Annals of Operations Research
https://doi.org/10.1007/s10479-025-06864-6

ORIGINAL RESEARCH

An analysis of short selling restrictions in factor model based portfolio immunization

F. Manzini[1] · **A. Ramponi**[1] · **S. Scarlatti**[1]

Received: 23 June 2023 / Accepted: 18 September 2025

Abstract

In this paper we consider the problem of hedging a portfolio of ZC bonds aiming to get a return close to a targeted one relative to a liability. For the time evolution of the YTM interest rate curve we assume a dynamic factor model. Moreover we adopt the criterium of minimum variance of the spread between the portfolio and liability returns under a generalized duration matching constraint in order to achieve an optimal portfolio choice, as proposed in Borup et al. (in: Immunization in the Treasury market with consistent term structure dynamics, SSRN. 2022. https://doi.org/10.2139/ssrn.4164195). We prove that, under non negativity constraints on the portfolio weights, the previous selection procedure might be unfeasible. More precisely, our unfeasibility result holds under a set of minimal hypothesis on the factor loadings which we show are verified by the Nelson–Siegel family. We also discuss an example for which the optimal portfolios can be easily determined, the shown instance clearly highlights how the shape of the factor loadings may influence the possibility of achieving an optimal hedge without short-selling.

Keywords Portfolio immunization · Factor models · Nelson–Siegel family · Generalized duration

JEL Classification C61 · G11 · G12

✉ S. Scarlatti
sergio.scarlatti@uniroma2.it

F. Manzini
francesco.manzini@uniroma2.it

A. Ramponi
alessandro.ramponi@uniroma2.it

[1] Department of Economics and Finance, University of Rome - Tor Vergata, Rome, Italy

1 Introduction

Immunization of a bond portfolio, which aims to hedge a single liability having time to maturity $\tau^* > 0$ or more liabilities having different maturities, has a long history. The foundational theory is due to Reddington (1952) and Fisher and Weil (1971). In the case of a single liability it was proven by Fisher and Weil that if the portfolio value matches the present value of the liability and the portfolio duration equalizes τ^* then immunization against parallel shifts of the interest rate term structure is achieved. To preserve it, the strategy must be dynamically updated by rebalancing the portfolio weights when a duration mismatch is detected. Subsequent improvements pointed to enlarge the set of the term structure perturbations which might be controlled by methods or techniques similar to the foundational ones, leading to immunization. These include relevant contributions by Shiu (1987, 1988, 1990), by Montrucchio and Peccati (1991) and by Uberti (1997). Although these developments were mathematically rigorous and elegant, they had limited impact on the management practice, notwithstanding with the fact that, by arbitrage arguments, it is not possible to preserve the portfolio value w.r.t. arbitrary perturbations of the term structure. However, for arbitrary additive shocks of the instantaneous forward-rate with bounded first derivative and under the duration matching condition, Fong and Vasicek (1984) found out a lower bound for the change in value of the portfolio. The bound was in terms of a new risk measure denoted by M^2, and the portfolio risk, that is the wideness of a possible shortfall, could be kept down by designing a portfolio minimizing M^2, see Fong and Fabozzi (1985) for further considerations. Later on Nawalkha and Chambers (1996) improved upon this viewpoint by introducing an alternative measure known as M-absolute. Subsequently, to efficiently hedge larger classes of variations in the term structure and the consequent changes in the bond prices, the previous approach was generalized by constructing a vector M of risk measures, see Nawalkha and Chambers (1997). The components of M were obtained by considering higher order Taylor polynomials of the function of the bond portfolio return. Antecedent to these works immunization based on a vector D of multiple duration measures was considered by Chambers et al. (1988) and Prisman and Shores (1988). We also want to mention that immunization is a relevant aspect of modern bond portfolio management, see e.g. Ortobelli et al. (2018) for results applied to US bond markets, and the management of pension funds, see e.g. Simoes et al. (2021), an important class of funds which require, in greater generality, the full specification of a multistage ALM model, Kopa et al. (2023) and ref.s therein.

We now turn our attention to works looking to variations in the term structure with the lens of the real-world, that is more empirically rather than theoretically. Among these, relevant contributions were given by Garbade (1986); Litterman and Scheinkman (1991); D'Ecclesia and Zenios (1994); Barber and Copper (1996); Willner (1996); Bliss (1997). These papers relied on PCA and factor models to describe variations in the term structure noticing that a limited number of factors suffice to describe a large portion of the changes. Nelson and Siegel (1987), shortly NS model, a very parsimonious functional form of modeling instantaneous forward-rates of return, and the subsequent Svensson extension Svensson (1994), shortly NSS model, were just tailored for this type of analysis.Nowadays these models represent flexible statistical tools which are used to fit real market data on a daily basis by important financial institutions, see e.g.Nymand-Andersen (2018); Banholzer et al. (2024). The previous stream of research had a direct impact on practical bond portfolio management, hinging on the belief that the more accurate the forecast of the dynamics of the forward-rates the more efficient the immunization procedure. In this work we consider a recent model introduced by Borup et al. (2022) which falls within the previous class of empirically positioned works. The optimality criterium adopted by the authors, to hedge a single liability, consisted in choosing portfolio weights which minimize the variance of the spread between the bond portfolio return and the liability return, seen as target, under a generalized duration matching condition. In this framework a generalized duration, a terminology first introduced by Diebold and Li (2006), is simply the sensitivity of the bond portfolio return w.r.t. one of the risk factors. The analysis of Borup et al. (2022) goes deeply on the empirical side exploiting the question of dynamic consistency among the family of selected curves and the dynamic term structure model (DTSM), investigated by Bjork and Bjork and Christensen (1999) for the first time, and the relative implications on the multi-period performance of the hedging strategy. The present paper does not touch these important aspects and has a much more narrow and different focus. Indeed, we only consider a one-period hedging model and prove that the result related to the existence of an optimal portfolio choice (Thm.1 in Borup et al. 2022) fails to be extended to a market with short-selling restrictions. In its essence this is a consequence of the Farkas Lemma (see e.g. Bachem and Kern (1992)) and of the shape of the curves which act as loadings for the random factors. The paper is organized as follows. In Sect. 2 we recall the general factor model considered by Borup et al. (2022). In Sect. 3 we formulate the one-period hedging problem determining explicitly the weights of the optimal portfolio under no short selling restrictions in a slightly more direct way w.r.t. Borup et al. (2022). In Sect. 4 we present a general formulation of our unfeasibility result under short selling restrictions and give a proof of it. In Sect. 5 we apply the result to the model previously presented, while in Sect. 6 we discuss an explicit example which highlights in a clear way how the shape of the factor loadings directly influence the possibility of having non negative portfolio weights. Conclusions are summarized at the end of the paper.

2 The factor model

Factor modeling is becoming pervasive in economics and finance, see Giglio et al. (2022) for a recent survey; moreover, the approach has already been applied to bond portfolio optimization by Caldeira et al. (2016), where results have been benchmarked against several bond portfolio strategies widely employed in bond desks.To introduce it for our purposes we assume that there are m ZC-bonds having different time to maturities $0 < \tau_1 < \ldots < \tau_m$ and that their log-prices at time t are given by

$$\log p_t(\tau_i) = -\tau_i y_t(\tau_i), \ \forall i = 1, \ldots, m, \tag{1}$$

so that $y_t \equiv (y_t(\tau_1), \ldots, y_t(\tau_m))' \in \mathbb{R}^m$ denotes the column vector of the corresponding yields to maturity. Given these instruments the portfolio manager must target the return of a single liability having time to maturity τ^*, with $\tau^* \neq \tau_i$ for $i = 1, \ldots, m$, and present value $p_t(\tau^*) = e^{-\tau^* y_t^*}$, with $y_t^* \in \mathbb{R}$.

Furthermore, it is assumed that the yields y_t and y_t^* both originate from an underlying dynamical k-factor linear model, with $k < m$. Henceforth for the yields we have

$$y_t = \mu + B\beta_t + \epsilon_t, \tag{2}$$

and

$$y_t^* = \mu^* + b^*\beta_t + \epsilon_t^*, \tag{3}$$

with $\mu \in \mathbb{R}^m$, $\mu^* \in \mathbb{R}$, B an $m \times k$ matrix of factor loadings of rank k, $b^* \in \mathbb{R}^k$, β_t an $\mathbb{R}^k$ -valued random vector, and $(\epsilon_t, \epsilon_t^*)$ an $\mathbb{R}^{m+1}$-valued random vector having uncorrelated components and independent on β_t. We also set the notation $\Sigma = cov(\beta_t, \beta_t)$ for the $k \times k$ variance-covariance matrix of the factors and $\Psi = cov(\epsilon_t, \epsilon_t)$ for the $m \times m$ variance-covariance diagonal matrix of the idiosyncratic shocks.

The log-return of the i-th ZC-bond at time $t+1$ is given by

$$r_{t+1,i} = \log \frac{p_{t+1}(\tau_i)}{p_t(\tau_i)} = -\tau_i \Delta y_{t+1}(\tau_i),$$

for $i = 1, \dots, m$, with $\Delta y_{t+1}(\tau_i) \equiv y_{t+1}(\tau_i) - y_t(\tau_i)$, or in vector notation

$$r_{t+1} = -\mathcal{T} \Delta y_{t+1}, \tag{4}$$

with $\mathcal{T} \equiv diag(\tau_1, \dots, \tau_m)$. Henceforth, inserting (2) into (4) leads to

$$r_{t+1} = -\mathcal{T}(B\Delta\beta_{t+1} + \Delta\epsilon_{t+1}). \tag{5}$$

We remark that the matrix $\mathcal{T}B$ appearing in (5), that is maturities times factor loadings, is such that its rows are the generalized durations of each one of the m ZC-bonds. A classical example of the present framework is the case $k = 3$ and the choice of the Nelson–Siegel family (NS) to fit the term structure of forward yields in a parsimonious way. By choosing the NS family in the representative form considered by Diebold and Li (2006), the components of the vector model (2) are given by

$$y_{t,i} = \mu_i + \underbrace{(1, \frac{1 - e^{-a\tau_i}}{a\tau_i}, \frac{1 - e^{-a\tau_i}}{a\tau_i} - e^{-a\tau_i})}_{(f_1(\tau_i), f_2(\tau_i), f_3(\tau_i))} \begin{pmatrix} \beta_{t,1} \\ \beta_{t,2} \\ \beta_{t,3} \end{pmatrix} + \epsilon_{t,i}, \tag{6}$$

for $i = 1, \dots, m$ and f_1, f_2, f_3 are the "factor loadings". In this case the three generalized durations associated in (6) with level, slope and curvature risk factors $(\beta_{t,1}, \beta_{t,2}, \beta_{t,3})$, are simply given by

$$D(t) = (D^1(t), D^2(t), D^3(t)) \equiv (t, \frac{1 - e^{-at}}{a}, \frac{1 - e^{-at}}{a} - te^{-at}), \tag{7}$$

see Fig. 1 that is for the *m* ZC-bonds considered we have the $m \times 3$ matrix

$$\mathcal{T}B = [D_i^1, D_i^2, D_i^3]_{i=1,\dots,m} \equiv [D^1(\tau_i), D^2(\tau_i), D^3(\tau_i)]_{i=1,\dots,m}. \tag{8}$$

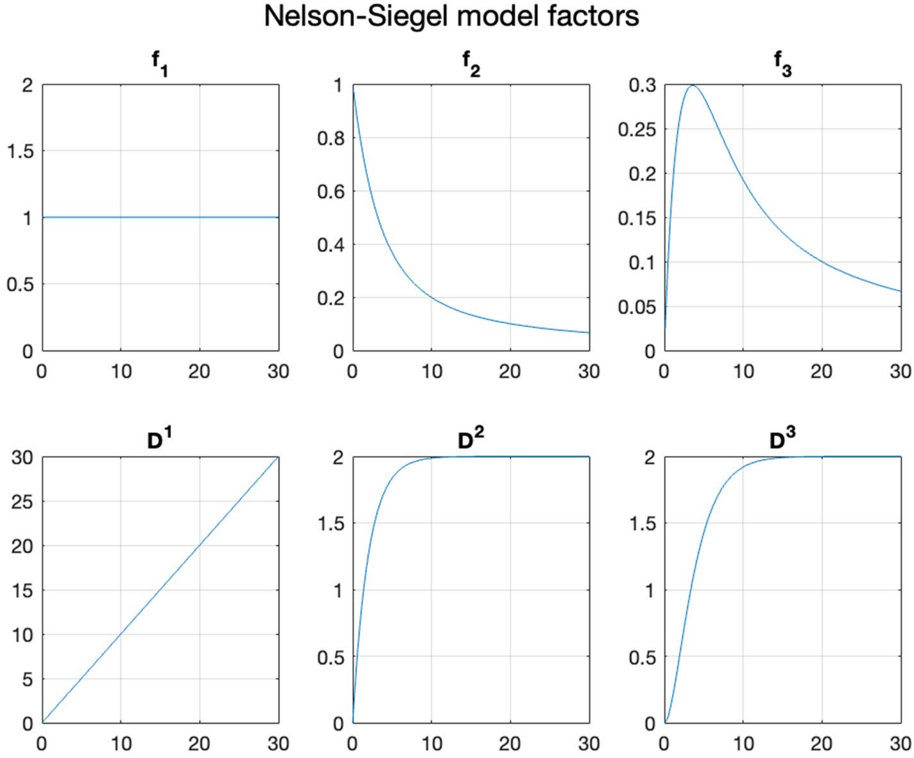

Fig. 1 Factor loadings of the Nelson-Siegel model f_i (upper row), and the corresponding generalized durations D^i (lower row), $i = 1, 2, 3$

We refer to Diebold and Li (2006) for a deep econometrical analysis of this approach, inclusive of the estimation procedures for the factors and their forecasting.

Furthermore, we wish to point out that many research papers in the financial literature fit the NS or NNS families to real fixed income market data and exploit the Diebold-Li approach to model yields dynamic evolution in order to improve the forecasting accuracy of forward rates. Among recent ones Ahi et al. (2018) discuss market data relative to U.S. Treasuries and to government bonds of important emerging economies, such as Brasil, Mexico and Turkey, which are then used to estimate the parameters of the NSS family by global optimization methods such as particle swarm algorithm; in Sojoudi et al. (2025) US municipal bond market data, including bonds of brown and green types, are taken under scrutiny showing that the NS model, distinguishing between level, slope, and curvature factors, allows to measure how much the greenium varies in relative value across maturities; in Umar et al. (2022), the authors show how the adoption of the dynamic Deibold-Li version of the NS model leads to the detection of diverse features of the interrelation of shocks in oil prices and the yield-curve parameters. From the portfolio immunization viewpoint to implement reliable hedging strategies it is important to achieve a high forecasting precision of forward rates , the previous cited stream of research supporting that accuracy.

3 The hedging problem

In this section we consider the one-period hedging problem. That is, we would like to invest at $t = 0$ in the ZC-bonds at disposal by choosing a portfolio, having weights $x = (x_1, \ldots, x_m)'$, in such a way that the spread among its return at $t = 1$, given by $r_1(x) = \sum_{i=1}^m x_i r_{1,i}$, and the return r_1^* would achieve the minimum variance, under a duration matching condition. For the returns we have the following formulas

$$r_1(x) = -x'\mathcal{T}(B\Delta\beta_1 + \Delta\epsilon_1) \tag{9}$$

and

$$r_1^* = -\tau^*(b^*\Delta\beta_1 + \Delta\epsilon_1^*). \tag{10}$$

Denoting by $\mathcal{F}_0$ the sigma-algebra of the informations available at time $t = 0$ it follows that

$$var(r_1^* - r_1(x)|\mathcal{F}_0) = var((x'\mathcal{T}B - \tau^* b^*)\beta_1 - \tau^*\epsilon_1^* + x'\mathcal{T}\epsilon_1). \tag{11}$$

Therefore, by imposing the constraint $x'\mathcal{T}B = \tau^* b^*$, that is the generalized durations vector of the portfolio matches the generalized durations vector of the target, we are left with the task of minimizing the following objective function

$$\varphi(x) \equiv \frac{1}{2}var(x'\mathcal{T}\epsilon_1 - \tau^*\epsilon_1^*) = \frac{1}{2}x'(\mathcal{T}\Psi\mathcal{T})x + \frac{1}{2}(\tau^*)^2 var(\epsilon_1^*), \tag{12}$$

over the set

$$\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \equiv \{x \in \mathbb{R}^m : A'x = d\} \cap \{x \in \mathbb{R}^m : x_1 + \ldots + x_m = 1\}, \tag{13}$$

with $A \equiv \mathcal{T}B$ an $m \times k$ matrix and $d \equiv (\tau^* b^*)' \in \mathbb{R}^k$. Clearly in (12) we can ignore the constant term and proceed to search for minimum points first on $\mathcal{F}_1$. To this aim we take $\lambda \in \mathbb{R}^k$, and consider the associated Lagrangian function

$$L(x, \lambda) = \varphi(x) - \sum_{j=1}^k \lambda_j[(A'x)_j - d_j].$$

By setting $\Omega \equiv \mathcal{T}\Psi\mathcal{T}$ the stationarity conditions w.r.t. the portfolio weights are given by

$$L_{x_i} = (\Omega x)_i - \sum_{j=1}^k a_{ij}\lambda_j = 0,$$

or equivalently

$$\Omega x = A\lambda, \quad that \;\; is \quad x = \Omega^{-1}A\lambda \tag{14}$$

Inserting (14) in $A'x = d$ we have

$$A'\Omega^{-1}A\lambda = d, \quad that \;\; is \quad \lambda^* = (A'\Omega^{-1}A)^{-1}d.$$

Therefore

$$\tilde{x} := \Omega^{-1}A\lambda^* = \Sigma^{-1}A(A'\Omega^{-1}A)^{-1}d = \mathcal{T}^{-1}\Psi^{-1}\mathcal{T}^{-1}\mathcal{T}B(B'\Psi^{-1}B)^{-1}d$$

$$= \mathcal{T}^{-1}\Psi^{-1}B(B'\Psi^{-1}B)^{-1}d$$

The vector $\tilde{x}$ is not a portfolio (its components do not sum up to one), however the new vector

$$x^* := \tilde{x} + (1 - 1'\tilde{x})y \tag{15}$$

is a portfolio if y is a portfolio, that is if $1'y = 1$. Moreover we must have $A'y = 0$. Given those conditions the components of y can be easily determined to be 5

$$y := \frac{\Lambda 1}{1'\Lambda 1}, \quad \text{with} \;\; \Lambda = \Omega^{-1}(I - A(A'\Omega^{-1}A)^{-1}A'\Omega^{-1}), \tag{16}$$

that is Λ is such that $A(A'\Omega^{-1}A)^{-1}A'\Omega^{-1} = I + \Omega\Lambda$.

To prove (16) we introduce the Lagrangian function

$$L(y, \lambda, \mu) = \varphi(x) - \sum_{j=1}^{k} \lambda_j[(A'y)_j - d'_j] - \mu(1'y - 1), \tag{17}$$

then the stationarity conditions become

$$L_{y_i} = (\Omega y)_i - \sum_{j=1}^{k} a_{ij}\lambda_j - \mu = 0,$$

which can be rewritten as

$$\Omega y = A\lambda - \mu 1, \quad that \;\; is \quad y = \Omega^{-1}A\lambda - \mu\Omega^{-1}1.$$

The requirement $A'y = 0$ leads to

$$A'\Omega^{-1}A\lambda = \mu A'\Omega^{-1}1,$$

henceforth

$$\lambda = \mu[(A'\Omega^{-1}A)^{-1}A'\Omega^{-1}1] =: \mu\tilde{\lambda}.$$

Furthermore we have

$$1 = 1'y = \mu 1'\Omega^{-1}A\tilde{\lambda} - \mu 1'\Omega^{-1}1 = \mu[1'\Omega^{-1}A\tilde{\lambda} - 1'\Omega^{-1}1],$$

and summing up

$$y = \mu[\Omega^{-1}A\tilde{\lambda} - \Omega^{-1}1] = \frac{\Omega^{-1}A\tilde{\lambda} - \Omega^{-1}1}{1'\Omega^{-1}A\tilde{\lambda} - 1'\Omega^{-1}1} = \frac{\Lambda 1}{1'\Lambda 1},$$

as it was previously stated.

4 A non-feasibility result

In this section it is assumed we are given: (i) a vectorial function $F : [0,\infty) \ni t \to F(t) = (F_1(t),\dots,F_k(t)) \in \mathbb{R}^k$; (ii) a set of positive numbers $\mathrm{T} = \{t_1,\dots,t_m\}$ such that $0 < t_1 < \dots < t_m$; (iii) a scalar $t^* > 0$, such that $t^* \notin \mathrm{T}$. Let $x \in \mathbb{R}^m$ and consider the following linear system

$$D'x = d \tag{18}$$

where the $m \times k$ matrix D and the column vector $d = (d_1,\dots,d_k)' \in \mathbb{R}^k$ have elements which are respectively defined, by means of the previous triplet (F, T, t^*), as

$$D_{ij} \equiv F_j(t_i) \ \text{ and } \ d_j \equiv F_j(t^*),$$

$i = 1,\dots,m, j = 1,\dots,k.$

Let $\Delta^{(m-1)} = \{x \in \mathbb{R}^m : x_i \geq 0, x_1 + \dots + x_m = 1\}$ denote the standard-simplex in $\mathbb{R}^m$ and $\mathcal{S}_{F,\ \mathrm{T},t^*}$ the set of solutions of (18). Assuming $\mathcal{S}_{F,\mathrm{T},t^*} \neq \emptyset$, or equivalently Rank($D'$)=Rank($D'\|d$), we wish to investigate under which conditions on the initial data (F, T, t^*) we may have $\Delta^{(m-1)} \cap \mathcal{S}_{F,\mathrm{T},t^*} = \emptyset$. We set $\underline{t} \equiv t_1 \wedge t^*$ and $\overline{t} \equiv t_n \vee t^*$, then the following result holds:

Theorem 1 *Suppose* $F_1(t) = t$ *and that there is a* $j_0 \in \{2,\dots,k\}$ *such that* F_{j_0} *is differentiable at* t^* *and* $K_{j_0}(t) \equiv \frac{F_{j_0}(t) - F_{j_0}(t^*)}{t - t^*}$ *is strictly monotone on the interval* $[\underline{t}, \overline{t}]$, *with* $K_{j_0}(t^*) \equiv F'_{j_0}(t^*)$. *Then* $\Delta^{(m-1)} \cap \mathcal{S}_{F,\mathrm{T},t^*} = \emptyset$.

Proof Let $1' = (1,\dots,1) \in \mathbb{R}^m$, consider the $(k+1) \times m$ matrix

$$A = \begin{pmatrix} D' \\ 1' \end{pmatrix}, \quad \text{and the (k+1)-column vector} \quad b = \begin{pmatrix} d \\ 1 \end{pmatrix}. \tag{19}$$

By Farkas Lemma the set $I_1 \equiv \{x \in \mathbb{R}^m | Ax = b, x \geq 0\}$ is empty if and only if the set $I_2 \equiv \{z \in \mathbb{R}^{k+1} | A'z \geq 0,\ b'z < 0\}$ is not empty. We claim that $I_2 \neq \emptyset$. To this aim we write more explicitly

$$A'z = (D, 1)\binom{z_k}{z_1} \geq 0 \text{ , where } z_k \in \mathbb{R}^k \text{ and } z_1 \in \mathbb{R}, \tag{20}$$

and

$$b'z = (d', 1)\binom{z_k}{z_1} = d'z_k + z_1 < 0, \tag{21}$$

so that we look for solutions to

$$\begin{cases} Dz_k + z_1 1 \geq 0 \\ d'z_k + z_1 < 0. \end{cases} \tag{22}$$

Suppose we can show that there exists z_k^* such that

$$(Dz_k^*)_i > d'z_k^* \quad \forall i = 1, ..., m \tag{23}$$

then

$$(Dz_k^*)_i \geq \hat{w}_1 \equiv \min_{1 \leq i \leq m} (Dz_k^*)_i > d'z_k^*.$$

Henceforth, by setting $\hat{z} \equiv (\hat{w}_1 + d'z_k^*)/2$, we have

$$(Dz_k^*)_i > \hat{z} > d'z_k^* \;\; \forall i,$$

or equivalently

$$(Dz_k^*)_i - \hat{z} > 0 \, \forall i$$

$$d'z_k^* - \hat{z} < 0$$

Therefore by choosing $z_1^* = -\hat{z}$ we obtain

$$\begin{cases} (Dz_k^*)_i + z_1^* > 0 \, \forall i \\ d'z_k^* + z_1^* < 0 \end{cases}$$

which shows that (22) holds true. Summing up, everything is reduced to prove that there is a solution z_k^* for the system of inequalities

$$(Dz_k)_i - d'z_k > 0 \quad \forall i = 1, ..., m, \tag{24}$$

which can be rewritten as

$$(t_i - t^*, F_2(t_i) - F_2(t^*), ..., F_k(t_i) - F_k(t^*))(z_{k1}, ..., z_{kk})' > 0 \quad \forall i = 1, ..., m, \tag{25}$$

where we have set $z'_k = (z_{k1}, ..., z_{kk})$. We now choose, w.l.o.g., $j_0 = 2$ and assume that $K_2(t)$ is strictly monotone. We have two cases:

(i) $K_2(t)$ is strictly increasing Let $J_1 = \{i : t_i < t^*\}$ and $J_2 = \{i : t_i > t^*\}$, clearly $J_1 \cup J_2 = \{1, \ldots, m\}$, $J_1 \cap J_2 = \emptyset$, but J_1 and J_2 cannot be both empty. In case both sets are not empty, we choose in (25) $z_{ki} \equiv z^*_{ki} = 0$ for $i = 3, \ldots, m$ and, after dividing (25) by $t_i - t^*$, we rewrite it as follows

$$\begin{cases} z_{k1} + z_{k2}K_2(t_i) > 0 & i \in J_2 \\ z_{k1} + z_{k2}K_2(t_i) < 0 & i \in J_1. \end{cases} \tag{26}$$

We set $m_1^{(+)} = \max_{i\in J_1} K_2(t_i)$ and $m_2^{(-)} = \min_{i\in J_2} K_2(t_i)$ and notice that, under our hypothesis, $m_1^{(+)} < m_2^{(-)}$. Clearly $K_2(t_i) \leq m_1^{(+)}$ on J_1 and $K_2(t_i) \geq m_2^{(-)}$ on J_2. If $\theta \in (m_1^{(+)}, m_2^{(-)})$, then $K_2(t_i) \leq m_1^{(+)} < \theta$ on J_1 and $K_2(t_i) \geq m_2^{(-)} > \theta$ on J_2, so $-\theta + K_2(t_i) < 0$ on J_1 and $-\theta + K_2(t_i) > 0$ on J_2. It follows that $z^*_k \equiv (-\theta, 1, 0, ..., 0)'$ is a solution of (25) since it holds

$$\begin{cases} -\theta + K_2(t_i) > 0 & i \in J_2 \\ -\theta + K_2(t_i) < 0 & i \in J_1. \end{cases} \tag{27}$$

Let us now suppose that $J_1 = \emptyset$, hence $J_2 = \{1, ..., m\}$. By setting $\hat{m} = \min_{i\in J_2} K_2(t_i)$ we have $K_2(t_i) \geq \hat{m}$ on J_2. Let $\theta < \hat{m}$, we have $K_2(t_i) \geq \hat{m} > \theta$, henceforth $-\theta + K_2(t_i) > 0 \ \forall i = 1, ..., m$. It follows that $(-\theta, 1, 0, ..., 0)$ is a solution of (26)(in this case there is only one inequality because $J_1 = \emptyset$). Now let us assume that $J_2 = \emptyset$, hence $J_1 = \{1, ..., m\}$. We set $M = \max_{i\in J_1} K_2(t_i)$, therefore $K_2(t_i) \leq M$ on J_1; let $\theta > M$, so $K(t_i) \leq M < \theta$, henceforth $-\theta + K_2(t_i) < 0 \ \forall i = 1, ..., m$. It follows that $(-\theta, 1, 0, ..., 0)$ is a solution of (26) (again only one inequality).

(ii) $K_2(t)$ is strictly decreasing. If $K_2(t)$ is strictly decreasing, $\overline{K}_2(t) \equiv -K_2(t)$ is strictly increasing. As previously we first consider the case both sets J_1 and J_2 are not empty and choose in (25) $z_{ki} \equiv z^*_{ki} = 0$ for $i = 3, \ldots, m$. Once again, by dividing it for the increment $t_i - t^*$, we may rewrite the conditions (26) in the form

$$\begin{cases} z_{k1} - z_{k2}\overline{K}_2(t_i) > 0 & i \in J_2 \\ z_{k1} - z_{k2}\overline{K}_2(t_i) < 0 & i \in J_1. \end{cases} \tag{28}$$

By mimicking the previous reasoning these conditions are clearly satisfied by choosing $z^*_{k2} = -1$, and $z^*_{k1} = -\theta$, with $\theta \in (\max_{i\in J_1} \overline{K}_2(t_i), \min_{i\in J_2} \overline{K}_2(t_i))$. The cases when $J_1 = \emptyset$ or when $J_2 = \emptyset$ can be treated in analogous fashion. □

5 Application to the hedging problem

In order to be able to apply the previous theorem to the portfolio hedging problem presented in Sect. 2 we need the following result:

Proposition 1 *Let $G : [a, b] \to \mathbb{R}$ be a continuous differentiable strictly convex function and fix $x_0 \in [a, b]$. Then the function $K_G(x) \equiv \frac{G(x)-G(x_0)}{x-x_0}$ for $x \neq x_0$, and $K_G(x_0) \equiv G'(x_0)$, is strictly increasing on [a, b].*

Proof G is strictly convex function so G' is a strictly increasing function. We will initially assume that $x_0 \in (a, b)$ and $x_1 < x_0 < x_2$.

$$\frac{K_G(x_1) - K_G(x_2)}{x_1 - x_2} = \frac{\frac{G(x_1)-G(x_0)}{x_1-x_0} - \frac{G(x_2)-G(x_0)}{x_2-x_0}}{x_1 - x_2} = \frac{G'(\xi_{1,0}) - G'(\xi_{2,0})}{x_1 - x_2}$$

with $x_1 < \xi_{1,0} < x_0 < \xi_{2,0} < x_2$, so the strictly growth of the G' imply

$$\frac{K_G(x_1) - K_G(x_2)}{x_1 - x_2} > 0.$$

Now assume that $x_1 < x_2 < x_0$

$$\frac{K_G(x_1) - K_G(x_2)}{x_1 - x_2} = \frac{\frac{G(x_1)-G(x_2)+G(x_2)-f(x_0)}{x_1-x_0} - \frac{G(x_2)-G(x_0)}{x_2-x_0}}{x_1 - x_2}$$

$$\begin{aligned}\frac{K_G(x_1) - K_G(x_2)}{x_1 - x_2} &= \frac{\frac{G(x_1)-G(x_2)+G(x_2)-f(x_0)}{x_1-x_0} - \frac{G(x_2)-G(x_0)}{x_2-x_0}}{x_1 - x_2} \\ &= \frac{\frac{G'(\xi_{1,2})(x_1-x_2)+G'(\xi_{0,2})(x_2-x_0)}{x_1-x_0} - G'(\xi_{0,2})}{x_1 - x_2} \\ &= \frac{G'(\xi_{1,2})(x_1 - x_2) + G'(\xi_{0,2})(x_2 - x_0) - G'(\xi_{0,2})(x_1 - x_0)}{(x_1 - x_2)(x_1 - x_0)} \\ &= \frac{G'(\xi_{1,2})(x_1 - x_2) + G'(\xi_{0,2})(x_2 - x_1)}{(x_1 - x_2)(x_1 - x_0)} \\ &= \frac{(x_1 - x_2)(G'(\xi_{1,2}) - G'(\xi_{0,2}))}{(x_1 - x_2)(x_1 - x_0)} \\ &= \frac{G'(\xi_{1,2}) - G'(\xi_{0,2})}{x_1 - x_0} > 0\end{aligned}$$

with $x_1 < \xi_{1,2} < x_2 < \xi_{2,0} < x_0$. In a similar way we can treat the case $x_0 < x_1 < x_2$. □

Corollary 1 *Let $F : [a, b] \to \mathbb{R}$ be a continuous differentiable strictly concave function and fix $x_0 \in [a, b]$. Then the function $K_F(x) \equiv \frac{F(x)-F(x_0)}{x-x_0}$ for $x \neq x_0$, and $K_F(x_0) \equiv F'(x_0)$, is strictly decreasing on [a, b].*

Proof the function $G(x) \equiv -F(x)$ verifies the hypothesis of the previous proposition, therefore $K_G(x) = -K_F(x)$ is is strictly increasing on [*a*, *b*]. Henceforth $K_F(x)$ is strictly decreasing on [*a*, *b*]. □

We now link the result of this last corollary to Theorem 1 of section 4 and to the approach to the hedging problem modeled in section 3 in order to hedge interest rate risk. To establish such a connection we choose the vectorial function $F(t) = (F_1(t), \ldots, F_k(t))$ of Theorem 1 in the following way

$$F_j(t) \equiv t f_j(t), \quad j = 1, \ldots, k \tag{29}$$

where $(f_1(t), \ldots, f_k(t))$ are the factor loadings of the econometric factor model. Furthermore the matrix $D = (D_{ij})$ of Theorem 1 is taken to be the matrix $A = \mathcal{T}B$ of section 3 having components $A_{ij} \equiv D^j(\tau_i) = F_j(\tau_i)$. Consider now the case of the Nelson Siegel family ($k = 3$) for which $F_2(\tau_i) \equiv \frac{1-e^{-a\tau_i}}{a}$ and $a > 0$. We notice that, in this case, $F_2(\cdot)$ is strictly concave henceforth its incremental ratio around a fixed time τ^* is strictly decreasing by applying the previous Corollary 1 to $F \equiv F_2$. By consequence Theorem 1 of Section 4 applies to the matrix $D \equiv \mathcal{T}B$ appearing in the hedging problem, designed by Borup et al. (2022), which has been set in Section 3. This leads to conclude that approach does not extend to a market where short selling is not permitted.

6 A simple analysis of the no-short selling constraints

The aim of this last section is to provide a more explicit analysis of the requirements on factor loadings which make the optimization problem viable or not with short selling restrictions in the market . To make this analysis as clear as possible we have chosen to illustrate it through a simplified, however very instructive, example. In particular we shall use such an example to highlight that the conditions enucleated in the previous sections leading to unfeasibility are sufficient but not necessary ones. That is, even when these conditions on generalized durations are not verified it can happen that optimality is reached only by short selling some bonds. We start by considering a market with $m = 3$ ZC bonds having maturities $t_i = i$ for $i = 1, 2, 3$ and a liability maturing at $t^* = \frac{3}{2}$ which must be hedged. We assume that the returns of the bonds and of the liability can be consistently modeled by the system of Eqs. (2) and (3). In the following example we shall fix $k = 2$ as number of risk factors and $(f_1(t), f_2(t)) = (1, f(t))$ as corresponding factor loadings, with $f(t)$ being an arbitrary non negative piece-wise smooth function. Let us consider the 3×2 matrix D and the vector d given by

$$D = \begin{pmatrix} 1 & g(1) \\ 2 & g(2) \\ 3 & g(3) \end{pmatrix}, \quad d = \begin{pmatrix} 3/2 \\ g(3/2) \end{pmatrix},$$

where $F_1(t) \equiv t$ and $F_2(t) \equiv g(t) \equiv t f(t)$ are the components of the vectorial field $F = (F_1, F_2)$. Hence the generalized duration matching condition is expressed by the relation $D'x = d$ as in (18). By adding to this linear system the additional condition $x_1 + x_2 + x_3 = 1$ it turns out that we must solve the system

$$\begin{array}{lcl} x_1 + 2x_2 + 3x_3 & = & 3/2 \\ g(1)x_1 + g(2)x_2 + g(3)x_3 & = & g(3/2) \\ x_1 + x_2 + x_3 & = & 1. \end{array} \tag{30}$$

By assuming the 3×3 matrix A of the coefficients of the above system has a non zero determinant, by Cramer's rule the unique solution admits the following representation

$$x_i^* = \frac{\det(B_i)}{\det(A)}, \quad i = 1, 2, 3 \tag{31}$$

where the matrix B_i is obtained by deleting column "i" from A and replacing it with the column vector $(\frac{3}{2}, g(\frac{3}{2}), 1)'$. Suppose the no-short selling constraints hold, these require the solutions x_1^*, x_2^* and x_3^* to be non negative. This can be achieved if it holds $\det(A) > 0$ and $\det(B_i) \geq 0,\ i = 1, 2, 3$ (or if the inequalities are all reversed in sign). It is readily seen that $\det(A) > 0$ if and only if $g(1) + g(3) > 2g(2)$ and that $\mathcal{D}_i \equiv \det(B_i) \geq 0,\ i = 1, 2, 3$ if and only if

$$\mathcal{D}_1 = \begin{vmatrix} 3/2 & 2 & 3 \\ g(3/2) & g(2) & g(3) \\ 1 & 1 & 1 \end{vmatrix} = g(\frac{3}{2}) - \frac{3}{2} g(2) + \frac{1}{2} g(3) \geq 0$$

$$\mathcal{D}_2 = \begin{vmatrix} 1 & 3/2 & 3 \\ g(1) & g(3/2) & g(3) \\ 1 & 1 & 1 \end{vmatrix} = \frac{3}{2} g(1) - 2\, g(\frac{3}{2}) + \frac{1}{2} g(3) \geq 0$$

$$\mathcal{D}_3 = \begin{vmatrix} 1 & 2 & 3/2 \\ g(1) & g(2) & g(3/2) \\ 1 & 1 & 1 \end{vmatrix} = -\frac{1}{2} g(1) + g(\frac{3}{2}) - \frac{1}{2} g(2) \geq 0.$$

In conclusion the function $g(t)$ must satisfies the following system of inequalities

$$\begin{array}{ll} (a) & \dfrac{g(1) + g(3)}{2} > g(2) \\ (b) & \dfrac{2g(\frac{3}{2}) + g(3)}{3} \geq g(2) \\ (c) & \dfrac{3g(1) + g(3)}{4} \geq g(\dfrac{3}{2}) \\ (d) & g(\dfrac{3}{2}) \geq \dfrac{g(1) + g(2)}{2}. \end{array} \tag{32}$$

There are infinitely many possible shapes of $g(t)$ which are compatible with the above system. Indeed it is sufficient to have

$$g(2) < g(1) \leq g(\frac{3}{2}) \leq \frac{3}{2} g(1) \leq \frac{1}{2} g(3) \leq g(3) \tag{33}$$

to satisfy all the inequalities.We conclude that, in such a case, it is possible to have optimal portfolios having non negative weights. We notice that this chain of inequalities shows that $g(t)$ changes concavity on the considered time interval [1, 3].This concavity change allows for the existence of optimal hedging portfolios having non negative weights. We remark that without this change of concavity Corollary 1 of the previous section would prevent such existence. Notice that, in this market, the feasible set $\mathcal{F}$ defined in (13) is a singleton, which makes the optimality problem not interesting. We therefore enlarge the market by adding a further ZC-bond with maturity $\tau_4 = 4$ which can be traded. Aiming to hedge the liability maturing at $t^* = 3/2$ all possible portfolio choices $x = (x_1, x_2, x_3, x_4)'$ are then determined by considering the following linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= \frac{3}{2} \\ g(1)x_1 + g(2)x_2 + g(3)x_3 + g(4)x_4 &= g\left(\frac{3}{2}\right) \\ x_1 + x_2 + x_3 + x_4 &= 1. \end{aligned} \tag{34}$$

We keep on going assuming the g term-structure (33). The above system has clearly a solution $x^* = (x_1^*, x_2^*, x_3^*, 0)$ given by (31) having all weights not negative. However there are now infinitely many solutions of (34), the linear system being rectangular. Following our approach the task of the portfolio manager is to select a specific solution of (34), among all the previous ones, by adopting the criterium of minimum variance of the spread of the returns previously illustrated. We shall show that according to the value of $g(4)$ the optimal solution can have or not have all the weights positive. This proves that when $F_2 = g$ is not concave (which is implied by (33)), so that the main hypothesis of Theorem 1 can not be granted, then optimality may be achieved by short-selling or not some of the bonds. To be more concrete we shall consider a specific g term-structure. To this aim and in agreement with the ordering (33), we suppose $(g(1), g(\frac{3}{2}), g(2), g(3), g(4)) = (1, \frac{5}{4}, \frac{1}{2}, 4, \epsilon)$, with ϵ spanning the interval [2/3, 10]. By implementing the minimization procedure detailed in Section 3 the weights of the corresponding minimum variance hedging portfolios can be easily found out (see Eq. 15), and the obtained results are reported in Fig. 2. In this example, the matrix Ψ, appearing in Sect. 2, has been set equal to σI, with $\sigma = 0.1$, I being the identity matrix 4×4. Moreover, with the notations of Sect. 2, $A = D = \begin{pmatrix} 1 & g(1) \\ 2 & g(2) \\ 3 & g(3) \\ 4 & g(4) \end{pmatrix}$,

$$B = \begin{pmatrix} 1 & g(1) \\ 1 & g(2)/2 \\ 1 & g(3)/3 \\ 1 & g(4)/4 \end{pmatrix}, d = \begin{pmatrix} \frac{3}{2} \\ g(\frac{3}{2}) \end{pmatrix}, \mathcal{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \text{ and } A = \mathcal{T}B.$$

As it was expected the first two assets are the more funded ones, furthermore, if the value of ϵ continues to be increased numerical results show that some weights become negative for $\epsilon < 7$. Figures 2 and 3 furnish a clear global picture of the sign changes occurring in some weights of the optimal portfolios as the value of $g(4)$ moves. We do not further address this point, indeed our main motivation was to provide numerical evidence of the interplay between the shape of the factor loadings and positiveness of the portfolio weights.

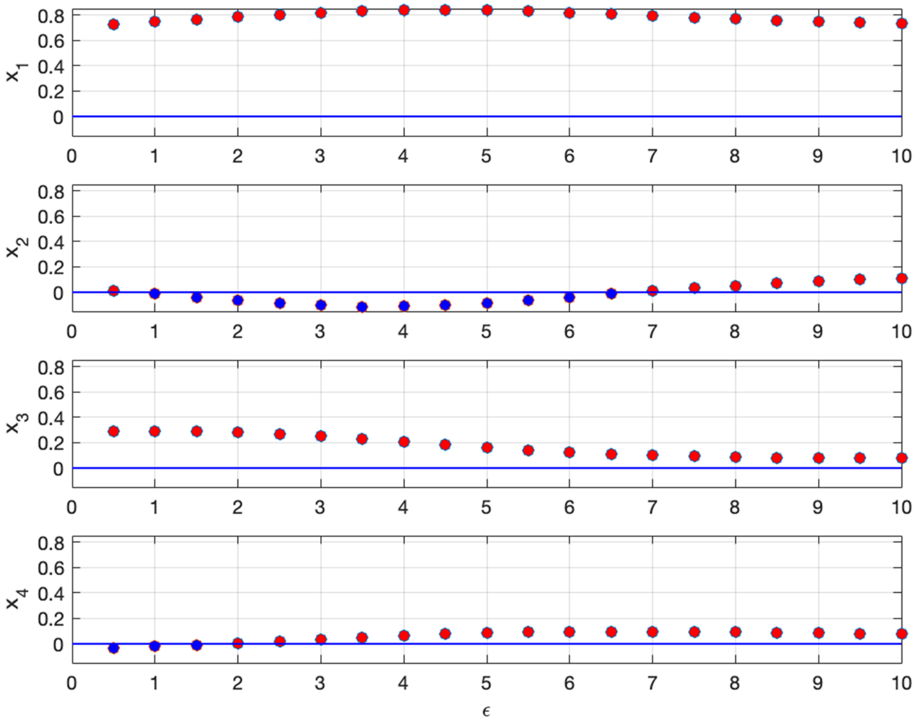

Fig. 2 The four components of the minimum variance portfolio $x^* = (x_1, x_2, x_3, x_4)$ (see Eq. 15) as a function of ϵ

7 Conclusions

In this paper a dynamical factor model is considered for modeling the time evolution of the yield to maturity of bonds and liabilities. Given this, to hedge a fixed liability a portfolio of ZC bonds may be chosen in an efficient way by imposing a generalized duration matching constraint and following the optimality criterium of Borup et al. (2022). We show that under a set of minimal requirements on the generalized durations arising from the factor model this procedure results to be unfeasible under no short selling conditions. A nice feature of the proven theorem is that the assumed hypothesis can be easily tested. For instance, if the factor loadings have the shape of the Nelson–Siegel family we have unfeasibility. Some related theoretical problem, such as full characterization of feasibility under no short selling, remains open for future research.

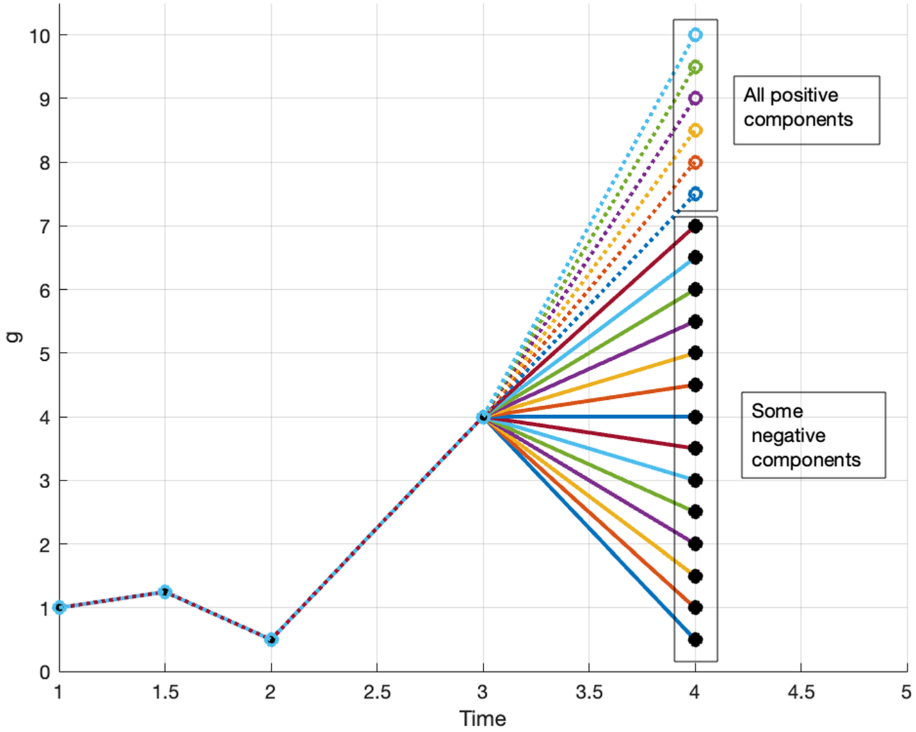

Fig. 3 The function $g(t)$ for different values of ϵ. The black dots identify values of $g(4) = \epsilon$ that result in optimal solutions with negative components

Funding Open access funding provided by Università degli Studi di Roma Tor Vergata within the CRUI-CARE Agreement.

References

Ahi, E., Akgiray, V., & Sener, E. (2018). Robust term structure estimation in developed and emerging markets. *Ann. Oper. Res., 260*, 23–49.

Bachem, A., & Kern, W. (1992). *Linear Programming Duality*. Springer.

Banholzer, D., Fliege, J., Werner, R.,(2024), Revisiting the fitting of the Nelson–Siegel and Svensson models, *Optimization*, n.73,3021-3053,

Barber, J., & Copper, M. (1996). Immunization using principal component analysis. *The Journal of Portfolio Management, 23*, 99–105.

Bjork, T., & Christensen, B. J. (1999). Interest rate dynamics and consistent forward rate curves. *Mathematical Finance, 9*(4), 323–348.

Bliss, R. (1997). Movements in the term structure of interest rates. *Economic Review - Federal Reserve Bank of Atlanta, 82*(4), 16–33.

Borup, B., Christensen, B. J., & Hansen, J. W. (2022). Immunization in the Treasury market with consistent term structure dynamics. *SSRN*. https://doi.org/10.2139/ssrn.4164195

Caldeira, J. F., Moura, G. V., & Santos, A. A. (2016). Bond portfolio optimization using dynamic factor models. *Journal of Empirical Finance, 37*, 128–158.

Chambers, D., Carleton, W., & McEnally, R. (1988). Immunizing default-free bond portfolios with a duration vector. *The Journal of Financial and Quantitative Analysis, 23*(1), 89–104.

D'Ecclesia, R. L., & Zenios, S. A. (1994). Risk factor analysis and portfolio immunization in the Italian bond market. *The Journal of Fixed Income, 4*(2), 51–58.

Diebold, F. X., & Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics, 130*, 337–364.

Fisher, L., & Weil, R. L. (1971). Coping with the risk of interest-rate fluctuations returns to bondholders from naive and optimal strategies. *Journal of Business, 44*(5), 408–431.

Fong, H. G., & Fabozzi, F. (1985). *Fixed Income Portfolio Management*. Homewood ILL Dow-Jones Irwin.

Fong, H. G., & Vasicek, O. A. (1984). A risk minimizing strategy for portfolio immunization. *The Journal of Finance, 39*(5), 1541–1546.

Garbade, K. (1986). *Modes of Fluctuation in Bond Yields: An Analysis of Principal Components*. New York, Bankers Trust Company: Topics in Money and Securities Markets.

Giglio, S., Kelly, B., & Xiu, D. (2022). Factor models, machine learning, and asset pricing. *Annual Rev. Financ. Econ., 14*, 337–68.

Kopa, M., Moriggia, V., & Vitali, S. (2023). *Multistage stochastic dominance: an application to pension fund management*. Ann: Oper. Res. published online.

Litterman, R., & Scheinkman, J. (1991). Common factors affecting bond returns. *Journal of Fixed Income, 1*, 54–61.

Montrucchio, L., & Peccati, L. (1991). Insurance: A note on Shiu -Fisher-Weil immunization theorem. *Mathematics and Economics, 10*(2), 125–131.

Nawalkha, S. K., & Chambers, D. R. (1996). An improved immunization strategy: m-absolute. *Financial Analysts Journal, 52*(5), 69–76.

Nawalkha, S. K., & Chambers, D. R. (1997). The m-vector model. *The Journal of Portfolio Management, 23*(2), 92–98.

Nelson, C., & Siegel, A. (1987). Parsimonious modeling of yield curves. *Journal of Business, 60*, 473–489.

Nymand-Andersen, P., (2018), Yield curve modeling and a conceptual framework for estimating yield curves: *evidence from the European Central Bank's yield curves, ECB, Statistical Paper Series*,27

Ortobelli, S., Vitali, S., Cassader, M., & Tichy, T. (2018). Portfolio selection strategy for fixed income markets with immunization on average. *Ann. Oper. Res., 260*, 395–415.

Prisman, E., & Shores, M. (1988). Duration models for specific term structure estimations and applications to bond portfolio immunization. *Journal of Banking and Finance, 12*, 493–504.

Reddington, F. M. (1952). Review of the principles of life office valuations. *Journal of the Institute of Actuaries, 78*, 286–340.

Shiu, E. S. W. (1987). On the Fisher-Weil immunization theorem. *Insurance: Mathematics and Economics, 6*(4), 259–266.

Shiu, E. S. W. (1988). Immunization of multiple liabilities. *Insurance: Mathematics and Economics, 7*(4), 219–224.

Shiu, E. S. W. (1990). On Redington's theory of immunization. *Insurance: Mathematics and Economics, 9*(2–3), 171–175.

Simoes, C., Oliveira, L., & Bravo, J. M. (2021). Immunization strategies for funding multiple inflation-linked retirement income benefits. *Risks, 9*, 60.

Sojoudi, M., Bernard, C., Dupuy, P., & Peters, G. W. (2025). *Green spread of US municipal bonds*. Ann: Oper. Res., published online.

Svensson, L. (1994). Estimating and interpreting forward interest rates: Sweden 1992-1994, NBER n.4871.

Uberti, M. (1997). A note on Shiu's immunization results. *Insurance: Mathematics and Economics, 21*(3), 195–200.

Umar, Z., Gubareva, M., Teplova, T., & Alwahedi, W. (2022). *Oil price shocks and the term structure of the US yield curve: a time–frequency analysis of spillovers and risk transmission*. Ann. Oper. Res., published online.

Willner, R. (1996). A new tool for portfolio managers: Level, slope, and curvature durations. *The Journal of Fixed Income, 6*, 48–59.