q}_m &= \left[L_g L_f^3 h(\eta_1 + h_d(z_1)) J\right]^{-1} \\
&\quad \cdot \left(\eta_4 + \frac{\partial^3 h_d}{\partial z_1^3}(z_1) \dot{\theta}^3(\eta_1, \eta_2, z_1, z_2) - E_2(\eta_1, \eta_2, \eta_3, z_1, z_2)\right)
\end{aligned}
\tag{4.36}
$$

where E_1 and E_2 are the expression of $\bar{E}_1$ and $\bar{E}_2$ in the new coordinates, respectively. Note that E_1 and E_2 are quadratic and cubic in z_2, respectively. Expressing the system in the new set of coordinates and applying the feedback $u = u_\epsilon = -(L_g L_f^3 h(x))^{-1}[L_f^4 h(x) + \sum_{i=0}^{3} \frac{K_i}{\epsilon^{4-i}} L_f^i h(x)]$ yields:

$$
\begin{cases}
\dot{\eta} = A(\epsilon)\eta \\
\dot{z}_1 = \left(1 + R(\eta_1 + h_d(z_1)) \frac{\partial h_d}{\partial z_1}(z_1)\right)^{-1} \left[\frac{z_2}{d_{NN}(\eta_1 + h_d(z_1))} - R(\eta_1 + h_d(z_1))\eta_2\right] \\
\dot{z}_2 = -\frac{\partial V}{\partial \theta}(\eta_1 + h_d(z_1), z_1),
\end{cases}
\tag{4.37}
$$

Moreover, let $\Delta(x) := \Phi \circ \bar{\Delta} \circ \Phi^{-1}(x)$ and $H(x) := \bar{H} \circ \Phi^{-1}(x)$. Throughout this section, assumptions RH1) – RH5), GH1) – GH5) and HH1) – HH5) of [92] (appropriately modified to take into account the vector relative degree 4 of the compliant model) are supposed to hold. It will be shown that, by choosing $h_d(\theta)$ as a Bézier polynomial of seventh degree, it is possible to achieve hybrid invariance of the zero dynamics manifold, namely $\Delta(S \cap Z) \subset Z$.

As a matter of fact, let the vector q_0^- introduced in hypothesis HH5) be denoted as $q_0^- := (q_a^-, \theta^-)$. Moreover, let $q_0^+ =: (q_a^+, \theta^+) = \Delta_q\, q_0^-$, $z^- = (z_1^-, z_2^-)$, with $z_1^- = \theta^-$ and z_2^- generic. Impact invariance of $S \cap Z$ is then equivalent to existence of $h_d(\theta)$ such that

$$\forall z_2^- \exists z_2^+ : \Phi^{-1}(0, z_1^+, z_2^+) = \bar{\Delta} \circ \Phi^{-1}(0, z_1^-, z_2^-). \tag{4.38}$$

In order to present the main result, some accessory notation will be introduced. Let

$$h_d(\theta) = \sum_{k=0}^{M} \alpha_k \frac{M!}{k!(M-k)!} s^k (1-s)^{M-k}, \qquad s = \frac{\theta - \theta^+}{\theta^- - \theta^+}. \tag{4.39}$$

Then, the values of $h_d(\theta)$ and its derivatives up to the 3-th order at the beginning and the end of a step may be written as:

$$\begin{aligned}
h_d(\theta^+) &= \alpha_0, \\
h_d(\theta^-) &= \alpha_M, \\
\frac{\partial h_d}{\partial \theta}(\theta^+) &= \frac{M}{\theta^- - \theta^+}(\alpha_1 - \alpha_0), \\
\frac{\partial h_d}{\partial \theta}(\theta^-) &= \frac{M}{\theta^- - \theta^+}(\alpha_M - \alpha_{M-1}), \\
\frac{\partial^2 h_d}{\partial \theta^2}(\theta^+) &= \frac{M(M-1)}{(\theta^- - \theta^+)^2}(\alpha_2 - 2\alpha_1 + \alpha_0), \\
\frac{\partial^2 h_d}{\partial \theta^2}(\theta^-) &= \frac{M(M-1)}{(\theta^- - \theta^+)^2}(\alpha_M - 2\alpha_{M-1} + \alpha_{M-2}), \\
\frac{\partial^3 h_d}{\partial \theta^3}(\theta^+) &= \frac{M(M-1)(M-2)}{(\theta^- - \theta^+)^3}(\alpha_3 - 3\alpha_2 + 3\alpha_1 - \alpha_0), \\
\frac{\partial^3 h_d}{\partial \theta^3}(\theta^-) &= \frac{M(M-1)(M-2)}{(\theta^- - \theta^+)^3}(\alpha_M - 3\alpha_{M-1} + 3\alpha_{M-2} - \alpha_{M-3}).
\end{aligned} \tag{4.40}$$

Moreover, let

$$M_0 := \frac{M}{\theta^- - \theta^+}, \qquad M_1 := \frac{M(M-1)}{(\theta^- - \theta^+)^2}, \qquad M_2 := \frac{M(M-1)(M-2)}{(\theta^- - \theta^+)^3},$$

and

$$Q_h^+ := L_g L_f^3 h(\alpha_0, z_1^+)J, \qquad Q_h^- := L_g L_f^3 h(\alpha_M, z_1^-)J,$$
$$c_{\dot\theta^-} := \frac{(1+M_0 R(\alpha_M)(\alpha_M-\alpha_{M-1}))}{d_{NN}(\alpha_M)}, \qquad c_{\dot\theta^+} := \frac{(1+M_0 R(\alpha_0)(\alpha_1-\alpha_0))}{d_{NN}(\alpha_0)},$$
$$\dot\theta_0^+ := \dot\theta(0,0,z_1^+,z_2^+) = c_{\dot\theta^+} z_2^+, \qquad \dot\theta_0^- := \dot\theta(0,0,z_1^-,z_2^-) = c_{\dot\theta^-} z_2^-.$$

A straightforward calculation shows that E_1 and E_2 may be written as follows (the dependence on h_d and its derivatives is shown for clarity):

$$\begin{aligned} E_1(\eta_1,\eta_2,z_1,z_2) &= a_1(\eta_1,\eta_2,z_1,h_d(z_1),\frac{\partial h_d}{\partial z_1})z_2^2 \\ &\quad + b_1(\eta_1,\eta_2,z_1,h_d(z_1),\frac{\partial h_d}{\partial z_1}), \end{aligned} \tag{4.41}$$

$$\begin{aligned} E_2(\eta_1,\eta_2,\eta_3,z_1,z_2) &= a_2(\eta_1,\eta_2,\eta_3,z_1,h_d(z_1),\frac{\partial h_d}{\partial z_1},\frac{\partial^2 h_d}{\partial z_1^2})z_2^3 \\ &\quad + b_2(\eta_1,\eta_2,\eta_3,z_1,h_d(z_1),\frac{\partial h_d}{\partial z_1},\frac{\partial^2 h_d}{\partial z_1^2})z_2. \end{aligned} \tag{4.42}$$

For ease of notation, let

$$E_1(0,0,z_1^\pm,z_2^\pm) =: a_1^\pm(z_2^\pm)^2 + b_1^\pm \tag{4.43}$$
$$E_2(0,0,0,z_1^\pm,z_2^\pm) =: a_2^\pm(z_2^\pm)^3 + b_2^\pm z_2^\pm. \tag{4.44}$$

Eq. (4.38) is then equivalent to the following system of equations:

$$\begin{bmatrix} \alpha_0 \\ z_1^+ \end{bmatrix} = \Delta_q \begin{bmatrix} \alpha_M \\ z_1^- \end{bmatrix} \tag{4.45}$$

$$\begin{bmatrix} M_0(\alpha_1-\alpha_0) \\ 1 \end{bmatrix} \dot\theta_0^+ = \Delta_{\dot q}(\alpha_M, z_1^-) \begin{bmatrix} M_0(\alpha_M-\alpha_{M-1}) \\ 1 \end{bmatrix} \dot\theta_0^- \tag{4.46}$$

$$\begin{aligned} &\left(Q_h^+\right)^{-1}\left[M_1(\alpha_2-2\alpha_1+\alpha_0)(\dot\theta_0^+)^2 - a_1^+(z_2^+)^2 - b_1^+\right] \\ &= R\left(Q_h^-\right)^{-1}\left[M_1(\alpha_M-2\alpha_{M-1}+\alpha_{M-2})(\dot\theta_0^-)^2 - a_1^-(z_2^-)^2 - b_1^-\right] \end{aligned} \tag{4.47}$$

$$\begin{aligned} &\left(Q_h^+\right)^{-1}\left[M_2(\alpha_3-3\alpha_2+3\alpha_1-\alpha_0)(\dot\theta_0^+)^3 - a_2^+(z_2^+)^3 - b_2^+ z_2^+\right] \\ &= R\left(Q_h^-\right)^{-1}\left[M_2(\alpha_M-3\alpha_{M-1}+3\alpha_{M-2}-\alpha_{M-3})(\dot\theta_0^-)^3 \right. \\ &\left. - a_2^-(z_2^-)^3 - b_2^- z_2^-\right], \end{aligned} \tag{4.48}$$

from which it is possible to express z_1^+, z_2^+ and α_0 through α_3 as function of z_1^-, z_2^- and α_{M-3} through α_M, by means of the following equations in echelon form:

$$\alpha_0 = R\alpha_M, \tag{4.49}$$

$$z_1^+ = -z_1^-, \tag{4.50}$$

$$\alpha_1 = \frac{1}{M_0\omega^-}\begin{bmatrix} I & 0 \end{bmatrix} \Delta_{\dot{q}}(\alpha_M, z_1^-) \begin{bmatrix} M_0(\alpha_M - \alpha_{M-1}) \\ 1 \end{bmatrix} + \alpha_0 \tag{4.51}$$

$$z_2^+ = \omega^- \frac{(1 + M_0 R(\alpha_0)(\alpha_1 - \alpha_0))d_{NN}(\alpha_M)}{(1 + M_0 R(\alpha_M)(\alpha_M - \alpha_{M-1}))d_{NN}(\alpha_0)} z_2^- =: \delta_{zero}\, z_2^- \tag{4.52}$$

$$\begin{aligned}\alpha_2 = {} & \frac{1}{M_1 c_{\dot{\theta}+}^2}\Bigg[\frac{1}{\delta_{zero}^2} Q_h^+ \, R \left(Q_h^-\right)^{-1} \\ & \cdot \Big(M_1(\alpha_M - 2\alpha_{M-1} + \alpha_{M-2})c_{\dot{\theta}-}^2 - a_1^-\Big) + a_1^+\Bigg] + 2\alpha_1 - \alpha_0 \\ & + \frac{1}{M_1 c_{\dot{\theta}+}^2 (z_2^+)^2}\left[Q_h^+ \, R \left(Q_h^-\right)^{-1} b_1^- - b_1^+\right]\end{aligned} \tag{4.53}$$

$$\begin{aligned}\alpha_3 = {} & \frac{1}{M_2 c_{\dot{\theta}+}^3}\Bigg[\frac{1}{\delta_{zero}^3} Q_h^+ \, R \left(Q_h^-\right)^{-1} \\ & \cdot \Big(M_2(\alpha_M - 3\alpha_{M-1} + 3\alpha_{M-2} - \alpha_{M-3})c_{\dot{\theta}-}^3 - a_2^-\Big) + a_2^+\Bigg] \\ & + 3\alpha_2 - 3\alpha_1 + \alpha_0 + \frac{1}{M_1 c_{\dot{\theta}+}^3 (z_2^+)^3}\left[Q_h^+ \, R \left(Q_h^-\right)^{-1} \frac{b_2^-}{\delta_{zero}} - b_2^+\right]\end{aligned} \tag{4.54}$$

where $\omega^- = e_N^T \Delta_{\dot{q}}(\alpha_M, z_1^-) \begin{bmatrix} M_0(\alpha_M - \alpha_{M-1}) \\ 1 \end{bmatrix}$. Note also that b_1^+ depends on α_{M-1} and α_M, and so does b_1^-, because α_1 and α_0 are given in terms of α_{M-1}, α_M by (4.49),(4.51). A similar reasoning shows that b_2^-, b_2^+ both depend on $\alpha_{M-2}, \alpha_{M-1}$ and α_M.

A close look to (4.53) and (4.54) shows that α_2 and α_3 do not depend on

z_2^+ if and only if the two following relations are satisfied:

$$Q_h^+ \; R \left(Q_h^-\right)^{-1} b_1^- \;\; = \;\; b_1^+, \qquad (4.55)$$

$$Q_h^+ \; R \left(Q_h^-\right)^{-1} b_2^- \;\; = \;\; \delta_{zero} b_2^+. \qquad (4.56)$$

The first relation involves α_{M-1} and α_M only, whereas the second one contains $\alpha_{M-2}, \alpha_{M-1}$ and α_M. As a consequence, whenever the solution exists, it is possible to express both $\alpha_{M-2}, \alpha_{M-1}$ as function of α_M, which can be considered as a free parameter. At this point, the values of α_{M-2} and α_{M-1} may be substituted back into equations (4.49)-(4.54) in order to find α_0 through α_3 (α_{M-3} is free).

Further insights on the above equations may be given by considering the conditions entailed by tangency of $\Delta(S \cap Z)$ to Z, as described in the previous section. As a matter of fact, under the usual transversality hypotheses, it is possible to define, analogously to [57, Sect. IV.A], a continuously differentiable function Γ such that $(\eta, z_1, z_2) \in \mathcal{S} \Leftrightarrow z_1 = \Gamma(\eta, z_2)$. Let $\hat{\Delta}(\eta, z_2) = \Delta(\eta, \Gamma(\eta, z_2), z_2)$ be the representation of Δ in local coordinates on $\mathcal{S}$. Then, tangency of $\Delta(S \cap Z)$ to Z is equivalent to

$$\frac{\partial \hat{\Delta}_\eta}{\partial z_2}(0, z_2^-) \;\; = \;\; 0. \qquad (4.57)$$

The above equation may be rewritten more explicitly as follows. For, note first that from the implicit function theorem, it follows that

$$\frac{\partial \Gamma}{\partial z_2}(0, z_2^-) = - \left[\frac{\partial \bar{H}}{\partial z_1}(0, \Gamma(0, z_2^-), z_2^-)\right]^{-1} \frac{\partial \bar{H}}{\partial z_2}(0, \Gamma(0, z_2^-), z_2^-) = 0,$$

because H is function of q only, and the component q of Φ^{-1} does not contain z_2. As a consequence,

$$\frac{\partial \hat{\Delta}_\eta}{\partial z_2}(0, z_2^-) = \frac{\partial \Delta_\eta}{\partial z_1}(0, \Gamma(0, z_2^-), z_2^-) \frac{\partial \Gamma}{\partial z_2}(0, z_2^-) + \frac{\partial \Delta_\eta}{\partial z_2}(0, \Gamma(0, z_2^-), z_2^-)$$
$$= \frac{\partial \Delta_\eta}{\partial z_2}(0, \Gamma(0, z_2^-), z_2^-)$$

Now, let $z_1^- = \Gamma(0, z_2^-)$. since

$$\frac{\partial \Phi}{\partial \xi}(\xi) = \left(\frac{\partial \Phi^{-1}}{\partial x}(\Phi(\xi)) \right)^{-1},$$

equation (4.57) may be written as follows:

$$\frac{\partial \Phi_\eta}{\partial \xi}(\bar{\Delta} \circ \Phi^{-1}(0, z_1^-, z_2^-)) \frac{\partial \bar{\Delta}}{\partial \xi}(\Phi^{-1}(0, z_1^-, z_2^-)) \frac{\partial \Phi^{-1}}{\partial z_2}(0, z_1^-, z_2^-) = 0,$$

or, equivalently:

$$\frac{\partial \bar{\Delta}}{\partial \xi}(\Phi^{-1}(0, z_1^-, z_2^-)) \frac{\partial \Phi^{-1}}{\partial z_2}(0, z_1^-, z_2^-) = \frac{\partial \Phi^{-1}}{\partial x}(\Phi \circ \bar{\Delta} \circ \Phi^{-1}(0, z_1^-, z_2^-)) \begin{bmatrix} 0 \\ \gamma_{z_1} \\ \gamma_{z_2} \end{bmatrix}, \tag{4.58}$$

where $\gamma_{z_1}, \gamma_{z_2}$ can be arbitrarily chosen. Suppose now that (4.49) through (4.54) are satisfied. Then, it holds that $\Phi \circ \bar{\Delta} \circ \Phi^{-1}(0, z_1^-, z_2^-) = (0, z_1^+, z_2^+)$, and equation (4.58) becomes

$$\frac{\partial \bar{\Delta}}{\partial \xi}(\Phi^{-1}(0, z_1^-, z_2^-)) \frac{\partial \Phi^{-1}}{\partial z_2}(0, z_1^-, z_2^-) = \frac{\partial \Phi^{-1}}{\partial x}(0, z_1^+, z_2^+) \begin{bmatrix} 0 \\ \gamma_{z_1} \\ \gamma_{z_2} \end{bmatrix}. \tag{4.59}$$

Taking $\gamma_{z_1} = 0$ in (4.59) ensures that the first component is satisfied. The second one reads

$$\Delta_{\dot{q}}(\alpha_M, z_1^-) \begin{bmatrix} M_0(\alpha_M - \alpha_{M-1}) \\ 1 \end{bmatrix} c_{\dot{\theta}}^- = \gamma_{z_2} \begin{bmatrix} M_0(\alpha_1 - \alpha_0) \\ 1 \end{bmatrix} c_{\dot{\theta}}^+,$$

This equation is satisfied for any value of α_M, α_{M-1} by choosing $\gamma_{z_2} = z_2^+/z_2^-$ and α_0, α_1 as in (4.49) and (4.51), respectively. The third component of (4.59) reads

$$\begin{aligned} &R\left(Q_h^-\right)^{-1}\left(2M_1\dot{\theta}_0^-\ c_{\dot{\theta}}^+(\alpha_M - 2\alpha_{M-1} + \alpha_{M-2}) - 2a_1^- z_2^-\right) \\ &= \gamma_{z_2}\left(Q_h^+\right)^{-1}\left(2M_1\dot{\theta}_0^+ c_{\dot{\theta}}^-(\alpha_2 - 2\alpha_1 + \alpha_0) - 2a_1^+ z_2^+\right), \end{aligned}$$

which, after substituting (4.53), yields

$$
\begin{aligned}
&R\left(Q_h^-\right)^{-1}\frac{2M_1\dot{\theta}_0^-}{z_2^-\dot{\theta}_0^+}(\dot{\theta}_0^+\theta_0^- - \theta_0^+\dot{\theta}_0^-)(\alpha_M - 2\alpha_{M-1} + \alpha_{M-2}) = 0 = \\
&= \frac{1}{z_2^-}R\left(Q_h^-\right)^{-1}\left(2a_1^-(z_2^-)^2 - 2a_1^-(z_2^-)^2 - 2b_1^-\right) \\
&+\frac{1}{z_2^-}(Q_h^+)^{-1}\left(2b_1^+ - 2a_1^+(z_2^-)^2 + 2a_1^+(z_2^+)^2\right) \\
&= -\frac{2}{z_2^-}\left[R\left(Q_h^-\right)^{-1}b_1^- - (Q_h^+)^{-1}b_1^+\right]. \qquad (4.60)
\end{aligned}
$$

The fourth component of (4.59) yields a similar result, namely:

$$
0 = -2\left[R\left(Q_h^-\right)^{-1}b_2^- - \delta_{zero}(Q_h^+)^{-1}b_2^+\right]. \qquad (4.61)
$$

If the above equations are satisfied, $\Delta(S \cap Z)$ will be tangent to Z for any value of z_2^-. So, if there exists one point of $S \cap Z$ such that its image through Δ belongs to Z, then $\Delta(S \cap Z)$ will be entirely contained in Z.

4.4.1 Simulations

In this section a simple application of the results introduced above to a one-legged mechanism is presented. It is not meant to be a thorough examination of all the results attainable by means of the proposed technique, yet it will show how impact invariance for mechanisms with compliant actuation can be achieved in a much simpler way than [58, 59].

Consider the 2-link mechanical system depicted in Figure 4.4, representing only the stance leg of a biped robot. As usual, the foot is considered as being pinned to the ground, with no actuation between the shin and the ground. The knee joint, on the contrary, contains a series compliant actuator. The dynamics of the “swing phase” are given by (4.31) with $q_a = q_1$, whereas the impact map is obtained by adding a fictitious massless leg to the mechanism and by using the usual algorithm as in [91]. The scalar output is chosen as in Section 4.4, namely $y = q_1 - h_d(\theta)$, with h_d being a Bézier polynomial of seventh degree. A searching routine similar to the one in [92] has been set up to

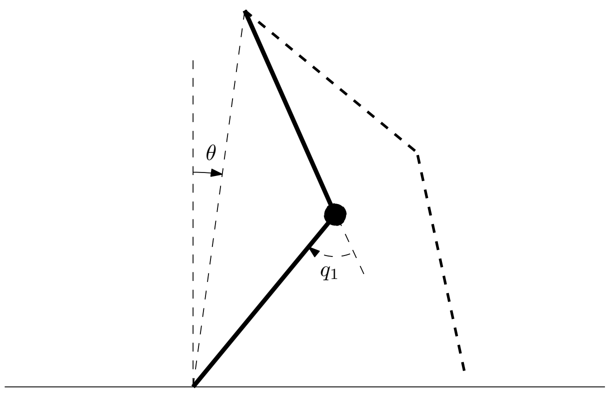

Figure 4.4: One-legged mechanism. The dashed line represents the fictitious leg used to derive the impact map.

find the values of the free parameters α_4, α_7 that give rise to an exponentially stable periodic orbit in the zero dynamics. Such a periodic orbit has then been stabilized in the full model by means of the linear feedback (4.10). The results are shown in Figures 4.5,4.6. Figure 4.5 shows the time plots of the robot's transformed coordinates, in order to emphasize invariance of the zero dynamics manifold across impacts. The initial condition is taken on $S \cap Z$, so that an impact occurs at $t = 0$. Since Z is impact invariant, the trajectory remains in Z after the first and subsequent impacts, that is, $\eta_i \equiv 0, i = 1, \ldots, 4$, while the z component converges exponentially towards the periodic orbit. Figure 4.6 shows how the periodic trajectory is exponentially stable in the full closed-loop system. Here, the initial condition is taken outside the zero dynamics manifold. The control law (4.10) makes the η component converge to zero exponentially fast during the swing phase, and even though at every impact the norm of η instantly increases, the overall trajectory exponentially converges to the periodic orbit. The value used for ϵ is 0.07; a value slightly greater than this would not be enough to stabilize the orbit.

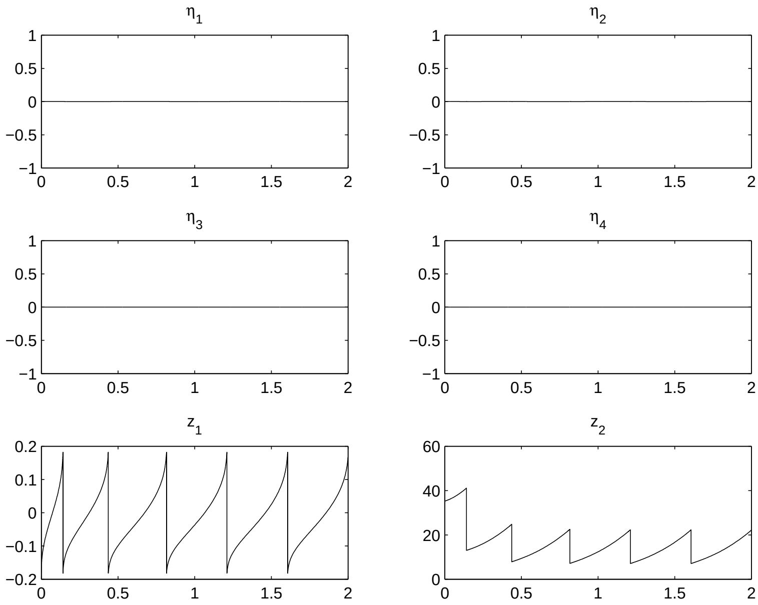

Figure 4.5: Impact invariance of the zero dynamics manifold.

4.4.2 Open issues

The extension of the hybrid zero dynamics technique to the case of compliant robots could lead to the design of control laws much simpler than [58]. Research is under way to apply the technique exposed above to the case of a compliant RABBIT. The main difficulty is represented by the numerical solution of equations (4.55) and (4.56), which may not always exist, and by the fact that ensuring impact invariance for a system with relative degree 4 causes many of the parameters α not to be free. This could have negative effects on the possibility of finding an exponentially stable periodic orbit in the hybrid zero dynamics and at the same time optimizing energy costs and other indices as in [92].

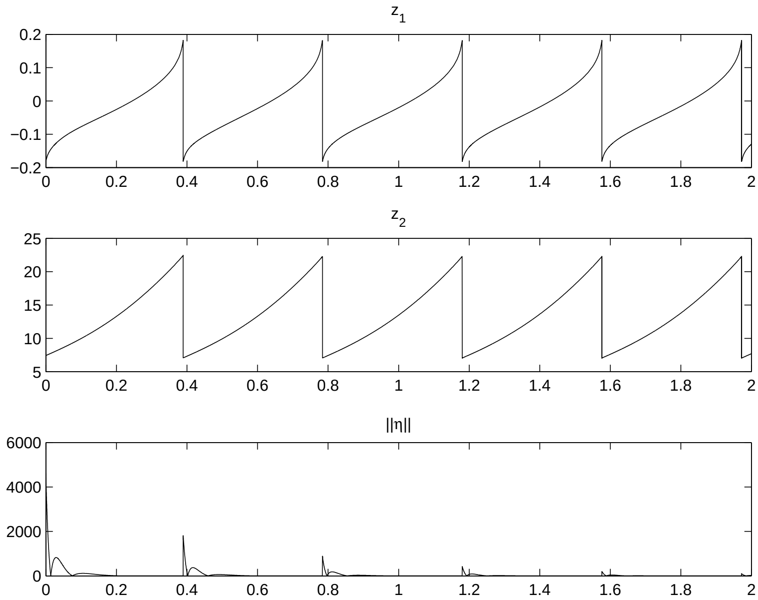

Figure 4.6: Convergence to the periodic orbit in the full model.

Chapter 5

Conclusions

This thesis focused on the control of underactuated mechanical systems, i.e. systems having less actuators than degrees of freedom. A set of different control problems for diverse classes of linear and nonlinear systems have been studied and solved by using different control techniques, which have recently been introduced in the control community. In this chapter, all of the results reported in this thesis will be summarized and directions for future research will be given.

5.1 Input-output decoupling with asymptotic stability

The classical problem of input-output decoupling of linear systems with asymptotic stability has been revisited by requiring that the controller be a mechanical system itself, to be physically connected to the given system in order to achieve the control goal. Such a problem has been studied by using a polynomial approach, namely, by modeling a linear mechanical system through its impedance matrix. In this way, the closed-loop system resulting from the interconnection of the given system and the controller may be seen as a sum of impedances, and results borrowed from the theory of positive realness and polynomial matrix descriptions can be used to prove stability properties. The work reported in this thesis is published in the following articles:

- L. Menini, A. Tornambè, and G. Viola, *Input-output decoupling for m-inputs m-outputs linear mechanical systems through interconnection*, 2005 Conference on Decision and Control, 2005;
- L. Menini, A. Tornambè, and G. Viola, *Input-Output Decoupling with Asymptotic Stability for Linear Mechanical Systems through Interconnection*, submitted to IEEE Transactions on Circuits and Systems.

It identifies a class of m-inputs m-outputs linear underactuated mechanical systems that can be decoupled with asymptotic stability. An algorithm is given that solves the proposed problem under some conditions on the structure of the given system. Further work can be devoted to enlarging the class of linear systems to cover cases for which the algorithm does not work, and it would be of undoubt interest to generalize such a problem to nonlinear mechanical systems, for instance by using techniques similar to [40, 9].

5.2 Stabilization of equilibria

Stabilization of open-loop unstable equilibrium points for nonlinear mechanical systems with underactuation degree one is dealt with in Chapter 3. The IDA–PBC framework is used to formalize and solve such a problem, by identifying a class of systems for which the matching PDE's between the given and the desired dynamics can be explicitly solved. The work presented in this thesis addresses the problem of extending the basic approach to include a more general class of systems, by explicitly solving and/or homogenizing the PDE's via coordinate changes and reparameterization of the target dynamics. The extended class includes many benchmark examples (The Acrobot, the pendulum on a cart, the Furuta pendulum, etc. . .), and gives deeper insights on the differences between IDA–PBC and the Controlled-Lagrangian approach. Here follows the list of publications that contain the present work:

- A. D. Mahindrakar, A. Astolfi, R. Ortega and G. Viola, *Further constructive results on interconnection and damping assignment control of mechanical systems: The acrobot example*, 2006 American Control Conference, Minneapolis, USA, 14-16 June 2006.

- Arun D. Mahindrakar, Romeo Ortega, A. Astolfi and Giuseppe Viola, *Further constructive results on interconnection and damping assignment control of mechanical systems: The acrobot example*, International Journal of Robust and Nonlinear Control, Vol. 16, No. 14, Sept. 2006, pp. 671-685.

- J. A. Acosta, G. Viola, R. Ortega, *Experimental results on IDA-PBC control of Furuta's pendulum*, Supélec–LSS internal report, March 2006

- G. Viola, R. Ortega, R. Banavar, J. A. Acosta, A. Astolfi, *Total Energy Shaping Control of Mechanical Systems: Simplifying the Matching Equations Via Coordinate Changes*, IEEE Transactions on Automatic Control, vol.52, no.6, pp.1093-1099, June 2007

- G. Viola, R. Ortega, R. Banavar, J. A. Acosta, A. Astolfi, *Some Remarks on Interconnection and Damping Assignment Passivity-Based Control of Mechanical Systems*, Taming Heterogeneity and Complexity of Embedded Control. Paris, International Scientific and Technical Encyclopedia (Iste) 2006

Further work in the field may be devoted to solving the stabilization problem for a higher order of underactuation, and to robustness and performance issues.

5.3 Stabilization of periodic orbits

In the third part of this thesis the problem of stabilizing a periodic orbit for an underactuated nonlinear mechanical system with impulse effects has been analyzed. Such a problem arise specifically in the field of biped robot locomotion, where underactuation is caused by the robot having point feet (or, equivalently, by the robot pivoting on the stance toe). The periodic orbit corresponds to a walking (or running) gait and contains by design impact events, which complicates by far the stability analysis. The work reported in the thesis extends the basic approach based on the concept of Hybrid Zero Dynamics to the case of compliant actuation, by providing tools for proving existence and asymptotical stability of periodic orbits through singular-perturbations and fixed-point analysis. Open issues still remain to be taken into account, and the results provided in the thesis will be submitted for publication as soon as they will be addressed.

Bibliography

[1] R. Abraham, J. Marsden, Foundations of Mechanics, Addison-Wesley, Reading, MA, 1978

[2] J. A. Acosta, R. Ortega, A. Astolfi and A. D. Mahindrakar: Interconnection and damping assignment passivity–based control of mechanical systems with underactuation degree one, *IEEE Trans. Automat. Contr.*, Vol. 50, No. 12, December 2005.

[3] J. A. Acosta, J. Aracil, F. Gordillo, On swinging the Furuta pendulum up based on Routh's reduction and SG method, *9th Mediterranean Conference on Control and Automation*, 2001 Dubrovnik (Croatia)

[4] J. A. Acosta, G. Viola, R. Ortega, Experimental results on IDA-PBC control of Furuta's pendulum, LSS internal report, March 2006

[5] A. Ailon, R. Ortega: An observer-based controller for robot manipulators with flexible joints, *Syst. & Cont. Letters*, Vol. 21, pp. 329-335, 1993.

[6] A. A. Agrachev, Y. L. Sachkov, Control theory from the geometric viewpoint, World Scientific Publishing, 2003

[7] R. McN. Alexander, Three uses for springs in legged locomotion, Int. J. Robot. Res, vol. 9, no. 2, pp. 5361, 1990.

[8] V. I. Arnold, Mathematical Methods of Classical Mechanics. New York: Springer Verlag, 1978

[9] A. Astolfi and L. Menini, Noninteractive control with stability for nonlinear m-inputs m-outputs Hamiltonian systems, 40th Conference on Decision and Control, Orlando (FL), December 2001.

[10] D. Auckly and L. Kapitanski and W. White, Control of nonlinear underactuated systems, *Comm. Pure Appl. Math.*, Vol. 3, 2000, pp. 354–369.

[11] D. Auckly and L. Kapitanski, On the λ–equations for matching control laws, *SIAM J. Control and Optimization*, Vol. 41, No. 5, 2002, pp. 1372–1388.

[12] K. Berns, The Walking Machine Catalogue, http://www.walking-machines.org/, 2007

[13] G. Blankenstein, R. Ortega and A.J. van der Schaft, The matching conditions of controlled Lagrangians and interconnection assigment passivity based control, *Int J of Control*, vol. 75, no. 9, pp. 645-665, 2002..

[14] A. Bloch, N. Leonard and J. Marsden, Controlled Lagrangians and the stabilization of mechanical systems, *IEEE Trans. Automat. Contr.*, Vol. 45, No. 12, December 2000.

[15] C. I. Byrnes, A. Isidori, J. C. Willems, Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems, IEEE Transactions on Automatic Control, vol.36, no.11, pp.1228-1240, Nov 1991

[16] F. Bullo and A. D. Lewis, Geometric Control of Mechanical Systems, Springer–Verlag, New York, 2004.

[17] G. Cabodevilla, G. Abba, and H. Sage, Energetically near optimal gait for a biped robot with double supporting phases, in Proc. Third

France-Japan Congr. First Europe-Asia Congr. Mechatronics, Besancon, France, Oct. 1996, pp. 958961.

[18] G. Cabodevilla, N. Chaillet, and G. Abba, Energy-minimized gait for a biped robot, in Autonome Mobile Systeme. Berlin, Germany: Springer-Verlag, 1995, pp. 9099.

[19] N. Chaillet, G. Abba, and E. Ostertag, Double dynamic modeling and computed-torque control of a biped robot, in Proc. IEEE/RSJ Int. Conf. Intelligent Robotics Systems, Munich, Germany, Sept. 1994, pp. 11491153.

[20] D.E. Chang, A.M. Bloch, N.E. Leonard, J.E. Marsden and C.A. Woolsey, The equivalence of controlled Lagrangian and controlled Hamiltonian systems for simple mechanical systems, *ESAIM: Control, Optimisation, and Calculus of Variations*, Vol. 8, pp. 393–422, 2002.

[21] D. E. Chang, The extended lambda-method for controlled Lagrangian systems, IFAC 2005, Prague, Czech, 2005.

[22] C. Chevallereau, A. Formalsky, and B. Perrin, Control of a walking robot with feet following a reference trajectory derived from ballistic motion, in Proc. IEEE Int. Conf. Robotics Automation, Albuquerque, NM, Apr. 1997, pp. 10941099.

[23] I. Fantoni and R. Lozano, Non-linear control for underactuated mechanical systems, Springer, London, 2002.

[24] C. Francois and C. Samson, A new approach to the control of the planar one-legged hopper, Int. J. Robot. Res., vol. 17, no. 11, pp. 11501166, 1998.

[25] J. Furusho and M. Masubuchi, Control of a dynamical biped locomotion system for steady walking, J. Dyn. Syst., Meas., Control, vol. 108, pp. 111118, 1986.

[26] J. Furusho and A. Sano, Sensor-based control of a nine-link biped, Int. J. Robot. Res., vol. 9, no. 2, pp. 8398, 1990.

[27] K. Fujimoto and T. Sugie, Canonical transformations and stabilization of generalized Hamiltonian systems, *Systems and Control Letters*, 2001, Vol 42, No. 3, pp. 217–227.

[28] E. Garcia-Canseco, A. Astolfi, and R. Ortega. Interconnection and damping assignment passivitybased control: Towards a constructive procedure: part II. 2004 Conference on Decision and Control, 2004.

[29] E. García-Canseco and R. Griñó and R. Ortega and M. Salich and A. Stankovic, Power factor compensation of electrical circuits: a control theory viewpoint. Control Systems Magazine, 27(4), April 2007

[30] E García-Canseco and R Ortega and R Griñó and A Stankovic, Power factor compensation is equivalent to cyclo-dissipativation. In American Control Conference, New York, USA, July 11-13 2007.

[31] E García-Canseco and R Ortega and J M A Scherpen and D Jeltsema, Power shaping control of nonlinear systems: A benchmark example. In 3rd Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Nagoya, Japan., July 19-21 2006.

[32] F. Gomez-Estern, R. Ortega, F. Rubio and J. Aracil, Stabilization of a class of underactuated mechanical systems via total energy shaping, *IEEE Conf on Decision and Control*, Orlando, FL, USA Dec. 4–7, 2001.

[33] F. Gordillo, J. A. Acosta, J. Aracil, A new swing–up law for the Furuta pendulum, International Journal of Control, Vol. 76, No. 8, pp. 836–844, 2003

[34] A. Goswami, Postural stability of biped robots and the foot-rotation indicator (FRI) point. *International Journal of Robotics Research*, 18(6):523–33, June 1999.

[35] A. Goswani, B. Espiau, and A. Keramane, Limit cycles and their stability in a passive bipedal gait, in Proc. IEEE Int. Conf. Robotics Automation, Minneapolis, MN, Apr. 1996, pp. 246251.

[36] J. W. Grizzle, G. Abba, F. Plestan, Asymptotically Stable Walking for Biped Robots: Analysis via Systems with Impulse Effects, IEEE TAC, Volume 46, No. 1, January 2001, pp. 51-64.

[37] K. Hirai, M. Hirose, Y. Haikawa, and T. Takenake, The development of Honda humanoid robot. In *Proc. of the 1998 IEEE Internatinonal Conference on Robotics and Automation*, Leuven, Belgium, pages 1321–26, 1998.

[38] Y. Hurmuzlu, Dynamics of bipedal gaitPart 1: Objective functions and the contact event of a planar five-link biped, J. Appl. Mech., vol. 60, pp. 331336, June 1993.

[39] Y. Hurmuzlu, Dynamics of bipedal gaitPart 2: Stability analysis of a planar five-link biped, J. Appl. Mech., vol. 60, pp. 337343, June 1993.

[40] A. Isidori, Nonlinear Control Systems, Springer-Verlag New York, 1995.

[41] A. Isidori and C. Moog. On the nonlinear equivalent of the notion of transmission zeros. In Byrnes C. and A. Kurzhanski, editors, Proc. of the IIASA Conference: Modeling and Adaptive Control, pages 146-157, Berlin, 1988. Springer-Verlag.

[42] B Jayawardhana and R Ortega and E García-Canseco and F Castaños, Passivity of nonlinear incremental systems: Application to PI stabilization of nonlinear RLC cricuits, Systems & Control Letters, 56(9-10):618-622, September-October 2007.

[43] T. Kailath. Linear systems. Prentice-Hall Information and System Sciences Series, Englewood Cliffs, N.J., 1980.

[44] R. Katoh and M. Mori, Control method of biped locomotion giving asymptotic stability of trajectory, Automatica, vol. 20, no. 4, pp. 405414, 1984.

[45] R. Kelly, V. Santibanez and A. Loria, Control of robot manipulators in joint space, Springer–Verlag, London, 2005.

[46] D. D. Koditschek and M. Buhler, Analysis of a simplified hopping robot, Int. J. Robot. Res., vol. 10, no. 6, pp. 587605, 1991.

[47] A. Lewis, Notes on energy shaping, *43rd IEEE Conf Decision and Control*, Dec 14–17, 2004, Paradise Island, Bahamas.

[48] A.G.J. MacFarlane. Dynamical system models. Harrap, London, 1970.

[49] A. D. Mahindrakar, A. Astolfi, R. Ortega and G. Viola, Further constructive results on interconnection and damping assignment control of mechanical systems: The acrobot example, 2006 American Control Conference, Minneapolis, USA, 14-16 June 2006.

[50] Arun D. Mahindrakar and Ravi N. Banavar, A Swing Up of the Acrobot Based on A Simple Pendulum Strategy, International Journal of Control, Vol. 78 No. 6, April 2005, pp. 424-429

[51] N. Manamani, N. N. Gauthier, and N. K. MSirdi, Sliding mode control for pneumatic robot leg, in Proc. European Control Conf., Brussels, Belgium, July 1997.

[52] B. M. Maschke, R. Ortega and A. J. van der Schaft, Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation, IEEE Conf. Dec. and Control, Tampa, FL, USA, Dec. 1998.

[53] T. McGeer, Passive dynamic walking, Int. J. Robot. Res., vol. 9, no. 2, pp. 6282, 1990.

[54] L. Menini, A. Tornambè, and G. Viola. Input-output decoupling for minputs m-outputs linear mechanical systems through interconnection. 2005 Conference on Decision and Control, 2005.

[55] L. Menini and A. Tornambè. Input-output decoupling with asymptotic stability of linear mechanical systems through connection with another mechanical system. 2005 American Control Conference, 2005.

[56] K. Mitobe, N. Mori, K. Aida, and Y. Nasu, Nonlinear feedback control of a bipedwalking robot, in Proc. IEEE Int. Conf. Robotics Automation, Nagoya, Japan, May 1995, pp. 28652870.

[57] B. Morris,J. W. Grizzle, A Restricted Poincaré Map for Determining Exponentially Stable Periodic Orbits in Systems with Impulse Effects: Application to Bipedal Robots, 2005-CDC, Seville, Spain, December, 2005.

[58] B. Morris and J. W. Grizzle, Hybrid Invariance in Bipedal Robots with Series Compliant Actuators, 2006-CDC, San Diego, California, December, 2006.

[59] B. Morris and J. W. Grizzle, Hybrid Invariant Manifolds in Systems with Impulse Effects with Application to Periodic Locomotion in Bipedal Robots, Submitted to IEEE Transaction on Automatic Control, February 2007.

[60] H. Nijmeijer and van der Schaft,A. J. Nonlinear Dynamical Control Systems. Springer-Verlag, Berlin, 1989.

[61] H. K. Khalil, Nonlinear Systems, Prentice Hall, 2002.

[62] R. Ortega, A. Astolfi, G. Bastin and H. Rodriguez, Output-feedback regulation of mass-balance systems, in New Directions in Nonlinear Observer Design, eds. H. Nijmeijer and T. Fossen, Springer-Verlag, Berlin, 1999.

[63] R. Ortega and E. Garcia-Canseco. Interconnection and damping assignment passivitybased control: Towards a constructive procedure: part I. 2004 Conference on Decision and Control, 2004.

[64] R. Ortega and E. Garcia-Canseco, Interconnection and damping assignment passivity–based control: A Survey, *European J of Control*, Vol. 10, pp. 432–450, 2004.

[65] R. Ortega, A. Loria, P.J. Nicklasson, and H. Sira-Ramirez. Passivity based control of euler-lagrange systems: Mechanical, electrical and electromechanical applications. London: Springer Verlag (Series Comunications and Control Engineering), 1998.

[66] R. Ortega and M. Spong, Adaptive motion control of rigid robots: A tutorial, Automatica, Vol. 25, No.6, pp. 877-888, 1989.

[67] R. Ortega, M. W. Spong, Stabilization of Underactuated Mechanical Systems via Interconnection and Damping Assignment, IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control, Princeton, NJ, March 16-18, 2000

[68] R. Ortega, M. W. Spong, F. Gomez-Estern, and G. Blankenstein. Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment. IEEE Transactions on Automatic Control, 47(8):12181233, 2002.

[69] R. Ortega, A. van der Schaft, B. Maschke and G. Escobar, Energy-shaping of port- controlled Hamiltonian systems by interconnection, IEEE Conf. Dec. and Control, Phoenix, AZ, USA, Dec. 1999, pp. 1646-1650.

[70] R. Ortega, A. van der Schaft, B. Maschke and G. Escobar, Stabilization of port- controlled Hamiltonian systems: Energy-balancing and passivation, Automatica

[71] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar. Interconnection and damping assignment passivity-based control of port-controlled hamiltonian systems. Automatica, 38(4):585596, 2002.

[72] R. Ortega, A. J. Van Der Schaft, I. Mareels, B. Maschke, Putting energy back in control, Control Systems Magazine, IEEE , vol.21, no.2, pp.18-33, Apr 2001

[73] J. H. Park and K. D. Kim, Biped robot walking using gravity-compensated inverted pendulum mode and computed torque control, in Proc. IEEE Int. Conf. Robotics Automation, Leuven, Belgium, May 1998, pp. 35283533.

[74] T. S. Parker and L. O. Chua, Practical Numerical Algorithms for Chaotic Systems. New York: Springer-Verlag, 1989.

[75] V. Petrovic, R. Ortega and A. Stankovic, A globally convergent energy-based controller for PM synchronous motors, CDC'99, Phoenix, AZ, USA, Dec. 7-10, 1999, pp. 334-339.

[76] J. Pratt and G. Pratt, Intuitive control of a planar bipedalwalking robot, in Proc. IEEE Int. Conf. Robotics Automation, Leuven, Belgium, May 1998, pp. 20142021.

[77] M. H. Raibert, Legged robots, Commun. ACM, vol. 29, no. 6, pp. 499514, 1986.

[78] M. H. Raibert, S. Tzafestas, and C. Tzafestas, Comparative simulation study of three control techniques applied to a biped robot, in Proc. IEEE Int. Conf. Systems, Man, Cybernetics-System Engineering Service Humans, Le Touquet, France, Oct. 1993, pp. 494502.

[79] H. Rodriguez, R. Ortega, and I. Mareels, Nonlinear control of magnetic levitation systems via energy balancing, in Proc. Amer. Control Conf. 2000, Chicago, June 2000.

[80] M. Rostami and G. Bessonnet, Impactless sagittal gait of a biped robot during the single support phase, in Proc. IEEE Int. Conf. Robotics Automation, Leuven, Belgium, May 1998, pp. 13851391.

[81] A. Sano and J. Furusho, Realization of natural dynamic walking using the angular momentum information, in Proc. IEEE Int. Conf. Robotics Automation, Cincinnati, OH, May 1990, pp. 14761481.

[82] M. Spong, The swing-up control problem for the Acrobot, *IEEE Control Systems Magazine*, Vol. 15, No. 1, pp. 49–55, Feb. 1995.

[83] M. Takegaki and S. Arimoto, A new feedback method for dynamic control of manipulators, ASME J. Dyn. Syst. Meas. Cont., Vol. 102, pp. 119-125, 1981.

[84] B. Thuilot, A. Goswani, and B. Espiau, Bifurcation and chaos in a simple passive bipedal gait, in Proc. IEEE Int. Conf. Robotics Automation, Albuquerque, NM, Apr. 1997, pp. 792798.

[85] R. Q. van der Linde, Active leg compliance for passive walking, in Proc. IEEE Int. Conf. Robotics Automation, Leuven, Belgium, May 1998, pp. 23392344.

[86] A. J. van der Schaft, L2-Gain and Passivity in Nonlinear Control, Springer-Verlag New York, Inc.

[87] G. Viola, R. Ortega, R. Banavar, J. A. Acosta, A. Astolfi, Total Energy Shaping Control of Mechanical Systems: Simplifying the Matching Equations Via Coordinate Changes, IEEE Transactions on Automatic Control, vol.52, no.6, pp.1093-1099, June 2007

[88] M. Vukobratovic and B. Borovac. Zero-moment point — thirty five years of its life. *International Journal of Humanoid Robotics*, 1(1): 157–73, 2004.

[89] M. Vukobratovic, B. Borovac, D. Surla, and D. Stokic. Biped locomotion. Springer-Verlag, Berlin, 1990.

[90] M. Vukobratovic, B. Borovac, D. Surla, and D. Stokic, Biped Locomotion. Berlin, Germany: Springer-Verlag, 1990.

[91] E. R. Westervelt, J. W. Grizzle, C. Chevallereau, J. H. Choi, B. J. Morris, Feedback Control of Dynamic Bipedal Robot Locomotion, Taylor & Francis/CRC Press, 2007.

[92] E. R. Westervelt, J. W. Grizzle, D. E. Koditschek, Hybrid Zero Dynamics of Planar Biped Walkers, IEEE Transactions on Automatic Control, Vol. 48, Issue 1 Jan 2003 p. 42–56.

[93] E.T. Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, Cambridge University Press, 1988