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Deterministic Asynchronous Threshold-Based Opinion Dynamics in Signed Weighted Graphs

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Abstract: Among the many (mostly randomized) models proposed in the last decades to study how opinions of a set of individuals interconnected by pairwise relations evolve, a novel deterministic model is introduced in this paper that is able to encompass individual choices, strength and sign of relations, and asynchronism. In particular, asynchronism has been considered until now only in randomized settings. It is here studied in which cases the behavior of the resulting dynamical network is predictable, that is, in which cases the number of opinion configurations encountered by the set of individuals before the dynamical network enters a loop is polynomially bounded by the network size.

Keywords: social networks; network analysis; graph theory; diffusion processes; computational complexity

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1. Introduction

A dynamical network is a set of entities connected by point-to-point links in which each entity is associated with a state that may change over time due to link-conveyed influences. Physics (interacting particle systems), biology (neural systems, bacteria, ...), computer science (distributed systems), as well as social sciences all provide a wide range of examples of dynamical networks. In particular, individuals in complex social environments typically form and change their opinion on the basis of the social influence they receive from the environment.

Consider the following scenario. A set of individuals is engaged in a social network, so that each of them is linked to a subset of other individuals. At a given time, all individuals in the network hear about information that can be true or false, and each of them takes his own prior opinion about the information being true or false. Then, each individual starts discussing (with the individuals he is related with) about the truth/falsity of the information, and the opinions of his friends and foes may make him change his opinion. Continuing the discussion, each individual possibly moves between true and false an indefinite number of times. Questions naturally arising in this scenario are: will the discussion ever end, that is, will the individuals opinions ever reach a point at which nobody changes his opinion any longer (*stability* or *equilibrium*)? Or will the discussion ever reach a point where all individuals have the same opinion (*consensus*)? Or will the discussion ever reach a point where more than half of the individuals have the same opinion (*majority*)?

The scenarios described above are suitable for several daily life problems: discovering fake news in a social environment, foreseeing a referendum results, and so on. But they are also able to describe somehow more technical issues that lie in community detection, leader election, and consensus establishment in (fully) distributed systems. Generally speaking, the common characteristic of all the problems cited here is the request that a set of (interrelated) entities reach a state with some given property when no centralized control is available.

The typical setting considered throughout this paper is that of a social environment in which each individual is engaged in a set of stronger or weaker friendship or hostility relations: simply speaking, a relation may range from strong hostility/distrusting to weak disliking (negative relations), as well as from weak liking to strong friendship/trusting (positive relations). Given a topic, every individual has an a priori (positive or negative) opinion about it and gets influenced over time by his relationships: a positive relationship provides him with some evidence in favor of the opinion received by his friend; a negative relation provides him with some evidence in contrast to the opinion received by his foe (that is, evidence in favor of the opposite opinion received by his foe). In a synchronous environment, at each (discrete) time step, every individual checks whether his social influence works in favor of keeping or changing his opinion at that step and behaves accordingly. If the environment is asynchronous, individuals check his social influence at some step only. Within this setting, the evolution of an initial opinion configuration in a given network under some opinion dynamics model is studied here.

Related studies. The interest in network dynamics dates several decades ago; recent surveys on dynamic models can be found in [1,2]. Several stochastic models for opinion dynamics in the case of positive-only relations have been considered starting with the French–DeGroot model [3,4] and continuing with the social impact model [5], the Voter model [6,7], and the majority rule model [8], among others. Some of these models have been adapted more recently to the case of positive/negative relations [9–12].

Generally speaking, when considering network dynamics, each individual might change his opinion an unlimited number of times, and the change can or must happen whenever some conditions are met. Models considered in more recent papers [8,13–20] are called *asynchronous*, meaning that, at each step, individuals that could change their opinion are not forced to actually do it, so that only a subset of the individuals that could change their opinion actually change it. A random choice is used to model this feature.

The study of deterministic opinion dynamics models, in which an individual allowed to change his opinion actually does it, has also been attempted. In [21], a framework for the definition of deterministic opinion dynamics is proposed. Deterministic dynamics have been studied in [22–26], ranging from the majority rule [22,23] to rules simplifying the Game of Life [24,27]. A framework encompassing both the deterministic majority rule and the underpopulation one introduced in [24] has been proposed in [25,26].

Paper contribution. Local threshold-based dynamics introduced in [25,26] are regulated by a pair of functions, $\theta^+ : \mathbb{N} \to \mathbb{N}$ and $\theta^- : \mathbb{N} \to \mathbb{N}$, that determine when an individual changes his opinion: an individual u in relation with k other individuals and having a positive (negative) opinion at some time-step changes to the negative (positive) opinion if and only if the number of his related individuals providing him evidence in favor of a positive (negative) opinion is at most $\theta^+(k) - 1$ (at least $\theta^-(k)$) and symmetrically for u with a negative opinion.

The local threshold-based rule, introduced to encompass the underpopulation and the deterministic majority rule, is here furtherly generalized in several aspects.

First of all, *heterogeneous* threshold-based rules (simply referred to as threshold-based rules) will be considered in which the functions θ^+ and θ^- are authonomously chosen by each individual: that is, by denoting as V the set of individuals, $\theta^+ : V \to \mathbb{N}$, and $\theta^- : V \to \mathbb{N}$. Hence, while in a local threshold-based rule two individuals with the same number of relations are associated with the same pair of thresholds, this is no longer the case in a heterogeous threshold-based rule.

Secondly, the existence of stronger and weaker relations is taken into account so that the network of individuals is modeled by a weighted graph $G = (V, E, \pi)$ with $\pi : E \to \mathbb{Z}$ (stronger positive or negative relations have larger absolute value weights).

Finally, an attempt to model asyncronism in a deterministic setting is pursued. A general setting for asynchronism may consist of a pair of functions $\lambda : V \times \mathbb{N} \to \{\texttt{true}, \texttt{false}\}$ and $\chi : V \times \mathbb{N} \to \mathbb{N}$ such that every node u listens at its neighbors' opinions only at times $\bar{t}$ such that $\lambda(u, \bar{t}) = \texttt{true}$, and then, if this is the case, it updates its opinion at time $\bar{t} + \chi(u, \bar{t})$ with the constraint that $\lambda(u, t) = \texttt{false}$ for all $\bar{t} < t < \bar{t} + \chi(u, \bar{t})$. In this paper, a special case of the model just introduced is considered in which $\chi(u, t) = 1$ for every node u and $t \in \mathbb{N}$ and each node u is associated with a pair of integer values, $\rho(u)$ and $\ell(u)$, so that $\lambda(t) = \texttt{true}$ if and only if $t = \ell(u) + i(\rho(u) + 1)$ for some $i \in \mathbb{N}$. Hence, u checks its neighbors' opinions to decide whether to keep or change its current opinion only at steps $\ell(u) + i(\rho(u) + 1)$, for $i \in \mathbb{N}$, (being in a quiescent state all the other time steps) and may change its opinion only at steps $\ell(u) + i(\rho(u) + 1) + 1$, for $i \in \mathbb{N}$. The resulting model will be called *periodic asynchronous* and closely resembles the model of neural systems with a refractory period [28].

Continuing the study performed in [25,26], the *orbit* of an initial opinion configuration in a graph evolving according to a given opinion dynamics was studied, that is, the set of distinct opinion configurations met by the graph while evolving from the initial one. This means that the interest is in studying how node opinions evolve over time starting from a setting in which each node has its own (initial) opinion. The study of the orbit of an initial configuration concerns questions like "will all the individuals ever reach a consensus configuration (a configuration in which all the individuals have the same opinion)?" or "will the individuals ever reach a stable configuration (a configuration in which no individual will change his opinion)?", which can be referred to as $\mathbb{P}$-*reachability questions*, where $\mathbb{P}$ is a polynomial-time checkable property that has to be satisfied by some opinion configuration received by a graph during its evolution from a given initial opinion configuration. There is a strong relation between $\mathbb{P}$-reachability questions and the size of the orbit: actually, in order to answer to any $\mathbb{P}$-reachability question, it is sufficient to let the graph evolve until the requested conguration is met or until it is possible to deduce that it will never be met. Hence, if the size of the orbit of an initial opinion configuration is polynomial in the size of the graph, then the above-mentioned $\mathbb{P}$-reachability questions can be answered in polynomial time. And, conversely, if it is proved that some of such question cannot be answered in polynomial time, then the orbit size is not polynomial in the size of G. In [25,26], it is proved that answering to a couple of $\mathbb{P}$-reachability questions in the case of directed unsigned graph evolving according, respectively, to the deterministic (synchronous) majority rule and to the (synchronous) underpopulation rule are PSPACE-complete problems; as a consequence, the size of the orbit of a configuration in a directed unsigned graph evolving according to the deterministic (synchronous) majority rule or to the (synchronous) underpopulation rule is not bounded by any polynomial in the size of the network unless P = PSPACE. Since the deterministic majority rule and the underpopulation rule are special cases of the dynamics considered in this paper, in what follows, the attention will be focused on undirected graphs only.

The first result of this paper presented in Section 3 concerns the impact of negative relations on the orbit of a configuration. Preliminary achievements regarding this issue can be found in [25,26], where the opinion evolution of a signed graph is simulated by the opinion evolution of a related size unsigned graph if the network is structurally balanced and the opinion dynamics are symmetric local threshold-based (that is, $\theta^+(k) + \theta^-(k) = k$) [25] or if the opinion dynamics are 1-symmetric local threshold-based (that is, $\theta^+(k) + \theta^-(k) = k + 1$) [26]. It was left as an open problem whether the results could be extended to more general topologies and/or more general opinion dynamics. The simulation of the opinion evolution of a signed graph by the opinion evolution of a related size unsigned graph is here extended to the weighted case and to any asyncronous (heterogeous) threshold-based rule independent of the network topology, so closing the mentioned open problem. All of this allows us to only consider unsigned weighted graphs.

Secondly, in Section 4, periodic asynchronous opinion dynamics are taken into account. It is here shown that the opinion evolution of a directed graph G according to any synchronous threshold-based opinion dynamics can be simulated by the opinion evolution of a related undirected graph $\overline{G}$ according to related periodic asynchronous threshold-based opinion dynamics. After the PSPACE-completeness proof in [26], this implies that deciding if an undirected unsigned weighted graph evolving from a given opinion configuration according to a periodic asynchronous threshold-based rule will ever reach an equilibrium configuration is a PSPACE-complete problem and, needless to say, such a result extends to the more general asynchronous dynamics setting introduced above. In more detail, since the weights in $\overline{G}$, the thresholds values, and the values of ℓ and ρ in the simulation are at most 4, it holds that the just reminded decision problem is strong PSPACE-complete. Hence, the size of the orbit of an opinion configuration in an undirected unsigned weighted graph evolving according to an asynchronous threshold-based opinion dynamics is not polynomial in the graph size and the numerical component of the instance (namely, edge weights, thresholds, and timing values) unless P = PSPACE.

The last contribution of this paper (Section 5) is proving that the size of the orbit of a configuration in an undirected (signed) weighted graph evolving according to any synchronous threshold-based opinion dynamics has a pseudo-polynomial upper bound in the graph size, that is, an upper bound which is a polynomial in the number of nodes and edges and in the value of the numerical component of the instance. We remark that the upper bound is polynomial when the graph is not weighted; hence, since the opinion dynamics according to which the unsigned graph simulating the opinion evolution of a signed graph evolves described in Section 3 is a (nonlocal) threshold-based one, this proves that the orbit of any opinion configuration of any signed unweighted graph evolving according to any local threshold-based dynamics (including the underpopulation rule) is upper bounded by a polynomial in the graph size, and this closes the open problem left in [25].

The state of the art concerning orbit size and reachability problems is finally summarized in Table 1.

Table 1. State of the art: a summary. Bold text describes graph topology and opinion dynamics features, the achievements are shown as plain text.

<table>
<tr><th rowspan="5">Graph</th><th colspan="4">Opinion Dynamics</th></tr>
<tr><th colspan="3">synchronous</th><th rowspan="2">periodic
asynchronous</th></tr>
<tr><th colspan="3">Threshold-based</th></tr>
<tr><th colspan="2">Local threshold-based</th><th rowspan="2"></th><th rowspan="4"></th></tr>
<tr><th>Underpopulation</th><th>Majority</th></tr>
<tr><td>undirected
unsigned unweighted</td><td>polynomial
orbit size [24]</td><td></td><td rowspan="4">pseudo-polynomial orbit size
Theorems 1 and 3, and Corollary 3</td></tr>
<tr><td>undirected
signed unweighted</td><td></td><td>polynomial
orbit size [26]</td></tr>
<tr><td>undirected
unsigned weighted</td><td colspan="2" rowspan="2"></td><td rowspan="2">ℙ-reachability strong
-PSPACE-complete
⇒ overpolynomial
orbit size
Theorem 2</td></tr>
<tr><td>undirected
signed weighted</td></tr>
<tr><td>directed
unsigned unweighted</td><td colspan="4">ℙ-reachability PSPACE-complete
⇒ overpolynomial orbit size [25,26]</td></tr>
</table>

2. Preliminary Definitions and Notations

A signed weighted graph $G = (V, E, \pi)$ is a graph together with an edge-weight function $\pi : E \to \mathbb{Z}$. A weighted graph $G = (V, E, \pi)$ is unsigned if $\pi : E \to \mathbb{N}$. For any node u of a signed weighted graph G, $N(u) = \{v \in V : (u, v) \in E\}$ is the set of neighbors of u.

An *opinion configuration* of a graph G is a node-labeling function $\omega : V \to \{0, 1\}$: for $u \in V$, $\omega(u) = 1$ if u endorses a specific topic, and $\omega(u) = 0$ if u contrasts that topic. The opinion of any node u may change over time due to the influences of u's neighbors opinions: for each $v \in N(u)$, if $\pi(v, u) > 0$, then v positively influences u, that is, v pushes u to obtain its same opinion; if $\pi(v, u) < 0$, then v negatively influences u, that is, v pushes u to get its opposite opinion. Furthermore, for any edge (u, v), the larger that $|\pi(u, v)|$ is, the stronger the influence u and v exert on each other: specifically, v *positively influences* u of an amount $|\pi(u, v)|$ if $\pi(u, v) > 0$ and $\omega(v) = 1$ or if $\pi(u, v) < 0$ and $\omega(v) = 0$: summarizing, v positively influences u of an amount $|\pi(u, v)|$ if $\pi(u, v)(2\omega(v) - 1) > 0$. The value

$$P(u) = \sum_{v \in N(u) : \pi(v,u)(2\omega(v)-1) > 0} |\pi(v, u)|$$

is the *positive influence* acting on u at configuration ω.

Opinion dynamics define a functional $\mathbf{d}$ which specifies, for a given opinion configuration ω of a graph G at some time step t, the opinion configuration $\mathbf{d}(G, \omega)$ of G at step $t + 1$. Hence, an opinion dynamics entails a (possibly infinite) discrete dynamic process during which, at each time step, each node might change the opinion it received at the previous step.

Periodic asynchronous threshold-based opinion dynamics $\mathbf{d}$ for a signed weighted graph $G = (V, E, \pi)$ are ruled by a quadruple $\langle \theta^-, \theta^+, \ell, \rho \rangle$, where each of θ^-, θ^+, ℓ, and ρ is a function defined on V associating to every $u \in V$ an integer value. For any node u and for any opinion configuration ω of G at step t, $\mathbf{d}(G,\omega) = \omega'$ is defined as

$$\omega'(u) = \begin{cases} 1 & \text{if } \exists i \in \mathbb{N} : t = \ell(u) + i(\rho(u)+1) \wedge \omega(u) = 1 \wedge P(u) \geq \theta^+(u) \\ & \text{or } \exists i \in \mathbb{N} : t = \ell(u) + i(\rho(u)+1) \wedge \omega(u) = 0 \wedge P(u) \geq \theta^-(u) \\ & \text{or } \forall i \in \mathbb{N}[t \neq \ell(u) + i(\rho(u)+1)] \wedge \omega(u) = 1, \\ 0 & \text{if } \exists i \in \mathbb{N} : t = \ell(u) + i(\rho(u)+1) \wedge \omega(u) = 1 \wedge P(u) < \theta^+(u) \\ & \text{or } \exists i \in \mathbb{N} : t = \ell(u) + i(\rho(u)+1) \wedge \omega(u) = 0 \wedge P(u) < \theta^-(u) \\ & \text{or } \forall i \in \mathbb{N}[t \neq \ell(u) + i(\rho(u)+1)] \wedge \omega(u) = 0. \end{cases}$$

Notice that, if $\Delta_\Pi = \max\{\sum_{v \in N(u)} |\Pi(u,v)| : u \in V\}$, without loss of generality, the bounds $\theta^+(u) \leq \Delta_\Pi + 1$ and $\theta^-(u) \leq \Delta_\Pi + 1$ can be assumed for every $u \in V$.

If $\ell(u) = \rho(u) = 0$ for every $u \in V$, the dynamics are *synchronous*. The *local* threshold-based dynamics introduced in [26] for unweighted graphs are synchronous threshold-based dynamics such that, for any pair of nodes u and v, it holds that $\theta^+(u) = \theta^+(v)$ and $\theta^-(u) = \theta^-(v)$ whenever $|N(u)| = |N(v)|$.

In the rest of this paper, any node u will be said to be *listening* at time steps $\ell(u) + i(\rho(u)+1)$, for some $i \in \mathbb{N}$, and to be *sleeping* all the remaining time steps.

The *evolution sequence* of the opinion configuration ω at time 0 of a graph G with respect to periodic asynchronous threshold-based dynamics $\mathbf{d}$ is the sequence $\mathcal{E}_\mathbf{d}(G,\omega) = \langle \omega_0 = \omega, \omega_1 \ldots, \omega_i, \ldots \rangle$ such that for $t \geq 1$ $\omega_t = \mathbf{d}(G, \omega_{t-1})$. The evolution sequence of a periodic opinion dynamics is periodic. Indeed, define the *state* of a node u as a pair (α, h), with $\omega \in \{0,1\}$ and $0 \leq h \leq \max\{\ell(u), \rho(u)\}$, which represents its opinion at some step and the number of steps it has to wait before checking the changing opinion conditions: hence, at any step, node u can be in one of $2(k_u+1)$ states, where $k_u = \max\{\ell(u), \rho(u)\}$. This implies that the total number of state configurations of G is at most $[2(k+1)]^{|V|}$, where $k = \max\{k_u : u \in V\}$. Once the evolution of an opinion configuration of a graph has covered all possible state configurations, it must necessarily end in a loop.

The *orbit* of an opinion configuration ω of a graph G at time 0 with respect to the periodic opinion dynamics $\mathbf{d}$ (or, in short, the $\mathbf{d}$-*orbit* of G at ω) $\mathcal{O}_\mathbf{d}(G,\omega)$ is the smallest prefix of $\mathcal{E}_\mathbf{d}(G,\omega)$ before a loop starts, that is, $\mathcal{O}_\mathbf{d}(G,\omega) = \langle \omega_0 = \omega, \omega_1, \ldots, \omega_T \rangle$, where $T \leq [2(k+1)]^{|V|}$ and the following holds:

- For $t = 1, \ldots, T$, $\omega_t = \mathbf{d}(G, \omega_{t-1})$;
- There exists $h \leq T$ such that $\omega_{T+ki} = \omega_{h+i-1}$ for $i = 1, \ldots, T-h+1$ and $k > 0$.

The opinion configuration ω_T is an *equilibrium configuration* if $h = T$.

For ease of notation, in what follows, the sequence $\mathcal{O}_\mathbf{d}(G,\omega)$ will be considered a set as well.

3. Removing Negative Relations

It is here shown the description the opinion evolution of a signed weighted graph according to any asynchronous threshold-based opinion dynamics by the opinion evolution of a related unsigned graph according to related asynchronous threshold-based opinion dynamics. Formally, it is as follows.

Theorem 1. *For any undirected signed weighted graph $G = (V, E, \pi)$ and for any asyncronous threshold-based opinion dynamics $\mathbf{d}$, we have an undirected unsigned weighted graph $\mathcal{D}(G) = (V_\mathcal{D}, E_\mathcal{D}, \pi_\mathcal{D})$, with $|V_\mathcal{D}| = 2|V|$ and $|E_\mathcal{D}| = 2|E|$, and the asyncronous threshold-based opinion dynamics $\mathbf{d}_\mathcal{D}$ can be derived from G and $\mathbf{d}$ such that, for any opinion configuration ω of G,*

$$|\mathcal{O}_\mathbf{d}(G,\omega)| = |\mathcal{O}_{\mathbf{d}_\mathcal{D}}(\mathcal{D}(G), \omega_\mathcal{D})|,$$

where $\omega_{\mathcal{D}}$ is an opinion configuration of $\mathcal{D}(G)$ that can be computed in polynomial time from ω.

The proof of the above theorem exploits the technique introduced in [26]. Let $G = (V, E, \pi)$ be an undirected signed weighted graph; the *display graph* $\mathcal{D}(G) = (V_{\mathcal{D}}, E_{\mathcal{D}}, \pi_{\mathcal{D}})$ of G is the undirected unsigned weighted graph derived from G as follows (Figure 1):

- For each $v \in V$, the pair of nodes α_v and β_v are contained in $V_{\mathcal{D}}$, that is, $V_{\mathcal{D}} = \{\alpha_v, \beta_v : v \in V\}$;
- For every $(u, v) \in E$, $E_{\mathcal{D}}$ contains the following:
  - Edges (α_u, α_v) and (β_u, β_v) if $\pi(u, v) > 0$;
  - Edges (α_u, β_v) and (β_u, α_v) if $\pi(u, v) < 0$.

  In both cases, the weight of all such arcs is $|\pi(u, v)|$.

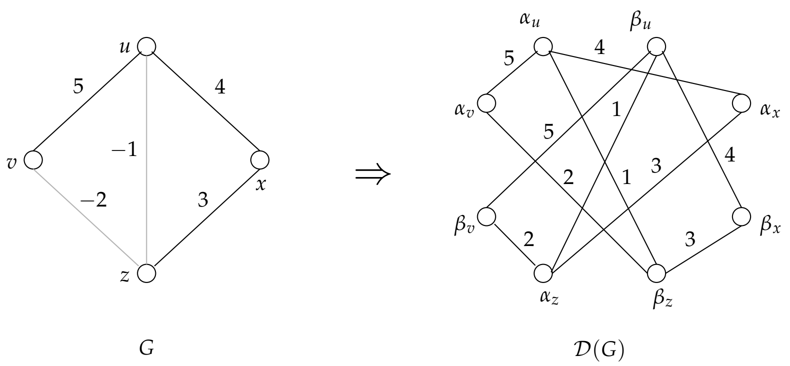

Figure 1. An undirected weighted signed graph G (for ease of readability, negative edges are depicted as gray) and its display graph $\mathcal{D}(G)$. A similar picture can be drawn in the directed case.

Let ω be an opinion configuration of G. The *extension* of ω is the opinion configuration $\omega_{\mathcal{D}}$ of $\mathcal{D}(G)$ such that, for every $v \in V$, $\omega_{\mathcal{D}}(\alpha_v) = \omega(v)$, and $\omega_{\mathcal{D}}(\beta_v) = 1 - \omega(v)$.

Let $\mathbf{d}$ be the asymmetric threshold-based opinion dynamics for G ruled by the quadruple $\langle \theta^-, \theta^+, \ell, \rho \rangle$. The *mirror* dynamics $\mathbf{d}_{\mathcal{D}}$ for $\mathcal{D}(G)$ are the asymmetric threshold-based opinion dynamics ruled by the quadruple $\langle \theta^-_{\mathcal{D}}, \theta^+_{\mathcal{D}}, \ell_{\mathcal{D}}, \rho_{\mathcal{D}} \rangle$ such that, for every $v \in V$,

$$
\begin{array}{ll}
\theta^+_{\mathcal{D}}(\alpha_v) = \theta^+(v), & \theta^+_{\mathcal{D}}(\beta_v) = \sum_{u \in N(v)} |\pi(u, v)| - \theta^-(v) + 1, \\
\theta^-_{\mathcal{D}}(\alpha_v) = \theta^-(v), & \theta^-_{\mathcal{D}}(\beta_v) = \sum_{u \in N(v)} |\pi(u, v)| - \theta^+(v) + 1, \\
\ell_{\mathcal{D}}(\alpha_v) = \ell_{\mathcal{D}}(\beta_v) = \ell(v) & \rho_{\mathcal{D}}(\alpha_v) = \rho_{\mathcal{D}}(\beta_v) = \rho(v).
\end{array}
$$

It remains to prove that, for any asynchronous threshold-based dynamics $\mathbf{d}$ for G, if $\mathcal{D}(G)$ is in an opinion configuration of $\mathcal{D}(G)$ extending an opinion configuration of G, then the opinion evolution of $\mathcal{D}(G)$ according to the mirror dynamics $\mathbf{d}_{\mathcal{D}}$ of $\mathbf{d}$ simulates the opinion evolution of G according to $\mathbf{d}$. This is the aim of the next lemma.

Lemma 1. *Let $G = (V, E, \pi)$ be an undirected signed weighted graph, and let $\mathcal{D}(G) = (V_{\mathcal{D}}, E_{\mathcal{D}})$ be the display graph of G. For any asynchronous threshold-based opinion dynamics* $\mathbf{d}$ *and for any opinion configuration ω of G at any time step t, if $\omega_{\mathcal{D}}$ is the extension of ω for $\mathcal{D}(G)$ and if* $\mathbf{d}_{\mathcal{D}}$ *is the asynchronous threshold-based opinion dynamics mirroring* $\mathbf{d}$, *then* $\mathbf{d}_{\mathcal{D}}(\mathcal{D}(G), \omega_{\mathcal{D}})$ *is the extension of* $\mathbf{d}(G, \omega)$.

Proof. Within this proof, the sets of neighbors of a generic node v in G and in $\mathcal{D}(G)$ will be denoted, respectively, as $N_G(v)$ and $N_{\mathcal{D}}(v)$; similarly, $P_G(v,\omega)$ is the positive influence at ω acting on node v in G and, since $\pi_{\mathcal{D}}(u,v) > 0$ for any $(u,v) \in E_{\mathcal{D}}$,

$$P_{\mathcal{D}}(v,\omega_{\mathcal{D}}) = \sum_{u \in N_{\mathcal{D}}(v):\omega_{\mathcal{D}}(u)>0} \pi_{\mathcal{D}}(u,v)$$

is the positive influence at $\omega_{\mathcal{D}}$ acting on node v in $\mathcal{D}(G)$. Finally, ω' and $\omega'_{\mathcal{D}}$ are the one-step evolutions of, respectively, ω and $\omega_{\mathcal{D}}$, that is, $\omega' = \mathbf{d}(G,\omega)$, and $\omega'_{\mathcal{D}} = \mathbf{d}_{\mathcal{D}}(\mathcal{D}(G),\omega_{\mathcal{D}})$.

For any $v,z \in V$, by construction, it holds that $z \in N_G(v)$ if and only if either $\alpha_z \in N_{\mathcal{D}}(\alpha_v)$ and $\beta_z \in N_{\mathcal{D}}(\beta_v)$ or $\alpha_z \in N_{\mathcal{D}}(\beta_v)$ and $\beta_z \in N_{\mathcal{D}}(\alpha_v)$; hence,

$$|N_{\mathcal{D}}(\alpha_v)| = |N_{\mathcal{D}}(\beta_v)| = |N_G(v)|.$$

Furthermore, since $\omega_{\mathcal{D}}$ is an extension of ω, for any $z \in N_G(v)$, the following hold:

- If z positively influences v at ω, then either $\omega(z) = 1$ and $\pi(z,v) > 0$ or $\omega(z) = 0$ and $\pi(z,v) < 0$. In the former case, $\omega_{\mathcal{D}}(\alpha_z) = 1$ and $(\alpha_z,\alpha_v) \in E_{\mathcal{D}}$, and symmetrically, $\omega_{\mathcal{D}}(\beta_z) = 0$ and $(\beta_z,\beta_v) \in E_{\mathcal{D}}$ so that α_z positively influences α_v, and β_z does not positively influence β_v. In the latter case, $\omega_{\mathcal{D}}(\alpha_z) = 0$ and $(\alpha_z,\beta_v) \in E_{\mathcal{D}}$, and symmetrically, $\omega_{\mathcal{D}}(\beta_z) = 1$ and $(\beta_z,\alpha_v) \in E_{\mathcal{D}}$ so that α_z does not positively influence β_v, and β_z positively influences α_v. In both cases, the neighbor of α_v in $\{\alpha_z,\beta_z\}$ positively influences α_v, and the neighbor of β_v in $\{\alpha_z,\beta_z\}$ does not positively influence β_v at $\omega_{\mathcal{D}}$.
- If z does not positively influence v at ω, then either $\omega(z) = 0$ and $\pi(z,v) > 0$, or $\omega(z) = 1$ and $\pi(z,v) < 0$. In the former case, $\omega_{\mathcal{D}}(\alpha_z) = 0$ and $(\alpha_z,\alpha_v) \in E_{\mathcal{D}}$, and symmetrically, $\omega_{\mathcal{D}}(\beta_z) = 1$ and $(\beta_z,\beta_v) \in E_{\mathcal{D}}$ so that α_z does not positively influence α_v, and β_z positively influences β_v. In the latter case, $\omega_{\mathcal{D}}(\alpha_z) = 1$ and $(\alpha_z,\beta_v) \in E_{\mathcal{D}}$, and symmetrically, $\omega_{\mathcal{D}}(\beta_z) = 0$ and $(\beta_z,\alpha_v) \in E_{\mathcal{D}}$ so that α_z positively influences β_v, and β_z does not positively influence α_v. In both cases, the neighbor of α_v in $\{\alpha_z,\beta_z\}$ does not positively influence α_v, and the neighbor of β_v in $\{\alpha_z,\beta_z\}$ positively influences β_v to 1 at $\omega_{\mathcal{D}}$.

Hence, $P_{\mathcal{D}}(\alpha_v) = P_G(v)$, and $P_{\mathcal{D}}(\beta_v) = \sum_{z \in N_G(v)} |\pi(z,v)| - P_G(v)$.

As a consequence, since $\omega_{\mathcal{D}}$ extends ω, then, for any $v \in V$, either v is sleeping at time t, as well as α_v and β_v, so that

$$\omega'_{\mathcal{D}}(\alpha_v) = \omega_{\mathcal{D}}(\alpha_v) = \omega(v) = \omega'(v) \text{ and } \omega'_{\mathcal{D}}(\beta_v) = \omega_{\mathcal{D}}(\beta_v) = -\omega(v) = -\omega'(v),$$

or v is listening at time t, as well as α_v and β_v, so that we have the following:

- If $\omega(v) = 1$ and $\omega'(v) = 1$, then $P_G(v) \geq \theta^+(v)$; hence, $P_{\mathcal{D}}(\alpha_v) \geq \theta^+(v) = \theta^+_{\mathcal{D}}(\alpha_v)$, and

$$\begin{aligned} P_{\mathcal{D}}(\beta_v) &= \sum_{z \in N_G(v)} |\pi(z,v)| - P_G(v) \leq \sum_{z \in N_G(v)} |\pi(z,v)| - \theta^+(v) \\ &< \theta^-_{\mathcal{D}}(\beta_v). \end{aligned}$$

  Since $\omega_{\mathcal{D}}(\alpha_v) = 1$ and $\omega_{\mathcal{D}}(\beta_v) = 0$, this implies that $\omega'_{\mathcal{D}}(\alpha_v) = 1$ and $\omega'_{\mathcal{D}}(\beta_v) = 0$.
- Similarly, it can be proved that the following hold:
  - Tf $\omega(v) = 1$ and $\omega'(v) = 0$, then $P_{\mathcal{D}}(\alpha_v) < \theta^+_{\mathcal{D}}(\alpha_v)$ and $P_{\mathcal{D}}(\beta_v) \geq \theta^-_{\mathcal{D}}(\beta_v)$ so that, since $\omega_{\mathcal{D}}(\alpha_v) = 1$ and $\omega_{\mathcal{D}}(\beta_v) = 0$, then $\omega'_{\mathcal{D}}(\alpha_v) = 0$, and $\omega'_{\mathcal{D}}(\beta_v) = 1$;
  - If $\omega(v) = 0$ and $\omega'(v) = 1$, then $P_{\mathcal{D}}(\alpha_v) \geq \theta^-_{\mathcal{D}}(\alpha_v)$ and $P_{\mathcal{D}}(\beta_v) < \theta^+_{\mathcal{D}}(\beta_v)$ so that, since $\omega_{\mathcal{D}}(\alpha_v) = 0$ and $\omega_{\mathcal{D}}(\beta_v) = 1$, then $\omega'_{\mathcal{D}}(\alpha_v) = 1$, and $\omega'_{\mathcal{D}}(\beta_v) = 0$;
  - If $\omega(v) = 0$ and $\omega'(v) = 0$, then $P_{\mathcal{D}}(\alpha_v) < \theta^-_{\mathcal{D}}(\alpha_v)$ and $P_{\mathcal{D}}(\beta_v) \geq \theta^+_{\mathcal{D}}(\beta_v)$ so that, since $\omega_{\mathcal{D}}(\alpha_v) = 0$ and $\omega_{\mathcal{D}}(\beta_v) = 1$, then $\omega'_{\mathcal{D}}(\alpha_v) = 0$, and $\omega'_{\mathcal{D}}(\beta_v) = 1$.

This proves the assertion. □

It is worthwhile to be noticed that the constructions of $\mathcal{D}(G)$ and $\mathbf{d}_{\mathcal{D}}$, as well as the proof of Lemma 1, are almost the same in the directed case.

This completes the proof of Theorem 1.

4. Periodic Asynchronous Threshold-Based Dynamics

The next theorem is the starting contribution of this section. In addition to its consequences, it has some interest on its own.

Theorem 2. *For any directed (unsigned unweighted) graph $G = (V, A)$, for any opinion configuration ω of G and for any synchronous threshold-based opinion dynamics $\mathbf{d}$, there exists an undirected weighted graph $\overline{G} = (\overline{V}, \overline{E}, \overline{\pi})$ with $V \subset \overline{V}$, an opinion configuration $\overline{\omega}$ of $\overline{G}$, and asyncronous threshold-based opinion dynamics $\overline{\mathbf{d}}$ such that, if $\mathcal{E}_{\mathbf{d}}(G, \omega) = \langle \omega_0 = \omega, \omega_1, \omega_2, \ldots \rangle$ and $\mathcal{E}_{\overline{\mathbf{d}}}(\overline{G}, \overline{\omega}) = \langle \overline{\omega}_0 = \overline{\omega}, \overline{\omega}_1, \overline{\omega}_2, \ldots \rangle$, it holds that*

$$\omega_t(u) = \overline{\omega}_{4t}(u) \qquad \text{for all } u \in V, \text{ and } t \geq 0.$$

Proof. The undirected graph $\overline{G} = (\overline{V}, \overline{E}, \overline{\pi})$ is derived from $G = (V, A)$ as follows (see Figure 2):

- $\overline{V} = V \cup \{u_v^1, u_v^2 : (u, v) \in A\}$, that is, $\overline{V}$ is obtained by adding to V a pair of new nodes for every arc in A;
- $\overline{E} = \{(u, u_v^1), (u_v^1, u_v^2), (u_v^2, v) : (u, v) \in A\}$, that is, $\overline{G}$ contains a chain of length two for every arc in A;
- $\pi(u, u_v^1) = 3$, $\pi(u_v^1, u_v^2) = 2$, and $\pi(u_v^2, v) = 1$ for every arc $(u, v) \in A$.

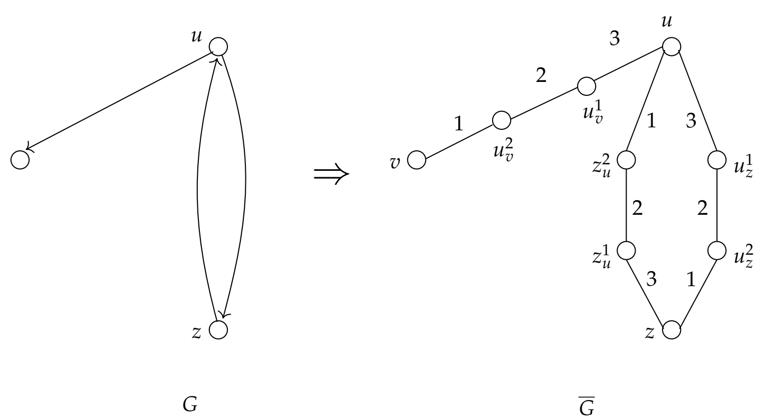

Figure 2. A directed unsigned unweighted graph G and its corresponding undirected unsigned weighted graph $\overline{G}$.

Let $\langle \theta^-, \theta^+ \rangle$ be the pair of functions ruling $\mathbf{d}$. The quadruple $\langle \overline{\theta}^-, \overline{\theta}^+, \overline{\ell}, \overline{\rho} \rangle$ ruling $\overline{\mathbf{d}}$ is defined in the following:

- For each $u \in V$, we have the following: $\overline{\theta}^-(u) = \theta^-(u)$, $\overline{\theta}^+(u) = \theta^+(u)$, $\overline{\ell}(u) = 3$, and $\overline{\rho}(u) = 3$.
- For each $(u, v) \in A$, we have the following: $\overline{\theta}^-(u_v^1) = \pi(u, u_v^1) = 3$, $\overline{\theta}^+(u_v^1) = \pi(u, u_v^1) + \pi(u_v^1, u_v^2) + 1 = 6$, $\overline{\ell}(u_v^1) = 1$, and $\overline{\rho}(u_v^1) = 1$.
- For each $(u, v) \in A$, we have the following: $\overline{\theta}^-(u_v^2) = \pi(u_v^1, u_v^2) = 2$, $\overline{\theta}^+(u_v^2) = \pi(u_v^1, u_v^2) + \pi(u_v^2, v) + 1 = 4$, $\overline{\ell}(u_v^2) = 2$, and $\overline{\rho}(u_v^2) = 1$.

Finally, the opinion configuration $\overline{\omega}$ of $\overline{G}$ is derived from ω as follows:

- For each $u \in V$, $\overline{\omega}(u) = \omega(u)$;

- For each $(u, v) \in A$, $\overline{\omega}(u_v^1) = \overline{\omega}(u_v^2) = 0$.

The assertion is proved by induction on t together with the claim that $\overline{\omega}_{4t}(u_v^1) = \overline{\omega}_{4t}(u_v^2) = 0$ for every $(u, v) \in A$, and $t \geq 0$.

By construction, the $t = 0$ assertion and claim both hold.

Assume that $\omega_i(u) = \overline{\omega}_{4i}(u)$ and $\overline{\omega}_{4i}(u_v^1) = \overline{\omega}_{4i}(u_v^2) = 0$ for all $u \in V$, that $(u, v) \in A$, and that $i \leq t$.

Let $(u, v) \in A$; by the inductive hypothesis, $\overline{\omega}_{4t}(u) = \omega(u)$, and $\overline{\omega}_{4t}(u_v^1) = \overline{\omega}_{4t}(u_v^2) = 0$. Hence, we have the following:

- Step $4t + 1$ is defined as follows:
  - Since u is sleeping at time-step $4t$, then $\overline{\omega}_{4t+1}(u) = \overline{\omega}_{4t}(u)$;
  - Since u_v^1 is listening at time-step $4t$, since its neighbors in $\overline{G}$ are u and u_v^2, since $\overline{\omega}_{4t}(u_v^1) = 0$, and since $\overline{\theta}^-(u_v^1) = \pi(u, u_v^1) > \pi(u_v^1, u_v^2)$, then $\overline{\omega}_{4t+1}(u_v^1) = \overline{\omega}_{4t}(u)$:
  - Since u_v^2 is sleeping at time-step $4t$, then $\overline{\omega}_{4t+1}(u_v^2) = 0$;
- Step $4t + 2$ is defined as follows:
  - Since u is sleeping at time-step $4t + 1$, then $\overline{\omega}_{4t+2}(u) = \overline{\omega}_{4t}(u)$;
  - Since u_v^1 is sleeping at time-step $4t + 1$, then $\overline{\omega}_{4t+2}(u_v^1) = \overline{\omega}_{4t}(u)$;
  - Since u_v^2 is listening at time-step $4t + 1$, since its neighbors in $\overline{G}$ are u_v^1 and v, since $\overline{\omega}_{4t}(u_v^2) = 0$, and since $\overline{\theta}^-(u_v^2) = \pi(u_v^1, u_v^2) > \pi(u_v^2, v)$, then $\overline{\omega}_{4t+2}(u_v^2) = \overline{\omega}_{4t}(u)$;
- Step $4t + 3$ is defined as follows:
  - Since u is sleeping at time-step $4t + 2$, then $\overline{\omega}_{4t+3}(u) = \overline{\omega}_{4t}(u)$;
  - Since u_v^1 is listening at time-step $4t + 2$ and since $\overline{\theta}^-(u_v^1) = \overline{\pi}(u, u_v^1)$ and $\overline{\theta}^+(u_v^1) > \overline{\pi}(u, u_v^1) + \overline{\pi}(u_v^1, u_v^2)$, then $\overline{\omega}_{4t+3}(u_v^1) = 0$ whatever its neighbors' opinions are;
  - Since u_v^2 is sleeping at time-step $4t + 2$, then $\overline{\omega}_{4t+3}(u_v^2) = \overline{\omega}_{4t}(u)$;
- Step $4t + 4 = 4(t + 1)$ is defined as follows:
  - Since u is listening at time-step $4t + 3$ and since, by the previous item, for each $(z, u) \in A$, $\overline{\omega}_{4t+3}(z_u^2) = \overline{\omega}_{4t}(z) = \omega_t(z)$, since $\overline{\omega}_{4t}(u) = \omega_t(u)$ and since $\overline{\theta}^-(u) = \theta^-(u)$ and $\overline{\theta}^+(u) = \theta^+(u)$, then $\overline{\omega}_{4t+4}(u) = \omega_{t+1}(u)$;
  - Since u_v^1 is sleeping at time-step $4t + 3$, then $\overline{\omega}_{4t+a}(u_v^1) = 0$;
  - Since u_v^2 is listening at time-step $4t + 3$ and since $\overline{\theta}^-(u_v^2) = \overline{\pi}(u_v 1, u_v^2)$ and $\overline{\theta}^+(u_v^2) > \overline{\pi}(u_v^1, u_v^2) + \overline{\pi}(u_v^2, v)$, then $\overline{\omega}_{4t+4}(u_v^2) = 0$ whatever its neighbors' opinions are.

This completes the induction and the proof. □

Deciding if a graph G is in a given opinion configuration ω and is evolving according to the deterministic majority rule $\mathbf{d}_M$ that will ever reach an equilibrium configuration has been proved to be a PSPACE-complete problem in [26] when G is directed, unsigned, and unweighted. Since the deterministic majority rule entails synchronous (local) threshold-based opinion dynamics such that $\theta^-(u) = \lfloor \frac{|N(u)|}{2} \rfloor + 1$ and $\theta^+(u) = \lceil \frac{|N(u)|}{2} \rceil$ for every node u in G, it can be easily verified that $\overline{G}$, $\overline{\omega}$, and $\overline{\mathbf{d}}$ are computable in polynomial time in the size of G. Hence, by noticing that the values of the functions $\overline{\pi}$, $\overline{\theta}^-$, $\overline{\theta}^+$, $\overline{\ell}$, and $\overline{\rho}$ in the proof of Theorem 2 are constant, from Theorem 2, the next corollary follows.

Corollary 1. *Deciding if an undirected unsigned weighted graph G in a given opinion configuration ω and if it is evolving according to periodic asynchronous threshold-based opinion dynamics* $\mathbf{d}$ *that will ever reach an equilibrium configuration is a strong PSPACE-complete problem.*

Recall that in order to decide the just introduced problem, it is sufficient to let the graph evolve until an equilibrium conguration is met or it is possible to deduce that it will never be met. As a consequence, the following corollary holds.

Corollary 2. *Unless $P = PSPACE$, there does not exist any polynomial $\mathcal{P}$ such that, for every undirected unsigned weighted graph $G = (V, E, \pi)$, for every opinion configuration $\omega : V ß\{0,1\}$ of G, and for all periodic asynchronous threshold-based opinion dynamics* **d** *ruled by the quadruple $\langle \theta^-, \theta^+, \ell, \rho \rangle$, it holds that $|\mathcal{O}_\mathbf{d}(G, \omega)| \leq \mathcal{P}(|G|, M)$, where M is the maximum value taken by π, θ^-, θ^+, ℓ, and ρ.*

5. Synchronous Threshold-Based Dynamics

The last section of this paper is devoted to proving that there exists a polynomial $\mathcal{P}$ such that, for any synchronous threshold-based dynamics **d** and for any undirected weighted graph G, $|\mathcal{O}_\mathbf{d}(G, \omega)| \leq \mathcal{P}(|G|, M_\pi)$, where M_π is the sum of the edge weights of G. After Theorem 1, is is sufficient to consider unsigned weighted graphs, that is, all edge weights are positive. The goal will be accomplished by exploiting the technique introduced in [24] for the Underpopulation rule suitably adapted to general threshold-based rules and weighted graphs.

Let $G = (V, E, \pi)$ be an undirected unsigned weighted graph, with $\pi : E \to \mathbb{N}$, let $\omega : V \to \mathbb{N}$ be an opinion configuration of G, and let **d** be synchronous threshold-based dynamics. Since **d** is synchronous, it is not needed to specify the starting time of the **d** evolution of G at ω. Furthermore, the **d** evolution set of G at ω is periodic. Denote as $T = |\mathcal{O}_\mathbf{d}(G, \omega)|$, that is, $\mathcal{O}_\mathbf{d}(G, \omega) = \langle \omega_0 = \omega, \omega_1, \ldots, \omega_{T-1} \rangle$. For any $v \in V$, the string $\omega_0(v)\omega_1(v)\ldots\omega_{T-1}(v)$ is called the *history* of v.

A sequence $y = y_0 y_2 \ldots y_{h-1} \in \{0,1,?\}^h$ will denote, in short, the set of sequences of length h in which every ? is replaced by 0 and by 1 so that, for instance, 0?1? is the set of sequences $\{0010, 0011, 0110, 0111\}$. For $y = y_0 y_1 \ldots y_{h-1} \in \{0,1,?\}^h$ and $z = z_0 z_1 \ldots z_{h-1} \in \{0,1\}^h$, y *matches* z (in short, $y \approx z$) if $y_i = z_i$ whenever $y_i \in \{0,1\}$ (no matter what happens if $y_i = ?$) for every $i = 0, \ldots, h-1$.

For any $v \in V$ and $y \in \{0,1,?\}^h$ with $h \leq T$, and for any $i = 0, \ldots, T - |y|$, we set

$$[y, v]_i = \begin{cases} 1 & \text{if } \omega_i(v)\omega_{i+2}(v)\ldots\omega_{i+|y|-1}(v) \approx y, \\ 0 & \text{otherwise} \end{cases}$$

and we denote as $[y, v]$ the number of matches of y inside the history of v, that is,

$$[y, v] = \sum_{i=0}^{T-|y|} [y, v]_i$$

and as $[y] = \sum_{v \in V} [y, v]$ the total number of matches of y inside the histories of all nodes in V.

The aim of the next lemma is proving an upper bound on the size of $|\mathcal{O}_\mathbf{d}(G, \omega)|$ for any synchronous dynamics, and it has already been proved in [24] for the Underpopulation rule and in [26] for local threshold-based dynamics. Although the proof for the general case is almost the same as that for the Underpopulation rule and the local threshold-based dynamics, for the sake of completeness, it is repeated here.

Lemma 2. *If* **d** *defines synchronous opinion dynamics, then, for any undirected unsigned weighted graph G and for any opinion configuration ω of G,*

$$|\mathcal{O}_\mathbf{d}(G, \omega)| \leq [110] + [100] + [011] + [001] + 2.$$

Proof. Let $\mathcal{O}_{\mathbf{d}}(G,\omega) = \{\omega_0, \omega_1, \ldots, \omega_{T-1}\}$. Since **d** defines synchronous dynamics, then, for every $0 \leq i, j \leq T-1$ with $i \neq j$, $\omega_i \neq \omega_j$, there exists a node $u_{i,j}$ in G such that $\omega_i(u_{i,j}) \neq \omega_j(u_{i,j})$. By choosing $j = i+2$ we obtain that, for any $0 \leq i \leq T-3$, the string

$$\omega_t(u_{i,i+2})\omega_{t+1}(u_{i,i+2})\omega_{t+2}(u_{i,i+2})$$

belongs to $\{001, 011, 100, 110\}$. This proves that $[001] + [011] + [100] + [110] \geq T-2$. $\square$

It remains to prove an upper bound on $[001] + [100] + [011] + [001]$ when the synchronous dynamics are threshold-based. This is the goal of the next lemma.

Lemma 3. *If* **d** *defines threshold-based opinion dynamics, then, for any undirected unsigned weighted graph* $G = (V, E, \pi)$ *and for any opinion configuration* ω *of* G *such that* $T = |\mathcal{O}_{\mathbf{d}}(G,\omega)| \geq 3$, *it holds that*

$$[110] + [100] + [011] + [001] \leq 4\Pi + 8|V|\Pi + 8|V||E| + 2|V|,$$

where Π *is the sum of edge weights in* G.

Proof. Since $[00] = [00]_0 + [000] + [100]$ and $[00] = [00]_{T-2} + [000] + [001]$, and since $[11] = [11]_0 + [011] + [111]$ and $[11] = [11]_{T-2} + [110] + [111]$, it follows that

$$|[100] - [001]| = |[00]_0 - [00]_{T-2}| \leq |V| \quad \text{and} \quad |[011] - [110]| \leq |V|. \tag{1}$$

Hence,

$$[001] + [011] \;\leq\; [100] + |V| + [110] + |V| \;=\; [1?0] + 2|V|. \tag{2}$$

Bounding $[1?0]$ requires taking into account the edge weights and introducing some more notation. Let $y, z \in \{0, 1, ?\}^*$ be such that $|y| = |z| \leq T$. Then, for any $u \in V$, we set

$$\langle y, z, u\rangle_i = [y, u]_i \sum_{v \in N(u)} \pi(u, v)[z, v]_i,$$

$$\langle y, z, u\rangle = \sum_{i=0}^{T-|y|} \langle y, z, u\rangle_i, \qquad \langle y, z\rangle_i = \sum_{u \in V} \langle y, z, u\rangle_i$$

and finally,

$$\langle y, z\rangle = \sum_{u \in V} \langle y, z, u\rangle = \sum_{i=0}^{T-|y|} \langle y, z\rangle_i$$

Notice that, for every $i = 0, \ldots, T-|y|$, it holds that

$$0 \leq \langle y, z\rangle_i \leq 2 \sum_{(u,v) \in E} \pi(u, v) = 2\Pi \tag{3}$$

(with the equality occurring when $y = z$). Furthermore, by the edge weights simmetry, for every $y, z \in \{0, 1, ?\}^*$ such that $|y| = |z| \leq T$, it holds that

$$\begin{aligned}
\langle y, z\rangle &= \textstyle\sum_{i=0}^{T-|y|} \sum_{u \in V} [y, u]_i \sum_{v \in N(u)} \pi(u, v)[z, v]_i \\
&= \textstyle\sum_{i=0}^{T-|y|} \sum_{u \in V} \sum_{v \in N(u)} \pi(u, v)[z, v]_i [y, u]_i \\
&= \textstyle\sum_{i=0}^{T-|y|} \sum_{v \in V} \sum_{u \in N(v)} \pi(v, u)[z, v]_i [y, u]_i \;=\; \langle z, y\rangle.
\end{aligned}$$

Hence, in particular, $\langle ?1, 1?\rangle = \langle 1?, ?1\rangle$ so that, since

$$\langle ?1, 1?\rangle = \langle ?1, 1?\rangle_0 + \langle 001, ?1?\rangle + \langle 011, ?1?\rangle + \langle 1?1, ?1?\rangle$$

and

$$\langle 1?, ?1\rangle = \langle 1?, ?1\rangle_{T-2} + \langle 100, ?1?\rangle + \langle 110, ?1?\rangle + \langle 1?1, ?1?\rangle,$$

we obtain

$$|\langle 001, ?1?\rangle + \langle 011, ?1?\rangle - \langle 100, ?1?\rangle - \langle 110, ?1?\rangle| = |\langle ?1, 1?\rangle_0 - \langle 1?, ?1\rangle_{T-2}|$$

and, by (3),

$$-2\Pi \le \langle 001, ?1?\rangle + \langle 011, ?1?\rangle - \langle 100, ?1?\rangle - \langle 110, ?1?\rangle \le 2\Pi. \quad (4)$$

Since the state of a node u changes from 0 to 1 if and only if the weighted number of its neighbors in state 1 is at least $\theta^-(u)$, and since a node u in state 0 remains in state 0 if and only if the weighted number of its neighbors in state 1 is less than $\theta^-(u)$, then, for $u \in V$,

$$\langle 001, ?1?, u\rangle \ge \theta^-(u)[001, u] \text{ and } \langle 100, ?1?, u\rangle \le \big(\theta^-(u) - 1\big)[100, u]$$

and, similarly,

$$\langle 011, ?1?, u\rangle \ge \theta^+(u)[011, u] \text{ and } \langle 110, ?1?, u\rangle \le \big(\theta^+(u) - 1\big)[110, u].$$

Hence,

$$\begin{aligned}
&\langle 001, ?1?\rangle - \langle 100, ?1?\rangle + \langle 011, ?1?\rangle - \langle 110, ?1?\rangle = \\
&\sum_{u\in V}\{\langle 001, ?1?, u\rangle - \langle 100, ?1?, u\rangle + \langle 011, ?1?, u\rangle - \langle 110, ?1?, u\rangle\} \ge \\
&\sum_{u\in V}\{\theta^-(u)[001, u] - \big(\theta^-(u) - 1\big)[100, u] + \theta^+(u)[011, u] - \big(\theta^+(u) - 1\big)[110, u]\} = \\
&\sum_{u\in V}\{\theta^-(u)([001, u] - [100, u]) + [100, u] + \theta^+(u)([011, u] - [110, u]) + [110, u]\} = \\
&[100] + [110] + \sum_{u\in V}\{\theta^-(u)([001, u] - [100, u]) + \theta^+(u)([011, u] - [110, u])\}.
\end{aligned}$$

And, by (4), this implies that

$$[100] + [110] \le 2\Pi + \sum_{u\in V}\{\theta^-(u)([100, u] - [001, u]) + \theta^+(u)([110, u] - [011, u]\}.$$

By Equations (1), it holds that $|[100, u] - [001, u]| \le |[100] - [001]| \le |V|$ and $|[110, u] - [011, u]| \le |[110] - [011]| \le |V|$ for any $u \in V$. Hence, by recalling that $\theta^+(u) \le \sum_{v\in N(u)} \pi(u, v) + 1$ and $\theta^-(u) \le \sum_{v\in N(u)} \pi(u, v) + 1$,

$$\begin{aligned}
[100] + [110] &\le 2\Pi + |V| \sum_{u\in V}\{\theta^-(u) + \theta^+(u)\} \\
&\le 2\Pi + |V| \sum_{u\in V}\sum_{v\in N(u)} 2[\pi(u, v) + 1] \le 2\Pi + 4|V|\Pi + 4|V||E|.
\end{aligned}$$

Finally, by (2), the assertion follows. □

The next theorem then follows from Lemmas 2 and 3.

Theorem 3. *For any synchronous threshold-based opinion dynamics* **d**, *for any unsigned weighted undirected graph* $G = (V, E, \pi)$, *and for any opinion configuration* ω *of* G, $|\mathcal{O}_{\mathbf{d}}(G, \omega)| \le 4\Pi + 8|V|\Pi + 8|V||E| + 2|V| + 2$.

As a consequence of the above theorem, by Theorem 1, the following corollary holds.

Corollary 3. *For any synchronous threshold-based opinion dynamics* **d**, *for any signed weighted undirected graph* $G = (V, E, \pi)$, *and for any opinion configuration* ω *of* G, $|\mathcal{O}_{\mathbf{d}}(G, \omega)| \le 2(4\Pi + 8|V|\Pi + 8|V||E| + 2|V| + 2)$.

6. Conclusions

In this paper, deterministic threshold-based opinion dynamics for signed weighted graphs have been introduced. They look to be the most general dynamics such that the decision of changing state at the next time step is taken accordingly to the state of the other nodes: in fact, by suitably adjusting the threshold functions and the edge weights, all neighborhood configuration and influence patterns may be represented (actually, it could be sufficient to only consider complete graphs with, eventually, weight edges of 0).

A deterministic model for asynchronism has also been defined, aiming to be the deterministic counterpart of the random asynchronous models widely studied in the literature and being inspired by the periodic refractory behavior of biological entities (organs, cells, neurons, etc.).

Within this framework, the following have been proved:

(a) The presence of negative relations in a graph G does not really matter in the study of the orbit of a given opinion configuration according to any periodic asynchronous threshold-based dynamics **d** in that the evolution of the opinions in G according to **d** can be simulated by the evolution of the opinions in a related nonsigned graph $\mathcal{D}(G)$ (whose size is polynomially related to that of G) according to related periodic asynchronous threshold-based dynamics $\mathbf{d}'$. This closes an open problem posed in [26].

(b) Unless P = PSPACE, the size of the orbits of an undirected signed weighted graph according to periodic asynchronous threshold-based dynamics cannot be upper-bounded by a polynomial in the size of the graph and in the sum of edge weights even when the edge weights are constant. As a consequence, possible asynchronous dynamics definitions generalizing the periodic threshold-based ones look not so meaningful if the interest is in the orbit size (or, equivalently, in the $\mathbb{P}$-reachability questions).

(c) The size of any orbit of an undirected signed weighted graph according to any synchronous threshold-based dynamics is upper-bounded by a polynomial in the size of the graph and in the sum of edge weights. By observing that the Underpopulation rule introduced in [24] is defined for unweighted graphs only, a consequence of the combination of (a) and (c) is that the extension of the Underpopulation rule to signed graph yields orbits whose sizes are upper-bounded by a polynomial in the size of the graph, thus closing an open problem in [25].

Since the study of the orbit sizes in directed graphs performed in [25,26] has ended in showing overpolynomial bounds (if P $\neq$ PSPACE), the above discussion shows that almost all has been said on this topic. An intriguing aspect to be investigated concerns the issue of pseudo-polynomiality versus polynomiality in the upper bound on the orbit size, that is, whether an upper bound which is polynomial in the size of the graph only (instead of in the size of the graph and in the sum of edge weights) exists or not.

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