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NONCENTRAL MODERATE DEVIATIONS FOR TIME-CHANGED MULTIVARIATE LÉVY PROCESSES WITH LINEAR COMBINATIONS OF INVERSE STABLE SUBORDINATORS

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Abstract. The term noncentral moderate deviations is used in the literature to mean a class of large deviation principles that, in some sense, fills the gap between the convergence in probability to a constant (governed by a reference large deviation principle) and a weak convergence to a non- Gaussian (and non-degenerating) distribution. Some noncentral moderate deviation results in the literature concern time-changed univariate Lévy processes, where the time-changes are given by inverse stable subordinators. In this paper we present analogue results for multivariate Lévy processes; in particular the random time-changes are suitable linear combinations of independent inverse stable subordinators.

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1. Introduction

The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale (see [1] as a reference of this topic). The term *moderate deviations* is used for a class of large deviation principles which fills the gap between a convergence to a constant x_0 (at least in probability), and a weak convergence to a non-constant centered Gaussian random variable. The convergence to x_0 is governed by a *reference large deviation principle* with speed v_t, and a rate function that uniquely vanishes at x_0. Moderate deviations provide a class of large deviation principles which depend on the choice of some positive scalings $\{a_t : t > 0\}$ such that $a_t \to 0$ and $v_t a_t \to \infty$ (as $t \to \infty$), each one with speed $1/a_t$ (thus the speed is slower than v_t).

The term *noncentral moderate deviations* has been recently introduced when one has the situation described above, but the weak convergence is towards a non-constant and *non-Gaussian* distributed random variable. A recent reference with some examples and several references is [2]. Several examples in the literature concern families of real valued random variables; a multivariate example is presented in [3].

In this paper we consider light-tailed h-variate Lévy processes

$$\{(S_1(t), \ldots, S_h(t)) : t \geq 0\}$$

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with independent components (see Asm. 1.1); obviously the univariate case $h = 1$ is also allowed. Our aim is to present noncentral moderate deviation results for two classes of independent time-changes of $\{(S_1(t), \dots, S_h(t)) : t \geq 0\}$ in terms of two classes of linear combinations of (independent) inverse stable subordinators; see Conditions 1.4 and 1.5 for a rigorous description.

The interest of time-changed processes is motivated by their potential applications in different areas (*e.g.*, finance, physics, and queueing theory) where random fluctuations can have an important impact on the system dynamics. Indeed random time-changes can add a layer of complexity and realism to models. Several examples of random time-changes are given by nondecreasing Lévy processes (called *subordinators*) or their inverses. Here we recall some references in which some inverse of subordinators (or their mixtures) are studied: [4] for inverse Gamma subordinators, [5] for inverse stable subordinators, [6] for inverse of mixtures of stable subordinators, [7] for inverse tempered stable subordinators, and [8] for inverse of mixtures of tempered stable subordinators. We also recall that the inverse stable subordinators have important connections with time-fractional processes.

We start by introducing the light-tailed h-variate Lévy processes $\{(S_1(t), \dots, S_h(t)) : t \geq 0\}$ with independent components; moreover we introduce some related notation.

Assumption 1.1. Let $\{S_1(t) : t \geq 0\}, \dots, \{S_h(t) : t \geq 0\}$ be independent real-valued Lévy processes; moreover they are assumed to be light-tailed, *i.e.* the functions $\kappa_{S_1}, \dots, \kappa_{S_h}$ defined by

$$\kappa_{S_i}(\theta) := \log \mathbb{E}[\mathrm{e}^{\theta S_i(1)}] \quad \text{(for all } i \in \{1, \dots, h\} \text{ and } \theta \in \mathbb{R})$$

are finite in a neighborhood of $\theta = 0$, and therefore the vector

$$m = (m_1, \dots, m_h) := (\kappa'_{S_1}(0), \dots, \kappa'_{S_h}(0)) = (\mathbb{E}[S_1(1)], \dots, \mathbb{E}[S_h(1)])$$

is well-defined.

We have the following remarks.

Remark 1.2. If $\kappa''_{S_i}(0) = 0$ for some $i \in \{1, \dots, h\}$, then $\{S_i(t) : t \geq 0\}$ is a deterministic Lévy process and, with the notation in Assumption 1.1, we have $S_i(t) = m_i t$. Thus we can say that, if $m = (m_1, \dots, m_h) = 0 \in \mathbb{R}^h$ and $\kappa''_{S_i}(0) = 0$ for every $i \in \{1, \dots, h\}$, then the processes in Conditions 1.4 and 1.5 below are identically equal to zero and all the results below will be trivial; moreover, in such a case, the weak convergence results in Propositions 3.2 and 5.2 are towards the constant random variable equal to $0 \in \mathbb{R}^h$.

Remark 1.3. Here we present a brief discussion on Assumption 1.1. The independence of the processes $\{S_1(t) : t \geq 0\}, \dots, \{S_h(t) : t \geq 0\}$ allows to make calculations on each component of the random vectors separately; therefore we can refer to the asymptotic behavior of the Mittag-Leffler functions involved, and we can refer to what happens for the univariate case. In our opinion this difficulty could be overcome at least under suitable hypotheses, and this could be studied in a successive work. Moreover $\{S_1(t) : t \geq 0\}, \dots, \{S_h(t) : t \geq 0\}$ are assumed to be light-tailed, and this allows to apply the Gärtner Ellis Theorem (Thm. 2.1). In general this problem would not be easy to face. For instance, as discussed in Section 4 in [9] (where the Lévy processes are compound Poisson processes), one could hope to find a connection with the results on empirical means of i.i.d. semi-exponential distributed random variables (see the references cited therein).

Now we present two classes of independent time-changes for $\{(S_1(t), \dots, S_h(t)) : t \geq 0\}$ studied in this paper. In particular we refer to the concept of inverse stable subordinator recalled below (see just after Eq. (2.2)). We remark that, actually, some of the positive coefficients c_i in Condition 1.4 and c_{ij} in Condition 1.5 could be equal to zero, and the statements of the results can be suitably modified (the details will be omitted).

Condition 1.4. Let $c_0, c_1, \dots, c_h > 0$ be arbitrarily fixed. Then let

$$\{(S_1(c_1 L_1^{\nu_1}(t) + c_0 L_0^{\nu_0}(t)), \dots, S_h(c_h L_h^{\nu_h}(t) + c_0 L_0^{\nu_0}(t))) : t \geq 0\}$$

be the stochastic process defined in terms of the following families of independent random variables:

- $\{S_1(t) : t \geq 0\}, \ldots, \{S_h(t) : t \geq 0\}$ are as in Assumption 1.1;
- $\{L_0^{\nu_0}(t) : t \geq 0\}, \{L_1^{\nu_1}(t) : t \geq 0\}, \ldots, \{L_h^{\nu_h}(t) : t \geq 0\}$ are independent inverse stable subordinators of parameters $\nu_0, \nu_1, \ldots, \nu_h \in (0,1)$, respectively.

Condition 1.5. Let $\{c_{ij} : i \in \{1, \ldots, h\}, j \in \{1, \ldots, k\}\}$ be arbitrarily fixed positive numbers. Then let

$$\left\{ \left(S_1\left(\sum_{j=1}^{k} c_{1j} L_j^{\nu}(t)\right), \ldots, S_h\left(\sum_{j=1}^{k} c_{hj} L_j^{\nu}(t)\right)\right) : t \geq 0 \right\}$$

be the stochastic process defined in terms of the following families of independent random variables:

- $\{S_1(t) : t \geq 0\}, \ldots, \{S_h(t) : t \geq 0\}$ are as in Assumption 1.1;
- $\{L_1^{\nu}(t) : t \geq 0\}, \ldots, \{L_k^{\nu}(t) : t \geq 0\}$ are independent inverse stable subordinators of parameter $\nu \in (0,1)$.

The results in this paper have some analogies with the ones for univariate time-changed Lévy processes in the literature: see *e.g.* [10], with some results for Lévy processes with bounded variation (and therefore for differences of two independent subordinators); see also [9] which concerns compound Poisson processes only, and [11] which concerns Skellam processes (which are differences between independent Poisson processes). This paper is the first contribution with results for the multivariate case.

Under both Conditions 1.4 and 1.5 we present the following results.

- A reference LDP with speed $v_t = t$ (which does not depend on m in Asm. 1.1), and we have a convergence in probability to $x_0 = 0$, where $0 \in \mathbb{R}^h$ is the null vector; see Proposition 3.1 under Condition 1.4, and Proposition 5.1 under Condition 1.5.
- Two weak convergence results (for $m = 0$ and $m \neq 0$, where $0 \in \mathbb{R}^h$ is the null vector) towards non-constant and non-Gaussian distributed random variables; see Proposition 3.2 under Condition 1.4, and Proposition 5.2 under Condition 1.5.
- Two noncentral moderate deviation results (for $m = 0$ and $m \neq 0$, again) concerning classes of LDPs which depend on the choices of some positive scalings $\{a_t : t \geq 0\}$ such that

$$a_t \to 0 \text{ and } ta_t \to \infty \quad (\text{as } t \to \infty); \tag{1.1}$$

  see Proposition 3.3 under Condition 1.4, and Proposition 5.3 under Condition 1.5.

We can say that the results in this paper have some features of the results for the univariate processes in the literature; see [10] (Sect. 3), or in [9] (for compound Poisson processes only). We mean that the reference LDP that does not depend on m in Assumption 1.1 while, for the other results, we have to distinguish the cases $m = 0$ and $m \neq 0$. To this aim it is useful to introduce the notation

$$\alpha_m(\nu) := \begin{cases} 1 - \nu/2 & \text{if } m = 0 \\ 1 - \nu & \text{if } m \neq 0, \end{cases} \quad \text{for } \nu \in (0,1). \tag{1.2}$$

Moreover, for the results concerning Condition 1.4, we use the notation $x \vee y$ to denote the maximum between $x, y \in \mathbb{R}$.

We conclude with the outline of the paper. Section 2 is devoted to recall some preliminaries. In Section 3 we prove the results under Condition 1.4, and in Section 4 we present some cases in which it is possible to get an explicit expression for the rate function in Proposition 3.3 (concerning moderate deviations). In Section 5 we prove the results under Condition 1.5.

2. Preliminaries

In this section we recall some preliminaries on large deviations, on the Lévy processes in Assumption 1.1, and on the inverse the stable subordinator, together with the Mittag-Leffler function.

2.1. Preliminaries on large deviations

We start with some basic definitions (see *e.g.* [1]). In view of what follows we present definitions and results for families of $\mathbb{R}^b$-valued random variables $\{Z(t) : t > 0\}$, for some b, defined on the same probability space $(\Omega, \mathcal{F}, P)$, where t goes to infinity. Throughout this paper we always have $b = h$.

A real-valued function $\{v_t : t > 0\}$ such that $v_t \to \infty$ (as $t \to \infty$) is called a *speed function*, and a lower semicontinuous function $I : \mathbb{R}^b \to [0, \infty]$ is called a *rate function*. Then $\{Z(t) : t > 0\}$ satisfies the LDP with speed v_t and a rate function I if

$$\limsup_{t\to\infty} \frac{1}{v_t} \log P(Z(t) \in C) \leq -\inf_{x\in C} I(x) \quad \text{for all closed sets } C,$$

and

$$\liminf_{t\to\infty} \frac{1}{v_t} \log P(Z(t) \in O) \geq -\inf_{x\in O} I(x) \quad \text{for all open sets } O.$$

The rate function I is said to be *good* if, for every $\beta \geq 0$, the level set $\{x \in \mathbb{R}^b : I(x) \leq \beta\}$ is compact. We also recall the following known result (see *e.g.* Thm. 2.3.6(c) in [1]).

Theorem 2.1 (Gärtner Ellis Theorem)**.** *Assume that, for all $\theta \in \mathbb{R}^b$, there exists*

$$\Lambda(\theta) := \lim_{t\to\infty} \frac{1}{v_t} \log \mathbb{E}\left[\mathrm{e}^{v_t\langle\theta, Z(t)\rangle}\right]$$

as an extended real number (here $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^b$). Assume that $\theta = 0 \in \mathbb{R}^b$ belongs to the interior of the set $\mathcal{D}(\Lambda) := \{\theta \in \mathbb{R}^b : \Lambda(\theta) < \infty\}$; moreover assume that the function Λ is essentially smooth and lower semi-continuous. Then the family of random variables $\{Z(t) : t > 0\}$ satisfies the LDP with good rate function Λ^* defined by

$$\Lambda^*(x) := \sup_{\theta\in\mathbb{R}^b} \{\langle\theta, x\rangle - \Lambda(\theta)\}$$

(i.e. Λ^* is the Legendre-Fenchel transform of Λ).

We also recall (see *e.g.* Def. 2.3.5 in [1]) that Λ is essentially smooth if the interior of $\mathcal{D}(\Lambda)$ is non-empty, the function Λ is differentiable throughout the interior of $\mathcal{D}(\Lambda)$, and Λ is steep, *i.e.* $\|\Lambda'(\theta_n)\| \to \infty$ whenever θ_n is a sequence of points in the interior of $\mathcal{D}(\Lambda)$ which converge to a boundary point of $\mathcal{D}(\Lambda)$.

2.2. Preliminaries on Lévy processes in Assumption 1.1

Some well-known properties on Lévy processes will be used for the processes in Assumption 1.1. Firstly, for every $t \geq 0$, we have

$$\mathbb{E}[\mathrm{e}^{\theta S_i(t)}] = \mathrm{e}^{t\kappa_{S_i}(\theta)} \quad (\text{for all } i \in \{1, \ldots, h\} \text{ and } \theta \in \mathbb{R}).$$

Moreover, for every $i \in \{1, \ldots, h\}$, we consider the following Taylor formulas for the function κ_{S_i}:

$$\kappa_{S_i}(\theta) = m_i\theta + o(\theta) \quad \text{as } \theta \to 0$$

and, if $m = 0 \in \mathbb{R}^h$, *i.e.* $m_1 = \cdots = m_h = 0$,

$$\kappa_{S_i}(\theta) = \kappa''_{S_i}(0)\frac{\theta^2}{2} + o(\theta^2) \quad \text{as } \theta \to 0.$$

We recall the discussion in Remark 1.2 on the case $\kappa''_{S_i}(0) = 0$.

2.3. Preliminaries on inverse stable subordinators

We start with the definition of the Mittag-Leffler function (see *e.g.* [12], Eq. (3.1.1))

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}.$$

It is known (see Prop. 3.6 in [12] for the case $\alpha \in (0,2)$; indeed here we consider the case $\alpha \in (0,1)$ only) that we have

$$\begin{cases} E_\alpha(x) \sim \frac{e^{x^{1/\alpha}}}{\alpha} \text{ as } x \to \infty \\ \text{if } y < 0, \text{ then } \frac{1}{x}\log E_\alpha(xy) \to 0 \text{ as } x \to \infty. \end{cases} \tag{2.1}$$

Then, for the stable subordinator of order $\nu \in (0,1)$, denoted by $\{S_\nu(t) : t \geq 0\}$, for all $t \geq 0$ we have

$$\mathbb{E}[e^{\theta S_\nu(t)}] = \begin{cases} e^{-t(-\theta)^\nu} & \text{if } \theta \leq 0 \\ \infty & \text{if } \theta > 0. \end{cases}$$

Moreover, for the inverse stable subordinator $\{L_\nu(t) : t \geq 0\}$ (defined by

$$L_\nu(t) := \inf\{s \geq 0 : S_\nu(s) > t\},$$

where $\{S_\nu(t) : t \geq 0\}$ is a stable subordinator of order $\nu \in (0,1)$), for all $t \geq 0$ we have

$$\mathbb{E}[e^{\theta L_\nu(t)}] = E_\nu(\theta t^\nu) \quad \text{for all } \theta \in \mathbb{R}. \tag{2.2}$$

3. Results under Condition 1.4

We start with the reference LDP and the result does not depend on m in Assumption 1.1. In particular one can check that $I_{\mathrm{LD}}(x_1, \ldots, x_h) = 0$ if and only if $(x_1, \ldots, x_h) = (0, \ldots, 0)$.

Proposition 3.1. *Assume that Condition 1.4 holds. Let Ψ be the function defined by*

$$\Psi(\theta_1, \ldots, \theta_h) := \sum_{i=1}^{h} (c_i \kappa_{S_i}(\theta_i))^{1/\nu_i} 1_{\{\kappa_{S_i}(\theta_i) \geq 0\}} + \left(c_0 \sum_{i=1}^{h} \kappa_{S_i}(\theta_i)\right)^{1/\nu_0} 1_{\{\sum_{i=1}^h \kappa_{S_i}(\theta_i) \geq 0\}} \quad (\textit{for all } \theta_1, \ldots, \theta_h \geq 0),$$

and assume that it is an essentially smooth and lower semicontinuous function. Then

$$\left\{\left(\frac{S_1(c_1 L_1^{\nu_1}(t) + c_0 L_0^{\nu_0}(t))}{t}, \ldots, \frac{S_h(c_h L_h^{\nu_h}(t) + c_0 L_0^{\nu_0}(t))}{t}\right) : t > 0\right\}$$

satisfies the LDP with speed function t and rate function I_{LD} defined by

$$I_{\mathrm{LD}}(x_1,\dots,x_h):=\sup_{(\theta_1,\dots,\theta_h)\in\mathbb{R}^h}\left\{\sum_{i=1}^h\theta_ix_i-\Psi(\theta_1,\dots,\theta_h)\right\}\quad(\textit{for all }(x_1,\dots,x_h)\in\mathbb{R}^h).$$

Proof. The desired LDP can be derived by applying the Gärtner Ellis Theorem (*i.e.* Thm. 2.1). In fact, for all $t>0$ (and for all $(\theta_1,\dots,\theta_h)\in\mathbb{R}^h$), we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}\right]&=\mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i(r_i+r_0)}\right]_{r_0=c_0L_0^{\nu_0}(t),r_1=c_1L_1^{\nu_1}(t),\dots,r_h=c_hL_h^{\nu_h}(t)}\right]\\&=\mathbb{E}\left[\prod_{i=1}^h\mathbb{E}\left[\mathrm{e}^{\theta_iS_i(1)}\right]^{c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t)}\right]=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))\kappa_{S_i}(\theta_i)}\right]\\&=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^hc_iL_i^{\nu_i}(t)\kappa_{S_i}(\theta_i)+c_0L_0^{\nu_0}(t)\sum_{i=1}^h\kappa_{S_i}(\theta_i)}\right]\end{aligned}$$

whence we obtain (by the independence of the involved inverse stable subordinators, and by the expressions of the involved moment generating functions given by (2.2) with one among $\nu_0,\nu_1,\dots,\nu_h$ in place of ν)

$$\begin{aligned}\frac{1}{t}\log\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}\right]&=\sum_{i=1}^h\frac{1}{t}\log\mathbb{E}\left[\mathrm{e}^{c_iL_i^{\nu_i}(t)\kappa_{S_i}(\theta_i)}\right]+\frac{1}{t}\log\mathbb{E}\left[\mathrm{e}^{c_0L_0^{\nu_0}(t)\sum_{i=1}^h\kappa_{S_i}(\theta_i)}\right]\\&=\sum_{i=1}^h\frac{1}{t}\log E_{\nu_i}(c_i\kappa_{S_i}(\theta_i)t^{\nu_i})+\frac{1}{t}\log E_{\nu_0}\left(c_0\sum_{i=1}^h\kappa_{S_i}(\theta_i)t^{\nu_0}\right).\end{aligned}$$

Finally, if we take the limit as t tends to infinity, we get

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}\right]=\Psi(\theta_1,\dots,\theta_h)$$

by taking into account equation (2.1) with $\alpha=\nu_i$ for all $i\in\{1,\dots,h\}$, and for $\alpha=\nu_0$. □

In the following results we have to distinguish the cases $m=0$ and $m\neq0$, where $0\in\mathbb{R}^h$ is the null vector. We start with the weak convergence result; in our setting the weak convergence can be proved by taking the limit of moment generating functions (instead of characteristic functions).

Proposition 3.2. *Assume that Condition 1.4 holds. Let $\alpha_m(\nu)$ be defined in equation (1.2). We have the following statements.*

- *If $m=0$, then $\left\{\left(t^{\alpha_m(\nu_0\vee\nu_i)}\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t}\right)_{i\in\{1,\dots,h\}}:t>0\right\}$ converges weakly to*

$$\left(\sqrt{c_iL_i^{\nu_i}(1)\kappa''_{S_i}(0)1_{\{\nu_0\leq\nu_i\}}}Z_i+\sqrt{c_0L_0^{\nu_0}(1)\kappa''_{S_i}(0)1_{\{\nu_i\leq\nu_0\}}}\widehat{Z}_i\right)_{i\in\{1,\dots,h\}},$$

where $Z_1,\dots,Z_h,\widehat{Z}_1,\dots,\widehat{Z}_h$ are independent standard Normal distributed random variables, and independent of all the other random variables.

- *If* $m \neq 0$, *then* $\left\{\left(t^{\alpha_m(\nu_0\vee\nu_i)}\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t}\right)_{i\in\{1,\dots,h\}}: t>0\right\}$ *converges weakly to*

$$\left(c_im_i1_{\{\nu_0\leq\nu_i\}}L_i^{\nu_i}(1)+c_0m_i1_{\{\nu_i\leq\nu_0\}}L_0^{\nu_0}(1)\right)_{i\in\{1,\dots,h\}}.$$

Proof. In both cases $m=0$ and $m\neq 0$ we study suitable limits (as $t\to\infty$) of suitable moment generating functions, which can be expressed in terms of Mittag-Leffler functions (see (2.2), with one among $\nu_0,\nu_1,\dots,\nu_h$ in place of ν). Moreover

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_it^{\alpha_m(\nu_0\vee\nu_i)}\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t}}\right]&=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t^{1-\alpha_m(\nu_0\vee\nu_i)}}}\right]\\
&=\mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\frac{S_i(r_i+r_0)}{t^{\alpha_m(\nu_0\vee\nu_i)}}}\right]_{r_0=c_0L_0^{\nu_0}(t),r_1=c_1L_1^{\nu_1}(t),\dots,r_h=c_hL_h^{\nu_h}(t)}\right]\\
&=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu_0\vee\nu_i)})}\right]\\
&=\prod_{i=1}^h\mathbb{E}\left[\mathrm{e}^{c_i\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu_0\vee\nu_i)})L_i^{\nu_i}(t)}\right]\mathbb{E}\left[\mathrm{e}^{c_0\sum_{i=1}^h\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu_0\vee\nu_i)})L_0^{\nu_0}(t)}\right]\\
&=\prod_{i=1}^hE_{\nu_i}\left(c_i\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu_0\vee\nu_i)})t^{\nu_i}\right)E_{\nu_0}\left(c_0\sum_{i=1}^h\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu_0\vee\nu_i)})t^{\nu_0}\right).
\end{aligned}$$

We start with the case $m=0$. We have

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_it^{\alpha_m(\nu_0\vee\nu_i)}\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t}}\right]\\
&\quad=\prod_{i=1}^hE_{\nu_i}\left(c_i\left(\frac{\kappa_{S_i}''(0)}{2}\frac{\theta_i^2}{t^{\nu_0\vee\nu_i}}+o\left(\frac{1}{t^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_i}\right)E_{\nu_0}\left(c_0\sum_{i=1}^h\left(\frac{\kappa_{S_i}''(0)}{2}\frac{\theta_i^2}{t^{\nu_0\vee\nu_i}}+o\left(\frac{1}{t^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_0}\right)
\end{aligned}$$

and, by (2.1), one can check that the limit of the moment generating function (as $t\to\infty$) is equal to

$$\begin{aligned}
&\prod_{i=1}^hE_{\nu_i}\left(\frac{c_i\kappa_{S_i}''(0)}{2}1_{\{\nu_0\leq\nu_i\}}\theta_i^2\right)E_{\nu_0}\left(c_0\sum_{i=1}^h\frac{\kappa_{S_i}''(0)}{2}1_{\{\nu_i\leq\nu_0\}}\theta_i^2\right)\\
&\quad=\prod_{i=1}^h\mathbb{E}\left[\mathrm{e}^{\theta_i\sqrt{c_iL_i^{\nu_i}(1)\kappa_{S_i}''(0)1_{\{\nu_0\leq\nu_i\}}}Z_i}\right]\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\sqrt{c_0L_0^{\nu_0}(1)\kappa_{S_i}''(0)1_{\{\nu_i\leq\nu_0\}}}\widehat{Z}_i}\right]\\
&\quad=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\left(\sqrt{c_iL_i^{\nu_i}(1)\kappa_{S_i}''(0)1_{\{\nu_0\leq\nu_i\}}}Z_i+\sqrt{c_0L_0^{\nu_0}(1)\kappa_{S_i}''(0)1_{\{\nu_i\leq\nu_0\}}}\widehat{Z}_i\right)}\right].
\end{aligned}$$

Finally the case $m\neq 0$. We have

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_it^{\alpha_m(\nu_0\vee\nu_i)}\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t}}\right]\\
&\quad=\prod_{i=1}^hE_{\nu_i}\left(c_i\left(\kappa_{S_i}'(0)\frac{\theta_i}{t^{\nu_0\vee\nu_i}}+o\left(\frac{1}{t^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_i}\right)E_{\nu_0}\left(c_0\sum_{i=1}^h\left(\kappa_{S_i}'(0)\frac{\theta_i}{t^{\nu_0\vee\nu_i}}+o\left(\frac{1}{t^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_0}\right)
\end{aligned}$$

and, by (2.1), one can check that the limit of the moment generating function (as $t\to\infty$) is equal to (we recall that $\kappa'_{S_i}(0)=m_i$)

$$\begin{aligned}\prod_{i=1}^h E_{\nu_i}\left(c_i\kappa'_{S_i}(0)1_{\{\nu_0\leq\nu_i\}}\theta_i\right)E_{\nu_0}\left(c_0\sum_{i=1}^h\kappa'_{S_i}(0)1_{\{\nu_i\leq\nu_0\}}\theta_i\right)\\ =\prod_{i=1}^h\mathbb{E}\left[\mathrm{e}^{c_i\theta_i m_i 1_{\{\nu_0\leq\nu_i\}}L_i^{\nu_i}(1)}\right]\mathbb{E}\left[\mathrm{e}^{c_0\sum_{i=1}^h\theta_i m_i 1_{\{\nu_i\leq\nu_0\}}L_0^{\nu_0}(1)}\right]\\ =\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\left(c_i m_i 1_{\{\nu_0\leq\nu_i\}}L_i^{\nu_i}(1)+c_0 m_i 1_{\{\nu_i\leq\nu_0\}}L_0^{\nu_0}(1)\right)}\right].\end{aligned}$$

□

Now we conclude with the noncentral moderate deviation result.

Proposition 3.3. *Assume that Condition 1.4 holds. Let $\alpha_m(\nu)$ be defined in equation (1.2). Then, for every family of positive numbers $\{a_t:t>0\}$ such that equation (1.1) holds, the family of random variables*

$$\left\{\left((ta_t)^{\alpha_m(\nu_0\vee\nu_i)}\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t}\right)_{i\in\{1,\dots,h\}}:t>0\right\}$$

satisfies the LDP with speed $1/a_t$ and good rate function $I_{\mathrm{MD}}(\cdot;m)$ defined by

$$I_{\mathrm{MD}}(x_1,\dots,x_h;m):=\sup_{(\theta_1,\dots,\theta_h)\in\mathbb{R}^h}\left\{\sum_{i=1}^h\theta_ix_i-\widetilde{\Psi}_m(\theta_1,\dots,\theta_h)\right\},$$

where

$$\widetilde{\Psi}_m(\theta_1,\dots,\theta_h)=\begin{cases}\sum_{i=1}^h\left(\frac{c_i\kappa''_{S_i}(0)}{2}1_{\{\nu_0\leq\nu_i\}}\theta_i^2\right)^{1/\nu_i}\\ +\left(c_0\sum_{i=1}^h\frac{\kappa''_{S_i}(0)}{2}1_{\{\nu_i\leq\nu_0\}}\theta_i^2\right)^{1/\nu_0} & \text{if } m=0\\ \sum_{i=1}^h\left(c_im_i1_{\{\nu_0\leq\nu_i\}}\theta_i1_{\{\theta_im_i\geq0\}}\right)^{1/\nu_i}\\ +\left(c_0\sum_{i=1}^h m_i1_{\{\nu_i\leq\nu_0\}}\theta_i1_{\{\theta_im_i\geq0\}}\right)^{1/\nu_0} & \text{if } m\neq0.\end{cases}\tag{3.1}$$

Proof. For every $m\in\mathbb{R}$ we apply the Gärtner Ellis Theorem (Thm. 2.1). So we have to show that

$$\lim_{t\to\infty}\underbrace{\frac{1}{1/a_t}\log\mathbb{E}\left[\mathrm{e}^{\frac{1}{a_t}\sum_{i=1}^h\theta_i(ta_t)^{\alpha_m(\nu_0\vee\nu_i)}\frac{S_i(c_iL_i^{\nu_i}(t)+c_0L_0^{\nu_0}(t))}{t}}\right]}_{=:\rho(t)}=\widetilde{\Psi}_m(\theta_1,\dots,\theta_h)\quad(\text{for all }(\theta_1,\dots,\theta_h)\in\mathbb{R}^h),$$

where $\widetilde{\Psi}_m(\theta_1,\dots,\theta_h)$ is defined by equation (3.1). Indeed, for every $m\in\mathbb{R}^h$, the function $(\theta_1,\dots,\theta_h)\mapsto\widetilde{\Psi}_m(\theta_1,\dots,\theta_h)$ is finite-valued and differentiable (thus the hypotheses of the Gärtner Ellis Theorem are satisfied).

In view of what follows we remark that

$$\rho(t) = a_t \log \mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h \theta_i \frac{S_i(c_i L_i^{\nu_i}(t)+c_0 L_0^{\nu_0}(t))}{(ta_t)^{1-\alpha_m(\nu_0\vee\nu_i)}}}\right] = a_t \log \mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h \kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu_0\vee\nu_i)})(c_i L_i^{\nu_i}(t)+c_0 L_0^{\nu_0}(t))}\right]$$
$$= a_t \left\{\sum_{i=1}^h \log \mathbb{E}\left[\mathrm{e}^{c_i\kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu_0\vee\nu_i)})L_i^{\nu_i}(t)}\right] + \log \mathbb{E}\left[\mathrm{e}^{c_0\sum_{i=1}^h \kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu_0\vee\nu_i)})L_0^{\nu_0}(t)}\right]\right\}$$
$$= a_t \left\{\sum_{i=1}^h \log E_{\nu_i}\left(c_i\kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu_0\vee\nu_i)})t^{\nu_i}\right) + \log E_{\nu_0}\left(c_0\sum_{i=1}^h \kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu_0\vee\nu_i)})t^{\nu_0}\right)\right\}.$$

We start with the case $m = 0$. We have

$$\rho(t) = a_t \left\{\sum_{i=1}^h \log E_{\nu_i}\left(\left(\frac{c_i\kappa''_{S_i}(0)}{2}\frac{\theta_i^2}{(ta_t)^{\nu_0\vee\nu_i}} + o\left(\frac{1}{(ta_t)^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_i}\right)\right.$$
$$\left. + \log E_{\nu_0}\left(\sum_{i=1}^h\left(\frac{c_0\kappa''_{S_i}(0)}{2}\frac{\theta_i^2}{(ta_t)^{\nu_0\vee\nu_i}} + o\left(\frac{1}{(ta_t)^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_0}\right)\right\}$$

and, by equation (2.1), we obtain

$$\lim_{t\to\infty}\rho(t) = \sum_{i=1}^h\left(\frac{c_i\kappa''_{S_i}(0)}{2}1_{\{\nu_0\leq\nu_i\}}\theta_i^2\right)^{1/\nu_i} + \left(c_0\sum_{i=1}^h\frac{\kappa''_{S_i}(0)}{2}1_{\{\nu_i\leq\nu_0\}}\theta_i^2\right)^{1/\nu_0}.$$

Finally the case $m \neq 0$. We have

$$\rho(t) = a_t \left\{\sum_{i=1}^h \log E_{\nu_i}\left(\left(c_i\kappa'_{S_i}(0)\frac{\theta_i}{(ta_t)^{\nu_0\vee\nu_i}} + o\left(\frac{1}{(ta_t)^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_i}\right)\right.$$
$$\left. + \log E_{\nu_0}\left(\sum_{i=1}^h\left(c_0\kappa'_{S_i}(0)\frac{\theta_i}{(ta_t)^{\nu_0\vee\nu_i}} + o\left(\frac{1}{(ta_t)^{\nu_0\vee\nu_i}}\right)\right)t^{\nu_0}\right)\right\},$$

and, by equation (2.1), we obtain

$$\lim_{t\to\infty}\rho(t) = \sum_{i=1}^h\left(c_i m_i 1_{\{\nu_0\leq\nu_i\}}\theta_i 1_{\{\theta_i m_i\geq 0\}}\right)^{1/\nu_i} + \left(c_0\sum_{i=1}^h m_i 1_{\{\nu_i\leq\nu_0\}}\theta_i 1_{\{\theta_i m_i\geq 0\}}\right)^{1/\nu_0}.$$

$\square$

Remark 3.4. The weak limit in Proposition 3.2 is a sum of two independent vectors

$$(B_1, \dots, B_h) + (\widehat{B}_1, \dots, \widehat{B}_h)$$

for some random variables $B_1, \dots, B_h, \widehat{B}_1, \dots, \widehat{B}_h$ (say), where $B_1, \dots, B_h$ are independent, and $\widehat{B}_1, \dots, \widehat{B}_h$ are conditionally independent given $L_0^{\nu_0}(1)$; actually, if $m \neq 0$, $(\widehat{B}_1, \dots, \widehat{B}_h)$ is a deterministic vector multiplied by

$L_0^{\nu_0}(1)$. This can be related with the expressions of the function $\widetilde{\Psi}_m$ defined by equation (3.1) (for both cases $m=0$ and $m\neq 0$), *i.e.*

$$\widetilde{\Psi}_m(\theta_1,\dots,\theta_h)=\sum_{i=1}^h f_i^{1/\nu_i}(\theta_i)+\left(\sum_{i=1}^h \widehat{f}_i(\theta_i)\right)^{1/\nu_0}$$

for some functions $f_1\dots,f_h,\widehat{f}_1,\dots,\widehat{f}_h$ (say).

We also remark that, for every $i\in\{1,\dots,h\}$, the factors $1_{\{\nu_0\leq\nu_i\}}$ and $1_{\{\nu_i\leq\nu_0\}}$ appear in B_i and $\widehat{B}_i$ (and also in f_i and $\widehat{f}_i$ as well), respectively. These two factors are both equal to 1 if and only if $\nu_0=\nu_i$; otherwise one factor is equal to 1, and the other one is equal to zero.

4. Some cases with an explicit expression for $I_{\mathrm{MD}}(\cdot;m)$

In general we do not have an explicit expression for the rate function I_{LD} concerning the reference LDP (see Prop. 3.1); this is not surprising because we have the same situation for the univariate processes studied in the literature (see *e.g.* Prop. 3.1 in [10]). On the contrary, by taking into account again what happens for the univariate processes (see *e.g.* Prop. 3.3 in [10]), we can wish to get an explicit expression for the rate function $I_{\mathrm{MD}}(\cdot;m)$. In this section we present some cases in which this is possible.

In view of what follows it is useful to consider the following notation.

- For $m=0$, we consider the function $U_{\nu_i,\kappa''_{S_i}(0)}:\mathbb{R}\to[0,\infty]$ defined by

  $$U_{\nu_i,\kappa''_{S_i}(0)}(x):=((\nu_i/2)^{\nu_i/(2-\nu_i)}-(\nu_i/2)^{2/(2-\nu_i)})\left(\frac{2x^2}{\kappa''_{S_i}(0)}\right)^{1/(2-\nu_i)}$$

  for $\kappa''_{S_i}(0)>0$, and

  $$U_{\nu_i,\kappa''_{S_i}(0)}(x):=\begin{cases}0 & \text{if } x=0\\ \infty & \text{if } x\neq 0.\end{cases}$$

  for $\kappa''_{S_i}(0)=0$.
- For $m\neq 0$ (note that, in general, we can have either $m_i=0$ or $m_i\neq 0$, but there exists $j\in\{1,\dots,h\}$ such that $m_j\neq 0$), we consider the function $V_{\nu_i,m_i}:\mathbb{R}\to[0,\infty]$ defined by

  $$V_{\nu_i,m_i}(x):=\begin{cases}(\nu_i^{\nu_i/(1-\nu_i)}-\nu_i^{1/(1-\nu_i)})\left(\frac{x_i}{m_i}\right)^{1/(1-\nu_i)} & \text{if } \frac{x_i}{m_i}\geq 0\\ \infty & \text{if } \frac{x_i}{m_i}<0\end{cases}$$

  for $m_i\neq 0$, and

  $$V_{\nu_i,m_i}(x):=\begin{cases}0 & \text{if } x=0\\ \infty & \text{if } x\neq 0\end{cases}$$

  for $m_i=0$.

4.1. On the case $\nu_0 < \min\{\nu_1, \ldots, \nu_h\}$

If $\nu_0 < \min\{\nu_1, \ldots, \nu_h\}$, then the function in (3.1) reads

$$\widetilde{\Psi}_m(\theta_1, \ldots, \theta_h) = \begin{cases} \sum_{i=1}^h \left(\frac{c_i\kappa''_{S_i}(0)}{2}\theta_i^2\right)^{1/\nu_i} & \text{if } m = 0 \\ \sum_{i=1}^h \left(c_i m_i \theta_i 1_{\{\theta_i m_i \geq 0\}}\right)^{1/\nu_i} & \text{if } m \neq 0. \end{cases}$$

So, in both cases $m = 0$ and $m \neq 0$, the function $\widetilde{\Psi}_m$ is a sum of h functions, and the i-th function depends on θ_i only. In this case $I_{\mathrm{MD}}(\cdot; m) = I_{\mathrm{MD}}(x_1, \ldots, x_h; m)$, which is the Legendre-Fenchel transform of $\widetilde{\Psi}_m$, is a sum of h functions, and the i-th function depends on x_i only; so in some sense we have the *asymptotic independence* of the components. In detail we have the following

Proposition 4.1. *Assume that* $\nu_0 < \min\{\nu_1, \ldots, \nu_h\}$. *Then*

$$I_{\mathrm{MD}}(x_1, \ldots, x_h; m) = \begin{cases} \sum_{i=1}^h U_{\nu_i, c_i\kappa''_{S_i}(0)}(x_i) & \textit{if } m = 0 \\ \sum_{i=1}^h V_{\nu_i, c_i m_i}(x_i) & \textit{if } m \neq 0. \end{cases}$$

Proof. By construction we have

$$I_{\mathrm{MD}}(x_1, \ldots, x_h; m) := \begin{cases} \sum_{i=1}^h \sup_{\theta_i \in \mathbb{R}} \left\{\theta_i x_i - \left(\frac{c_i\kappa''_{S_i}(0)}{2}\theta_i^2\right)^{1/\nu_i}\right\} & \text{if } m = 0 \\ \sum_{i=1}^h \sup_{\theta_i \in \mathbb{R}} \left\{\theta_i x_i - \left(c_i m_i \theta_i 1_{\{\theta_i m_i \geq 0\}}\right)^{1/\nu_i}\right\} & \text{if } m \neq 0. \end{cases}$$

We start with the case $m = 0$. For every $i \in \{1, \ldots, h\}$ we have to check that

$$\sup_{\theta_i \in \mathbb{R}} \left\{\theta_i x_i - \left(\frac{c_i\kappa''_{S_i}(0)}{2}\theta_i^2\right)^{1/\nu_i}\right\} = U_{\nu_i, c_i\kappa''_{S_i}(0)}(x_i).$$

The case $\kappa''_{S_i}(0) = 0$ is immediate. For $\kappa''_{S_i}(0) > 0$ it is easy to check with some computations that the supremum is attained at

$$\theta_i = \left(\frac{2}{c_i\kappa''_{S_i}(0)}\right)^{1/(2-\nu_i)} \left(\frac{\nu_i}{2}x_i\right)^{\nu_i/(2-\nu_i)},$$

and we get

$$\begin{aligned} &\sup_{\theta_i \in \mathbb{R}} \left\{\theta_i x_i - \left(\frac{c_i\kappa''_{S_i}(0)}{2}\theta_i^2\right)^{1/\nu_i}\right\} \\ &\quad = \left(\frac{2}{c_i\kappa''_{S_i}(0)}\right)^{1/(2-\nu_i)} \left(\frac{\nu_i}{2}x_i\right)^{\nu_i/(2-\nu_i)} x_i - \left(\frac{c_i\kappa''_{S_i}(0)}{2}\left(\frac{2}{c_i\kappa''_{S_i}(0)}\right)^{2/(2-\nu_i)} \left(\frac{\nu_i}{2}x_i\right)^{2\nu_i/(2-\nu_i)}\right)^{1/\nu_i} \\ &\quad = \left(\frac{2}{c_i\kappa''_{S_i}(0)}\right)^{1/(2-\nu_i)} \left(\frac{\nu_i}{2}\right)^{\nu_i/(2-\nu_i)} x_i^{2/(2-\nu_i)} - \left(\frac{2}{c_i\kappa''_{S_i}(0)}\right)^{1/(2-\nu_i)} \left(\frac{\nu_i}{2}\right)^{2/(2-\nu_i)} x_i^{2/(2-\nu_i)} \\ &\quad = \left(\left(\frac{\nu_i}{2}\right)^{\nu_i/(2-\nu_i)} - \left(\frac{\nu_i}{2}\right)^{2/(2-\nu_i)}\right) \left(\frac{2x_i^2}{c_i\kappa''_{S_i}(0)}\right) = U_{\nu_i, c_i\kappa''_{S_i}(0)}(x_i). \end{aligned}$$

Finally the case $m \neq 0$. For every $i \in \{1, \ldots, h\}$ we have to check that

$$\sup_{\theta_i \in \mathbb{R}} \left\{ \theta_i x_i - \left(c_i m_i \theta_i 1_{\{\theta_i m_i \geq 0\}}\right)^{1/\nu_i} \right\} = V_{\nu_i, c_i m_i}(x_i).$$

The case $m_i = 0$ is immediate. For $m_i > 0$ one can check that: for $x_i < 0$ the supremum is equal to infinity by taking the limit as $\theta_i \to -\infty$; for $x_i > 0$ the supremum is attained at

$$\theta_i = \frac{(\nu_i x_i)^{\nu_i/(1-\nu_i)}}{(c_i m_i)^{1/(1-\nu_i)}}$$

(after some computations), and we get

$$\begin{aligned}
&\sup_{\theta_i \in \mathbb{R}} \left\{ \theta_i x_i - \left(c_i m_i \theta_i 1_{\{\theta_i m_i \geq 0\}}\right)^{1/\nu_i} \right\} \\
&\quad = \frac{(\nu_i x_i)^{\nu_i/(1-\nu_i)}}{(c_i m_i)^{1/(1-\nu_i)}} x_i - \left(c_i m_i \frac{(\nu_i x_i)^{\nu_i/(1-\nu_i)}}{(c_i m_i)^{1/(1-\nu_i)}} \right)^{1/\nu_i} = \frac{\nu_i^{\nu_i/(1-\nu_i)}}{(c_i m_i)^{1/(1-\nu_i)}} x_i^{1/(1-\nu_i)} - \frac{\nu_i^{1/(1-\nu_i)}}{(c_i m_i)^{1/(1-\nu_i)}} x_i^{1/(1-\nu_i)} \\
&\quad = \left(\nu_i^{\nu_i/(1-\nu_i)} - \nu_i^{1/(1-\nu_i)}\right) \left(\frac{x_i}{c_i m_i} \right)^{1/(1-\nu_i)} = V_{\nu_i, c_i m_i}(x_i).
\end{aligned}$$

For $m_i < 0$ we remark that

$$\sup_{\theta_i \in \mathbb{R}} \left\{ \theta_i x_i - \left(c_i m_i \theta_i 1_{\{\theta_i m_i \geq 0\}}\right)^{1/\nu_i} \right\} = \sup_{\theta_i \in \mathbb{R}} \left\{ -\theta_i(-x_i) - \left(c_i(-m_i)(-\theta_i) 1_{\{(-\theta_i)(-m_i) \geq 0\}}\right)^{1/\nu_i} \right\}$$

and, by referring to the previous part of the proof with $-m_i > 0$ in place of m_i, we get

$$\sup_{\theta_i \in \mathbb{R}} \left\{ -\theta_i(-x_i) - \left(c_i(-m_i)(-\theta_i) 1_{\{(-\theta_i)(-m_i) \geq 0\}}\right)^{1/\nu_i} \right\} = V_{\nu_i, c_i(-m_i)}(-x_i).$$

Then, since $V_{\nu_i, c_i(-m_i)}(-x_i) = V_{\nu_i, c_i m_i}(x_i)$ by construction, the desired equality holds also for $m_i < 0$. $\square$

Remark 4.2. We do not have the asymptotic independence of the components for the rate function $I_{\mathrm{LD}} = I_{\mathrm{LD}}(x_1, \ldots, x_h)$ (as we said above for $I_{\mathrm{MD}}(\cdot; m)$). Indeed, in general (if $\nu_0 < \min\{\nu_1, \ldots, \nu_h\}$ holds or not), the function Ψ in Proposition 3.1 cannot be expressed as a sum of h functions such that the i-th function depends on θ_i only.

4.2. On the case $\nu_0 > \max\{\nu_1, \ldots, \nu_h\}$ with $m \neq 0$

If $\nu_0 > \max\{\nu_1, \ldots, \nu_h\}$, then the function in (3.1) reads

$$\widetilde{\Psi}_m(\theta_1, \ldots, \theta_h) = \begin{cases} \left(c_0 \sum_{i=1}^h \frac{\kappa''_{S_i}(0)}{2} \theta_i^2 \right)^{1/\nu_0} & \text{if } m = 0 \\ \left(c_0 \sum_{i=1}^h m_i \theta_i 1_{\{\theta_i m_i \geq 0\}} \right)^{1/\nu_0} & \text{if } m \neq 0. \end{cases}$$

Actually here we concentrate our attention on the case $m \neq 0$ only. We have the following

Proposition 4.3. *Assume that $\nu_0 > \max\{\nu_1, \ldots, \nu_h\}$ and $m \neq 0$. Moreover consider the following conditions:*
(i) *$x_i m_i \geq 0$ for every $i \in \{1, \ldots, h\}$;*

(ii) *if* $m_j = 0$ *some* $j \in \{1, \dots, h\}$*, then* $x_j = 0$.
Then, if we consider the set

$$\mathcal{A} := \{(x_1, \dots, x_h) \in \mathbb{R}^h : \text{(i) } \textit{and } \text{(ii) } \textit{holds}\},$$

we have

$$I_{\mathrm{MD}}(x_1, \dots, x_h; m) = \begin{cases} \max\{V_{\nu_0, c_0 m_i}(x_i) : i \in \{1, \dots, h\}\} & (x_1, \dots, x_h) \in \mathcal{A} \\ \infty & (x_1, \dots, x_h) \notin \mathcal{A}. \end{cases}$$

Proof. We start noting that, if $h = 1$, we can refer to a slight modification of Proposition 3.3 in [10] (here we have to consider $c_0 m_1$ in place of m_1, and $m_1 \neq 0$), and therefore

$$I_{\mathrm{MD}}(x_1; m_1) = \begin{cases} V_{\nu_0, c_0 m_1}(x_1) & x_1 m_1 \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

So, from now on, we essentially refer to the case $h \geq 2$. We start with $(x_1, \dots, x_h) \notin \mathcal{A}$, and we have two cases.

- If (i) fails, then there exists $j \in \{1, \dots, h\}$ such that $x_j m_j < 0$. So we have two cases: if $x_j > 0$ and $m_j < 0$, then we have

$$\lim_{\theta_j \to +\infty} \theta_j x_j - \left(c_0 m_j \theta_j 1_{\{\theta_j m_j \geq 0\}}\right)^{1/\nu_0} = \lim_{\theta_j \to +\infty} \theta_j x_j = +\infty;$$

  if $x_j < 0$ and $m_j > 0$, then have

$$\lim_{\theta_j \to -\infty} \theta_j x_j - \left(c_0 m_j \theta_j 1_{\{\theta_j m_j \geq 0\}}\right)^{1/\nu_0} = \lim_{\theta_j \to -\infty} \theta_j x_j = +\infty.$$

- If (ii) fails, then we have $m_j = 0$ some $j \in \{1, \dots, h\}$ and $x_j \neq 0$; so we immediately get

$$\sup_{\theta_j \in \mathbb{R}} \left\{\theta_j x_j - \left(c_0 m_j \theta_j 1_{\{\theta_j m_j \geq 0\}}\right)^{1/\nu_0}\right\} = \sup_{\theta_j \in \mathbb{R}} \{\theta_j x_j\} = +\infty.$$

Now we consider $(x_1, \dots, x_h) \in \mathcal{A}$. By taking into account what we have shown above, we do not lose of generality if we assume that $m_i \neq 0$ for every $i \in \{1, \dots, h\}$; moreover we can say that

$$I_{\mathrm{MD}}(x_1, \dots, x_h; m) = \sup\left\{\sum_{i=1}^h \theta_i x_i - \left(c_0 \sum_{i=1}^h \theta_i m_i\right)^{1/\nu_0} : \theta_i x_i \geq 0 \text{ for every } i \in \{1, \dots, h\}\right\}. \tag{4.1}$$

Then we can consider the equations for the stationary points

$$x_j = \frac{c_0^{1/\nu_0}}{\nu_0} \left(\sum_{i=1}^h \theta_i m_i\right)^{1/\nu_0 - 1} m_j \quad \text{for every } j \in \{1, \dots, h\}, \tag{4.2}$$

which admit solutions if

$$\text{there exists } \alpha = \alpha(x_1, \dots, x_h) \geq 0 \text{ such that } x_i = \alpha m_i \text{ for every } i \in \{1, \dots, h\}. \tag{4.3}$$

So we have two cases: (4.3) holds, or (4.3) fails. If (4.3) holds, then (after some computations, in which we set $\eta = \sum_{i=1}^{h} \theta_i m_i$) we get

$$I_{\mathrm{MD}}(\alpha m_1, \ldots, \alpha m_h; m) = \sup_{\eta \geq 0}\{\alpha\eta - (c_0\eta)^{1/\nu_0}\} = \{\alpha\eta - (c_0\eta)^{1/\nu_0}\}\Big|_{\eta=(\frac{\alpha\nu_0}{c})^{\nu_0/(1-\nu_0)}} = V_{\nu_0,1}\left(\frac{\alpha}{c_0}\right).$$

Then, in this case, the desired result is proved because we have

$$V_{\nu_0,c_0 m_i}(x_i) = V_{\nu_0,1}\left(\frac{\alpha}{c_0}\right) \quad \text{for every } i \in \{1, \ldots, h\}.$$

If (4.3) fails, then the supremum in (4.1) is attained when we have $\theta_j = 0$ for some $j \in \{1, \ldots, h\}$. Thus

$$I_{\mathrm{MD}}(x_1, \ldots, x_h; m) = \max\{\varrho_1, \ldots, \varrho_h\},$$

where

$$\varrho_j = \sup\left\{\sum_{i=1}^{h} \theta_i x_i - \left(c_0 \sum_{i=1}^{h} \theta_i m_i\right)^{1/\nu_0} : \theta_i x_i \geq 0 \text{ for every } i \in \{1, \ldots, h\},\ \text{and } \theta_j = 0\right\}$$

for every $j \in \{1, \ldots, h\}$. Then, by induction on h (we already know that the case $h = 1$ works well), we get

$$\varrho_j = \max\{V_{\nu_0,c_0 m_i}(x_i) : i \in \{1, \ldots, h\} \setminus \{j\}\} \quad \text{for every } j \in \{1, \ldots, h\};$$

thus we easily obtain

$$I_{\mathrm{MD}}(x_1, \ldots, x_h; m) = \max\{V_{\nu_0,c_0 m_i}(x_i) : i \in \{1, \ldots, h\}\}$$

as desired. □

5. Results under Condition 1.5

We start with the reference LDP and the result does not depend on m in Assumption 1.1. In particular one can check that $J_{\mathrm{LD}}(x_1, \ldots, x_h) = 0$ if and only if $(x_1, \ldots, x_h) = (0, \ldots, 0)$.

Proposition 5.1. *Assume that Condition 1.5 holds. Let Υ be the function defined by*

$$\Upsilon(\theta_1, \ldots, \theta_h) := \sum_{j=1}^{k}\left(\sum_{i=1}^{h} c_{ij}\kappa_{S_i}(\theta_i)\right)^{1/\nu} 1_{\{\sum_{i=1}^{h} c_{ij}\kappa_{S_i}(\theta_i) \geq 0\}} \quad (\text{for all } \theta_1, \ldots, \theta_h \geq 0),$$

and assume that it is an essentially smooth and lower semicontinuous function. Then

$$\left\{\left(\frac{S_1\left(\sum_{j=1}^{k} c_{1j} L_j^{\nu}(t)\right)}{t}, \ldots, \frac{S_h\left(\sum_{j=1}^{k} c_{hj} L_j^{\nu}(t)\right)}{t}\right) : t > 0\right\}$$

satisfies the LDP with speed function t and rate function J_{LD} defined by

$$J_{\mathrm{LD}}(x_1,\dots,x_h):=\sup_{(\theta_1,\dots,\theta_h)\in\mathbb{R}^h}\left\{\sum_{i=1}^h\theta_ix_i-\Upsilon(\theta_1,\dots,\theta_h)\right\}\quad (\textit{for all } (x_1,\dots,x_h)\in\mathbb{R}^h).$$

Proof. The desired LDP can be derived by applying the Gärtner Ellis Theorem (*i.e.* Thm. 2.1). In fact, for all $t>0$ (and for all $(\theta_1,\dots,\theta_h)\in\mathbb{R}^h$), we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i\left(\sum_{j=1}^kc_{ij}L_j^\nu(t)\right)}\right]&=\mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i\left(\sum_{j=1}^kc_{ij}r_j\right)}\right]_{r_1=L_1^\nu(t),\dots,r_k=L_k^\nu(t)}\right]\\&=\mathbb{E}\left[\prod_{i=1}^h\mathbb{E}\left[\mathrm{e}^{\theta_iS_i(1)}\right]^{\sum_{j=1}^kc_{ij}L_j^\nu(t)}\right]=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h(\sum_{j=1}^kc_{ij}L_j^\nu(t))\kappa_{S_i}(\theta_i)}\right]\\&=\mathbb{E}\left[\mathrm{e}^{\sum_{j=1}^k(\sum_{i=1}^hc_{ij}\kappa_{S_i}(\theta_i))L_j^\nu(t)}\right]\end{aligned}$$

whence we obtain (by the independence of the involved inverse stable subordinators, and by the expressions of the involved moment generating functions given by (2.2))

$$\frac{1}{t}\log\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i\left(\sum_{j=1}^kc_{ij}L_j^\nu(t)\right)}\right]=\sum_{j=1}^k\frac{1}{t}\log\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^hc_{ij}\kappa_{S_i}(\theta_i)L_j^\nu(t)}\right]=\sum_{j=1}^k\frac{1}{t}\log E_\nu\left(\sum_{i=1}^hc_{ij}\kappa_{S_i}(\theta_i)t^\nu\right).$$

Finally, if we take the limit as t tends to infinity, we get

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_iS_i\left(\sum_{j=1}^kc_{ij}L_j^\nu(t)\right)}\right]=\Upsilon(\theta_1,\dots,\theta_h)$$

by taking into account equation (2.1) with $\alpha=\nu$. □

As happens for the analogue results in Section 3, in the following results we have to distinguish the cases $m=0$ and $m\neq0$, where $0\in\mathbb{R}^h$ is the null vector. We start with the weak convergence result and, again, we take the limit of moment generating functions (instead of characteristic functions).

Proposition 5.2. *Assume that Condition 1.5 holds. Let $\alpha_m(\nu)$ be defined in equation (1.2). We have the following statements.*

- *If $m=0$, then $\left\{\left(t^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^kc_{ij}L_j^\nu(t)\right)}{t}\right)_{i\in\{1,\dots,h\}}:t>0\right\}$ converges weakly to*

$$\left(\sqrt{\sum_{j=1}^kc_{ij}L_j^\nu(1)\kappa''_{S_i}(0)}Z_i\right)_{i\in\{1,\dots,h\}},$$

  where $Z_1,\dots,Z_h$ are independent standard Normal distributed random variables, and independent of all the other random variables.

- *If* $m\neq 0$, *then* $\left\{\left(t^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}\right)_{i\in\{1,\dots,h\}}:t>0\right\}$ *converges weakly to*

$$\left(m_i\sum_{j=1}^k c_{ij}L_j^\nu(1)\right)_{i\in\{1,\dots,h\}}.$$

Proof. In both cases $m=0$ and $m\neq 0$ we study suitable limits (as $t\to\infty$) of suitable moment generating functions, which can be expressed in terms of Mittag-Leffler functions (see (2.2)). Moreover

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i t^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}}\right]&=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t^{1-\alpha_m(\nu)}}}\right]\\&=\mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\frac{S_i\left(\sum_{j=1}^k c_{ij}r_j\right)}{t^{1-\alpha_m(\nu)}}}\right]_{r_1=L_1^\nu(t),\dots,r_k=L_k^\nu(t)}\right]=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu)})}\right]\\&=\prod_{j=1}^k\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h c_{ij}\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu)})L_j^\nu(t)}\right]=\prod_{j=1}^k E_\nu\left(\sum_{i=1}^h c_{ij}\kappa_{S_i}(\theta_i/t^{1-\alpha_m(\nu)})t^\nu\right).\end{aligned}$$

We start with the case $m=0$. We have

$$\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i t^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}}\right]=\prod_{j=1}^k E_\nu\left(\sum_{i=1}^h c_{ij}\left(\frac{\kappa''_{S_i}(0)}{2}\frac{\theta_i^2}{t^\nu}+o\left(\frac{1}{t^\nu}\right)\right)t^\nu\right)$$

and, by equation (2.1), we get

$$\lim_{t\to\infty}\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i t^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}}\right]=\prod_{j=1}^k E_\nu\left(\sum_{i=1}^h\frac{c_{ij}\kappa''_{S_i}(0)}{2}\theta_i^2\right)=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i\sqrt{\sum_{j=1}^k c_{ij}L_j^\nu(1)\kappa''_{S_i}(0)}Z_i}\right]$$

(the second equality can be easily checked).

Finally the case $m\neq 0$. We have

$$\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i t^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}}\right]=\prod_{j=1}^k E_\nu\left(\sum_{i=1}^h c_{ij}\left(\kappa'_{S_i}(0)\frac{\theta_i}{t^\nu}+o\left(\frac{1}{t^\nu}\right)\right)t^\nu\right)$$

and, by equation (2.1), we get

$$\lim_{t\to\infty}\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i t^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}}\right]=\prod_{j=1}^k E_\nu\left(\sum_{i=1}^h c_{ij}\kappa'_{S_i}(0)\theta_i\right)=\mathbb{E}\left[\mathrm{e}^{\sum_{i=1}^h\theta_i m_i\sum_{j=1}^k c_{ij}L_j^\nu(1)}\right]$$

(the second equality can be easily checked; we recall that $\kappa'_{S_i}(0)=m_i$). $\square$

Now we conclude with the noncentral moderate deviation result.

Proposition 5.3. *Assume that Condition 1.5 holds. Let $\alpha_m(\nu)$ be defined in equation (1.2). Then, for every family of positive numbers $\{a_t : t > 0\}$ such that equation (1.1) holds, the family of random variables*

$$\left\{\left((ta_t)^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}\right)_{i\in\{1,\dots,h\}} : t > 0\right\}$$

satisfies the LDP with speed $1/a_t$ and good rate function $J_{\mathrm{MD}}(\cdot;m)$ defined by

$$J_{\mathrm{MD}}(x_1,\dots,x_h;m) := \sup_{(\theta_1,\dots,\theta_h)\in\mathbb{R}^h}\left\{\sum_{i=1}^h \theta_i x_i - \widetilde{\Upsilon}_m(\theta_1,\dots,\theta_h)\right\},$$

where

$$\widetilde{\Upsilon}_m(\theta_1,\dots,\theta_h) = \begin{cases} \sum_{j=1}^k\left(\sum_{i=1}^h \frac{c_{ij}\kappa''_{S_i}(0)}{2}\theta_i^2\right)^{1/\nu} & \text{if } m = 0\\ \sum_{j=1}^k\left(\sum_{i=1}^h c_{ij}m_i\theta_i 1_{\{\theta_i m_i\geq 0\}}\right)^{1/\nu} & \text{if } m \neq 0. \end{cases} \tag{5.1}$$

Proof. As in the proof of Proposition 3.3, for every $m \in \mathbb{R}$ we apply the Gärtner Ellis Theorem (Thm. 2.1). So we have to show that

$$\lim_{t\to\infty}\underbrace{\frac{1}{1/a_t}\log\mathbb{E}\left[e^{\frac{1}{a_t}\sum_{i=1}^h\theta_i(ta_t)^{\alpha_m(\nu)}\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{t}}\right]}_{=:\rho(t)} = \widetilde{\Upsilon}_m(\theta_1,\dots,\theta_h) \quad (\text{for all } (\theta_1,\dots,\theta_h)\in\mathbb{R}^h),$$

where $\widetilde{\Upsilon}_m(\theta_1,\dots,\theta_h)$ is defined by equation (5.1). Indeed, for every $m \in \mathbb{R}^h$, the function $(\theta_1,\dots,\theta_h) \mapsto \widetilde{\Upsilon}_m(\theta_1,\dots,\theta_h)$ is finite-valued and differentiable (thus the hypotheses of the Gärtner Ellis Theorem are satisfied).

In view of what follows we remark that

$$\begin{aligned}\rho(t) &= a_t\log\mathbb{E}\left[e^{\sum_{i=1}^h\theta_i\frac{S_i\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)}{(ta_t)^{1-\alpha_m(\nu)}}}\right]\\ &= a_t\log\mathbb{E}\left[e^{\sum_{i=1}^h\left(\sum_{j=1}^k c_{ij}L_j^\nu(t)\right)\kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu)})}\right] = a_t\log\mathbb{E}\left[e^{\sum_{j=1}^k\left(\sum_{i=1}^h c_{ij}\kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu)})\right)L_j^\nu(t)}\right]\\ &= a_t\sum_{j=1}^k\log\mathbb{E}\left[e^{\sum_{i=1}^h c_{ij}\kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu)})L_j^\nu(t)}\right] = a_t\sum_{j=1}^k\log E_\nu\left(\sum_{i=1}^h c_{ij}\kappa_{S_i}(\theta_i/(ta_t)^{1-\alpha_m(\nu)})t^\nu\right).\end{aligned}$$

We start with the case $m = 0$. We have

$$\rho(t) = a_t\sum_{j=1}^k\log E_\nu\left(\sum_{i=1}^h c_{ij}\left(\frac{\kappa''_{S_i}(0)}{2}\frac{\theta_i^2}{(ta_t)^\nu} + o\left(\frac{1}{(ta_t)^\nu}\right)\right)t^\nu\right)$$

and, by equation (2.1), we obtain

$$\lim_{t\to\infty}\rho(t)=\sum_{j=1}^{k}\left(\sum_{i=1}^{h}c_{ij}\frac{\kappa''_{S_i}(0)}{2}\theta_i^2\right)^{1/\nu}.$$

Finally the case $m\neq 0$. We have

$$\rho(t)=a_t\sum_{j=1}^{k}\log E_\nu\left(\sum_{i=1}^{h}c_{ij}\left(\kappa'_{S_i}(0)\frac{\theta_i}{(ta_t)^\nu}+o\left(\frac{1}{(ta_t)^\nu}\right)\right)t^\nu\right)$$

and, by equation (2.1), we obtain (we recall that $\kappa'_{S_i}(0)=m_i$)

$$\lim_{t\to\infty}\rho(t)=\sum_{j=1}^{k}\left(\sum_{i=1}^{h}c_{ij}m_i\theta_i 1_{\{\theta_i m_i\geq 0\}}\right)^{1/\nu}.$$

□

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Data availability statement

The research data associated with this article are included in the article.

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