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A CHARACTERISATION OF WIGNER–YANASE SKEW INFORMATION AMONG STATISTICALLY MONOTONE METRICS

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Let $M_n = M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices endowed with the Hilbert–Schmidt scalar product, let S_n be the unit sphere of M_n and let $D_n \subset M_n$ be the space of strictly positive density matrices. We show that the scalar product over D_n introduced by Gibilisco and Isola[3] (that is the scalar product induced by the map $D_n \ni \rho \to \sqrt{\rho} \in S_n$) coincides with the Wigner–Yanase monotone metric.

Keywords: Monotone metrics, Fisher–Rao metric, Wigner–Yanase information

1. Introduction

In commutative information geometry the Fisher–Rao metric can be characterised in (at least) three ways: (i) it is the unique statistically monotone metric (Chentsov theorem); (ii) it is the Hessian of the Kullback–Leibler relative entropy; (iii) it is obtained by division of square root of densities. In noncommutative information geometry, the classification theorem of Petz shows that there exists a whole family of statistically monotone metrics parametrised by the family of operator monotone functions.[9] Nevertheless the results of Lesniewski and Ruskai[7] and Gibilisco and Isola[4] prove that each monotone metric is the Hessian of a suitable generalised relative entropy and is obtained by division of a generalised square-root operator. In view of these results it is important to have characterisations that single out a particular monotone metric (for an example see Dittmann[2]). Indeed it is sometimes difficult to decide which monotone metric is the good one for a certain application in quantum physics (Refs. 10 and 11). In a previous paper[3] we considered a scalar product on density matrices derived by the pull-back of the map $\rho \to \sqrt{\rho}$. It is

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natural to ask if this pull-back metric is a statistically monotone one. We show in this note that the pull-back of the square-root embedding is the Wigner–Yanase monotone metric introduced by Hasegawa and Petz.[5,6]

2. Pull-Back of Riemannian Metrics

Let $\mathcal{M}$ be a differentiable manifold and $(\mathcal{N}, g)$ a Riemannian manifold (see Ref. 1 for differential geometric concepts). Suppose $\varphi\colon \mathcal{M} \to \mathcal{N}$ is an immersion, that is a differentiable map such that its differential $D_\rho\varphi\colon T_\rho\mathcal{M} \to T_{\varphi(\rho)}\mathcal{N}$ is injective, for any $\rho \in \mathcal{M}$. Then

Proposition 2.1. *On $\mathcal{M}$ there exists a unique Riemannian scalar product $g^\varphi := \varphi^* g$ compatible with the differential structure of $\mathcal{M}$ such that $\varphi\colon (\mathcal{M}, g^\varphi) \to (\mathcal{N}, g)$ is an isometry.*

Proof. If ρ is an arbitrary point of $\mathcal{M}$, and $u, v \in T_\rho\mathcal{M}$, define

$$g_\rho^\varphi(u, v) := g_{\varphi(\rho)}(D_\rho\varphi(u), D_\rho\varphi(v))\,.$$

Since φ is differentiable, g^φ is compatible with the differential structure of $\mathcal{M}$ and makes φ an isometry. The uniqueness is obvious. □

Definition 2.2. Under the above hypothesis g is said the pull-back metric induced by φ.

Remark 2.3. Let $\gamma\colon [0,1] \to \mathcal{M}$ be a curve, and denote by $L(\gamma)$ the length of γ. Then $L(\gamma) = L(\varphi \circ \gamma)$.

3. The Fisher–Rao Metric and the Square Root

Let $\mathcal{P}_n \subset \mathbb{R}^n$ be the simplex of strictly positive probability vectors, i.e. $\mathcal{P}_n := \{\rho \in \mathbb{R}^n\colon \sum_{i=1}^n \rho_i = 1, \rho_i > 0, i = 1, \ldots, n\}$.

Definition 3.1. The Fisher–Rao Riemannian metric on $T\mathcal{P}_n \equiv \{u \in \mathbb{R}^n\colon \sum_{i=1}^n u_i = 0\}$ is given by

$$M_\rho^{\mathrm{FR}}(u, v) := \sum_{i=1}^n \frac{u_i v_i}{\rho_i}\,,$$

for $u,\ v \in T_\rho\mathcal{P}_n$.

Consider the map $\varphi\colon \rho \in \mathcal{P}_n \to \sqrt{\rho} \in \tilde{S}_n$, where $\tilde{S}_n$ is the unit sphere of $\mathbb{R}^n$, endowed with the natural metric as a Riemannian submanifold of $\mathbb{R}^n$. Then, the following result is well known.

Theorem 3.2. *The pull-back by the map φ of the natural metric on $\tilde{S}_n$ coincides with the Fisher–Rao metric (namely the unique commutative statistically monotone metric).*

Proof. An easy calculation shows that, up to a scalar, the differential of φ is given by $D_\rho\varphi = M_\rho^{-1/2}$, where $M_\rho(u) := (\rho_1 u_1, \dots, \rho_n u_n)$. Therefore

$$\begin{aligned} g_\rho^\varphi(u,v) &:= g_{\varphi(\rho)}(D_\rho\varphi(u), D_\rho\varphi(v)) \\ &= \langle M_\rho^{-1/2}(u), M_\rho^{-1/2}(v)\rangle \\ &= \langle u, M_\rho^{-1}(v)\rangle \\ &= \sum_{i=1}^n \frac{u_i v_i}{\rho_i} = M_\rho^{\mathrm{FR}}(u,v) . \end{aligned}$$

□

4. The Wigner–Yanase Skew Information

Let $\rho \in D_n$ be a density matrix and let A be a self-adjoint matrix. The Wigner–Yanase information (or skew information, information content relative to A) is defined as

$$I(\rho, A) := -\mathrm{Tr}([\rho^{1/2}, A]^2) ,$$

where $[\cdot,\cdot]$ denotes the commutator. The tangent space to D_n at ρ is given by $T_\rho D_n \equiv \{A \in M_n\colon A = A^*, \mathrm{Tr}(A) = 0\}$, and decomposes as $T_\rho D_n = (T_\rho D_n)^{\mathrm{c}} \oplus (T_\rho D_n)^{\mathrm{o}}$, where $(T_\rho D_n)^{\mathrm{c}} := \{A \in T_\rho D_n\colon [A,\rho] = 0\}$, and $(T_\rho D_n)^{\mathrm{o}}$ is the orthogonal complement of $(T_\rho D_n)^{\mathrm{c}}$, with respect to the Hilbert–Schmidt scalar product $\langle A, B\rangle := \mathrm{Tr}(A^*B)$. Let f be a symmetric operator monotone function and $c_f(x,y) := \frac{1}{yf(x/y)}$ the associated Chentsov–Morotsova function.

Petz classification theorem states that each statistically monotone metric on TD_n has the form $M_\rho^f(A,B) := \mathrm{Tr}(Ac_f(L_\rho, R_\rho)(B))$, where $L_\rho(A) := \rho A$ and $R_\rho(A) := A\rho$. Each statistically monotone metric has a unique expression (up to a constant) given by $\mathrm{Tr}(\rho^{-1}A^2)$, for $A \in (T_\rho D_n)^{\mathrm{c}}$, because of the Chentsov uniqueness theorem. Now consider the function

$$f_{\mathrm{WY}}(x) := (\sqrt{x} + 1)^2 ,$$

which is operator monotone.[5] The associated Chentsov–Morotsova function is

$$c_{\mathrm{WY}}(x,y) := \frac{1}{yf_{\mathrm{WY}}(x/y)} = \frac{1}{(\sqrt{x} + \sqrt{y})^2} .$$

Let us consider the monotone metric

$$M_\rho^{\mathrm{WY}}(A,B) := \mathrm{Tr}(Ac_{\mathrm{WY}}(L_p, R_\rho)(B)) .$$

A typical element of $(T_\rho D_n)^{\mathrm{o}}$ has the form $i[\rho, A]$, where A is self-adjoint. We have

$$\begin{aligned} &M^{\mathrm{WY}}(i[\rho,A], i[\rho,A]) \\ &\quad = \mathrm{Tr}(i[\rho,A](L_\rho^{1/2} + R_\rho^{1/2})^{-2}(i[\rho,A])) \\ &\quad = -\langle (L_\rho^{1/2} + R_\rho^{1/2})^{-1}[\rho,A], (L_\rho^{1/2} + R_\rho^{1/2})^{-1}[\rho,A]\rangle \end{aligned}$$

$$= -\langle (L_\rho^{1/2} + R_\rho^{1/2})^{-1}(L_\rho - R_\rho)(A), (L_\rho^{1/2} + R_\rho^{1/2})^{-1}(L_\rho - R_\rho)(A)\rangle$$
$$= -\langle (L_\rho^{1/2} - R_\rho^{1/2})(A), (L_\rho^{1/2} - R_\rho^{1/2})(A)\rangle$$
$$= -\langle [\rho^{1/2}, A], [\rho^{1/2}, A]\rangle$$
$$= -\mathrm{Tr}([\rho^{1/2}, A]^2) = I(\rho, A)\,.$$

5. The Main Result

Let us consider the unit sphere of M_n, denoted by S_n, as a real Riemannian submanifold of M_n. The natural metric on S_n is the one induced by the Hilbert–Schmidt scalar product of M_n.

Let $D_n \subset M_n$ be the manifold of strictly positive definite matrices. The map φ: $\rho \in D_n \to \sqrt{\rho} \in S_n$ is differentiable so we can apply the results of Sec. 2. We have the following:

Theorem 5.1. *The pull-back by the map φ of the natural metric on S_n coincides with the Wigner–Yanase monotone metric.*

Proof. The differential of φ at the point ρ is given by $D_\rho\varphi := (L_\rho^{1/2} + R_\rho^{1/2})^{-1}$ (see Ref. 8 for example). Therefore the pull-back metric is

$$\begin{aligned} g_\rho^\varphi(A, B) &:= g_{\varphi(\rho)}(D_\rho\varphi(A), D_\rho\varphi(B)) \\ &= \langle (L_\rho^{1/2} + R_\rho^{1/2})^{-1}(A), (L_\rho^{1/2} + R_\rho^{1/2})^{-1}(B)\rangle \\ &= \mathrm{Tr}(A(L_\rho^{1/2} + R_\rho^{1/2})^{-2}(B)) \\ &= \mathrm{Tr}(Ac_{\mathrm{WY}}(L_\rho, R_\rho)(B)) = M_\rho^{\mathrm{WY}}(A, B)\,, \end{aligned}$$

which was to be proved. □

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