

GENERIC q -MARKOV SEMIGROUPS AND SPEED OF CONVERGENCE OF q -ALGORITHMS

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We study a special class of generic quantum Markov semigroups, on the algebra of all bounded operators on a Hilbert space \mathcal{H}_S , arising in the stochastic limit of a generic system interacting with a boson–Fock reservoir. This class depends on an orthonormal basis of \mathcal{H}_S . We obtain a new estimate for the trace distance of a state from a pure state and use this estimate to prove that, under the action of a semigroup of this class, states with finite support with respect to the given basis converge to equilibrium with a speed which is exponential, but with a polynomial correction which makes the convergence increasingly worse as the dimension of the support increases (Theorem 5.1). We interpret the semigroup as an algorithm, its initial state as input and, following Belavkin and Ohya,¹⁰ the dimension of the support of a state as a measure of complexity of the input. With this interpretation, the above results mean that the complexity of the input “slows down” the convergence of the algorithm. Even if the convergence is exponential and the slow down the polynomial, the constants involved may be such that the convergence times become unacceptable from a computational standpoint. This suggests that, in the absence of estimates of the constants involved, distinctions such as “exponentially fast” and “polynomially slow” may become meaningless from a constructive point of view.

We also show that, for arbitrary states, the speed of convergence to equilibrium is controlled by the rate of decoherence and the rate of purification (i.e. of concentration of the probability on a single pure state). We construct examples showing that the order of magnitude of these two decays can be quite different.

Keywords: Generic quantum Markov semigroup; convergence to invariant state; complexity.

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1. Introduction

Let \mathcal{B}_S be a C^* -algebra. A quantum Markov semigroup $(\mathcal{T}_t)_{t \geq 0}$ on \mathcal{B}_S is said to converge to equilibrium (in a given topology) if there exists a subset \mathcal{S}_∞ , of the set $\mathcal{S}(\mathcal{B}_S)$ of states on \mathcal{B}_S , and a map $\rho_\infty: \mathcal{S}(\mathcal{B}_S) \rightarrow \mathcal{S}_\infty$ such that, denoting $\mathcal{T}_t^*: \mathcal{B}_S \rightarrow \mathcal{B}_S$ the dual of $(\mathcal{T}_t)_{t \geq 0}$, one has

$$\lim_{t \rightarrow +\infty} \mathcal{T}_t^* \rho = \rho_\infty(\rho) \quad \forall \rho \in \mathcal{S}(\mathcal{B}_S),$$

where the limit is meant in the given topology.

It is worth emphasizing that the term “equilibrium” here refers to dynamics, and not restricted to thermodynamics. It is clear that any equilibrium state for $(\mathcal{T}_t)_{t \geq 0}$ is stationary and that the converse is not true in general. The problem of proving convergence to equilibrium of a dynamical evolution and of estimating its speed has an established tradition in quantum physics and in quantum probability (see the survey Ref. 17 for more information). Recent developments of quantum optics and quantum information provided a stimulus for the study of those q -Markov semigroups, acting on the bounded operators $\mathcal{B}(\mathcal{H}_S)$ of a Hilbert space \mathcal{H}_S , such that:

- (i) they enjoy the property of convergence to equilibrium,
- (ii) the limit set \mathcal{S}_∞ consists only of pure states,
- (iii) they leave invariant several operator spaces (not necessarily subalgebras) contained in the algebra where they are defined, for example: the diagonal and off-diagonal spaces, with respect to a given orthonormal basis $(e_j)_j$ of \mathcal{H}_S , or the “partial off-diagonal spaces”, generated by the flip operators $|e_j\rangle\langle e_k|$, $|e_k\rangle\langle e_j|$ for some fixed pair j, k with $j \neq k$.

These semigroups are remarkable because they combine two quite distinct effects whose time scales can be quite different (cf. comment at the end of the present section), namely:

- (1) decoherence, which is measured by the speed of convergence to zero of the off-diagonal terms (with respect to a given basis),
- (2) purification, i.e. concentration of the probability in a single pure state of the given basis, which is measured by the speed of convergence to a stationary pure state.

Important physical examples of such semigroups arise from the study of 3-level atoms interacting with laser fields, and are discussed in Refs. 4 and 5.

In the following section we will describe some problems of quantum information which naturally lead to the study of the above described class of quantum Markov semigroups and which constitute the motivation of this paper.

The reader who is only interested in the mathematical results and not in their information theoretical motivations, can proceed directly to Sec. 3.

The paper is organized as follows. In Sec. 3 we define the generic vacuum q -Markov semigroups. We outline their construction starting from the form generator (3.1) and recall the explicit formula allowing us to write them as the difference of two classical sub-Markov semigroup plus a conjugation with a contraction semigroup. Finally we write the natural hypotheses **R1** and **R2** that guarantee the existence of a unique ground state.

In Sec. 4 we prove two estimates (Theorems 4.1 and 4.2) of the trace-norm distance of a state ρ from a pure state in terms of some matrix elements. These formulas are the key tools for our results on the speed of convergence towards the ground state described in Sec. 5.

We compute the exact exponential rate for states with sub-Poisson type tails (a natural generalization of states with finite support, e.g. coherent states) and give an example showing that states with heavier tails (Example 6.1) might converge towards the ground state even with polynomial speed.

The examples discussed at the end of Sec. 4 and in Sec. 5 show that the speed of convergence of states with sub-Poisson tails is determined by the decay of the off-diagonal elements which is typically of the type $e^{-gt/2}$ while, for diagonal elements, it is of the type e^{-gt} : thus, although both exponential, the decoherence time is twice the purification time. On the other hand, the diagonal part of states with heavier tails might decay like t^{-1} (polynomial purification time) while the decoherence time (decay of the off-diagonal part) is still exponential.

2. Driving a System to a Ground State: Information Theoretical Motivations

The reason why these states are of interest in quantum information is that on the one hand the typical output states of quantum algorithms are pure states; on the other hand there is a natural analogy between the convergence of an algorithm, considered as a discrete time dynamical system and convergence to equilibrium in the usual sense of q -Markov semigroups.^{8,9}

This analogy is the starting point of the so-called *control through decoherence* technique. The realizability of this program, from the mathematical point of view, was proved in Ref. 6 for general finite-dimensional quantum systems. The idea to combine this technique with the chameleon effect (more precisely its dual) was used in Ref. 9 to build an amplification effect for the output state of the Ohya–Masuda q -SAT algorithm.

The idea to replace quantum annealing procedures by control through decoherence in the approach of q -ground state computation with kinematical gates, is being pursued in Ref. 1 as an attempt to concretely realize the program of kinematical q -computation, based on triods (3-level systems), described in Refs. 12 and 13.

In these papers the output of a quantum algorithm is represented by the ground state of a Hamiltonian H_S (the network Hamiltonian) and the network itself is

represented by a set of 3-level atoms. The geometry of the network is described by the “wires”, i.e. by an interaction term in H_S , describing the coupling of two atoms. In another language we can say that the network is a graph whose vertices are the atoms and whose edges are the wires.

Castagnoli and Finkelstein associate with each Boolean circuit a network and a positive Hamiltonian with the property that the Boolean circuit is satisfiable if and only if the ground state of the corresponding Hamiltonian is associated with the eigenvalue zero.

On the other hand, from the “driving principle” of the stochastic limit of quantum theory, we know that, if a control quantum field in the vacuum state is weakly coupled to the network, then for a generic coupling, the field will drive the system to one of its ground states with an exponential speed (Refs. 4 and 5 for physical examples of this situation).

In this picture, the q -algorithm is represented by,

- (i) the network Hamiltonian (including the interaction with the control field)
- (ii) the initial state, for a given Hamiltonian, different input states will correspond to different algorithms.

We expect that more complex algorithms will converge more slowly than simpler ones. But in finite dimensions every ergodic Markov semigroup has a mass gap, which implies exponential decay. Thus, if we want to distinguish between algorithms of different complexity, we need a finer analysis which allows to account for *state-dependent corrections to the exponential decay*.

The goal of this paper is to develop this finer analysis for the special class of generic vacuum Markov semigroups which have the property of driving the network to a ground state of a given Hamiltonian. Since the dimension of the state space of a network with N atoms grows exponentially with N and, for complex circuits, N can be very large, a natural thing to do is to embed the finite systems into an infinite one. In this embedding the states of finite systems will correspond to states whose support is a finite-dimensional projection.

In the estimates on the speed of convergence, that we are going to develop, this finite dimension will play an explicit role (Theorem 5.1).

3. The Generic Vacuum Quantum Markov Semigroup

Let S be a discrete system with Hamiltonian

$$H_S = \sum_{\sigma \in V} \varepsilon_\sigma |\sigma\rangle\langle\sigma|,$$

where V is a finite or countable set, $(|\sigma\rangle)_{\sigma \in V}$ is an orthonormal basis of the complex separable Hilbert space \mathcal{H}_S of the system and $(\varepsilon_\sigma)_{\sigma \in V}$ are the eigenvalues of H_S . We denote by \mathcal{V}_S the linear manifold generated by finite linear combinations of vectors $|\sigma\rangle$.

Following Accardi and Kozyrev,³ p. 34, we call the Hamiltonian H_S *generic* if the eigenspace associated with each eigenvalue ε_σ is one-dimensional and one has $\varepsilon_\sigma - \varepsilon_\tau = \varepsilon_{\sigma'} - \varepsilon_{\tau'}$ for $\sigma \neq \tau$ if and only if $\sigma = \sigma'$ and $\tau = \tau'$.

The *generic* quantum Markov semigroup was obtained by Accardi and Kozyrev³ (p. 27, (1.1.85)) in the stochastic limit of a discrete system with generic free Hamiltonian H_S interacting with a mean zero, gauge invariant, 0-temperature, Gaussian field. The interaction between the system and the field has the dipole type form

$$H_I = D \otimes A^+(g) + D^+ \otimes A(g),$$

where D, D^+ are system operators, i.e. an operator on \mathcal{H}_S , with domain containing \mathcal{V}_S , such that $\langle v, Du \rangle = \langle D^+v, u \rangle$ for $u, v \in \mathcal{V}_S$ and satisfying the analyticity condition (Γ is the Euler Gamma function)

$$\sum_{n \geq 1} \frac{|\langle \sigma', D^n \sigma \rangle|}{\Gamma(\theta n)} < \infty$$

for all σ, σ' and some $\theta \in]0, 1[$ (a bounded operator, for instance), and $A^+(g), A(g)$ are the creation and annihilation operators on the boson-Fock space over a Hilbert space with test function $g \in \mathfrak{h}_1$. The function g is called the form factor (or cutoff) of the interaction.

The form generator of the *generic* quantum Markov semigroup is

$$\mathcal{L}(x) = \frac{1}{2} \sum_{\sigma, \sigma' \in V, \varepsilon_{\sigma'} < \varepsilon_\sigma} (\gamma_{\sigma\sigma'} (2|\sigma\rangle\langle\sigma'|x|\sigma'\rangle\langle\sigma| - \{|\sigma\rangle\langle\sigma|, x\}) + i\xi_{\sigma\sigma'}[x, |\sigma\rangle\langle\sigma|]). \quad (3.1)$$

Here $\mathcal{L}(x)$ has a meaning as a quadratic form on $\mathcal{V}_S \times \mathcal{V}_S$ because the sums are always weakly converging on this domain, the constants $\gamma_{\sigma\sigma'}$ are non-negative and one has (see Accardi and Kozyrev,³ p. 36 (1.1.99))

$$\gamma_{\sigma\sigma'} = 2\Re(g|g)_\omega^- |\langle\sigma', D\sigma\rangle|^2, \quad \xi_{\sigma\sigma'} = \Im(g|g)_\omega^- |\langle\sigma', D\sigma\rangle|^2, \quad \omega = \varepsilon_\sigma - \varepsilon_{\sigma'}. \quad (3.2)$$

The complex constants $(g|g)_\omega^-$, for several choices of g (for instance continuous and rapidly decreasing at infinity when $\mathfrak{h}_1 = L^2(\mathbb{R}^d)$ with $d \geq 3$) and reasonable choices of the free evolution of the field, satisfy

$$|g|_\sigma = \sup_{\sigma' \in V} \Re(g|g)_{\varepsilon_\sigma - \varepsilon_{\sigma'}}^- < \infty$$

for all $\sigma \in V$. In this case we find from (3.2)

$$\sum_{\sigma' \in V, \varepsilon_{\sigma'} < \varepsilon_\sigma} \gamma_{\sigma\sigma'} \leq 2|g|_\sigma \cdot \sum_{\sigma' \in V, \varepsilon_{\sigma'} < \varepsilon_\sigma} |\langle\sigma', D\sigma\rangle|^2 \leq 2|g|_\sigma \cdot \|D\sigma\|^2 < \infty$$

for all $\sigma \in V$. In the same way, if the constants $(g|g)_\omega^-$ are uniformly bounded in ω , we can show that $\sum_{\varepsilon_{\sigma'} < \varepsilon_\sigma} |\xi_{\sigma\sigma'}| < \infty$. Therefore, throughout this paper, we shall assume that the following summability conditions

$$\mu_\sigma := \sum_{\{\sigma' \in V | \varepsilon_{\sigma'} < \varepsilon_\sigma\}} \gamma_{\sigma\sigma'} < +\infty, \quad (3.3)$$

$$\sum_{\{\sigma' \in V | \varepsilon_{\sigma'} < \varepsilon_\sigma\}} |\xi_{\sigma\sigma'}| < \infty, \quad \kappa_\sigma := \sum_{\{\sigma' \in V | \varepsilon_{\sigma'} < \varepsilon_\sigma\}} \xi_{\sigma\sigma'} \in \mathbb{R}, \quad (3.4)$$

hold for all $\sigma \in V$. If, for some $\sigma \in V$, the set $\{\sigma' \in V | \varepsilon_{\sigma'} < \varepsilon_\sigma\}$ is empty, then the corresponding sums are 0 by definition.

Under this summability condition, the form generator (3.1) can be represented in the generalized canonical GKLS (Gorini, Kossakowski, Sudarshan, Lindblad) form

$$\mathcal{L}(x) = G^*x + \sum_{\sigma, \sigma'} L_{\sigma\sigma'}^* x L_{\sigma\sigma'} + xG,$$

where

$$G = \sum_{\sigma \in V} \left(-\frac{\mu_\sigma}{2} - i\kappa_\sigma \right) |\sigma\rangle\langle\sigma|, \quad L_{\sigma\sigma'} = \sqrt{\gamma_{\sigma\sigma'}} |\sigma'\rangle\langle\sigma|. \quad (3.5)$$

Notice that G is a normal dissipative operator. We denote by $(P_t)_{t \geq 0}$ the strongly continuous contraction semigroup generated by G .

The *minimal semigroup associated with G and $L_{\sigma\sigma'}$* (see, for instance, Chebotarev and Fagnola¹⁴ or Ref. 15) is defined, on elements x of $\mathcal{B}(\mathcal{H}_S)$, by means of the non-decreasing sequence of positive maps $(\mathcal{T}_t^{(n)})_{n \geq 0}$ defined, by recurrence, as follows

$$\begin{aligned} \mathcal{T}_t^{(0)}(x) &= P_t^* x P_t, \\ \langle v, \mathcal{T}_t^{(n+1)}(x)u \rangle &= \langle P_t v, x P_t u \rangle + \sum_{\sigma\sigma'} \int_0^t \langle L_{\sigma\sigma'} P_{t-s} v, \mathcal{T}_s^{(n)}(x) L_{\sigma\sigma'} P_{t-s} u \rangle ds \end{aligned} \quad (3.6)$$

for all $x \in \mathcal{B}(\mathcal{H}_S)$, $t \geq 0$, $v, u \in \text{Dom}(G)$. Indeed, we have

$$\mathcal{T}_t(x) = \sup_{n \geq 0} \mathcal{T}_t^{(n)}(x)$$

for all positive $x \in \mathcal{B}(\mathcal{H}_S)$ and all $t \geq 0$. The definition of positive maps \mathcal{T}_t is then extended to all the elements of $\mathcal{B}(\mathcal{H}_S)$ by linearity. The minimal semigroup associated with G and $L_{\sigma\sigma'}$ satisfies the integral equation

$$\begin{aligned} \langle v, \mathcal{T}_t(x)u \rangle &= \langle v, x u \rangle \\ &+ \int_0^t \left\{ \langle Gv, \mathcal{T}_s(x)u \rangle + \sum_{\sigma\sigma'} \langle L_{\sigma\sigma'} v, \mathcal{T}_s(x) L_{\sigma\sigma'} u \rangle + \langle v, \mathcal{T}_s(x) G u \rangle \right\} ds \end{aligned} \quad (3.7)$$

for all $x \in \mathcal{B}(\mathcal{H}_S)$, $t \geq 0$, $v, u \in \text{Dom}(G)$. Moreover, it is the unique solution to the above equation if and only if it is conservative (or Markov), i.e. $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ for all $t \geq 0$ (see e.g. Ref. 15).

Theorem 3.1. *Assume that (3.3) and (3.4) hold. Then the minimal semigroup associated with the form generator (3.1) is conservative. In particular it is the unique contraction semigroup on $\mathcal{B}(\mathcal{H}_S)$ satisfying (3.7).*

Proof. Let $\text{sp}(H_S)$ be the spectrum of H_S . In order to verify conservativity, it suffices to observe that, for any increasing function $f : \text{sp}(H_S) \rightarrow [0, +\infty[$ one has

$$\mathcal{L}(f(H_S)) = \sum_{\sigma \in V} \sum_{\{\sigma' \in V \mid \varepsilon_{\sigma'} < \varepsilon_{\sigma}\}} \gamma_{\sigma\sigma'} (f(\varepsilon_{\sigma'}) - f(\varepsilon_{\sigma})) |\sigma\rangle\langle\sigma| \leq 0 \tag{3.8}$$

and apply the well-known criteria (see, for instance, Ref. 15 Corollary 3.41 p. 73) taking as self-adjoint operator C in the inequality $2\Re\langle Cu, Gu \rangle + \sum_{\sigma, \sigma'} \langle L_{\sigma\sigma'} u, CL_{\sigma\sigma'} u \rangle \leq b\langle u, Cu \rangle$ ($u \in \mathcal{V}_S$) the operator $C = f(H_S)$ for a function f such that, for all σ ,

$$f(\varepsilon_{\sigma}) \geq \sum_{\{\sigma' \mid \varepsilon_{\sigma'} < \varepsilon_{\sigma}\}} \gamma_{\sigma\sigma'} = \mu_{\sigma}. \tag{3.9} \quad \square$$

As a consequence (see Fagnola and Rebolledo¹⁶) there exists a unique quantum Markov semigroup \mathcal{T} on $\mathcal{B}(\mathcal{H}_S)$ associated with the form generator (3.1) whose generator \mathcal{L} is characterized by

$$\text{Dom}(\mathcal{L}) = \{x \in \mathcal{B}(\mathcal{H}_S) \mid \mathcal{L}(x) \text{ is bounded}\}, \quad \mathcal{L}(x) = \mathcal{L}(x)$$

for all $x \in \text{Dom}(\mathcal{L})$.

Formula (3.8) also shows that the Abelian subalgebra $L^{\infty}(H_S)$ of $\mathcal{B}(\mathcal{H}_S)$ generated by the system Hamiltonian H_S is invariant under the action of the form generator; a well-known fact in the stochastic limit of quantum theory (see Accardi and Kozyrev,³ Sec. 1.1.9, p. 29). Moreover, it is clear from (3.8) that the restriction of the form generator to this Abelian algebra is the generator of a classical jump process with state space $\{\varepsilon_{\sigma} \mid \sigma \in V\}$ jumping from levels ε_{σ} to lower levels $\varepsilon_{\sigma'}$ with intensity $\gamma_{\sigma\sigma'}$.

We shall refer to the Abelian algebra $L^{\infty}(H_S)$ as the *diagonal* subalgebra and denote it by \mathcal{D} . Moreover, we shall denote by \mathcal{D}_{off} the operator space of off-diagonal operators namely the closed (in the norm, strong and weak* topologies) subspace of $x \in \mathcal{B}(\mathcal{H}_S)$ such that $\langle \sigma, x\sigma \rangle = 0$ for all $\sigma \in V$.

Theorem 3.2. *The Abelian subalgebra \mathcal{D} and the operator space \mathcal{D}_{off} are \mathcal{T}_t -invariant for all $t \geq 0$. Moreover, $\mathcal{T}_t(x) = P_t^* x P_t$ for all $x \in \mathcal{D}_{\text{off}}$.*

Proof. We check first that \mathcal{D}_{off} is \mathcal{T}_t -invariant. Indeed, an element of \mathcal{D}_{off} can be written as a sum $\sum_{\tau \neq \tau'} x_{\tau\tau'} |\tau\rangle\langle\tau'|$ convergent in the weak operator topology and

$$P_t^* x P_t = \sum_{\tau \neq \tau'} e^{-((\mu_{\tau} + \mu_{\tau'})/2 + i(\kappa_{\tau} - \kappa_{\tau'})t)} x_{\tau\tau'} |\tau\rangle\langle\tau'|.$$

Computing the second iteration step according to the definition of the minimal semigroup (3.6) we find

$$L_{\sigma\sigma'}^* P_t^* x P_t L_{\sigma\sigma'} = \gamma_{\sigma\sigma'} \sum_{\tau \neq \tau'} e^{-((\mu_{\tau} + \mu_{\tau'})/2 + i(\kappa_{\tau} - \kappa_{\tau'})t)} x_{\tau\tau'} |\sigma\rangle\langle\sigma'| \cdot |\tau\rangle\langle\tau'| \cdot |\sigma'\rangle\langle\sigma| = 0$$

because $\tau \neq \tau'$ and then there are no terms with $\tau = \sigma' = \tau'$. Therefore we have $\mathcal{T}_t^{(n)}(x) = P_t^* x P_t$ for all $n \geq 2$ and, letting n tend to infinity, $\mathcal{T}_t(x) = P_t^* x P_t$.

We check then by induction on n that \mathcal{D} is $\mathcal{T}_t^{(n)}$ -invariant for all n . By formula (3.6) defining the minimal semigroup this is clear for $n = 0$. Indeed, if $x = \sum_{\sigma} x_{\sigma} |\sigma\rangle\langle\sigma|$, then

$$P_t^* x P_t = \sum_{\sigma} e^{-\mu_{\sigma} t} x_{\sigma} |\sigma\rangle\langle\sigma|.$$

Therefore $P_t^* x P_t$ belongs to \mathcal{D} . Suppose that $\mathcal{T}_s^{(n)}(x)$ belongs to \mathcal{D} . Then, for all σ, σ' we have

$$L_{\sigma\sigma'}^* \mathcal{T}_s^{(n)}(x) L_{\sigma\sigma'} = \gamma_{\sigma\sigma'} \langle\sigma', \mathcal{T}_s^{(n)}(x) \sigma'\rangle |\sigma\rangle\langle\sigma|.$$

This shows that

$$L_{\sigma\sigma'}^* \mathcal{T}_s^{(n)}(x) L_{\sigma\sigma'}, \quad P_{t-s}^* L_{\sigma\sigma'}^* \mathcal{T}_s^{(n)}(x) L_{\sigma\sigma'} P_{t-s}$$

are bounded operators in \mathcal{D} as well as the operator defined by the weak* integral

$$\int_0^t P_{t-s}^* L_{\sigma\sigma'}^* \mathcal{T}_s^{(n)}(x) L_{\sigma\sigma'} P_{t-s} ds.$$

Writing now x as a linear combination of four non-negative operators we can assume that x is non-negative. It follows then from the recursion formula (3.6) that the sum of positive operators

$$\sum_{\{\sigma, \sigma' | \varepsilon_{\sigma'} < \varepsilon_{\sigma}\}} \int_0^t P_{t-s}^* L_{\sigma\sigma'}^* \mathcal{T}_s^{(n)}(x) L_{\sigma\sigma'} P_{t-s} ds$$

is strongly convergent. Moreover, its limit is an element of \mathcal{D} . As a consequence $\mathcal{T}_t^{(n+1)}(x)$, which is the sum of this limit and $P_t^* x P_t$, also belongs to \mathcal{D} . This completes the induction argument. □

Denote by T_t the restriction of \mathcal{T}_t to \mathcal{D} . Then $T = (T_t)_{t \geq 0}$ is a weakly*-continuous classical Markov semigroup on \mathcal{D} whose generator A is characterized (see, for example, Lemma 2.19 of Ref. 17) by

$$\text{Dom}(A) = \text{Dom}(\mathcal{L}) \cap \mathcal{D}, \quad Af = \mathcal{L}(f)$$

for all $f \in \text{Dom}(A)$. A straightforward computation shows that the operator A satisfies

$$A_{\sigma\sigma'} = \gamma_{\sigma\sigma'}, \text{ for all } \sigma, \sigma' \text{ with } \varepsilon_{\sigma'} < \varepsilon_{\sigma}, \quad A_{\sigma\sigma} = - \sum_{\{\sigma, \sigma' \in V | \varepsilon_{\sigma'} < \varepsilon_{\sigma}\}} \gamma_{\sigma\sigma'} = -\mu_{\sigma}. \tag{3.9}$$

We denote by A_d the diagonal part of A . Clearly it generates a sub-Markov semigroup $(e^{tA_d})_{t \geq 0}$ on $\ell^{\infty}(V)$ given explicitly by

$$(e^{tA_d} f)(\sigma) = e^{-\mu_{\sigma} t} f(\sigma). \tag{3.10}$$

Notice that A_d coincides formally with $G^* + G$. However, the former is the generator of a semigroup on $\ell^\infty(V)$ space and the latter, a negative self-adjoint operator on \mathcal{H}_S , is the generator of a contraction semigroup on the Hilbert space. Clearly \mathcal{H}_S is isometrically isomorphic to $\ell^2(V)$ and, up to this isomorphism, A_d and $G + G^*$ coincide on $\ell^2(V) \subseteq \ell^\infty(V)$.

The generic quantum Markov semigroup admits a quite explicit representation formula due to Accardi, Hachicha and Ouerdiane⁸ (in the case when $\kappa_\sigma = 0$ for all σ). In order to recall briefly this formula we start by the following:

Lemma 3.1. *Let M be a classical sub-Markov transition operator on $L^\infty(H_S)$. The linear map $\Phi[M] : \mathcal{B}(\mathcal{H}_S) \rightarrow L^\infty(H_S)$ defined by*

$$\Phi[M](x) = \sum_{\sigma, \sigma' \in V} M_{\sigma\sigma'} \langle \sigma', x\sigma' \rangle |\sigma\rangle \langle \sigma|,$$

is a completely positive contraction on $L^\infty(H_S)$ vanishing on \mathcal{D}_{off} . If $M \mathbb{1} = \mathbb{1}$, then $\Phi[M](\mathbb{1}) = \mathbb{1}$.

Proof. The map $\Phi[M]$ is well-defined. Indeed, since M is a classical sub-Markov transition operator, we have $M_{\sigma\sigma'} \geq 0$ for all σ, σ' and $\sum_{\sigma'} M_{\sigma\sigma'} \leq 1$. Therefore, for each $\sigma \in V$, we have

$$\left| \sum_{\sigma'} M_{\sigma\sigma'} \langle \sigma', x\sigma' \rangle \right| \leq \|x\| \sum_{\sigma'} M_{\sigma\sigma'} \leq \|x\|.$$

It follows that $\Phi[M](x)$ is an element of $L^\infty(H_S)$ with norm bounded by $\|x\|$. Moreover, $\Phi[M]$ is completely positive because of the identity

$$\Phi[M](x) = \sum_{\sigma, \sigma' \in V} M_{\sigma\sigma'} |\sigma\rangle \langle \sigma' | x | \sigma'\rangle \langle \sigma|$$

showing that $\Phi[M]$ is a sum of positive multiples of the completely positive maps $x \rightarrow |\sigma\rangle \langle \sigma' | x | \sigma'\rangle \langle \sigma|$. This formula also shows that $\Phi[M]$ vanishes on \mathcal{D}_{off} .

A straightforward computation shows that $\Phi[M](\mathbb{1}) = \mathbb{1}$ whenever $M \mathbb{1} = \mathbb{1}$. □

We are now in a position to prove the representation formula for the generic semigroup.

Theorem 3.3. *The generic quantum Markov semigroup \mathcal{T} satisfies*

$$\mathcal{T}_t(x) = \Phi[e^{tA}](x) - \Phi[e^{tA_d}](x) + P_t^* x P_t \tag{3.11}$$

for all $x \in \mathcal{B}(\mathcal{H}_S)$ and $t \geq 0$. Moreover, $\mathcal{T}_t(x) = \Phi[e^{tA}](x)$ for all $x \in \mathcal{D}$ and $\mathcal{T}_t(x) = P_t^ x P_t$ for all $x \in \mathcal{D}_{\text{off}}$.*

Proof. Every element of $\mathcal{B}(\mathcal{H}_S)$ can be decomposed into the sum of its diagonal and off-diagonal part. Therefore it suffices to prove the identity (3.11) separately for all $x \in \mathcal{D}$ and all $x \in \mathcal{D}_{\text{off}}$.

The identity holds for all $x \in \mathcal{D}_{\text{off}}$. Indeed, by Lemma 3.1, both $\Phi[e^{tA}]$ and $\Phi[e^{tA_d}]$ vanish on \mathcal{D}_{off} .

We consider then $x \in \mathcal{D}$. Writing x in the form $x = \sum_{\sigma} x_{\sigma} |\sigma\rangle\langle\sigma|$ we find immediately

$$P_t^* x P_t = \sum_{\sigma} e^{-\mu_{\sigma} t} x_{\sigma} |\sigma\rangle\langle\sigma| = \Phi[e^{tA_d}](x).$$

Therefore the right-hand side of (3.11) coincides with $\Phi[e^{tA}](x)$. On the other hand, the formula

$$\Phi[e^{tA}](x) = \sum_{\sigma\sigma' \in V} (e^{tA})_{\sigma\sigma'} x_{\sigma'} |\sigma\rangle\langle\sigma|$$

shows that $\Phi[e^{tA}]$ coincides with the restriction of \mathcal{T}_t to the Abelian subalgebra \mathcal{D} . It follows then from Theorem 3.2 that (3.11) holds for all $x \in \mathcal{D}$. □

Thinking of the classical jump process obtained by restriction to \mathcal{D} it is clear that the generic semigroup can have several different behaviors according to the values of the ε_{σ} and $\gamma_{\sigma\sigma'}$. In this paper, since we are interested in the speed of convergence towards a pure state we suppose that the following hypothesis on “regularity” of the Hamiltonian holds

R1. For every $\sigma \in V$ there exists only a finite number of $\sigma' \in V$ with $\varepsilon_{\sigma'} < \varepsilon_{\sigma}$.

Under this regularity assumption it is clear that the generic system Hamiltonian H_S has a unique ground state. We shall denote it by $|e_0\rangle$ and denote by ε_0 the corresponding lowest energy level. Clearly, **R1** implies conditions (3.3) and (3.4).

Moreover, since we are interested in those cases when the quantum Markov semigroup converges to the pure invariant state $|e_0\rangle\langle e_0|$ we suppose that this state can actually be reached in the evolution. This means that the following obviously necessary (and, as we shall see later, sufficient) condition for convergence to this invariant state holds

R2. For every $\sigma \in V$ with $\sigma \neq 0$, there exists a σ' with $\varepsilon_{\sigma'} < \varepsilon_{\sigma}$ and $\gamma_{\sigma\sigma'} > 0$.

Notice that this hypothesis implies that

$$\mu_{\sigma} = \sum_{\{\sigma' \in V | \varepsilon_{\sigma'} < \varepsilon_{\sigma}\}} \gamma_{\sigma\sigma'} > 0, \quad \text{for all } \sigma > 0,$$

and, by definition $\mu_0 = 0$.

Under the hypotheses **R1** and **R2** it is quite natural to take as V the set of natural numbers and denote by $(e_n)_{n \geq 0}$ the canonical orthonormal basis of the system space.

We shall denote by \mathcal{T}_* the predual semigroup of \mathcal{T} acting on the trace class operators on \mathcal{H}_S and by \mathcal{L}_* its generator.

Proposition 3.1. *Suppose that the hypotheses **R1**, **R2** hold. The quantum Markov semigroup \mathcal{T} has the invariant state $|e_0\rangle\langle e_0|$.*

Proof. Finite rank operators belong to the domain of the generator of the predual semigroup \mathcal{T}_* . Therefore $|e_0\rangle\langle e_0|$ belongs to the domain of \mathcal{L}_* . A straightforward computation yields $\mathcal{L}_*(|e_0\rangle\langle e_0|) = 0$ proving that $|e_0\rangle\langle e_0|$ is an invariant state. \square

Remark. In Sec. 5, as a corollary of Theorem 5.1, we shall prove that the above invariant state. Moreover, given any initial state ρ , its time evolution $\mathcal{T}_{*t}(\rho)$ converges towards $|e_0\rangle\langle e_0|$ for the trace norm as t goes to ∞ .

4. Distance from a Pure State

In this section we prove a useful inequality for estimating the trace distance of any state ρ from a pure state in terms of a single matrix element of ρ .

Theorem 4.1. *Let ρ be a normal state and e_0 a unit vector. Then one has*

$$\|\rho - |e_0\rangle\langle e_0|\|_1 \leq 2(1 - \rho_{00})^{1/2},$$

where $\rho_{00} = \langle e_0, \rho e_0 \rangle$.

Proof. Denoting by E the projection $|e_0\rangle\langle e_0|$ we can write

$$\rho = \rho_{00}E + E^\perp \rho E + E \rho E^\perp + E^\perp \rho E^\perp. \tag{4.1}$$

Let v be the vector $E^\perp \rho e_0$ which is orthogonal to e_0 . The normalization $\text{tr}(\rho) = 1$ yields $\rho_{00} + \text{tr}(E^\perp \rho E^\perp) = 1$. The positivity of ρ gives immediately the inequalities

$$\rho_{00}E^\perp \rho E^\perp \geq |v\rangle\langle v|, \quad \|v\|^2 \leq \rho_{00}\text{tr}(E^\perp \rho E^\perp) = \rho_{00}(1 - \rho_{00}).$$

The triangular inequality yields

$$\begin{aligned} \|\rho - |e_0\rangle\langle e_0|\|_1 &\leq \|(\rho_{00} - 1)E + E^\perp \rho E + E \rho E^\perp\|_1 + \|E^\perp \rho E^\perp\|_1 \\ &= \|(\rho_{00} - 1)E + E^\perp \rho E + E \rho E^\perp\|_1 + (1 - \rho_{00}). \end{aligned}$$

Notice that, if $v = 0$, then $E^\perp \rho E = E \rho E^\perp = 0$ and we have the classical commutative identity

$$\|\rho - |e_0\rangle\langle e_0|\|_1 = (1 - \rho_{00}) + \sum_{\sigma \neq 0} \rho_{\sigma\sigma} = 2(1 - \rho_{00})$$

which implies the claimed inequality. Therefore we can assume $v \neq 0$. In order to compute the norm of $(\rho_{00} - 1)E + E^\perp \rho E + E \rho E^\perp$ note that, in any orthonormal basis of the form $(e_0, v/\|v\|, \dots)$ it is representable as 2×2 matrix B

$$B = \begin{pmatrix} \rho_{00} - 1 & \|v\| \\ \|v\| & 0 \end{pmatrix}.$$

Put $r = 1 - \rho_{00}$. Elementary matrix computations yield

$$B^*B = \begin{pmatrix} r^2 + \|v\|^2 & -r\|v\| \\ -r\|v\| & \|v\|^2 \end{pmatrix}$$

$$|B| = \sqrt{B^*B} = \frac{1}{\sqrt{r^2 + 4\|v\|^2}} \begin{pmatrix} r^2 + 2\|v\|^2 & -r\|v\| \\ -r\|v\| & 2\|v\|^2 \end{pmatrix}.$$

It follows that the trace norm of B is equal to $\sqrt{r^2 + 4\|v\|^2}$.

Therefore, since $\|v\|^2 \leq \rho_{00}(1 - \rho_{00})$, we find

$$\begin{aligned} \|\rho - |e_0\rangle\langle e_0|\|_1 &\leq ((1 - \rho_{00})^2 + 4\|v\|^2)^{1/2} + (1 - \rho_{00}) \\ &\leq (1 - \rho_{00})^{1/2}((1 - \rho_{00})^{1/2} + (1 + 3\rho_{00})^{1/2}). \end{aligned}$$

Observing that $x \mapsto \sqrt{1 - x} + \sqrt{1 + 3x}$ is increasing in $[0, 1]$ we deduce that

$$\max_{0 \leq x \leq 1} (\sqrt{1 - x} + \sqrt{1 + 3x}) = 2$$

which proves our inequality. □

Remark. The inequality of Theorem 4.1 is sharp. Indeed, taking the 2×2 matrices

$$|e_0\rangle\langle e_0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho = \frac{\mathbb{1} + u \cdot \sigma}{2},$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, $u_1^2 + u_2^2 + u_3^2 \leq 1$, $u \cdot \rho = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3$, we have $\rho - |e_0\rangle\langle e_0| = (v \cdot \sigma)/2$ with $v = (u_1, u_2, u_3 - 1)$. Therefore we find the identity $|\rho - |e_0\rangle\langle e_0||^2 = \|v\|^2 \mathbb{1}/4$. It follows that

$$\begin{aligned} \|\rho - |e_0\rangle\langle e_0|\|_1 &= \|v\| = ((1 - u_3)^2 + u_1^2 + u_2^2)^{1/2} \\ &\leq ((1 - u_3)^2 + 1 - u_3^2)^{1/2} = \sqrt{2}(1 - u_3)^{1/2}. \end{aligned}$$

Now, since $\rho_{00} = (1 + u_3)/2$, we have $1 - \rho_{00} = (1 - u_3)/2$ and

$$\|\rho - |e_0\rangle\langle e_0|\|_1 = 2(1 - \rho_{00})^{1/2}.$$

This shows that the inequality of Theorem 4.1 is sharp.

Remark. In this paper we will only consider convergence to pure states. However, if $p_j \geq 0$, $\sum_j p_j = 1$ and $(e_j)_{j \geq 0}$ is an orthonormal basis of \mathcal{H}_S , then the inequality

$$\left\| \rho - \sum_j p_j |e_j\rangle\langle e_j| \right\|_1 \leq \sum_j p_j \|\rho - |e_j\rangle\langle e_j|\|_1 \leq 2 \sum_j p_j (1 - \langle e_j, \rho e_j \rangle)^{1/2}$$

allows us to extend Theorem 4.1 to non-pure states.

Theorem 4.1 shows that, in the study of convergence to the pure state $|e_0\rangle\langle e_0|$, it suffices to find estimates of

$$1 - \langle e_0, \mathcal{T}_{*t}(\rho)e_0 \rangle = \sum_{n>0} \langle e_n, \mathcal{T}_{*t}(\rho)e_n \rangle.$$

However, if the off-diagonal part decays twice more quickly than the diagonal part it might be useful to use a different estimate.

Theorem 4.2. *Let ρ be a normal state and e_0 a unit vector. Then one has*

$$\|\rho - |e_0\rangle\langle e_0|\|_1 \leq 2(1 - \rho_{00}) + 2 \left(\sum_{k>0} |\rho_{0k}|^2 \right)^{1/2}, \tag{4.2}$$

where $\rho_{0k} = \langle e_0, \rho e_k \rangle$.

Proof. Indeed, with the notation of the proof of Theorem 4.1, writing ρ as in (4.1) we clearly have the inequality

$$\begin{aligned} \|\rho - |e_0\rangle\langle e_0|\|_1 &\leq \|(\rho_{00} - 1)E + E^\perp \rho E^\perp\|_1 + \|E^\perp \rho E + E \rho E^\perp\|_1 \\ &= 2(1 - \rho_{00}) + \||v\rangle\langle e_0| + |e_0\rangle\langle v|\|_1 \\ &= 2(1 - \rho_{00}) + 2\|v\|. \end{aligned} \quad \square$$

As a consequence, denoting $\rho(t)$ the state at time t , if $(1 - \rho_{00}(t)) \leq e^{-k_0 t}$ and $\|v(t)\| \leq e^{-k_d t}$ with $k_d > k_0/2$, then the estimate of Theorem 4.2 is more convenient than that of Theorem 4.1 for large times t .

5. Convergence Towards the Ground State

In this section we shall estimate the speed of convergence towards the invariant state $|e_0\rangle\langle e_0|$ finding that it depends on the γ_{ml} and on the initial state ρ .

The off-diagonal elements

$$\langle e_j, \mathcal{T}_{*t}(\rho) e_m \rangle = \langle P_t^* e_j, \rho P_t^* e_m \rangle = e^{-\left(\frac{\mu_j + \mu_m}{2} + i(\kappa_j - \kappa_m)\right)t} \langle e_j, \rho e_m \rangle \tag{5.1}$$

with $j \neq m$ are explicit. Moreover, the last term on the right-hand side of (4.2) for the state $\mathcal{T}_{*t}(\rho)$ satisfies the inequality

$$\left(\sum_{k>0} |\langle e_0, \mathcal{T}_{*t}(\rho) e_k \rangle|^2 \right)^{1/2} = \left(\sum_{k>0} e^{-\mu_k t} |\rho_{0k}|^2 \right)^{1/2} \leq e^{-gt/2} \left(\sum_{k>0} |\rho_{0k}|^2 \right)^{1/2},$$

where $g = \inf_{m>0} \mu_m$. This shows that the off-diagonal part of any initial states ρ converges to 0 with exponential rate $g/2$.

We now concentrate on the diagonal elements for $j = m > 0$.

Lemma 5.1. *Let $\rho = \sum_{h,k \geq 0} \rho_{hk} |e_h\rangle\langle e_k|$ be a normal state. Then, for all $m \geq 0$, we have*

$$\langle e_m, \mathcal{T}_{*t}(\rho) e_m \rangle = \sum_{k \geq 0} \rho_{kk} \langle e_m, \mathcal{T}_{*t}(|e_k\rangle\langle e_k|) e_m \rangle.$$

Proof. Indeed, since $\langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle = \text{tr}(\mathcal{T}_{*t}(\rho)|e_m\rangle\langle e_m|)$, we have

$$\langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle = \text{tr}(\rho \mathcal{T}_t(|e_m\rangle\langle e_m|)) = \sum_{k \geq 0} \rho_{kk} \langle e_k, \mathcal{T}_t(|e_m\rangle\langle e_m|)e_k \rangle. \quad \square$$

This elementary lemma shows that it suffices to restrict ourselves to pure initial states. In this case the problem of estimating the decay of diagonal matrix elements is finite-dimensional as shown:

Lemma 5.2. *For every $d \geq 0$ we have*

$$\langle e_m, \mathcal{T}_{*t}(|e_d\rangle\langle e_d|)e_m \rangle = \langle e_d, \mathcal{T}_t(|e_m\rangle\langle e_m|)e_d \rangle = 0$$

for all $m > d$ and all $t \geq 0$.

Proof. Clearly $\langle e_d, P_t^*(|e_m\rangle\langle e_m|)P_t e_d \rangle = e^{-\mu_m t} |\langle e_d, e_m \rangle|^2 = 0$. Recalling the identity (3.6) defining the minimal semigroup, assume that we proved $\langle e_d, \mathcal{T}_t^{(n)}(|e_m\rangle\langle e_m|)e_d \rangle = 0$ for all $m > d$. Then we have

$$\langle e_d, \mathcal{T}_t^{(n+1)}(|e_m\rangle\langle e_m|)e_d \rangle = \sum_{j>d} \int_0^t e^{-\mu_d(t-s)} \langle L_{jd}e_d, \mathcal{T}_s^{(n)}(|e_m\rangle\langle e_m|)L_{jd}e_d \rangle ds.$$

Since $L_{jd}e_d = 0$, the conclusion follows. □

We shall now prove our key lemma for estimating the decay of the diagonal elements of a state.

Definition 5.1. Given two $d \times d$ matrices A, B with non-negative entries, we say that A is smaller than B entry wise and write $A \preceq B$ if $A_{jk} \leq B_{jk}$ for all $1 \leq j, k \leq d$.

This definition can obviously be extended in the same way to rectangular matrices and vectors. Notice that, if $A \preceq B$, then for every vector $u \in \mathbb{R}^d$ with non-negative entries we have also $Au \preceq Bu$.

The following lemma is the key step of our estimates.

Lemma 5.3. *Let M, N be $d \times d$ matrices ($d \geq 1$) with $M \leq 0$ diagonal and N lower triangular with non-negative entries and zero diagonal entries. Then one has*

$$e^{t(M+N)} \preceq e^{-gt} \sum_{k=0}^{d-1} \frac{t^k N^k}{k!},$$

where $g = \inf_{1 \leq k \leq d} \{-M_{kk}\}$.

Proof. Starting from the identity

$$\frac{d}{ds} e^{(t-s)M} e^{s(M+N)} = e^{(t-s)M} N e^{s(M+N)}$$

and integrating it on $[0, t]$ we find

$$e^{t(M+N)} = e^{tM} + \int_0^t e^{(t-s)M} N e^{s(M+N)} ds.$$

Iterating d times we have

$$\begin{aligned} e^{t(M+N)} &= e^{tM} + \int_0^t e^{(t-s_1)M} N e^{s_1 M} ds_1 + \dots \\ &+ \int_0^t e^{(t-s_1)M} N ds_1 \dots \int_0^{s_{d-2}} e^{(s_{d-2}-s_{d-1})M} N e^{s_{d-1} M} ds_{d-1} \\ &+ \int_0^t e^{(t-s_1)M} N ds_1 \dots \int_0^{s_{d-1}} e^{(s_{d-1}-s_d)M} N e^{s_d(M+N)} ds_d. \end{aligned}$$

Notice that the last term vanishes because it contains the product of d lower triangular (with zero diagonal) $d \times d$ matrices $e^{rM} N$.

Observing that $e^{rM} \preceq e^{-gr}$ and that the entries of N are non-negative, we find the inequality

$$e^{t(M+N)} \preceq e^{-gt} \sum_{k=0}^{d-1} N^k \int_0^t ds_1 \dots \int_0^{s_{k-1}} ds_k.$$

The conclusion then follows. □

Theorem 5.1. *For any state ρ with $\langle e_k, \rho e_k \rangle = 0$ for all $k > d$ we have*

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho) e_m \rangle \leq e^{-g_d t} \sum_{k=0}^{d-1} \frac{(C_d t)^k}{k!}, \tag{5.2}$$

where $g_d = \inf_{0 < m \leq d} \mu_m$ and $C_d = \sup_{0 < m \leq d} \mu_m$.

Proof. Put $\pi_m(t) = \langle e_m, \mathcal{T}_{*t}(\rho) e_m \rangle$ and recall that $\langle e_m, \mathcal{T}_{*t}(\rho) e_m \rangle = 0$ for $m > d$ by Lemma 5.2. Since $|e_m\rangle\langle e_m|$ belongs to $\text{Dom}(\mathcal{L})$, differentiating we find

$$\pi'_m(t) = -\mu_m \pi_m(t) + \sum_{m < h \leq d} \gamma_{hm} \pi_h(t) \quad \text{for } 1 \leq m \leq d \tag{5.3}$$

with the initial condition $\pi_m(0) = \langle e_m, \rho e_m \rangle$. Denoting $\Pi(t)$ the d -dimensional row vector $(\pi_1(t), \dots, \pi_d(t))$, Eq. (5.3) can be written in the form $\Pi'(t) = \Pi(t)(M + N)$ where M is the $d \times d$ diagonal matrix $\text{diag}(-\mu_1, \dots, -\mu_d)$ and N the lower triangular matrix $N_{hm} = \gamma_{hm}$ for $h > m$, $N_{hm} = 0$ for $h \leq m$. The initial condition is the d -dimensional row vector $\Pi(0)$. From Lemma 5.3 we then have

$$\Pi(t) \preceq e^{-gt} \sum_{k=0}^{d-1} \frac{t^k}{k!} \Pi(0) N^k.$$

It follows that

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle = \sum_{m=1}^d \pi_m(t) \leq e^{-gt} \sum_{k=0}^{d-1} \frac{t^k}{k!} \sum_{m=1}^d (\Pi(0)N^k)_m.$$

Denote $\ell^1(1, \dots, d)$ the Banach space of d -dimensional row vectors Π endowed with the norm $\|\Pi\|_1 = \sum_{m=1}^d |\Pi_m|$. Clearly N leaves invariant the cone $\ell^1_+(1, \dots, d)$ of vectors with non-negative components. A straightforward computation shows that $\|\Pi N\|_1$ is equal to

$$\sup_{\Pi, \|\Pi\|_1 \leq 1} \|\Pi N\|_1 = \sup_{\Pi, \|\Pi\|_1 \leq 1} \left| \sum_{m=1}^d \sum_{h=m+1}^d \Pi_h \gamma_{hm} \right| = \sup_{\Pi, \|\Pi\|_1 \leq 1} \left| \sum_{h=2}^d \Pi_h \sum_{m=1}^{h-1} \gamma_{hm} \right|.$$

Recalling that $\mu_h = \sum_{m=1}^{h-1} \gamma_{hm}$ we have

$$\sup_{\Pi, \|\Pi\|_1 \leq 1} \|\Pi N\|_1 \leq \sup_{2 \leq h \leq d} \mu_h \sup_{\Pi, \|\Pi\|_1 \leq 1} \sum_{h=2}^d \Pi_h \leq \sup_{2 \leq h \leq d} \mu_h \leq C_d.$$

It follows that

$$\sum_{m=1}^d (\Pi(0)N^k)_m = \|\Pi(0)N^k\|_1 \leq C_d \|\Pi(0)N^{k-1}\|_1 \leq \dots \leq \|\Pi(0)\|_1 \cdot C_d^k.$$

This proves the inequality (5.2). □

Corollary 5.1. *Suppose that R1 and R2 hold. Then, for all normal state ρ ,*

$$\lim_{t \rightarrow \infty} \|\mathcal{T}_{*t}(\rho) - |e_0\rangle\langle e_0|\|_1 = 0.$$

In particular $|e_0\rangle\langle e_0|$ is the unique normal \mathcal{T} -invariant state.

Proof. Let ρ be a normal invariant state. For all $\varepsilon > 0$ let ρ' be a normal state with $\langle e_k, \rho e_k \rangle = 0$ for all k larger than some integer $d > 1$ such that $\|\rho - \rho'\|_1 < \varepsilon$. By Theorems 4.1 and 5.1 we then have

$$\begin{aligned} \|\mathcal{T}_{*t}(\rho) - |e_0\rangle\langle e_0|\|_1 &\leq \|\mathcal{T}_{*t}(\rho) - \mathcal{T}_{*t}(\rho')\|_1 + \|\mathcal{T}_{*t}(\rho') - |e_0\rangle\langle e_0|\|_1 \\ &\leq \varepsilon + \left(\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho')e_m \rangle \right)^{1/2} \\ &\leq \varepsilon + e^{-gat} \sum_{k=0}^{d-1} \frac{(C_d t)^k}{k!}. \end{aligned}$$

Since ε is arbitrary, the claimed limit is proved by letting t tend to infinity.

It also follows that the only invariant state is $|e_0\rangle\langle e_0|$. Indeed, if ρ and $\tilde{\rho}$ are two invariant states, then $\rho - \tilde{\rho} = \mathcal{T}_{*t}(\rho) - \mathcal{T}_{*t}(\tilde{\rho})$ and, letting t tend to infinity, we find $\rho - \tilde{\rho} = |e_0\rangle\langle e_0| - |e_0\rangle\langle e_0| = 0$. □

Remark. The inequality (5.2) is sharp. Indeed, taking

$$\gamma_{mm-1} = 1 \text{ for } 1 \leq m \leq d, \quad \rho = |e_d\rangle\langle e_d|,$$

with the notation of Lemma 5.3 we have

$$M = \text{diag}(-1, \dots, -1), \quad N = \sum_{m=1}^{d-1} |e_m\rangle\langle e_{m+1}|, \quad \rho = (0, \dots, 0, 1).$$

Therefore we can compute explicitly

$$(0, \dots, 0, 1)e^{t(M+N)} = e^{-t}(t^{d-1}/(d-1)!, t^{d-2}/(d-2)!, \dots, t, 1).$$

Since $g_d = C_d = 1$, it follows that the right- and left-hand sides of (5.2) coincide.

The estimate (5.2) shows that the speed of convergence towards the invariant state can depend on the initial state. Indeed, the constants g_d and C_d depend on d . We shall see later that this is unavoidable therefore this is not a real drawback. The inequality (5.2) however, has two drawbacks:

- (a) depends only the “size” of the support of the initial state,
- (b) involves the constants C_d that, for big d , behave essentially as the operator norm of the generator \mathcal{L} .

We shall now try to find estimates (Theorem 5.2 here below) for arbitrary initial states requiring weaker properties of the generator \mathcal{L} and depending on the “tail” of the initial state ρ .

As a preliminary step we prove the following:

Lemma 5.4. *Let N be the lower triangular matrix with non-negative entries $N_{dm} = \gamma_{dm}$ for $d > m$ and $N_{dm} = 0$ for $d \leq m$ and let $c = \sup_{\{d,m|d>m\}} \gamma_{dm}$. Then, for all $d > m$ and all k with $1 \leq k \leq d - m$, the following inequality holds*

$$(N^k)_{dm} \leq c^k \binom{d - m - 1}{k - 1}.$$

Proof. The inequality obviously holds when $d = m + 1$ and $k = 1$. Suppose then $d > m + 1$. In this case, since N is lower triangular with zero diagonal, we have

$$\begin{aligned} (N^k)_{dm} &= \sum_{m < i_1 < i_2 < \dots < i_{k-1} < d} \gamma_{di_1} \gamma_{i_1 i_2} \dots \gamma_{i_{k-2} i_{k-1}} \gamma_{i_{k-1} m} \\ &\leq c^k \sum_{m < i_1 < i_2 < \dots < i_{k-1} < d} 1. \end{aligned}$$

The conclusion follows because the number of terms of the above sum is equal to the number of subsets of the set $\{m + 1, \dots, d - 1\}$ containing $k - 1$ elements. \square

Theorem 5.2. *Let ρ be a normal state. Then*

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle \leq e^{-gt} \left(1 - \rho_{00} + \sum_{k=1}^{\infty} \frac{(ct)^k R_k(\rho)}{k!} \right), \tag{5.4}$$

where

$$g = \inf_{m>0} \mu_m, \quad c = \sup_{\{d,m|d>m\}} \gamma_{dm}, \quad R_k(\rho) = \sum_{h \geq k} \binom{h-1}{k-1} \sum_{d>h} \rho_{dd}.$$

Proof. Take $d \geq 1$ and consider the truncation of the state ρ defined by $\rho' = \sum_{0 \leq j,k \leq d} \rho_{jk} |e_j\rangle \langle e_k|$. For all $t \geq 0$ and $m \geq 1$ put

$$\pi_m(t) = \langle e_m, \mathcal{T}_{*t}(\rho')e_m \rangle = \sum_{j=1}^d \rho_{jj} \langle e_m, \mathcal{T}_{*t}(|e_j\rangle \langle e_j|)e_m \rangle.$$

Arguing as in the proof of Theorem 5.1 we find

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho')e_m \rangle = \sum_{m=1}^d \pi_m(t) \leq e^{-gt} \sum_{k=0}^{d-1} \frac{t^k}{k!} \sum_{m=1}^d (\Pi(0)N^k)_m,$$

where $\Pi(0) = (\rho_{11}, \dots, \rho_{nn})$. Letting d tend to infinity we have the inequality

$$\begin{aligned} \sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle &\leq e^{-gt} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{m=1}^{\infty} (\Pi(0)N^k)_m \\ &= e^{-gt} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \rho_{dd} (N^k)_{dm}. \end{aligned}$$

By Lemma 5.2 and we have then

$$\begin{aligned} \sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle &\leq e^{-gt} \left(\sum_{d>0} \rho_{dd} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{d=1}^{\infty} \sum_{m=1}^{\infty} \rho_{dd} (N^k)_{dm} \right) \\ &= e^{-gt} \left((1 - \rho_{00}) + \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{d>k} \sum_{m=1}^{d-k} \rho_{dd} (N^k)_{dm} \right) \\ &\leq e^{-gt} \left(1 - \rho_{00} + \sum_{k=1}^{\infty} \frac{(ct)^k}{k!} \sum_{d>k} \rho_{dd} \sum_{m=1}^{d-k} \binom{d-m-1}{k-1} \right). \end{aligned}$$

The conclusion follows changing the index m by $h = d - m$ and exchanging summation in h and d . □

The series on the right-hand side of (5.4) can be divergent. This is not the case, however, if the tails of the state ρ are small enough.

Corollary 5.2. *Suppose that the state ρ has sub-Poissonian tails, i.e.*

$$\sum_{d>h} \rho_{dd} \leq \frac{r^h}{h!}$$

for some constant $r > 0$ and all $h \geq 0$. Then the following inequality holds

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle \leq e^{-gt/2} \left(1 + e^r \sum_{k=1}^{\infty} \frac{(crt)^k}{(k!)^2} \right), \tag{5.5}$$

where $g = \inf_{m>0} \mu_m$ and $c = \sup_{0<m<n} \gamma_{nm}$. If ρ has sub-exponential tails, i.e.

$$\sum_{d>h} \rho_{dd} \leq r\theta^h$$

for a $\theta \in]0, 1[$, an $r > 0$, and all $h \geq 0$ then

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle \leq e^{-gt/2} (1 + r\theta(e^{-c\theta t/(1-\theta)} - 1)). \tag{5.6}$$

In particular, if $\theta < g(g + 2c)^{-1}$, then

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(\rho)e_m \rangle \leq e^{-gt/2} (r\theta + e^{-gt/2}). \tag{5.7}$$

Proof. Indeed, if the first hypothesis on the tails of the state ρ yields

$$R_k(\rho) \leq \sum_{h \geq k} \frac{(h-1)!}{(k-1)!(h-k)!} \frac{r^h}{h!} \leq \frac{r^k}{k!} \sum_{h \geq k} \frac{r^{h-k}}{(h-k)!} = e^r \frac{r^k}{k!}.$$

On the other hand, if the state ρ satisfies $\rho_{dd} \leq r\theta^d$, then $\sum_{d>h} \rho_{dd} \leq r\theta^{h+1}/(1-\theta)$. Moreover, by a well-known formula on the Pascal (or negative binomial) distribution, we have

$$R_k(\rho) \leq r \sum_{h \geq k} \binom{h-1}{k-1} \theta^{h+1} = \frac{r\theta^{k+1}}{(1-\theta)^k} \sum_{h \geq k} \binom{h-1}{k-1} (1-\theta)^k \theta^{h-k} = \frac{r\theta^{k+1}}{(1-\theta)^k}.$$

It follows that, denoting $\eta = \theta(1-\theta)^{-1}$, the series

$$\sum_{k=1}^{\infty} \frac{(ct)^k}{k!} R_k(\rho) \leq r\theta \sum_{k=1}^{\infty} \frac{(c\eta t)^k}{k!}$$

and the inequality (5.6) follows.

It is now easy to check (5.7). □

Remark. Notice that, for states with exponential tails, the exponential rate g on the right-hand side of (5.6) becomes $g - c\theta/(1-\theta)$ which is positive if and only if $\theta < g(g+c)^{-1}$.

We shall see in the next section that the decrease of the exponential rate g in the right-hand side of (5.4) is unavoidable for states with exponential tails.

Coherent states $\rho = |v\rangle\langle v|$ with

$$v = e^{-|z|^2/2} \sum_{n \geq 0} \frac{z^n}{\sqrt{n!}} e_n,$$

($z \in \mathbb{C}$) (normalized exponential vector) have sub-Poissonian tails. Indeed $\rho_{dd} = |\langle e_d, v \rangle|^2 = e^{-|z|^2} |z|^{2d} / d!$ and

$$\sum_{d > k} \rho_{dd} = e^{-|z|^2} \sum_{d > k} \frac{|z|^{2d}}{d!} \leq e^{-|z|^2} \frac{|z|^{2(k+1)}}{(k+1)!} \sum_{d > k} \frac{|z|^{2(d-k-1)}}{(d-k-1)!} \leq \frac{|z|^{2(k+1)}}{(k+1)!}.$$

Clearly also states with finite support in the basis $(e_n)_{n \geq 0}$ (i.e. such that $\langle e_k, \rho e_k \rangle = 0$ for all k larger than a certain integer d) satisfy this hypothesis.

In order to determine the asymptotic behavior of the series on the right-hand side of (5.5) recall that, for all $x \geq 0$,

$$\sum_{k \geq 0} \frac{x^k}{(k!)^2} = I_0(2\sqrt{x}),$$

where $I_0(r)$ is the modified Bessel function that solves the differential equation $r^2 y''(r) + y'(r) - r^2 y(r) = 0$.

Proposition 5.1. *The following inequality holds*

$$\frac{e^x}{\sqrt{2\pi x + 1}} \leq I_0(x) \leq \frac{e^x}{\sqrt{2\pi x + 1}} \left(1 + \frac{2}{2\pi x + 1} \right)$$

for every $x > 0$.

The factor $(k!)^{-2}$ in (5.5) improves drastically the factor $(k!)^{-1}$ in (5.4). Indeed, we have the following:

Theorem 5.3. *For any state ρ we have*

$$\|\mathcal{T}_{*t}(\rho) - |e_0\rangle\langle e_0|\|_1 \leq 2e^{-gt} \left(1 - \rho_{00} + \sum_{k=1}^{\infty} \frac{(ct)^k R_k(\rho)}{k!} \right) + 2e^{-gt/2} \left(\sum_{k > 0} |\rho_{0k}|^2 \right)^{1/2},$$

where

$$g = \inf_{m > 0} \mu_m, \quad c = \sup_{\{d, m \mid d > m\}} \gamma_{dm}, \quad R_k(\rho) = \sum_{d > k} \rho_{dd} \sum_{h=k}^{d-1} \binom{h-1}{k-1}.$$

In particular, if the state ρ has sub-Poissonian tails, i.e. $\sum_{d > h} \rho_{dd} \leq r^h / h!$ for some constant $r > 0$ and all $h \geq 0$, then

$$\|\mathcal{T}_{*t}(\rho) - |e_0\rangle\langle e_0|\|_1 \leq e^{-gt/2} + 2e^{-gt} (1 + e^r (3e^{2\sqrt{crt}} - 1)).$$

Proof. The first inequality follows immediately from Theorem 4.2, applied to the state $\mathcal{T}_{*t}(\rho)$, identity (5.1) and Theorem 5.2.

The second inequality, for states with sub-Poissonian tails, follows from the first noting that, by the positivity of ρ , we have

$$\sum_{k>0} |\rho_{0k}|^2 \leq \sum_{k>0} \rho_{00}\rho_{kk} \leq \rho_{00}(1 - \rho_{00}) \leq 1/4$$

and, by Proposition 5.1, the inequality (5.5) yields

$$\sum_{k=1}^{\infty} \frac{(crt)^k}{(k!)^2} = I_0(2\sqrt{crt}) - 1 \leq 3e^{2\sqrt{crt}} - 1. \quad \square$$

Remark. The exponential rate $g/2$ in Theorem 5.3 cannot be improved. Indeed, let $\gamma_{mm-1} = 1$ for all $m \geq 1$ so that $g = 1$ and let $\rho = |v\rangle\langle v|$, for the unit vector $v = (e_0 + e_1)/\sqrt{2}$. We can compute explicitly $\mathcal{T}_{*t}(\rho)$ and find that it is representable as the 2×2 matrix

$$\frac{1}{2} \begin{pmatrix} 2 - e^{-t} & e^{-t/2} \\ e^{-t/2} & e^{-t} \end{pmatrix}.$$

Therefore, since the eigenvalues of $|\mathcal{T}_{*t}(\rho) - |e_0\rangle\langle e_0|^2$ are $e^{-t/2}(1 - e^{-t/2})/2$ and $e^{-t/2}(1 + e^{-t/2})/2$, we have

$$\|\mathcal{T}_{*t}(\rho) - |e_0\rangle\langle e_0|\|_1 = e^{-t/2}.$$

This example also shows that the off-diagonal terms can decay slower than diagonal terms.

6. A Nearest Neighbour Generic Semigroup

In this section we compute the exact decay rate of the diagonal part for a special choice of the γ_{hk} and three pure initial states. These examples show that we can find initial states converging to the unique invariant state as slow as we want. Here we call our generic semigroup *nearest neighbour* because transitions are possible only from level ε_{h+1} to level ε_h at constant rate

$$\gamma_{h+1h} = 1, \quad \gamma_{kh} = 0, \quad k > h.$$

With this choice of the γ_{kh} , we computed in the previous section (see the remark following Theorem 5.1)

$$\langle e_m, \mathcal{T}_{*t}(|e_d\rangle\langle e_d|)e_m \rangle = \begin{cases} 0 & \text{if } m > d, \\ \frac{e^{-t}t^{d-m}}{(d-m)!} & \text{if } m \leq d. \end{cases}$$

Therefore, if the initial state is a pure state $|v\rangle\langle v|$ for some unit vector $v \in \mathcal{H}_S$, we have

$$\begin{aligned} \langle e_m, \mathcal{T}_{*t}(|v\rangle\langle v|)e_m \rangle &= \sum_{k>m} |\langle v, e_k \rangle|^2 \langle e_m, \mathcal{T}_{*t}(|e_k\rangle\langle e_k|)e_m \rangle \\ &= e^{-t} \sum_{k \geq m} |\langle v, e_k \rangle|^2 \frac{t^{k-m}}{(k-m)!}. \end{aligned}$$

6.1. Slowly convergent states

Let v be the vector

$$v = \sum_{k \geq 0} ((k+1)(k+2))^{-1/2} e_k.$$

Clearly it has norm one because

$$\|v\|^2 = \sum_{k \geq 0} (k+1)^{-1}(k+2)^{-1} = \sum_{k \geq 0} ((k+1)^{-1} - (k+2)^{-1}) = 1.$$

For the pure state $|v\rangle\langle v|$ one finds

$$\begin{aligned} \sum_{m>0} \langle e_m, \mathcal{T}_{*t}(|v\rangle\langle v|)e_m \rangle &= e^{-t} \sum_{m>0} \sum_{k \geq m} \frac{t^{k-m}}{(k+1)(k+2)(k-m)!} \\ &= e^{-t} \sum_{m>0} t^{-(n+2)} \sum_{k \geq m} \int_0^t ds \int_0^s \frac{r^k}{(k-m)!} dr \\ &= e^{-t} \sum_{n>0} t^{-(n+2)} \int_0^t ds \int_0^s r^n e^r dr \\ &= t^{-2} e^{-t} \int_0^t ds \int_0^s \frac{r/t}{1-r/t} e^r dr \\ &= t^{-2} e^{-t} \int_0^t \frac{r e^r}{t-r} dr \int_r^t ds = t^{-2} e^{-t} \int_0^t r e^r dr \\ &= t^{-2} e^{-t} (t e^{-t} - e^t + 1) = t^{-1} (1 - t^{-1} + t^{-1} e^{-t}). \end{aligned}$$

The off-diagonal elements, however, decay faster at an exponential speed. Indeed $\langle e_0, \mathcal{T}_{*t}(|v\rangle\langle v|)e_k \rangle = e^{-t/2}$ for $k > 0$ and $\langle e_h, \mathcal{T}_{*t}(|v\rangle\langle v|)e_k \rangle = e^{-t}$ for $h \neq k$ and $h, k > 0$.

6.2. Number states

These are the pure states $|v\rangle\langle v|$ with v unit vector

$$v = (1 - \theta^2)^{1/2} \sum_{k \geq 0} \theta^k e_k = (1 - \theta^2)^{1/2} \theta^{a^* a}$$

with $|\theta| < 1$ and a^*a the number operator defined by $(a^*a)e_k = ke_k$. In this case $|\langle v, e_k \rangle|^2 = (1 - \theta^2)\theta^{2k}$, therefore one finds

$$\begin{aligned} \sum_{k>0} \langle e_n, \mathcal{T}_{*t}(|v\rangle\langle v|)e_n \rangle &= (1 - \theta^2)e^{-t} \sum_{k>0} \sum_{k \geq n} \theta^{2k} \frac{t^{k-n}}{(k-n)!} \\ &= (1 - \theta^2)e^{-t} \sum_{n>0} \theta^{2n} \sum_{k \geq n} \frac{(\theta^2 t)^{k-n}}{(k-n)!} \\ &= (1 - \theta^2)e^{-t} \sum_{n>0} \theta^{2n} e^{\theta^2 t} = \theta^2 e^{-t(1-\theta^2)}. \end{aligned}$$

Then the diagonal part of ρ decays at the exponential rate $(1 - \theta^2)$.

6.3. Coherent states

Let $v = e^{-|z|^2/2}e(z)$ be the normalized exponential vector with $z \in \mathbb{C}$. In this case we have immediately

$$\langle e_m, \mathcal{T}_{*t}(|v\rangle\langle v|)e_m \rangle = e^{-|z|^2} e^{-t} \sum_{k \geq m} \frac{|z|^{2k} t^{k-m}}{k!(k-m)!} = e^{-|z|^2} |z|^{2m} e^{-t} \sum_{j \geq 0} \frac{(|z|^2 t)^j}{j!(j+m)!}.$$

Therefore we find the identity

$$\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(|v\rangle\langle v|)e_m \rangle = e^{-|z|^2} e^{-t} \sum_{j \geq 0} \frac{(|z|^2 t)^j}{j!} \sum_{m>0} \frac{|z|^{2m}}{(j+m)!}.$$

Lemma 6.1. *For every $j \geq 0, x \geq 0$ we have*

$$\sum_{n>0} \frac{x^n}{(j+n)!} = \frac{e^x}{j!x^j} \int_0^x s^j e^{-s} ds.$$

Proof. The above formula holds for $j = 0$. We assume that it is true for j and we prove it for $j + 1$. Note that

$$\begin{aligned} \frac{d}{dx} \sum_{n>0} \frac{x^{n+j+1}}{(j+1+n)!} &= \sum_{n>0} \frac{x^{j+n}}{(j+n)!}, \\ \frac{d}{dx} \frac{e^x}{(j+1)!} \int_0^x s^{j+1} e^{-s} ds &= \frac{x^{j+1}}{(j+1)!} + \frac{e^x}{(j+1)!} \int_0^x s^{j+1} e^{-s} ds \end{aligned}$$

integrating by parts one gets

$$= \frac{x^{j+1}}{(j+1)!} + \frac{e^x}{(j+1)!} [s^{j+1} e^{-s}]_0^x + \frac{e^x}{j!} \int_0^x s^j e^{-s} ds.$$

From the induction assumption for j the equality of the derivatives follow and therefore, since the functions coincide for $x = 0$,

$$\sum_{n>0} \frac{x^{j+1+n}}{(j+1+n)!} = \frac{e^x}{(j+1)!} \int_0^x s^{j+1} e^{-s} ds.$$

This proves the lemma. □

Therefore we find the identities

$$\begin{aligned} \sum_{m>0} \langle e_m, \mathcal{T}_{*t}(|v\rangle\langle v|)e_m \rangle &= e^{-t} \sum_{j\geq 0} \frac{t^j}{(j!)^2} \int_0^{|z|^2} s^j e^{-s} ds \\ &= e^{-t} \int_0^{|z|^2} \sum_{j\geq 0} \frac{(ts)^j}{(j!)^2} e^{-s} ds \\ &= e^{-t} \int_0^{|z|^2} I_0(2\sqrt{ts})e^{-s} ds. \end{aligned} \tag{6.1}$$

We now prove a standard estimate on Gaussian integrals.

Lemma 6.2. *For all $a > b > 0$ we have*

$$\frac{e^{-a^2}}{2a} (1 - e^{-b(b+2a)}) \left(1 - \frac{1}{2a^2}\right) \leq \int_a^{a+b} e^{-s^2} ds \leq \frac{e^{-a^2}}{2a} (1 - e^{-b(b+2a)}). \tag{6.2}$$

Proof. In order to prove the upper bound notice that

$$\int_a^{a+b} e^{-s^2} ds \leq \int_a^{a+b} \frac{s}{a} e^{-s^2} ds = \frac{e^{-a^2}}{2a} (1 - e^{-b(b+2a)}).$$

We now prove the lower bound

$$\int_a^{a+b} e^{-s^2} ds = \int_a^{a+b} \frac{1}{s} (se^{-s^2}) ds.$$

Integrating by parts we have

$$\begin{aligned} \int_a^{a+b} e^{-s^2} ds &= \left[-\frac{e^{-s^2}}{2s} \right]_a^{a+b} - \frac{1}{2} \int_a^{a+b} \frac{e^{-s^2}}{s^2} ds \\ &\geq \frac{e^{-a^2}}{2a} - \frac{e^{-(a+b)^2}}{2(a+b)} - \frac{1}{2} \int_a^{a+b} \left(\frac{s}{a}\right)^3 \frac{e^{-s^2}}{s^2} ds \\ &= \frac{e^{-a^2}}{2a} - \frac{e^{-(a+b)^2}}{2(a+b)} - \frac{1}{4a^3} (e^{-a^2} - e^{-(a+b)^2}) \\ &= \frac{e^{-a^2}}{2a} \left(1 - \frac{1}{2a^2}\right) - \frac{e^{-(a+b)^2}}{2} \left(\frac{1}{a+b} - \frac{1}{2a^3}\right) \\ &\geq \frac{e^{-a^2}}{2a} \left(1 - \frac{1}{2a^2}\right) - \frac{e^{-(a+b)^2}}{2(a+b)} \left(1 - \frac{1}{2a^2}\right). \end{aligned}$$

Therefore, if $a^2 \geq 1/2$, we have

$$\int_a^{a+b} e^{-s^2} ds \geq \frac{e^{-a^2}}{2a} \left(1 - \frac{1}{2a^2}\right) - \frac{e^{-(a+b)^2}}{2a} \left(1 - \frac{1}{2a^2}\right).$$

If $a^2 < 1/2$, then the left-hand side of (6.2) is negative and the claimed inequality obviously holds. This proves the lemma. \square

Proposition 6.1. *For all $t > |z|^2$ we have*

$$\sum_{m>0} \langle e_m, \mathcal{I}_{*t}(|v\rangle\langle v|)e_m \rangle \leq \frac{|z|^{1/2} e^{-(\sqrt{t}-|z|)^2}}{2\pi^{1/2} t^{1/4} (t^{1/2} - |z|)} \frac{1 + 3(4\pi|z|\sqrt{t})^{-1}}{1 + (4\pi|z|\sqrt{t})^{-1}}, \tag{6.3}$$

$$\begin{aligned} \sum_{m>0} \langle e_m, \mathcal{I}_{*t}(|v\rangle\langle v|)e_m \rangle &\geq \frac{|z|^{1/2} e^{-(\sqrt{t}-|z|)^2}}{2\pi^{1/2} t^{1/4} (t^{1/2} - |z|)} (1 + (4\pi t^{1/2} |z|)^{-1})^{-1/2} \\ &\quad - \frac{e^{-(\sqrt{t}-|z|)^2}}{2(1 - |z|t^{-1/2})(\sqrt{t} - |z|)^2}. \end{aligned} \tag{6.4}$$

Proof. Recalling formula (6.1) and Proposition 5.1 the left-hand side of (6.3) is dominated by

$$\begin{aligned} e^{-t} \int_0^{|z|^2} \frac{e^{2\sqrt{ts}}}{\sqrt{4\pi(ts)^{1/2} + 1}} e^{-s} \left(1 + \frac{2}{4\pi\sqrt{ts} + 1}\right) ds \\ = \int_0^{|z|^2} e^{-(\sqrt{s}-\sqrt{t})^2} \frac{4\pi\sqrt{ts} + 3}{(4\pi\sqrt{ts} + 1)^{3/2}} ds \\ = \int_{\sqrt{t}-|z|}^{\sqrt{t}} \frac{2(\sqrt{t}-r)(4\pi\sqrt{t}(\sqrt{t}-r) + 3)}{(4\pi\sqrt{t}(\sqrt{t}-r) + 1)^{3/2}} e^{-r^2} dr, \end{aligned}$$

where the last integral has been obtained by the change of variables $r = \sqrt{t} - \sqrt{s}$. Notice that the function on $[0, +\infty[$

$$y \rightarrow \frac{y(4\pi\sqrt{t}y + 3)}{(4\pi\sqrt{t}y + 1)^{3/2}}$$

is increasing since

$$\frac{d}{dy} \frac{y(4\pi\sqrt{t}y + 3)}{(4\pi\sqrt{t}y + 1)^{3/2}} = \frac{8\pi^2 t y^2 + 2\pi\sqrt{t}y + 3}{(4\pi\sqrt{t}y + 1)^{5/2}} > 0.$$

Therefore, for each $r \in [\sqrt{t} - |z|, \sqrt{t}]$, we have

$$\frac{2(\sqrt{t}-r)(4\pi\sqrt{t}(\sqrt{t}-r) + 3)}{(4\pi\sqrt{t}(\sqrt{t}-r) + 1)^{3/2}} \leq \frac{2|z|(4\pi\sqrt{t}|z| + 3)}{(4\pi\sqrt{t}|z| + 1)^{3/2}}.$$

It follows then from Lemma 6.2 that the left-hand side of (6.3) is not larger than

$$\frac{2|z|(4\pi\sqrt{t}|z| + 3)}{(4\pi\sqrt{t}|z| + 1)^{3/2}} e^{|z|^2} \frac{e^{-(\sqrt{t}-|z|)^2}}{2(\sqrt{t} - |z|)}$$

from which (6.3) follows.

The left-hand side of (6.4), by Proposition 5.1, is greater than or equal to

$$\begin{aligned} e^{-t} \int_0^{|z|^2} \frac{e^{-2\sqrt{ts}}}{\sqrt{4\pi(ts)^{1/2} + 1}} e^{-s} ds &= \int_0^{|z|^2} \frac{e^{-(\sqrt{t}-\sqrt{s})^2}}{\sqrt{4\pi(ts)^{1/2} + 1}} ds \\ &= \int_{\sqrt{t}-|z|}^{\sqrt{t}} \frac{2(\sqrt{t}-r)e^{-r^2}}{\sqrt{4\pi t^{1/2}(\sqrt{t}-r) + 1}} dr, \end{aligned}$$

where the last integral has been obtained by the change of variables $r = \sqrt{t} - \sqrt{s}$. Let $\varphi : [\sqrt{t} - |z|, \sqrt{t}] \rightarrow [0, +\infty[$ be the function

$$\varphi(r) = \frac{(\sqrt{t} - r)}{r\sqrt{4\pi t^{1/2}(\sqrt{t} - r) + 1}}.$$

Integrating by parts we find

$$\begin{aligned} \int_{\sqrt{t}-|z|}^{\sqrt{t}} \frac{2(\sqrt{t}-r)e^{-r^2}}{\sqrt{4\pi t^{1/2}(\sqrt{t}-r) + 1}} dr &= \int_{\sqrt{t}-|z|}^{\sqrt{t}} \varphi(r) 2re^{-r^2} dr \\ &= \frac{|z|e^{-(\sqrt{t}-|z|)^2}}{(\sqrt{t} - |z|)\sqrt{4\pi t^{1/2}|z| + 1}} + \int_{\sqrt{t}-|z|}^{\sqrt{t}} \varphi'(r)e^{-r^2} dr. \end{aligned}$$

A straightforward computation yields

$$\varphi'(r) = -\frac{\sqrt{t}}{r^2\sqrt{4\pi t^{1/2}(\sqrt{t}-r) + 1}} + \frac{2\pi\sqrt{t}(\sqrt{t}-r)}{r(4\pi t^{1/2}(\sqrt{t}-r) + 1)^{3/2}}.$$

Therefore the left-hand side of (6.4) is larger than or equal to

$$\begin{aligned} &\frac{|z|e^{-(\sqrt{t}-|z|)^2}}{(\sqrt{t} - |z|)\sqrt{4\pi t^{1/2}|z| + 1}} - \int_{\sqrt{t}-|z|}^{\sqrt{t}} \frac{\sqrt{t}e^{-r^2}}{r^2\sqrt{4\pi t^{1/2}(\sqrt{t}-r) + 1}} dr \\ &\geq \frac{|z|e^{-(\sqrt{t}-|z|)^2}}{(\sqrt{t} - |z|)\sqrt{4\pi t^{1/2}|z| + 1}} - \frac{\sqrt{t}}{(\sqrt{t} - r)^2} \int_{\sqrt{t}-|z|}^{\sqrt{t}} e^{-r^2} dr \\ &\geq \frac{|z|e^{-(\sqrt{t}-|z|)^2}}{(\sqrt{t} - |z|)\sqrt{4\pi t^{1/2}|z| + 1}} - \frac{e^{-(\sqrt{t}-|z|)^2}\sqrt{t}}{2(\sqrt{t} - |z|)^3} \end{aligned}$$

where the last inequality follows from Lemma 6.2. □

The following exact asymptotic result now follows immediately.

Corollary 6.1. For $v = e^{-|z|^2/2}e(z)$ we have

$$\lim_{t \rightarrow \infty} \frac{\sum_{m>0} \langle e_m, \mathcal{T}_{*t}(|v\rangle\langle v|)e_m \rangle}{(2\sqrt{\pi})^{-1}t^{3/4}|z|^{1/2}e^{-(\sqrt{t}-|z|)^2}} = 1.$$

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References

1. L. Accardi, G. Castagnoli and D. R. Finkelstein, Triode networks, unpublished manuscript (2004).
2. L. Accardi, A. Frigerio and Y. G. Lu, The weak coupling limit as a quantum functional central limit, *Comm. Math. Phys.* **131** (1990) 537–570.
3. L. Accardi and S. Kozyrev, Lectures on quantum interacting particle systems, *Quantum Interacting Particle Systems* (Trento, 2000), QP–PQ: Quantum Probab. White Noise Anal., Vol. 14 (World Scientific, 2002), pp. 1–195.
4. L. Accardi and S. V. Kozyrev, Coherent population trapping in the stochastic limit, quant-ph/0303185.
5. L. Accardi, S. V. Kozyrev and A. N. Pechen, Coherent quantum control of Λ -atoms through the stochastic limit, *Quantum Information: Mathematical, Physical Engineering and Industrial Aspects* (Tokio, 2003), QP–PQ: Quantum Probab. White Noise Anal., Vol. 19 (World Scientific, 2005).
6. L. Accardi and K. Imafuku, Control of quantum states by decoherence, *Quantum Information: Mathematical, Physical Engineering and Industrial Aspects* (Tokio, 2003), QP–PQ: Quantum Probab. White Noise Anal., Vol. 19 (World Scientific, 2005).
7. L. Accardi, Y. G. Lu and I. Volovich, *Quantum Theory and Its Stochastic Limit* (Springer-Verlag, 2002).
8. L. Accardi, S. Hachicha and H. Ouerdiane, Generic quantum Markov semigroups: The Fock case, *Open Syst. Inf. Dyn.* **12** (2005) 385–399.
9. L. Accardi and M. Ohya, A stochastic limit approach to the SAT problem, *Open Syst. Inf. Dyn.* **11** (2004) 219–233, <http://dx.doi.org/10.1023/B:OPSY.0000047567.88377.74>.
10. V. P. Belavkin and M. Ohya, Entanglement, quantum entropy, mutual information, *R. Soc. Lond. Proc. Ser. A: Math. Phys. Eng. Sci.* **458** (2002) 209–231.
11. R. Carbone and F. Fagnola, Exponential L^2 -convergence of quantum Markov semigroups on $\mathcal{B}(h)$, *Math. Notes* **68** (2000) 452–463.
12. G. Castagnoli and D. R. Finkelstein, Quantum ground-state computation with kinematical gates, *R. Soc. Lond. Proc. Ser. A: Math. Phys. Eng. Sci.* **459** (2003) 3099–3108.
13. G. Castagnoli and D. R. Finkelstein, Kinematical quantum computation, preprint (2003).
14. A. M. Chebotarev and F. Fagnola, Sufficient conditions for conservativity of minimal quantum dynamical semigroups, *J. Funct. Anal.* **153** (1998) 382–404.
15. F. Fagnola, Quantum Markov semigroups and quantum Markov flows, *Proyecciones* **18** (1999) 1–144.

16. F. Fagnola and R. Rebolledo, The approach to equilibrium of a class of quantum dynamical semigroups, *Inf. Dim. Anal. Quantum Probab. Rel. Topics* **1** (1998) 561–572.
17. F. Fagnola and R. Rebolledo, Lectures on the qualitative analysis of quantum Markov semigroups, in *Quantum Interacting Particle Systems* (Trento, 2000), QP–PQ: Quantum Probab. White Noise Anal., Vol. 14, eds. L. Accardi and F. Fagnola (World Scientific, 2002), pp. 197–239.
18. J. C. García and R. Quezada, Hille-Yosida estimate and nonconservativity criteria for quantum dynamical semigroups, *Inf. Dim. Anal. Quantum Probab. Rel. Topics* **7** (2004) 383–394.