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RESEARCH ARTICLE

Corrigenda to “Curves on surfaces of mixed characteristic”

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Abstract

We correct the description of p-closed foliations on bi-disc quotients in McQuillan (Eur J Math 3(3):433–470, 2017) occasioned by the omission of modèles étranges. As such, the geometry of curves remains exactly that of op. cit., but its relationship with the arithmetic is richer.

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1 Introduction

A corollary, cf. [3, Section 3], of Marguils' arithmeticity theorem is the classification of rigid bi-disc quotients, to wit: there are a totally real number field K with $(K:\mathbb{Q}) = d+2$, $d \geqslant 0$, and a quaternion algebra B/K which is un-ramified at exactly 2 real places, τ', τ'', i.e.

$$B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathrm{End}_{\mathbb{R}}(\mathbb{C})^2 \times \mathbb{H}^d \tag{1}$$

such that, up to finite étale coverings, any bi-disc quotient which is not a product of curves is the (connected) Shimura variety, $M(G, h)$, defined by the arithmetic group,

$$G \coloneqq \mathrm{Res}_{K/\mathbb{Q}} B^\times, \quad \text{Weil's restriction of scalars,} \tag{2}$$

with, a Hodge structure on (2) defined via the obvious embedding $\mathbb{C} \hookrightarrow \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ and the isomorphism (1),

$$h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G\colon \left(z \mapsto z^{-1} \times z^{-1} \times \mathrm{id}^{\oplus d}\right); \tag{3}$$

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or if one prefers, on choosing an order $\mathcal{O}_B \hookrightarrow B$ the quotient of the bi-disc by the norm 1-elements in $\mathcal{O}_B^\times$ acting by,

$$(\mathcal{O}^\times)^1 \times (\Delta \times \Delta) \colon \gamma \times z_{\tau'} \times z_{\tau''} \mapsto z_{\tau'}^{\tau'(\gamma)} \times z_{\tau''}^{\tau''(\gamma)}; \tag{4}$$

while, should B be a division algebra, the reflex field, $E(G, h)$, over which $M(G, h)$ has a canonical model, is K, and the quotient, along with the foliations defined by the projections in (4) are smooth. Otherwise, the reflex field is $\mathbb{Q}$, and $\mathcal{O}_B$ contains uni-potent elements giving rise to cusps at which the quotient (by definition a Hilbert modular surface) together with the aforesaid foliations are singular. Irrespectively, the cases where K is not quadratic, i.e. $d > 0$, are, exactly the bi-disc quotients which Deligne, [2, Section 6], describes as *modèles étranges*. In particular, although the decomposition of primes in K continues to give rise to a dichotomy between characteristic zero like behaviour, so inter alia, hyperbolicity, which was the focus of [11], versus the appearance of rational curves whose degree grows with the characteristic, the way in which this occurs on modèles étranges is more subtle. Ultimately, we only care about sufficiently large characteristics p, so, without loss of generality p is unramified in K, and the simple expedient of choosing an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ suffices to give a well-defined action of Frobenius on the set of real embeddings of K, with an orbit of Frobenius the same thing as a prime of K lying over p. In the current context, however, orbits rather than primes are what naturally occur, so we'll eschew the latter language in favour of the former, beginning with:

Corrigendum 1.1 (Lemma 3.2 et seq.) *For $M(G, h)$ a bi-disc quotient defined by* (4)*, and $\mathcal{F}'$, resp. $\mathcal{F}''$, the foliation defined by the projection onto the factor τ'', resp. τ'* (*so $\mathcal{F}'$ is parallel to τ'*) *at any sufficiently large prime, p, either:*

(A) *τ' and τ'' belong to the same orbit of Frobenius, F, and for $e+1 \geqslant 1$, the smallest integer such that $F^{e+1}(\tau') = \tau''$, the foliation $\mathcal{F}'$ has height e, but fails to have height $e+1$. In particular $\mathcal{F}'$ is not p-closed iff $F(\tau') = \tau''$.*
(B) *Otherwise, both $\mathcal{F}'$ and $\mathcal{F}''$ are height e foliations for all $e \geqslant 0$.*

wherein, following, [4], albeit [12, Section 3] is more precise, a *height e foliation* is one that in a (generic) complete local ring can be written,

$$(x, y) \mapsto (x^{p^e}, y) \tag{5}$$

so, height 1 is p-closed. Regardless, should K be quadratic, Corrigendum 3.5 simplifies, [11, Theorems 1.2 & 1.5], to p inert, resp. split, in K. Plainly, however, the probability of (A) occurring goes to zero as d goes to infinity, while, the easier property of p closure is more common still, and for $d > 0$ really different from the p-adic integrability of (B), so, as far as the appearance of rational curves goes,

Corrigendum 1.2 (Corollary 3.5) *Should case* (A) *of Corrigendum* 1.1 *occur, there are rational curves tangent to $\mathcal{F}'$ whose canonical degree admits the lower, resp. upper, bound $p^{e+1}/2$, resp. $(p^{e+1} - 1) \cdot \mathrm{c}_2(M(G, h))$.*

which in the Hilbert modular case, should, as it is in [11, Theorem 1.5], be understood logarithmically. Arguably, however, the more interesting phenomenon associated to modèles étranges occurs in the opposite situation, i.e.

Corrigendum 1.3 (Facts 4.1–4.2 & Corollary 4.7) *Suppose case* (B) *of Corrigendum 1.1 occurs, then there is a section of* $K_{\mathcal{F}'}^{p-1/2}$, *resp.* $K_{\mathcal{F}''}^{p-1/2}$, *defining an irreducible smooth* $\mathcal{F}''$ *invariant, resp.* $\mathcal{F}'$ *invariant, divisor,* Δ', *resp.* Δ'', *everywhere transverse to* $\mathcal{F}'$, *resp.* $\mathcal{F}''$, *such that no other power of* $K_{\mathcal{F}'}$, *resp.* $K_{\mathcal{F}''}$, *has a section, i.e. both foliations have Kodaira dimension 0, while over* $\overline{\mathbb{F}}_p$, *on the complement of* Δ', *resp.* Δ'', $\mathcal{F}'$ *resp.* $\mathcal{F}''$, *is an action of the unique one dimensional formal group of height,* r', *resp.* r'', *the period of the orbit under Frobenius of* τ', *resp.* τ''.

Here, under the hypothesis of (B), either foliation is defined by a formal groupoid, [12], $\mathfrak{F}'$, resp. $\mathfrak{F}''$, $\rightrightarrows M(G,h)$, and invariant in Corrigendum 1.3 means $\mathfrak{F}'$, resp. $\mathfrak{F}''$, invariant, or, equivalently, invariant by all height e foliations in the language of [4] or [11], while the action of Corrigendum 1.3 is in the complete local rings, or étale locally for the p^eth truncation, Remarks 3.7 & 4.3. In particular, [11, Complement 3.13], asserted there were no such invariant curves, whereas, since both $K_{\mathcal{F}'}$ & $K_{\mathcal{F}''}$ are nef, whenever (B) holds, there is exactly one invariant curve, i.e. Δ'', resp. Δ'—the mistake in op. cit. is in (73) wherein the order of vanishing got confused with the vanishing of the leading term on applying the Cartier operator. In any case, as Corrigendum 1.3 suggests, the basic problem occasioned by modèles étranges is that a generic point isn't represented by an ordinary abelian variant. Indeed, already in characteristic zero, the Hodge structure, (3) isn't rational iff we have a modèle étrange. After a choice, however, of a purely imaginary quadratic extension $K \hookrightarrow Z$, there is, as Deligne says, a miracle, and one finds a modular description, Revision/Claim 2.1. Nevertheless this depends on a further choice, h_Z in (6), of an embedding $Z \hookrightarrow \mathbb{C}$ above every real place where the algebra B is Hamilton's quaternions, and while the resulting Shimura varieties, (8), are isomorphic, the corresponding moduli problems are not because a different choice of h_Z gives a different relation between the Hodge structure and the action of Frobenius, (22). Moreover,

Complimet 1.4 (Remark 3.6) If an orbit, under Frobenius, of embeddings of $Z \hookrightarrow \mathbb{C}$ has a cardinality different from its restriction to K, then the generic element of the moduli problem of Revision/Claim 2.1 associated to a modèle étrange, is never ordinary unless all such orbits contain only embeddings lying over τ' or τ'', and should this occur there are, for d_p the number of orbits of the ramified (in (2)) places, 2^{d_p} out of the 2^d choices for h_Z such that the generic point of the moduli problem is ordinary. Otherwise, there are, again 2^{d_p} out of the 2^d choices for h_Z, such that the generic point of the aforesaid moduli problem is an ordinary abelian variety iff $r' = r'' = 1$ whenever case (B) of Corrigendum 1.1 occurs, and iff Frobenius permutes τ' and τ'' in case (A).

To put this in context, there are natural (independent of any choices) moduli problems when K is quadratic, and, unsurprisingly, at least for $p \gg 0$, they're generically ordinary for the same reasons encountered in Compliment 1.4, i,e, $d = 0$, so there's no Z, and the orbits of Frobenius are wholly filled by τ' and/or τ''. Now while this isn't critical to knowing which primes are p-closed (in the Hilbert modular case this can just be read off, [11], from the singularities) generic ordinarity is how, following [5, pg. 23 remark (iii)], [11] proves that p-closure is equivalent (for $d = 0$) to the far stronger condition of an action by a formal groupoid, which, over the ordinary locus, is even a $\widehat{\mathbb{G}}_m$ action. Fortunately, however, [14], following [16], provides a plugin substitute, μ-ordinary, i.e. as ordinary as the action of Frobenius on crystalline cohomology allows, so, the content of this note is, mostly, to keep book on the choices in the moduli problem(s) associated to a modèle étrange, and to describe the resulting μ-ordinary locus, albeit the statements, Corrigendum 1.3 & Remark 4.8, on the Kodaira dimension are a lot trickier than characteristic 0, and need a variant on Szpiro's proof, [15, Théorème 1], of Arakelov's theorem.

2 Modèle étrange and moduli

We can safely confine the case of K quadratic to the parenthetical Remarks 3.8 & 4.5, and concentrate on $d > 0$ in (3). To this end, recall, for convenience, [2, Section 6], that to construct the modèle étrange of $M(G, h)$ (where for ease of notation we'll systematically ignore level structure) one first makes the aforesaid choice of a totally imaginary quadratic extension Z/K and, as a function of this choice, there is a K-algebraic group,

$$1 \to \mathbb{G}_m \to G \times_K \mathrm{Res}_{Z/K}\, \mathbb{G}_m \to G' \to 1 \tag{6}$$

so that the further choice of an embedding $Z \hookrightarrow \mathbb{C}$ for every embedding $K \hookrightarrow \mathbb{R}$ yields a Hodge-structure,

$$h_Z\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \to (\mathrm{Res}_{Z/K}\, \mathbb{G}_m) \otimes_K \mathbb{R}\colon\ \big(z \mapsto (z)^2 \times (z^{-1})^d\big). \tag{7}$$

As such, for h' the image of $h \times h_Z$, we have a map,

$$M(G, h) \times M(\mathrm{Res}_{Z/K}\, \mathbb{G}_m, h_Z) \to M(G', h') \tag{8}$$

exhibiting, $M(G, h)/\mathbb{C}$ as an étale cover of $M(G', h')$, while $B' := B \otimes_K Z$, acting faithfully on itself on the left, preserves the (imaginary valued) alternating form,

$$(\_, \_)' \in \mathrm{Sym}^2 B \otimes \Lambda^2 Z \subset \Lambda^2 B' \to \Lambda^2 Z\colon\ x \otimes \xi \times y \otimes \eta \mapsto \mathrm{Tr}(x^* y)\, \xi \wedge \eta \tag{9}$$

which, independent of the choice of i, in turn yields a positive definite form,

$$\langle X, Y \rangle := -i(X, h(i)Y)'_{b'}, \quad (X, Y)'_{b'} = (X, b'Y)' \tag{10}$$

for a suitable choice of $b' \in B'$, [2, Lemma 6.4.2]. As such, G' is contained in the group of symplectic similitudes of $(\_, \_)'_{b'}$ and one concludes, op. cit., to the existence of a canonical model of $M(G', h')$. However, cf. [17, Proposition 1.1.3], somewhat more is true, to wit:

Revision/Claim 2.1 Any complex structure on $B'\otimes_{\mathbb{Q}}\mathbb{R}$ respecting left multiplication by $B^\times Z^\times$ is right multiplication by a square root, J, of -1 in $B'\otimes_{\mathbb{Q}}\mathbb{R}$. In particular the formal deformation space (which includes complex tori) of such $B^\times Z^\times$ invariant complex structures at h' is 2 copies, $\widehat{M}\times\widehat{M}$, of the completion, $\widehat{M}$, of $M(G',h')$ at h' (or, indeed, $M(G.h)$ at h) with the latter embedded as the diagonal in the former. Furthermore, if J preserves the alternating form (9) then at each of the $1\leqslant i\leqslant d+2$ components of the product in (1),

$$J_i=\alpha_i\otimes\beta_i\in B_i\otimes_{\mathbb{R}}\mathbb{C},\quad \text{where}\ \ \alpha_i^2=-1,\ \ \beta_i=1,\quad \text{or}\ \ \alpha_i=1,\ \ \beta_i^2=-1, \tag{11}$$

so that if, in addition, the complex structure respects the positivity condition of (10) then we’re in the former case of (11) for $i\leqslant 2$, and the latter for $i>2$. In particular therefore $M(G',h')$ is exactly one of the 2^d irreducible connected components of the moduli space of complex tori (A,α',α'') of dimension $4d=2(2d)$ with multiplication by B', admitting a transposition α', resp. α'', which swaps (and otherwise acts identically) the two foliations in the two directions lying over τ', resp. τ'', every point of which is an abelian variety, and amongst the different components it is exactly that which is polarised by (10), while any other component corresponds to a different choice of h_Z in (7), equivalently b' in (10).

Proof That J is right multiplication by a root of -1 in $B'\otimes_{\mathbb{Q}}\mathbb{R}$ is clear. However, infinitesimally, the tangent space of such a given h' is the anti-commutator in the Lie algebra, and since the choice of h' is central at the ramified places, it can only move in the τ' or τ'' directions. At such factors, therefore, the tangent space is 2 copies of the anti-commutator of, say, $h_{\tau'}$, in $B_{\tau'}$, into which the tangent space of $M(G,h')$ at h' is embedded diagonally. As to the global description, modulo the irrelevance (since one argues one component at a time in (11) et seq.) that it is concerned with one un-ramified place at infinity, while we’re concerned with two (indeed any number r as per [2, Section 6] or Remark 4.9 works) (11) is exactly [17, Proposition 1.1.3], cf. [2, Scholie 4.11 & Section 6], albeit there’s minor notational issues since the role of left and right multiplication here follow [2], rather than the opposite choice of [17]. Finally, therefore, one can put the local and global propositions together to conclude how $M(G',h')$ sits inside the space of tori, together with its relation to the choices of (7) and (8). □

Plainly, Hamilton’s quaternions enjoy an S^3 of roots of -1, so it’s worth making,

Remark 2.2 If instead of making the choice of a purely imaginary extension $K\hookrightarrow Z$, we simply chose a square root of -1 in the quaternions at every ramified place to get an honest complex structure, θ, on $B\otimes_{\mathfrak{F}}\mathbb{R}$ rather than the irrational Hodge structure of (3), then the result is a moduli space, $M(G,\theta)$ of complex tori, up to isogeny, with multiplication by an order in B which wholly fail to be abelian varieties since (1) doesn’t admit a B-linear alternating form. Nevertheless, rigidity guarantees that,

$$M(G,\theta)\to M(G,h) \tag{12}$$

is a local system of $(\mathbb{P}^1)^d$'s defined over a number field, but the lack of a modular interpretation as a scheme (rather than a formal scheme) makes the (myriad of) choice(s) of (8) preferable to the seemingly more natural (12).

3 *p*-Closure and curvature

Once we know that we have a space of abelian varieties we need to be able to calculate the p-curvature of the Gauss–Manin connection on their moduli. In the considerably greater generality of Picard–Fuch's equations, i.e. Gauss–Manin for any family of varieties, this is the key theorem, 3.2, of [9]. In our current context, it is much easier. To wit,

Set Up/Fact 3.1 Denoting by F the absolute Frobenius of any variety we may encounter, let $\pi : A \to S$ be a smooth family of abelian varieties of dimension g in characteristic p, then the moduli map affords,

$$\mathcal{T}_S \to \mathcal{H}om_S(R^1\pi_*\Omega_{A/S}, R^1\pi_*\mathcal{O}_A) : \ \partial \mapsto \bar{\partial}$$

so that for $\Theta : \mathcal{T}_S \to \mathcal{E}nd_S(\mathcal{H}^1) : \partial \to \Theta_{\bar{\partial}}$ the p-curvature of the Gauss–Manin connection, ∇, on $\mathcal{H}^1 := \mathcal{H}^1_{\mathrm{DR}}(A/S)$, $\partial \in \mathcal{T}_S$, the following diagram commutes,

$$\begin{tikzcd} R^1\pi_*\Omega_{A/S} \arrow[r, "\Pi\cdot\Theta_{\bar{\partial}}"'] \arrow[d, "\text{Cartier}"'] & R^1\pi_*\mathcal{O}_A \\ F^*R^1\pi_*\Omega_{A/S} \arrow[r, "F^*\bar{\partial}"] & F^*R^1\pi_*\mathcal{O}_A \arrow[u, "\text{Frobenius}"'] \end{tikzcd} \tag{13}$$

wherein the first horizontal, $\Pi \cdot \Theta_{\bar{\partial}}$, is obtained by restricting the endomorphism $\Theta_{\bar{\partial}}$ (to the kernel) and projecting (to the co-kernel) of the exact sequence,

$$0 \to R^1\pi_*\Omega_{A/S} \to \mathcal{H}^1 \to R^1\pi_*\mathcal{O}_A \to 0. \tag{14}$$

As such, it doesn't quite make sense to say that $\Theta_{\bar{\partial}}$ is the left, bottom, right composition in (13), however the map,

$$\Theta_\partial \mapsto \Pi \cdot \Theta_{\bar{\partial}} \tag{15}$$

is injective, and, better still,

$$\Pi \cdot (\nabla_{\bar{\partial}})^p = 0 \bmod (\bar{\partial} = \Pi \cdot \nabla_{\bar{\partial}}). \tag{16}$$

Proof Certainly one can unravel this from the larger diagram of [9, Theorem 3.1] in which Ψ is our Θ. Alternatively, we can usefully see the utility of ordinary abelian varieties and prove (15) & (16) into the bargain. Specifically, everything follows a fortiori from the corresponding statements on the moduli space. As such although an arbitrary S may well not contain any ordinary abelian varieties, we can, since all of our assertions are identities of F-linear maps on bundles, without loss of generality suppose that S is the ordinary locus. Consequently, it suffices to check (13) for ∂ a Serre–Tate canonical vector field on the formal deformation space, for which (13) commutes by [7, 4.3.1], and, indeed, $\Pi\cdot\Theta_\partial=\partial$, which proves that (15) is injective. It is, however, even legitimately an identity on identifying an endomorphism with a 2×2 block matrix via the isomorphism resulting from the composition,

$$F^*R^1\pi_*\mathcal{O}_A\to\mathcal{H}^1\to R^1\pi_*\mathcal{O}_A$$

in the ordinary case. In particular therefore $\Pi\cdot(\nabla_\partial)^2=0$ if ∂ is Serre–Tate, albeit this is neither a linear nor an F-linear statement, whereas (16) is. □

To relate this to real and complex embeddings, observe that if a ring, R, acts linearly on a module N over another ring $\mathcal{O}$, in such a way that the identity of R is the identity of N, then a decomposition,

$$\mathrm{id}_{R\otimes_{\mathbb{Z}}\mathcal{O}}=\sum_\tau e_\tau \tag{17}$$

implies, for $R\otimes_{\mathbb{Z}}\mathcal{O}\hookrightarrow\mathrm{End}_{\mathcal{O}}(N)$, a decomposition of $\mathrm{id}_N=\sum e_\tau$ into idempotents. For example, the action of B, resp. B', on the left of $B\otimes_{\mathbb{Q}}\mathbb{C}$, resp. $B'\otimes_{\mathbb{Q}}\mathbb{C}$ yields an action of the same on the resulting local system, $\mathcal{V}$, resp. $\mathcal{V}'$, on $M(G,h)$, resp. $M(G',h')$, respecting the Hodge structures, $V^{p,q}$, resp. $(V')^{p,q}$, identified with vector bundles on $M(G,h)$, resp. $M(G',h')$. In particular, even though it isn't even a rational Hodge structure, the tangent bundle of $M(G,h)$ can still be identified with the B-equivariant sub-bundle,

$$\mathcal{H}om^B_{M(G,h)}\big(V^{0,1},V^{1,0}\big)\hookrightarrow\mathcal{H}om_{M(G,h)}\big(V^{0,1},V^{1,0}\big), \tag{18}$$

so that for $\tau'\colon K\to\mathbb{R}$ one of the un-ramified places in (1), the foliation, $\mathcal{F}'$, corresponds to a rank 1 sub-bundles of the left of (18) defined by elements on the right admitting B-equivariant factorisations of the form,

$$V^{0,1}\xrightarrow{e_{\tau'}}V^{0,1}_{\tau'}:=e_{\tau'}V^{0,1}\to V^{1,0}_{\tau'}:=e_{\tau'}V^{1,0}\hookrightarrow V^{1,0}. \tag{19}$$

As such if $\tau'_i\colon Z\to\mathbb{C}$, $1\leqslant i\leqslant 2$, are the 2-places lying over τ' then, on pulling back along (8), we have a canonical decomposition,

$$V^{0,1}_{\tau'}\otimes_K Z\xrightarrow{\sim}e_{\tau'_1}(V')^{0,1}\prod e_{\tau'_2}(V')^{0,1}|_{M(G,h)} \tag{20}$$

into isomorphic sub-bundles, or, alternatively, we can identify $V^{0,1}$ with the sub-bundle,

$$e_{\tau'_1}(V')^{0,1}\hookrightarrow e_{\tau'_1}(V')^{0,1}\prod e_{\tau'_2}(V')^{0,1}|_{M(G,h)}\colon\ v\mapsto(v,\bar v) \tag{21}$$

where $v \mapsto \bar{v}$ is the $\mathbb{C}$-linear (on the left) map afforded by conjugation in Z (on the right), and, similarly whether in (20) or (21) for the $(1,0)$ terms. For places, $\tau: K \to \mathbb{R}$ where (1) is ramified, however, the Hodge structure on $\mathcal{V}'$ of (7) first chooses a place, τ_1, from the places τ_i, $1 \leqslant i \leqslant 2$, lying over it, then acts by its conjugate, so, here we have canonical identifications,

$$e_{\tau_2} V_\tau^{0,0} \otimes_K Z \xrightarrow{\sim} (V')^{0,-1}|_{M(G,h)}, \quad e_{\tau_1} V_\tau^{0,0} \otimes_K Z \xrightarrow{\sim} (V')^{-1,0}|_{M(G,h)}. \tag{22}$$

In particular, therefore, although every embedding $\tau: Z \to \mathbb{C}$ defines a non-zero projector, e_τ, of $\mathcal{H}^1$, for those lying over the ramified places of K one of them will be zero on the kernel of (14) while the other will be zero on the cokernel, which one should bear in mind in observing that (13), for ∂ a derivation on $M(G', h')$, refines to a commutative square,

$$\begin{array}{ccc} e_{F(\tau)}(V')^{0,1} & \xrightarrow[\Pi\cdot\Theta_\partial]{} & e_{F(\tau)}(V')^{1,0} \\ {\scriptstyle \text{Cartier}}\big\downarrow & & \big\uparrow{\scriptstyle \text{Frobenius}} \\ e_\tau(V')^{0,1} & \xrightarrow{\partial} & e_\tau(V')^{1,0} \end{array} \tag{23}$$

Similarly, if following [14], we denote by $\mathfrak{f}(\tau)$ the rank of the kernel of,

$$F_\tau : e_\tau(V')^1 \to e_{F(\tau)}(V')^1 \tag{24}$$

then, [8, Theorem 7.4], for $\tau: Z \hookrightarrow \mathbb{C}$ we have the following possibilities,

$$\mathfrak{f}(\tau) = \begin{cases} 2 & \text{if } \tau \text{ lies over } \tau' \text{ or } \tau'' \\ 4 & \text{if } e_\tau(V')^{1,0} = 0 \\ 0 & \text{if } e_\tau(V')^{0,1} = 0 \end{cases} \tag{25}$$

and one should (modulo a typographical confusion between kernels and co-kernels) recall [14, 2.1.4] that the tangent space to the deformation of the implied Barsotti–Tate groups with B'-action, for F_τ as per (24), is,

$$\coprod_\tau \mathrm{Hom}_{B'}\big(\mathrm{Ker}\, F_\tau, e(\tau)(V')^1/\mathrm{Ker}\, F_\tau\big) \tag{26}$$

to which, by (25), each place over τ', or τ'' contributes exactly one dimension, i.e. we have exactly the two copies of $\widehat{M}$ encountered in Revision/Claim 2.1, inside which the tangent space to $M(G', h')$ is, exactly as in (21), the identification of the two copies by way of conjugation. This said we have a weak version of Corrigendum 1.1,

Lemma 3.2 *The foliation defined by projection onto the factor τ'' in (4) is, apart from finitely many characteristics, not p-closed iff τ'' is the composite of τ' with Frobenius.*

Proof We only do the more difficult case where the places τ_i', $1 \leqslant i \leqslant 2$, over τ' belong to the same orbit. Irrespectively, by Fact/Set Up 3.1 we have, for any vector field, ∂, on $M(G', h')$,

$$\partial^p = \Pi \cdot \nabla_{\partial^p} = \Pi \cdot (\nabla_\partial)^p + \Pi \cdot \Theta_\partial = \Pi \cdot \Theta_\partial \mod (\partial). \tag{27}$$

Equally we can identify a vector field ∂ with a pair of conjugate, (21), fields ∂_i defined via a factorisation of the form (19) for τ_i'. Now if $F(\tau')$ is anything other than τ' or τ'' then one of the verticals in (23) is zero by (22), so each Θ_{∂_i}, and whence Θ_∂ is 0, and we're p-closed by (27). Otherwise if $F(\tau_1') = \tau_2'$ then Θ_{∂_i} are conjugate by (23), and the right hand side of (27) is parallel to ∂, so $F(\tau') = \tau''$ is certainly necessary for p-closure to fail. Conversely, supposing this holds, then its sufficiency amounts to the non-vanishing of the conjugate Θ_{∂_i} in (27), so, a fortiori, the vertical arrows in (23), with $\tau = \tau_i'$, being generic isomorphisms on $M(G', h')$ will do. This is, however, a part of the definition μ-ordinary. Specifically, the cohomology, $\mathcal{H}^1_{\bar{\mathbb{F}}_p}$, of (14) extended over $\overline{\mathbb{F}}_p$, decomposes into idempotents according to the general prescription of (17) for $\overline{\mathbb{F}}_p \otimes \mathcal{O}_Z$, i.e.

$$\mathcal{H}^1_{\bar{\mathbb{F}}_p} = \prod_{\tau \hookrightarrow \mathbb{C}} e_\tau(\mathcal{H}^1)$$

and μ-ordinary implies (or indeed means), [14, 1.2.3], that Frobenius, F, and Verschiebung, $\mathfrak{V}$, are as disjoint as possible, i.e. each $e_\tau(\mathcal{H}^1)$ has a basis x_τ^a, $1 \leqslant a \leqslant 4$, such that,

$$F(x_\tau^a),\ \text{resp. } \mathfrak{V}\big(x_{F(\tau)}^a\big) = \begin{cases} x_{F(\tau)}^a,\ \text{resp. } p x_\tau^a,\ a \leqslant 4 - \mathfrak{f}(\tau) \\ p x_{F(\tau)}^a,\ \text{resp. } x_\tau^a,\ a > 4 - \mathfrak{f}(\tau) \end{cases} \tag{28}$$

which is the same as vertical isomorphisms in (23) as soon as $\mathfrak{f}(F(\tau)) = \mathfrak{f}(\tau)$ whenever the idempotents respect the Hodge structure. Consequently we'll be done if,

Claim 3.3 *The generic point of every irreducible component of the moduli problem of Revision/Claim 2.1 is μ-ordinary.*

The moduli problem of Revision/Claim 2.1 admits both a PEL description, and the non-polarised description via the transpositions. In the former case, one can appeal to the general PEL methodology of [14, Theorem 4.1.7]. The non-polarised route is, however, much more direct. Indeed, over $\overline{\mathbb{F}}_p$ the isomorphism classes of p-torsion Barsotti–Tate groups with B'-action admitting (25) as the kernel of Frobenius on the projectors (afforded by the idempotents) of its Dieudonné-module, are canonically, [14, 1.1.5], the cosets,

$$W_{(B')^\times} / W_P \tag{29}$$

of the Weyl group of $(B')^\times$ modulo that of the parabolic sub-group P fixing the kernel of Frobenius, and, [14, 1.2.2], the μ-ordinary class is that whose minimal length (as a word in a Coxeter system) is longest. Similarly, [13, Theorem 2.1.2], if the class of the p-torsion of a Barsotti–Tate is w in (29), then the dimension of its deformation space fixing w is the length of w. In our current situation, after an identification of B' as a product of 2×2 matrix algebras, the cosets of (29) are, for $\mathfrak{f}$ of (25), just, [13, 2.2.2], the cokernel of,

$$\prod_{\tau:\, Z\to\mathbb{C}} \mathrm{S}_{\mathfrak{f}(\tau)/2}\times \mathrm{S}_{2-\mathfrak{f}(\tau)/2} \hookrightarrow \prod_{\tau:\, Z\to\mathbb{C}} \mathrm{S}_2 \tag{30}$$

and, the word length is the cardinality of the set of τ' where we find a generator of S_2. Better still, the transpositions in the moduli problem of Revision/Claim 2.1, imply that the resulting word in (30) must be symmetric in the places τ_i', resp. τ_i'', over τ', resp. τ'', so, identifying S_2 with $\mathbb{Z}/2$, the only way that $M(G', h')$ could fail to be μ-ordinary would be if, in (30), we found, generically, either the word,

$$w' := 1_{\tau_1'}\times 1_{\tau_2'}\times 0_{\tau_1''}\times 0_{\tau_2''}, \quad \text{or,} \quad w'' := 0_{\tau_1'}\times 0_{\tau_2'}\times 1_{\tau_1''}\times 1_{\tau_2''}, \tag{31}$$

and, generically, $M(G', h')$ were the 2-dimensional deformation space fixing, say, w'. Plainly, however, the 2-dimensional projector of (26) in the τ' direction doesn't move the τ'' direction, so this would have to be the generic tangent space to $M(G', h')$ which it definitely is not. □

At which juncture we may proceed to give,

Proof of Corrigendum 1.1 We've done case (A) with $e = 0$, so, without loss of generality, $\mathcal{F}'$ is p-closed, and $\tau'' = F(\sigma)$ for a place $\sigma: K \hookrightarrow \mathbb{R}$ other than τ', while for each embedding $\tau_i''\colon Z \hookrightarrow \mathbb{C}$ there is a corresponding σ_i such that $F(\sigma_i) = \tau_i''$. Furthermore (24) is a map of flat bundles of determinant 1, so, as soon as it's a generic isomorphism, it's an isomorphism. Thus by, (28) & Claim 3.3, without loss of generality, (24) is an isomorphism for $\tau = \sigma_1$, while at the conjugate place the dual conclusion holds, i.e. the Verschiebung,

$$\mathfrak{V}\colon e(\tau_2'')(V')^1 \to e(\sigma_2)(V')^1 \tag{32}$$

is an isomorphism. In particular, if for $W = W(\overline{\mathbb{F}}_p)$ the Witt vectors over $\mathbb{F}_p$, and i the embedding of the closed fibre of any W-scheme, we identify the flat bundle afforded by $i^{-1}(V')^\vee\otimes_{\mathbb{Q}} W$ with the relative crystalline cohomology of our family of abelian varieties mod p then, for $1\leqslant i\leqslant 2$, we can define a bundle, D, of Dieudonné modules with B' action by the formulae,

$$e(\tau)(D) := \begin{cases} e(\tau)(V')^1, & \tau \neq \tau_i'', \sigma_i \\ (F^{-1})^*\big(e(\tau_i'')(V')^1\big), & \tau = \sigma_i \\ F^*\big(e(\sigma_i)(V')^1\big), & \tau = \tau_i'' \end{cases} \tag{33}$$

wherein $(F^i)^*$ is the same abelian group but W-module structure twisted by F^{-i}, while Frobenius and Verschiebung are unchanged except for σ_i, τ_i'', and their immediate successors and predecessors, where,

$$
\begin{array}{c}
e(F^{-1}(\sigma_1))(D) \underset{F^2}{\overset{\mathfrak{V}F^{-1}}{\rightleftarrows}} e(\sigma_1)(D) \underset{F^{-1}}{\overset{F\mathfrak{V}F=pF}{\rightleftarrows}} e(\tau_1'')(D) \underset{F^2}{\overset{F^{-1}\mathfrak{V}}{\rightleftarrows}} e(F(\tau_1''))(D) \\
e(F^{-1}(\sigma_2))(D) \underset{\mathfrak{V}^{-1}F}{\overset{\mathfrak{V}^2}{\rightleftarrows}} e(\sigma_2)(D) \underset{\mathfrak{V}F\mathfrak{V}=p\mathfrak{V}}{\overset{\mathfrak{V}^{-1}}{\rightleftarrows}} e(\tau_2''(D) \underset{F\mathfrak{V}^{-1}}{\overset{\mathfrak{V}^2}{\rightleftarrows}} e(F(\tau_2''))(D)
\end{array}
\tag{34}
$$

with F, $\mathfrak{V}$ Frobenius & Verschiebung on $(V')^1$, and the possibility that the rightmost and leftmost terms in (34) coincide is allowed. In particular, the geometric Frobenius (whose component in τ is that of $(V')^1$ at $F^{-1}(\tau)$) factors through D, which apart from the critical range of (34) is just the identity followed by F, and otherwise, in for example, the more difficult case where (32) rather than (24) is the isomorphism, is given by,

$$
\begin{tikzcd}
e(F^{-1}(\sigma_2))((V')^1) \arrow[r, equal] \arrow[d, "F"'] & e(F^{-1}(\sigma_2))(D) \arrow[r, "F"'] \arrow[d, "F_{\mathrm{new}}=" ', "\mathfrak{V}^{-1}F"] & e(\sigma_2)((V')^1) \arrow[d, "F"] \\
e(\sigma_2)((V')^1) \arrow[r, "\mathfrak{V}^{-1}"] \arrow[d, "F"'] & e(\sigma_2)(D) \arrow[r, "F\mathfrak{V}", "=p"'] \arrow[d, "F_{\mathrm{new}}=" ', "F\mathfrak{V}F"] & e(\tau_2'')((V')^1) \arrow[d, "F"] \\
e(\tau_2''((V')^1) \arrow[r, "\mathfrak{V}"] \arrow[d, "F"'] & e(\tau_2''(D) \arrow[r, "F\mathfrak{V}^{-1}", "=F_{\mathrm{new}}"'] \arrow[d, "F_{\mathrm{new}}=" ', "F\mathfrak{V}^{-1}"] & e(F(\tau_2'')((V')^1) \arrow[d, "F"] \\
e(F(\tau_2''))((V')^1) \arrow[r, equal] & e(F(\tau_2''))(D) \arrow[r, "F"] & e(F(\tau_2''))((V')^1)
\end{tikzcd}
\tag{35}
$$

Now once one has the formulae (34)–(35) it's easy to conclude,

Claim 3.4 *Supposing $\tau'' \neq F(\tau')$, the quotient $\rho\colon M(G', h') \to M(G', h')/\mathcal{F}'$ represents an irreducible connected component of the same moduli problem as Revision/Claim 2.1 except that the transposition α'' switches the factors at the place $F^{-1}(\tau'') = \sigma$ rather than τ''.*

Sub-proof of Claim 3.4 The bundle of Dieudonné modules defined by the co-kernel of (35), are the Dieudonné module of a subgroup of the kernel of Frobenius on each abelian variety in $M(G', h')$, as such the moduli of the abelian varieties quotiented by this sub-group is a factorisation of Frobenius on $M(G', h')$ which, by construction, kills exactly the vectors tangent to $\mathcal{F}'$, so it is $M(G', h')/\mathcal{F}'$, and the pull-back, under ρ, of the cohomology of the universal family is the Dieudonné module of (33)–(34).□

As such, if we take, in case (A), of Corrigendum 1.1 an equivalent definition of e as the number of places, in an orbit of Frobenius, between the transpositions α', α'' in the moduli problem of Revision/Claim 2.1 & Claim 3.4, then the effect of replacing

$M(G', h')$ by $M(G', h')/\mathcal{F}'$ is to reduce e by 1, and we arrive, by induction, to the case $e = 0$ of Lemma 3.2, while in case (B) (or if one prefers $e = \infty$) there is no obstruction to continuing indefinitely. □

In the language of [12] this has the wholly desirable,

Corollary 3.5 *In* (B) *of Corrigendum* 1.1 *there are no absolutely singular curves. Indeed, better, even in case* (A) *of Corrigendum* 1.1 *with* $e \geqslant 2$, *the foliation is un-ramified in the sense of op. cit.* (3.15)–(3.16), *i.e. the canonical bundle of the new foliation defined by* τ' *on* $M(G', h')/\mathcal{F}'$, *pulled back to* $M(G', h')$ *is* $K_{\mathcal{F}'}^{\otimes p}$. *In particular, Corrigendum* 1.2 *follows from* [11, Proposition 3.6].

Proof The lack of ramification, i.e. we have the the above formula, cf. (46), for the canonical bundle on the quotient, is immediate from the explicit description (33)–(35), while the reference of the in particular is to the case $e = 0$, to which one reduces by the simple expedient of taking the quotient in the τ' direction e-times. □

Similarly, with the definition of μ-ordinary to hand we can confine the proof of Compliment 1.4 to,

Remark 3.6 The μ-ordinary condition is, [14, 1.3.10], the same as ordinary iff $\mathfrak{f}$ of (25) is constant on orbits, which will fail whenever an orbit in Z has a different cardinality from its restriction to K, except for the particular circumstances of Compliment 1.4. Even here, however, $\mathfrak{f}$ of (25) still won't be constant unless the choice of Hodge structure, h_Z, of (7) is compatible with the action of Frobenius, and for each orbit there are exactly 2 choices which work. Similarly, once every place of K is split in Z, constant $\mathfrak{f}$ implies that the orbits of ramified and un-ramified $\tau : K \hookrightarrow \mathbb{R}$ must be disjoint, and this, again, implies ordinary for the right choice of h_Z.

Equally the final part of Corrigendum 1.3 merits no more than,

Remark 3.7 In case (B) of Corrigendum 1.1, Frobenius, on the Dieudonné module of each orbit containing a place of Z over τ' or τ'', has 2 slopes, [14, 1.2.5], by (25). As such, we're in op. cit.'s strict generalisation, [14, 2.3.3], of Serre–Tate coordinates, and even the worked example of op. cit. 0.4. Following this through in the case where the orbit restricted to K has the same cardinality, one finds for (25) (whose notation is that of op. cit.) that the formal deformations of a μ-ordinary object with Z action, over $\overline{\mathbb{F}}_p$, in, say, a direction τ_i' lying over τ', are 4 copies of the formal group, $\mathfrak{G}_{r'}$, of height the period, r', of τ' under Frobenius. We are, however, only concerned with B'-equivariant deformations, so, just as in Revision/Claim 2.1, the formal deformation in the τ' direction of abelian varieties with multiplication by B', are $\mathfrak{G}_{r'} \times \mathfrak{G}_{r'}$, and since we furthermore have the swapping condition on α' and α'' of op. cit., we find that the fibre of the formal groupoid $\mathfrak{F}'$ at a μ-ordinary point of, not just $M(G', h')$, but even $M(G, h)$, is a $\mathfrak{G}_{r'}$ action on the complete local ring. Should, however, the cardinality of the orbit of τ' of Frobenius in Z be different from that in K, the formal deformations in the τ'-direction of abelian varieties with multiplication by B' are, for the same reasons, canonically still a formal group, but it no longer comes with a 1st and 2nd projection like the previous case. Nevertheless it does still split as diagonal by anti-diagonal (i.e. dual of the dual diagonal), and the diagonal copy is again our deformation space and isomorphic to $\mathfrak{G}_{r'}$. In either case, therefore, on completing in an ordinary point the quotient by, say $\mathcal{F}'$, is exactly,

$$F \times \mathrm{id} \colon \mathfrak{G}_{r'} \times \mathfrak{G}_{r''} \to \mathfrak{G}_{r'} \times \mathfrak{G}_{r''} \tag{36}$$

and if one repeats this r' times, the result is $p \times \mathrm{id}$, which is equally the situation in which Claim 3.4 repeats itself.

Should the modèle not be étrange, we may usefully add,

Remark 3.8 In the quadratic case everything is valid as stated but radically simpler, since we're p-closed iff both orbits have cardinality 1. Better still the modular interpretation, (33)–(35), of the factorisation (36) over the ordinary locus is particularly simple, i.e. for each direction there is a copy of $\boldsymbol{\mu}_p \hookrightarrow A[p]$, which is dual to the co-kernel of the final vertical arrow in (35). In particular, therefore, the only point for caution is the cusps in the Hilbert modular sub-case, and, specifically, that either foliation is un-ramified, in the sense of Corollary 3.5. Fortunately, however, the points of a cusp correspond to tori of the form, $I \otimes_{\mathbb{Z}} \mathbb{G}_m$, for I a fractional ideal in K (i.e. the obvious structure on $\mathbb{G}_m^2$ that inherits. from I, multiplication by $\mathcal{O}_K$) so that when p splits in K, we get a canonical splitting of $I \otimes_{\mathbb{Z}} \mathbb{G}_m$, and the resulting copies of $\boldsymbol{\mu}_p$ extend the previous modular description. Arguably, therefore, cusps are ordinary.

4 Kodaira dimension and modular forms

Already in characteristic zero, the Hodge decomposition of the flat bundle $e(\tau')V^1$, with trivial determinant, and (19), imply,

$$K_{\mathcal{F}'} = \mathrm{c}_1\big(e(\tau')V^{0,1}\big), \quad T_{\mathcal{F}'} = \mathrm{c}_1\big(e(\tau')V^{1,0}\big) \tag{37}$$

and similarly for τ''. In addition, B is un-ramified at τ', so, $e(\tau')V^{0,1}$ and $e(\tau')V^{1,0}$ are direct sums of isomorphic line bundles, which, following (37) we write, for example, as,

$$e(\tau')V^{0,1} = K_{\mathcal{F}'}^{1/2} \oplus K_{\mathcal{F}'}^{1/2}. \tag{38}$$

As such, following standard usage in the Hilbert modular case, we will describe global sections of,

$$K_{\mathcal{F}'}^{i/2} \otimes K_{\mathcal{F}''}^{j/2}, \quad i, j \in \mathbb{Z},$$

as *modular forms of weight* (i, j), and we assert,

Fact 4.1 *Should we find ourselves in case* (A) *of Corrigendum 1.1, albeit with e', resp. e'', the value of e for $\mathcal{F}'$, resp. $\mathcal{F}''$, then for $e' > 0$, resp. $e'' > 0$, amongst the $2^{e'}$, resp. $2^{e''}$, choices for h_Z of* (7) *along the orbit between a place over τ' to that over τ'', resp. vice versa, there are 2 choices in either direction, while if e', resp. e'', is zero there is* *no choice to make, such that the locus of non μ-ordinary varieties is defined by the vanishing of a modular form of weight* $(p^{e'+1}, -1)$*, resp.* $(-1, p^{e''+1})$*, whose double is the section afforded by the failure of* $p^{e'+1}$*, resp.* $p^{e''+1}$*-closure. Similarly, in case* (B)*, if* $r' > 1$*, resp.* $r'' > 1$*, we only assert that amongst the* $2^{r'-1}$*, resp.* $2^{r''-1}$*, choices for* h_Z *of* (7) *along the orbit of* τ'*, resp.* τ''*, there are* 2*, in either direction, while if* r'*, resp.* r''*, is* 1 *there is no choice to be made, such the non-*μ*-ordinary locus is defined by a modular form of weight,* $(p^{r'} - 1, 0)$*, and another of weight* $(0, p^{r''} - 1)$.

Proof Consider first the case $e' = e'' = 0$, then orbits under Frobenius of embeddings of $Z \hookrightarrow \mathbb{C}$ are formed either of places wholly over τ' or τ'', or have an empty intersection with such. In the latter case, proceeding as we began the proof of Corrigendum 1.1, the μ-ordinary condition at a place τ amounts to exactly one of,

$$F: e(\tau)(V')^1 \to e(F(\tau))(V')^1, \quad \mathfrak{V}: e(F(\tau))(V')^1 \to e(\tau)(V')^1$$

being an isomorphism, while these are flat bundles with trivial determinant, and the condition holds generically, so it holds everywhere. Otherwise, the p-closure map is given by (27), which is certainly where a modular form of weight $(2p, -2)$, resp. $(-2, 2p)$, vanishes. These are however, the product of the two vertical arrows in (27) each of which is a form of weight $(p, -1)$, resp. $(-1, p)$, and whose vanishing is identically the failure to be μ-ordinary by (28). However, not only do conjugate places over τ', resp. τ'', define the same modular form in either vertical of (27) because of transposition condition in Revision/Claim 2.1, but the verticals themselves are dual by way of the polarisation of op. cit., so there's only really one modular form in play in either direction.

Now, as soon as, say $e' > 0$, one certainly finds by induction and Claim 3.4, that the failure of $\mathcal{F}'$ to be a height $e' + 1$ foliation is the double of a modular form of weight $(p^{e'+1}, -1)$. However, it's exact relation with the μ-ordinary locus isn't so clear since sometimes the induction will use the top line in (34), and sometimes the bottom. If, however, it always does one or the other, then for places τ_i', resp. τ_i'', over τ', resp. τ'', we have that for a given i exactly one of,

$$\begin{array}{c} e(F(\tau_i'))(V')^{1,0} \xrightarrow{F} \cdots \xrightarrow{F} e(F^{e'}(\tau_i'))(V')^{1,0} \xrightarrow{F} e(\tau_i'')(V')^{1,0} \\ e(\tau_i')(V')^{0,1} \xleftarrow{\mathfrak{V}} e(F(\tau_i'))(V')^{0,1} \xleftarrow{\mathfrak{V}} \cdots \xleftarrow{\mathfrak{V}} e(F^{e'}(\tau_i'))(V')^{0,1} \end{array} \tag{39}$$

is a chain of isomorphisms, while the alternative occurs at the other value of i, so that along this part of the orbit, the μ-ordinary condition becomes, for i given, that the composition of all the maps in exactly one of,

$$\begin{array}{c} e(\tau_i')(V')^{1,0} \xrightarrow{F} \cdots \xrightarrow{F} e(F^{e'}(\tau_i'))(V')^{1,0} \xrightarrow{F} e(\tau_i'')(V')^{1,0} \\ e(\tau_i')(V')^{0.1} \xleftarrow{\mathfrak{V}} e(F(\tau_i'))(V')^{0,1} \xleftarrow{\mathfrak{V}} \cdots \xleftarrow{\mathfrak{V}} e(\tau_i'')(V')^{0,1} \end{array} \tag{40}$$

is an isomorphism, with the opposite at the other value of i. Better still, for $p \gg 0$, B is un-ramified as a $\mathbb{Q}_p$ algebra, so the splittings of (38) can be chosen compatibly along an orbit of F, and even with a compatible isomorphism between either factor, while the maps on the top in (40) are F linear, those on the bottom F^{-1} linear; whence, the failure of the composites to be an isomorphism are cut out by 2 (one for the top, resp. bottom, line for one value of i, resp. the other) modular forms of weight $(p^{e'+1}, -1)$, with the respective possibilities again being dual via the polarisation of Revision/Claim 2.1, so, there is one form whose double is that afforded by the failure of $\mathcal{F}'$ to be height $e'+1$, and, should this work for one choice of h_Z, the conjugate choice switches the role of top and bottom for a given i.

The situation in case (B) is much the same, with of course $r' > 1$ and each τ_i' in the same orbit the most difficult case, where, for example, we'll have chains,

$$\begin{array}{c} e(\tau_1')(V')^{1,0} \xrightarrow{F} \cdots \xrightarrow{F} e(F^{r'-1}(\tau_1'))(V')^{1,0} \xrightarrow{F} e(\tau_2')(V')^{1,0} \\ e(\tau_1')(V')^{0.1} \xleftarrow{\mathfrak{V}} e(F(\tau_1'))(V')^{0,1} \xleftarrow{\mathfrak{V}} \cdots \xleftarrow{\mathfrak{V}} e(\tau_2')(V')^{0,1} \end{array} \tag{41}$$

the total composition of exactly one of which is generically non-zero, and defines a modular form of weight $(p^{r'}-1, 0)$. The alternative chains wherein we switch $\tau_1' \leftrightarrow \tau_2'$ in (41) only have a duality, between say, the top line of (41) and the bottom line of the switch, in the intermediary terms (i.e. the analogue here of (39)), while the transposition condition of Revision/Claim 2.1 furnishes an isomorphism between the end terms, so, again, we only find one modular form. □

In case (A) of Corrigendum 1.1, our analysis of the divisor, Δ', say, defined by the form of weight $(p^{e'+1}, -1)$ is limited to Corollary 3.5, i.e. it's where the rational curves of Corrigendum 1.2 lie. As such, from here till the end, we're always in case (B), and we assert,

Fact 4.2 *The divisor Δ' defined by the modular form of Fact 4.1 of weight $(p^{r'}-1, 0)$ is smooth and everywhere transverse to $\mathcal{F}'$, but fully invariant by the groupoid $\mathfrak{F}''$ defining $\mathcal{F}''$.*

Proof The operator Π of (13)–(14) may be applied to any endomorphism, and, by definition, $\Pi \cdot \nabla_\partial$, applied to the Gauss–Manin connection evaluated at a field parallel to a foliation defined by an idempotent τ, is an endomorphism factoring as in (23). Thus, for example, Gauss–Manin restricts to a leaf-wise connection,

$$\nabla : e(\tau')V^{0,1} \to e(\tau')(V^{0,1}) \otimes K_{\mathcal{F}''}.$$

At the same time, cf. [8, Theorem 7.4], for any power F^r of Frobenius there is a commutative diagram,

$$\begin{tikzcd} e(\tau')V^{0,1} \arrow[r] \arrow[d, "\nabla"'] & V^1 \arrow[r] \arrow[d, "\nabla|_{\mathcal{F}''}"'] & e(\tau')(F^r)^* V^{0,1} \arrow[d, "\nabla"] \\ e(\tau')V^{0,1} \otimes K_{\mathcal{F}''} \arrow[r] & V^1 \arrow[r] & e(\tau')(F^r)^* V^{0,1} \otimes K_{\mathcal{F}''} \end{tikzcd} \tag{42}$$

wherein the composed horiztonals are r-fold compositions of Cartier operators, so, in particular, under the hypothesis of case (B) of Corrigendum 1.1, the modular form of Fact 4.1 when $r = r'$. In terms of a local generator v of either factor in (38), and ∂ a field along $\mathcal{F}''$, this means that a local equation, x, for Δ' satisfies,

$$\partial(x) = x\nabla_\partial v \otimes v^\vee \in (x) \trianglelefteq \mathcal{O}_{M(G',h')}$$

so, it's invariant for the associated 1-foliation, but this reasoning can be repeated whether on $M(G', h')/\mathcal{F}''$ or subsequent quotients thereof by Claim 3.4, so it's invariant by the full groupoid.

In the other direction, we, can only assert, for example, that, Gauss–Manin affords,

$$\nabla\colon e(\tau')V^1 \to e(\tau')V^1 \otimes K_{\mathcal{F}'}.$$

On the other hand this still fits into the diagram (42), albeit the dual diagram where we find,

$$e(\tau')(F^{r'})^* V^{1,0} \to V^1$$

is more convenient. In particular if we choose a local lifting $v^{1,0} \in V^1$ of a generator (of a factor in the dual of (38)) of $V^{1,0}$, and similarly for $v^{0,1}$, then we can write (38) locally as,

$$(v^{1,0})^{\otimes p^{r'}} \mapsto xv^{1,0} + yv^{0,1} \tag{43}$$

so, $x = 0$, is still a local equation for Δ', while, for ∂ now in the $\mathcal{F}'$ direction, the commutativity of the dual diagram to (42), and the nullity of ∇ on the left of (43) implies,

$$\partial(x)\, v^{1,0} = -\, y\mathrm{pr}^{1,0}(\nabla_\partial v^{0,1}) \mod (x). \tag{44}$$

Now modulo (x), both $\partial(x)$, i.e. the tangency with $\mathcal{F}'$, cf. (47), and y, the projection of (38)–(43) to $V^{0.1}$, are local equations for modular forms of weight $(p^{r'} + 1, 0)$ over Δ', which even in the Hilbert modular case is complete by [11, Lemma 3.14], so, they're the same up to a constant. The form defined, mod (x), by y, however is everywhere non-zero by [8, Theorem 7.4] so, either $\partial(x)$ is everywhere non-zero mod (x) as asserted, or it's identically zero. It is, however, already $\mathcal{F}''$ invariant, so if it's zero x is a unit by a pth power, and $K_{\mathcal{F}'}$ is divisible by p in Pic, whence so is $A \cdot K_{\mathcal{F}'}$ for A a fixed ample, which is nonsense. □

In turn, we may relate this to the deformation theory, of Remark 3.7, by way of,

Remark 4.3 Following, [11, (67) et seq.], one can certainly take an $\ell' := p^{r'} - 1$th root of Δ' to find a $\mu_{\ell'}$-cover $M(G,h)_{\ell'} \to M(G,h)$ ramified in Δ' over which $K_{\mathcal{F}'}^{1/2}$ trivialises, and everything pulls back nicely. A priori this isn't enough to identify $\mathfrak{F}' \rightrightarrows M(G', h')_{\ell'} \backslash \Delta'$ with a $\mathfrak{G}_{r'}$-action, but, it is sufficient to identify the height 1-foliation with an action of $\mathfrak{G}_{r'}[p]$, so, at worst Zariski locally on $M(G', h')_{\ell'}\backslash\Delta'$, the height e foliations are $\mathfrak{G}_{r'}[p^e]$ actions. An action of $\mathfrak{G}_{r'}[p^e]$ everywhere on $M(G', h')_{\ell'}\backslash\Delta'$, however, is very similar to asserting that Δ' lifts mod p^e.

At this juncture we should be finished since [11, Lemma 3.17] asserts that Kodaira dimension is 1 impossible. Unfortunately, however, in the case that the base of the Kodaira fibration is $\mathbb{P}^1$ it's only valid if there are at least 3 multiple fibres, while 2 multiple fibres with at least one wild is a possibility. To correct this we'll need,

Revision/Notation 4.4 (cf. [11, (41)], [12, Section 3]) Denoting by, $X^{(e)}$, $e \in \mathbb{Z}$, the eth conjugate under Frobenius of an algebraic space over $\overline{\mathbb{F}}_p$, and, for $e \geqslant 0$, $F_X^e \colon X \to X^{(e)}$ the e-fold geometric Frobenius we write the scheme quotient defined by the height e-foliation $\mathcal{F}'$, resp. $\mathcal{F}''$, as,

$$\rho_e' \colon M(G,h) \to M(G,h)/\mathcal{F}_e', \quad \text{resp.} \quad \rho_e'' \colon M(G,h) \to M(G,h)/\mathcal{F}_e''. \tag{45}$$

Consequently, we have, for example a factorisation,

$$F^e_{M(G,h)} \colon M(G,h) \xrightarrow{\rho_e'} M(G,h)/(\mathcal{F}')^e \xrightarrow{\sigma_e'} M(G,h)^{(e)}$$

along with formulae, such as,

$$(\rho_e')^* K_{\mathcal{F}_e'} = p^e K_{\mathcal{F}'}, \quad (\rho_e')^* K_{\mathcal{F}_e''} = K_{\mathcal{F}'}, \tag{46}$$

for $\mathcal{F}_e'$, resp. $\mathcal{F}_e''$, the foliation afforded by $\mathcal{F}'$, resp. $\mathcal{F}''$, on the quotient.

With this in hand, we may, conveniently, make a preliminary,

Remark 4.5 If a curve is generically transverse to the 1-foliation defined by say, $\mathcal{F}'$, then (employ the coordinates of (5)) it stays transverse to the new $\mathcal{F}'$ in $M(G,h)/\mathcal{F}'$. If however it's only invariant by the 1-foliation, then it may become transverse in the quotient. The alternative that it's invariant by every height e foliation is the same as being invariant by the formal groupoid $\mathfrak{F}'$, which, apart from the singularities at cusps in the Hilbert modular case is even a formal relation so, [12, Fact 2.2], outwith the same, a fully invariant curve is smooth. On the other hand, if a curve, $\rho_e' \colon C \to C_e$, becomes transverse in a height e quotient, then we have a non-trivial tangency,

$$(K_{\mathcal{F}_e'} + C_e) \cdot C_e = \text{ tangency with } \mathcal{F}_e' \geqslant 0, \tag{47}$$

so a contractible curve on which $K_{\mathcal{F}'}$ is nil must be fully invariant by (46)–(47). Better still, $K_{\mathcal{F}'}$ is nef., [11, Lemma 3.16], so, just as in characteristic zero, cf. [10, Definition III.3.1], after contracting the cusps in the Hilbert modular case, we have, the definition of a canonical model:

$$K_{\mathcal{F}'}, \text{ is nef., and, } K_{\mathcal{F}'} \cdot C = 0 \Rightarrow C^2 \geqslant 0. \tag{48}$$

Now, an advantage of characteristic p in the Hilbert Modular case, is that it's $\mathbb{Q}$-factorial, [1, 2.11], so, for convenience of notation, we can eliminate it since if the Kodaira dimension, of say, $K_{\mathcal{F}'}$, were equal to 1, we'd have a base point free linear system, and through any cusp, a fibre, which, viewed in $M(G,h)$ would look like,

$$C \text{ (supported on cusps)} + D \text{ (not fully } \mathcal{F}' \text{ invariant)}$$

but, since, without loss of generality, the fibre is connected, D is contractible, and whence, the absurdity that $K_{\mathcal{F}'} \cdot D > 0$ by (47) et seq.

As such, we may suppose that everything is smooth, and we assert:

Claim 4.6 *If $K_{\mathcal{F}'}$, has Kodaira dimension* 1, (*equivalently, since $M(G,h)$ has nil irregularity for $p \gg 0$, &* (48)*, there is not exactly one irreducible curve parallel to it in Néron–Severi*) *then, up to a twist by Frobenius, the Kodaira fibration is iso-trivial.*

Proof A priori, the map, $\pi : M(G,h) \to B$, afforded by the Kodaira fibration may not be separable, but, cf. [11, Lemma 3.17], some $M(G,h) \to B^{(-n)}$ is, so replacing B by $B^{(-n)}$ and Stein factorising, we may suppose that π is separable with connected fibres. Now, if the generic fibre of π isn't transverse to $\mathcal{F}'$, then either there is a maximal e such that π factors through ρ'_e of (45), or it's fully $\mathcal{F}'$ invariant. However, in the latter case it must be a smooth elliptic curve, which is nonsense whether by [11, Proposition 3.9], or, employing adjunction to lift it to characteristic zero. As such, by way of the coordinates (5), we have that for every $e \gg 0$, on the quotient $M(G,H)/\mathcal{F}'_e$ of (45), the reduction of every fibre afforded by π is generically transverse to $\mathcal{F}'$, and whence, smooth and everywhere transverse by (47). In particular, therefore, the Kodaira fibration on $M(G,h)/\mathcal{F}'_e$ doesn't descend to $M(G,h)/\mathcal{F}'_{e+1}$, so, in the obvious notation, once $e \gg 0$, for any $\epsilon > 0$, there's a commutative square,

$$\begin{tikzcd}
M(G,h)/\mathcal{F}'_e \arrow[r, "\rho''_{e,e+\epsilon}"'] \arrow[d, "\pi_e", "\text{separable, connected fibres}"'] & M(G,h)/\mathcal{F}'_{e+\epsilon} \arrow[d, "\text{separable connected fibres}", "\pi_{e+\epsilon}"'] \\
B_e \arrow[r, "F^{\epsilon}_{B_e}"] & B_e^{(\epsilon)} = B_{e+\epsilon}
\end{tikzcd} \tag{49}$$

wherein B_e is a curve over a finite field, k, so, provided $(k : \mathbb{F}_p)$ divides ϵ, we may identify B_e with $B_{e+\epsilon}$. Similarly, in a variant of [15, Théorème 1], observe, [11, Lemma 3.16] & Remark 4.5, that the canonical bundle of every $M(G,h)/\mathcal{F}'_e$ is ample, while by (46), neither of the Chern classes,

$$\mathrm{c}_1^2(M(G,h)/\mathcal{F}'_e) = 2\mathrm{c}_2(M(G,h/\mathcal{F}'_e) \tag{50}$$

depends on e, and everything is defined over a finite field. We may, therefore, find a sequence of e's going to infinity such that both the B_e's and the $M(G,h)/\mathcal{F}_e$'s, are isomorphic. Now, continuing to follow op. cit., once the Kodaira fibration is everywhere transverse to $\mathcal{F}'$, the genus $g > 1$ of the fibres never changes, so, for e, $e + \epsilon$

in our sequence of isomorphisms, there is, for $b_\bullet$ the generic point of $B_\bullet$, by (49), a commutative diagram,

$$\begin{tikzcd} b_e \arrow[r, "F^\epsilon"'] \arrow[d, "\text{moduli}"', "\beta_e"] & b_{e+\epsilon} = B_e \arrow[d, "\beta_e"', "=\beta_{e+\epsilon}\ \text{moduli}"] \\ \mathcal{M}_g \arrow[r, equal] & \mathcal{M}_g \end{tikzcd}$$

and, whence, the image of β is a closed point. □

Rather immediately, therefore, we may deduce,

Corollary 4.7 *The Kodaira dimension whether of $\mathcal{F}'$, or $\mathcal{F}''$, is zero and the divisor Δ', resp. Δ'', cut out by the modular forms of weight $(p^{r'}-1, 0)$, resp. $(0, p^{r''}-1)$, of Fact 4.2 is irreducible.*

Proof If the Kodaira dimension is zero, then the index theorem implies that the said divisors are irreducible. Regardless, while, we may have had to pass to some inseparable quotient ρ'_e in the course of the proof of Claim 4.6, the pull-back of an étale cover along an inseparable map is still an étale cover, and rational maps from smooth varieties to higher genus curves are everywhere defined, so we've found a curve C of genus $g \geqslant 2$ such that,

$$\begin{tikzcd} X \arrow[r, "\text{not trivial}"'] \arrow[d, "\text{finite étale cover}"'] & C \\ M(G, h) & \end{tikzcd}$$

Now this doesn't quite lift to characteristic zero, but étale covers of proper varieties do lift, [6, Exposé X, Théorème 2.1], Consequently, not only would X lift, but so would all the covers of it afforded by C, but then $\mathrm{H}_1(X)$ is finite and infinite at the same time. □

To conclude we may usefully make,

Remark 4.8 The proof of Claim 4.6 made no use of the other foliation $\mathcal{F}''$, and works in considerable generality. Indeed, it proves a very tight analogue of a key lemma, [10, Case IV.4.4.(iv)], in the classification of foliations in characteristic zero. Specifically: if on a surface over a finite field, $X \to X/\mathcal{F}$ is an un-ramified foliation defined by a formal groupoid (i.e. height e, for all e, satisfying (46), albeit with $K_{X/\mathcal{F}}$ instead of $K_{\mathcal{F}''}$) such that the canonical model condition (48) holds then if the Kodaira dimension is 1 and $\mathrm{c}_2(X) > 0$, after a twist by Frobenius, the Kodaira fibration is iso-trivial in a curve of genus > 1. Indeed, modulo being careful about what it means if X is singular, (50) remains true independent of e, so by (48), and Matasuka, some $mK_{\mathcal{F}_e} + nK_{X/\mathcal{F}_e}$ is very ample with degree independent of e.

Remark 4.9 On a general modèle étrange of any dimension as defined in [2, Section 6], pretty much everything works. Specifically, in the notation of op. cit., we have a quaternion algebra B over a totally real field K which is unramified at the places, $\tau_i : K \hookrightarrow \mathbb{R}$, $1 \leqslant i \leqslant r$, and G/K generalises from (2) to the resulting K-form of Gp_{2n}. As such, the modèle is étrange if $r < (K:\mathbb{Q})$, and what we've done is the $r = 2$, $n = 1$ case. Nevertheless, there are still foliations, $\mathcal{F}_i$ tangent to each τ_i, i.e. (19) holds, and $\mathcal{F}_i$ is height e iff $F^e(\tau_i) \neq \tau_j$, for some $j \neq i$. In particular, when the formal groupoid, i.e. e unbounded, case occurs the principle invariant becomes the period r_i, with Remark 3.7, and Fact 4.1 valid as stated. After that, i.e. Fact 4.2 on, one needs to suppose that our foliations are in curves, i.e. $n = 1$ in Deligne's notation, with already the Shmura curves $r = 1$ being a non-empty example, and, again, everything continues to work, albeit one should work logarithmically so as to treat the cases of B a division algebra or otherwise in a uniform way.

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