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DEPENDENCE AND AGING PROPERTIES OF LIFETIMES WITH SCHUR-CONSTANT SURVIVAL FUNCTIONS

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For n-dimensional survival functions, we study some probabilistic aspects of the Schur-constant property. The latter is of interest in that it extends the "lack-of-memory" property in a Bayesian context. Some general facts are studied in detail, and related results about interdependence, aging, and extendibility are presented.

1. INTRODUCTION

We consider n non-negative random variables $T_1, \ldots, T_n$ with a *Schur-constant* joint survival function $\bar{F}_n(t_1, \ldots, t_n)$; i.e.,

$$\bar{F}_n(t_1, \ldots, t_n) = P(T_1 > t_1, \ldots, T_n > t_n) = \phi\left(\sum_{i=1}^{n} t_i\right), \qquad (t_1, \ldots, t_n) \in \mathbb{R}_+^n, \tag{1.1}$$

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where ϕ is a non-increasing function, continuous from the right and such that

$\phi(0) = 1$

$\lim_{t\to+\infty} \phi(t) = 0$

for any pair $\mathbf{x},\mathbf{y} \in \mathbb{R}_+^n$ with $x_i \leq y_i$,

$$\sum_{\xi} \phi\left(\sum_{i=1}^{n} \xi_i\right)(-1)^n(-1)^{w(\xi)} \geq 0,$$

where ξ are the vertexes of the n-dimensional interval $U \equiv \{\mathbf{u} \in \mathbb{R}_+^n \mid x_i \leq u_i \leq y_i\}$ and $w(\xi)$ is the number of components of ξ, which are equal to components of x.

Of course, Eq. (1.1) is a special case of exchangeability; by putting, in particular, for a fixed $\lambda > 0$,

$$\phi(t) = \exp\{-\lambda \cdot t\}, \tag{1.2}$$

we obtain that $T_1, \ldots, T_n$ are i.i.d. exponentially distributed.

Note that the h-dimensional marginal survival function of any h variables $T_{j_1}, \ldots, T_{j_h}$ $(h < n)$ is $\bar{F}_h(t_1, \ldots, t_h) = P(T_1 > t_1, \ldots, T_h > t_h, T_{h+1} > 0, \ldots, T_n > 0) = \phi\left(\sum_{i=1}^{h} t_i\right)$.

The vector $\mathbf{T} \equiv (T_1, \ldots, T_n)$ is *N-extendible* $(N > n)$ if $\bar{F}_n(t)$ can be seen as the n-dimensional marginal survival function of some N-dimensional survival function $\bar{F}_N$:

$$\bar{F}_n(t) = \bar{F}_N \quad (t_1, \ldots, t_n, 0, \ldots, 0).$$

$\mathbf{T}$ is *(S.C.)-N-extendible* if it is N-extendible with $\bar{F}_N$ Schur-constant; obviously this happens if and only if ϕ is such that $\phi\left(\sum_{i=1}^{N} t_i\right)$ is a (N-dimensional) survival function.

N^* is the *maximum rank* if $\mathbf{T}$ is N^*-extendible but not $(N^* + 1)$-extendible. If N^* is finite, $\mathbf{T}$ is a vector of *finitely extendible* r.v.'s; if, on the contrary, $\mathbf{T}$ is N-extendible for any N, $T_1, \ldots, T_n$ are *infinitely extendible.* Analogously, we shall denote by $N^*_{\text{S.C.}}$ the maximum rank relative to (S.C.)-extendibility.

It can be immediately seen that if $T_1, \ldots, T_n, \ldots$, is a denumerable sequence of i.i.d. or conditionally i.i.d. random variables, such that the joint n-dimensional survival function is Schur-constant for any n, then $T_1, \ldots, T_n, \ldots$, are necessarily exponential or conditionally exponential, respectively; we can then conclude that $N^*_{\text{S.C.}}$ is infinite if and only if $\mathbf{T}$ is a vector of n i.i.d. or a mixture of n i.i.d. exponentially distributed r.v.'s.

Having in mind applications in the field of reliability, we interpret the r.v.'s $T_1, \ldots, T_n$ as lifetimes of similar units and, for a given $s > 0$, the quantity $T_i - s$ is seen as the *residual lifetime of a unit of age s.* In this field, and in the related field of survival analysis, the interest of Eq. (1.1) is in that it provides, in a subjectivist context, a multidimensional version of the lack-of-memory property or, in other words, it expresses a condition of *no-aging*, as argued by Barlow and

Mendel [1] and Spizzichino [8]. In particular, indeed, Eq. (1.1) holds if and only if for any $\tau > 0$, for any possible vector of ages $(s_1, \ldots, s_n) \in \mathbb{R}_+^n$ and for $i \neq j$,

$$P(T_i - s_i > \tau | T_1 > s_1, \ldots, T_i > s_i, \ldots, T_n > s_n) = P(T_j - s_j > \tau | T_1 > s_1, \ldots, T_j > s_j, \ldots, T_n > s_n);$$

i.e., the residual lifetimes $(T_i - s_i)$ and $(T_j - s_j)$ of two units of different ages s_i and s_j, respectively, have the same conditional distribution.

In applications the following properties (for joint survival functions) may be of interest: interdependence properties, aging, and extendibility. Lifetimes with Schur-constant survival function may present different forms of interdependence, aging, and extendibility, and our aim is to illustrate some relations among them.

This will be done in Section 2, after showing some general properties of Schur-constant survival functions.

2. BASIC PROPERTIES OF SCHUR-CONSTANT SURVIVAL FUNCTION

From now on we shall suppose, if not otherwise stated, that the survival function $\bar{F}_n(t) = \phi(\sum t_i)$ is absolutely continuous; in such a case, the k-dimensional marginal density function ($k \leq n$) is given by

$$f_k(t_1, \ldots, t_k) = (-1)^k \frac{\partial^k}{\partial t_1 \ldots \partial t_k} \bar{F}_k(t_1, \ldots, t_k) = (-1)^k \phi^{(k)}\left(\sum_{i=1}^k t_i\right),$$

where $\phi^{(k)}(\cdot)$ is the kth order derivative of ϕ.

As it is immediate to verify, we have the following proposition.

PROPOSITION 2.1: *If $\bar{F}_n(t) = \phi(\sum t_i)$ is absolutely continuous, then the following conditions hold*:

(i) $\forall h \in \{1,2,\ldots,n\}$ $(-1)^h \phi^{(h)}(t) \geq 0$ $\forall t \geq 0$.

(ii) $\forall h \in \{1,2,\ldots,n\} \lim_{t \to +\infty} \phi^{(h)}(t) = 0$.

Remark 2.2 (see, e.g., Barlow and Mendel [1]): An absolutely continuous survival function $\bar{F}_n(\cdot)$ is Schur-constant if and only if its density $f_n(\cdot)$ is Schur-constant.

Fix now $s > 0$, and put

$$\bar{G}_n(t_1, \ldots, t_n) = \begin{cases} \left(1 - \frac{1}{s}\sum_{i=1}^n t_i\right)^{n-1} & \text{if } 0 \leq \sum_{i=1}^n t_i \leq s, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

$\bar{G}_n(\cdot)$ provides an example of a (singular) Schur-constant survival function: as it can be easily shown, r.v.'s $T_1, \ldots, T_n$ with a survival function of the form of Eq. (2.1) can only take values in the *simplex* $\varphi_s = \{\xi \in \mathbb{R}_+^n | \sum_{i=1}^n \xi_i = s\}$ and the probability distribution defined by Eq. (2.1) is the *uniform distribution on the simplex* φ_s. A distribution of the form of Eq. (2.1) turns out to be the joint laws of the spacings of $(n-1)$ points dropped at random into intervals of fixed length [5]; such distributions are significant because they can be used to give an integral representation for any absolutely continuous Schur-constant survival function.

PROPOSITION 2.3: *Let* $S_n = \sum_{i=1}^n T_i$ *and* $\bar{F}_n(\cdot)$ *be absolutely continuous;* $\bar{F}_n(\cdot)$ *is Schur-constant if and only if the conditional survival function of* $T_1, \ldots, T_n$ *given* $(S_n = s)$ *is the uniform distribution on the simplex* φ_s.

PROOF: Let $\bar{F}_n(\cdot)$ be Schur-constant; by Proposition 4 in Barlow and Mendel [1], $\forall k < n$

$$\bar{F}_k(t_1, \ldots, t_k \mid S_n = s) = \begin{cases} \left(1 - \frac{1}{s}\sum_{i=1}^k t_i\right)^{n-1} & \text{if } 0 \le \sum_{i=1}^k t_i \le s, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Here, we show that Eq. (2.2) holds also for $k = n$. Indeed,

$$\begin{aligned} &\bar{F}_n(t_1, \ldots, t_n \mid S_n = s) \\ &\quad = P(\{T_n > t_n\} \cap \{T_1 > t_1, \ldots, T_{n-1} > t_{n-1}\} \mid S_n = s) \\ &\quad = \int_{t_1}^{\infty} d\xi_1 \ldots \int_{t_{n-1}}^{\infty} d\xi_{n-1} P(T_n > t_n \mid S_n = s,\ T_1 = \xi_1, \ldots, T_{n-1} = \xi_{n-1}) \\ &\quad\quad \times f_{n-1}(\xi \mid S_n = s) \end{aligned}$$

where $f_{n-1}(\xi \mid S_n = s)$ is the joint conditional density function of $T_1, \ldots, T_{n-1}$ and $0 \le \sum_{i=1}^n t_i \le s$. By Eq. (2.2)

$$f_{n-1}(\xi \mid S_n = s) = (n-1)!/s^{n-1} \mathbf{1}_{\left(0 \le \sum_{i=1}^{n-1} \xi_i \le s\right)}.$$

Moreover, we can write

$$\begin{aligned} &P(T_n > t_n \mid S_n = s, T_1 = \xi_1, \ldots, T_{n-1} = \xi_{n-1}) \\ &\quad = P\left(T_n > t_n \mid T_n = s - \sum_{i=1}^{n-1} \xi_i, T_1 = \xi_1, \ldots, T_{n-1} = \xi_{n-1}\right) \\ &\quad = \mathbf{1}_{\left(t_n \le s - \sum_{i=1}^{n-1} \xi_i\right)} = \mathbf{1}_{\left(\sum_{i=1}^{n-1} \xi_i \le s - t_n\right)}. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{F}_n(t_1, \ldots, t_n \mid S_n = s) &= \int_{t_1}^{\infty} d\xi_1 \ldots \int_{t_{n-1}}^{\infty} d\xi_{n-1} \mathbf{1}_{\left(\sum_{i=1}^{n-1} \xi_i \le s - t_n\right)} \frac{(n-1)!}{s^{n-1}} \\ &= \left(1 - \frac{1}{s}\sum_{i=1}^n t_i\right)^{n-1}. \end{aligned}$$

■

As an immediate application of Proposition 2.3, we have the following corollary.

COROLLARY 2.4 (Integral Representation): *An absolutely continuous survival function $\bar{F}_n(\cdot)$ is Schur-constant if and only if there exists a probability measure μ_n on $[0,+\infty)$ such that $\forall (t_1,\ldots,t_n) \in \mathbb{R}_+^n$*

$$\bar{F}_n(t_1,\ldots,t_n) = \int_0^\infty \left(1 - \frac{1}{s}\sum_{i=1}^n t_i\right)_+^{n-1} \mu_n(ds),$$

where $f_+ = 0$ if $f < 0$.

Of course, μ_n is the measure induced by $S_n = \sum_{i=1}^n T_i$.

Let us consider now the special case where $\bar{F}_n(t_1,\ldots,t_n) = \psi(\sum t_i)$ and

$$\psi(y) = \begin{cases} \sum_{i=0}^k c_i y^i & y \in I, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

where $c_k, c_{k-1} \neq 0$ and I is a closed bounded interval of $[0,+\infty)$.

It can be seen [3] that $\bar{F}_n(t) = \psi(\sum t_i)$ is actually a joint survival function iff $I = [0,p]$ and

$$\psi(y) = \begin{cases} \left(1 - \frac{y}{p}\right)^k & \text{if } 0 \le y \le p, \\ 0 & \text{otherwise,} \end{cases}$$

where $p = -c_{k-1}/kc_k$. Note that if $k < n$, the joint survival function $\bar{F}_n(t) = \phi(\sum_{i=1}^n t_i)$ is not absolutely continuous, and the r.v.'s $T_1,\ldots,T_{k+1}$ have a uniform distribution on the particular simplex φ_p.

By Proposition 2.3 the class of the (n-dimensional) distributions with Schur-constant absolutely continuous survival functions is a subclass of the class $\mathcal{C}_n$ of the distributions of non-negative random variables $X_1,\ldots,X_n$, whose conditional joint distribution, given the event $(\sum_{i=1}^n X_i = s)$, is uniform on the simplex.

The class $\mathcal{C}_n$ has been studied in detail by Diaconis and Freedman [6]; in particular, they proved that if the maximum (S.C.)-rank is finite, for any $k \le N^*_{\text{S.C.}}$ the r.v.'s $X_1,\ldots,X_k$, $k < n$, are nearly a mixture of k i.i.d. and exponential r.v.'s and the variation error is at most $2(k+1)/(N^*_{\text{S.C.}} - k - 1)$.

As already mentioned, Eq. (1.1) combined with the condition of infinite (S.C.)-extendibility means that $T_1,\ldots,T_n$ are i.i.d. exponentially distributed or conditionally i.i.d. exponentially distributed. It is well known [2] that, in the latter case, the (*predictive*) one-dimensional marginal distribution is DFR.

We now remark that Eq. (1.1) is equivalent to

$$\bar{F}_n(t) = \bar{F}_1\left(\sum_{i=1}^n t_i\right). \tag{2.4}$$

The latter identity shows the dependence of $\bar{F}_n$ on t in terms of the one-dimensional marginal survival function. In principle, by using Eq. (2.4), we might combine an arbitrary one-dimensional survival function $\bar{F}_1$ with the function $\psi(t) = \sum_{i=1}^{n} t_i$ in order to build a Schur-constant survival function with the prescribed one-dimensional survival function $\bar{F}_1$. As a matter of fact, such a procedure may lead to a function that is not an n-dimensional survival function (being an n-dimensional survival function requires of course that any n-dimensional interval has a non-negative probability). This argument shows that Eq. (2.4) may imply some constraint on the form of $\bar{F}_1$. In what follows we shall present two simple but interesting results in this direction. First of all we recall that the r.v.'s $T_1, \ldots, T_n$ are called *positively upper orthant dependent* if

$$P(T_1 > t_1, \ldots, T_n > t_n) \geq \prod_{i=1}^{n} P(T_i > t_i)$$

and *negatively upper orthant dependent* if the opposite unequality holds.

We now recall that a single lifetime T is NWU (*New Worse than Used*) if

$$P(T > t + s \mid T > s) \geq P(T > t).$$

and NBU (*New Better than Used*) if the opposite unequality holds.

PROPOSITION 2.5: *Let $\bar{F}_n(\cdot)$ be Schur-constant. $T_1, \ldots, T_n$ are positively (negatively) upper orthant dependent if and only if the one-dimensional marginal distribution of T_i is* NWU (NBU).

PROOF: If $T_1, \ldots, T_n$ are positively dependent $P(T_1 > t + s) = P(T_i > t, T_j > s) \geq P(T_1 > t)P(T_1 > s)$; i.e.,

$$P(T_1 > t + s \mid T_1 > s) \geq P(T_1 > t).$$

Vice versa if T_i is NWU:

$$P(T_1 > t_1, \ldots, T_n > t_n) = \bar{F}_1\left(\sum_{i=1}^{n} t_i\right) = P\left(T_1 > \sum_{i=1}^{n} t_i\right) \geq P(T_1 > t_1)$$

$$\times P\left(T_1 > \sum_{i=2}^{n} t_i\right) \geq \cdots \geq \prod_{i=1}^{n} P(T_1 > t_i)$$

$$= \prod_{i=1}^{n} P(T_i > t_i).$$

To obtain the equivalence between NBU and negative dependence, $\geq$ is to be replaced by $\leq$ in the preceding proof. ■

In what follows we shall give a result concerning extendibility of Schur-constant survival functions. Actually, it is to be noticed that also extendibility properties are strictly related with interdependence properties: positive dependence is a necessary condition for infinite extendibility, whereas negative dependence is a sufficient condition for finite extendibility. On the other hand, we remark that N-extendibility means that $\bar{F}_1\left(\sum_{i=1}^{N} t_i\right)$ is still a (N-dimensional) joint survival function.

By the integral representation in Corollary 2.4, the measure μ_n induced by S_n characterizes any Schur-constant absolutely continuous survival function, and so it can, in particular, be used to study the maximum rank $N^*_{\text{S.C.}}$; a connection between $N^*_{\text{S.C.}}$ and properties of μ_n is obtained in the following result.

PROPOSITION 2.6: *Let $\bar{F}_n(\cdot)$ be Schur-constant.*

(a) *$T_1, \ldots, T_n$ are (S.C.)-infinitely extendible if and only if μ_n is a gamma or a mixture (with respect to λ) of gammas with parameters n and λ.*

(b) *if $n > 2$ and $\mathbb{E}[S_n^2] \le ((n+1)/n)\mathbb{E}^2[S_n]$, then the r.v.'s are (S.C.)-finitely extendible with maximum rank $N^*_{\text{S.C.}}$ given by*

$$N^*_{\text{S.C.}} = \max\left\{k \in \mathbb{N} : k \le \frac{n\mathbb{E}[S_n^2]}{(n+1)\mathbb{E}^2[S_n] - n\mathbb{E}[S_n^2]}\right\}. \tag{2.5}$$

PROOF:

(a) The proof is obvious: it is sufficient to recall that in such a case $T_1, \ldots, T_n$ are i.i.d. or conditional i.i.d. exponentially distributed.

(b) Let us compute the correlation coefficient $\rho = (\text{Cov}(T_1, T_2))/(\text{Var}(T_1))$. By the integral representation of Eq. (2.2), we can compute the joint marginal density function of any $k < n$ r.v.'s $T_1, \ldots, T_k$

$$f_k(t_1, \ldots, t_k) = \int_0^\infty \frac{(n-1)(n-2)\cdots(n-k)}{s^k} \times \left(1 - \frac{1}{s}\sum_{i=1}^{k} t_i\right)_+^{n-k-1} \mu_n(ds)$$

so that

$$\mathbb{E}[T_1] = \int_0^\infty dt \int_0^\infty t\,\frac{(n-1)}{s}\left(1 - \frac{t}{s}\right)_+^{n-2} \mu_n(ds)$$

$$\mathbb{E}[T_1 \cdot T_2] = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty t_1 t_2 \frac{(n-1)(n-2)}{s^2} \times \left(1 - \frac{1}{s}\sum_{i=1}^{2} t_i\right)_+^{n-3} \mu_n(ds)$$

$$\mathbb{E}(T_1^2] = \int_0^\infty dt \int_0^\infty t^2\,\frac{(n-1)}{s}\left(1 - \frac{t}{s}\right)_+^{n-2} \mu_n(ds).$$

By computing such integrals, it follows that

$$\mathbb{E}[T_1] = \frac{1}{n}\,\mathbb{E}[S_n], \qquad \mathbb{E}[T_1 \cdot T_2] = \frac{1}{n(n+1)}\,\mathbb{E}[S_n^2],$$

$$\mathbb{E}[T_1^2] = \frac{2}{n(n+1)}\,\mathbb{E}[S_n^2].$$

Thus,

$$\mathrm{Cov}(T_1, T_2) = \frac{1}{n(n+1)}\,\mathbb{E}[S_n^2] - \frac{1}{n^2}\,\mathbb{E}^2[S_n]$$

and

$$\mathrm{Var}(T_1) = \frac{2}{n(n+1)}\,\mathbb{E}[S_n^2] - \frac{1}{n^2}\,\mathbb{E}^2[S_n].$$

Therefore,

$$\rho = \frac{n\mathbb{E}[S_n^2] - (n+1)\mathbb{E}^2[S_n]}{2n\mathbb{E}[S_n^2] - (n+1)\mathbb{E}^2[S_n]}$$

and $\rho \le 0$ iff $\mathbb{E}[S_n^2] \le ((n+1)/n)\mathbb{E}^2[S_n]$. This condition implies finite extendibility, and in this case the maximum rank N^* must satisfy: $\rho \ge -(1/(N^* - 1))$. From this condition and by obvious computations, the identity of Eq. (2.5) follows. ■

Remark 2.7: If $\bar{F}_n(t) = \bar{F}_1(\sum t_i)$ is an absolutely continuous Schur-constant survival function, then, for $k \le n$, the density $f_{S_k}(s)$ of $S_k = \sum_1^k t_i$ is related to the function $\phi \equiv \bar{F}_1$ through the equation

$$f_{S_k}(s) = (-1)^k \frac{\phi^{(k)}(s)}{(n-1)!}\, s^{k-1}.$$

By combining this formula with Proposition 2.6, we can translate extendibility conditions for $\bar{F}_n(t)$ into constraints on the function $\bar{F}_1$.

In applications it is of interest to consider life-testing experiments on n units $U_1, \ldots, U_n$ with lifetimes $T_1, \ldots, T_n$. Think now of a life-testing experiment in which $U_1, \ldots, U_n$ are new and start working at time 0. As time elapses the units progressively fail, and suppose that we can observe progressively all the failure times. For $t > 0$, let us then denote by H_t the random number of failures that will be observed up to time t:

$$H_t = \sum_{i=1}^{n} \mathbf{1}_{(T_i \le t)}.$$

We also denote by Y_t the *total time on test* process:

$$Y_t = \sum_{i=1}^{n} T_i \wedge t.$$

Consider now the two-dimensional process $Z_t = (H_t, Y_t)$, which takes its values on the set $E = \{0,1,\ldots,n\} \times [0,+\infty)$.

The interest of Eq. (1.1) lies in that it allows the pair (H_t, Y_t) to be sufficient in the life-testing experiment [8]; this fact implies the Markov property of the process Z_t. As will be shown in a subsequent article, interdependence and aging arguments about **T** can be used for obtaining useful monotonicity properties of Z_t.

References

1. Barlow, R.E. & Mendel, M.B. (1992). Similarity as a characteristic of wear-out. In C.A. Clarotti, R.E. Barlow, & F. Spizzichino (eds.), *Reliability and decision making.* Amsterdam: Elsevier Science Publisher.
2. Barlow, R.E. & Proschan, F. (1975). *Statistical theory of reliability and life testing.* New York: Holt, Rinehart and Winston.
3. Caramellino, L. (1991). *Variabili aleatorie non negative: Proprieta' di Schur e di ageing delle distribuzioni.* Unpublished graduation thesis in Mathematics, University "La Sapienza," Rome.
4. de Finetti, B. (1937). Le prevision: Ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincare* 1: 1–68.
5. de Finetti, B. (1964). Alcune osservazioni sulla suddivisione casuale di un intervallo. *Giornale dell'Istituto Italiano degli Attuari.*
6. Diaconis, P. & Freedman, D. (1987). A dozen de Finetti-style results in search of a theory. *Annales de l'Institut Henri Poincare* 23: 397–423.
7. Marshall, A.W. & Olkin, I. (1979). *Inequalities: Theory of majorization and its applications.* New York: Academic Press.
8. Spizzichino, F. (1992). Reliability decision problems under condition of ageing. In Bernardo, Berger, Dawid, & Smith (eds.), *Bayesian statistics 4.* Oxford: Clarendon Press.