Infinite Dimensional Analysis, Quantum Probability and Related Topics
Vol. 2, No. 1 (1999) 169–178

CONNECTIONS ON STATISTICAL MANIFOLDS OF DENSITY OPERATORS BY GEOMETRY OF NONCOMMUTATIVE L^p-SPACES

PAOLO GIBILISCO*

Centro Vito Volterra, Università di Roma "Tor Vergata", I–00133 Roma, Italy
Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, I–10129 Torino, Italy

TOMMASO ISOLA†

Dipartimento di Matematica, Università di Roma "Tor Vergata", I-00133 Roma, Italy

Received 10 June 1998

Let $\mathcal{N}$ be a statistical manifold of density operators, with respect to an n.s.f. trace τ on a semifinite von Neumann algebra M. If S^p is the unit sphere of the noncommutative space $L^p(M,\tau)$, using the noncommutative Amari embedding $\rho \in \mathcal{N} \to \rho^{1/p} \in S^p$, we define a noncommutative α-bundle-connection pair $(\mathcal{F}^\alpha, \nabla^\alpha)$, by the pullback technique. In the commutative case we show that it coincides with the construction of nonparametric Amari–Čentsov α-connection made in Ref. 8 by Gibilisco and Pistone.

1. Introduction

Information geometry is the theory of statistical manifolds, that is of manifolds whose points ρ can be identified with density functions with respect to a certain measure μ. The classical references for the theory can be found in Refs. 1, 2, 4, 15 and 19.

The noncommutative version of the theory has been developed by some authors. For example noncommutative versions of Amari–Čentsov α-connections have been proposed in the literature.[10–13,20,21]

Recently a nonparametric version of the commutative theory has been proposed (see Refs. 8, 24 and 25). One of the most important results obtained in Ref. 8 is that the α-connections can be defined for $\alpha \in [-1,1]$ also in the nonparametric infinite-dimensional case. More precisely one shows that the right definition is that of α-bundle-connection pair $(\mathcal{F}^\alpha, \nabla^\alpha)$: this means that, generally speaking, the

*E-mail: gibilisco@polito.it
†E-mail: isola@mat.uniroma2.it

α-connection is not defined on the tangent space of the statistical manifold $\mathcal{N}$ but on a suitable vector bundle $\mathcal{F}^\alpha \to \mathcal{N}$. For $\alpha \in (-1,1)$, and $p := 2/(1-\alpha)$, the pair $(\mathcal{F}^\alpha, \nabla^\alpha)$ is simply (isomorphic to) the pullback of the Amari embedding $\rho \to \rho^{1/p} \in S^p$, where S^p is the unit sphere of the (commutative) L^p space equipped with the natural connection that S^p has as a submanifold of L^p.

One of the merits of this approach (besides the nonparametric feature) is that it shows that the notion of duality introduced by Amari is exactly the L^p-space duality (or Orlicz space duality, if one has to deal with exponential and mixture connections).

The purpose of this paper is twofold. On the hand, we show that the construction of the α-connection, for $\alpha \in (-1,1)$, made in Ref. 8 is based on the fact that the commutative L^p space is uniformly convex with dual space uniformly convex. On the other hand, when the construction of Ref. 8 is seen at this abstract level, it is natural to conjecture that a similar construction can be made for statistical manifolds of density operators. Indeed this is the case: we show that the α-bundle-connection pair can also be defined on an arbitrary statistical manifold of density operators. One should also note that in the noncommutative case our approach is fully general and nonparametric: this means that we do not have to restrict to the matrix case but we can deal with manifolds of density operators with respect to a normal, semifinite, faithful trace τ on a semifinite von Neumann algebra M.

In a subsequent paper we will compare our approach to noncommutative α-connections with the others appearing in the literature.

2. Uniformly Convex Banach Spaces

In this section we review, for the reader's convenience, some results on the geometry of uniformly convex Banach spaces, needed in the sequel. In the first part of this section we consider real Banach spaces. $\widetilde{X}$ will denote the dual space of X and S^X the unit sphere of X. If $L \in \widetilde{X}$ and $x \in X$ we will write $\langle L, x\rangle = L(x)$.

Definition 2.1. We say that x is *orthogonal* to y, and denote it by $x \perp y$, if $\|x\| \le \|x + \lambda y\|$, for any $\lambda \in \mathbb{R}$. Moreover, if $A \subset X$, $x \perp A$ means $x \perp y$, for any $y \in A$.

Definition 2.2. The duality mapping $J : X \to \mathcal{P}(\widetilde{X})$ is defined by

$$J(x) := \{v \in \widetilde{X} : \langle v, x\rangle = \|x\|^2 = \|v\|^2\}.$$

By the Hahn–Banach theorem $J(x) \neq \emptyset$, for any $x \in X$. We say that X has the *duality map property* if J is single-valued. In this case we set $\tilde{x} := J(x)$.

Definition 2.3. We say that X has the *projection property* if for any closed convex $M \subset X$ and any $x \in X$ there is a unique $m \in M$ s.t.

$$\|x - m\| = \inf\{\|x - z\| : z \in M\} \equiv d(x, M).$$

In this case we define $\pi_M(x) := m$.

Definition 2.4. (i) X is *strictly convex* if all the points of S^X are extreme points (i.e. on S^X there are no intervals).

(ii) X is *uniformly convex* if for any $\varepsilon > 0$ there is $\delta > 0$ s.t. $x, y \in S^X$ and $\|(x+y)/2\| > 1 - \delta$ implies $\|x - y\| < \varepsilon$.

(iii) X is *uniformly smooth* if for any $\varepsilon > 0$ there is $\delta > 0$ s.t. $\|x\| \geq 1$, $\|y\| \geq 1$, and $\|x - y\| \leq \varepsilon$ implies $\|x + y\| \geq \|x\| + \|y\| - \varepsilon\|x - y\|$.

Proposition 2.1. (i) *X uniformly convex implies X strictly convex.*

(ii) *X uniformly convex (resp. uniformly smooth) implies $\widetilde{X}$ uniformly smooth (resp. uniformly convex).*

Proposition 2.2. (p. 25 of Ref. 6) *Let $x,\ y \in X$, $f \in \widetilde{X}$, $\alpha \in \mathbb{R}$. Then*

(i) $x \perp \ker(f)$ *is equivalent to* $|f(x)| = \|f\|\|x\|$.
(ii) $x \perp (\alpha x + y) \Leftrightarrow$ *there is* $f \in S^{\widetilde{X}}$ *s.t.* $f(x) = \|x\|$ *and* $\alpha = -f(y)/f(x)$.
(iii) $x \perp (\alpha x + y) \Rightarrow |\alpha| \leq \|y\|/\|x\|$.

Proposition 2.3. *Let X and $\widetilde{X}$ be uniformly convex Banach spaces. Then*

(i) *X has the projection property.*
(ii) *X has the duality map property.*
(iii) $x \perp \ker(\tilde{x})$.
(iv) *If* $M := \ker(\tilde{x})$, *then* $\pi_M(v) = v - \frac{\langle \tilde{x}, v\rangle}{\langle \tilde{x}, x\rangle} x$.

Proof. (i) See p. 363 of Ref. 22.

(ii) $\widetilde{X}$ is strictly convex, and this implies that J is single-valued.

(iii) As $|\langle \tilde{x}, x\rangle| = \|\tilde{x}\|\|x\|$, applying Proposition 2.2(i), we get $x \perp \ker(\tilde{x})$.

(iv) We want to prove that $\|v - \pi_M(v)\| \leq \|v - z\|$, for any $z \in M$. Since

$$\|v - \pi_M(v)\| = \left\| \langle \tilde{x}, v\rangle \frac{x}{\|x\|^2} \right\| = \frac{|\langle \tilde{x}, v\rangle|}{\|x\|} = \left| \left\langle \frac{\tilde{x}}{\|\tilde{x}\|}, v \right\rangle \right|$$

we may reduce to the case $\|x\| = \|\tilde{x}\| = 1$. So we have to show that $|\langle \tilde{x}, v\rangle| \leq \|v - z\|$, for any $z \in M$. Fix $z \in M$, and set $\alpha := -\langle \tilde{x}, v\rangle$, $y := v - z$. Then $\alpha x + y = -\langle \tilde{x}, v\rangle x + v - z = \pi_M(x) - z \in \ker(\tilde{x})$, since $\pi_M(x)$, $z \in \ker(\tilde{x}) \equiv M$. It follows from (iii) that $x \perp \alpha x + y$, so that, by Proposition 2.2(iii), we have $|\alpha| \leq \|y\|/\|x\|$, i.e. $|\langle \tilde{x}, v\rangle| \leq \|v - z\|$. □

Remark 2.1. Now remember that if $\mathcal{M}$ is a Banach manifold and $\mathcal{N} \subset \mathcal{M}$ is a submanifold, then for any $p \in \mathcal{N}$ there is a splitting of the tangent space $T_p\mathcal{M} = T_p\mathcal{N} \oplus V$ and a projection operator $\pi_p : T_p\mathcal{M} \to T_p\mathcal{N}$. Moreover if there is a connection ∇ on $\mathcal{M}$, one gets a connection ∇' on the submanifold $\mathcal{N}$, by setting $\nabla' := \pi \circ \nabla$.

Proposition 2.4. *Let X, $\widetilde{X}$ be uniformly convex Banach spaces. Then*

(i) *S^X is a Banach submanifold of X.*
(ii) *T_xS^X, the tangent space to S^X at $x \in S^X$, can be identified with* $\ker(\tilde{x})$.

(iii) *The projection operator* $\pi_x : T_x X \to T_x S^X$ *is given by* $\pi_x(v) = v - \langle \tilde{x}, v \rangle x$. *Using this projection, the trivial connection on* X *induces a connection on* S^X, *that we call the natural connection on* S^X.

Proof. (i) Since $\widetilde{X}$ is uniformly convex, we have that X is uniformly smooth, so that the norm is a uniformly strongly differentiable function (p. 364 of Ref. 16).

(ii) The hyperplane $\{v \in X : \langle \tilde{x}, v \rangle = 1\}$ is evidently the unique supporting hyperplane at S^X in x. Therefore $\ker(\tilde{x}) := \{v \in X : \langle \tilde{x}, v \rangle = 0\}$ can be identified with the tangent vector space $T_x S^X$.

(iii) This is simply a rewriting of Proposition 2.3(iv) in a particular case, and of Remark 2.1. □

Now suppose that X is a complex Banach space. We denote by $X_{\mathbb{R}}$ the same space considered as a real Banach space. Let $L \in \widetilde{X}$, then $v \in X_{\mathbb{R}} \to \mathrm{Re}L(v) \in \mathbb{R}$ defines an element of $\widetilde{X_{\mathbb{R}}}$. The map $L \in \widetilde{X} \to \mathrm{Re}L \in \widetilde{X_{\mathbb{R}}}$ is a bijective linear isometry (pp. 179 and 344 of Ref. 16). We have therefore a complex duality mapping $x \to \tilde{x}$ on X, and the real duality mapping is given by $x \to \mathrm{Re}\tilde{x}$. Correspondingly we have that the supporting hyperplane at $x \in S^X$ is given by $\{v \in X : \mathrm{Re}\langle \tilde{x}, v \rangle = 1\}$, and therefore the tangent space $T_x S^X$ is given by the real Banach space $M \equiv T_x S^X \cong \ker(\mathrm{Re}\tilde{x}) = \{v \in X : \mathrm{Re}\langle \tilde{x}, v \rangle = 0\}$. The projection formula is $\pi_M(v) = v - \mathrm{Re}\langle \tilde{x}, v \rangle x$.

3. α-Connections for Commutative Statistical Manifolds

In this section we summarize some of the results of Ref. 8 in the light of the abstract setting of Sec. 2. Let $(X, \mathfrak{X}, \mu)$ be a measure space. We give the following:

Definition 3.1. If $\alpha \in (-1, 1)$, set $p := 2/(1-\alpha)$. $L^p_{\mathbb{R}} \equiv L^p_{\mathbb{R}}(X, \mathfrak{X}, \mu) := \{u : X \to \mathbb{R} : u \text{ is } \mathfrak{X}\text{-measurable}, \int_X |u|^p \, d\mu < \infty\}$, for $p \in [1, \infty)$. The unit sphere is denoted by $S^p := \{f \in L^p_{\mathbb{R}} : \|u\|_p = 1\}$. $\mathfrak{M}_\mu := \{\rho \in L^1_{\mathbb{R}} : \rho > 0, \int \rho = 1\}$. For any $\rho \in \mathfrak{M}_\mu$ we set $\mathcal{F}^\alpha_\rho \equiv L^p_0(\rho) := \{u \in L^p_{\mathbb{R}}(X, \mathfrak{X}, \rho\mu) : \int_X u\rho \, d\mu = 0\}$. If $p > 1$ we define $\tilde{p}$ by $1/p + 1/\tilde{p} = 1$.

A calculation shows that the duality map is given by $u \in L^p_{\mathbb{R}} \to \tilde{u} := \|u\|_p^{2-p} \operatorname{sgn} u |u|^{p/\tilde{p}} \in L^{\tilde{p}}_{\mathbb{R}}$. Therefore, if $\rho \in \mathfrak{M}_\mu$, we have that $\rho^{1/p} \in S^p$ and $\widetilde{\rho^{1/p}} = \rho^{1/\tilde{p}} \in S^{\tilde{p}}$. The spaces $L^p_{\mathbb{R}}$ are uniformly convex, so the results of Sec. 2 are applicable. For the tangent space of S^p at $\rho^{1/p}$ we have $T_{\rho^{1/p}} S^p = \{u \in L^p_{\mathbb{R}} : \int u \rho^{1/\tilde{p}} \, d\mu = 0\}$. We denote by ∇^p the natural connection on S^p induced by the trivial connection on $L^p_{\mathbb{R}}$. Observe that the isometric isomorphism $I^p_\rho : u \in L^p_{\mathbb{R}}(X, \mathfrak{X}, \mu) \to u\rho^{-1/p} \in L^p_{\mathbb{R}}(X, \mathfrak{X}, \rho\mu)$ sets up a bijection between $L^p_0(\rho)$ and $T_{\rho^{1/p}} S^p$.

Let $\mathcal{N} \subset \mathfrak{M}_\mu$ be a statistical model, equipped with a structure of a differential manifold. Consider the bundle-connection pair on S^p given by the tangent bundle and the natural connection (TS^p, ∇^p). Making use of the Amari embedding A^α :

$\rho \in \mathcal{N} \to \rho^{1/p} \in S^p$, we may construct the pullback $((A^\alpha)^* TS^p, (A^\alpha)^* \nabla^p)$ of the bundle-connection pair (TS^p, ∇^p) to $\mathcal{N}$. This means that the fiber over $\rho \in \mathcal{N}$ of the pullback bundle is given by $T_{\rho^{1/p}} S^p$. Consider now $\mathcal{F}^\alpha := \bigcup_{\rho \in \mathcal{N}} \mathcal{F}^\alpha_\rho$. Using the family of isomorphisms I^p_ρ, $\rho \in \mathcal{N}$, it is possible to identify $\mathcal{F}^\alpha$ with the pullback bundle $(A^\alpha)^* TS^p$. One can also transfer the pullback connection $(A^\alpha)^* \nabla^p$ using this isomorphism. We denote by ∇^α this last connection on the bundle $\mathcal{F}^\alpha$.

Theorem 3.1. *Consider the bundle-connection pair $(\mathcal{F}^\alpha, \nabla^\alpha)$, $\alpha \in (-1, 1)$, on the statistical manifold $\mathcal{N}$. Then ∇^α coincides with the Amari–Čentsov α-connection.*

Proof. See Ref. 8. □

Obviously one may also define a "complex" version of the α-connections. Let $L^p \equiv L^p(X, \mathfrak{X}, \mu) := \{u : X \to \mathbb{C} : u \text{ is } \mathfrak{X}\text{-measurable}, \int_X |u|^p < \infty\}$. Introduce the function

$$\operatorname{sgn} z := \begin{cases} \dfrac{z}{|z|} & z \in \mathbb{C}, z \neq 0 \\ 0 & z = 0\,. \end{cases}$$

The duality mapping in this case has the form $\tilde{u} := \|u\|_p^{2-p} \overline{\operatorname{sgn} u} |u|^{p/\tilde{p}}$. The tangent space is $T_{\rho^{1/p}} S^p = \{u \in L^p(\mu) : \operatorname{Re} \int_X u \rho^{1/\tilde{p}} \, d\mu = 0\}$. In this case we set $L^p_0(\rho) := \{u \in L^p(X, \mathfrak{X}, \rho\mu) : \operatorname{Re} \int_X u \rho \, d\mu = 0\}$, with the isomorphism still given by $I^p_\rho(u) = u \rho^{-1/p}$. The rest of the construction applies directly and we have therefore a "complex" bundle-connection pair $(\mathcal{F}^\alpha, \nabla^\alpha)$, on any statistical manifold $\mathcal{N} \subset \mathfrak{M}_\mu$.

4. Noncommutative L^p-Spaces

We recall in this section the construction of noncommutative L^p-spaces on a general von Neumann algebra, following the approach by Araki and Masuda.[3,18] Moreover we prove a result that we need in the next section. Observe that there are different approaches to noncommutative integration.[5,9,14,17,30] Therefore let M be a von Neumann algebra, which is standardly represented on $\mathcal{H}$, i.e. (10.23 of Ref. 28) there are a conjugation $J : \mathcal{H} \to \mathcal{H}$ and a self-polar convex cone $\mathfrak{P} \subset \mathcal{H}$ s.t.

(i) the mapping $j(x) := Jx^*J$ is a *-anti-isomorphism $j : M \to M'$, which acts identically on the center of M,
(ii) $\xi \in \mathfrak{P} \Rightarrow J\xi = \xi$,
(iii) $xJxJ\mathfrak{P} \subset \mathfrak{P}$ for any $x \in M$.

Recall that two standard representations of M are unitarily equivalent (10.26 of Ref. 28), and if φ is a (normal semifinite) faithful weight on M, then its GNS representation is a standard representation of M. Let us denote by $W(M)$ the set of normal semifinite weights on M, and $W_f(M)$ the subset of the faithful ones. Take a $\varphi \in W_f(M)$, and denote by $\mathfrak{N}_\varphi := \{x \in M : \varphi(x^*x) < \infty\}$, and by $(\pi_\varphi, \mathcal{H}_\varphi, \eta_\varphi)$ the GNS triple. Then the L^p-space w.r.t. φ, denoted by $L^p(M, \varphi)$, consists of the closed densely defined linear operators T on $\mathcal{H}$ s.t.

(i) $TJ\sigma^{\varphi}_{-i/p}(y)J \supset JyJT$, for any $y \in \mathfrak{N}^{an}_{\varphi} := \{y \in \mathfrak{N}_{\varphi} : t \in \mathbb{R} \to \sigma^{\varphi}_{t}(y)$ is an analytic function$\}$,

(ii) $\|T\|_{p,\varphi} := \sup_{x\in\mathfrak{N}^{an}_{\varphi}, \|x\|\leq 1} \||T|^{p/2}\eta_{\varphi}(x)\|^{2/p} < \infty$.

For any $T \in L^p(M,\varphi)$ there is a unique normal positive linear functional $\psi \in M_{*+}$, and a partial isometry $w \in M$ s.t. $w^*w = s(\psi)$, the support projection of ψ, and $T = w\Delta^{1/p}_{\psi\varphi}$, where $\Delta_{\psi\varphi}$ is the relative modular operator. We have that $L^p(M,\varphi)$ is a uniformly convex Banach space, if $p \in (1,\infty)$, and $L^1(M,\varphi) \cong M_*$, $L^2(M,\varphi) \cong \mathcal{H}$, $L^\infty(M,\varphi) \cong M$, and $L^p(M,\varphi)\check{} \cong L^{\bar{p}}(M,\varphi)$, where $\frac{1}{p} + \frac{1}{\bar{p}} = 1$. Besides, if $\varphi_0 \in W_f(M)$ is a different nsf weight, then $L^p(M,\varphi_0)$ and $L^p(M,\varphi)$ are isometrically isomorphic, and the isomorphism is given by $I^p_{\varphi_0\varphi} : w\Delta^{1/p}_{\psi\varphi} \in L^p(M,\varphi) \to w\Delta^{1/p}_{\psi\varphi_0} \in L^p(M,\varphi_0)$, $\psi \in M_{*+}$, $w \in M$ a partial isometry.

We want to give an explicit formula, that we use in Sec. 5, for this isomorphism, in the particular case when φ_0 commutes with φ, which means that there is a positive self-adjoint operator $\rho\hat{\in} M^{\varphi}$, with $\mathrm{supp}(\rho) = 1$, s.t. $\varphi_0 = \varphi_\rho$, where $\varphi_\rho(x) := \lim_{\varepsilon\to 0}\varphi(\rho_\varepsilon^{1/2}x\rho_\varepsilon^{1/2})$ and $\rho_\varepsilon := \rho(1+\varepsilon\rho)^{-1} \in M^{\varphi}$. Then

Proposition 4.1. *$\Delta_{\psi\varphi_\rho} = \Delta_{\psi\varphi}J\rho^{-1}J$, so that $I^p_\rho \equiv I^p_{\varphi_\rho\varphi} : T \in L^p(M,\varphi) \to TJ\rho^{-1/p}J \in L^p(M,\varphi_\rho)$.*

Proof. We will be using Theorem C.1 of Ref. 3 repeatedly. It follows from Eq. (C.5) (*loc. cit.*) that $\Delta^{it}_{\varphi\psi}\Delta^{-it}_{\varphi_\rho\psi} = (D\varphi : D\varphi_\rho)_t Js(\psi)J$, and $(D\varphi : D\varphi_\rho)_t = \rho^{-it}$, by 4.8 of Ref. 27. As $\mathrm{supp}(\Delta_{\varphi\psi}) = \mathrm{supp}(\Delta_{\varphi_\rho\psi}) = Js(\psi)J$, we get $\Delta^{it}_{\varphi\psi} = \Delta^{it}_{\varphi_\rho\psi}\rho^{it}$. Observe that ρ and $\Delta_{\varphi\psi}$ commute, as $\Delta^{it}_{\varphi\psi}\rho^{is}\Delta^{-it}_{\varphi\psi} = \sigma^{\varphi}_t(\rho^{is})Js(\psi)J = \rho^{is}Js(\psi)J$, so that $\Delta^{it}_{\varphi\psi}\rho^{is} = \rho^{is}Js(\psi)J\Delta^{it}_{\varphi\psi} = \rho^{is}\Delta^{it}_{\varphi\psi}$. Therefore $\Delta_{\varphi_\rho\psi} = \Delta_{\varphi\psi}\rho$. Now from Eq. ($\beta$5) (*loc. cit.*) it follows that $\Delta^{-1}_{\psi\varphi} = J_{\varphi\psi}\Delta_{\varphi\psi}J^*_{\varphi\psi}$ and analogously with φ_ρ in place of φ. As from Eq. (C.12) (*loc. cit.*) that $J_{\varphi\psi} = J_{\varphi_\rho\psi} = s(\psi)J$, we get

$$\begin{aligned}
\Delta^{-1}_{\psi\varphi_\rho} &= J_{\varphi_\rho\psi}\Delta_{\varphi_\rho\psi}J^*_{\varphi_\rho\psi} \\
&= s(\psi)J\Delta_{\varphi\psi}Js(\psi)J\rho J = J_{\varphi\psi}\Delta_{\varphi\psi}J^*_{\varphi\psi}J\rho J \\
&= \Delta^{-1}_{\psi\varphi}J\rho J
\end{aligned}$$

and the thesis follows. □

Example 4.1. If M is a semifinite von Neumann algebra and $\varphi = \tau$ is an nsf trace on M, then any φ_0 commutes with τ, and $L^p(M,\tau)$ coincides with the L^p-space defined in Refs. 7, 23 and 26. Besides $\varphi_0 \equiv \tau_\rho \in M_{*+}$ iff $\rho \in L^1(M,\tau)_+$ (see Sec. 7.1 of Ref. 18). Moreover for any $\psi = \tau_T \in M_{*+}$, with $T \in L^1(M,\tau)_+$, one has that $\Delta_{\psi\tau} = T$. Indeed $\Delta^{it}_{\psi\tau}\Delta^{-it}_{\tau\tau} = (D\psi : D\tau)_t = (D\tau_T : D\tau)_t = T^{it}$. Therefore, if $\varphi_0 = \tau_\rho \in M_{*+}$, then the isometric isomorphism is given by the map $I^p_\rho : u \in L^p(M,\tau) \to uJ\rho^{-1/p}J \in L^p(M,\tau_\rho)$.

Example 4.2. Let us assume now that $M = \mathcal{B}(\mathcal{H})$ is a type I factor, and τ is the ordinary trace. Then $L^p(\mathcal{B}(\mathcal{H}),\tau)$ is the von Neumann–Schatten class $L^p(\mathcal{H})$.

Let $(\mathcal{H}_\tau, \pi_\tau, \eta_\tau)$ be the GNS triple of τ, $\mathcal{H}_\tau \equiv L^2(\mathcal{H})$, $J_\tau \eta_\tau(x) = \eta_\tau(x^*)$, for any $x \in \mathfrak{N}_\tau \equiv L^2(\mathcal{H})$. We want to express the modular operators relative to a normal positive linear functional $\psi \in M_{*+}$. Recall that $\psi = \tau_\sigma$ with $\sigma \in L^1(\mathcal{H})_+ \subset M$. Then its GNS representation is $(\mathcal{H}_{\tau_\sigma}, \pi_{\tau_\sigma}, \eta_{\tau_\sigma})$, where $\mathcal{H}_{\tau_\sigma} := \mathcal{H}_\tau$, $\eta_{\tau_\sigma}(x) = \eta_\tau(x\sigma^{1/2}) = J_\tau \sigma^{1/2} J_\tau \eta_\tau(x)$, $x \in \mathfrak{N}_{\tau_\sigma} \equiv \{x \in M : \tau(x^* x \sigma) < \infty\}$, $J_{\tau_\sigma} = J_\tau \operatorname{supp}(\sigma)$, and, if $\varphi = \tau_\rho \in M_{*+}$ is a normal faithful positive linear functional, then $\Delta_{\tau_\sigma \tau_\rho} = \sigma J_\tau \rho^{-1} J_\tau$, as

$$\begin{aligned}\Delta^{1/2}_{\tau_\sigma \tau_\rho} \eta_{\tau_\rho}(x) &= J_\tau \eta_{\tau_\sigma}(x^*) = J_\tau \eta_\tau(x^* \sigma^{1/2}) \\ &= \sigma^{1/2} J_\tau \eta_\tau(x^*) = \sigma^{1/2} \eta_\tau(x) \\ &= \sigma^{1/2} J_\tau \rho^{-1/2} J_\tau \eta_{\tau_\rho}(x)\,.\end{aligned}$$

Therefore the proof of the above proposition simplifies considerably.

Example 4.3. In case $\mathcal{H} = \mathbb{C}^n$ is finite-dimensional, i.e. M is the full matrix algebra of $n \times n$ complex matrices, and τ is the ordinary normalized trace, the Hilbert space of the GNS representation is given by $\mathcal{H}_\tau \equiv \mathbb{C}^{n^2}$, with orthonormal basis $\{e_{ij}\}$, whereas M (which is generated by the matrix units $\{u_{hk}\}$) acts on $\mathcal{H}_\tau$ as $\pi_\tau(u_{hk})e_{ij} = \delta_{ki} e_{hj}$, and the cyclic vector is $\xi_\tau = \sum_{i=1}^n e_{ii}$. Then J_τ is given by the antilinear extension of the map $e_{hk} \to e_{kh}$. The L^p-spaces are given by $L^p(M, \tau) = \pi_\tau(M)$ with the L^p-norm, whereas, for $\varphi = \tau_\rho \in M_{*+}$ a faithful (normal) positive linear functional, $L^p(M, \tau_\rho) = \{\pi_\tau(X) J_\tau \rho^{-1/p} J_\tau : X \in M\}$.

5. α-Connections for Statistical Manifolds of Density Operators

This section contains the main result of the paper.

Definition 5.1. (Theorem 1 of Ref. 18) On $L^1(M, \varphi)$ one defines an integral as

$$\int T\, d\varphi := \lim_{y \to 1} (\eta_\varphi(y), T\eta_\varphi(y))\,,$$

where the limit is taken in the *-strong operator topology of the unit ball of $\mathfrak{N}^{an}_\varphi$. Observe that, if $\varphi \in M_{*+}$, the previous formula simplifies in $\int T\, d\varphi = (|T|^{1/2} w^* \xi_\varphi, |T|^{1/2} \xi_\varphi)$, where $T = w|T|$ is the polar decomposition, and $\xi_\varphi \in \mathcal{H}$ is the GNS vector of φ.

Remark 5.1. For $p \in (1, \infty)$, $T \in L^p(M, \varphi)$, $S \in L^{\tilde{p}}(M, \varphi)$, then $TS \in L^1(M, \varphi)$ and the duality between $L^p(M, \varphi)$ and $L^{\tilde{p}}(M, \varphi)$ is given by $\langle T, S \rangle = \int T^* S\, d\varphi$.

Let M be a semifinite von Neumann algebra, τ an nsf trace on M, and $\tau_\rho \in M_{*+}$, as in Example 4.1.

Definition 5.2. Introduce the set $\mathfrak{M}_\tau := \{\rho \in L^1(M, \tau)_+ : \operatorname{supp}(\rho) = 1,\ \|\rho\|_1 = \tau(\rho) = 1\}$, which, by the Pedersen–Takesaki theorem (4.10 of Ref. 27), is in bijective correspondence with the set of normal faithful states of M.

Definition 5.3. We call any $\mathcal{N} \subset \mathfrak{M}_\tau$ a statistical model, whereas we call statistical manifold any statistical model which is also a Banach manifold. Let $\mathcal{N}$ be a statistical manifold, and define the Amari map $A^\alpha : \rho \in N \to \rho^{1/p} \in S^p$, where $p = 2/(1-\alpha)$, and $\alpha \in (-1,1)$. Define $\mathcal{F}^\alpha := \bigcup_{\rho\in\mathcal{N}} \mathcal{F}^\alpha_\rho$, where $\mathcal{F}^\alpha_\rho := \{v \in L^p(M,\tau_\rho) : \mathrm{Re} \int v\, d\tau_\rho = 0\}$.

Theorem 5.1. *Let $\mathcal{N}$ be a statistical manifold of density operators w.r.t. (M,τ). Then $\mathcal{F}^\alpha$ is a vector bundle on $\mathcal{N}$, and, using the pullback by the Amari embedding, we may define an α-connection ∇^α on $\mathcal{F}^\alpha$. In this way we obtain the noncommutative α-bundle-connection pair $(\mathcal{F}^\alpha, \nabla^\alpha)$.*

If M is commutative, this construction reduces to the construction of the non-parametric Amari–Čentsov α-bundle-connection pair $(\mathcal{F}^\alpha, \nabla^\alpha)$ of Theorem 3.1.

Proof. Denote by $S^p := \{T \in L^p(M,\tau) : \|T\|_p = 1\}$ the unit sphere of $L^p(M,\tau)$. As we have seen, the noncommutative L^p-spaces are uniformly convex, with uniformly convex duals, if $p \in (1,\infty)$, so that the results of Sec. 2 apply. As S^p is a Banach submanifold of $L^p(M,\tau)$, there is a splitting of the tangent space $T_xL^p = T_xS^p \oplus V$, as in Remark 2.1, and a continuous projection $\pi_x : T_xL^p \to T_xS^p$. Using π we define the natural connection ∇^p on S^p by the formula $\nabla^p := \pi \circ \nabla$, where ∇ is the trivial connection on $L^p(M,\tau)$ (see Proposition 2.4).

Using the Amari map, we can pull the natural connection on S^p back to $\mathcal{N}$, and obtain a bundle-connection pair $((A^\alpha)^*TS^p, (A^\alpha)^*\nabla^p)$. The fiber of $(A^p)^*TS^p$ at $\rho \in \mathcal{N}$ is isomorphic to $T_{\rho^{1/p}}S^p$. We have in general the duality mapping in $L^p(M,\tau)$ given by $T = w|T| \in L^p(M,\tau) \to \widetilde{T} := \|T\|_p^{1-p/\bar{p}} w|T|^{p/\bar{p}} \in L^{\bar{p}}(M,\tau)$. Indeed $\|\widetilde{T}\|_{\bar{p}} = \|T\|_p^{1-p/\bar{p}} \tau(|T|^p)^{1/\bar{p}} = \|T\|_p$, and $\tau(\widetilde{T}^*T) = \|T\|_p^{1-p/\bar{p}}\tau(|T|^{p/\bar{p}}w^*w|T|) = \|T\|_p^{1-p/\bar{p}}\tau(|T|^p) = \|T\|_p^2$. Therefore $\widetilde{\rho^{1/p}} = \rho^{1/\bar{p}}$, and $T_{\rho^{1/p}}S^p = \{u \in L^p(M,\tau) : \mathrm{Re} \int u\rho^{1/\bar{p}}\, d\tau = 0\}$. Now we need the following:

Lemma 5.1. *The isometric isomorphism $I^p_\rho(u) = uJ\rho^{-1/p}J$ of Sec. 4 sets up a bijective correspondence between $\{u \in L^p(M,\tau) : \mathrm{Re} \int u\rho^{1/\bar{p}}\, d\tau = 0\}$ and $\{v \in L^p(M,\tau_\rho) : \mathrm{Re} \int v\, d\tau_\rho = 0\}$.*

Proof. Observe that the statement follows from the formula $\int v\, d\tau_\rho = \int u\rho^{1/\bar{p}}\, d\tau$, if $v = uJ\rho^{-1/p}J$, that is what we are going to prove. On identifying $\mathcal{H}_\tau$ with $L^2(M,\tau)$, we get $\xi_{\tau_\rho} = \rho^{1/2}$, and $J\rho^{-1/2p}J\xi_{\tau_\rho} = J\rho^{-1/2p}J\rho^{1/2} = \rho^{1/2\bar{p}}$. Indeed $\rho^{-1/2p}$ is τ-measurable, and J becomes the *-operation on L^2. Therefore

$$\begin{aligned}\int v\, d\tau_\rho &= (|u|^{1/2}w^*J\rho^{-1/2p}J\xi_{\tau_\rho}, |u|^{1/2}J\rho^{-1/2p}J\xi_{\tau_\rho}) \\ &= (|u|^{1/2}w^*\rho^{1/2\bar{p}}, |u|^{1/2}\rho^{1/2\bar{p}}) \\ &= \tau(\rho^{1/2\bar{p}}u\rho^{1/2\bar{p}}) = \tau(u\rho^{1/\bar{p}}) = \int u\rho^{1/\bar{p}}\, d\tau\,. \end{aligned}$$

□

Using the previous lemma, the fiber of $(A^p)^*TS^p$ at $\rho \in N$ is isomorphic to $\mathcal{F}^\alpha_\rho := \{v \in L^p(M, \tau_\rho) : \mathrm{Re} \int v\, d\tau_\rho = 0\}$. Using this isomorphism we may transfer the pullback connection $(A^\alpha)^*\nabla^p$ on the bundle $\mathcal{F}^\alpha$ to get a bundle-connection pair $(\mathcal{F}^\alpha, \nabla^\alpha)$ over $\mathcal{N}$, for any $\alpha \in (-1, 1)$.

If M is commutative, then, by e.g. Ref. 29, there is a measure space $(X, \mathfrak{X}, \mu)$ s.t. $L^p(M, \tau) \cong L^p(X, \mathfrak{X}, \mu)$, for $p \in [1, \infty]$, and $\tau(T) \equiv \int T\, d\tau = \int_X T(x)\, d\mu(x)$, for $T \in L^1(M, \tau)$. Therefore the previous construction reduces to that of Theorem 3.1, and this concludes the proof of Theorem 5.1. $\square$

References

1. S. Amari, *Differential-Geometrical Methods in Statistics, Lecture Notes in Statistics,* Vol. 28 (Springer, 1990).
2. S. Amari, O. Barndorff-Nielsen, R. Kass, S. Lauritzen and C. R. Rao, *Differential Geometry in Statistical Inference, Institute of Mathematical Statistics Lecture Notes — Monograph Series,* Vol. 10 (Hayward, 1987).
3. H. Araki and T. Masuda, *Positive cones and L^p-spaces for von Neumann algebras, Publ. R.I.M.S. Kyoto Univ.* **18** (1982) 339–411.
4. N. N. Čentsov, *Statistical Decision Rules and Optimal Inference, Transl. Math. Monographs,* Vol. 53 (AMS, 1982).
5. A. Connes, *On the spatial theory of von Neumann algebras, J. Funct. Anal.* **35** (1980) 153–164.
6. J. Diestel, *Geometry of Banach Spaces — Selected Topics, Lecture Notes in Math.,* Vol. 475 (Springer, 1975).
7. T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators, Pacific J. Math.* **123** (1986) 269.
8. P. Gibilisco and G. Pistone, *Connections on non-parametric statistical manifolds by Orlicz space geometry, Infinite Dimensional Anal. Quantum Prob. Related Topics* **1** (1998) 325–347.
9. U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra, Colloques Internationaux C.N.R.S.* **274** (1979) 175–184.
10. H. Hasegawa, *α-divergence of the non-commutative information geometry, Rep. Math. Phys.* **33** (1993) 87–93.
11. H. Hasegawa, *Non-commutative extension of the Information geometry. Proc. Int. Workshop on Quantum Communications and Measurement,* Nottingham 1994, eds. V. P. Belavkin, O. Hirota and R. L. Hudson (Plenum Press, 1995).
12. H. Hasegawa and D. Petz, *On the Riemannian metric of α-entropies of density matrices, Lett. Math. Phys.* **38** (1996) 221–255.
13. H. Hasegawa and D. Petz, *Non-commutative extension of Information geometry II,* preprint 1997.
14. M. Hilsum, *Les espaces L^p d'une algèbre de von Neumann définies par la derivée spatiale, J. Funct. Anal.* **40** (1981) 151–169.
15. R. E. Kass and P. W. Vos, *Geometrical Foundations of Asymptotic Inference* (Wiley, 1997).
16. G. Köthe, *Topological Vector Spaces I* (Springer, 1969).
17. H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra: non-commutative L^p-spaces, J. Funct. Anal.* **56** (1984) 29–78.
18. T. Masuda, *L^p-spaces for von Neumann algebra with reference to a faithful normal semifinite weight, Publ. R.I.M.S. Kyoto Univ.* **19** (1983) 673–727.

19. M. K. Murray and J. W. Rice, *Differential Geometry and Statistics, Monographs on Statistics and Applied Probability*, Vol. 48 (Chapman & Hall, 1993).
20. H. Nagaoka, *IEICE Tech. Rep.* **89** (1989) 9.
21. H. Nagaoka, *Differential geometrical aspects of quantum state estimation and relative entropy*, in *Proc. Int. Workshop on Quantum Communications and Measurement*, Nottingham 1994, eds. V. P. Belavkin, O. Hirota and R. L. Hudson (Plenum Press, 1995).
22. L. Narici and E. Beckenstein, *Topological Vector Spaces* (Marcel Dekker, 1985).
23. E. Nelson, *Notes on non-commutative integration*, *J. Funct. Anal.* **15** (1974) 103.
24. G. Pistone and M. P. Rogantin, *The exponential statistical manifold: mean parameters, orthogonality, and space transformation*, to appear in *Bernoulli*.
25. G. Pistone and C. Sempi, *An infinite dimensional geometric structure on the space of all the probability measures equivalent to a given one*, *Ann. Stat.* **33** (1995) 1543–1561.
26. I. E. Segal, *A non-commutative extension of abstract integration*, *Ann. Math.* **57** (1953) 401.
27. S. Stratila, *Modular Theory in Operator Algebras* (Abacus Press, 1981).
28. S. Stratila and L. Zsido, *Lectures on von Neumann Algebras* (Abacus Press, 1979).
29. M. Takesaki, *Theory of Operator Algebras, I* (Springer-Verlag, 1979).
30. M. Terp, *L^p spaces associated with von Neumann algebras*, København Universitet, Matematisk Institut, preprint series, June 1981, Report No. 3a, 3b.